Internship report M2
Parisian Master of Research in Computer Science Level 2

Non-singleton Elimination

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Abstract

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1 Introduction

1.1 The outline

2 Sheaves

2.1 Topoï and sheaves in topoï

Proofs, results and details for this section can often be found in *Sheaves in geometry and logic: A first introduction to topos theory* by Saunders Maclane and Ieke Moerdijk [1].

Definition 2.1 (Subobject). *In a category, a* subobject *of X is an equivalence class of monomorphism $m : A \rightarrowtail X$, where the equivalence comes from the preorder where $m : A \rightarrowtail X$ is smaller than $m' : A' \rightarrowtail X$ when there is a map $f : A \to A'$ with $m' \circ f = m$.*

We deduce a presheaf **Sub** where $\mathbf{Sub}(X)$ is the the set of subobjects of X, and $\mathbf{Sub}(f) : \mathbf{Sub}(Y) \to \mathbf{Sub}(X)$ sends $m : A \to Y$ to its pullback by $f : X \to Y$.

Definition 2.2 (Topos). *A* topos *is a cartesian closed category with all finite limits and a suboject* classifier Ω *and an isomorphism* $\mathbf{Sub}(X) \cong \mathbf{Hom}(X, \Omega)$ *natural in X.*

We note that topoï also have finite colimits.

A topos serves to give models of intuitionistic logic in classical mathematical language. It has an internal logic which is higher order.

For example, **Set** is a topos, and given a topos $\mathcal{E}$, $\mathcal{E}/X$, the category of maps with codomain X and commuting triangles, as well as $\mathcal{E}^{\mathbf{C}^{op}}$, the category of contravariant functors from a small category $\mathbf{C}$ and natural transformation, are all topoï. In particular categories of presheaves are topoï, and correspond to Kripke models.

The subobject classifier Ω is equipped with an internal meet-semilattice structure inherited from the meet-semilattice structure on each $\mathbf{Sub}(X)$, which is natural in X.

Definition 2.3 (Lawvere-Tierney topology). *A* Lawvere-Tierney topology *is a left exact idempotent monad j on the internal meet-semilattice on Ω.*

- $id_\Omega \leq j$,
- $j \circ j \leq j$
- $j \circ \wedge = \wedge \circ j \times j$

From a topology j we extract a closure operator J_X of $\mathbf{Sub}(X)$ for any X.

Definition 2.4 (Dense subobject). *A suboject U of X is dense if $J_X U = X$*

A topology can be lifted to a left exact idempotent monad on the entirety of the topos, the sheafification monad.

Definition 2.5 (j-Sheaf in topos). *An object F is a j-*sheaf *in a topos if for any dense subobject U of any object X, the morphism* $\mathbf{Hom}(X, F) \to \mathbf{Hom}(U, F)$ *obtained by precomposition is an isomorphism.*

A j-Sheaf is up to isomorphism the result of sheafifying an object.

j-Sheaves form a topos. The sheaves on a presheaf topos correspond to Beth semantics.

2.2 Sheaves in type theory

Consider a type theory with a notion of proof irrelevant propositions Prop, *e.g.* book-HoTT with mere propositions, or Rocq with SProp.

In this case, a Lawvere-Tierney topology may be similarily defined, as a monad:

- $\mathsf{J} : \mathsf{Prop} \to \mathsf{Prop}$
- $\eta : \Pi(P : \mathsf{Prop}).P \to \mathsf{J}\ P$
- $\mathsf{bind} : \forall(PQ : \mathsf{Prop}).\mathsf{J}\ P \to (P \to \mathsf{J}\ Q) \to \mathsf{J}\ Q$

Then a sheaf is just a type T with

- A map $\mathsf{ask}_T : \Pi(P : \mathsf{Prop}).\ \mathsf{J}\ P \to (P \to T) \to T$
- A coherence $\varepsilon_T : \Pi(P : \mathsf{Prop})\ (j : \mathsf{J}\ P)\ (x : T).\ \mathsf{ask}_T\ \ P\ j\ (\lambda p : P.x) = x$

Now, the sheafified of a type doesn't exists in general, if the theory admits quotient inductive types, it can then be defined as follow :

$$
\begin{array}{l}
\mathsf{Inductive}\ \mathcal{S}_{\mathsf{J}}\ T : \mathsf{Type} := \\
|\ \mathsf{ret} : T \to \mathcal{S}_{\mathsf{J}}\ T \\
|\ \mathsf{ask} : \Pi(P : \mathsf{Prop}), \mathsf{J}\ P \to (P \to \mathcal{S}_{\mathsf{J}}\ T) \to \mathcal{S}_{\mathsf{J}}\ T \\
|\ \varepsilon : \Pi(P : \mathsf{Prop})\ (j : J\ P)\ (x : \mathcal{S}_{\mathsf{J}}\ T).\, \mathsf{ask}\ \ P\ j\ (\lambda p : P.x) = x
\end{array}
$$

We note that by taking $I := \Sigma(P : \mathsf{Prop}).\mathsf{J}\ P$ and $O\ (P, j) : P$, a sheaf is then a

- A type T
- A map $\mathsf{ask}_T : \Pi(i : I), (O\ i \to T) \to T$
- A coherence map $\varepsilon_T : \Pi(i : I)\ (x : T), \mathsf{ask}_T\ \ i\ (\lambda o : O\ i.x) = x$

and simimlarly for sheafification. This is marginally simpler, and make sheaves appear as quotient dialogue trees, hence why we will henceforth consider (I, O)-*sheaves* instead of J-*sheaves*.

3 Models

A significant part of my internship was dedicated to contructing models of type theory in Rocq.

1. A model of a variant of Baclofen TT using dialogue trees. Predicates must be linearized before eliminating an inductive into them.
2. An exceptional model, with a type of exceptions E. A special type of dialogue trees where $I = E$ and $Oi = \mathbf{0}$, the resulting theory is inconsistent (when E is inhabited), as always when $Oi \to \mathbf{0}$ for some i.
3. A model using sheaves, which requires univalence, and quotient inductive types to model positive types.
4. A incomplete model using presheaves.

4 System T and MLTT

4.1 MLTT

We sought to extend the following variant of MLTT.

We consider a type of levels containing two elements $\mathbf{s}$ and $\mathbf{l}$ for small and large, with generic ℓ

With terms :

$$M, N ::= x|\lambda x.M|MN|0|S|\mathbf{N}_{\mathrm{rec}}|\bot_{\mathrm{rec}}|\mathbf{N}|\bot|\Pi x : A.B|\square_{\mathbf{s}}|\square_{\mathbf{l}}$$

Contexts :

$$\Gamma ::= \Gamma, x : A|\cdot$$

And conversion rules :

$$\textsc{Wf-Empty}\ \frac{}{\cdot \vdash \mathsf{well\text{-}formed}} \qquad \textsc{Wf-Ext}\ \frac{\Gamma \vdash A \equiv A \qquad \Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma, x : A \vdash \mathsf{well\text{-}formed}}$$

$$\textsc{Int-Typ}\ \frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash \mathbf{N} \equiv \mathbf{N} : \square_\ell} \qquad \textsc{Emp-Typ}\ \frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash \bot \equiv \bot : \square_\ell}$$

$$\textsc{Fun-Typ}\ \frac{\Gamma \vdash A \equiv A' : \square_\ell \qquad \Gamma, x : A \vdash B \equiv B' : \square_\ell}{\Gamma \vdash \Pi x : A, B \equiv \Pi x : A', B' : \square_\ell} \qquad \textsc{Typ-Typ}\ \frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash \square_{\mathbf{s}} : \square_{\mathbf{l}}}$$

$$\textsc{Fun-Intro}\ \frac{\Gamma, x : A \vdash M \equiv M' : B \qquad \Gamma \vdash A \equiv A : \square_\ell}{\Gamma \vdash \lambda x.M \equiv \lambda x.M' : \Pi x : A, B} \qquad \textsc{Fun-Elim}\ \frac{\Gamma \vdash M \equiv M' : \Pi x : A, B \qquad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash MN \equiv M'N' : B(N/x)}$$

$$\textsc{Axiom}\ \frac{\Gamma \vdash \mathsf{well\text{-}formed} \qquad x : A \in \Gamma}{\Gamma \vdash x \equiv x : A} \qquad \textsc{Beta}\ \frac{\Gamma, x : A \vdash M \equiv M' : B \qquad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash (\lambda x.M)N \equiv M'(N'/x) : B(N/x)}$$

$$\textsc{Int-Zero}\ \frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash 0 \equiv 0 : \mathbf{N}} \qquad \textsc{Int-Succ}\ \frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash S \equiv S : \mathbf{N} \to \mathbf{N}}$$

$$\textsc{Int-Rec}\ \frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash \mathbf{N}_{\mathrm{rec}} \equiv \mathbf{N}_{\mathrm{rec}} : \Pi A : \mathbf{N} \to \square_{\mathbf{s}}, A0 \to (\Pi n : \mathbf{N}, An \to A(Sn)) \to \Pi n : \mathbf{N}, An}$$

$$\textsc{Int-Rec-Zero}\frac{\Gamma \vdash A \equiv A : \mathbf{N} \to square_{\mathbf{s}} \quad \Gamma \vdash N_0 \equiv N_0' \equiv A0 \quad \Gamma \vdash N_S \equiv N_S : \Pi n : \mathbf{N}, An \to A(Sn)}{\Gamma \vdash \mathbf{N}_{\text{rec}} A N_0 N_S 0 \equiv N_0' : A0}$$

$$\textsc{Int-Rec-Succ}\frac{\Gamma \vdash A \equiv A' : \mathbf{N} \to square_{\mathbf{s}} \quad \Gamma \vdash N_0 \equiv N_0' \equiv A0 \quad \Gamma \vdash N_S \equiv N_S' : \Pi n : \mathbf{N}, An \to A(Sn) \quad \Gamma \vdash N \equiv N' : \mathbf{N}}{\Gamma \vdash \mathbf{N}_{\text{rec}} A N_0 N_S (SN) \equiv N_S' N' (\mathbf{N}_{\text{rec}} A' N_0' N_S' N') : A(SN)}$$

$$\textsc{Emp-Rec}\frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash \bot_{\text{rec}} \equiv \bot_{\text{rec}} : \Pi A : \bot \to \square_{\mathbf{s}}, \Pi e : \bot, Ae}$$

$$\textsc{Sym}\frac{\Gamma \vdash M \equiv M' : A}{\Gamma \vdash M' \equiv M : A} \qquad \textsc{Trans}\frac{\Gamma \vdash M \equiv M' : A \quad \Gamma \vdash M' \equiv M'' : A}{\Gamma \vdash M \equiv M'' : A}$$

$$\textsc{Conv}\frac{\Gamma \vdash M \equiv M' : A \quad \Gamma \vdash A \equiv A' : \square_{\ell}}{\Gamma M \equiv M' : A'}$$

4.2 System T

To identify and solve problems in a simpler envirronement, we studied a modified System T before, based on the following variant.

With types :

$$A ::= A \to A | \mathbf{N} | \bot$$

Terms :

$$M, N ::= x | \lambda x.M | MN | 0 | S | \mathbf{N}_{\text{rec}} | \bot_{\text{rec}}$$

Contexts :

$$\Gamma ::= \Gamma, x : A | \cdot$$

And conversion rules :

$$\textsc{Fun-Intro}\frac{\Gamma, x : A \vdash M \equiv M' : B}{\Gamma \vdash \lambda x.M \equiv \lambda x.M' : A \to B} \qquad \textsc{Fun-Elim}\frac{\Gamma \vdash M \equiv M' : A \to B \quad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash MN \equiv M'N' : B}$$

$$\textsc{Axiom}\frac{x : A \in \Gamma}{\Gamma \vdash x \equiv x : A} \qquad \textsc{Beta}\frac{\Gamma, x : A \vdash M \equiv M' : B \quad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash (\lambda x.M)N \equiv M' : B}$$

$$\textsc{Int-Zero}\frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash 0 \equiv 0 : \mathbf{N}} \qquad \textsc{Int-Succ}\frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash S \equiv S : \mathbf{N} \to \mathbf{N}}$$

$$\textsc{Int-Rec}\frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash \mathbf{N}_{\text{rec}} \equiv \mathbf{N}_{\text{rec}} : A \to (\mathbf{N} \to A \to A) \to \mathbf{N} \to A}$$

$$\textsc{Int-Rec-Zero}\frac{\Gamma \vdash N_0 \equiv N_0' \equiv A \quad \Gamma \vdash N_S \equiv N_S : \mathbf{N} \to A \to A}{\Gamma \vdash \mathbf{N}_{\text{rec}} N_0 N_S 0 \equiv N_0' : A}$$

$$\textsc{Int-Rec-Succ}\frac{\Gamma \vdash N_0 \equiv N_0' \equiv A \quad \Gamma \vdash N_S \equiv N_S' : \mathbf{N} \to A \to A \quad \Gamma \vdash N \equiv N' : \mathbf{N}}{\Gamma \vdash \mathbf{N}_{\text{rec}} N_0 N_S (SN) \equiv N_S' N' (\mathbf{N}_{\text{rec}} N_0' N_S' N') : A}$$

$$\textsc{Emp-Rec}\frac{}{\Gamma \vdash \bot_{\text{rec}} \equiv \bot_{\text{rec}} : \bot \to A}$$

$$\textsc{Sym}\frac{\Gamma \vdash M \equiv M' : A}{\Gamma \vdash M' \equiv M : A} \qquad \textsc{Trans}\frac{\Gamma \vdash M \equiv M' : A \quad \Gamma \vdash M' \equiv M'' : A}{\Gamma \vdash M \equiv M'' : A}$$

5 Martin Baillon's ShTT

My work was meant to generalize the work of my supervisor past PhD. student, Martin Baillon. He worked on a roughly similar MLTT extended with

$$M, N ::= \dots | \alpha$$

with secondary contexts, with n and b an integer and boolean respectively

$$\mathcal{L}, \overline{n} \mapsto \overline{b} | \cdot$$

6 Logical relations

6.1 System T extension

7 Meta-informations

7.1 Time expenditure

7.2 Difficulties

The subject is rather vast and a bit unclear.

Doing logical relations for MLTT is a large task, even before adding sheaves. My next attempt will probably start from logrel rocq insted of nothing

8 Conclusion

Appendix

A Dummy

References

[1] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.