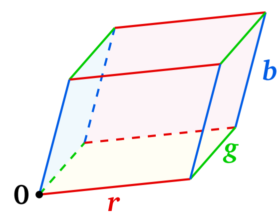
Mathematics for Machine Learning

Exercises 3: Analytic geometry



3.1 Show that $\langle \cdot, \cdot \rangle$, defined for all $\boldsymbol{x} = [x_1 \; x_2]^\top \in \mathbb{R}^2$ and $\boldsymbol{y} = [y_1 \; y_2]^\top \in \mathbb{R}^2$ by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2 x_2 y_2,$$

is an inner product.

An inner product must be all of:

Symmetric

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2 : \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle.$$

Bilinear

$$\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^2,\ \forall \lambda, \phi \in \mathbb{R} : \begin{cases} \langle \lambda \boldsymbol{x} + \phi \boldsymbol{y}, \boldsymbol{z} \rangle = \lambda \langle \boldsymbol{x}, \boldsymbol{z} \rangle + \phi \langle \boldsymbol{y}, \boldsymbol{z} \rangle, \\ \langle \boldsymbol{x}, \lambda \boldsymbol{y} + \phi \boldsymbol{z} \rangle = \lambda \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \phi \langle \boldsymbol{x}, \boldsymbol{z} \rangle. \end{cases}$$

Positive definite

$$\forall \boldsymbol{x} \in \mathbb{R}^2 \setminus \{\boldsymbol{0}\} : \langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, \qquad \langle \boldsymbol{0}, \boldsymbol{0} \rangle = 0.$$

That the given $\langle \cdot, \cdot \rangle$ is symmetric is obvious. For bilinearity, consider that fixing one argument while holding the other constant, there are only linear terms in the components of each (no quadratic terms). For positive definiteness, consider

$$\begin{aligned} \langle \boldsymbol{x}, \boldsymbol{x} \rangle &= x_1^2 - 2 x_1 x_2 + 2 x_2^2 \\ &= (x_1 - x_2)^2 + x_2^2 \end{aligned}$$

which is always positive for non-zero x_1, x_2, and $\langle \boldsymbol{0}, \boldsymbol{0} \rangle = 0$, as required.

This inner product may be represented with the following symmetric positive definite matrix:

$$\mathbf{A} := \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \qquad \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\top \mathbf{A} \boldsymbol{y}$$

3.2 Consider $\mathbb{R}^2$ with $\langle\cdot,\cdot\rangle$ defined for all $\boldsymbol{x},\boldsymbol{y}$ in $\mathbb{R}^2$ as

$$\langle \boldsymbol{x},\boldsymbol{y}\rangle = \boldsymbol{x}^\top \underbrace{\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}}_{\mathbf{A}:=} \boldsymbol{y}$$

Is $\langle\cdot,\cdot\rangle$ an inner product?

No. An inner product may only be defined by a symmetric positive definite matrix. Since $\mathbf{A}$ is not symmetric, $\langle\cdot,\cdot\rangle$ is not symmetric in its arguments.

3.3 Compute the distance between

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

a. Using $\langle \boldsymbol{x},\boldsymbol{y}\rangle = \boldsymbol{x}^\top \boldsymbol{y}$

$$\boldsymbol{x}-\boldsymbol{y} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \qquad \begin{aligned} d(\boldsymbol{x},\boldsymbol{y}) &= \|\boldsymbol{x}-\boldsymbol{y}\| \\ &= \sqrt{\langle \boldsymbol{x}-\boldsymbol{y}, \boldsymbol{x}-\boldsymbol{y}\rangle} \\ &= \sqrt{22} \\ &\approx 4.69 \end{aligned}$$

b. Using $\langle \boldsymbol{x},\boldsymbol{y}\rangle = \boldsymbol{x}^\top \mathbf{A}\boldsymbol{y}, \quad \mathbf{A} := \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$\mathbf{A}(\boldsymbol{x}-\boldsymbol{y}) = \begin{bmatrix} 7 \\ 8 \\ 3 \end{bmatrix} \qquad (\boldsymbol{x}-\boldsymbol{y})^\top \mathbf{A}(\boldsymbol{x}-\boldsymbol{y}) = 47 \qquad d(\boldsymbol{x},\boldsymbol{y}) = \sqrt{47} \approx 6.86$$

3.4 Compute the angle between

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

a. Using $\langle \boldsymbol{x},\boldsymbol{y}\rangle = \boldsymbol{x}^\top \boldsymbol{y}$

$$\cos\theta = \frac{\langle \boldsymbol{x},\boldsymbol{y}\rangle}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|} \qquad \langle \boldsymbol{x},\boldsymbol{y}\rangle = -3 \qquad \|\boldsymbol{x}\| = \sqrt{5} \qquad \|\boldsymbol{y}\| = \sqrt{2} \qquad \theta \approx 161.6^\circ$$

b. Using $\langle \boldsymbol{x},\boldsymbol{y}\rangle = \boldsymbol{x}^\top \mathbf{B}\boldsymbol{y}, \quad \mathbf{B} := \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

$$\mathbf{B}\boldsymbol{x} = \begin{bmatrix} 4 \\ 7 \end{bmatrix} \quad \mathbf{B}\boldsymbol{y} = \begin{bmatrix} -3 \\ -4 \end{bmatrix} \qquad \langle \boldsymbol{x},\boldsymbol{y}\rangle = -11 \qquad \|\boldsymbol{x}\| = \sqrt{18} \qquad \|\boldsymbol{y}\| = \sqrt{7} \qquad \theta \approx 168.5^\circ$$

3.5 Consider the Euclidean vector space $\mathbb{R}^5$ with the dot product. A subspace $U \subseteq \mathbb{R}^5$ and $\boldsymbol{x} \in \mathbb{R}^5$ are given by

$$U = \operatorname{span}\left(\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 5 \\ 0 \\ 7 \end{bmatrix}\right) \qquad \boldsymbol{x} = \begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

a. Determine the orthogonal projection $\pi_U(\boldsymbol{x})$ of $\boldsymbol{x}$ onto U.

b. Determine the distance $d(\boldsymbol{x}, U)$.

Let $\mathbf{B} = [\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3\ \boldsymbol{b}_4]$, where the respective $\boldsymbol{b}_i$ are the vectors in the span of U as above, and let $\boldsymbol{\lambda} = [\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4]^\top$, such that $\pi_U(\boldsymbol{x}) = \mathbf{B}\boldsymbol{\lambda}$, where $\boldsymbol{\lambda}$ gives the coordinates of the projection of $\boldsymbol{x}$ onto U, relative to the ordered basis of U.

The residual, $\boldsymbol{x} - \mathbf{B}\boldsymbol{\lambda}$, must be orthogonal to each basis vector:

$$\begin{aligned} \langle \boldsymbol{b}_1, \boldsymbol{x} - \mathbf{B}\boldsymbol{\lambda} \rangle &= 0 \\ &\vdots \\ \langle \boldsymbol{b}_4, \boldsymbol{x} - \mathbf{B}\boldsymbol{\lambda} \rangle &= 0 \end{aligned}$$

which, given bilinearity and our use of the standard dot product, is equivalently:

$$\begin{aligned} \langle \boldsymbol{b}_1, \mathbf{B}\boldsymbol{\lambda} \rangle &= \langle \boldsymbol{b}_1, \boldsymbol{x} \rangle \\ &\vdots \\ \langle \boldsymbol{b}_4, \mathbf{B}\boldsymbol{\lambda} \rangle &= \langle \boldsymbol{b}_4, \boldsymbol{x} \rangle \end{aligned} \quad \iff \quad \begin{aligned} \boldsymbol{b}_1^\top \mathbf{B}\boldsymbol{\lambda} &= \boldsymbol{b}_1^\top \boldsymbol{x} \\ &\vdots \\ \boldsymbol{b}_4^\top \mathbf{B}\boldsymbol{\lambda} &= \boldsymbol{b}_4^\top \boldsymbol{x} \end{aligned} \quad \iff \quad \mathbf{B}^\top \mathbf{B}\boldsymbol{\lambda} = \mathbf{B}^\top \boldsymbol{x}$$

Note, we find $\mathbf{B}$ is rank 3 — its columns are linearly dependent. Consequently, its Gram matrix ($\mathbf{B}^\top\mathbf{B}$) is singular. Below, we proceed to find $\boldsymbol{\lambda}'$ with Gaussian elimination, but note it may instead be computed directly by inverting the Gram matrix in the expression on the right above — though only if the Gram matrix is invertible.

So, discarding $\boldsymbol{b}_4$ and λ_4, let $\mathbf{B}' = [\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3]$, eliminate $[\,\mathbf{B}'^\top\mathbf{B} \mid \mathbf{B}'^\top\boldsymbol{x}\,] \rightsquigarrow [\,\mathbf{I} \mid \boldsymbol{\lambda}'\,]$, and finally find $\pi_U(\boldsymbol{x}) = \mathbf{B}\boldsymbol{\lambda}'$, and the corresponding residual:

$$\left[\begin{array}{ccc|c} 9 & 9 & 0 & 9 \\ 9 & 16 & -14 & 23 \\ 0 & -14 & 31 & -25 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 7 & -14 & 14 \\ 0 & -14 & 31 & -25 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 3 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\pi_U(\boldsymbol{x}) = \begin{bmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{bmatrix} \qquad \boldsymbol{x} - \pi_U(\boldsymbol{x}) = \begin{bmatrix} -2 \\ -4 \\ 0 \\ 6 \\ -2 \end{bmatrix}$$

Compute the distance, $d(\boldsymbol{x}, U) = \|\boldsymbol{x} - \pi_U(\boldsymbol{x})\| = \sqrt{60} = 2\sqrt{15} \approx 7.75$.

3.6 Consider $\mathbb{R}^3$ with the inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^\top \mathbf{A} \boldsymbol{y}$, where

$$\mathbf{A} := \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Furthermore, define $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ to be the standard / canonical basis in $\mathbb{R}^3$.

a. Determine the orthogonal projection $\pi_U(\boldsymbol{e}_2)$ of $\boldsymbol{e}_2$ onto

$$U = \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_3).$$

b. Compute the distance $d(\boldsymbol{e}_2, U)$.

c. Draw the scenario, standard basis vectors and $\pi_U(\boldsymbol{e}_2)$.

Let $\mathbf{B} = [\boldsymbol{e}_1\ \boldsymbol{e}_3]$, and let $\boldsymbol{\lambda} = [\lambda_1\ \lambda_2]^\top$ be the coordinates of $\pi_U(\boldsymbol{e}_2)$ in the ordered basis of U, such that $\pi_U(\boldsymbol{e}_2) = \mathbf{B}\boldsymbol{\lambda}$. Then

$$\begin{matrix} \langle \boldsymbol{e}_1, \boldsymbol{e}_2 - \mathbf{B}\boldsymbol{\lambda} \rangle = 0 \\ \langle \boldsymbol{e}_3, \boldsymbol{e}_2 - \mathbf{B}\boldsymbol{\lambda} \rangle = 0 \end{matrix} \iff \begin{matrix} \boldsymbol{e}_1^\top \mathbf{A} \boldsymbol{e}_2 = \boldsymbol{e}_1^\top \mathbf{A}\mathbf{B}\boldsymbol{\lambda} \\ \boldsymbol{e}_3^\top \mathbf{A} \boldsymbol{e}_2 = \boldsymbol{e}_3^\top \mathbf{A}\mathbf{B}\boldsymbol{\lambda} \end{matrix} \iff \mathbf{B}^\top \mathbf{A} \boldsymbol{e}_2 = \mathbf{B}^\top \mathbf{A}\mathbf{B}\boldsymbol{\lambda}$$

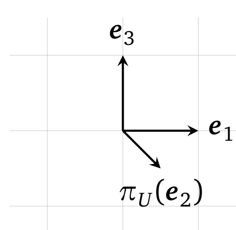
Computing

$$\mathbf{B}^\top \mathbf{A} \boldsymbol{e}_2 = \mathbf{B}^\top \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \mathbf{B}^\top \mathbf{A}\mathbf{B} = \mathbf{B}^\top \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\boldsymbol{\lambda} = \tfrac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \pi_U(\boldsymbol{e}_2) = \tfrac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \boldsymbol{e}_2 - \pi_U(\boldsymbol{e}_2) = \tfrac{1}{2}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Finally,

$$\begin{aligned} d(\boldsymbol{e}_2, U) &= \|\boldsymbol{e}_2 - \pi_U(\boldsymbol{e}_2)\| \\ &= \sqrt{(\boldsymbol{e}_2 - \pi_U(\boldsymbol{e}_2))^\top \mathbf{A} (\boldsymbol{e}_2 - \pi_U(\boldsymbol{e}_2))} \\ &= \sqrt{\tfrac{1}{2}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}^\top \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \\ &= 1. \end{aligned}$$



3.7 Let V be a vector space and π an endomorphism of V.

a. Prove that π is a projection if and only if $\mathrm{id}_V - \pi$ is a projection, where id_V is the identity endomorphism on V.

Proof. Define $\rho = \mathrm{id}_V - \pi$. Then

$$\rho^2 = \mathrm{id}_V - 2\pi + \pi^2.$$

$$\rho^2 = \rho \quad \begin{aligned} &\iff \quad \mathrm{id}_V - 2\pi + \pi^2 = \mathrm{id}_V - \pi \\ &\iff \quad \pi^2 = \pi \end{aligned}$$

□

b. Assume now that π is a projection. Calculate $\mathrm{Im}(\mathrm{id}_V - \pi)$ and $\ker(\mathrm{id}_V - \pi)$ as a function of $\mathrm{Im}(\pi)$ and $\ker(\pi)$.

$$\mathrm{Im}(\mathrm{id}_V - \pi) = \ker(\pi)$$

Proof. Let $\boldsymbol{v} \in \mathrm{Im}(\mathrm{id}_V - \pi)$, let $\boldsymbol{u} \in V$ be a preimage of $\boldsymbol{v}$ under $\mathrm{id}_V - \pi$, and consider:

$$\begin{aligned} \boldsymbol{v} &= (\mathrm{id}_V - \pi)(\boldsymbol{u}) \\ &= \boldsymbol{u} - \pi(\boldsymbol{u}), \\ \pi(\boldsymbol{v}) &= \pi(\boldsymbol{u}) - \pi(\pi(\boldsymbol{u})) \\ &= \pi(\boldsymbol{u}) - \pi(\boldsymbol{u}) \\ &= \boldsymbol{0} \implies \boldsymbol{v} \in \ker(\pi). \end{aligned}$$

Then let $\boldsymbol{w} \in \ker(\pi)$, and consider:

$$\begin{aligned} \pi(\boldsymbol{w}) &= \boldsymbol{0}, \\ (\mathrm{id}_V - \pi)(\boldsymbol{w}) &= \boldsymbol{w} - \pi(\boldsymbol{w}) \\ &= \boldsymbol{w} \implies \boldsymbol{w} \in \mathrm{Im}(\mathrm{id}_V - \pi). \end{aligned}$$

□

$$\ker(\mathrm{id}_V - \pi) = \mathrm{Im}(\pi)$$

Proof. Let $\boldsymbol{w} \in \ker(\mathrm{id}_V - \pi)$, and consider:

$$\begin{aligned} &(\mathrm{id}_V - \pi)(\boldsymbol{w}) = \boldsymbol{w} - \pi(\boldsymbol{w}) = \boldsymbol{0} \\ \iff \quad &\pi(\boldsymbol{w}) = \boldsymbol{w} \\ \implies \quad &\boldsymbol{w} \in \mathrm{Im}(\pi). \end{aligned}$$

Then let $\boldsymbol{v} \in \mathrm{Im}(\pi)$, let $\boldsymbol{u} \in V$ be a preimage of $\boldsymbol{v}$ under $\mathrm{id}_V - \pi$, and consider:

$$\begin{aligned} \pi(\boldsymbol{u}) &= \boldsymbol{v}, \\ (\mathrm{id}_V - \pi)(\boldsymbol{v}) &= \boldsymbol{v} - \pi(\boldsymbol{v}) \\ &= \pi(\boldsymbol{u}) - \pi(\pi(\boldsymbol{u})) \\ &= \pi(\boldsymbol{u}) - \pi(\boldsymbol{u}) \\ &= \boldsymbol{0} \implies \boldsymbol{v} \in \ker(\mathrm{id}_V - \pi). \end{aligned}$$

□

3.8 Using the Gram-Schmidt method, turn the basis $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ of a two-dimensional subspace $U \subseteq \mathbb{R}^3$ into an orthonormal basis, $C = (\boldsymbol{c}_1, \boldsymbol{c}_2)$ of U, where

$$\boldsymbol{b}_1 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{b}_2 := \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

Let $\boldsymbol{u}_1 = \boldsymbol{b}_1$. Then let $\boldsymbol{u}_2 = \boldsymbol{b}_2 - \pi_{\boldsymbol{u}_1}(\boldsymbol{b}_2)$, where $\pi_{\boldsymbol{u}_1}$ is the projection operator onto the one-dimensional subspace spanned by $\boldsymbol{u}_1$. Assuming the standard dot product,

$$\pi_{\boldsymbol{u}_1}(\boldsymbol{b}_2) = \frac{\boldsymbol{u}_1 \boldsymbol{u}_1^\top}{\|\boldsymbol{u}_1\|^2} \boldsymbol{b}_2 = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{u}_2 = \frac{1}{3} \begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix}.$$

Normalizing each of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, we define our orthonormal basis vectors to be:

$$\boldsymbol{c}_1 := \frac{\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|} = \frac{\sqrt{3}}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{c}_2 := \frac{\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|} = \frac{\sqrt{42}}{42} \begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix}.$$

3.9 Let $n \in \mathbb{N}$ and let $a_1, \ldots, a_n > 0$ be n positive real numbers so that $\sum_{i=1}^n a_i = 1$. Use the Cauchy-Schwarz inequality to show that:

$$\text{a. } \sum_{i=1}^n a_i^2 \geq \frac{1}{n} \qquad \qquad \text{b. } \sum_{i=1}^n \frac{1}{a_i} \geq n^2$$

$$\lim_{k \to \infty} \left(\frac{(k - (n-1))^2}{k^2} + \sum_{i=1}^{n-1} \frac{1}{k^2} \right) = \lim_{k \to \infty} \left(\frac{(k - (n-1))^2}{k^2} + \frac{n-1}{k^2} \right) = 1.$$

a. Let $\boldsymbol{x} = [a_1 \ \cdots \ a_n]^\top$ and let $\boldsymbol{y} = [1 \ \cdots \ 1]^\top \in \mathbb{R}^n$. Then, using the standard inner product:

$$\|\boldsymbol{x}\|^2 = \boldsymbol{x}^\top \boldsymbol{x} = \sum_{i=1}^n a_i^2 \qquad \|\boldsymbol{y}\|^2 = \boldsymbol{y}^\top \boldsymbol{y} = \sum_{i=1}^n 1 = n$$

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\top \boldsymbol{y} = \sum_{i=1}^n a_i = 1$$

Cauchy-Schwarz:

$$\begin{aligned} & |\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leq \|\boldsymbol{x}\| \|\boldsymbol{y}\| \\ \iff \quad & \langle \boldsymbol{x}, \boldsymbol{y} \rangle^2 \leq \|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2 \end{aligned}$$

Substituting:

$$1 \leq n \sum_{i=1}^n a_i^2 \quad \iff \quad \frac{1}{n} \leq \sum_{i=1}^n a_i^2.$$

b. Let $\boldsymbol{x} = \begin{bmatrix} \frac{1}{\sqrt{a_1}} & \cdots & \frac{1}{\sqrt{a_n}} \end{bmatrix}^\top$ and let $\boldsymbol{y} = \begin{bmatrix} \sqrt{a_1} & \cdots & \sqrt{a_n} \end{bmatrix}^\top$. Then:

$$\|\boldsymbol{x}\|^2 = \boldsymbol{x}^\top \boldsymbol{x} = \sum_{i=1}^n \left(\frac{1}{\sqrt{a_i}} \right)^2 = \sum_{i=1}^n \frac{1}{a_i} \qquad \|\boldsymbol{y}\|^2 = \boldsymbol{y}^\top \boldsymbol{y} = \sum_{i=1}^n \left(\sqrt{a_i} \right)^2 = \sum_{i=1}^n a_i = 1$$

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\top \boldsymbol{y} = \sum_{i=1}^n \frac{\left(\sqrt{a_i} \right)^2}{a_i} = \sum_{i=1}^n 1 = n.$$

Substituting:

$$n^2 \leq \sum_{i=1}^n \frac{1}{a_i}.$$

3.10 Rotate the following vectors by 30°:

$$\boldsymbol{x}_1 := \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \boldsymbol{x}_2 := \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\mathbf{R}_{30^\circ} = \begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

$$\mathbf{R}_{30^\circ} \boldsymbol{x}_1 = \tfrac{1}{2} \begin{bmatrix} 2\sqrt{3} - 3 \\ 3\sqrt{3} + 2 \end{bmatrix} \approx \begin{bmatrix} 0.23 \\ 3.60 \end{bmatrix} \qquad \mathbf{R}_{30^\circ} \boldsymbol{x}_2 = \tfrac{1}{2} \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} \approx \begin{bmatrix} 0.50 \\ -0.87 \end{bmatrix}$$