

Internship report M2
Parisian Master of Research in Computer Science Level 2

Non-singleton Elimination

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1 Introduction

1.1 The outline

2 Sheaves

2.1 Topoï and sheaves in topoï

Proofs, results and details for this section can often be found in *Sheaves in geometry and logic: A first introduction to topos theory* by Saunders MacLane and Ieke Moerdijk [1].

Definition 2.1 (Subobject). *In a category, a subobject of X is an equivalence class of monomorphism $m : A \rightarrowtail X$, where the equivalence comes from the preorder where $m : A \rightarrowtail X$ is smaller than $m' : A' \rightarrowtail X$ when there is a map $f : A \rightarrow A'$ with $m' \circ f = m$.*

We deduce a presheaf **Sub** where **Sub**(X) is the set of subobjects of X , and **Sub**(f) : **Sub**(Y) \rightarrow **Sub**(X) sends $m : A \rightarrow Y$ to its pullback by $f : X \rightarrow Y$.

Definition 2.2 (Topos). *A topos is a cartesian closed category with all finite limits and a subobject classifier Ω and an isomorphism **Sub**(X) \cong **Hom**(X, Ω) natural in X .*

We note that topoï also have finite colimits.

A topos serves to give models of intuitionistic logic in classical mathematical language. It has an internal logic which is higher order.

For example, **Set** is a topos, and given a topos \mathcal{E} , \mathcal{E}/X , the category of maps with codomain X and commuting triangles, as well as $\mathcal{E}^{\mathbf{C}^{op}}$, the category of contravariant functors from a small category **C** and natural transformation, are all topoï. In particular categories of presheaves are topoï, and correspond to Kripke models.

The subobject classifier Ω is equipped with an internal meet-semilattice structure inherited from the meet-semilattice structure on each **Sub**(X), which is natural in X .

Definition 2.3 (Lawvere-Tierney topology). *A Lawvere-Tierney topology is a left exact idempotent monad j on the internal meet-semilattice on Ω .*

- $id_{\Omega} \leq j$,
- $j \circ j \leq j$
- $j \circ \wedge = \wedge \circ j \times j$

From a topology j we extract a closure operator J_X of **Sub**(X) for any X .

Definition 2.4 (Dense subobject). *A subobject U of X is dense if $J_X U = X$*

A topology can be lifted to a left exact idempotent monad on the entirety of the topos, the sheafification monad.

Definition 2.5 (j -Sheaf in topos). *An object F is a j -sheaf in a topos if for any dense subobject U of any object X , the morphism **Hom**(X, F) \rightarrow **Hom**(U, F) obtained by precomposition is an isomorphism.*

A j -Sheaf is up to isomorphism the result of sheafifying an object.

j -Sheaves form a topos. The sheaves on a presheaf topos correspond to Beth semantics.

2.2 Sheaves in type theory

Consider a type theory with a notion of proof irrelevant propositions **Prop**, e.g. book-HoTT with mere propositions, or ROCQ with **SProp**.

In this case, a Lawvere-Tierney topology may be similarly defined, as a monad:

- $J : \mathbf{Prop} \rightarrow \mathbf{Prop}$
- $\eta : \Pi(P : \mathbf{Prop}). P \rightarrow J P$
- $\text{bind} : \forall(PQ : \mathbf{Prop}). J P \rightarrow (P \rightarrow J Q) \rightarrow J Q$

Then a sheaf is just a type T with

- A map $\text{ask}_T : \Pi(P : \mathbf{Prop}). J P \rightarrow (P \rightarrow T) \rightarrow T$
- A coherence $\varepsilon_T : \Pi(P : \mathbf{Prop}) (j : J P) (x : T). \text{ask}_T P j (\lambda p : P.x) = x$

Now, the sheafified of a type doesn't exists in general, if the theory admits quotient inductive types, it can then be defined as follow :

$$\begin{aligned} \text{Inductive } \mathcal{S}_J T : \text{Type} := \\ & | \text{ret} : T \rightarrow \mathcal{S}_J T \\ & | \text{ask} : \Pi(P : \text{Prop}), J P \rightarrow (P \rightarrow \mathcal{S}_J T) \rightarrow \mathcal{S}_J T \\ & | \varepsilon : \Pi(P : \text{Prop}) (j : J P) (x : \mathcal{S}_J T). \text{ask } P j (\lambda p : P.x) = x \end{aligned}$$

We note that by taking $I := \Sigma(P : \text{Prop}). J P$ and $O (P, j) : P$, a sheaf is then a

- A type T
- A map $\text{ask}_T : \Pi(i : I), (O i \rightarrow T) \rightarrow T$
- A coherence map $\varepsilon_T : \Pi(i : I) (x : T), \text{ask}_T i (\lambda o : O i.x) = x$

and simimlarly for sheafification. This is marginally simpler, and make sheaves appear as quotient dialogue trees, hence why we will henceforth consider (I, O) -sheaves instead of J -sheaves.

3 Models

A significant part of my internship was dedicated to contructing models of type theory in ROCQ.

1. A model of a variant of Baclofen TT using dialogue trees. Predicates must be linearized before eliminating an inductive into them.
2. An exceptional model, with a type of exceptions E . A special type of dialogue trees where $I = E$ and $Oi = \mathbf{0}$, the resulting theory is inconsistent (when E is inhabited), as always when $Oi \rightarrow \mathbf{0}$ for some i .
3. A model using sheaves, which requires univalence, and quotient inductive types to model positive types.
4. A incomplete model using presheaves.

4 System T and MLTT

4.1 MLTT

We sought to extend the following variant of MLTT.

We consider a type of levels containing two elements **s** and **l** for small and large, with generic ℓ

With terms :

$$M, N ::= x | \lambda x. M | MN | 0 | S | \mathbf{N}_{\text{rec}} | \perp_{\text{rec}} | \mathbf{N} | \perp | \Pi x : A. B | \Box_{\mathbf{s}} | \Box_{\mathbf{l}}$$

Contexts :

$$\Gamma ::= \Gamma, x : A | \cdot$$

And conversion rules :

$$\begin{aligned} & \text{WF-EMPTY} \frac{}{\cdot \vdash \text{well-formed}} \quad \text{WF-EXT} \frac{\Gamma \vdash A \equiv A \quad \Gamma \vdash \text{well-formed}}{\Gamma, x : A \vdash \text{well-formed}} \\ & \text{INT-TYP} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \mathbf{N} \equiv \mathbf{N} : \Box_{\ell}} \quad \text{EMP-TYP} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \perp \equiv \perp : \Box_{\ell}} \\ & \text{FUN-TYP} \frac{\Gamma \vdash A \equiv A' : \Box_{\ell} \quad \Gamma, x : A \vdash B \equiv B' : \Box_{\ell}}{\Gamma \vdash \Pi x : A, B \equiv \Pi x : A', B' : \Box_{\ell}} \quad \text{TYP-TYP} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \Box_{\mathbf{s}} : \Box_{\mathbf{l}}} \\ & \text{FUN-INTRO} \frac{\Gamma, x : A \vdash M \equiv M' : B \quad \Gamma \vdash A \equiv A : \Box_{\ell}}{\Gamma \vdash \lambda x. M \equiv \lambda x. M' : \Pi x : A, B} \quad \text{FUN-ELIM} \frac{\Gamma \vdash M \equiv M' : \Pi x : A, B \quad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash MN \equiv M'N' : B(N/x)} \\ & \text{AXIOM} \frac{\Gamma \vdash \text{well-formed} \quad x : A \in \Gamma}{\Gamma \vdash x \equiv x : A} \quad \text{BETA} \frac{\Gamma, x : A \vdash M \equiv M' : B \quad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash (\lambda x. M)N \equiv M'(N'/x) : B(N/x)} \\ & \text{INT-ZERO} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash 0 \equiv 0 : \mathbf{N}} \quad \text{INT-SUCC} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash S \equiv S : \mathbf{N} \rightarrow \mathbf{N}} \\ & \text{INT-REC} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \mathbf{N}_{\text{rec}} \equiv \mathbf{N}_{\text{rec}} : \Pi A : \mathbf{N} \rightarrow \Box_{\mathbf{s}}, A0 \rightarrow (\Pi n : \mathbf{N}, An \rightarrow A(Sn)) \rightarrow \Pi n : \mathbf{N}, An} \end{aligned}$$

$$\begin{array}{c}
\text{INT-REC-ZERO} \frac{\Gamma \vdash A \equiv A : \mathbf{N} \rightarrow \text{square}_{\mathbf{s}} \quad \Gamma \vdash N_0 \equiv N'_0 \equiv A0 \quad \Gamma \vdash N_S \equiv N_S : \Pi n : \mathbf{N}, An \rightarrow A(Sn)}{\Gamma \vdash \mathbf{N}_{\text{rec}} A N_0 N_S 0 \equiv N'_0 : A0} \\
\\
\text{INT-REC-SUCC} \frac{\Gamma \vdash A \equiv A' : \mathbf{N} \rightarrow \text{square}_{\mathbf{s}} \quad \Gamma \vdash N_0 \equiv N'_0 \equiv A0 \quad \Gamma \vdash N_S \equiv N'_S : \Pi n : \mathbf{N}, An \rightarrow A(Sn) \quad \Gamma \vdash N \equiv N' : \mathbf{N}}{\Gamma \vdash \mathbf{N}_{\text{rec}} A N_0 N_S (SN) \equiv N'_S N' (\mathbf{N}_{\text{rec}} A' N'_0 N'_S N') : A(SN)} \\
\\
\text{EMP-REC} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \perp_{\text{rec}} \equiv \perp_{\text{rec}} : \Pi A : \perp \rightarrow \Box_{\mathbf{s}}, \Pi e : \perp, Ae} \\
\\
\text{SYM} \frac{\Gamma \vdash M \equiv M' : A}{\Gamma \vdash M' \equiv M : A} \quad \text{TRANS} \frac{\Gamma \vdash M \equiv M' : A \quad \Gamma \vdash M' \equiv M'' : A}{\Gamma \vdash M \equiv M'' : A} \\
\\
\text{CONV} \frac{\Gamma \vdash M \equiv M' : A \quad \Gamma \vdash A \equiv A' : \Box_{\ell}}{\Gamma M \equiv M' : A'}
\end{array}$$

But it is useful to consider the extension, with new terms : $M, N ::= \dots | \mathbf{B} | \mathbf{B}_{\text{rec}} | \overline{\text{tt}} | \overline{\text{ff}}$
And conversion rules

$$\begin{array}{c}
\text{BOOL-TRUE} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \overline{\text{tt}} \equiv \overline{\text{tt}} : \mathbf{B}} \quad \text{BOOL-FALSE} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \overline{\text{ff}} \equiv \overline{\text{ff}} : \mathbf{B}} \\
\\
\text{BOOL-REC} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \mathbf{B}_{\text{rec}} \equiv \mathbf{B}_{\text{rec}} : \Pi A : \mathbf{B} \rightarrow \Box_{\mathbf{s}}, A \overline{\text{tt}} \rightarrow A \overline{\text{ff}} \rightarrow \Pi b : \mathbf{B}, Ab} \\
\\
\text{BOOL-REC-TRUE} \frac{\Gamma \vdash A \equiv A : \mathbf{B} \rightarrow \Box_{\mathbf{s}} \quad \Gamma \vdash M_{\overline{\text{tt}}} \equiv M'_{\overline{\text{tt}}} \equiv A \overline{\text{tt}} \quad \Gamma \vdash M_{\overline{\text{ff}}} \equiv M_{\overline{\text{ff}}} : A \overline{\text{ff}}}{\Gamma \vdash \mathbf{B}_{\text{rec}} A M_{\overline{\text{tt}}} M_{\overline{\text{ff}}} \overline{\text{tt}} \equiv M'_{\overline{\text{tt}}} : A \overline{\text{tt}}} \\
\\
\text{BOOL-REC-FALSE} \frac{\Gamma \vdash A \equiv A : \mathbf{B} \rightarrow \Box_{\mathbf{s}} \quad \Gamma \vdash M_{\overline{\text{tt}}} \equiv M_{\overline{\text{tt}}} \equiv A \overline{\text{tt}} \quad \Gamma \vdash M_{\overline{\text{ff}}} \equiv M'_{\overline{\text{ff}}} : A \overline{\text{ff}}}{\Gamma \vdash \mathbf{B}_{\text{rec}} A M_{\overline{\text{tt}}} M_{\overline{\text{ff}}} \overline{\text{ff}} \equiv M'_{\overline{\text{ff}}} : A \overline{\text{ff}}}
\end{array}$$

4.2 System T

To identify and solve problems in a simpler environnement, we studied a modified System T before, based on the following variant.

With types :

$$A ::= A \rightarrow A | \mathbf{N} | \perp$$

Terms :

$$M, N ::= x | \lambda x. M | MN | 0 | S | \mathbf{N}_{\text{rec}} | \perp_{\text{rec}}$$

Contexts :

$$\Gamma ::= \Gamma, x : A | \cdot$$

And conversion rules :

$$\begin{array}{c}
\text{FUN-INTRO} \frac{\Gamma, x : A \vdash M \equiv M' : B}{\Gamma \vdash \lambda x. M \equiv \lambda x. M' : A \rightarrow B} \quad \text{FUN-ELIM} \frac{\Gamma \vdash M \equiv M' : A \rightarrow B \quad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash MN \equiv M' N' : B} \\
\\
\text{AXIOM} \frac{x : A \in \Gamma}{\Gamma \vdash x \equiv x : A} \quad \text{BETA} \frac{\Gamma, x : A \vdash M \equiv M' : B \quad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash (\lambda x. M) N \equiv M' : B} \\
\\
\text{INT-ZERO} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash 0 \equiv 0 : \mathbf{N}} \quad \text{INT-SUCC} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash S \equiv S : \mathbf{N} \rightarrow \mathbf{N}} \\
\\
\text{INT-REC} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \mathbf{N}_{\text{rec}} \equiv \mathbf{N}_{\text{rec}} : A \rightarrow (\mathbf{N} \rightarrow A \rightarrow A) \rightarrow \mathbf{N} \rightarrow A} \\
\\
\text{INT-REC-ZERO} \frac{\Gamma \vdash N_0 \equiv N'_0 \equiv A \quad \Gamma \vdash N_S \equiv N_S : \mathbf{N} \rightarrow A \rightarrow A}{\Gamma \vdash \mathbf{N}_{\text{rec}} N_0 N_S 0 \equiv N'_0 : A} \\
\\
\text{INT-REC-SUCC} \frac{\Gamma \vdash N_0 \equiv N'_0 \equiv A \quad \Gamma \vdash N_S \equiv N'_S : \mathbf{N} \rightarrow A \rightarrow A \quad \Gamma \vdash N \equiv N' : \mathbf{N}}{\Gamma \vdash \mathbf{N}_{\text{rec}} N_0 N_S (SN) \equiv N'_S N' (\mathbf{N}_{\text{rec}} N'_0 N'_S N') : A}
\end{array}$$

$$\begin{array}{c}
\text{EMP-REC} \frac{}{\Gamma \vdash \perp_{\text{rec}} \equiv \perp_{\text{rec}} : \perp \rightarrow A} \\
\text{SYM} \frac{\Gamma \vdash M \equiv M' : A}{\Gamma \vdash M' \equiv M : A} \quad \text{TRANS} \frac{\Gamma \vdash M \equiv M' : A \quad \Gamma \vdash M' \equiv M'' : A}{\Gamma \vdash M \equiv M'' : A}
\end{array}$$

5 Martin Baillon's ShTT

My work was meant to generalize the work of my supervisor past PhD. student, Martin Baillon. He worked on a roughly similar MLTT extended with boolean and a generic function α .

$$M, N ::= \dots | \alpha$$

with secondary *forcing contexts*, with n and b an integer and boolean respectively

$$\mathcal{L} ::= \mathcal{L}, n \mapsto b | \cdot$$

We write $n \mapsto_{\mathcal{L}} b$ when $n \mapsto b$ appears in \mathcal{L} , and $n \not\mapsto_{\mathcal{L}}$ when neither $n \mapsto \text{tt}$ nor $n \mapsto \text{ff}$ do.

And conversion rules :

$$\text{name} \frac{\mathcal{L}, \Gamma \vdash ?_0 \quad \dots \quad \mathcal{L}, \Gamma \vdash ?_n}{\mathcal{L}, \Gamma \vdash ?}$$

whenever the following is a rule of MLTT (with booleans)

$$\text{name} \frac{\Gamma \vdash ?_0 \quad \dots \quad \Gamma \vdash ?_n}{\Gamma \vdash ?}$$

Exception made of WF-EMPTY, which becomes

$$\text{WF-EMPTY} \frac{}{\cdot, \cdot \vdash \text{well-formed}}$$

The new conversion rules are :

$$\begin{array}{c}
\text{WF-EXT-FORC} \frac{\mathcal{L}, \Gamma \vdash \text{well-formed} \quad n \not\mapsto_{\mathcal{L}}}{\mathcal{L}, n \mapsto b, \Gamma \vdash \text{well-formed}} \\
\text{GEN} \frac{\mathcal{L}, \Gamma \vdash \text{well-formed}}{\mathcal{L}, \Gamma \vdash \alpha \equiv \alpha : \mathbf{N} \rightarrow \mathbf{B}} \quad \text{ASK} \frac{\mathcal{L}, \Gamma \vdash \text{well-formed} \quad n \mapsto_{\mathcal{L}} b}{\mathcal{L}, \Gamma \vdash \alpha \bar{n} \equiv \bar{b}} \\
\text{SPLIT} \frac{\mathcal{L}, n \mapsto \text{tt}, \Gamma \vdash M \equiv M' : A \quad \mathcal{L}, n \mapsto \text{ff}, \Gamma \vdash M \equiv M' : A \quad n \not\mapsto_{\mathcal{L}}}{\mathcal{L}, \Gamma \vdash M \equiv M' : A}
\end{array}$$

6 Logical relations

6.1 System T extension

7 Meta-informations

7.1 Time expenditure

7.2 Difficulties

The subject is rather vast and a bit unclear.

Doing logical relations for MLTT is a large task, even before adding sheaves. My next attempt will probably start from logrel rocq insted of nothing

8 Conclusion

A Dummy

References

- [1] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.