Internship defense M2
Parisian Master of Research in Computer Science
Élimination non-singleton :
Avec les faisceaux, c'est pas faux

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The starting point

Cartesian closed categories	$\Leftrightarrow$	Simply typed λ-calculus	$\Leftrightarrow$	Negative fragment of intuitionistic propositional logic
$\mathrm{Hom}(A \times B, C) \simeq \mathrm{Hom}(A, B \to C)$		$\lambda x : A.t : A \to B$		$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$

Subobject classifier

$$\mathrm{Sub}(A) \simeq \mathrm{Hom}(A, \Omega)$$

Sub(A) := equivalences classes of monomorphisms $X \rightarrowtail A$.
Introduces a type of proposition in the λ-calculus, as well as subobjects

- All finite limits as well. Finite colimits are recovered[1].

Formulas

$\wedge, \vee, \forall, \exists, \neg, \bot, \top, \Rightarrow$ can all be expressed *e.g.*

$$\begin{array}{ccc} X \wedge Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & A \end{array}$$

So can $=$ and $\in$. Equality is the predicate $A \times A \to \Omega$ given by the diagonal $A \rightarrowtail A \times A$.

Objects similar to the set theoritic $\{x \in A | \phi(x)\}$ can be constructed.

Internal logic of Ω

- $\wedge : \mathrm{Sub}(A) \times \mathrm{Sub}(A) \to \mathrm{Sub}(A)$
- $\wedge : \mathrm{Hom}(A, \Omega) \times \mathrm{Hom}(A, \Omega) \to \mathrm{Hom}(A, \Omega)$
- $\wedge : \mathrm{Hom}(A, \Omega \times \Omega) \to \mathrm{Hom}(A, \Omega)$
- $\wedge : \Omega \times \Omega \to \Omega$

Similarily for $\vee, \bot, \top, \Rightarrow$.

Topoi of presheaves

For a small category $\mathbf{C}$, its category of presheaves is noted $\widehat{\mathbf{C}} := \mathrm{Set}^{\mathbf{C}^{op}}$.

$$(P \times Q)(C) = P(C) \times Q(C)$$

$$P^Q(C) \simeq \mathrm{Nat}(\mathrm{y}C, P^Q) \simeq \mathrm{Nat}(\mathrm{y}C \times Q, P)$$

$\Omega(C)$ is the set of subpresheaves of $\mathrm{y}C$ (also called sieves on C).
(A subpresheaf is pointwise a subset, with function restricted)

Small limits are pointwise.

Sheaves

A sheaf = A presheaf with a compatibility property.

More precisely : certain sieves are considered special, they are *covering sieves*.
The rule that selects special sieves is called a grothendieck topology :

- The full sieve $\mathrm{y}C$ is covering.
- Pullbacks along $\mathrm{y}f$ preserves covering.
- For any sieve R, if pullback along any map of a covering sieve S yields a covering sieve, then R is covering.

A sheaf is a presheaf where a map in $\mathrm{Nat}(S, F)$, with S a covering sieve, correspond to exactly one element of $F(C) \simeq \mathrm{Nat}(\mathrm{y}C, F)$.

Example : Sheaves on R^2

$$\mathbf{C} := \mathcal{O}(\mathsf{R}^2)$$

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Bibliography I

[1] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.