Calculus

Exercises 1: Basic properties of numbers

1. Prove the following:

(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$.

$$\begin{aligned} & ax = a \\ \iff \quad & (a^{-1})ax = (a^{-1})a \\ \iff \quad & 1 \cdot x = 1 \\ \iff \quad & x = 1. \end{aligned}$$

(ii)

$$\begin{aligned} x^2 - y^2 &= (x-y)(x+y) \\ &= x(x+y) - y(x+y) \\ &= x^2 + xy - yx - y^2 \\ &= x^2 + xy - xy - y^2 \\ &= x^2 - y^2. \end{aligned}$$

(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$.

Taking the square root of both sides:

$$\begin{aligned} & \sqrt{x^2} = \sqrt{y^2} \\ \iff \quad & |x| = |y|. \end{aligned}$$

(iv)

$$\begin{aligned} x^3 - y^3 &= (x-y)(x^2 + xy + y^2) \\ &= x(x^2 + xy + y^2) - y(x^2 + xy + y^2) \\ &= x^3 + x^2y + xy^2 - (x^2 + xy + y^2)y \\ &= x^3 + x^2y + xy^2 - x^2y + xy^2 + y^3 \\ &= x^3 + y^3. \end{aligned}$$

(v)

$$\begin{aligned} x^n - y^n &= (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \\ &= x(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \\ &- y(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \\ &= x^n + x^{n-1}y + x^{n-2}y^2 + \cdots + x^2y^{n-2} + xy^{n-1} \\ &\qquad -(x^{n-1}y + x^{n-2}y^2 + \cdots + x^2y^{n-2} + xy^{n-1}) - y^n \\ &= x^n - y^n. \end{aligned}$$

More succinctly:

$$x^n - y^n = (x-y)\sum_{i=1}^{n} x^{n-i}y^{i-1}$$

(vi) Taking the result that:

$$x^3 - y^3 = (x-y)(x^2+xy+y^2),$$

substitute $(-y)$ for y, which gives us:

$$\begin{aligned} & x^3 - (-y)^3 = (x-(-y))(x^2+x(-y)+(-y)^2), \\ \iff \quad & x^3 + y^3 = (x+y)(x^2-xy+y^2). \end{aligned}$$

Making the same substitution into the result from (v), we have:

$$x^n + y^n = (x+y)\sum_{i=1}^{n}(-1)^{i-1}x^{n-i}y^{i-1} \quad \text{for odd } n.$$

2. In the 'proof' that $2=1$, since $x=y$ by definition, dividing out the factor $(x-y)$ is equivalent to dividing by zero, the result of which is not defined.

3.

(-i) Prove $a^{-1}b^{-1} = (ab)^{-1}$.

By the definition of inverses,

$$\begin{aligned} ab\cdot(ab)^{-1} &= 1 \\ \iff \quad a^{-1}\cdot ab\cdot(ab)^{-1} &= a^{-1} \\ \iff \quad b\cdot(ab)^{-1} &= a^{-1} \\ \iff \quad b^{-1}\cdot b\cdot(ab)^{-1} &= b^{-1}a^{-1} \\ \iff \quad (ab)^{-1} &= a^{-1}b^{-1}. \end{aligned}$$

(i) Prove:

$$\frac{a}{b} = \frac{ac}{bc}, \quad \text{if } b, c \neq 0.$$

$$\frac{a}{b} = a\cdot\frac{1}{b} = a\cdot\frac{c}{c}\cdot\frac{1}{b} = a\cdot c\cdot\frac{1}{c}\cdot\frac{1}{b} = ac\cdot\frac{1}{bc} = \frac{ac}{bc}.$$

(ii) Prove:

$$\frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd}, \quad \text{if } b, d \neq 0.$$

$$\frac{a}{b}+\frac{c}{d} = \frac{a}{b}\cdot\frac{d}{d}+\frac{b}{b}\cdot\frac{c}{d} = \frac{ad}{bd}+\frac{bc}{bd} = ad\cdot\frac{1}{bd}+bc\cdot\frac{1}{bd} = (ad+bc)\cdot\frac{1}{bd} = \frac{ad+bc}{bd}.$$

(iii) See (-i).

(iv)
$$\frac{a}{b}\cdot\frac{c}{d} = a\cdot\frac{1}{b}\cdot c\cdot\frac{1}{d} = ac\cdot\frac{1}{d}\cdot\frac{1}{b} = ac\cdot\frac{1}{db} = \frac{ac}{db}.$$

(v)
$$\frac{a}{b}\Big/\frac{c}{d} = \frac{ad}{bc}, \qquad \text{if } b, c, d \neq 0.$$

$$\begin{aligned}
\frac{a}{b}\Big/\frac{c}{d} &= \frac{a}{b}\cdot\left(c\cdot\frac{1}{d}\right)^{-1} && \text{(definition of division)}\\
&= a\cdot\frac{1}{b}\cdot c^{-1}\cdot\left(\frac{1}{d}\right)^{-1} && \text{(by (iii) inverse of a product)}\\
&= a\cdot\frac{1}{b}\cdot\frac{1}{c}\cdot d && \text{(definition of inverses)}\\
&= ad\cdot\frac{1}{bc} && \text{(commutativity \& grouping)}\\
&= \frac{ad}{bc}. && \text{(definition of division)}
\end{aligned}$$

(vi) If $b, d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Also determine when $\frac{a}{b} = \frac{b}{a}$.

$$\begin{aligned}
& \frac{a}{b} = \frac{c}{d} && \\
\iff\quad & a\cdot\frac{1}{b} = c\cdot\frac{1}{d} && \text{(definition of division)}\\
\iff\quad & a\cdot\frac{1}{b}\cdot b = c\cdot\frac{1}{d}\cdot b && \text{(multiply by } b\text{)}\\
\iff\quad & a\cdot 1 = b\cdot c\cdot\frac{1}{d} && \text{(multiplicative inverse; commutativity)}\\
\iff\quad & ad = bc\cdot\frac{1}{d}\cdot d && \text{(multiply by } d\text{)}\\
\iff\quad & ad = bc. && \text{(multiplicative inverse)}
\end{aligned}$$

Therefore, we find that
$$\frac{a}{b} = \frac{b}{a} \quad\iff\quad a^2 = b^2,$$
which from (iii) holds when either $a = b$ or $a = -b$.

4. Find all numbers for which

(i)
$$\{x \in \mathbb{R} \mid 4 - x < 3 - 2x\} = (-\infty, -1).$$

(ii)
$$\left\{x \in \mathbb{R} \;\middle|\; 5 - x^2 < 8\right\} = \mathbb{R}.$$

(iii)
$$\left\{x \in \mathbb{R} \;\middle|\; 5 - x^2 < -2\right\} = \left(-\infty, -\sqrt{7}\right) \cup \left(\sqrt{7}, \infty\right).$$

(iv)
$$\{x \in \mathbb{R} \mid (x-1)(x-3) > 0\} = (-\infty, 1) \cup (3, \infty).$$

(v)
$$\left\{x \in \mathbb{R} \;\middle|\; x^2 - 2x + 2 > 0\right\} = \mathbb{R}.$$

(vi)
$$\left\{x \in \mathbb{R} \;\middle|\; x^2 + x + 1 > 2\right\} = \left(-\infty, \tfrac{-1-\sqrt{5}}{2}\right) \cup \left(\tfrac{-1+\sqrt{5}}{2}, \infty\right).$$

(vii)
$$\left\{x \in \mathbb{R} \;\middle|\; x^2 - x + 10 > 16\right\} = (-\infty, -2) \cup (3, \infty).$$

(viii)
$$\left\{x \in \mathbb{R} \;\middle|\; x^2 + x + 1 > 0\right\} = \mathbb{R}.$$

(ix)
$$\{x \in \mathbb{R} \mid (x-\pi)(x+5)(x-3) > 0\} = (-5, 3) \cup (\pi, \infty).$$

(x)
$$\left\{x \in \mathbb{R} \;\middle|\; (x - \sqrt[3]{2})(x - \sqrt{2}) > 0\right\} = (-\infty, \sqrt[3]{2}) \cup (\sqrt{2}, \infty).$$

(xi)
$$\{x \in \mathbb{R} \mid 2^x < 8\} = (-\infty, 3).$$

(xii)
$$\{x \in \mathbb{R} \mid x + 3^x < 4\} = (-\infty, 1).$$

(xiii)
$$\left\{x \in \mathbb{R} \;\middle|\; \frac{1}{x} + \frac{1}{1-x} > 0\right\} = (0, 1).$$

(xiv)
$$\left\{x \in \mathbb{R} \;\middle|\; \frac{x-1}{x+1} > 0\right\} = (-\infty, -1) \cup (1, \infty).$$

5. Prove the following:

(i) If $a < b$ and $c < d$, then $a + c < b + d$.

$$\begin{aligned}
\text{Since} \quad & a < b \iff (a-b) < 0, \\
\text{and} \quad & c < b \iff (c-d) < 0, \\
\text{then} \quad & (a-b) + (c-d) < 0, \\
\text{therefore} \quad & a + c - d < b, \\
\text{finally} \quad & a + c < b + d.
\end{aligned}$$

7. Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a+b}{2} < b.$$

Proof. Since $a, b > 0$, from $a < b$ find (multiplying by a resp. b preserves sign):

$$a^2 < ab \implies a < \sqrt{ab},$$
$$ab < b^2 \implies \sqrt{ab} < b.$$

Thus, the geometric mean, $a < \sqrt{ab} < b$. For the arithmetic mean, consider:

$$a < b \iff 2a < a+b \iff a < \frac{a+b}{2},$$
$$a < b \iff a+b < 2b \iff \frac{a+b}{2} < b.$$

So $a < \frac{a+b}{2} < b$. Finally, to compare the two means:

$$\begin{aligned} & \sqrt{ab} < \frac{a+b}{2} \\ \iff \quad & 4ab < (a+b)^2 \\ \iff \quad & 0 < a^2 - 2ab + b^2 \\ \iff \quad & 0 < (b-a)^2, \end{aligned}$$

which holds since $a \neq b$. Thus, as claimed,

$$a < \sqrt{ab} < \frac{a+b}{2} < b.$$

□

Proof (terse). For $a, b > 0$ with $a < b$:

$$a^2 < ab \implies a < \sqrt{ab}, \qquad ab < b^2 \implies \sqrt{ab} < b,$$

so $a < \sqrt{ab} < b$. Similarly,

$$2a < a+b < 2b \implies a < \frac{a+b}{2} < b.$$

Finally,

$$\sqrt{ab} < \frac{a+b}{2} \iff 0 < (b-a)^2,$$

which is true since $a < b$. Hence

$$a < \sqrt{ab} < \frac{a+b}{2} < b.$$

□

9. Express each of the following with at least one less pair of absolute value signs:

(i) $|\sqrt{2}+\sqrt{3}-\sqrt{5}+\sqrt{7}| = \sqrt{2}+\sqrt{3}-\sqrt{5}+\sqrt{7}$.

(ii) $|(|a+b|-|a|-|b|)| = |a|+|b|-|a+b|$.

(iii) $|(|a+b|+|c|-|a+b+c|)| = |a+b|+|c|-|a+b+c|$.

(iv) $|x^2-2xy+y^2| = (x-y)^2$.

(v) $|(|\sqrt{2}+\sqrt{3}|-|\sqrt{5}-\sqrt{7}|)| = \sqrt{2}+\sqrt{3}+\sqrt{5}-\sqrt{7}$.

10. Express each of the following without absolute value signs, treating various cases separately when necessary.

(i) $|a+b|-|b| = \begin{cases} a & b \geq 0,\ a \geq -b, \\ -a-2b & b \geq 0,\ a < -b, \\ -a & b < 0,\ a < -b, \\ a+2b & b < 0,\ a \geq -b. \end{cases}$

(ii) $|(|x|-1)| = \begin{cases} -x-1 & -1 \leq x, \\ x+1 & -1 < x \leq 0, \\ -x+1 & 0 < x \leq 1, \\ x-1 & 1 < x. \end{cases}$

(iii) $|x|-|x^2| = \begin{cases} -x-x^2 & x < 0, \\ x-x^2 & x \geq 0. \end{cases}$

(iv) $a-|(a-|a|)| = \begin{cases} 3a & a < 0, \\ a & a \geq 0. \end{cases}$

11. Find all numbers for which:

(i) $|x-3| = 8 \implies x \in \{-5, 11\}$.

(ii) $|x-3| < 8 \implies x \in (-5, 11)$.

(iii) $|x+4| < 2 \implies x \in (-6, -2)$.

(iv) $|x-1|+|x-2| > 1 \implies x \in (-\infty, 1) \cup (2, \infty)$

$$|x-1|+|x-2|-1 = \begin{cases} -2x+2 & x \leq 1, \\ -1 & 1 < x \leq 2, \\ 2x-4 & 2 < x. \end{cases}$$

(v) $|x-1|+|x-2|<2 \implies x \in \left(-\infty, \frac{1}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$

$$|x-1|+|x-2|-2 = \begin{cases} -2x+1 & x \leq 1, \\ -1 & 1 < x \leq 2, \\ 2x-5 & 2 < x. \end{cases}$$

(vi) $|x-1|+|x+1|<1 \implies x \in \emptyset.$

$$|x-1|+|x+1|-1 = \begin{cases} -2x-1 & x < -1, \\ 1 & -1 \leq x \leq 1, \\ 2x-1 & x > 1. \end{cases}$$

(vii) $|x-1| \cdot |x+1| = 0 \implies x \in \{-1, 1\}.$

(viii) $|x-1| \cdot |x+2| = 3 \implies x \in \left\{\frac{-1\pm\sqrt{21}}{2}\right\}.$

($\star$) Given $a < b$ and $k \geq 0$, solve for x:

$$|x-a| \cdot |x-b| = k.$$

This divides the number line into three segments:

$$(-\infty, a], [a, b], [b, \infty).$$

Analysing piecewise:

$$|x-a| \cdot |x-b| - k = \begin{cases} -x^2 + (a-b)x - (ab+k) & a \leq x \leq b, \\ x^2 - (a+b)x + ab - k & \text{otherwise.} \end{cases}$$

Evaluate at the midpoint of the interior interval, $\frac{1}{2}(a+b)$:

$$\begin{aligned} \mathcal{D} =& |\tfrac{1}{2}(a+b) - a| \cdot |\tfrac{1}{2}(a+b) - b| \\ =& |\tfrac{1}{2}(b-a)| \cdot |\tfrac{1}{2}(a-b)| \\ =& \tfrac{1}{4}(b-a)^2. \end{aligned}$$

So, for $k > \mathcal{D}$, the interior interval has no solutions; for $k = \mathcal{D}$, one repeated solution; for $0 \leq k < \mathcal{D}$, two distinct solutions; and for $k < 0$ there are no solutions at all, since the entire function lies above the line $y = k$ (if we plot the function).

Solving for $a \leq x \leq b$:

$$\tfrac{1}{2}\left(a - b \pm \sqrt{a^2 + b^2 - 6ab - 4k}\right).$$

Solving for $x < a$ and $b < x$:

$$\tfrac{1}{2}\left(a + b \pm \sqrt{a^2 + b^2 - 2ab - 4k}\right).$$

12. Prove the following:

(i) $|xy| = |x| \cdot |y|$.

Proof. For any real number, t, we have $|t|^2 = t^2$. Consider,

$$|xy|^2 = x^2y^2 = |x|^2 \cdot |y|^2.$$

Since both sides are non-negative, taking square roots gives:

$$|xy| = |x| \cdot |y|.$$

$\square$

(ii) $\left|\frac{1}{x}\right| = \frac{1}{|x|}$, if $x \neq 0$.

Proof. As before, for any real number, t, we have $|t|^2 = t^2$. Therefore,

$$\left|\frac{1}{x}\right|^2 = \frac{1}{x^2} = \frac{1}{|x|^2}.$$

Once again, both sides are non-negative; taking square roots gives the desired result:

$$\left|\frac{1}{x}\right| = \frac{1}{|x|}.$$

$\square$

(iii) $\frac{|x|}{|y|} = \left|\frac{x}{y}\right|$, if $y \neq 0$.

Proof.

$$\left|\frac{x}{y}\right|^2 = \frac{x^2}{y^2} = \frac{|x|^2}{|y|^2}.$$

Since both sides are non-negative, taking square roots gives:

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

$\square$

(iv) $|x - y| \leq |x| + |y|$.

Proof. By definition, $\pm x \leq |x|$ and $\pm y \leq |y|$. Then, since addition preserves inequality order:

$$\begin{aligned} & x - y \leq |x| + |y|, \\ \text{and} \qquad & -(x - y) \leq |x| + |y|, \end{aligned}$$

which bounds $x - y$ from above and below:

$$-(|x| + |y|) \leq x - y \leq |x| + |y|.$$

Since $|a| \leq b$ is equivalent to $-b \leq a \leq b$, we have:

$$|x - y| \leq |x| + |y|.$$

□

(v) $|x| - |y| \leq |x - y|$.

Proof. From (iv), $|a - b| \leq |a| + |b|$, or equivalently $|a - b| - |b| \leq |a|$. Let $x = a - b$ and let $y = b$. Then $|x| - |y| \leq |x - y|$, as desired. □

(vi) $\big|\, |x| - |y| \,\big| \leq |x - y|$.

Proof. From (v), $|x| - |y| \leq |x - y|$. Multiplying by -1, reversing the direction of the inequality, permuting x and y, and noting that $|y - x| = |x - y|$, we have

$$-|x - y| \leq |x| - |y|.$$

As before, since $|a| \leq b$ is equivalent to $-b \leq a \leq b$, and $|x| - |y|$ is bounded from above and below, we find

$$\big|\, |x| - |y| \,\big| \leq |x - y|.$$

□

(vii) $|x + y + z| \leq |x| + |y| + |z|$.

Proof. Since $\pm a \leq |a|$,

$$\begin{aligned} & x + y + z \;\leq\; |x| + y + z \;\leq\; |x| + |y| + z \;\leq\; |x| + |y| + |z|, \\ \text{and} \qquad & -x - y - z \;\leq\; |x| - y - z \;\leq\; |x| + |y| - z \;\leq\; |x| + |y| + |z|. \end{aligned}$$

So $x + y + z$ is bounded from above and below, or equivalently

$$|x + y + z| \leq |x| + |y| + |z|.$$

□

If and only if $x, y, z \leq 0$, or $x, y, z \geq 0$, then $|x+y+z| = |x|+|y|+|z|$.

Proof. For non-negative x, y, z, $|x| = x$, $|y| = y$, $|z| = z$, so

$$|x|+|y|+|z| = x+y+z = |x+y+z|.$$

For non-positive x, y, z, $|x| = -x$, $|y| = -y$, $|z| = -z$, so

$$|x|+|y|+|z| = -(x+y+z) = |x+y+z|.$$

For only if, consider non-negative x, y but negative z,

$$|x|+|y|+|z| = x+y-z < |x+y+z|.$$

The same inequality holds for any other permutation where the signs of x, y, z are not all the same. $\square$

($\star$) $|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|$.

13. The maximum of two numbers x and y is denoted by $\max(x, y)$. The minimum of x and y is denoted by $\min(x, y)$.

$\max(x, y) = \frac{1}{2}\,(x + y + |y - x|)$.

Proof. If $x > y$, then

$$\begin{aligned}\max(x, y) &= x\\ &= \tfrac{1}{2}(x + y + |y - x|)\\ &= \tfrac{1}{2}(x + y + x - y)\\ &= x.\end{aligned}$$

Since $|x - y| = |y - x|$, the corresponding result holds for the second argument. In the case $x = y$, the equality above holds. □

$\min(x, y) = \frac{1}{2}\,(x + y - |y - x|)$.

Proof. If $x < y$, then

$$\begin{aligned}\min(x, y) &= x\\ &= \tfrac{1}{2}(x + y - |y - x|)\\ &= \tfrac{1}{2}(x + y - y + x)\\ &= x.\end{aligned}$$

Since $|x - y| = |y - x|$, the corresponding result again holds for the second argument. Likewise, in the case $x = y$, the equality above holds. □

Derive a formula for $\max(x, y, z)$ and $\min(x, y, z)$.

$$\begin{aligned}\max(x, y, z) &= \max(x, \max(y, z))\\ &= \tfrac{1}{2}(x + \max(y, z) + |\max(y, z) - x|)\\ &= \tfrac{1}{2}\left(x + \tfrac{1}{2}(y + z + |z - y|) + \left|\tfrac{1}{2}(y + z + |z - y|) - x\right|\right)\\ &= \tfrac{1}{4}\left(2x + y + z + |z - y| + \left|y + z + |z - y| - 2x\right|\right)\end{aligned}$$

14.

(a) Prove that $|a| = |-a|$. (The trick is not to become confused by too many cases. First prove the statement for $a \geq 0$. Why is it then obvious for $a \leq 0$)?

(b) Prove that $-b \leq a \leq b$ if and only if $|a| \leq b$. In particular it follows that $-|a| \leq a \leq |a|$.

(c) Use this fact to give a new proof that $|a + b| \leq |a| + |b|$.

*15. Prove that if x and y are not both 0, then

$$\begin{aligned} x^2 + xy + y^2 &> 0, \\ x^4 + x^3y + x^2y^2 + xy^3 + y^4 &> 0. \end{aligned}$$

*16.

(a) Show that

$$(x+y)^2 = x^2 + y^2 \quad \text{only when } x = 0 \text{ or } y = 0,$$
$$(x+y)^3 = x^3 + y^3 \quad \text{only when } x = 0,\ y = 0 \text{ or } x = -y.$$

(b) Using the fact that

$$x^2 + 2xy + y^2 = (x+y)^2 \geq 0,$$

show that $4x^2 + 6xy + 4y^2 > 0$ unless x and y are both 0.

(c) Use part (b) to find out when $(x+y)^4 = x^4 + y^4$.

(d) Find out when $(x+y)^5 = x^5 + y^5$.

Hint: From the assumption $(x+y)^5 = x^5 + y^5$ you should be able to derive the equation $x^3 + 2x^2y + 2xy^2 + y^3 = 0$, if $xy \neq 0$. This implies that $(x+y)^3 = x^2y + xy^2 = xy(x+y)$.

You should now be able to make a good guess as to when $(x+y)^n = x^n + y^n$.