Internship report M2
Parisian Master of Research in Computer Science Level 2

Non-singleton Elimination

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Abstract

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1 Introduction

1.1 The outline

2 Sheaves

2.1 Topoï and sheaves in topoï

Proofs, results and details for this section can often be found in *Sheaves in geometry and logic: A first introduction to topos theory* by Saunders Maclane and Ieke Moerdijk [1].

Definition 2.1 (Subobject). *In a category, a subobject of X is an equivalence class of monomorphism $m : A \rightarrowtail X$, where the equivalence comes from the preorder where $m : A \rightarrowtail X$ is smaller than $m' : A' \rightarrowtail X$ when there is a map $f : A \to A'$ with $m' \circ f = m$.*

We deduce a presheaf **Sub** where **Sub**(X) is the the set of subobjects of X, and **Sub**$(f) : $ **Sub**$(Y) \to $ **Sub**(X) sends $m : A \to Y$ to its pullback by $f : X \to Y$.

Definition 2.2 (Topos). *A topos is a cartesian closed category with all finite limits and a suboject classifier Ω and an isomorphism* **Sub**$(X) \cong$ **Hom**(X, Ω) *natural in X.*

We note that topoï also have finite colimits.

A topos serves to give models of intuitionistic logic in classical mathematical language. It has an internal logic which is higher order.

For example, **Set** is a topos, and given a topos $\mathcal{E}$, $\mathcal{E}/X$, the category of maps with codomain X and commuting triangles, as well as $\mathcal{E}^{\mathbf{C}^{op}}$, the category of contravariant functors from a small category $\mathbf{C}$ and natural transformation, are all topoï. In particular categories of presheaves are topoï, and correspond to Kripke models.

The subobject classifier Ω is equipped with an internal meet-semilattice structure inherited from the meet-semilattice structure on each **Sub**(X), which is natural in X.

Definition 2.3 (Lawvere-Tierney topology). *A Lawvere-Tierney topology is a left exact idempotent monad j on the internal meet-semilattice on Ω.*

- $id_\Omega \leq j$,

- $j \circ j \leq j$

- $j \circ \wedge = \wedge \circ j \times j$

From a topology j we extract a closure operator J_X of **Sub**(X) for any X.

Definition 2.4 (Dense subobject). *A suboject U of X is dense if $J_X U = X$*

A topology can be lifted to a left exact idempotent monad on the entirety of the topos, the sheafification monad.

Definition 2.5 (j-Sheaf in topos). *An object F is a j-sheaf in a topos if for any dense subobject U of any object X, the morphism* **Hom**$(X, F) \to$ **Hom**(U, F) *obtained by precomposition is an isomorphism.*

A j-Sheaf is up to isomorphism the result of sheafifying an object.

j-Sheaves form a topos. The sheaves on a presheaf topos correspond to Beth semantics.

2.2 Sheaves in type theory

Consider a type theory with a notion of proof irrelevant propositions Prop, *e.g.* book-HoTT with mere propositions, or Rocq with SProp.

In this case, a Lawvere-Tierney topology may be similarly defined, as a monad:

- J : Prop $\to$ Prop

- $\eta : \Pi(P : \mathsf{Prop}).P \to \mathsf{J}\ P$

- bind : $\forall(PQ : \mathsf{Prop}).\mathsf{J}\ P \to (P \to \mathsf{J}\ Q) \to \mathsf{J}\ Q$

Then a sheaf is just a type T with

- A map $\mathsf{ask}_T : \Pi(P : \mathsf{Prop}).\ \mathsf{J}\ P \to (P \to T) \to T$

- A coherence $\varepsilon_T : \Pi(P : \mathsf{Prop})\ (j : \mathsf{J}\ P)\ (x : T).\ \mathsf{ask}_T\ P\ j\ (\lambda p : P.x) = x$

Now, the sheafified of a type doesn't exists in general, if the theory admits quotient inductive types, it can then be defined as follow :

$$
\begin{aligned}
&\text{Inductive } \mathcal{S}_\mathsf{J}\ T : \mathsf{Type} := \\
&\mid \mathsf{ret} : T \to \mathcal{S}_\mathsf{J}\ T \\
&\mid \mathsf{ask} : \Pi(P : \mathsf{Prop}), \mathsf{J}\ P \to (P \to \mathcal{S}_\mathsf{J}\ T) \to \mathcal{S}_\mathsf{J}\ T \\
&\mid \varepsilon : \Pi(P : \mathsf{Prop})\ (j : J\ P)\ (x : \mathcal{S}_\mathsf{J}\ T).\ \mathsf{ask}\ P\ j\ (\lambda p : P.x) = x
\end{aligned}
$$

We note that by taking $I := \Sigma(P : \mathsf{Prop}).\mathsf{J}\ P$ and $O\ (P,j) : P$, a sheaf is then a

- A type T

- A map $\mathsf{ask}_T : \Pi(i : I), (O\ i \to T) \to T$

- A coherence map $\varepsilon_T : \Pi(i : I)\ (x : T), \mathsf{ask}_T\ \ i\ (\lambda o : O\ i.x) = x$

and simimlarily for sheafification. This is marginally simpler, and make sheaves appear as quotient dialogue trees, hence why we will henceforth consider (I, O)-*sheaves* instead of J-*sheaves*.

3   Models

A significant part of my internship was dedicated to contructing models of type theory in Rocq.

1. A model of a variant of Baclofen TT using dialogue trees. Predicates must be linearized before eliminating an inductive into them.

2. An exceptional model, with a type of exceptions E. A special type of dialogue trees where $I = E$ and $Oi = \mathbf{0}$, the resulting theory is inconsistent (when E is inhabited), as always when $Oi \to \mathbf{0}$ for some i.

3. A model using sheaves

4   System T and MLTT

4.1   MLTT

We sought to extend the following variant of MLTT.

We consider a type of levels containing two elements $\mathbf{s}$ and $\mathbf{l}$ for small and large, with generic ℓ

With terms :

$$
M, N ::= x \mid \lambda x.M \mid MN \mid 0 \mid S \mid \mathbf{N}_{\mathrm{rec}} \mid \bot_{\mathrm{rec}} \mid \mathbf{N} \mid \bot \mid \Pi x : A.B \mid \Box_{\mathbf{s}} \mid \Box_{\mathbf{l}}
$$

Contexts :

$$
\Gamma ::= \Gamma, x : A \mid \cdot
$$

And conversion rules :

$$
\text{W\textsc{f}-E\textsc{mpty}} \frac{}{\cdot \vdash \text{well-formed}} \qquad \text{W\textsc{f}-E\textsc{xt}} \frac{\Gamma \vdash A \equiv A \quad \Gamma \vdash \text{well-formed}}{\Gamma, x : A \vdash \text{well-formed}}
$$

$$
\text{I\textsc{nt}-T\textsc{yp}} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \mathbf{N} \equiv \mathbf{N} : \Box_\ell} \qquad \text{E\textsc{mp}-T\textsc{yp}} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \bot \equiv \bot : \Box_\ell}
$$

$$
\text{F\textsc{un}-T\textsc{yp}} \frac{\Gamma \vdash A \equiv A' : \Box_\ell \quad \Gamma, x : A \vdash B \equiv B' : \Box_\ell}{\Gamma \vdash \Pi x : A, B \equiv \Pi x : A', B' : \Box_\ell} \qquad \text{T\textsc{yp}-T\textsc{yp}} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \Box_{\mathbf{s}} : \Box_{\mathbf{l}}}
$$

$$
\text{F\textsc{un}-I\textsc{ntro}} \frac{\Gamma, x : A \vdash M \equiv M' : B \quad \Gamma \vdash A \equiv A : \Box_\ell}{\Gamma \vdash \lambda x.M \equiv \lambda x.M' : \Pi x : A, B} \qquad \text{F\textsc{un}-E\textsc{lim}} \frac{\Gamma \vdash M \equiv M' : \Pi x : A, B \quad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash MN \equiv M'N' : B(N/x)}
$$

$$
\text{A\textsc{xiom}} \frac{\Gamma \vdash \text{well-formed} \quad x : A \in \Gamma}{\Gamma \vdash x \equiv x : A} \qquad \text{B\textsc{eta}} \frac{\Gamma, x : A \vdash M \equiv M' : B \quad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash (\lambda x.M)N \equiv M'(N'/x) : B(N/x)}
$$

$$
\text{I\textsc{nt}-Z\textsc{ero}} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash 0 \equiv 0 : \mathbf{N}} \qquad \text{I\textsc{nt}-S\textsc{ucc}} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash S \equiv S : \mathbf{N} \to \mathbf{N}}
$$

$$
\text{I\textsc{nt}-R\textsc{ec}} \frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \mathbf{N}_{\mathrm{rec}} \equiv \mathbf{N}_{\mathrm{rec}} : \Pi A : \mathbf{N} \to \Box_{\mathbf{s}}, A0 \to (\Pi n : \mathbf{N}, An \to A(Sn)) \to \Pi n : \mathbf{N}, An}
$$

$$
\text{I\textsc{nt}-R\textsc{ec}-Z\textsc{ero}} \frac{\Gamma \vdash A \equiv A : \mathbf{N} \to square_{\mathbf{s}} \quad \Gamma \vdash N_0 \equiv N_0' \equiv A0 \quad \Gamma \vdash N_S \equiv N_S : \Pi n : \mathbf{N}, An \to A(Sn)}{\Gamma \vdash \mathbf{N}_{\mathrm{rec}} A N_0 N_S 0 \equiv N_0' : A0}
$$

$$\text{Int-Rec-Succ}\frac{\Gamma \vdash A \equiv A' : \mathbf{N} \to square_{\mathbf{s}} \qquad \Gamma \vdash N_0 \equiv N_0' \equiv A0}{\Gamma \vdash N_S \equiv N_S' : \Pi n : \mathbf{N}, An \to A(Sn) \qquad \Gamma \vdash N \equiv N' : \mathbf{N}}$$
$$\text{Int-Rec-Succ}\frac{}{\Gamma \vdash \mathbf{N}_{\text{rec}} AN_0 N_S(SN) \equiv N_S' N'(\mathbf{N}_{\text{rec}} A' N_0' N_S' N') : A(SN)}$$

$$\text{Emp-Rec}\frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \bot_{\text{rec}} \equiv \bot_{\text{rec}} : \Pi A : \bot \to \square_{\mathbf{s}}, \Pi e : \bot, Ae}$$

$$\text{Sym}\frac{\Gamma \vdash M \equiv M' : A}{\Gamma \vdash M' \equiv M : A} \qquad\qquad \text{Trans}\frac{\Gamma \vdash M \equiv M' : A \qquad \Gamma \vdash M' \equiv M'' : A}{\Gamma \vdash M \equiv M'' : A}$$

$$\text{Conv}\frac{\Gamma \vdash M \equiv M' : A \qquad \Gamma \vdash A \equiv A' : \square_\ell}{\Gamma M \equiv M' : A'}$$

4.2 System T

To identify and solve problems in a simpler envirronement, we studied a modified System T before, based on the following variant.

With types :

$$A ::= A \to A | \mathbf{N} | \bot$$

Terms :

$$M, N ::= x | \lambda x.M | MN | 0 | S | \mathbf{N}_{\text{rec}} | \bot_{\text{rec}}$$

Contexts :

$$\Gamma ::= \Gamma, x : A | \cdot$$

And conversion rules :

$$\text{Fun-Intro}\frac{\Gamma, x : A \vdash M \equiv M' : B}{\Gamma \vdash \lambda x.M \equiv \lambda x.M' : A \to B} \qquad \text{Fun-Elim}\frac{\Gamma \vdash M \equiv M' : A \to B \qquad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash MN \equiv M'N' : B}$$

$$\text{Axiom}\frac{x : A \in \Gamma}{\Gamma \vdash x \equiv x : A} \qquad \text{Beta}\frac{\Gamma, x : A \vdash M \equiv M' : B \qquad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash (\lambda x.M)N \equiv M' : B}$$

$$\text{Int-Zero}\frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash 0 \equiv 0 : \mathbf{N}} \qquad \text{Int-Succ}\frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash S \equiv S : \mathbf{N} \to \mathbf{N}}$$

$$\text{Int-Rec}\frac{\Gamma \vdash \text{well-formed}}{\Gamma \vdash \mathbf{N}_{\text{rec}} \equiv \mathbf{N}_{\text{rec}} : A \to (\mathbf{N} \to A \to A) \to \mathbf{N} \to A}$$

$$\text{Int-Rec-Zero}\frac{\Gamma \vdash N_0 \equiv N_0' \equiv A \qquad \Gamma \vdash N_S \equiv N_S : \mathbf{N} \to A \to A}{\Gamma \vdash \mathbf{N}_{\text{rec}} N_0 N_S 0 \equiv N_0' : A}$$

$$\text{Int-Rec-Succ}\frac{\Gamma \vdash N_0 \equiv N_0' \equiv A \qquad \Gamma \vdash N_S \equiv N_S' : \mathbf{N} \to A \to A \qquad \Gamma \vdash N \equiv N' : \mathbf{N}}{\Gamma \vdash \mathbf{N}_{\text{rec}} N_0 N_S(SN) \equiv N_S' N'(\mathbf{N}_{\text{rec}} N_0' N_S' N') : A}$$

$$\text{Emp-Rec}\frac{}{\Gamma \vdash \bot_{\text{rec}} \equiv \bot_{\text{rec}} : \bot \to A}$$

$$\text{Sym}\frac{\Gamma \vdash M \equiv M' : A}{\Gamma \vdash M' \equiv M : A} \qquad\qquad \text{Trans}\frac{\Gamma \vdash M \equiv M' : A \qquad \Gamma \vdash M' \equiv M'' : A}{\Gamma \vdash M \equiv M'' : A}$$

5 Meta-informations

5.1 Time expenditure

5.2 Difficulties

The subject is rather vast and a bit unclear.

Doing logical relations for MLTT is a large task, even before adding sheaves. My next attempt will probably start from logrel rocq insted of nothing

6 Conclusion

A   Dummy

References

[1] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory.* Springer Science & Business Media, 2012.