Internship defense M2
Parisian Master of Research in Computer Science
Élimination non-singleton :
Avec les faisceaux, c'est pas faux

Soudant Léo, fourth year student at ENS Paris Saclay

March to July 2025

Supervisor
Pierre-Marie Pédrot

Table of contents

The starting point

Cartesian closed categories	$\Leftrightarrow$	Simply typed λ-calculus	$\Leftrightarrow$	Negative fragment of intuitionistic propositional logic
$\mathrm{Hom}(A \times B, C) \simeq \mathrm{Hom}(A, B \to C)$		$\lambda x : A.t : A \to B$		$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$

Subobject classifier

$$\mathrm{Sub}(A) \simeq \mathrm{Hom}(A, \Omega)$$

Sub(A) := equivalences classes of monomorphisms $X \rightarrowtail A$.
Introduces a type of proposition in the λ-calculus, as well as subobjects

- All finite limits as well. Finite colimits are recovered[1].

Formulas

$\wedge, \vee, \forall, \exists, \neg, \perp, \top, \Rightarrow$ can all be expressed *e.g.*

$$\begin{tikzcd} X \wedge Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & A \end{tikzcd}$$

So can $=$ and $\in$. Equality is the predicate $A \times A \to \Omega$ given by the diagonal $A \rightarrowtail A \times A$.

Objects similar to the set theoritic $\{x \in A | \phi(x)\}$ can be constructed.

Internal logic of Ω

- $\wedge : \mathrm{Sub}(A) \times \mathrm{Sub}(A) \to \mathrm{Sub}(A)$
- $\wedge : \mathrm{Hom}(A, \Omega) \times \mathrm{Hom}(A, \Omega) \to \mathrm{Hom}(A, \Omega)$
- $\wedge : \mathrm{Hom}(A, \Omega \times \Omega) \to \mathrm{Hom}(A, \Omega)$
- $\wedge : \Omega \times \Omega \to \Omega$

Similarily for $\vee, \perp, \top, \Rightarrow$.

Topoi of presheaves

For a small category $\mathbf{C}$, its category of presheaves is noted $\widehat{\mathbf{C}} := \mathrm{Set}^{\mathbf{C}^{op}}$.

$$(P \times Q)(C) = P(C) \times Q(C)$$

$$P^Q(C) \simeq \mathrm{Nat}(\mathrm{y}C, P^Q) \simeq \mathrm{Nat}(\mathrm{y}C \times Q, P)$$

$\Omega(C)$ is the set of subpresheaves of $\mathrm{y}C$ (also called sieves on C).
(A subpresheaf is pointwise a subset, with function restricted)

Small limits are pointwise.

Sheaves

A sheaf = A presheaf with a compatibility property.

A Grothendieck topology introduces a notion of cover.

- If $f : D \to C$ is in $S(D)$, then it is covered by S.
- If f is covered then $f \circ g$ is covered.
- If S covers f and R covers any map of $S(D)$ for all D, then R covers f

We note the sieve of all maps covered by S as $\overline{S}$.

A sheaf is a presheaf where the inclusion of $\mathrm{Nat}(\overline{S}, F)$ in $\mathrm{Nat}(S, F)$ is an equivalence.

Example : Sheaves on R^2

$\mathbf{C} := \mathcal{O}(\mathrm{R}^2)$ (poset : at most one map between object).
We fix $C := \mathrm{R}^2$.
S covers $U := (U \subseteq \bigcup S)$:

- if $U \in S$ then $U \subseteq \bigcup S$
- if $U \subseteq \bigcup S$, and $V \subset U$, then $V \subseteq \bigcup S$
- if $U \subseteq \bigcup S$, and for all $V \in S$, $V \subseteq \bigcup R$, then $U = \bigcup R$

The topos of sheaves

Finite limits of sheaves as presheaves are sheaves and are finite limits for sheaves.

F^P is a sheaf whenever F is.

Ω contains the sieves which are also sheaves, that is which only cover themselves.

Lawvere-Tierney topology

Sheaves are defined through an operator $j : \Omega \to \Omega$ for which

- $id \leq j$, since $S \subseteq \overline{S}$
- $\Rightarrow \circ id \times j \leq \Rightarrow \circ j \times j : \Omega \times \Omega \to \Omega$, since $\overline{S} \subseteq \overline{R}$ whenever $S \subseteq \overline{R}$

This can be defined in any topos.

Or, rather, any type theory with a type of (strict) propositions.

Table of contents

J-Sheaves

Given J : Prop $\rightarrow$ Prop with

- $\text{ret}_J : \Pi P, P \rightarrow JP$
- $\text{bind}_J : \Pi QP, JP \rightarrow (P \rightarrow JQ) \rightarrow JQ$

A sheaf is then a type F together with

- A map ask $: \Pi P, JP \rightarrow (P \rightarrow F) \rightarrow F$
- Such that for all proposition P and any proof j of JP, ask $P\ j\ (\lambda\_, f) = f$

IO-sheaves

Noting $I := \Sigma P.JP$: Type and $O(P, j) := P : I \to \text{Prop}$,

A sheaf is equivalently given by:

$$\text{Shf} \quad := \left\{ \begin{array}{cl} F & : \text{Type} \\ \text{ask} & : \Pi i : I, (O\ i \to F) \to F \\ \varepsilon & : \Pi(i : I)(f : F), \text{ask}\ i\ (\lambda\_, f) = f \end{array} \right\}$$

A type has at most one sheaf structure (with univalence).

Table of contents

The sheaf model

$$
\begin{aligned}
[\mathrm{Type}] &:= \{F := \mathrm{Shf}; \dots\} \\
[\Pi a : A, B] &:= \{F := \Pi a : [\![A]\!], [\![B]\!]; \dots\} \\
[\![A]\!] &:= F\ [A]
\end{aligned}
$$

Positive types

Interpreting positive types requires higher inductive types.

$$
\begin{aligned}
&\text{Inductive B : Type} := \\
&|\ \text{true : B} \\
&|\ \text{false : B} \\
&|\ \text{ask}_\text{B} : \Pi(i : I).(O\ i \to \text{B}) \to \text{B} \\
&|\ \varepsilon_\text{B} : \Pi(i : I)\ (b : \text{B}).\text{ask}\ i\ (\lambda\_.b) = b
\end{aligned}
$$

$$[B] \quad := F := B; \text{ask} := \text{ask}_B; \varepsilon := \varepsilon_B$$

Basic sheaf theory

$$[\mathsf{ask}] \quad := \mathsf{ask}$$

But ask has type $\Pi(A : \mathsf{Shf})(i : I), (O\ i \to A) \to A$, should be $[\![?]\!]$

An approximation :

$$\frac{O_i, \Gamma \vdash k : A}{\Gamma \vdash \mathsf{ask}_i\ k : A} \qquad \frac{}{O_i, \Gamma \vdash \mathsf{ask}_i\ k \equiv k}$$

With $[\mathsf{ask}_i\ k] := \mathsf{ask}\ [A]\ i\ (\lambda(o_i : O_i), [k])$

Geometric formulas

$$\bigwedge_{i:I}\left(\forall\overrightarrow{x}, O_i \Rightarrow \bigvee_{j:J_i}\exists\overrightarrow{y}, Q_{i,j}\right)$$

Up to curryfication, of the shape $\Pi i, Oi$.

$$\frac{Q_{i,1}, \overrightarrow{y}, \Gamma \vdash k_1 : A \qquad \cdots \qquad Q_{i,n}, \overrightarrow{y}, \Gamma \vdash k_n : A}{O_i, \Gamma \vdash \mathsf{ask}_i\ A\ \overrightarrow{x}\ \overrightarrow{k} : A}$$

Compatibility

Using $\bigwedge_{i:I} \bigvee_{j:J_i} O_{i,j}$:

$$\frac{(O_{i,j}, O_{i,j'}, \Gamma \vdash k_j \equiv k_{j'} : A)_{j,j'}}{\Gamma \vdash \mathsf{ask}_i \ A \ (k_j)_{j:J_i} : A}$$

Normalisation of a modified system T

- No types-as-terms
- Natural numbers, functions, empty type

$$\bigwedge_{i:I} \bigvee_{j:J_i} O_{i,j}$$

Proof of normalisation through logical relations.

Example of working theory

With an extra cohen real $\alpha : \mathsf{N} \to \mathsf{B}$

$$\begin{array}{ll} & (\bigwedge_n \top \to (n \mapsto_\alpha \mathsf{true}) \vee (n \mapsto_\alpha \mathsf{false})) \\ \wedge & (\bigwedge_n (n \mapsto_\alpha \mathsf{true}) \wedge (n \mapsto_\alpha \mathsf{false}) \to \bot) \end{array}$$

$$\frac{}{n \mapsto_\alpha b \vdash \alpha \; n \equiv b : \mathsf{B}} \text{ and } \mathsf{ask}_n \; A := \mathsf{B}_{rec} \; A \; (\alpha \; n)$$

Can be used to show continuity of functionals $(\mathsf{N} \to \mathsf{B}) \to \mathsf{N}$

Table of contents

Conclusion

- Generalise to all geometric formulas
- Generalise normalisation to type theory with types-as-terms
- Study specific theories.

Table of contents

Bibliography I

[1] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.