JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 16, No. 3, May–June 1993

Quasi-Closed-Form Solution to the Time-Optimal Rigid Spacecraft Reorientation Problem

R. M. Byers*
University of Central Florida, Orlando, Florida 32816
and
S. R. Vadali†
Texas A&M University, College Station, Texas 77843

The problem of slewing a rigid body from an arbitrary initial orientation to a desired target orientation in minimum time is addressed. The nature of the time-optimal solution is observed via an open-loop solution using the switch time-optimization algorithm developed by Meier and Bryson. Conclusions as to the number and timing of control switches are drawn and substantiated analytically. The solution of the kinematic differential equations for Euler parameters is examined for systems in which the applied torque is much greater than the nonlinear terms in Euler's equations. An approximate solution to these equations is used to construct the state transition matrix as a function of a given control sequence and control intervals. This allows a rapid solution for the required switch times for all admissible control sequences. Uncoupled switching functions can be generated given the approximate switch times for the optimal sequence. The resulting feedforward/feedback control is suitable for online computation.

Introduction

THE problem of transferring a dynamical system, initially at rest, from an arbitrary initial attitude to a desired target attitude, also at rest, in minimum time is of fundamental interest as an optimal control problem. In particular, several current and anticipated spacecraft have a requirement to perform such maneuvers.

The necessary conditions for optimality yield a two point boundary-value problem (TPBVP), which, when solved, gives the control history that minimizes the specified performance index. When elapsed time is the performance measure to be minimized, the necessary conditions for the rest-to-rest maneuver typically dictate that the control be saturated throughout the duration of the maneuver; the controls are "bang-bang" in nature with instantaneous switches between extreme admissible (saturation) values. Although not exactly realizable in physical systems, such controls are approximated by spacecraft equipped with nonthrottleable thrusters that operate in on/off modes.

Substantial attention has been paid to time-optimal controls of linear systems in the literature. Several authors[1-3] have examined the single-axis time-optimal rotation problem for a rigid body. In the single-axis case, the state-space trajectory is determined to be parabolic. It is found that a maximum of one control switch is necessary to intercept a trajectory which passes through the origin of the state-space. The equation for the switching line can be found analytically without numerically solving a TPBVP.

The multiaxis attitude/angular velocity control problem for an arbitrary rigid body has not been amenable to a simple closed-form solution. Closed-loop schemes which use bang-bang controls have generally depended on some constraint on the maneuver path. Vadali and Kobberdahl[4] develop robust switching functions for multiaxis rotational maneuvers, using "sliding" controls but these are not time optimal. Etter[5] presents a feedback control for rotation about a fixed axis. The so-called "Euler axis" rotation is also used as the basis for the bang-bang feedforward/feedback control by Redding and Adams.[6] Euler axis rotations are used for other nontime-optimal controls by Vadali,[7] Wie et al.,[8] and D'Amario and Stubbs.[9]

Li and Bainum[10] present a numerical method, using a quadratic performance index, whereby the final time is iteratively reduced until the controls are saturated. Similarly Ercoli-Finzi et al.[11] attempt to solve the minimum time problem using a Newton-Raphson iterative technique. Neither method gives true bang-bang controls and the computation burden associated with these methods exceeds the capability of onboard processors. Chowdhry and Cliff[12] study the nature of minimum time controls in the absence of direct control over one of the angular velocity components. Bilimoria and Wie[13] numerically obtain the time-optimal solution for a spherical body with three-axis control performing rest-to-rest rotations which can be otherwise accomplished by a rotation about a control axis. They make the observation that the optimal rotation is not about a fixed axis, i.e., Euler axis rotations are not time-optimal in general. They determine that either five or seven control switches are necessary for the minimum time solution for rest-to-rest maneuvers of a spherical body, depending on the magnitude of the desired rotation.

In the present paper, the nature of time-optimal rest-to-rest reorientation is examined. For bodies which have near-linear dynamics, a piecewise solution to the state transition matrix is generated. This allows estimation of the optimal control switch times in an open-loop fashion. The open-loop approximate solution in turn permits the synthesis of uncoupled switching functions that are essential for constructing feedback control laws. This technique is computationally straightforward and can be executed in real time.

Equations of Motion

The differential equations for the angular velocity of a rigid body are given by

$$\dot{\underline{\omega}} = -\boldsymbol{I}^{-1}(\underline{\omega} \times \boldsymbol{I}\underline{\omega}) + \boldsymbol{I}^{-1}\boldsymbol{B}\underline{u} \tag{1}$$

where $\boldsymbol{I}$ is the rigid body inertia matrix, $\boldsymbol{B}$ the control influence matrix, $\underline{\omega}$ the 3×1 angular velocity and $\underline{u}$ the $m \times 1$ control

Received April 30, 1991; revision received July 5, 1992; accepted for publication July 21, 1992.

*Assistant Professor, Department of Mechanical and Aerospace Engineering. Member AIAA.

†Associate Professor, Department of Aerospace Engineering. Associate Fellow AIAA.

vector. The normalized control is constrained such that

$$|u_i| \leq 1 \qquad i = 1, 2, 3, \ldots, m \tag{2}$$

The rigid-body attitude is expressed in terms of the 4×1 vector of Euler parameters $\underline{\beta} = [\beta_1\ \beta_2\ \beta_3\ \beta_0]^T$ which are defined by

$$\beta_0 = \cos(\theta/2)$$

$$\beta_i = l_i \sin(\theta/2) \qquad i = 1, 2, 3 \tag{3}$$

where $\underline{l}$ is a unit vector about which a rotation by an angle θ will align a body-fixed frame with a space-fixed frame. Euler parameters have many advantages as attitude coordinates, the most noteworthy being the absence of singularities. Their differential equations are given by

$$\dot{\underline{\beta}} = \tfrac{1}{2} \boldsymbol{G}(\underline{\omega}) \underline{\beta} \tag{4}$$

where

$$\boldsymbol{G}(\underline{\omega}) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix} \tag{5}$$

To render the problem tractable, only systems for which

$$\|\underline{\omega} \times \boldsymbol{I}\underline{\omega}\| \ll \|\boldsymbol{B}\underline{u}\| \tag{6}$$

are considered. The dynamical equations may then be approximated by

$$\dot{\underline{\omega}} \approx \boldsymbol{K}\underline{u} \tag{7}$$

where $\boldsymbol{K} = \boldsymbol{I}^{-1}\boldsymbol{B}$. For the purpose of this paper, $\underline{u}$ is considered to be a 3×1 vector. However, $\boldsymbol{K}$ need not be diagonal.

Time-Optimal Control Switching

The linear single-axis minimum time control problem is well understood. The equation of motion is

$$\ddot{\theta} = Ku \qquad |u| \leq 1 \tag{8}$$

The rigid-body states can be brought to the state-space origin with a maximum of one control switch. The final time for the rest-to-rest case can be computed in closed form

$$t_f = \sqrt{|\theta_0|/K} \tag{9}$$

where θ_0 is the initial angular displacement. The feedback control law is given by:

$$u = \begin{cases} -\operatorname{sign}(s) \\ -\operatorname{sign}(\dot{\theta}), & \text{if } s = 0 \end{cases} \tag{10}$$

where

$$s = \theta + \frac{\dot{\theta}|\dot{\theta}|}{2K} = 0 \tag{11}$$

is the parabolic switching line passing through the state-space origin.

The multiaxis case is extremely nonlinear even for a spherical body due to the kinematic coupling between the axes and must be solved numerically. Meier and Bryson[14] developed the switch time-optimization (STO) algorithm, a first-order gradient method, to arrive at time-optimal control histories for a two-link manipulator. Herein, it is applied to the rigid-body rotation problem. STO achieves computational efficiencies by assuming that the controls are always saturated. Rather than computing an incremental control over the entire maneuver period, only the times at which control switches occur are iteratively adjusted. The initial guess is obtained via a first-order gradient method using the performance index

$$J = \int_0^{t_f} \left\{ 1 + C \sum_{i=1}^{m} (u_i^2 - 1)^2 \right\} \mathrm{d}t \tag{12}$$

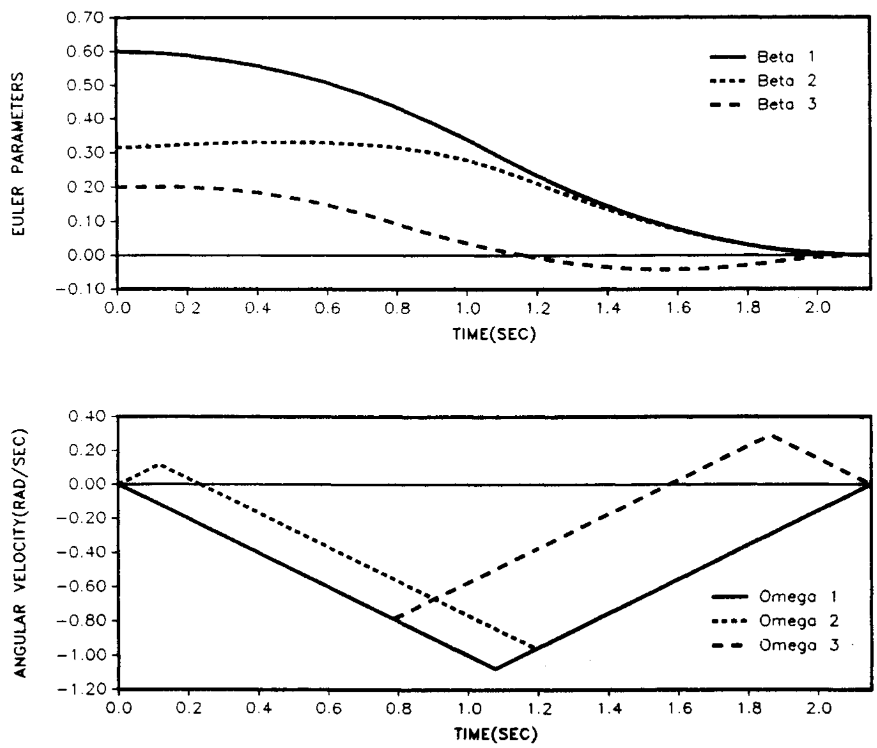

Fig. 1 90-deg rest-to-rest maneuver ($t_f = 2.153$ s).

which penalizes the control for values of u_i other than ± 1. C is a positive scalar.

Consider a spherical body for which $K = KI_D$, with diagonal inertia and control influence matrices, where $K = B/I$ and I_D is the 3×3 identity matrix. A typical time-optimal, rest-to-rest maneuver with $K = 1$ is shown in Fig. 1. In this example the initial Euler parameters are $\underline{\beta} = [0.6\ 0.316228\ 0.2\ 0.707107]^T$. The final time is $t_f = 2.153$ s.

From extensive numerical studies, it is observed that, in general, five control switches are required for the rest-to-rest maneuver. Furthermore, the controls switch sequentially; no control switches a second time until all other controls have switched. These two attributes are interdependent as illustrated subsequently.

From the necessary conditions it is observed that the control vector $\underline{u}$ is constant at the saturation limit between switches. This leads to the angular velocity vector being a linear function of time between control switches

$$\underline{\omega}(t) = K\underline{u}[t - t_0] + \underline{\omega}(0) \tag{13}$$

where t_0 is the time of the previous control switch and $\underline{\omega}_0$ is the angular velocity vector evaluated at $t = t_0$. The following notational convention is adopted

$$\Delta_i = t_i - t_{i-1}$$

$$\underline{\omega}_i = \underline{\omega}(t = t_i)$$

$$\underline{u}_i = \underline{u}(t_i > t > t_{i-1})$$

$$\underline{v}_i = K\underline{u}_i$$

where t_i are the instants of time at which switching occurs. For $K = KI_D$, the angular velocity at $t = t_i$ is

$$\begin{aligned}\underline{\omega}_i &= K\underline{u}_i\Delta_i + \underline{\omega}_{i-1} \\ &= \sum_{j=1}^{i} \underline{v}_j\Delta_j\end{aligned} \tag{14}$$

The squared norm of the angular velocity at $t = t_i$ is given by

$$\|\underline{\omega}_i\|^2 = K^2\langle \sum_{j=1}^{i} \underline{u}_j\Delta_j,\ \sum_{j=1}^{i} \underline{u}_j\Delta_j \rangle \tag{15}$$

where $\langle\ ,\ \rangle$ is the inner product operator. Sequential switching yields the inner product values

$$\langle \underline{u}_i, \underline{u}_i \rangle = 3$$

$$\langle \underline{u}_i, \underline{u}_{i-1} \rangle = 1$$

$$\langle \underline{u}_i, \underline{u}_{i-2} \rangle = -1$$

$$\langle \underline{u}_i, \underline{u}_{i-3} \rangle = -3$$

$$\langle \underline{u}_i, \underline{u}_{i-4} \rangle = -1$$

$$\langle \underline{u}_i, \underline{u}_{i-5} \rangle = 1$$

$$\langle \underline{u}_i, \underline{u}_{i-6} \rangle = 3$$

Letting $c = 1/K^2$ and examining Eq. (15) at the switch times $t_1, t_2, \ldots, t_{ns}$, where ns is not known a priori, it is found that, except for the trivial case of $\Delta_i \equiv 0$ for $i \leq 5$, the angular velocity cannot be brought to zero. Furthermore $\|\underline{\omega}_3\| > \|\underline{\omega}_2\| > \|\underline{\omega}_1\|$ for all $\Delta_i > 0$. However, at $t = t_6$

$$\begin{aligned}c\|\underline{\omega}_6\|^2 = 3\big[&(\Delta_1 - \Delta_4)^2 + (\Delta_2 - \Delta_5)^2 + (\Delta_3 - \Delta_6)^2\big] \\ &+ 2\{(\Delta_1 - \Delta_4)[(\Delta_2 - \Delta_5) + (\Delta_6 - \Delta_3)] \\ &+ (\Delta_2 - \Delta_5)(\Delta_3 - \Delta_6)\}\end{aligned}$$

Table 1 Maneuver times for control-axis reorientations

Initial angle, deg	Maneuver times, s: 5 switches	7 switches (Ref. 13)	Euler rotation
90	2.4211	—	2.5066
45	1.7539	1.7499	1.7724
10	0.8351	0.8334	0.8355

Clearly, besides the trivial case where $\Delta_i \equiv 0$, the magnitude of the angular velocity can be brought to zero if the following symmetry is imposed:

$$\begin{aligned}\Delta_i &= \Delta_{i+3} \\ \underline{u}_i &= -\underline{u}_{i+3} \qquad i = 1, 2, 3\end{aligned} \tag{16}$$

That is, a minimum of six sequential control intervals, separated by five control switches are required for the general rest-to-rest maneuver. Two of the controls will switch twice and one will switch only once at $t_3 = \Delta_1 + \Delta_2 + \Delta_3 = t_f/2$. Although it is intuitively satisfying to assume that the general minimum time rest-to-rest maneuver solution requires the minimum number of control switches, namely five, this assumption cannot be rigorously justified.

There exists a special case for which fewer than five switches are observed. When the optimal maneuver corresponds to an Euler rotation $\{$e.g., $\underline{\beta}(0) = [1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, 0]^T\}$ the solution consists of three simultaneous switches halfway through the maneuver. Furthermore, for certain boundary conditions $\{$e.g., $\underline{\beta}(0) = [0.5, 0.5, 1/\sqrt{2}, 0]^T\}$, the solution of the TPBVP appears to require four switches. Both of these cases can be resolved as five switch maneuvers in which one or two of the unique switch intervals have zero duration.

In addition, the five switch solution is not always optimal, as shown by Bilimoria and Wie.[13] They consider only boundary conditions that correspond to a control-axis reorientation. That is, they compare the multiaxis solution for a spherical body to the Euler axis rotation about a control axis. For maneuvers where the initial Euler angle is greater than 73 deg, five switches are required. For angles less than 73 deg they observe seven switches; two of the controls switch three times while the control corresponding to the axis of rotation switches only once. In the seven switch maneuver, switching is no longer sequential. Table 1 shows a comparison of the final times achieved using seven switches[13] and those achieved using the switch time optimization program with five switches for a rotation about the z axis. In addition, the final time computed from Eq. (9) is shown.

The five switch solution to the control-axis boundary conditions is not unique; there are four solutions with exactly the same final time. The two controls which have multiple switches can have any combination. It is observed for these boundary conditions that $\Delta_1 = \Delta_3$. The most remarkable observation from Table 1 is that the time-optimal maneuver is not achieved by an Euler-axis rotation. While the seven switch minimum time solution is correct for certain control-axis boundary conditions, it has not been shown to be a requirement for general maneuvers. On the other hand, the symmetry observed in Eq. (16) is true for any rest-to-rest maneuver. In addition, the improvement of the seven switch solution over the five switch solution is negligible. Therefore, for simplicity, five switches are used to approximate the optimal solution for the remainder of this paper.

The preceding analysis, although explaining the number of control switches for a rest-to-rest maneuver, says nothing about time-optimality. In fact, there are several paths, using only five bang-bang control switches, which will bring the rigid body to the desired final state, albeit not in minimum time. There are four possible paths for which $\dot{\beta}_0(0) > 0$, one of which is the minimum time solution. For the control-axis reorientation all four paths all require exactly the same time.

Approximate Solution to the State Transition Matrix

For the linear system

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) \tag{17}$$

where $\underline{x}$ is the $n \times 1$ state vector, and $A(t)$ is the $n \times n$ state matrix. The general solution is given by

$$\underline{x}(t) = \Phi(t, t_0)\underline{x}(t_0) \tag{18}$$

where $\Phi(t, t_0)$ is the $n \times n$ state transition matrix which gives the correspondence of the state at time t to the initial state at t_0. The state transition matrix has the following important properties:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, t_0) = A(t)\Phi(t, t_0) \tag{19}$$

$$\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0) \tag{20}$$

If $A(t)$ possess the commutative property

$$A(t)\left(\int_{t_0}^{t} A(\tau)\mathrm{d}\tau\right) = \left(\int_{t_0}^{t} A(\tau)\mathrm{d}\tau\right)A(t) \tag{21}$$

then the state transition matrix has the solution

$$\Phi(t, t_0) = \exp\left[\int_{t_0}^{t} A(\tau)\mathrm{d}\tau\right] \tag{22}$$

Equation (4) has the same form as Eq. (17) and has the general solution

$$\underline{\beta}(t) = \Phi(t, t_0)\underline{\beta}(t_0) \tag{23}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, t_0) = \frac{1}{2}G(\underline{\omega})\Phi(t, t_0) \tag{24}$$

Detailed analysis[15] of the state transition matrix reveals that there are only four unique elements giving $\Phi(t, t_0)$ the anti-symmetric structure

$$\Phi(t, t_0) = \begin{bmatrix} \phi_0 & -\phi_3 & \phi_2 & -\phi_1 \\ \phi_3 & \phi_0 & -\phi_1 & -\phi_2 \\ -\phi_2 & \phi_1 & \phi_0 & -\phi_3 \\ \phi_0 & \phi_2 & \phi_3 & \phi_0 \end{bmatrix} \tag{25}$$

Solving Eq. (23) for the instantaneous values of the four elements $\underline{\phi}$ gives

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_0 \end{Bmatrix} = \begin{bmatrix} -\beta_0(t_0) & -\beta_3(t_0) & \beta_2(t_0) & \beta_1(t_0) \\ \beta_3(t_0) & -\beta_0(t_0) & -\beta_1(t_0) & \beta_2(t_0) \\ -\beta_2(t_0) & \beta_1(t_0) & -\beta_0(t_0) & \beta_3(t_0) \\ \beta_1(t_0) & \beta_2(t_0) & \beta_3(t_0) & \beta_0(t_0) \end{bmatrix} \begin{Bmatrix} \beta_1(t) \\ \beta_2(t) \\ \beta_3(t) \\ \beta_0(t) \end{Bmatrix} \tag{26}$$

Choosing $\underline{\beta}(t_f) = [0\ \ 0\ \ 0\ \ 1]^T$ results in

$$\underline{\phi}(t_f, t_0) = \underline{\beta}(t_0) \tag{27}$$

Given a system described approximately by Eq. (7), over a time interval between control switches, $t_i > t \geq t_{i-1}$, Eq. (4) becomes

$$\dot{\underline{\beta}}(t) = ½G(\underline{\Omega}_i)\underline{\beta}(t) \tag{28}$$

where

$$\underline{\Omega}_i = \underline{\nu}_i[t - t_{i-1}] + \underline{\omega}_{i-1}$$

If a solution to the state transition matrix exists for Eq. (28) over the interval Δ_i then the property of Eq. (20) can be applied to determine the elements of Eq. (25) in terms of the control vector sequence and the time intervals between switches.

Assuming that $G(\underline{\Omega}_i)$ satisfies the commutative property of Eq. (21) then

$$\Phi(t_i, t_{i-1}) = \exp\frac{1}{2}G\left(\underline{\nu}_i\frac{\Delta_i^2}{2} + \underline{\omega}_{i-1}\Delta_i\right) \tag{29}$$

Carrying out a power series expansion, the elements of the state transmission matrix are found to be

$$\phi_0(t_i, t_{i-1}) = \cos\left(\|\underline{M}_i\|\frac{\Delta_i}{2}\right)$$

$$\phi_j(t_i, t_{i-1}) = -\frac{m_{ij}}{\|\underline{M}_i\|}\sin\left(\|\underline{M}_i\|\frac{\Delta_i}{2}\right) \qquad j = 1, 2, 3 \tag{30}$$

where

$$\underline{M}_i = \underline{\nu}_i\frac{\Delta_i}{2} + \omega_{i-1}$$

$$= [m_{i1}, m_{i2}, m_{i3}]^T \tag{31}$$

Unfortunately, the commutative property holds exactly only when $\underline{\nu}_i$ and $\underline{\omega}_{i-1}$ are parallel. However, over short periods of time the components of the state transition matrix for the interval Δ_i are given approximately by Eq. (30).

The matrix exponential has the well known property that if $AB = BA$ then

$$e^{At}e^{Bt} = e^{(A+B)t} \tag{32}$$

For systems for which this is true, the matrix multiplication implicit in Eq. (20) is no longer necessary. Let $\underline{a} = \underline{M}_i$, $\underline{b} = \underline{M}_{i-1}$ and $A = G(\underline{M}_i)$, $B = G(\underline{M}_{i-1})$. Carrying out the matrix multiplication results in

$$AB = \langle\underline{a}, \underline{b}\rangle I_D + G(\underline{a} \times \underline{b}) \tag{33}$$

$$BA = \langle\underline{a}, \underline{b}\rangle I_D - G(\underline{a} \times \underline{b}) \tag{34}$$

Because the G matrix is skew symmetric it is clear that the diagonal terms in Eqs. (33) and (34) do, in fact, commute. The conclusion is that, for the diagonal elements of the state transition matrix ϕ_0, Eq. (32) is approximately correct.

Knowing that there are six switch intervals, and having an approximate solution to the state transition matrix for each interval, the state transition matrix for the entire maneuver follows from Eq. (20)

$$\Phi(t_f, t_0) = \prod_{k=6}^{1}\Phi(\Delta_k) \tag{35}$$

where $\Phi(t_i, t_{i-1}) = \Phi(\Delta_i)$.

Applying the principle of Eq. (32) to the equation for ϕ_0 in Eq. (30) gives

$$\phi_0(t_f, t_0) = \cos\left(\|\sum_{k=1}^{6}\underline{M}_k\frac{\Delta_k}{2}\|\right) \tag{36}$$

Applying the symmetry of Eq. (16) and carrying out the summation results in

$$\phi_0(t_f, t_0) = \cos\left(\|\underline{\omega}_3\|\frac{t_3}{2}\right) \tag{37}$$

where $t_3 = \Delta_1 + \Delta_2 + \Delta_3$. Equating Eqs. (27) with (37) and recalling the definition of β_0 in Eq. (3), the approximate solution is obtained

$$\theta_0 = \|\underline{\omega}_3\|\, t_3 \tag{38}$$

Equation (38) can be used to quantify the difference between the single-axis maneuver time from Eq. (9). When control switching is sequential, as observed, the norm of the angular velocity at $t = t_3$, is given by

$$\|\underline{\omega}_3\|^2 = K^2\left[3(\Delta_1^2 + \Delta_2^2 + \Delta_3^2) + 2(\Delta_1\Delta_2 + \Delta_2\Delta_3 - \Delta_1\Delta_3)\right] \tag{39}$$

It has also been observed that for the "single-axis" boundary conditions, $\Delta_1 = \Delta_3$ which corresponds to the minimum value of $\|\underline{\omega}_3\|$. Making this substitution results in

$$\|\underline{\omega}_3\|^2 = K^2[t_3^2 + 2\Delta_2^2] \tag{40}$$

Equation (38) thus becomes

$$\theta_0 = Kt_3\sqrt{t_3^2 + 2\Delta_2^2} \tag{41}$$

Solving Eq. (9) for θ_0 and substituting the result in Eq. (41) gives

$$T^2 = t_3\sqrt{t_3^2 + 2\Delta_2^2} \tag{42}$$

where $T = t_f/2$ for the single-axis control maneuver. Equation (42) gives a quantitative description of the time improvement achieved by the multiaxis control over the single-axis control maneuver. Noting that t_3 and T, respectively, are exactly half of the maneuver time, Eq. (42) can be written

$$(T^4 - t_3^4)/t_3^2 = 2\Delta_2^2 \tag{43}$$

It is apparent for the single-axis case that $\Delta_2 > 0$ leads to $t_3 < T$. This quantifies the improvement in maneuver time observed by Bilimoria and Wie. Table 2 shows that Eq. (41) gives a good approximation of the maneuver angle, especially for smaller angles. Similar results are obtained for the general multiaxis maneuver using Eq. (38). This equation is also valid for the switch times associated with nontime-optimal paths.

Estimation of Control Switch Times

Equation (27) leads to four nonlinear equations of the form

$$\underline{f} = \underline{\phi}(t_f, t_0) - \underline{\beta}(t_0) = \underline{0} \tag{44}$$

where the elements of $\underline{\phi}(t_f, t_0)$ are found via Eq. (35). These equations are functions of the sequence of three unique control vectors and the three unique time intervals. If a control sequence is assumed, the associated time intervals can be found.

Because these equations are extremely nonlinear, an iterative method such as Newton's method is needed to solve them. Thus, it is necessary to numerically generate the 4×3 Jacobian $\boldsymbol{P}$ using the chain rule so that

$$\underline{f} \approx \boldsymbol{P}\underline{\Delta} \tag{45}$$

where

$$\boldsymbol{P} = \frac{\partial \underline{f}}{\partial \underline{\Delta}} \tag{46}$$

and $\underline{\Delta} = [\Delta_1, \Delta_2, \Delta_3]^T$.

Because Eq. (44) is not independent, it is possible to solve for the three unknown Δ using any three of the equations. However, it is desirable to use all four equations to avoid any possible ambiguities caused by an arbitrary selection of equa-

Table 2 Numerical check of Eq. (42)

θ_0, deg	T^2	$t_3(t_3^2 + 2\Delta_2^2)^{1/2}$	Error, %
10	0.1745	0.1749	0.229
30	0.5236	0.5227	0.172
45	0.7855	0.7831	0.305
90	1.5710	1.5602	0.687
120	2.0944	2.0750	0.926
135	2.3564	2.3342	0.942
180	3.1416	3.1185	0.735

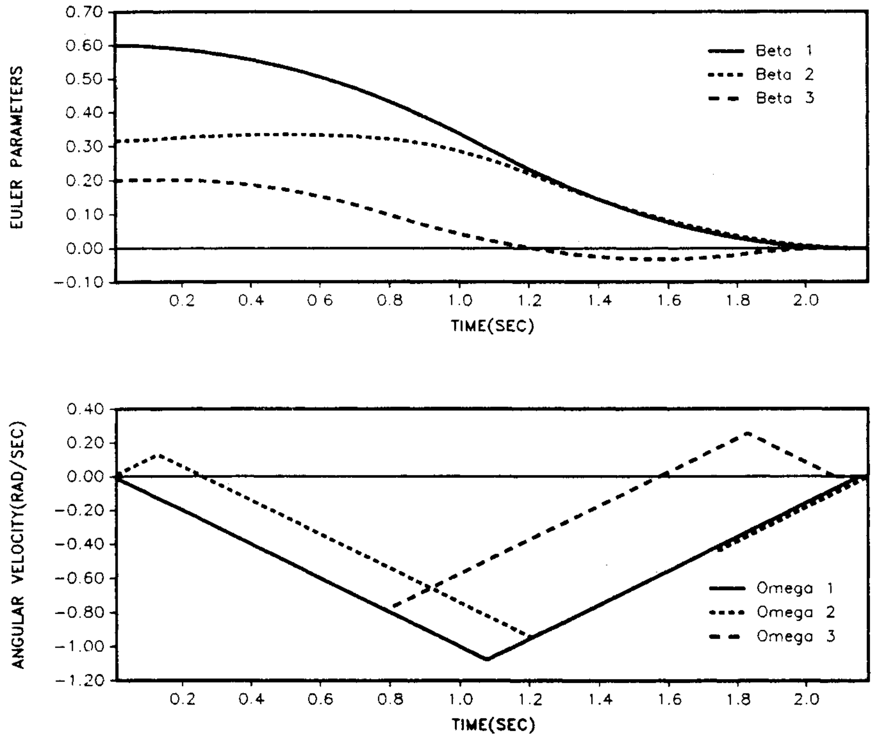

Fig. 2 Feedforward/feedback control for 90-deg rotation (t_f = 2.182 s).

Table 3 Optimal vs ELVIS switch times

Time	Switch times, s ELVIS	STO
t_1	0.129	0.117
t_2	0.787	0.786
t_3	1.077	1.077
t_4	1.206	1.194
t_5	1.864	1.863
t_f	2.154	2.153

Table 4 Predicted vs actual control switch times

Optimal/predicted/actual switch times, s u_1	u_2	u_3
1.077/1.077/1.078	0.117/0.129/0.129[a]	0.786/0.787/0.787[a]
	1.194/1.206/1.207	1.863/1.864/1.829
	—/—/1.731	—/—/2.090[b]
	—/—/1.745	

[a]Open-loop switch. [b]Shut off.

tions. Therefore, the pseudoinverse is generated so that the iterative equations of Newton's method become

$$\underline{\Delta}_{\text{new}} = \underline{\Delta}_{\text{old}} - \epsilon(\boldsymbol{P}^T\boldsymbol{P})^{-1}\boldsymbol{P}^T\underline{f} \tag{47}$$

where $(\boldsymbol{P}^T\boldsymbol{P})^{-1}\boldsymbol{P}^T$ is the pseudoinverse and ϵ is a positive scalar less than or equal to one. Equation (47) is iterated until $\underline{\Delta}_{\text{new}} - \underline{\Delta}_{\text{old}}$ is less than some desired tolerance. The control sequence and the associate switch intervals which give the least total time is an approximation of the optimal solution.

It was observed that the time-optimal, five switch solution required three unique control vectors. Each control vector consists of three elements of ± 1. There are eight possible combinations of ± 1 in groups of three

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix} \tag{48}$$

Note that the last four column vectors are the negative of the first four. If each of these vectors is considered as the initial control for the system, four of them will cause $\ddot{\beta}_0(0) < 0$ and can be discarded as time-optimal possibilities. Of the remaining four vectors, there are 24 combinations of three of them, that is, there are 24 possible control sequences to consider associated with 24 sets of switch times that can potentially perform the desired maneuver. In fact, each of the four admissible initial control vectors has only a single solution associated with it.

The five switch solution requires that one of the controls switch only once, exactly halfway through the maneuver. If this axis is identified, there are only eight control sequences to consider. For a body in which $\boldsymbol{K} = K\boldsymbol{I}_D$, this control is the one associated with the largest initial component of $\underline{l}$. For cases in which the controls are coupled or the moments of inertia are not identical the critical (single switch) axis is identified by

$$\max(\xi_i) \qquad \underline{\xi} = \boldsymbol{K}^{-1}\underline{l}(0) \tag{49}$$

Although Eq. (32) does not hold in general for the key symmetric matrix $\boldsymbol{G}(\underline{\Omega}_i)$, it was shown that a useful approximation for $\phi_0(t_f, t_0)$ could be obtained. This approximation is further extended to arrive at

$$\phi_j(t_f, t_0) \approx -\frac{\omega_{3j}}{\|\underline{\omega}_3\|}\sin\left(\|\underline{\omega}_3\|\frac{t_3}{2}\right) = \beta_j(t_0) \qquad j = 1, 2, 3 \tag{50}$$

where

$$\underline{\omega}_3 = \underline{v}_1\Delta_1 + \underline{v}_2\Delta_2 + \underline{v}_3\Delta_3 = \boldsymbol{V}\underline{\Delta}$$

$$t_3 = \Delta_1 + \Delta_2 + \Delta_3$$

Using the definition of the Euler parameters [Eq. (3)] gives the set of equations

$$t_3\underline{\Delta} = -\theta_0\boldsymbol{V}^{-1}\underline{l} \tag{51}$$

which yield the initial guess for $\underline{\Delta}$ to start the iteration of Eq. (47).

A nonfeasible sequence will result in one or more of the corresponding Δ being less than zero. The eight likely sequences can be tested, and a maximum of four are found which will perform the desired maneuver. Because the solution for the switching intervals obtained is approximate, and depends on the linearity of the system dynamics, this algorithm is called estimation of linear velocity interval switching, or ELVIS.

Table 3 gives a comparison of the optimal switch times with those generated by the ELVIS algorithm for the initial conditions of the maneuver shown in Fig. 1.

Numerically integrating the equation of motion using these switch times and the optimal control sequence gives a final state with $\underline{\omega}(t_f) = \underline{0}$ and $\underline{\beta}(t_f) = [0.000527\ 0.006284\ 0.00957\ 0.999934]$ which translates to a terminal pointing error of approximately 1.32 deg. For smaller rotations, the terminal error is less.

The ELVIS algorithm with $\epsilon = 0.5$, takes 0.37 s of CPU time on a VAX 8650 computer to obtain the solution. It is likely that a mission specialized processor would be able to perform the required computations quickly enough for approximately real time application.

Feedforward/Feedback Control Law

Although the ELVIS algorithm provides excellent estimates of control switch times for the minimum time problem, a small terminal error will result if these switch times are applied without modification. This error is exacerbated if there is any gyroscopic coupling due to differences in moments of inertia. Finally, the open-loop solution cannot react to disturbances or modeling errors. It is desirable to develop a feedback control that will correct for these shortcomings.

For the linear, single-axis case, a feedback control law using parabolic switching functions can be used. Unfortunately, this approach cannot be applied directly to the multiaxis system because the attitude states are not direct integrals of the angular velocity states as they are in the single-axis case. However, it is attractive in its simplicity and makes a useful starting point.

The quasicoordinates, the attitude and momentum (A&M) vector $\underline{\alpha}$, is introduced

$$\dot{\underline{\alpha}}(t) = \boldsymbol{K}^{-1}\underline{\omega}(t)$$

$$\underline{\alpha}(t) = \boldsymbol{K}^{-1}\int_0^t \underline{\omega}(\tau)\,\mathrm{d}\tau + \underline{\alpha}(0) \tag{52}$$

Furthermore, for the case of linear dynamics

$$\ddot{\underline{\alpha}}(t) = \underline{u}(t) \tag{53}$$

and $\dot{\underline{\alpha}}$ corresponds to the scaled momentum of the system, whereas $\underline{\alpha}$ can be interpreted as an attitude vector which is the integral of momentum. It is important to note, however, that $\underline{\alpha}$ has no physical sense.

In general, $\underline{\alpha}$ may be impossible to evaluate directly. However, near the state-space origin, where the kinematic equations become uncoupled, Eq. (4) becomes

$$\dot{\underline{\beta}}_i \approx \omega_i/2 \qquad i = 1, 2, 3 \tag{54}$$

Integrating backward in time from the final time leads to

$$\alpha_i = 2\sum_{j=1}^{3} K_{ij}^{-1}\beta_j \qquad i = 1, 2, 3 \tag{55}$$

By posing the problem in terms of these parameters, the three axes can be uncoupled to yield the equivalent of three single-axis control switching curves

$$s_i = \alpha_i + \frac{\dot{\alpha}_i\,|\dot{\alpha}_i|}{2} \qquad i = 1, 2, 3 \tag{56}$$

where the parabolic trajectories defined by $s_i = 0$ pass through the α_i, $\dot{\alpha}_i$ space origin. The feedback control law is given by

$$u_i = \begin{cases} -\operatorname{sign}(s_i) & \\ -\operatorname{sign}(\dot{\alpha}_i), & \text{if } s_i = 0 \end{cases} \qquad i = 1, 2, 3 \tag{57}$$

This feedback control is completely analogous to that in Eqs. (10) and (11). As in the single-axis system, each of these switching curves will command a maximum of one control switch to intersect the origin.

There are two problems with the direct implementation of this logic to the multiaxis system. Although $\dot{\underline{\alpha}}(0) = \underline{0}$ for the rest-to-rest maneuver, the initial conditions on $\underline{\alpha}(0)$ are unknown. In addition, more than one control switch is required for two of the controls.

Fortunately, a very good approximation of $\underline{\alpha}(0)$ can be obtained. Setting $\underline{\alpha}(t_f) = \underline{0}$ in Eq. (52) gives

$$\underline{\alpha}(0) = -\boldsymbol{K}^{-1}\int_0^{t_f} \underline{\omega}(\tau)\,\mathrm{d}\tau \tag{58}$$

where

$$\int_0^{t} \underline{\omega}(\tau)\,\mathrm{d}\tau = \sum_{j=1}^{6}\sum_{k=1}^{j-1} \underline{v}_k \Delta_k \Delta_j = t_3 \boldsymbol{V}\underline{\Delta} \tag{59}$$

Thus the initial conditions are given by

$$\underline{\alpha}(0) = t_3 \boldsymbol{U}\underline{\Delta} \tag{60}$$

Where $\boldsymbol{U} = \boldsymbol{K}^{-1}\boldsymbol{V} = [\underline{u}_1, \underline{u}_2, \underline{u}_3]$. The ELVIS algorithm selects the optimal $\boldsymbol{U}$ when it calculates the estimates of $\underline{\Delta}$ and t_3. The time varying values of $\underline{\alpha}$ can then be obtained by numerical integration of Eq. (52).

It should be obvious that the initial values of $\underline{\alpha}$ depend on the particular path taken. Furthermore, the initial control given by Eq. (56) will rarely give the optimal initial control vector and will accommodate only a single control switch per axis. This problem is overcome by performing the first two switches in the open loop at the times generated by the ELVIS algorithm. When only a single switch for each control remains, the switching curves of Eq. (56) may be invoked.

Because the values used for $\underline{\alpha}(0)$ are approximate, the state-space trajectories will not pass exactly through the origin. However, as the state approaches the origin, Eq. (55) becomes valid, and can be used to generate Eq. (56).

An example maneuver is shown in Fig. 2. The initial conditions are the same as for the maneuver in Fig. 1. The maneuver is stopped when $\sqrt{\theta^2 + \dot{\theta}^2} < 0.002$ at 2.182 s. This corresponds to a terminal pointing error of 0.1 deg and compares very favorably with the optimal time of 2.153 s generated by STO.

The first two switches are carried out in an open-loop fashion. Table 4 gives a summary of the switch times predicted by the ELVIS algorithm and the actual switch times generated by the switching curves. The validity of the switching curves, and the initial values of $\underline{\alpha}$ is demonstrated by the close correspondence of the control switches at t_3 and t_4 with the optimal and

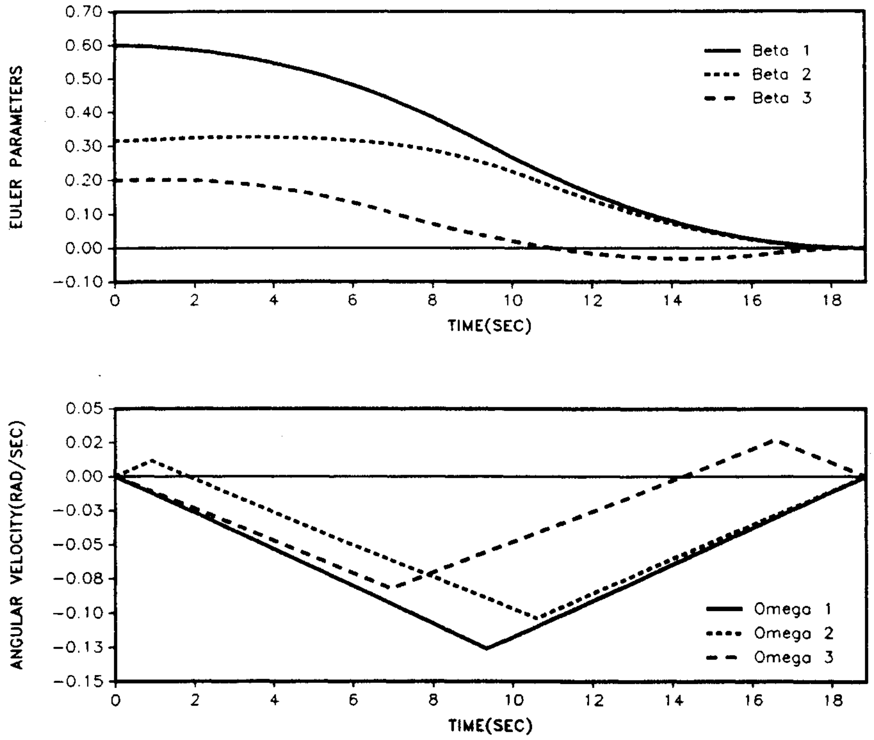

Fig. 3 90-deg maneuver of precessing spacecraft (t_f = 189.86 s).

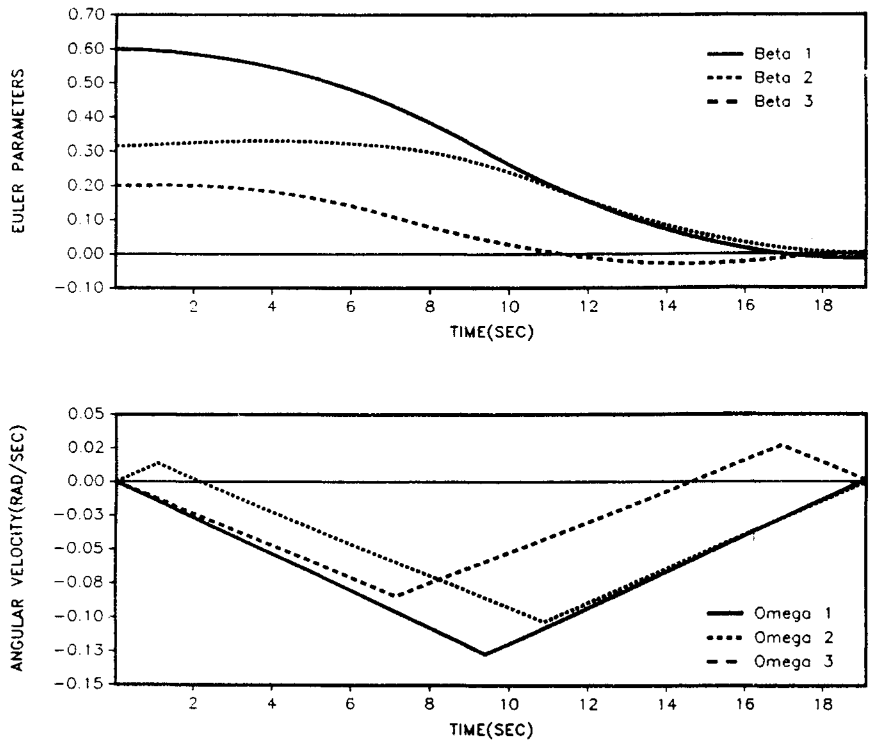

Fig. 4 Feedforward/feedback control for 90-deg rotation of precessing spacecraft (t_f = 19.11).

Table 5 Optimal vs ELVIS switch times for nonsymmetric spacecraft

Time	Switch times, s ELVIS	STO
t_1	1.0894	0.9261
t_2	7.1386	6.9518
t_3	9.4667	9.3462
t_4	10.5561	10.5947
t_5	16.6053	16.5713
t_f	18.9335	18.8565

predicted times. Likewise, the switch of u_3 at $t = 1.83$ s is approximately correct for the predicted time of 1.87 s. The error in the first open-loop switch and the increasing sensitivity to the approximate values of $\underline{\alpha}$ result in the perceived error. The u_3 is set to zero to avoid chattering when β_3 and ω_3 arrive within a target region close to the origin. The remaining switches in u_2 are the closed-loop correction of the trajectory error caused by the open-loop switching.

It is more likely, for an actual spacecraft, that the principal moments of inertia will not be identical. In addition, torque available about a given axis will generally be much smaller than the corresponding moment of inertia. The gyroscopic coupling associated with differing moments of inertia, although small, distorts the symmetry of the switch times observed earlier. As an illustration of this, consider a spacecraft where

$$I = \begin{bmatrix} 75 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 85 \end{bmatrix}, \qquad B = I_D \tag{61}$$

The time-optimal maneuver, obtained using the STO algorithm, for the initial conditions: $\underline{\beta}(0) = [0.6 \ 0.316228 \ 0.2 \ 0.707107]$, is shown in Fig. 3. Table 5 shows the skewing of the optimal switch times with respect to the symmetry of the switch times assumed in the ELVIS algorithm. Interestingly, the ELVIS algorithm predicts the final time with a high degree of accuracy.

The feedforward/feedback control brings the state very close to the target state at approximately the optimal time, as shown in Fig. 4, despite the errors in the predicted switch times. The pointing error is approximately 0.8 deg with drift of about 0.12 deg/s. Because the boundary conditions are not satisfied exactly, it is necessary to modulate the bang-bang torques with other actuators (e.g., reaction wheels), so that the effects of gyroscopic precession may be overcome.

Conclusions

A practical method of computing approximate solutions to the time-optimal control switch times for rigid body reorientation has been developed. In the process, the nature and number of control switches has been explained. The approximate solution of the switch times allows the synthesis of a feedforward/feedback control law which approximates the time-optimal solution.

Acknowledgments

This research was made possible by the Texas Advanced Research and Technology Program (Project 4193/1987 and 231/1989) and by the Air Force Office of Scientific Research Summer Research Fellowship Program (Contract F49620-88-C-0053). We are very grateful for their support. In addition, we wish to acknowledge the invaluable contribution of Alok Das of the Air Force Phillips Laboratory.

References

[1]Athans, M., and Falb, P. L., *Optimal Control,* McGraw-Hill, New York, 1966.

[2]Ryan, E. P., *Optimal Relay and Saturating Control Systems Synthesis,* Peter Peregrinus, New York, 1982.

[3]Junkins, J. L., and Turner, J. D., *Optimal Spacecraft Rotational Maneuvers,* Elsevier Scientific, New York, 1985.

[4]Vadali, S. R., and Kobberdahl, D. W., "Robust Control of Spacecraft Rotational Maneuvers," AIAA 24th Aerospace Sciences Meeting, Reno, NV, Jan. 1986 (AIAA Paper 86-0252).

[5]Etter, J. R., "A Solution of the Time Optimal Euler Rotation

Problem," *Proceedings of the AIAA Guidance, Navigation, and Control Conference* (Boston, MA), AIAA, Washington, DC, Aug. 1989 (AIAA Paper 89-3601).

[6]Redding, D. C., and Adams, N. J., "Optimized Rotation-Axis Attitude Maneuver Controller for the Space Shuttle Orbiter," *Journal of Guidance, Control, and Dynamics,* Vol. 10, No. 1, 1987, pp. 4–13.

[7]Vadali, S. R., "Variable-Structure Control of Large-Angle Maneuvers," *Journal of Guidance, Control, and Dynamics,* Vol. 9, No. 2, 1986, pp. 235–239.

[8]Wie, B., Weiss, H., and Arapostathis, A., "Quaternion Feedback Regulator for Spacecraft Eigenaxis Rotations," *Journal of Guidance, Control, and Dynamics,* Vol. 12, No. 3, 1989, pp. 375–380.

[9]D'Amario, L. A., and Stubbs, G. S., "New Single-Rotation-Axis Autopilot for Rapid Spacecraft Attitude Manuevers," *Journal of Guidance and Control,* Vol. 2, No. 4, 1979, pp. 339–346.

[10]Li, F., and Bainum, P. M., "Numerical Approach for Solving Rigid Spacecraft Minimum Time Attitude Maneuvers," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 1, 1990, pp. 38–45.

[11]Ercoli-Finzi, A., Mantegazza, P., and Oliva, F., "Numerical Solution for Minimum Time Attitude Maneuvers," University of Milan, Preprint n.d., (private collection of R. Byers).

[12]Chowdhry, R. S., and Cliff, E. M., "Optimal Rigid Body Reorientation Problem," *Proceedings of the AIAA Guidance, Navigation, and Control Conference* (Portland, OR), AIAA, Washington, DC, Aug. 1990 (AIAA Paper 90-3485).

[13]Bilimoria, K. D., and Wie, B., "Minimum-Time Large-Angle Reorientation of a Rigid Spacecraft," *Proceedings of the AIAA Guidance, Navigation, and Control Conference* (Portland, OR), AIAA, Washington, DC, Aug. 1990 (AIAA Paper 90-3486); also *Journal of Guidance, Control, and Dynamics* (to be published).

[14]Meier, E. B., and Bryson, A. E., "Efficient Algorithm for Time Optimal Control of a Two-Link Manipulator," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 5, 1990, pp. 859–866.

[15]Morton, H. S., Junkins, J. L., and Blanton, J. N., "Analytical Solutions for Euler Parameters," *Celestial Mechanics,* Vol. 10, 1974, pp. 287–301.