

Non-singleton Elimination

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Abstract

Contents

1	Introduction 1.1 The outline	2 2
2	Sheaves 2.1 Topoï and sheaves in topoï 2.2 Sheaves in type theory	2 2 2
3	System T and MLTT 3.1 MLTT 3.2 System T	2 2 3
4	Meta-informations4.1 Time expenditure4.2 Difficulties	4 4
5	Conclusion	4
Δ	Dummy	5

1 Introduction

1.1 The outline

2 Sheaves

2.1 Topoï and sheaves in topoï

Proofs, results and details for this section can often be found in *Sheaves in geometry and logic: A first introduction to topos theory* by Saunders Maclane and Ieke Moerdijk [1].

Definition 2.1 (Subobject). In a category, a subobject of X is an equivalence class of monomorphism $m: A \rightarrow X$, where the equivalence comes from the preorder where $m: A \rightarrow X$ is smaller than $m': A' \rightarrow X$ when there is a map $f: A \rightarrow A'$ with $m' \circ f = m$.

We deduce a presheaf **Sub** where **Sub**(X) is the set of subobjects of X, and **Sub**(f): **Sub**(Y) \rightarrow **Sub**(X) sends $m: A \rightarrow Y$ to its pullback by $f: X \rightarrow Y$.

Definition 2.2 (Topos). A topos is a cartesian closed category with all finite limits and a suboject classifier Ω and an isomorphism $\mathbf{Sub}(X) \cong \mathbf{Hom}(X,\Omega)$ natural in X.

We note that topoï also have finite colimits.

A topos serves to give models of intuitionistic logic in classical mathematical language. It has an internal logic which is higher order.

For example, **Set** is a topos, and given a topos \mathcal{E} , \mathcal{E}/X , the category of maps with codomain X and commuting triangles, as well as $\mathcal{E}^{\mathbf{C}^{op}}$, the category of contravariant functors from a small category \mathbf{C} and natural transformation, are all topoï. In particular categories of presheaves are topoï, and correspond to Kripke models.

The subobject classifier Ω is equipped with an internal meet-semilattice structure inherited from the meet-semilattice structure on each $\mathbf{Sub}(X)$, which is natural in X.

Definition 2.3 (Lawvere-Tierney topology). A Lawvere-Tierney topology is a left exact idempotent monad j on the internal meet-semilattice on Ω .

- $id_{\Omega} \leq j$,
- $j \circ j \leq j$
- $j \circ \land = \land \circ j \times j$

From a topology j we extract a closure operator J_X of $\mathbf{Sub}(X)$ for any X.

Definition 2.4 (Dense subobject). A suboject U of X is dense if $J_XU = X$

A topology can be lifted to a left exact idempotent monad on the entirety of the topos, the sheafification monad.

Definition 2.5 (j-Sheaf in topos). An object F is a j-sheaf in a topos if for any dense subobject U of any object X, the morphism $\mathbf{Hom}(X,F) \to \mathbf{Hom}(U,F)$ obtained by precomposition is an isomorphism.

A *j*-Sheaf is up to isomorphism the result of sheafifying an object.

j-Sheaves form a topos. The sheaves on a presheaf topos correspond to Beth semantics.

2.2 Sheaves in type theory

Consider a type theory with a notion of proof irrelevant propositions Prop, e.g. book-HoTT with mere propositions, or Rocq with SProp.

Then, a Lawvere-Tiernev topology may be similarly defined, as a map $j: \mathsf{Prop} \to \mathsf{Prop}$

3 System T and MLTT

3.1 MLTT

We sought to extend the following variant of MLTT.

We consider a type of levels containing two elements ${\bf s}$ and ${\bf l}$ for small and large, with generic ℓ With terms :

$$M, N ::= x | \lambda x. M | MN | 0 | S | \mathbf{N}_{rec} | \bot_{rec} | \mathbf{N} | \bot | \Pi x : A. B | \Box_{\mathbf{s}} | \Box_{\mathbf{l}}$$

Contexts:

$$\Gamma ::= \Gamma, x : A|\cdot$$

And conversion rules:

3.2 System T

To identify and solve problems in a simpler envirronement, we studied a modified System T before, based on the following variant.

With types :

 $A ::= A \rightarrow A|\mathbf{N}| \perp$

Terms:

 $M, N ::= x |\lambda x. M| MN |0| S |\mathbf{N}_{rec}| \perp_{rec}$

Contexts:

$$\Gamma ::= \Gamma, x : A|\cdot$$

And conversion rules:

$$\Gamma \vdash N_0 \equiv N_0' \equiv A$$

$$\Gamma \vdash N_S \equiv N_S' : \mathbf{N} \to A \to A \qquad \Gamma \vdash N \equiv N' : \mathbf{N}$$

$$\Gamma \vdash \mathbf{N}_{rec} N_0 N_S(SN) \equiv N_S' N' (\mathbf{N}_{rec} N_0' N_S' N') : A$$

$$E_{\text{MP-Rec}} \overline{ \Gamma \vdash \bot_{\text{rec}} \equiv \bot_{\text{rec}} : \bot \to A }$$

$$\operatorname{Sym} \frac{\Gamma \vdash M \equiv M' : A}{\Gamma \vdash M' \equiv M : A} \qquad \operatorname{Trans} \frac{\Gamma \vdash M \equiv M' : A}{\Gamma \vdash M \equiv M'' : A}$$

- 4 Meta-informations
- 4.1 Time expenditure
- 4.2 Difficulties
- 5 Conclusion

Appendix

A Dummy

References

[1] Saunders MacLane and Ieke Moerdijk. Sheaves in geometry and logic: A first introduction to topos theory. Springer Science & Business Media, 2012.