HAMT

Heat and Mass Transfer

Contents

1 Introduction

Text [1].

2 FEM

2.1 General Statements

The simulation domain is seen in Fig.

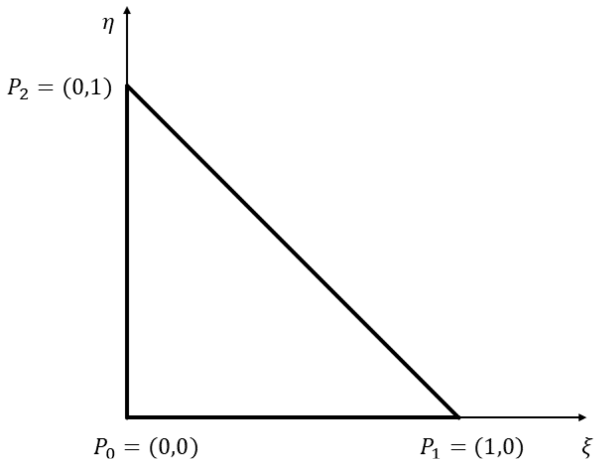

Figure 2.1: Radiation.

Using the definitions

$$x_i \equiv P_{i,x} \tag{2.1}$$
$$y_i \equiv P_{i,y} \tag{2.2}$$

and

$$\Delta x_k = x_0 - x_k \tag{2.3}$$
$$\Delta y_k = y_0 - y_k \tag{2.4}$$

the cartesian coordinates can be expressed in terms of the new coordinates as

$$x(\xi, \eta) = x_0 + \Delta x_1 \xi + \Delta x_2 \eta \tag{2.5}$$
$$y(\xi, \eta) = y_0 + \Delta y_1 \xi + \Delta y_2 \eta \tag{2.6}$$

and the new coordinates as

$$\xi(x,y) = \frac{1}{\det\left(\mathrm{J}(x,y)\right)}\big(\Delta y_2(x-x_0) - \Delta x_2(y-y_0)\big) \tag{2.7}$$

$$\eta(x,y) = \frac{1}{\det\left(\mathrm{J}(x,y)\right)}\big(-\Delta y_1(x-x_0) + \Delta x_1(y-y_0)\big) \tag{2.8}$$

with the Jacobian matrix

$$\mathrm{J}(x,y) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} = \begin{bmatrix} \Delta x_1 & \Delta x_2 \\ \Delta y_1 & \Delta y_2 \end{bmatrix} \tag{2.9}$$

and its determinant

$$\det\left(\mathrm{J}(x,y)\right) = \Delta x_1 \Delta y_2 - \Delta x_2 \Delta y_1. \tag{2.10}$$

The Jacobian for the backward transformation can be expressed as

$$\mathrm{J}(\xi,\eta) = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{\det\left(\mathrm{J}(x,y)\right)} \begin{bmatrix} \Delta y_2 & -\Delta x_2 \\ -\Delta y_1 & \Delta x_1 \end{bmatrix} \tag{2.11}$$

and its determinant

$$\det\left(\mathrm{J}(\xi,\eta)\right) = \frac{\Delta x_1 \Delta y_2 + \Delta x_2 \Delta y_1}{\Delta x_1 \Delta y_2 - \Delta x_2 \Delta y_1}. \tag{2.12}$$

Required derivatives can be transformed as

$$\frac{\partial u(\xi,\eta)}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \tag{2.13}$$

$$\frac{\partial u(\xi,\eta)}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}. \tag{2.14}$$

With use tetrahedral ansatz functions $\phi(\xi,\eta)$ with the definition

$$\phi_i \equiv \phi(P_i) \tag{2.15}$$

which can be explicitly given as for the new coordinate system as

$$\phi_0(\xi,\eta) = -\xi - \eta + 1 \tag{2.16}$$

$$\phi_1(\xi,\eta) = \xi \tag{2.17}$$

$$\phi_2(\xi,\eta) = \eta. \tag{2.18}$$

Using this ansatz function the temperature is defined in the support space as

$$T(\xi,\theta) = \sum_i^3 T_i \phi_i \tag{2.19}$$

and its gradient

$$\begin{aligned} \nabla T(\xi,\eta) &= \begin{pmatrix} \frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y} \end{pmatrix} \\ &= \frac{1}{\det\left(\mathrm{J}(x,y)\right)} \begin{pmatrix} T_0(\Delta y_1 - \Delta y_2) + T_1 \Delta y_2 - T_2 \Delta y_1 \\ T_0(\Delta x_2 - \Delta x_1) - T_1 \Delta x_2 + T_2 \Delta x_1 \end{pmatrix}. \end{aligned} \tag{2.20}$$

3 Heat Equation

3.1 General Statements

Given is the heat equation

$$\frac{\partial\left(\rho c_p T\right)}{\partial t} = \nabla \cdot (\lambda \nabla T) + \dot{q}_V \tag{3.21}$$

where ρ is the density, c_p the heat capacity at constant pressure, λ the thermal conductivity, T the temperature and $\dot{q}_V$ a volumetric heat source. Assuming piecewise constant material properties and no changes in time Eqn. 3.21 can be rewritten as

$$\alpha \nabla^2 T + \beta\, \dot{q}_V = 0 \tag{3.22}$$

where $\alpha = \frac{\lambda}{\rho c_p}$ and $\beta = \frac{1}{\rho c_p}$.

3.2 2D FEM Discretization Using a Triangular Grid

Using a tetrahedral base function as defined in the previous chapter and assuming a volumetric heat flux which is constant per cell we can write

$$\sum_c \alpha_c \iint \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \phi_c \,\mathrm{d}x\,\mathrm{d}y = \tag{3.23}$$

$$-\sum_c \beta_c \dot{q}_c \iint \phi_c \,\mathrm{d}x\,\mathrm{d}y. \tag{3.24}$$

Using integration by parts and the fact that the support is defined such that is is zero and at the boundary we get for the left hand side looking at a single cell and dropping the summation

$$\mathrm{LHS} = \alpha \iint \left(\frac{\partial T}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial T}{\partial y}\frac{\partial \phi}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y \tag{3.25}$$

$$\mathrm{RHS} = \beta \dot{q} \iint \phi \,\mathrm{d}x\,\mathrm{d}y. \tag{3.26}$$

The missing minus sign on the RHS comes from the minus sign in the integration by parts method.

We will now using coordinate transformation to unit triangle. The right hand side can imidiatly be evaluated as

$$\mathrm{RHS} = \beta \dot{q} \int_0^1 \int_0^{1-\eta} \phi \det\left(\mathrm{J}(x,y)\right) \mathrm{d}\xi\,\mathrm{d}\eta = \frac{1}{6} \beta \dot{q} \det\left(\mathrm{J}(x,y)\right). \tag{3.27}$$

For the right had side we need to transform the derivatives. We get

$$\frac{\partial T}{\partial x}\frac{\partial \phi}{\partial x} = \left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial x}\right) \tag{3.28}$$

$$= \frac{\Delta y_1 - \Delta y_2}{\det\left(\mathrm{J}(x,y)\right)^2}\Big(T_0(\Delta y_1 - \Delta y_2) + T_1\Delta y_2 - T_2\Delta y_1\Big) \tag{3.29}$$

and

$$\frac{\partial T}{\partial y}\frac{\partial \phi}{\partial y} = \left(\frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial y}\right) \tag{3.30}$$

$$= \frac{\Delta x_2 - \Delta x_1}{\det\left(\mathrm{J}(x,y)\right)^2}\Big(T_0(\Delta x_2 - \Delta x_1) - T_1\Delta x_2 + T_2\Delta x_1\Big) \tag{3.31}$$

Putting back into LHS we get

$$\alpha\int_0^1\int_0^{1-\eta}\frac{\partial T}{\partial x}\frac{\partial \phi}{\partial x}\det\left(\mathrm{J}(x,y)\right)\mathrm{d}\xi\,\mathrm{d}\eta = \tag{3.32}$$

$$\frac{1}{2}\alpha\frac{\Delta y_1 - \Delta y_2}{\det\left(\mathrm{J}(x,y)\right)}\Big(T_0(\Delta y_1 - \Delta y_2) + T_1\Delta y_2 - T_2\Delta y_1\Big) \tag{3.33}$$

and

$$\alpha\int_0^1\int_0^{1-\eta}\frac{\partial T}{\partial y}\frac{\partial \phi}{\partial y}\det\left(\mathrm{J}(x,y)\right)\mathrm{d}\xi\,\mathrm{d}\eta = \tag{3.34}$$

$$\frac{1}{2}\alpha\frac{\Delta x_2 - \Delta x_1}{\det\left(\mathrm{J}(x,y)\right)}\Big(T_0(\Delta x_2 - \Delta x_1) - T_1\Delta x_2 + T_2\Delta x_1\Big). \tag{3.35}$$

4 Boundary Conditions

4.1 Heat Flux - radiation branch

Given a boundary segment we can prescribe the a heat flux normal to that boundary as

$$-\lambda\,(\vec{n}\cdot\nabla T)=\dot{q}_{\perp} \tag{4.36}$$

where λ is the thermal conductivity, $\vec{n}$ the normal vector of the boundary, ∇T the temperature gradient at a given point in the cell and $\dot{q}_{\perp}$ the differential heat flux perpendicular to the boundary.

We use linear element as our ansatz function for T and triangles as cells which means that $\vec{n}$ and ∇T are constants. If we now assume a per cell constant thermal conductivity λ this means that the differential heat flux $\dot{q}_{\perp}$ must be constant over a given cell side.

Using the FEM ansatz with linear tetrahedral elements we can write

$$-\lambda\int_0^1\int_0^{1-\eta}(\vec{n}\cdot\nabla T)\,\phi_0\det\left(\mathrm{J}(x,y)\right)\mathrm{d}\xi\,\mathrm{d}\eta=\dot{q}_{\perp}\int_0^1\int_0^{1-\eta}\phi_0\det\left(\mathrm{J}(x,y)\right)\mathrm{d}\xi\,\mathrm{d}\eta. \tag{4.37}$$

Assuming a normal vector we get

$$\vec{n}\cdot\nabla T=\frac{1}{\det\left(\mathrm{J}(x,y)\right)}\vec{n}\cdot\begin{pmatrix}T_0(\Delta y_1-\Delta y_2)+T_1\Delta y_2-T_2\Delta y_1\\ T_0(\Delta x_2-\Delta x_1)-T_1\Delta x_2+T_2\Delta x_1\end{pmatrix} \tag{4.38}$$

which leads to

$$\vec{n}\cdot\begin{pmatrix}T_0(\Delta y_1-\Delta y_2)+T_1\Delta y_2-T_2\Delta y_1\\ T_0(\Delta x_2-\Delta x_1)-T_1\Delta x_2+T_2\Delta x_1\end{pmatrix}\iint\phi_0\,\mathrm{d}\xi\,\mathrm{d}\eta= \tag{4.39}$$

$$-\frac{\dot{q}_{\perp}\det\left(\mathrm{J}(x,y)\right)}{\lambda}\iint\phi_0\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{4.40}$$

and therefor

$$\boxed{\vec{n}\cdot\begin{pmatrix}T_0(\Delta y_1-\Delta y_2)+T_1\Delta y_2-T_2\Delta y_1\\ T_0(\Delta x_2-\Delta x_1)-T_1\Delta x_2+T_2\Delta x_1\end{pmatrix}=-\frac{\dot{q}_{\perp}\det\left(\mathrm{J}(x,y)\right)}{\lambda}} \tag{4.41}$$

4.2 Heat Flux

Given a boundary segment we can prescribe the a heat flux normal to that boundary as

$$-\lambda\,(\vec{n}\cdot\nabla T)=\dot{q}_{\perp} \tag{4.42}$$

where λ is the thermal conductivity, $\vec{n}$ the normal vector of the boundary, ∇T the temperature gradient at a given point in the cell and $\dot{q}_\perp$ the differential heat flux perpendicular to the boundary.

We use linear element as our ansatz function for T and triangles as cells which means that $\vec{n}$ and ∇T are constants. If we now assume a per cell constant thermal conductivity λ this means that the differential heat flux $\dot{q}_\perp$ must be constant over a given cell side. Using Eqn. 2.20 we can write

$$\begin{aligned} T_0\left[\vec{n}_x(\Delta y_1 - \Delta y_2) + \vec{n}_y(\Delta x_2 - \Delta x_1)\right] + \\ T_1(\vec{n}_x\Delta y_2 - \vec{n}_y\Delta x_2) + T_2(\vec{n}_y\Delta x_1 - \vec{n}_x\Delta y_1) = -\frac{\dot{q}_\perp}{\lambda}\det\left(\mathrm{J}(x,y)\right). \end{aligned} \tag{4.43}$$

4.3 Radiation

The net radiation heat flux from surface 1 to surface 2 using grey body radiation can be calculated as

$$\dot{Q}_{1\to 2} = A_1 F_{1\to 2} E_1 - A_2 F_{2\to 1} E_2. \tag{4.44}$$

using the formula for emission of grey bodies

$$E_i = \epsilon_i \sigma T_i^4 \tag{4.45}$$

and the reciprocity rule for configuration factors $A_1 F_{1\to 2} = A_2 F_{2\to 1}$ we can write

$$\dot{Q}_{1\to 2} = \sigma A_1 F_{1\to 2}\left(\epsilon_1 T1^4 - \epsilon_2 T_2^4\right) \tag{4.46}$$

and $\dot{Q}_{1\to 2} = -\dot{Q}_{2\to 1}$. Given two line segments as seen in Fig. 4.1

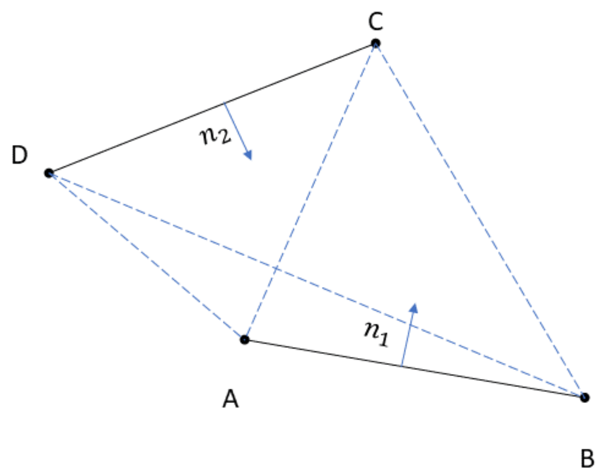

Figure 4.1: Radiation.

the configuration factor from surface $\overline{\mathrm{AB}}$ to surface $\overline{\mathrm{CD}}$ can be calculated as

$$F_{\overline{\mathrm{AB}}\to\overline{\mathrm{CD}}} = \frac{\overline{\mathrm{AC}} + \overline{\mathrm{BD}} - \overline{\mathrm{AD}} - \overline{\mathrm{BC}}}{2\overline{\mathrm{AB}}} \tag{4.47}$$

where $\overline{\mathrm{XY}}$ is the distance from X to Y. The total heat flux from or to a single surface is the sum of the heat fluxes to other surfaces plus the heat flux to the background

$$\dot{Q}_{tot} = \sigma A_1 \left\{ \epsilon_1 T_1^4 \sum_i F_{1\to i} - \sum_i \epsilon_i F_{1\to i} T_i^4 \right\} + \dot{Q}_{backgr} \tag{4.48}$$

$$\dot{Q}_{tot} = \sigma A_1 \left\{ \epsilon_1 T_1^4 \sum_i F_{1\to i} - \sum_i \epsilon_i F_{1\to i} T_i^4 + F_{1\to bg}(\epsilon_i T_1^4 - \epsilon_{bg} T_{bg}^4) \right\} \tag{4.49}$$

Since $F_{1\to bg} = 1 - \sum_i F_{1\to i}$ we get

$$\dot{Q}_{tot} = \sigma A_1 \left\{ \epsilon_1 T_1^4 - \sum_i \epsilon_i F_{1\to i} T_i^4 - F_{1\to bg} \epsilon_{bg} T_{bg}^4 \right\} \tag{4.50}$$

In terms of boundary conditions, using $\dot{q}_{1\to 2} = \dot{Q}_{1\to 2}/A_1$,we ca write

$$\lambda\left(\vec{n} \cdot \nabla T\right) = \dot{q}_{1\to 2} \tag{4.51}$$

5 Euler Equations

5.1 General formulations

The Euler equations in the conservative form can be stated using the Einstein summations as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0 \tag{5.52}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j + p\delta_{ij}) = 0 \tag{5.53}$$

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_j}[(\rho E + p)u_j] = 0 \tag{5.54}$$

those are the continuum equation, momentum equation and energy equation. They are close using the ideal gas equation

$$p = \rho R T \tag{5.55}$$

and the equation for internal energy

$$\rho E = \frac{\gamma}{\gamma - 1}p + \frac{1}{2}\rho|u|^2. \tag{5.56}$$

It can also be written in vector form as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{5.57}$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}^T) + \nabla p - \mathbf{f} = 0 \tag{5.58}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{u}(E + p)) = 0 \tag{5.59}$$

where

$$E = \rho e + \frac{1}{2}\rho\|\mathbf{u}\|^2 \tag{5.60}$$

and

$$e = T c_v. \tag{5.61}$$

According to Anderson, JR. 6th edition: Conservation of mass (continuity equation)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{5.62}$$

Momentum equation

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{u}) = -\frac{\partial p}{\partial x} + \rho f_x + (F_x)_{\text{viscous}} \tag{5.63}$$

$$\frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v \mathbf{u}) = -\frac{\partial p}{\partial y} + \rho f_y + (F_y)_{\text{viscous}} \tag{5.64}$$

$$\frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w \mathbf{u}) = -\frac{\partial p}{\partial z} + \rho f_z + (F_z)_{\text{viscous}} \tag{5.65}$$

where f are body forces like gravitation and electromagnetic and F are forces due to viscosity. The energy equation

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{1}{2}\|\mathbf{u}\|^2\right)\right] + \nabla \cdot \left[\rho\left(e + \frac{1}{2}\|\mathbf{u}\|^2\right)\mathbf{u}\right] = \rho\dot{q} - \nabla \cdot (p\mathbf{u}) + \rho(\mathbf{f}\cdot\mathbf{u}) + \dot{Q}_{\text{viscous}} + \dot{w}_{\text{viscous}} \tag{5.66}$$

where $\dot{Q}_{\text{viscous}}$ and $\dot{w}_{\text{viscous}}$ are the viscous heat and work respectively. Here again we have

$$e = Tc_v \tag{5.67}$$

which is equivalent for noble gases to

$$e = \frac{3}{2}\frac{k_B}{m}T \tag{5.68}$$

where m is the per molecule mass.

5.2 Inviscid steady state form

Disregarding body forces in vector form

$$\nabla \cdot (\rho \mathbf{u}) = 0 \tag{5.69}$$

$$\nabla \cdot (\rho \mathbf{u}\mathbf{u}^T) + \nabla p = 0 \tag{5.70}$$

$$\nabla \cdot (\mathbf{u}(E + p)) = 0 \tag{5.71}$$

or in the Einstein notation

$$\frac{\partial \rho u_j}{\partial x_j} = 0 \tag{5.72}$$

$$\frac{\partial}{\partial x_j}\left(\rho u_i u_j + p\delta_{ij}\right) = 0 \tag{5.73}$$

$$\frac{\partial}{\partial x_j}\left[(E + p)u_j\right] = 0 \tag{5.74}$$

again with

$$E = \rho e + \frac{1}{2}\rho\|\mathbf{u}\|^2 \tag{5.75}$$

$$e = c_v T \tag{5.76}$$

$$e = \frac{3}{2}k_B T. \tag{5.77}$$

5.3 Discritization

Using finite difference methods with non–equidistant spacing as seen in Fig. 5.1.

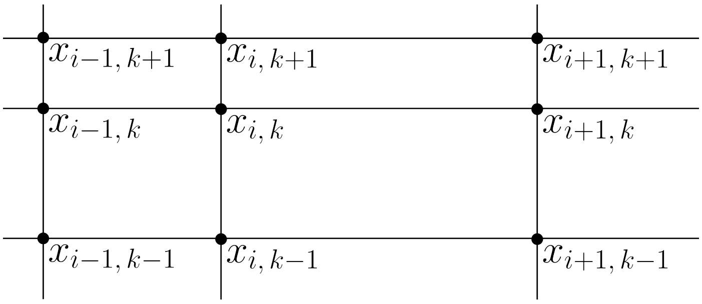

Figure 5.1: Non–equidistant spacing on a regular grid.

Forward differential coefficient and backward differential coefficient

$$\frac{\partial y}{\partial x} = \frac{y(x_{i+1}) - y(x_i)}{x_{i+1} - x_i} \tag{5.78}$$

$$\frac{\partial y}{\partial x} = \frac{y(x_i) - y(x_{i-1})}{x_i - x_{i-i}} \tag{5.79}$$

using Taylor expansion

$$Ty(x;a) = y(a) + \left.\frac{\partial y}{\partial x}\right|_{x=a}(x-a) \tag{5.80}$$

we get

$$y(x_{i+1}) = y(x_i) + \left.\frac{\partial y}{\partial x}\right|_{x=x_i}(x_{i+1} - x_i) \tag{5.81}$$

$$y(x_{i-1}) = y(x_i) + \left.\frac{\partial y}{\partial x}\right|_{x=x_i}(x_{i-1} - x_i) \tag{5.82}$$

we get

$$y(x_{i+1}) - y(x_{i-1}) = y'(x_i)(x_{i+1} - x_i) - y'(x_i)(x_{i-1} - x_i) \tag{5.83}$$

$$y(x_{i+1}) - y(x_{i-1}) = y'(x_i)(x_{i+1} - x_{i-1}) \tag{5.84}$$

and therefore

$$\frac{y(x_{i+1}) - y(x_{i-1})}{x_{i+1} - x_{i-1}} = y'(x_i). \tag{5.85}$$

For second order we have

$$y''(x_i) = \frac{\frac{y(x_{i+1})-y(x_i)}{x_{i+1}-x_i} - \frac{y(x_i)-y(x_{i-1})}{x_i-x_{i-1}}}{x_{i+1} - x_i} \tag{5.86}$$

using the definitions

$$\Delta x^+ = x_{i+1} - x_i \tag{5.87}$$

$$\Delta x^- = x_i - x_{i-1} \tag{5.88}$$

$$\Delta \overline{x} = \Delta x^+ - \Delta x^- \tag{5.89}$$

and therefore

$$y''(x_i) = \frac{y(x_{i+1}) - y(x_i)}{\Delta x^+ \Delta x^+} - \frac{y(x_i) - y(x_{i-1})}{\Delta x^- \Delta x^+} \tag{5.90}$$

$$y''(x_i) = \frac{\Delta x^- y(x_{i+1}) - \Delta x^- y(x_i) - \Delta x^+ y(x_i) + \Delta x^+ y(x_{i-1})}{\Delta x^+ \Delta x^- \Delta x^+} \tag{5.91}$$

$$y''(x_i) = \frac{\Delta x^- \, y(x_{i+1}) - \Delta \overline{x} \, y(x_i) + \Delta x^+ \, y(x_{i-1})}{\Delta x^+ \, \Delta x^- \, \Delta x^+}. \tag{5.92}$$

For second order, second derivative we have

$$y''(x_i) = \frac{\frac{y(x_{i+2})-y(x_i)}{x_{i+2}-x_i} - \frac{y(x_i)-y(x_{i-2})}{x_i-x_{i-2}}}{x_{i+1} - x_{i-1}}. \tag{5.93}$$

Bibliography

[1] Hans Dieter Baehr and Karl Stephan. *Wärme- und Stoffübertragung.* Berlin, Heidelberg: Springer Berlin Heidelberg, 2019. ISBN: 978-3-662-58440-8 978-3-662-58441-5. DOI: 10.1007/978-3-662-58441-5. URL: http://link.springer.com/10.1007/978-3-662-58441-5 (visited on 10/20/2024).