Internship report M2
Parisian Master of Research in Computer Science Level 2

Non-singleton Elimination

Soudant Léo
fourth year student at ENS Paris Saclay

2025

Supervisor
Pierre-Merie Pédrot, chargé de recherche at Inria Rennes-Bretagne-Atlantique,

Abstract

Contents

1 Introduction

1.1 The outline

2 Sheaves

2.1 Topoï and sheaves in topoï

Proofs, results and details for this section can often be found in *Sheaves in geometry and logic: A first introduction to topos theory* by Saunders Maclane and Ieke Moerdijk [1].

Definition 2.1 (Subobject)**.** *In a category, a* subobject *of X is an equivalence class of monomorphism $m : A \rightarrowtail X$, where the equivalence comes from the preorder where $m : A \rightarrowtail X$ is smaller than $m' : A' \rightarrowtail X$ when there is a map $f : A \to A'$ with $m' \circ f = m$.*

We deduce a presheaf **Sub** where $\mathbf{Sub}(X)$ is the the set of subobjects of X, and $\mathbf{Sub}(f) : \mathbf{Sub}(Y) \to \mathbf{Sub}(X)$ sends $m : A \to Y$ to its pullback by $f : X \to Y$.

Definition 2.2 (Topos)**.** *A* topos *is a cartesian closed category with all finite limits and a suboject* classifier Ω *and an isomorphism* $\mathbf{Sub}(X) \cong \mathbf{Hom}(X, \Omega)$ *natural in X.*

We note that topoï also have finite colimits.
A topos serves to give models of intuitionistic logic in classical mathematical language. It has an internal logic which is higher order.
For example, **Set** is a topos, and given a topos $\mathcal{E}$, $\mathcal{E}/X$, the category of maps with codomain X and commuting triangles, as well as $\mathcal{E}^{\mathbf{C}^{op}}$, the category of contravariant functors from a small category $\mathbf{C}$ and natural transformation, are all topoï. In particular categories of presheaves are topoï, and correspond to Kripke models.
The subobject classifier Ω is equipped with an internal meet-semilattice structure inherited from the meet-semilattice structure on each $\mathbf{Sub}(X)$, which is natural in X.

Definition 2.3 (Lawvere-Tierney topology)**.** *A* Lawvere-Tierney topology *is a left exact idempotent monad j on the internal meet-semilattice on Ω.*

- $id_\Omega \leq j$,
- $j \circ j \leq j$
- $j \circ \wedge = \wedge \circ j \times j$

From a topology j we extract a closure operator J_X of $\mathbf{Sub}(X)$ for any X.

Definition 2.4 (Dense subobject)**.** *A suboject U of X is dense if $JU = X$*

A topology can be lifted to a left exact idempotent monad on the entirety of the topos, the sheafification monad.

Definition 2.5 (j-Sheaf in topos)**.** *An object F is a j-*sheaf *in a topos if for any dense subobject U of any object X, the morphism* $\mathbf{Hom}(X, F) \to \mathbf{Hom}(U, F)$ *obtained by precomposition is an isomorphism.*

A j-Sheaf is up to isomorphism the result of sheafifying an object.
j-Sheaves form a topos. The sheaves on a presheaf topos correspond to Beth semantics.

2.2 Sheaves in type theory

Consider a type theory with a notion of proof irrelevant propositions Prop, *e.g.* book-HoTT with mere propositions, or Rocq with SProp.
Then, a Lawvere-Tierney topology may be similarily defined, as a map $j : \mathsf{Prop} \to \mathsf{Prop}$

3 System T and MLTT

3.1 MLTT

We sought to extend the following variant of MLTT.
We consider a type of levels containing two elements $\mathbf{s}$ and $\mathbf{l}$ for small and large, with generic ℓ
With terms :

$$M, N ::= x|\lambda x.M|MN|0|S|\mathbf{N}_{\mathrm{rec}}|\bot_{\mathrm{rec}}|\mathbf{N}|\bot|\Pi x : A.B|\Box_{\mathbf{s}}|\Box_{\mathbf{l}}$$

Contexts :

$$\Gamma ::= \Gamma, x : A|\cdot$$

And conversion rules :

$$\textsc{Wf-Empty}\,\frac{}{\cdot \vdash \mathsf{well\text{-}formed}} \qquad \textsc{Wf-Ext}\,\frac{\Gamma \vdash A \equiv A \quad \Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma, x : A \vdash \mathsf{well\text{-}formed}}$$

$$\textsc{Int-Typ}\,\frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash \mathbf{N} \equiv \mathbf{N} : \Box_\ell} \qquad \textsc{Emp-Typ}\,\frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash \bot \equiv \bot : \Box_\ell}$$

$$\textsc{Fun-Typ}\,\frac{\Gamma \vdash A \equiv A' : \Box_\ell \quad \Gamma, x : A \vdash B \equiv B' : \Box_\ell}{\Gamma \vdash \Pi x : A, B \equiv \Pi x : A', B' : \Box_\ell} \qquad \textsc{Typ-Typ}\,\frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash \Box_\mathbf{s} : \Box_\mathbf{l}}$$

$$\textsc{Fun-Intro}\,\frac{\Gamma, x : A \vdash M \equiv M' : B \quad \Gamma \vdash A \equiv A : \Box_\ell}{\Gamma \vdash \lambda x.M \equiv \lambda x.M' : \Pi x : A, B} \qquad \textsc{Fun-Elim}\,\frac{\Gamma \vdash M \equiv M' : \Pi x : A, B \quad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash MN \equiv M'N' : B(N/x)}$$

$$\textsc{Axiom}\,\frac{\Gamma \vdash \mathsf{well\text{-}formed} \quad x : A \in \Gamma}{\Gamma \vdash x \equiv x : A} \qquad \textsc{Beta}\,\frac{\Gamma, x : A \vdash M \equiv M' : B \quad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash (\lambda x.M)N \equiv M'(N'/x) : B(N/x)}$$

$$\textsc{Int-Zero}\,\frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash 0 \equiv 0 : \mathbf{N}} \qquad \textsc{Int-Succ}\,\frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash S \equiv S : \mathbf{N} \to \mathbf{N}}$$

$$\textsc{Int-Rec}\,\frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash \mathbf{N}_{\text{rec}} \equiv \mathbf{N}_{\text{rec}} : \Pi A : \mathbf{N} \to \Box_\mathbf{s}, A0 \to (\Pi n : \mathbf{N}, An \to A(Sn)) \to \Pi n : \mathbf{N}, An}$$

$$\textsc{Int-Rec-Zero}\,\frac{\Gamma \vdash A \equiv A : \mathbf{N} \to square_\mathbf{s} \quad \Gamma \vdash N_0 \equiv N_0' \equiv A0 \qquad \Gamma \vdash N_S \equiv N_S : \Pi n : \mathbf{N}, An \to A(Sn)}{\Gamma \vdash \mathbf{N}_{\text{rec}} A N_0 N_S 0 \equiv N_0' : A0}$$

$$\textsc{Int-Rec-Succ}\,\frac{\Gamma \vdash A \equiv A' : \mathbf{N} \to square_\mathbf{s} \quad \Gamma \vdash N_0 \equiv N_0' \equiv A0 \qquad \Gamma \vdash N_S \equiv N_S' : \Pi n : \mathbf{N}, An \to A(Sn) \quad \Gamma \vdash N \equiv N' : \mathbf{N}}{\Gamma \vdash \mathbf{N}_{\text{rec}} A N_0 N_S (SN) \equiv N_S' N' (\mathbf{N}_{\text{rec}} A' N_0' N_S' N') : A(SN)}$$

$$\textsc{Emp-Rec}\,\frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash \bot_{\text{rec}} \equiv \bot_{\text{rec}} : \Pi A : \bot \to \Box_\mathbf{s}, \Pi e : \bot, Ae}$$

$$\textsc{Sym}\,\frac{\Gamma \vdash M \equiv M' : A}{\Gamma \vdash M' \equiv M : A} \qquad \textsc{Trans}\,\frac{\Gamma \vdash M \equiv M' : A \quad \Gamma \vdash M' \equiv M'' : A}{\Gamma \vdash M \equiv M'' : A}$$

$$\textsc{Conv}\,\frac{\Gamma \vdash M \equiv M' : A \quad \Gamma \vdash A \equiv A' : \Box_\ell}{\Gamma M \equiv M' : A'}$$

3.2 System T

To identify and solve problems in a simpler envirronement, we studied a modified System T before, based on the following variant.

With types :

$$A ::= A \to A | \mathbf{N} | \bot$$

Terms :

$$M, N ::= x | \lambda x.M | MN | 0 | S | \mathbf{N}_{\text{rec}} | \bot_{\text{rec}}$$

Contexts :

$$\Gamma ::= \Gamma, x : A | \cdot$$

And conversion rules :

$$\textsc{Fun-Intro}\,\frac{\Gamma, x : A \vdash M \equiv M' : B}{\Gamma \vdash \lambda x.M \equiv \lambda x.M' : A \to B} \qquad \textsc{Fun-Elim}\,\frac{\Gamma \vdash M \equiv M' : A \to B \quad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash MN \equiv M'N' : B}$$

$$\textsc{Axiom}\,\frac{x : A \in \Gamma}{\Gamma \vdash x \equiv x : A} \qquad \textsc{Beta}\,\frac{\Gamma, x : A \vdash M \equiv M' : B \quad \Gamma \vdash N \equiv N' : A}{\Gamma \vdash (\lambda x.M)N \equiv M' : B}$$

$$\textsc{Int-Zero}\,\frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash 0 \equiv 0 : \mathbf{N}} \qquad \textsc{Int-Succ}\,\frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash S \equiv S : \mathbf{N} \to \mathbf{N}}$$

$$\textsc{Int-Rec}\,\frac{\Gamma \vdash \mathsf{well\text{-}formed}}{\Gamma \vdash \mathbf{N}_{\text{rec}} \equiv \mathbf{N}_{\text{rec}} : A \to (\mathbf{N} \to A \to A) \to \mathbf{N} \to A}$$

$$\textsc{Int-Rec-Zero}\,\frac{\Gamma \vdash N_0 \equiv N_0' \equiv A \qquad \Gamma \vdash N_S \equiv N_S : \mathbf{N} \to A \to A}{\Gamma \vdash \mathbf{N}_{\text{rec}} N_0 N_S 0 \equiv N_0' : A}$$

$$\textsc{Int-Rec-Succ}\ \frac{\Gamma \vdash N_0 \equiv N_0' \equiv A \quad \Gamma \vdash N_S \equiv N_S' : \mathbf{N} \to A \to A \quad \Gamma \vdash N \equiv N' : \mathbf{N}}{\Gamma \vdash \mathbf{N}_{\mathrm{rec}} N_0 N_S (SN) \equiv N_S' N' (\mathbf{N}_{\mathrm{rec}} N_0' N_S' N') : A}$$

$$\textsc{Emp-Rec}\ \frac{}{\Gamma \vdash \bot_{\mathrm{rec}} \equiv \bot_{\mathrm{rec}} : \bot \to A}$$

$$\textsc{Sym}\ \frac{\Gamma \vdash M \equiv M' : A}{\Gamma \vdash M' \equiv M : A} \qquad \textsc{Trans}\ \frac{\Gamma \vdash M \equiv M' : A \quad \Gamma \vdash M' \equiv M'' : A}{\Gamma \vdash M \equiv M'' : A}$$

4 Meta-informations

4.1 Time expenditure

4.2 Difficulties

5 Conclusion

Appendix

A Dummy

References

[1] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.