# Internship defense M2
# Parisian Master of Research in Computer Science
# Élimination non-singleton :
# Avec les faisceaux, c'est pas faux

**Soudant Léo**, second year student at ENS Paris Saclay

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Supervisor
**Pierre-Marie Pédrot**

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# The starting point

| Cartesian closed categories | $\Leftrightarrow$ | Simply typed $\lambda$-calculus | $\Leftrightarrow$ | Negative fragment of intuitionistic propositional logic |
|---|---|---|---|---|
| $\mathrm{Hom}(A \times B, C) \simeq \mathrm{Hom}(A, B \to C)$ | | $\lambda x : A.t : A \to B$ | | $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$ |

# Subobject classifier

$$\mathrm{Sub}(A) \simeq \mathrm{Hom}(A, \Omega)$$

Sub(A) := equivalences classes of monomorphisms $X \rightarrowtail A$.
Introduces a type of proposition in the $\lambda$-calculus, as well as subobjects

- All finite limits as well. Finite colimits are recovered[1].

# Formulas

$\wedge, \vee, \forall, \exists, \neg, \bot, \top, \Rightarrow$ can all be expressed *e.g.*

$$\begin{tikzcd} X \wedge Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & A \end{tikzcd}$$

So can $=$ and $\in$. Equality is the predicate $A \times A \to \Omega$ given by the diagonal $A \rightarrowtail A \times A$.

Objects similar to the set theoritic $\{x \in A | \phi(x)\}$ can be constructed.

# Internal logic of $\Omega$

- $\wedge : \mathrm{Sub}(A) \times \mathrm{Sub}(A) \to \mathrm{Sub}(A)$
- $\wedge : \mathrm{Hom}(A, \Omega) \times \mathrm{Hom}(A, \Omega) \to \mathrm{Hom}(A, \Omega)$
- $\wedge : \mathrm{Hom}(A, \Omega \times \Omega) \to \mathrm{Hom}(A, \Omega)$
- $\wedge : \Omega \times \Omega \to \Omega$

Similarily for $\vee, \perp, \top, \Rightarrow$.

# Topoi of presheaves

For a small category $\mathbf{C}$, its category of presheaves is noted $\widehat{\mathbf{C}} := \mathrm{Set}^{\mathbf{C}^{op}}$.

$$(P \times Q)(C) = P(C) \times Q(C)$$

$$P^Q(C) \simeq \mathrm{Nat}(\mathrm{y}C, P^Q) \simeq \mathrm{Nat}(\mathrm{y}C \times Q, P)$$

$\Omega(C)$ is the set of subpresheaves of $\mathrm{y}C$ (also called sieves on $C$).
(A subpresheaf is pointwise a subset, with function restricted)

Small limits are pointwise.

# Sheaves

A sheaf = A presheaf with a compatibility property.

A Grothendieck topology introduces a notion of cover.

- If $f : D \to C$ is in $S(D)$ , then it is covered by $S$.
- If $f$ is covered then $f \circ g$ is covered.
- If $S$ covers $f$ and $R$ covers any map of $S(D)$ for all $D$, then $R$ covers $f$

We note the sieve of all maps covered by $S$ as $\overline{S}$.

A sheaf is a presheaf where a map in $\mathrm{Nat}(S, F)$ correspond to exactly one map of $\mathrm{Nat}(\overline{S}, F)$.

# Example : Sheaves on $\mathrm{R}^2$

$\mathbf{C} := \mathcal{O}(\mathrm{R}^2)$ (poset : at most one map between object).
We fix $C := \mathrm{R}^2$.
$S$ covers $U := (U \subseteq \bigcup S)$ :

- if $U \in S$ then $U \subseteq \bigcup S$
- if $U \subseteq \bigcup S$, and $V \subset U$, then $V \subseteq \bigcup S$
- if $U \subseteq \bigcup S$, and for all $V \in S$, $V \subseteq \bigcup R$, then $U = \bigcup R$

# The topos of sheaves

Finite limits of sheaves as presheaves are sheaves and are finite limits for sheaves.

$F^P$ is a sheaf whenever $F$ is.

$\Omega$ contains the sieves which are also sheaves, that is which only cover themselves.

# Lawvere-Tierney topology

Sheaves are defined through an operator $j : \Omega \to \Omega$ for which

- $id \leq j$, since $S \subseteq \overline{S}$
- $jj \leq j$, since $\overline{S} \subseteq \overline{R}$ whenever $S \subseteq \overline{R}$
- $j \circ \wedge = \wedge \circ j \times j : \Omega \times \Omega \to \Omega$, since for $f$ covered by $S$ and $R$,

This can be defined in any topos.

Or, rather, any type theory with a type of (strict) proposition.

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# Bibliography I

[1] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.