Mathematics for Machine Learning

Exercises 2: Linear algebra

2.1 We consider $(R \setminus \{-1\}, \star)$, where:

$$a \star b = ab + a + b \qquad a, b \in R \setminus \{-1\}.$$

a. Show that $(R \setminus \{-1\}, \star)$ is an Abelian group.

Neutral element

We have $0 \in R \setminus \{-1\}$, and for all $a \in R \setminus \{-1\}$:

$$\begin{aligned} a \star 0 &= a0 + a + 0 = a, \quad \text{and} \\ 0 \star a &= 0a + 0 + a = a. \end{aligned}$$

Commutativity

For all $a, b \in R \setminus \{-1\}$, we have:

$$\begin{aligned} a \star b &= ab + a + b \\ &= ba + b + a \\ &= b \star a. \end{aligned}$$

Associativity

For all $a, b, c \in R \setminus \{-1\}$, we have:

$$\begin{aligned} (a \star b) \star c &= (ab + a + b) \star c \\ &= (abc + ac + bc) + (ab + a + b) + c \\ &= a(bc + b + c) + a + (bc + b + c) \\ &= a(b \star c) + a + (b \star c) \\ &= a \star (b \star c). \end{aligned}$$

Existence of inverses

For all $a \in R \setminus \{-1\}$, we require the existence of an element b such that:

$$\begin{aligned} & a \star b = b \star a &= 0 \\ \iff \quad & ab + a + b &= 0 \\ \iff \quad & b(a+1) + a &= 0 \\ \iff \quad & b &= \frac{-a}{a+1} \end{aligned}$$

This expression for b is always defined, since a cannot be -1, and the denominator is always non-zero.

Closure under $\star$

For contradiction, assume that there exist $a, b \in R \setminus \{-1\}$, such that:

$$\begin{aligned} a \star b &= -1 \\ \iff \quad ab + a + b &= -1 \\ \iff \quad a(1+b) &= -(1+b) \\ \iff \quad a &= -\frac{1+b}{1+b} \\ \iff \quad a &= -1. \end{aligned}$$

b. In the Abelian group $(R \setminus \{-1\}, \star)$, solve

$$3 \star x \star x = 15.$$

We have

$$\begin{aligned} 3 \star x \star x &= 15 \\ \iff \quad (3x + 3 + x) \star x &= 15 \\ \iff \quad (4x + 3) \star x &= 15 \\ \iff \quad (4x^2 + 3x) + (4x + 3) + x &= 15 \\ \iff \quad 4x^2 + 8x &= 12 \\ \iff \quad x^2 + 2x - 3 &= 0 \\ \iff \quad (x+3)(x-1) &= 0 \end{aligned}$$

which yields the solutions $x \in \{1, -3\} \subset R \setminus \{-1\}$.

2.2 Let n be in $N \setminus \{0\}$. Let k, x be in Z. We define the congruence class $\overline{k}$ of the integer k as the set

$$\begin{aligned} \overline{k} &= \{x \in Z \mid x - k \equiv 0 \mod n\} \\ &= \{x \in Z \mid \exists a \in Z : x - k = n \cdot a\} \end{aligned}$$

We now define Z/nZ (also Z_n) as the set of all congruence classes modulo n. Euclidean division implies that this is a finite set of n elements:

$$Z_n = \left\{\overline{0}, \overline{1}, \ldots, \overline{n-1}\right\}.$$

For all $a, b \in Z_n$, we define:

$$\overline{a} \oplus \overline{b} = \overline{a+b}$$

a. Show that $(Z_n, \oplus)$ is a group. Is it Abelian?

Neutral element

We have $\overline{0} \in Z_n$ such that:

$$\begin{aligned} &\overline{a} \oplus \overline{0} = \overline{a+0} = \overline{a}, \quad \text{and} \\ &\overline{0} \oplus \overline{a} = \overline{0+a} = \overline{a}. \end{aligned}$$

Commutativity

For all $\overline{a}, \overline{b} \in Z_n$, we have:

$$\begin{aligned}\overline{a} \oplus \overline{b} &= \overline{a+b} \\ &= \overline{b+a} \\ &= \overline{b} \oplus \overline{a}.\end{aligned}$$

Associativity

For all $a, b, c \in Z_n$, we have:

$$\begin{aligned}(\overline{a} \oplus \overline{b}) \oplus \overline{c} &= \overline{a+b} \oplus \overline{c} \\ &= \overline{(a+b)+c} \\ &= \overline{a+(b+c)} \\ &= \overline{a} \oplus \overline{b+c} \\ &= \overline{a} \oplus (\overline{b} \oplus \overline{c}).\end{aligned}$$

Existence of inverses

For all $\overline{a} \in Z_n$, we require the existence of an element, $\overline{b} \in Z_n$, such that:

$$\overline{a} \oplus \overline{b} = \overline{b} \oplus \overline{a} = \overline{0}.$$

We first note that in Z_n, $\overline{n} = \overline{0}$, and since $n - a \in Z$, its congruence class $\overline{n-a} \in Z_n$. Supposing then that $\overline{b} = \overline{n-a}$, we have:

$$\begin{aligned}\overline{a} \oplus \overline{b} &= \overline{a} \oplus \overline{n-a} \\ &= \overline{a+n-a} \\ &= \overline{n} \\ &= \overline{0}\end{aligned}$$

as required, and commutativity gives us $\overline{b} \oplus \overline{a} = \overline{0}$.

Closure under $\oplus$

By definition, we have that

$$\overline{a} \oplus \overline{b} = \overline{a+b}$$

Since Z_n is the set of congruence classes $\overline{0}, \overline{1}, \ldots, \overline{n-1}$, and every integer has a unique representation modulo n, then given $a + b \in Z$, their congruence class $\overline{a+b} \in Z_n$.

b. We now define another operation $\otimes$ for all $\overline{a}, \overline{b} \in Z_n$,

$$\overline{a} \otimes \overline{b} = \overline{a \times b},$$

where $\times$ represents the usual multiplication in Z.

We then have the following multiplication table for $Z_5 \setminus \{\overline{0}\}$ under $\otimes$:

$\otimes$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{2}$	$\overline{2}$	$\overline{4}$	$\overline{1}$	$\overline{3}$
$\overline{3}$	$\overline{3}$	$\overline{1}$	$\overline{4}$	$\overline{2}$
$\overline{4}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

It follows that $Z_5 \setminus \{\overline{0}\}$ is closed under $\otimes$, with the neutral element $\overline{1}$. From the symmetry about the diagonal, we can immediately conclude that $\otimes$ commutes. For the inverse, we find the column (resp. row) that yields $\overline{1}$ for a given row (resp. column), noting that $\overline{1}$ appears in every row (resp. column). For associativity, we note that for any three $\overline{a}, \overline{b}, \overline{c} \in Z_n \setminus \{\overline{0}\}$, both $\overline{a} \otimes (\overline{b} \otimes \overline{c})$ and $(\overline{a} \otimes \overline{b}) \otimes \overline{c}$ yield the same result. Hence, $(Z_5 \setminus \{\overline{0}\}, \otimes)$ forms an Abelian group.

c. We find that $(Z_8 \setminus \{\overline{0}\}, \otimes)$ does not form a group, since $\overline{2} \otimes \overline{4} = \overline{0}$, so closure is not satisfied.

d. Bézout's lemma tells us that two integers a and b are relatively prime (that is, $\gcd(a, b) = 1$) if and only if there exist two integers u and v such that $au + bv = 1$.

   Show that $(Z_n \setminus \{\overline{0}\}, \otimes)$ is a group if and only if $n \in N \setminus \{0\}$ is prime.

   Proof. The neutral element is $\overline{1} \in Z_n \setminus \{\overline{0}\}$, given that for all $\overline{a} \in Z_n \setminus \{\overline{0}\}$:

$$\overline{a} \otimes \overline{1} = \overline{a \times 1} = \overline{a} \quad \text{and}$$
$$\overline{1} \otimes \overline{a} = \overline{1 \times a} = \overline{a}.$$

   For all $\overline{a}, \overline{b}, \overline{c} \in Z_n \setminus \{\overline{0}\}$, we have associativity (which follows directly from associativity of integer multiplication):

$$(\overline{a} \otimes \overline{b}) \otimes \overline{c} = \overline{a \times b \times c} = \overline{a} \otimes (\overline{b} \otimes \overline{c}).$$

   If n is composite, then there exist $\overline{p}, \overline{q} \in Z_n \setminus \{\overline{0}\}$ such that $\overline{p} \otimes \overline{q} = \overline{0}$; that is, $Z_n \setminus \{\overline{0}\}$ contains zero divisors, and is not closed under $\otimes$. Conversely, if n is prime then no such $\overline{p}$ or $\overline{q}$ exist; $Z_n \setminus \{\overline{0}\}$ contains no zero divisors and is closed under $\otimes$.

   Finally, for the existence of an inverse for all $\overline{a} \in Z_n \setminus \{\overline{0}\}$, we have from Bézout's lemma that a and n are relatively prime if and only if there exist two integers u and v such that $au + nv = 1$. Reducing both sides modulo n, we have

$$au \equiv 1 \mod n,$$

   which tells us that u exists if and only if n is prime. Restated:

$$\forall \overline{a} \in Z_n \setminus \{\overline{0}\}, \exists \overline{u} \in Z_n \setminus \{\overline{0}\} : \overline{a} \otimes \overline{u} = \overline{1} \quad \iff \quad n \text{ is prime.}$$

   Concluding, a neutral element always exists; associativity always holds; closure and the existence of inverses hold if and only if n is prime. Therefore, $(Z_n \setminus \{\overline{0}\}, \otimes)$ forms a group if and only if n is prime.

$\square$

2.3 Consider the set $\mathcal{G}$ of 3×3 matrices, defined as follows

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in R^3 \,\middle|\, x, y, z \in R \right\}$$

We define $\cdot$ as the standard matrix multiplication. Is $(\mathcal{G}, \cdot)$ a group? If yes, is it Abelian?

For $a, b, c, x, y, z \in R$, let:

$$\mathbf{A} = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

Then we have closure:

$$\mathbf{B} \cdot \mathbf{A} = \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{G},$$

and

$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 1 & a+x & c+ay+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{G}.$$

Associativity follows from associativity of standard matrix multiplication. Similarly, we have the standard identity matrix, $\mathbf{I} \in \mathcal{G}$. For $\mathbf{A}$ defined as above, we have its inverse:

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -a & ab-c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{G}.$$

Verifying, we have

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \begin{bmatrix} 1 & a-a & ab-c-ab+c \\ 0 & 1 & b-b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

and

$$\mathbf{A}^{-1} \cdot \mathbf{A} = \begin{bmatrix} 1 & a-a & c-ab+ab-c \\ 0 & 1 & b-b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

as desired. We saw above that, in general, multiplication of $\mathbf{A}, \mathbf{B} \in \mathcal{G}$ does not commute. We therefore conclude that $(\mathcal{G}, \cdot)$ is a group, but not Abelian.

2.4 a.

$$\begin{bmatrix}1 & 2\\4 & 5\\7 & 8\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix} \text{ is not defined.}$$

b.

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix} = \begin{bmatrix}4 & 3 & 5\\10 & 9 & 11\\16 & 15 & 17\end{bmatrix}$$

c.

$$\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} = \begin{bmatrix}5 & 7 & 9\\11 & 13 & 15\\8 & 10 & 12\end{bmatrix}$$

d.

$$\begin{bmatrix}1 & 2 & 1 & 2\\4 & 1 & -1 & -4\end{bmatrix}\begin{bmatrix}0 & 3\\1 & -1\\2 & 1\\5 & 2\end{bmatrix} = \begin{bmatrix}14 & 2\\-21 & 2\end{bmatrix}$$

e.

$$\begin{bmatrix}0 & 3\\1 & -1\\2 & 1\\5 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & 1 & 2\\4 & 1 & -1 & -4\end{bmatrix} = \begin{bmatrix}12 & 3 & -3 & -12\\-3 & 1 & 2 & 6\\6 & 5 & 1 & 0\\13 & 12 & 3 & 2\end{bmatrix}$$

2.5 Find the set $\mathcal{S}$ of all solutions in x of the following inhomogeneous systems $\mathbf{A}x = b$, where $\mathbf{A}$ and b are defined as follows:

a.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 5 & -7 & -5 \\ 2 & -1 & 1 & 3 \\ 5 & 2 & -4 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 6 \end{bmatrix}$$

Constructing the augmented matrix, $\tilde{\mathbf{A}}$, and row-reducing,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right] \quad \begin{array}{l} r_2 \mapsto r_2 - 2r_1 \\ r_3 \mapsto r_3 - r_2 \\ r_4 \mapsto r_4 - 5r_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -6 & 8 & 8 & 6 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right] \quad \begin{array}{l} r_3 \mapsto r_3 + 2r_2 \\ r_4 \mapsto r_4 + r_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & -4 & 4 & 2 \end{array}\right] \quad r_4 \mapsto r_4 - 2r_3$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right]$$

we find that this system $\mathbf{A}x = b$, has no solutions for x, since the final row of the augmented matrix corresponds to the inconsistent equation in the entries of x, $0 = 6$. This tells us that $b \notin \text{Im}(\mathbf{A})$.

If instead we were solving the *homogeneous* system, $\mathbf{A}x = 0$, then we *would* find solutions for x, namely the null space of $\mathbf{A}$, which is nontrivial since we know $\mathbf{A}$ is less than full rank.

Similarly, if instead $b \in \text{Im}(\mathbf{A})$, the solutions for x in $\mathbf{A}x = b$ would form an affine subspace in R^4, namely the null space of $\mathbf{A}$ shifted by any particular solution; that is

$$\{x \in R^4 \mid \mathbf{A}x = b\} = x_0 + \text{Null}(\mathbf{A})$$

b.

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -3 & 0 \\ 2 & -1 & 0 & 1 & -1 \\ -1 & 2 & 0 & -2 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 6 \\ 5 \\ -1 \end{bmatrix}$$

Constructing the augmented matrix, $\tilde{\mathbf{A}}$, and row-reducing,

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right] \quad \begin{array}{l} r_2 \mapsto r_2 - r_1 \\ r_3 \mapsto r_3 - 2r_2 \\ r_4 \mapsto r_4 + r_2 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & -3 & -1 & 3 \\ 0 & -3 & 0 & 7 & -1 & -7 \\ 0 & 3 & 0 & -5 & -1 & 5 \end{array}\right] \quad \begin{array}{l} r_3 \mapsto (2r_3 + 3r_2)/5 \\ r_4 \mapsto (r_4 + r_3)/2 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & -3 & -1 & 3 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right] \quad r_4 \mapsto r_4 - r_3$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & -3 & -1 & 3 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Equivalently, in the components of x,

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -1 \\ 0 \end{bmatrix}$$

This gives us the following particular solution and solution set,

$$x_0 = \begin{bmatrix} 6 \\ 6 \\ 0 \\ 2 \\ 3 \end{bmatrix} \quad \left\{ x \in R^5 \;\middle|\; x = x_0 + \lambda_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix},\ \lambda_1, \lambda_2 \in R \right\}$$

2.6 Using Gaussian elimination, find all solutions of the inhomogeneous system $\mathbf{A}x = b$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Constructing the augmented matrix, $\tilde{\mathbf{A}}$, and eliminating,

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad r_3 \mapsto r_1 - r_3$$

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

yields the following solution set

$$\left\{ x \in R^6 \,\middle|\, x = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix},\ \lambda_1, \lambda_2, \lambda_3 \in R \right\}$$

2.7 Find all solutions in $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in R^3$ of the equation system $\mathbf{A}x = 12x$, where

$$\mathbf{A} = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix} \quad \text{and} \quad \sum_{i=1}^{3} x_i = 1.$$

Rearranging, we wish to solve $(\mathbf{A} - 12\mathbf{I})x = 0$, subject to the sum over the components of x. Let $\mathbf{B} = \mathbf{A} - 12\mathbf{I}$, then we can construct an augmented matrix, $\tilde{\mathbf{B}}$, which captures all the constraints (in particular, the first row encodes the constraint that $x_1 + x_2 + x_3 = 1$):

$$\tilde{\mathbf{B}} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -6 & 4 & 3 & 0 \\ 6 & -12 & 9 & 0 \\ 0 & 8 & -12 & 0 \end{array}\right] \quad r_3 \mapsto r_3 + r_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -6 & 4 & 3 & 0 \\ 0 & -8 & 12 & 0 \\ 0 & 8 & -12 & 0 \end{array}\right] \quad \begin{array}{l} r_2 \mapsto r_2 + 6r_1 \\ r_3 \mapsto -r_3/4 \\ r_4 \mapsto r_4 + r_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 10 & 9 & 6 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad r_3 \mapsto (r_2 - 5r_3)/6$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 10 & 9 & 6 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From this, we find a single solution, $x_0 = \frac{1}{8} \begin{bmatrix} 3 & 3 & 2 \end{bmatrix}^\top$.

Examining the last three rows of the reduced $\tilde{\mathbf{B}}$, we can conclude that $\text{rank}(\mathbf{B}) = 2$, and therefore $\dim(\text{Null}(\mathbf{B})) = 1$. That is, the null space of $\mathbf{B}$ is a line, which intersects with $1 \in R^3$ at a single point, x_0. Specifically, we have

$$\text{Null}(\mathbf{B}) = \left\{ x \in R^3 : x = \lambda \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}, \ \lambda \in R \right\}.$$

2.8

a. Given

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

we find that $\mathbf{A}$ has no inverse, since $\det(\mathbf{A}) = 0$.

b. Given

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

we find

$$\det(\mathbf{A}) = \det\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\right) + \det\left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = 0 + 1 = 1.$$

We look for solutions to the following:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} = \begin{bmatrix} a+i & b+j & c+k & d+l \\ e+i & f+j & g+k & h+l \\ a+e+m & b+f+n & c+g+o & d+h+p \\ a+e+i & b+f+j & c+g+k & d+h+l \end{bmatrix} = \mathbf{I}$$

Solving each system of four equations in four variables gives us:

$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -2 \end{bmatrix}$$

Alternatively, we can augment $\mathbf{A}$ with the identity matrix $\mathbf{I}$ and reduce:

$$\begin{aligned}
\tilde{\mathbf{A}} &= \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \\
&\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{array}\right] \\
&\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -2 \end{array}\right] \\
&\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -2 \end{array}\right] = \left[\,\mathbf{I} \mid \mathbf{A}^{-1}\,\right]
\end{aligned}$$

2.9 Which of the following are subspaces of R^3?

$A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) \mid \lambda, \mu \in R\}$

$B = \{(\lambda^2, -\lambda^2, 0) \mid \lambda \in R\}$

$C = \{(\eta_1, \eta_2, \eta_3) \in R^3 \mid \eta_1 - 2\eta_2 + 3\eta_3 = \gamma,\ \gamma \in R\}$

$D = \{(\kappa_1, \kappa_2, \kappa_3) \in R^3 \mid \kappa_2 \in Z\}$

a. A is *not* a subspace of R^3, since it is not possible to express all members of A as a linear combination of a basis.

b. B is also *not* a subspace of R^3, for the same reason.

c. C is a subspace of R^3 if and only if $\gamma = 0$, since otherwise C does not contain the origin. In the case that γ *does* equal 0, then $C = \text{span}((1, 2, 1))$.

d. D is *not* a subspace of R^3, since D is not closed under scalar multiplication by $\lambda \in R$.

2.10 a. The set of vectors x_1, x_2, x_3 is linearly dependent, as found by constructing the matrix $\begin{bmatrix} x_1 \ x_2 \ x_3 \end{bmatrix}$ and row-reducing:

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The final row of all zeros indicates the matrix has rank $2 < 3$, so the columns are linearly dependent.

b. The set of vectors x_1, x_2, x_3 is linearly independent, as found by constructing the matrix $\begin{bmatrix} x_1 \ x_2 \ x_3 \end{bmatrix}$ and row-reducing:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

revealing three pivot columns. The matrix therefore has full column rank 3, and its column vectors are linearly independent.

2.11 Given the following

$$y = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} \quad x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad x_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

we can express

$$y = -6x_1 + 3x_2 + 2x_3.$$

as found by forming the augmented matrix

$$[\ x_1\ x_2\ x_3 \mid y\ ] = [\ \mathbf{A} \mid y\ ]$$

and row-reducing to $[\ \mathbf{I} \mid v\ ]$, where $\mathbf{A}v = y$.

2.12 Consider two subspaces of R^4:

$$U = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right), \quad V = \text{span}\left(\begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -2 \\ -1 \end{bmatrix}\right).$$

Determine a basis of $U \cap V$.

If we denote the spanning vectors of U as u_1, u_2, u_3, we find $u_1 = 2u_2 + 3u_3$, and likewise denoting the spanning vectors of V as v_1, v_2, v_3, we find $v_1 = -2v_2 - v_3$, so

$$U = \text{span}(u_2, u_3) \quad V = \text{span}(v_2, v_3).$$

We therefore wish to find the intersection of two planes, which amounts to solving $\mathbf{U}x = \mathbf{V}y$, where

$$\mathbf{U} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} 2 & -3 \\ -2 & 6 \\ 0 & -2 \\ 0 & -1 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We can reframe this as solving

$$\begin{bmatrix} 2 & -1 & 2 & -3 \\ -1 & 1 & -2 & 6 \\ 0 & -1 & 0 & -2 \\ -1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ -y_1 \\ -y_2 \end{bmatrix} = 0$$

allowing us to row-reduce:

$$\begin{bmatrix} 2 & -1 & 2 & -3 \\ -1 & 1 & -2 & 6 \\ 0 & -1 & 0 & -2 \\ -1 & 1 & 0 & -1 \end{bmatrix} \qquad \begin{array}{l} r_2 \mapsto 2r_2 + r_1 \\ r_4 \mapsto r_4 - r_2 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 2 & -3 \\ 0 & 1 & -2 & 9 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 2 & -7 \end{bmatrix} \qquad r_3 \mapsto -(r_3 + r_2)$$

$$\begin{bmatrix} 2 & -1 & 2 & -3 \\ 0 & 1 & -2 & 9 \\ 0 & 0 & 2 & -7 \\ 0 & 0 & 2 & -7 \end{bmatrix} \qquad \begin{array}{l} r_1 \mapsto (r_1 + r_2)/2 \\ r_2 \mapsto r_2 + r_3 \\ r_4 \mapsto r_4 - r_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From which we can determine that $6u_2 + 4u_3 = 7v_2 + 2v_3$, and that

$$U \cap V = \text{span}\left(\begin{bmatrix} 6 \\ 4 \\ -7 \\ -2 \end{bmatrix}\right)$$

2.13 Consider two subspaces U and V, where U is the solution of the homogeneous equation system $\mathbf{A}x = 0$ and V is the solution of the homogeneous equation system $\mathbf{B}x = 0$, with

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}$$

a. Determine the dimension of U and of V.

b. Determine the bases of U and V.

c. Determine a basis of $U \cap V$.

We first row-reduce $\mathbf{A}$:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} r_2 \mapsto -\frac{1}{2}(r_2 - r_1) \\ r_3 \mapsto r_3 - 2r_1 \\ r_4 \mapsto r_4 - r_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad r_3 \mapsto r_3 - r_2$$

finding that $\mathbf{A}$ is of rank 2. Since $x \in R^3$, we have $\dim(U) = 3 - \operatorname{rank}(\mathbf{A}) = 1$.

Row-reducing $\mathbf{B}$,

$$\begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -7 & -6 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} r_1 \mapsto r_1 - 2r_2 \\ r_3 \mapsto -(r_3 - 2r_1 - r_2) \\ r_4 \mapsto \frac{1}{2}(r_4 - r_1) \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} r_1 \mapsto r_1 + 7r_3 \\ r_2 \mapsto r_2 - 2r_3 \\ r_4 \mapsto r_4 - r_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} r_2 \mapsto r_3 \\ r_3 \mapsto r_2 - r_1 \end{matrix}$$

we find an identical reduced form as we found for $\mathbf{A}$, so likewise $\dim(V) = 1$.

We therefore find the same solution set to both systems, $\mathbf{A}x = 0$ and $\mathbf{B}x = 0$,

$$\left\{ x \in R^3 \;\middle|\; x = \lambda \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},\ \lambda \in R \right\}$$

Finally, we conclude that U and V span the same line in R^3:

$$U = V = U \cap V = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right)$$

2.14 Consider two subspaces U and V, where U is spanned by the columns of $\mathbf{A}$, and V is spanned by the columns of $\mathbf{B}$, with

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}$$

a. Determine the dimension of U and of V.

b. Determine the bases of U and V.

c. Determine a basis of $U \cap V$.

We can reformulate $\mathbf{A}$ as follows

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & -2 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad c_3 \mapsto c_3 - c_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{matrix} c_1 \mapsto c_1 - 2c_2 \\ c_3 \mapsto c_3 - c_2 \end{matrix}$$

which is to say, $\dim(U) = 2$, since the two remaining non-zero columns cannot be expressed as a linear combination of each other, and these two column vectors form a basis of U.

Similarly, we reformulate $\mathbf{B}$ as follows

$$\begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 3 \\ 7 & 2 & 2 \\ 3 & 2 & 2 \end{bmatrix} \quad c_2 \mapsto c_2 + c_1$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 7 & 2 & 0 \\ 3 & 2 & 0 \end{bmatrix} \quad c_3 \mapsto c_3 - c_2$$

and likewise, the two remaining non-zero columns form a linearly independent basis for V, and $\dim(V) = 2$.

Denoting the respective column vectors of the reduced bases of U and V as u_1, u_2, v_1, v_2, let

$$\mathbf{A}' = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \quad \mathbf{B}' = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Then to determine a basis for the intersection $U \cap V$, we can solve $\mathbf{A}'x - \mathbf{B}'y = 0$. That is, we look for solutions where some linear combination of the basis of U equals some linear combination of the basis of V. We do this by forming a single augmented matrix $\begin{bmatrix} \mathbf{A}' & -\mathbf{B}' \end{bmatrix}$

and row-reducing:

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 5 & -2 & -1 & -3 \\ 0 & 1 & -7 & -2 \\ 1 & 0 & -3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 2 & -14 & 3 \\ 0 & 1 & -7 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} r_2 \mapsto -(r_2 - 5r_1) \\ r_4 \mapsto -\frac{1}{2}(r_4 - r_1) \end{array}$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -7 & 5 \\ 0 & 1 & -7 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad r_2 \mapsto r_2 - r_3$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} r_2 \mapsto r_2 - 5r_4 \\ r_3 \mapsto -\frac{1}{7}(r_3 - r_2) \end{array}$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad r_4 \mapsto r_4 - r_3$$

Seeking to determine the intersection of the two planes U and V, we find three pivot variables (x_1, x_2, y_2) and one free variable (y_1) as expected, and the following solution set

$$\left\{ v \in R^4 \,\middle|\, v = \lambda \begin{bmatrix} 3 \\ 7 \\ 1 \\ 0 \end{bmatrix},\ \lambda \in R \right\} \qquad \text{and} \qquad U \cap V = \operatorname{span}\left(\begin{bmatrix} 3 \\ 7 \\ 1 \\ 0 \end{bmatrix} \right)$$

2.15 Let $F = \{(x, y, z) \in R^3 \mid x + y - z = 0\}$ and $G = \{(a - b, a + b, a - 3b) \mid a, b \in R\}$.

a. Show that F and G are subspaces of R^3.

b. Calculate $F \cap G$ without resorting to any basis vector.

c. Find one basis for F and one basis for G. Calculate $F \cap G$ using the basis vectors previously found and check your result with the previous question.

F is a subspace of R^3

Additive identity

F contains $(0, 0, 0)$, since $0 + 0 - 0 = 0$.

Closure under scalar multiplication

Let $(a, b, c) \in F$, $\lambda \in R$.
Then $\lambda(a, b, c) = (\lambda a, \lambda b, \lambda c)$
Since $a + b - c = 0$, it follows that $\lambda a + \lambda b - \lambda c = 0$, as required.

Closure under addition

Let $(a, b, c), (x, y, z) \in F$.
Then $(a, b, c) + (x, y, z) = (a + x, b + y, c + z)$.
Since $a + b - c = 0$ and $x + y - z = 0$, it follows that $(a + x) + (b + y) - (c + y) = 0$.

G is a subspace of R^3

Additive identity

Given $(a - b, a + b, a - 3b) \in G$, choose $a = b = 0$:

$$(0 - 0, 0 + 0, 0 - 3 \cdot 0) = (0, 0, 0) \in G.$$

Closure under scalar multiplication

Let $(\alpha - \beta, \alpha + \beta, \alpha - 3\beta) \in G$, $\lambda \in R$. Then

$$\lambda(\alpha - \beta, \alpha + \beta, \alpha - 3\beta) = (\lambda\alpha - \lambda\beta, \lambda\alpha + \lambda\beta, \lambda\alpha - 3\lambda\beta)$$

which is in G with $a = \lambda\alpha$, $b = \lambda\beta$.

Closure under addition

Let $(\alpha - \beta, \alpha + \beta, \alpha - 3\beta), (\gamma - \delta, \gamma + \delta, \gamma - 3\delta) \in G$. Then

$$\begin{aligned} & (\alpha - \beta, \alpha + \beta, \alpha - 3\beta) + (\gamma - \delta, \gamma + \delta, \gamma - 3\delta) \\ = \ & ((\alpha + \gamma) - (\beta + \delta), (\alpha + \gamma) + (\beta + \delta), (\alpha + \gamma) - 3(\beta + \delta)) \end{aligned}$$

which is in G with $a = \alpha + \gamma$, $b = \beta + \delta$.

Intersection $F \cap G$

$$\begin{aligned} F \cap G &= \left\{(a-b, a+b, a-3b) \mid (a-b)+(a+b)-(a-3b)=0 \mid a, b \in R^3\right\} \\ &= \left\{(a-b, a+b, a-3b) \mid a+3b=0 \mid a, b \in R^3\right\} \end{aligned}$$

Basis for F

Let $f_1 = (1, 0, 1)$, $f_2 = (0, 1, 1)$; both $f_1, f_2 \in F$.

Given any $(a, b, c) \in F$, $c = a + b$, and $af_1 + bf_2 = (a, b, a + b) = (a, b, c)$.

So $F = \text{span}(f_1, f_2)$.

Basis for G

Choose three members of G, $(1, 2, -2), (1, 1, 1), (2, 4, 0) \in G$ (for (a, b), choose $(1, 1)$, $(1, 0)$, $(1, 3)$, respectively).

Form a matrix with these columns and row-reduce:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 4 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix} \qquad r_3 \mapsto r_3 + r_2$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 4 \end{bmatrix} \qquad r_2 \mapsto -(r_2 - 2r_1)$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad r_3 \mapsto \tfrac{1}{4}(r_3 - 2r_2)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad r_1 \mapsto r_1 - r_2 - 2r_3$$

The result has full rank, so $G = R^3$.

Basis for $F \cap G$

Since $G = R^3$ and $F \subset R^3$, $F \cap G = F = \text{span}(f_1, f_2)$.

Revisiting G

Considering any $(x, y, z) \in R^3$ and given the definition

$$G = \{(a-b, a+b, a-3b) \mid a, b \in R\}$$

solving for a and b,

$$a = \tfrac{1}{2}(x+y) \qquad b = \tfrac{1}{6}(x+y-2z)$$

which are always defined. It follows that $G = R^3$.

2.16 Are the following mappings linear?

a. Let $a, b \in R^3$.

$$\Phi : L^1([a, b]) \to R$$
$$f \mapsto \Phi(f) = \int_a^b f(x)\, dx$$

where $L^1([a, b])$ denotes the set of integrable functions on $[a, b]$.
Linear.
Additivity: $\int_a^b f + g = \int_a^b f + \int_a^b g$
Homogeneity: $\int_a^b \lambda f = \lambda \int_a^b f$

b.

$$\Phi : C^1 \to C^0$$
$$f \mapsto \Phi(f) = f'$$

where for $k \geq 1$, C^k denotes the set of k times continuously differentiable functions, and C^0 denotes the set of continuous functions.
Linear.
Additivity: $(f + g)' = f' + g'$.
Homogeneity: $(\lambda f)' = \lambda f'$.

c.

$$\Phi : R \to R$$
$$x \mapsto \Phi(x) = \cos(x)$$

Non-linear. Additivity required, but in general,

$$\cos(x + y) \neq \cos(x) + \cos(y).$$

d.

$$\Phi : R^3 \to R^2$$
$$x \mapsto \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} x$$

Linear. Matrix multiplication is linear.

e. Let $\theta \in [0, 2\pi]$ and

$$\Phi : R^2 \to R^2$$
$$x \mapsto \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} x$$

Linear. Matrix multiplication is linear.

2.17 Consider the linear mapping

$$\Phi : R^3 \to R^4$$
$$\Phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$

a. Find the transformation matrix $\mathbf{A}_\Phi$

b. Determine rank($\mathbf{A}_\Phi$)

c. Compute the kernel and image of Φ. What are dim(ker(Φ)) and dim(Im(Φ))?

$$\mathbf{A}_\Phi = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Row-reducing, we have

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{matrix} r_1 \mapsto \frac{1}{2}(r_1 - r_3 - r_4) \\ r_4 \mapsto r_4 - 2r_2 \end{matrix}$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad r_3 \mapsto r_3 + 3r_4$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} r_1 \mapsto r_3 \\ r_2 \mapsto r_1 \\ r_3 \mapsto r_2 - r_1 - r_4 \\ r_4 \mapsto r_4 - r_1 \end{matrix}$$

So $\mathbf{A}_\Phi$ has rank 3, $\ker(\Phi) = \{0\} \subset R^3$.

By the rank-nullity theorem,

$$\dim(\mathrm{Dom}\,\Phi) = \dim(\mathrm{Im}\,\Phi) + \dim(\ker \Phi)$$

so $\dim(\ker \Phi) = 0$ gives $\dim(\mathrm{Im}\,\Phi) = 3$.

2.18 Let E be a vector space.

Let f and g be two automorphisms on E such that $f \circ g = \mathrm{id}_E$ (the identity mapping on E).

Show that:

a. $\ker(f) = \ker(g \circ f)$

b. $\mathrm{Im}(g) = \mathrm{Im}(g \circ f)$

c. $\ker(f) \cap \mathrm{Im}(g) = \{0_E\}$

Since $f \circ g = \mathrm{id}_E$, both f and g are invertible: $g = f^{-1}$ and $f = g^{-1}$.

a. $\ker(f) = \ker(g \circ f)$

Since f is invertible, it is surjective (maps onto all of E), so

$$\ker(f) = \{0_E\}.$$

Similarly, $g \circ f = \mathrm{id}_E$, so

$$\ker(g \circ f) = \ker(\mathrm{id}_E) = \{0_E\}.$$

Therefore

$$\ker(f) = \ker(g \circ f) = \{0_E\}.$$

b. $\mathrm{Im}(g) = \mathrm{Im}(g \circ f)$

Since g is invertible, it is surjective, so

$$\mathrm{Im}(g) = E.$$

Likewise, $g \circ f = \mathrm{id}_E$, so

$$\mathrm{Im}(g \circ f) = \mathrm{Im}(\mathrm{id}_E) = E.$$

Therefore,

$$\mathrm{Im}(g) = \mathrm{Im}(g \circ f) = E.$$

c. $\ker(f) \cap \mathrm{Im}(g) = \{0_E\}$

Taking the results from a. and b.,

$$\ker(f) \cap \mathrm{Im}(g) = \{0_E\} \cap E = \{0_E\}.$$

2.19 Consider an endomorphism $\Phi : R^3 \to R^3$ whose transformation matrix (with respect to the standard basis in R^3) is

$$\mathbf{A}_\Phi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

a. Determine $\ker(\Phi)$ and $\text{Im}(\Phi)$

$\mathbf{A}_\Phi$ row-reduces to $\mathbf{I} \in R^3$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore $\ker(\Phi) = \{0\} \in R^3$ and $\text{Im}(\Phi) = R^3$

b. Determine the transformation matrix $\tilde{\mathbf{A}}_\Phi$ with respect to the basis

$$B = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

Construct and row-reduce the augmented matrix, $[\ \mathbf{S}_B \mid \mathbf{I}\ ][\ \mathbf{I} \mid \mathbf{S}_B^{-1}\ ]$ (where $\mathbf{S}_B$ is the change-of-basis matrix for B) as follows

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & -1 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & -1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

This gives

$$\mathbf{S}_B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad \mathbf{S}_B^{-1} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

The transformation matrix with respect to the basis B, is then given by

$$\tilde{\mathbf{A}}_\Phi = \mathbf{S}_B^{-1} \mathbf{A}_\Phi \mathbf{S}_B$$

$$\tilde{\mathbf{A}}_\Phi = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 1 \\ -3 & -5 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

2.20 Let us consider b_1, b_2, b'_1, b'_2 as vectors of R^2, expressed in the standard basis of R^2 as

$$b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad b_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad b'_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \quad b'_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

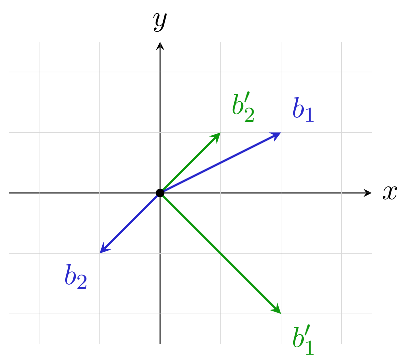
and let us define two ordered bases $B = (b_1, b_2)$ and $B' = (b'_1, b'_2)$.

a. Show that B and B' are two bases of R^2 and draw those basis vectors.

We find

$$\det\left(\begin{bmatrix} b_1 & b_2 \end{bmatrix}\right) = -1 \qquad \det\left(\begin{bmatrix} b'_1 & b'_2 \end{bmatrix}\right) = 4$$

so b_1 and b_2 are linearly independent, likewise b'_1 and b'_2, so both pairs of vectors span the whole space, and B and B' are bases of R^2:



b. Compute the matrix $\mathbf{P}_1$ that performs a basis change from B' to B.

Denote $\mathbf{S}_B$ and $\mathbf{S}_{B'}$ as the change-of-basis matrices from B and B' to E_2, the standard basis of R^2, respectively:

$$\mathbf{S}_B = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{S}_{B'} = \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}$$

Then

$$\mathbf{P}_1 = \mathbf{S}_B^{-1}\mathbf{S}_{B'} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}$$

c. Let $C = (c_1, c_2, c_3)$, where c_1, c_2, c_2 are three vectors of R^3 defined in E_3, the standard basis of R^3:

$$c_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad c_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \quad c_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

i. Show that C is a basis of R^3 (for example, by using determinants).

The matrix with columns c_1, c_2, c_3 has full rank, so C is a basis of R^3:

$$\det\left(\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}\right) + \det\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right) = 4$$

ii. Let us call $C' = (c'_1, c'_2, c'_3)$ the standard basis of R^3. Determine the matrix $\mathbf{P}_2$ that performs the change of basis from C to C'.

$\mathbf{P}_2$ is exactly the matrix above, with columns c_1, c_2, c_3:

$$\mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$$

d. We consider a homomorphism $\Phi : R^2 \to R^3$, such that

$$\Phi(b_1 + b_2) = c_2 + c_3$$
$$\Phi(b_1 - b_2) = 2c_1 - c_2 + 3c_3$$

where $B = (b_1, b_2)$ and $C = (c_1, c_2, c_3)$ are ordered bases of R^2 and R^3, respectively. Determine the transformation matrix $\mathbf{A}_\Phi$ of Φ with respect to the ordered bases B and C.

Reformulating, we have

$$\mathbf{A}_\Phi \begin{bmatrix} 1 \\ 1 \end{bmatrix}_B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_C \quad \text{and} \quad \mathbf{A}_\Phi \begin{bmatrix} 1 \\ -1 \end{bmatrix}_B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}_C$$

which we combine and solve:

$$\begin{aligned} \mathbf{A}_\Phi \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} &= \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} \\ \iff \quad \mathbf{A}_\Phi &= \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \end{aligned}$$

e. Determine $\mathbf{A}'_\Phi$, the transformation matrix with respect to the bases B' and C'.

We already computed $\mathbf{P}_1$, the change of basis matrix from B' to B, and $\mathbf{P}_2$, the change of basis matrix from C to C', so composing these with $\mathbf{A}_\Phi$ we have

$$\mathbf{A}'_\Phi = \mathbf{P}_2 \mathbf{A}_\Phi \mathbf{P}_1$$

Spelling this out,

$$\underbrace{[x]_{B'}}_{\text{coords in } B'} \xrightarrow{\mathbf{P}_1} \underbrace{[x]_B}_{\text{coords in } B} \xrightarrow{\mathbf{A}_\Phi} \underbrace{[\Phi(x)]_C}_{\text{coords in } C} \xrightarrow{\mathbf{P}_2} \underbrace{[\Phi(x)]_{C'}}_{\text{coords in } C'}$$

$$
\begin{tikzcd}
R^2 \arrow[r, "\Phi"] & R^3 \\
B \arrow[r, "\mathbf{A}_\Phi"] \arrow[d, bend right, "\mathbf{S}_B"'] & C \arrow[d, "\mathbf{P}_2"] \\
E_2 \arrow[u, bend right, "\mathbf{S}_B^{-1}"'] \arrow[d, bend left, "\mathbf{S}_{B'}^{-1}"] & C' = E_3 \\
B' \arrow[u, bend left, "\mathbf{S}_{B'}"] \arrow[uu, bend left=60, "\mathbf{P}_1"] \arrow[ur, bend right, "\mathbf{A}'_\Phi"'] &
\end{tikzcd}
$$

Figure 1: Bases and matrices in Exercise 2.20

We compute $\mathbf{A}'_\Phi$:

$$\mathbf{A}'_\Phi = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix}$$

f. Let us consider the vector $x \in R^2$, whose coordinates in B' are $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

In other words, $x = 2b'_1 + 3b'_2$.

i. Calculate the coordinates of x in B.

$$[x]_B = \mathbf{P}_1 [x]_{B'} = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

ii. Based on that, compute the coordinates of $\Phi(x)$ expressed in C.

$$[\Phi(x)]_C = \mathbf{A}_\Phi [x]_B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix}$$

iii. Then write $\Phi(x)$ in terms of c'_1, c'_2, c'_3.

$$[\Phi(x)]_{C'} = \mathbf{P}_2 [\Phi(x)]_C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix}$$

iv. Use the representation of x in B' and the matrix $\mathbf{A}'_\Phi$ to find this result directly.

$$[\Phi(x)]_{C'} = \mathbf{A}'_\Phi [x]_{B'} = \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix}$$