

Internship defense M2
Parisian Master of Research in Computer Science
Élimination non-singleton :
Avec les faisceaux, c'est pas faux

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The starting point

Cartesian closed categories	\Leftrightarrow	Simply typed λ -calculus	\Leftrightarrow	Negative fragment of intuitionistic propositional logic
$\text{Hom}(A \times B, C)$ \simeq $\text{Hom}(A, B \rightarrow C)$		$\lambda x : A. t : A \rightarrow B$		$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$

Subobject classifier

$$\text{Sub}(A) \simeq \text{Hom}(A, \Omega)$$

$\text{Sub}(A) :=$ equivalence classes of monomorphisms $X \rightarrowtail A$.

Introduces a type of proposition in the λ -calculus, as well as subobjects

- All finite limits as well. Finite colimits are recovered[1].

Formulas

$\wedge, \vee, \forall, \exists, \neg, \perp, \top, \Rightarrow$ can all be expressed e.g.

$$\begin{array}{ccc} X \wedge Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & A \end{array}$$

So can $=$ and \in . Equality is the predicate $A \times A \rightarrow \Omega$ given by the diagonal $A \rightarrowtail A \times A$.

Objects similar to the set theoretic $\{x \in A \mid \phi(x)\}$ can be constructed.

Internal logic of Ω

- $\wedge : \text{Sub}(A) \times \text{Sub}(A) \rightarrow \text{Sub}(A)$
- $\wedge : \text{Hom}(A, \Omega) \times \text{Hom}(A, \Omega) \rightarrow \text{Hom}(A, \Omega)$
- $\wedge : \text{Hom}(A, \Omega \times \Omega) \rightarrow \text{Hom}(A, \Omega)$
- $\wedge : \Omega \times \Omega \rightarrow \Omega$

Similarly for $\vee, \perp, \top, \Rightarrow$.

Topoi of presheaves

For a small category \mathbf{C} , its category of presheaves is noted $\widehat{\mathbf{C}} := \mathbf{Set}^{\mathbf{C}^{op}}$.

$$(P \times Q)(C) = P(C) \times Q(C)$$

$$P^Q(C) \simeq \mathbf{Nat}(yC, P^Q) \simeq \mathbf{Nat}(yC \times Q, P)$$

$\Omega(C)$ is the set of subpresheaves of yC (also called sieves on C).
(A subpresheaf is pointwise a subset, with function restricted)

Small limits are pointwise.

Sheaves

A sheaf = A presheaf with a compatibility property.

A Grothendieck topology introduces a notion of cover.

- If $f : D \rightarrow C$ is in $S(D)$, then it is covered by S .
- If f is covered then $f \circ g$ is covered.
- If S covers f and R covers any map of $S(D)$ for all D , then R covers f

We note the sieve of all maps covered by S as \overline{S} .

A sheaf is a presheaf where the inclusion of $\text{Nat}(\overline{S}, F)$ in $\text{Nat}(S, F)$ is an equivalence.

Example : Sheaves on \mathbb{R}^2

$\mathbf{C} := \mathcal{O}(\mathbb{R}^2)$ (poset : at most one map between object).

We fix $C := \mathbb{R}^2$.

S covers $U := (U \subseteq \bigcup S)$:

- if $U \in S$ then $U \subseteq \bigcup S$
- if $U \subseteq \bigcup S$, and $V \subset U$, then $V \subseteq \bigcup S$
- if $U \subseteq \bigcup S$, and for all $V \in S$, $V \subseteq \bigcup R$, then $U = \bigcup R$

The topos of sheaves

Finite limits of sheaves as presheaves are sheaves and are finite limits for sheaves.

F^P is a sheaf whenever F is.

Ω contains the sieves which are also sheaves, that is which only cover themselves.

Lawvere-Tierney topology

Sheaves are defined through an operator $j : \Omega \rightarrow \Omega$ for which

- $id \leq j$, since $S \subseteq \overline{S}$
- $\Rightarrow \circ id \times j \leq \Rightarrow \circ j \times j : \Omega \times \Omega \rightarrow \Omega$, since $\overline{S} \subseteq \overline{\overline{R}}$ whenever $S \subseteq \overline{R}$

This can be defined in any topos.

Or, rather, any type theory with a type of (strict) propositions.

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J -Sheaves

Given $J : \text{Prop} \rightarrow \text{Prop}$ with

- $\text{ret}_J : \prod P, P \rightarrow JP$
- $\text{bind}_J : \prod QP, JP \rightarrow (P \rightarrow JQ) \rightarrow JQ$

A sheaf is then a type F together with

- A map $\text{ask} : \prod P, JP \rightarrow (P \rightarrow F) \rightarrow F$
- Such that for all proposition P and any proof j of JP ,
 $\text{ask } P \ j \ (\lambda _, f) = f$

IO -sheaves

Noting $I := \Sigma P. JP : \text{Type}$ and $O(P, j) := P : I \rightarrow \text{Prop}$,

A sheaf is equivalently given by:

$$\text{Shf} := \left\{ \begin{array}{l} F : \text{Type} \\ \text{ask} : \Pi i : I, (O\ i \rightarrow F) \rightarrow F \\ \varepsilon : \Pi (i : I)(f : F), \text{ask}\ i\ (\lambda _, f) = f \end{array} \right\}$$

A type has at most one sheaf structure (with univalence).

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The sheaf model

$$\begin{aligned} [\text{Type}] &:= \{F := \text{Shf}; \dots\} \\ \llbracket A \rrbracket &:= F \llbracket A \rrbracket \end{aligned}$$

Positive types

Interpreting positive types requires higher inductive types.

Inductive $B : \text{Type} :=$

| true : B

| false : B

| ask_B : $\prod(i : I).(O\ i \rightarrow B) \rightarrow B$

| $\varepsilon_B : \prod(i : I) (b : B). \text{ask } i (\lambda. O\ i.b) = b$

Basic sheaf theory

$$[\text{ask}] := \text{ask}$$

But ask has type $\prod(A : \text{Shf})(i : I), (O \ i \rightarrow A) \rightarrow A$, should be $\llbracket ? \rrbracket$

An approximation :

$$\frac{O_i, \Gamma \vdash k : A}{\Gamma \vdash \text{ask}_i k : A} \qquad \frac{}{O_i, \Gamma \vdash \text{ask}_i k \equiv k}$$

With $[\text{ask}_i k] := \text{ask } [A] \ i \ (\lambda(o_i : O_i), [k])$

Geometric formulas

$$\bigwedge_{i:I} \left(\forall \vec{x}, O_i \Rightarrow \bigvee_{k:K_i} \exists \vec{y}, Q_{i,k} \right)$$

Up to currying, of the shape $\prod i, O_i$.

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Bibliography I

- [1] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.