Internship defense M2
Parisian Master of Research in Computer Science
Élimination non-singleton :
Avec les faisceaux, c'est pas faux

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February 28th to July 30th 2023

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The starting point

Cartesian closed categories	$\Leftrightarrow$	Simply typed λ-calculus	$\Leftrightarrow$	Negative fragment of intuitionistic propositional logic
$\mathrm{Hom}(A \times B, C) \simeq \mathrm{Hom}(A, B \to C)$		$\lambda x : A.t : A \to B$		$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$

Subobject classifier

$$\mathrm{Sub}(A) \simeq \mathrm{Hom}(A, \Omega)$$

Sub(A) $:=$ equivalences classes of monomorphisms $X \rightarrowtail A$.
Introduces a type of proposition in the λ-calculus, as well as subobjects

- All finite limits as well. Finite colimits are recovered[1].

Formulas

$\wedge, \vee, \forall, \exists, \neg, \perp, \top, \Rightarrow$ can all be expressed *e.g.*

$$\begin{tikzcd} X \wedge Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & A \end{tikzcd}$$

So can $=$ and $\in$. Equality is the predicate $A \times A \to \Omega$ given by the diagonal $A \rightarrowtail A \times A$.

Objects similar to the set theoritic $\{x \in A | \phi(x)\}$ can be constructed.

Internal logic of Ω

- $\wedge : \mathrm{Sub}(A) \times \mathrm{Sub}(A) \to \mathrm{Sub}(A)$
- $\wedge : \mathrm{Hom}(A, \Omega) \times \mathrm{Hom}(A, \Omega) \to \mathrm{Hom}(A, \Omega)$
- $\wedge : \mathrm{Hom}(A, \Omega \times \Omega) \to \mathrm{Hom}(A, \Omega)$
- $\wedge : \Omega \times \Omega \to \Omega$

Similarily for $\vee, \perp, \top, \Rightarrow$.

Topoi of presheaves

For a small category $\mathbf{C}$, its category of presheaves is noted $\widehat{\mathbf{C}} := \mathrm{Set}^{\mathbf{C}^{op}}$.

$$(P \times Q)(C) = P(C) \times Q(C)$$

$$P^Q(C) \simeq \mathrm{Nat}(\mathrm{y}C, P^Q) \simeq \mathrm{Nat}(\mathrm{y}C \times Q, P)$$

$\Omega(C)$ is the set of subpresheaves of $\mathrm{y}C$ (also called sieves on C).
(A subpresheaf is pointwise a subset, with function restricted)

Small limits are pointwise.

Sheaves

A sheaf = A presheaf with a compatibility property.

A Grothendieck topology introduces a notion of cover.

- If $f : D \to C$ is in $S(D)$, then it is covered by S.
- If f is covered then $f \circ g$ is covered.
- If S covers f and R covers any map of $S(D)$ for all D, then R covers f

We note the sieve of all maps covered by S as $\overline{S}$.

A sheaf is a presheaf where the inclusion of $\mathrm{Nat}(\overline{S}, F)$ in $\mathrm{Nat}(S, F)$ is an equivalence.

Example : Sheaves on R^2

$\mathbf{C} := \mathcal{O}(\mathrm{R}^2)$ (poset : at most one map between object).
We fix $C := \mathrm{R}^2$.
S covers $U := (U \subseteq \bigcup S)$:

- if $U \in S$ then $U \subseteq \bigcup S$
- if $U \subseteq \bigcup S$, and $V \subset U$, then $V \subseteq \bigcup S$
- if $U \subseteq \bigcup S$, and for all $V \in S$, $V \subseteq \bigcup R$, then $U = \bigcup R$

The topos of sheaves

Finite limits of sheaves as presheaves are sheaves and are finite limits for sheaves.

F^P is a sheaf whenever F is.

Ω contains the sieves which are also sheaves, that is which only cover themselves.

Lawvere-Tierney topology

Sheaves are defined through an operator $j : \Omega \to \Omega$ for which

- $id \leq j$, since $S \subseteq \overline{S}$
- $\Rightarrow \circ id \times j \leq \Rightarrow \circ j \times j : \Omega \times \Omega \to \Omega$, since $\overline{S} \subseteq \overline{R}$ whenever $S \subseteq \overline{R}$

This can be defined in any topos.

Or, rather, any type theory with a type of (strict) propositions.

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J-Sheaves

Given $J : \mathrm{Prop} \to \mathrm{Prop}$ with

- $\mathrm{ret}_J : \Pi P, P \to JP$
- $\mathrm{bind}_J : \Pi QP, JP \to (P \to JQ) \to JQ$

A sheaf is then a type F together with

- A map $\mathrm{ask} : \Pi P, JP \to (P \to F) \to F$
- Such that $\Pi P\ j, \mathrm{ask}\ P\ j\ (\lambda\_, f) = f$

IO-sheaves

Noting $I := \Sigma P.JP$: Type and $O(P, j) := P : I \to \text{Prop}$,

A sheaf is equivalently a type F where:

- ask : $\Pi i : I, (O\ i \to F) \to F$
- $\Pi i : I, \text{ask}\ i\ (\lambda\_, f) = f$

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Bibliography I

[1] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.