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Time-Optimal Three-Axis Reorientation of a Rigid Spacecraft

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New results are presented for the minimum-time rest-to-rest reorientation control problem of a rigid spacecraft with independent three-axis control. It is shown that in general the eigenaxis rotation maneuver is not time optimal. An inertially symmetric (e.g., spherical or cubical) rigid body is examined to demonstrate that an eigenaxis rotation about a control axis of even such a simple body is not time optimal. The computed optimal solution is bang-bang in all three control components and has a significant nutational component of motion. The total number of switches is found to be a function of the specified reorientation angle.

Introduction

TIME-OPTIMAL control of dynamical systems is a class of optimization problems of interest in many different research areas.[1-6] The problem of spacecraft reorientation or slewing maneuvers has been extensively studied by several researchers[6-12] with the objective of minimizing the maneuver time, control effort, or structural vibrations.

The so-called Euler's principal axis (eigenaxis) rotation has often been considered as a natural approach to a rapid rotational maneuver.[10,11] In a recent study by Li and Bainum,[12] the eigenaxis rotation about a control axis is claimed to be the time-optimal maneuver for the special case of an inertially symmetric body with independent three-axis control, and the eigenaxis rotation is considered to be generally "near" the time-optimal solution for other cases. This paper investigates the minimum-time reorientation of a rigid, inertially symmetric spacecraft with three independent bounded controls. It will be shown that in general the eigenaxis rotation is not the optimal solution to the problem of minimum-time rest-to-rest reorientation of such a spacecraft. It will also be shown that the computed optimal solution is bang-bang in all three control components and has a significant nutational component of motion. These new results for the minimum-time reorientation control problem do not appear to have been previously published in the open literature.

Li and Bainum[12] studied minimum-time rigid-body slewing maneuvers by using an approximate continuation-type approach to determine control histories and trajectories. The goal of the present work is to analyze this problem using a mathematically rigorous approach and to solve the "exact" time-optimal problem. A simple rigid body, which is inertially symmetric, is used in this study where the objective is to identify the solution structure and the overall nature of the optimal controls and trajectories.

Problem Formulation

The Euler's rotational equations of motion for a rigid body, referenced to body-fixed principal axes with origin at the center of mass, are

$$I_1 \frac{d\Omega_1}{dt} - (I_2 - I_3)\Omega_2\Omega_3 = \tau_1 \tag{1a}$$

$$I_2 \frac{d\Omega_2}{dt} - (I_3 - I_1)\Omega_3\Omega_1 = \tau_2 \tag{1b}$$

$$I_3 \frac{d\Omega_3}{dt} - (I_1 - I_2)\Omega_1\Omega_2 = \tau_3 \tag{1c}$$

In Eqs. (1), Ω_1, Ω_2, and Ω_3 are angular velocity components along the orthogonal body-fixed axes; τ_1, τ_2, and τ_3 are bounded independent control torques about these axes; and I_i $(i = 1, 2, 3)$ are the moments of inertia about the body-fixed axes $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$. A schematic representation is shown in Fig. 1 along with the inertial axes (X, Y, Z).

Equations (1) can be simplified for the special case of an inertially symmetric body by setting $I_i = I_0$. A time-scale factor equal to $\sqrt{I_0/\tau_{\max}}$ is introduced, where I_0 is the moment of inertia of the inertially symmetric body about any principal axis, and $\tau_{\max}$ is the maximum available torque about each control axis. Nondimensional angular velocity components $\omega_i = (\Omega_i\sqrt{I_0/\tau_{\max}})$ and nondimensional control components $u_i = (\tau_i/\tau_{\max})$ are defined. It is clear that $|u_i| \le 1$. The Euler's rotational equations of motion are now written as

$$\dot{\omega}_1 = u_1 \tag{2a}$$

$$\dot{\omega}_2 = u_2 \tag{2b}$$

$$\dot{\omega}_3 = u_3 \tag{2c}$$

In Eqs. (2), the quantities $\dot{\omega}_i$ represent nondimensional angular acceleration components. Figure 2 shows the nondimensional angular velocity components ω_i and the nondimensional control components u_i, along with the body axes $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$.

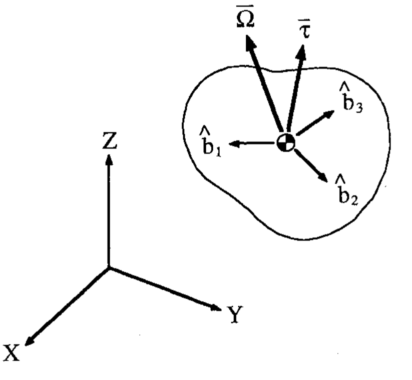

Fig. 1 Inertial and body axes.

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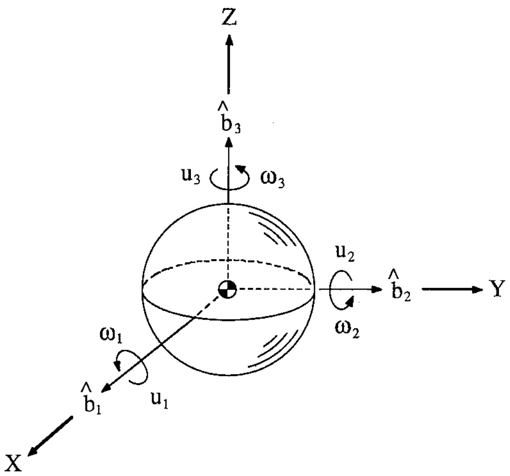

Fig. 2 Inertially symmetric body.

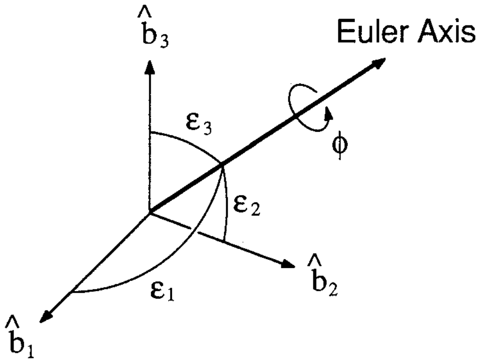

Fig. 3 Euler axis and principal rotation angle.

The kinematic equations of angular motion are

$$\dot{q}_1 = \tfrac{1}{2}(\omega_1 q_4 - \omega_2 q_3 + \omega_3 q_2) \tag{3a}$$

$$\dot{q}_2 = \tfrac{1}{2}(\omega_1 q_3 + \omega_2 q_4 - \omega_3 q_1) \tag{3b}$$

$$\dot{q}_3 = \tfrac{1}{2}(-\omega_1 q_2 + \omega_2 q_1 + \omega_3 q_4) \tag{3c}$$

$$\dot{q}_4 = \tfrac{1}{2}(-\omega_1 q_1 - \omega_2 q_2 - \omega_3 q_3) \tag{3d}$$

where q_1, q_2, q_3, and q_4 are the quaternions used to describe the angular orientation of the spacecraft. In Eqs. (3), the angular velocity components ω_i and the quaternion rates $\dot{q}_i$ are dimensionless. The quaternions are defined by

$$q_i = \cos(\epsilon_i)\sin(\phi/2), \qquad i = 1, 2, 3$$

$$q_4 = \cos(\phi/2) \tag{4}$$

where ϵ_1, ϵ_2, and ϵ_3 are the orientation angles of the Euler axis (i.e., equivalent rotation axis or eigenaxis) relative to the body-fixed reference frame, and ϕ is the principal rotation angle (i.e., the magnitude of the Euler axis rotation), as shown in Fig. 3.

The minimum-time rest-to-rest reorientation problem may be formally stated as follows:

Determine the controls (u_1^*, u_2^*, u_3^*) that drive the dynamic system described by Eqs. (2) and (3) from rest at its initial orientation described by the quaternions $(q_1, q_2, q_3, q_4)_o$ to rest at its final orientation described by the quaternions $(q_1, q_2, q_3, q_4)_f$, while minimizing the cost function

$$J = \int_0^{t_f} \mathrm{d}t \tag{5a}$$

subject to the control constraint

$$-1 \le u_i \le 1, \qquad i = 1, 2, 3 \tag{5b}$$

The Hamiltonian[1] is formed by adjoining the right-hand sides of the state equations, i.e., Eqs. (2a–2c) and (3a–3d), with the appropriate adjoint variables $(\lambda_{\omega_1}, \lambda_{\omega_2}, \lambda_{\omega_3}, \lambda_{q_1}, \lambda_{q_2}, \lambda_{q_3}, \lambda_{q_4})$ and is given by

$$\begin{aligned} H = {} & 1 + \lambda_{\omega_1} u_1 + \lambda_{\omega_2} u_2 + \lambda_{\omega_3} u_3 + \lambda_{q_1} \tfrac{1}{2}(\omega_1 q_4 - \omega_2 q_3 + \omega_3 q_2) \\ & + \lambda_{q_2} \tfrac{1}{2}(\omega_1 q_3 + \omega_2 q_4 - \omega_3 q_1) + \lambda_{q_3} \tfrac{1}{2}(-\omega_1 q_2 + \omega_2 q_1 + \omega_3 q_4) \\ & + \lambda_{q_4} \tfrac{1}{2}(-\omega_1 q_1 - \omega_2 q_2 - \omega_3 q_3) \end{aligned} \tag{6}$$

The adjoint equations are obtained in the standard fashion by differentiating the negative of the Hamiltonian with respect to the states and are given by

$$\dot{\lambda}_{\omega_1} = -\tfrac{1}{2}(\lambda_{q_1} q_4 + \lambda_{q_2} q_3 - \lambda_{q_3} q_2 - \lambda_{q_4} q_1) \tag{7a}$$

$$\dot{\lambda}_{\omega_2} = -\tfrac{1}{2}(-\lambda_{q_1} q_3 + \lambda_{q_2} q_4 + \lambda_{q_3} q_1 - \lambda_{q_4} q_2) \tag{7b}$$

$$\dot{\lambda}_{\omega_3} = -\tfrac{1}{2}(\lambda_{q_1} q_2 - \lambda_{q_2} q_1 + \lambda_{q_3} q_4 - \lambda_{q_4} q_3) \tag{7c}$$

$$\dot{\lambda}_{q_1} = -\tfrac{1}{2}(-\lambda_{q_2}\omega_3 + \lambda_{q_3}\omega_2 - \lambda_{q_4}\omega_1) \tag{8a}$$

$$\dot{\lambda}_{q_2} = -\tfrac{1}{2}(\lambda_{q_1}\omega_3 - \lambda_{q_3}\omega_1 - \lambda_{q_4}\omega_2) \tag{8b}$$

$$\dot{\lambda}_{q_3} = -\tfrac{1}{2}(-\lambda_{q_1}\omega_2 + \lambda_{q_2}\omega_1 - \lambda_{q_4}\omega_3) \tag{8c}$$

$$\dot{\lambda}_{q_4} = -\tfrac{1}{2}(\lambda_{q_1}\omega_1 + \lambda_{q_2}\omega_2 + \lambda_{q_3}\omega_3) \tag{8d}$$

Addressing the issue of boundary conditions, it is clear that the adjoint variables will be "free" (i.e., unspecified) at both the initial and final times since the corresponding state variables are specified. The set of quaternion adjoint variables $(\lambda_{q_1}, \lambda_{q_2}, \lambda_{q_3}, \lambda_{q_4})$ is not unique; hence any one of these variables can be normalized to unity at the initial (or final) time.

For the problem under consideration, the Hamiltonian is constant along an optimal trajectory, and from the transversality condition[1] its value along an optimal trajectory is given by

$$H^* \equiv H^*(t_f) = 0 \tag{9}$$

Optimal Controls

According to the Minimum Principle,[1] the optimal controls must minimize the Hamiltonian. Applying this condition to a time-optimal problem with linearly appearing bounded controls, one first defines the switching function S_i associated with the control component u_i as follows:

$$S_i = \frac{\partial H}{\partial u_i}, \qquad i = 1, 2, 3 \tag{10}$$

The optimal control logic is then given by the following relationships:

$$u_i^* = +1, \qquad \text{if } S_i < 0 \tag{11a}$$

$$u_i^* = -1, \qquad \text{if } S_i > 0 \tag{11b}$$

$$u_i^* = u_{i_s}, \qquad \text{if } S_i \equiv 0 \tag{11c}$$

where $-1 < u_{i_s} < +1$. In Eq. (11c), u_{i_s} is a singular control component defined on the finite time interval over which the corresponding switching function S_i is identically zero.

Singular Controls

Consider the switching functions S_i defined in Eq. (10). If a singular control component exists, it may be determined[13] by differentiating the corresponding switching function S_i with respect to time until the appropriate control component appears and then solving

$$u_{i_s} = \arg\left[\left(\frac{\mathrm{d}^{(2k_i)}S_i}{\mathrm{d}t^{(2k_i)}}\right) = 0\right] \tag{12}$$

If $2k_i$ differentiations of the switching function are required, then the corresponding singular control component is of order k_i. The switching functions and their time derivatives are required to be zero along a singular subarc. Therefore,

$$\left[\frac{\mathrm{d}^{(j)}S_i}{\mathrm{d}t^{(j)}}\right] = 0, \qquad j = 0, 1, \ldots, (2k_i - 1) \tag{13}$$

For the problem under consideration, there are three independent control components, and hence the following control structures are possible at any given instant of time:

1) All three control components are saturated.
2) Two control components are saturated and one control component is singular.
3) One control component is saturated and two control components are singular.
4) All three control components are singular.

It will be shown here that control structure 4 is not optimal for the problem under study, i.e., all three control components cannot be simultaneously singular at any point on an optimal path.

Substituting Eq. (6) into Eq. (10) and differentiating, one obtains

$$S_i = \lambda_{\omega_i}, \qquad i = 1, 2, 3 \tag{14}$$

$$\dot{S}_i = \dot{\lambda}_{\omega_i}, \qquad i = 1, 2, 3 \tag{15}$$

$$\ddot{S}_i = \ddot{\lambda}_{\omega_i}, \qquad i = 1, 2, 3 \tag{16}$$

Differentiating Eqs. (7) with respect to dimensionless time and using Eqs. (3) and (8) yield a result that may be expressed as[5]

$$\ddot{\lambda}_{\omega_1} = -\dot{\lambda}_{\omega_3}\omega_2 + \dot{\lambda}_{\omega_2}\omega_3 \tag{17a}$$

$$\ddot{\lambda}_{\omega_2} = \dot{\lambda}_{\omega_3}\omega_1 - \dot{\lambda}_{\omega_1}\omega_3 \tag{17b}$$

$$\ddot{\lambda}_{\omega_3} = -\dot{\lambda}_{\omega_2}\omega_1 + \dot{\lambda}_{\omega_1}\omega_2 \tag{17c}$$

Using Eqs. (15) and (17) in Eq. (16), one obtains

$$\begin{bmatrix} \ddot{S}_1 \\ \ddot{S}_2 \\ \ddot{S}_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{S}_1 \\ \dot{S}_2 \\ \dot{S}_3 \end{bmatrix} \tag{18}$$

Consider a subarc that is first-order singular in all three control components. Along such a subarc, the first time derivatives of all three switching functions must be zero as required by Eq. (13). Substitution of this requirement into Eq. (18) indicates that the second time derivatives of all three switching functions are intrinsically zero, and hence setting them equal to zero will not yield first-order singular optimal controls from Eq. (12). From the structure of Eq. (18) and the requirements of Eq. (13), it is clear that all higher time derivatives of the switching functions will also be intrinsically zero and cannot yield finite order singular optimal controls. Infinite order singular subarcs can be ruled out if Euler angles are used instead of quaternions to describe the orientation of the body.[5] The nonoptimality of a subarc that is simultaneously singular in all three control components has been established. Therefore, it may be concluded that, for the problem under study, *at least* one control component must be saturated at any given instant of time.

In geometric terms, the control constraint given by Eq. (5b) may be visualized as a unit cube in the three-dimensional control space. The control constraint requires that an admissible control vector lie inside or on the surface of this cube. From the discussion of the preceding paragraph, it is clear that the optimal control vector cannot lie inside the cube since this would imply that all three components of the optimal control vector are simultaneously singular. Therefore, the optimal control vector must lie *on* the cubical constraint surface.

Numerical Approach—Modified Control Constraint

The approach taken in this study is to investigate the bang-bang type control structure (i.e., all three control components saturated), while monitoring the switching functions for the possible occurrence of singular subarcs in one or two of the control components. A multiple-shooting algorithm[14] that solves two-point boundary-value problems is used along with the state-adjoint equations and the appropriate control logic to determine candidate optimal trajectories and controls.

Since no assumptions are made about the control structure, the structure itself needs to be determined. This is accomplished by using a modified control constraint approach. In this approach, one starts with a modified (and smooth) constraint surface characterized by a parameter ζ. In mathematical terms, the modified control constraint surface is represented by

$$|u_1|^{2\zeta} + |u_2|^{2\zeta} + |u_3|^{2\zeta} = 1, \qquad \zeta \geq 1 \tag{19}$$

Note that with $\zeta = 1$ the surface is a sphere, and with $\zeta \to \infty$ the surface approaches a cube (i.e., the original control constraint surface). This approach provides a "limit passage" from a modified problem to the original problem.[15]

To obtain a solution to the modified problem, the Hamiltonian given by Eq. (6) must be minimized subject to the modified control constraint described by Eq. (19). Because of the nature of this control constraint, there are only two independent control components, say u_1 and u_2. Hence

$$u_3 = \mathrm{sgn}(u_3)\left[1 - |u_1|^{2\zeta} - |u_2|^{2\zeta}\right]^{\frac{1}{2\zeta}} \tag{20}$$

Substituting Eq. (20) into Eq. (6), evaluating the partials of the Hamiltonian with respect to u_1 and u_2, and setting them equal to zero yield

$$u_1 = \left[\mathrm{sgn}\left(\frac{\lambda_{\omega_1}}{\lambda_{\omega_3}}\right)\left|\frac{\lambda_{\omega_1}}{\lambda_{\omega_3}}\right|^{\frac{1}{2\zeta-1}}\right] u_3 \tag{21a}$$

$$u_2 = \left[\mathrm{sgn}\left(\frac{\lambda_{\omega_2}}{\lambda_{\omega_3}}\right)\left|\frac{\lambda_{\omega_2}}{\lambda_{\omega_3}}\right|^{\frac{1}{2\zeta-1}}\right] u_3 \tag{21b}$$

Substituting Eqs. (21) into Eq. (20) and solving, one obtains

$$|u_3| = \frac{|\lambda_{\omega_3}|^{\frac{1}{2\zeta-1}}}{\left[|\lambda_{\omega_1}|^{\frac{2\zeta}{2\zeta-1}} + |\lambda_{\omega_2}|^{\frac{2\zeta}{2\zeta-1}} + |\lambda_{\omega_3}|^{\frac{2\zeta}{2\zeta-1}}\right]^{\frac{1}{2\zeta}}} \tag{22}$$

To determine the sign of u_3, the Legendre-Clebsch condition[1] is examined. For the problem under consideration, this condition requires that the 2×2 matrix of second partials of the Hamiltonian with respect to the controls u_1 and u_2 be positive semidefinite. It can be shown that this requirement is satisfied if

$$\mathrm{sgn}(u_3) = -\mathrm{sgn}(\lambda_{\omega_3}) \tag{23}$$

Combining Eqs. (22) and (23) and substituting into Eqs. (21) yield the minimizing control components:

$$u_i^* = \frac{-\mathrm{sgn}(\lambda_{\omega_i})|\lambda_{\omega_i}|^{\frac{1}{2\zeta-1}}}{\left[|\lambda_{\omega_1}|^{\frac{2\zeta}{2\zeta-1}} + |\lambda_{\omega_2}|^{\frac{2\zeta}{2\zeta-1}} + |\lambda_{\omega_3}|^{\frac{2\zeta}{2\zeta-1}}\right]^{\frac{1}{2\zeta}}}, \quad i = 1, 2, 3 \tag{24}$$

The state-adjoint equations can now be solved by a multiple-shooting algorithm using the control logic of Eq. (24) and the appropriate boundary conditions. Increasing the parameter ζ will yield control histories that approach the bang-bang (or possibly piecewise singular) control structure associated with the original cubical constraint surface. Once this control structure has been established, the bang-bang (or possibly piecewise singular) control components may be computed using a multiple-shooting algorithm and the control logic of Eqs. (11).

Results and Discussion

The maneuver addressed in this study is a minimum-time rest-to-rest reorientation that provides a *net* change in yaw angle, with the spacecraft yaw axis parallel to the inertial Z axis at the start and end of the maneuver. Mathematically, this requirement translates into the following boundary conditions:

$$\omega_1(0) = \omega_2(0) = \omega_3(0) = 0$$

$$q_1(0) = q_2(0) = q_3(0) = 0, \qquad q_4(0) = 1$$

$$\omega_1(t_f) = \omega_2(t_f) = \omega_3(t_f) = 0$$

$$q_1(t_f) = q_2(t_f) = 0, \qquad q_3(t_f) = \sin\phi/2, \qquad q_4(t_f) = \cos\phi/2$$

The angle ϕ is the principal rotation angle; in this study it is also called the reorientation angle.

For these boundary conditions, the Euler axis is aligned with the principal body axis $\hat{b}_3$ (see Fig. 2), and the eigenaxis (equivalent) rotation is about the spacecraft yaw axis. It is emphasized that the actual rotation is not required to be performed about the yaw axis. The objective here is to determine the true minimum-time maneuver and its characteristics.

Initial investigations focused on a reorientation angle of 180 deg. To determine the control strucure, the modified control constraint was used with several values of the parameter ζ ranging from 1 to 14. After extensive computational work, including guess trajectories, trial and error, and continuation

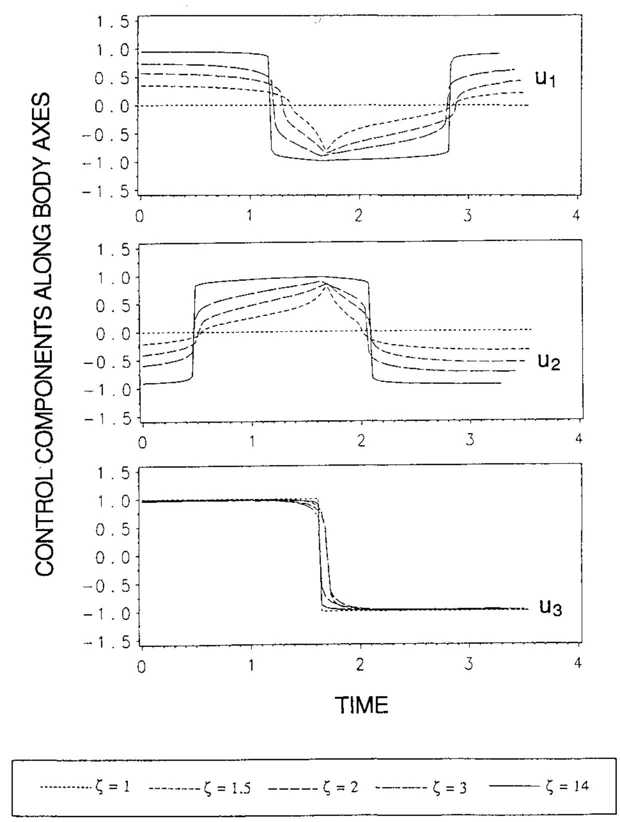

Fig. 4 Control histories with modified control constraint (ζ = 1–14).

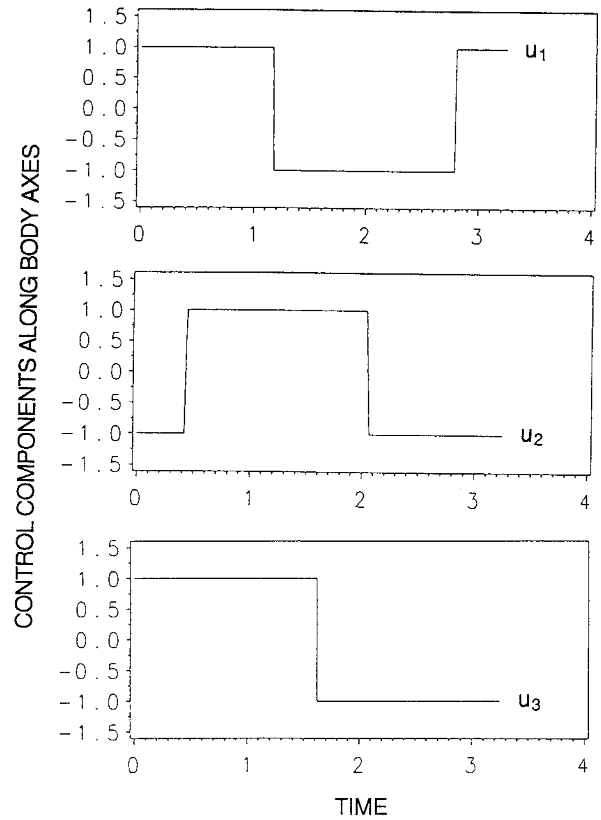

Fig. 5 Minimum-time control histories along body axes for ϕ = 180 deg.

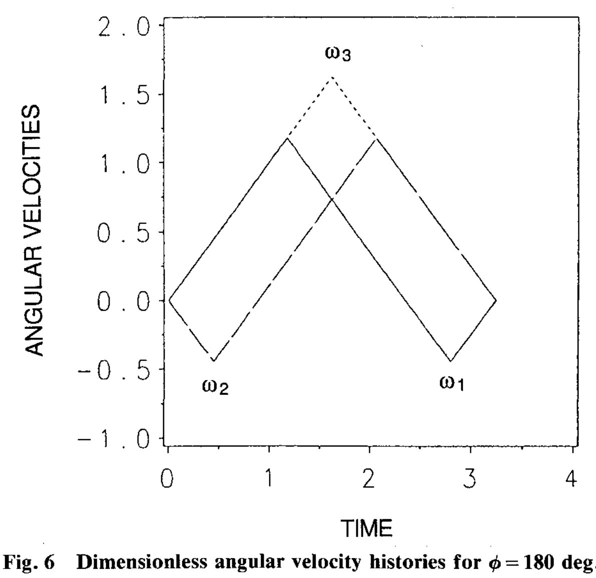

Fig. 6 Dimensionless angular velocity histories for ϕ = 180 deg.

approaches, a family of solutions was obtained. The computed control histories are presented in Fig. 4 for selected values of the parameter ζ.

It is parenthetically noted that, for the case where $\zeta = 1$, the modified control constraint surface is a sphere, and the analysis of Athans and Falb[16] on time-optimal problems with control vector constrained to a hypersphere is applicable here. Reference 16 (p. 819) states that, because of the required continuity of the adjoint variables, "a very important property of the time-optimal control ... is that its components ... are continuous functions of time." However, Fig. 4 shows that, for $\zeta = 1$, the control components u_1 and u_2 are identically zero (and continuous) but u_3 is bang-bang and discontinuous. This anomaly occurs because the adjoint variables λ_{ω_1}, λ_{ω_2}, and λ_{ω_3} are exactly zero at the switching time, providing an exception to the "rule" cited in Ref. 16.

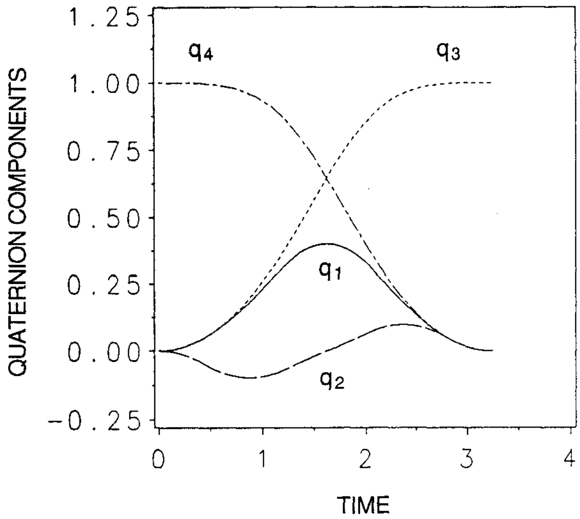

Fig. 7 Quaternion histories for $\phi = 180$ deg.

From Fig. 4 it is clear that, as $\zeta \to \infty$, all control components will approach a bang-bang type structure with u_1 and u_2 having two switches each and u_3 having a single switch. It is also evident that there will be no singular subarcs in any of the control components. Using the switching logic of Eqs. (11a) and (11b), the solution associated with the original control constraint of Eq. (5b) was computed. The bang-bang control histories are shown in Fig. 5, and the corresponding dimensionless angular velocity and quaternion histories are shown in Figs. 6 and 7, respectively. The switching functions S_i were examined and were found to have signs consistent with the requirements of Eqs. (11a) and (11b). It was also observed that the slopes of the switching functions were nonzero at the switching points; this feature is consistent with the requirements of "pure" bang-bang controls with no singular subarcs.

To gain an understanding of the geometry of the motion, angles α, β, and γ (see Fig. 8a) were computed from the quaternions and plotted as functions of time (Fig. 8b). These angles represent the orientations of the body axes with respect to the inertial axes at any given instant of time; they are not the standard Euler angles. The angles are defined by the following expressions:

$$\alpha = \cos^{-1}(1 - 2q_2^2 - 2q_3^2) \tag{25a}$$

$$\beta = \cos^{-1}(1 - 2q_3^2 - 2q_1^2) \tag{25b}$$

$$\gamma = \cos^{-1}(1 - 2q_1^2 - 2q_2^2) \tag{25c}$$

Consider the eigenaxis rotation maneuver, performed about the spacecraft yaw axis, that yields the desired final orientation. This maneuver would require the control components u_1 and u_2 to be identically zero (i.e., singular) and u_3 to be bang-bang with one switch. Also, the maneuver would be performed with the body axis $\hat{b}_3$ aligned with the inertial Z axis ($\gamma = 0$) at all times and with angle α equal to angle β at all times. Figures 5 and 8b clearly show that the computed optimal maneuver is very different from the eigenaxis rotation maneuver, both in terms of the geometry of the motion and the controls required to command this motion. A noteworthy feature of the computed optimal maneuver is the significant nutational component of motion evident from the large values of the angle γ in Fig. 8b.

The minimum time required to perform an eigenaxis rotation about the control axis of an inertially symmetric body is simply $(2\sqrt{\phi})$ dimensionless-time units, where ϕ is the reorientation angle. For the reorientation maneuver under study, ϕ is equal to 180 deg, and the eigenaxis rotation time is 3.545 dimensionless-time units. The time required for the computed optimal maneuver is 3.243 dimensionless-time units, which

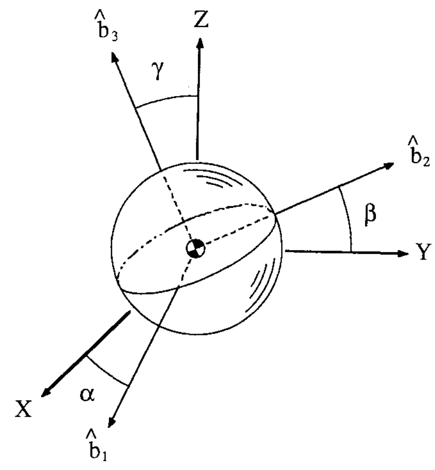

Fig. 8a Definition of orientation angles α, β, and γ.

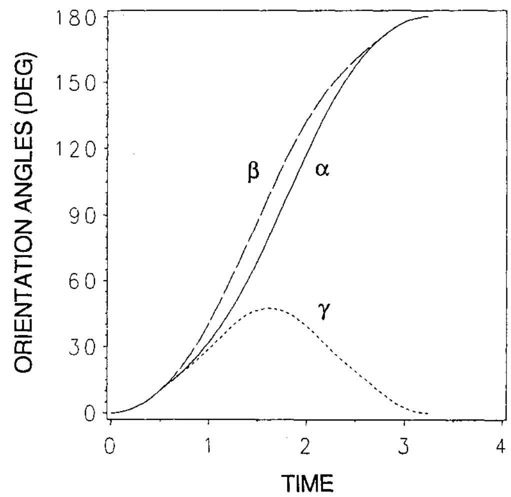

Fig. 8b Orientation angle histories for $\phi = 180$ deg.

is a reduction of 8.514%. From the preceding discussion, one may conclude that in general the eigenaxis rotation maneuver is not time optimal.

Why is the computed optimal maneuver faster than the eigenaxis rotation maneuver? In an attempt to answer this question, control components u_X, u_Y, and u_Z along the inertial axes were computed and plotted as shown in Fig. 9. Recalling that the desired reorientation is a *net* change in yaw angle with the spacecraft yaw axis parallel to the inertial Z axis at the start and end of the maneuver, the control component u_Z is examined. Figure 9b shows that this control component achieves values significantly in excess of 1.0 over a substantial portion of the total time interval (region A). Although there is a small time interval where u_Z is less than 1.0 (region B), it is clear that region B is significantly smaller than region A; this feature is sufficient (but not necessary) to insure a shorter maneuver time. It appears that the computed optimal motion with its strong nutational component is able to provide more torque along the inertial Z axis and can therefore complete the desired maneuver in less time.

The investigation was continued, using reorientation angle ϕ values of 135 and 90 deg. It was found that the control structure and the motion characteristics remained the same, and the computed optimal reorientation times remained less than the times for the corresponding eigenaxis rotations. The percent difference between these times decreased with reorientation angle ϕ as shown in Table 1.

The same trends were observed for smaller reorientation angles up to and including 73 deg. A further decrease in reorientation angle to 72 deg resulted in a different and more complex control structure, which then remained unchanged for

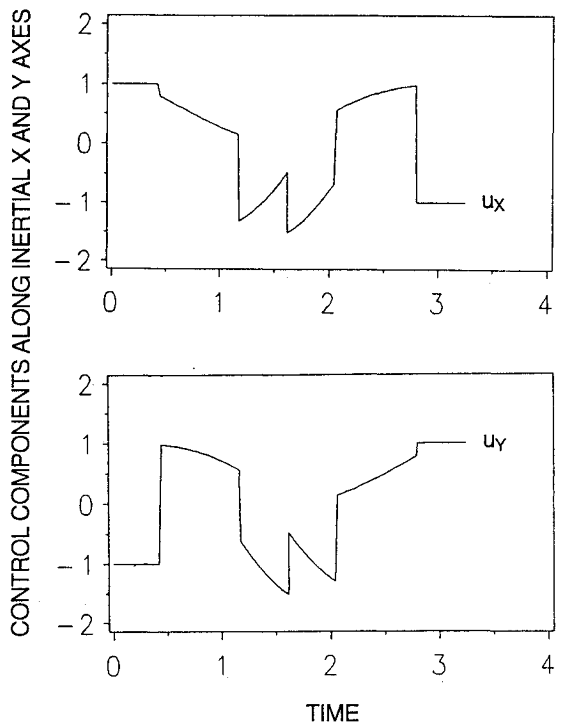

Fig. 9a Minimum-time control histories along inertial X and Y axes for $\phi = 180$ deg.

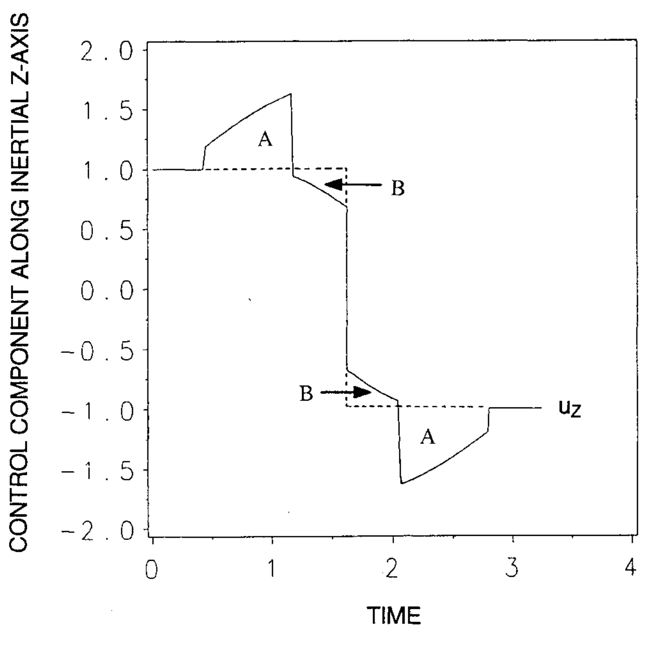

Fig. 9b Minimum-time control history along inertial Z axis for $\phi = 180$ deg.

Table 1 Comparison of reorientation times for eigenaxis rotation and optimal maneuver

ϕ, deg	Eigenaxis	Optimal	Switches	Reduction, %
180	3.5449	3.2431	5	8.514
135	3.0700	2.8845	5	6.041
90	2.5066	2.4211	5	3.411
73	2.2575	2.2024	5	2.442
72	2.2420	2.1885	7	2.386
45	1.7724	1.7499	7	1.274
10	0.8355	0.8334	7	0.257
1	0.2642	0.2641	7	0.025

Note: Dimensionless time may be multiplied by the scale factor $\sqrt{I_0/\tau_{max}}$ to convert to units of seconds.

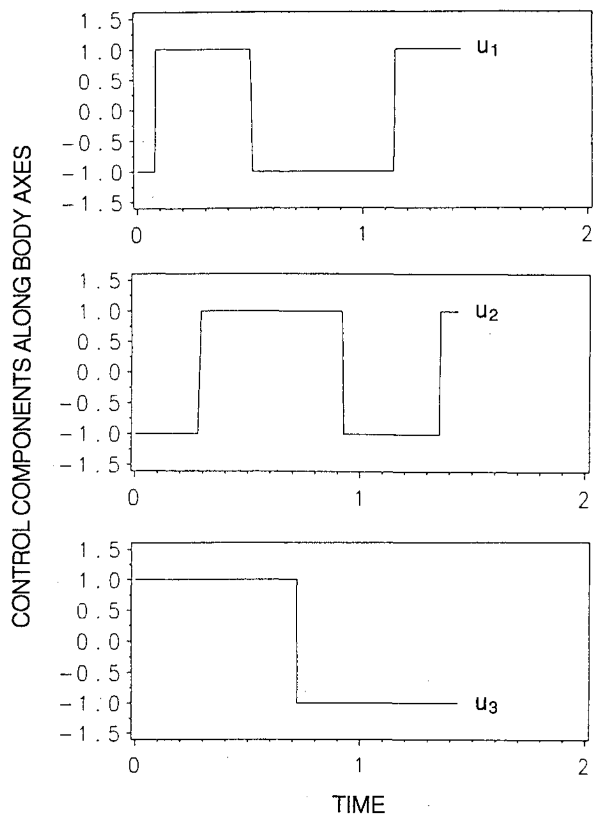

Fig. 10 Minimum-time control histories along body axes for $\phi = 30$ deg.

smaller angles (computations were made for reorientation angles as small as 0.001 deg). Figure 10 shows the control components for a reorientation angle of 30 deg. It is observed that the control components u_1 and u_2 now exhibit three switches each, whereas u_3 has a single switch as before. The new control structure and motion characteristics continue to be significantly different from the eigenaxis rotation maneuver, and the computed optimal maneuver times are also correspondingly smaller as shown in Table 1.

The control structures that provide solutions to the optimal control problem under study are not unique; a finite number of additional control structures may be obtained by simple permutations of the primary control structure. (The question of whether additional solutions can be constructed by interchanging components of the primary solution was posed by one of the reviewers.) Let the primary control structure, shown in Fig. 5, be represented by ($U1$, $U2$, $U3$). For maneuvers involving five control switches and reorientation angles less than 180 deg, there are four possible control structures: ($U1$, $U2$, $U3$), ($-U1$, $-U2$, $U3$), ($-U2$, $U1$, $U3$), and ($U2$, $-U1$, $U3$). For the special case where the reorientation angle is equal to 180 deg, there are eight possible control structures: the four structures just described plus ($U1$, $-U2$, $-U3$), ($-U1$, $U2$, $-U3$), ($U2$, $U1$, $-U3$), and ($-U2$, $-U1$, $-U3$).

A limited number of additional studies were conducted for reorientations about axes other than the control axes, specifically for axes in the vicinity of the spacecraft yaw axis. Again, the eigenaxis rotation maneuvers were found to be nonoptimal, and the computed optimal motion and controls had the same general characteristics as before. It was found, however, that the eigenaxis rotation maneuver *is* time optimal for reorientations about an axis that makes equal angles with the control axes.

Conclusions

The minimum-time reorientation problem has been studied for an inertially symmetric rigid spacecraft with independent three-axis controls. It has been analytically established that all three control components cannot be simultaneously singular at any instant during a minimum-time maneuver. Time-optimal solutions were computed for several maneuvers corresponding to reorientations about a control axis. The computed optimal controls and motion geometry are very different from those corresponding to an eigenaxis rotation about the control axis, and the maneuver time is less than the eigenaxis rotation time. Therefore, in general, the eigenaxis rotation maneuver is not time optimal. The geometric nature of the computed optimal motion is quite complex and has a significant nutational component. It appears that this complex motion is able to provide more torque along the desired inertial axis of reorientation and can therefore complete the maneuver in less time. Two control structures were discovered: one for large reorientation angles ranging from 180 to approximately 73 deg and another for reorientation angles ranging from approximately 72 deg to very small angles.

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