

HAMT

Heat and Mass Transfer

Contents

1	Introduction	3
2	FEM	4
3	Boundary Conditions	6
3.1	Heat Flux	6
3.2	Radiation	7

1 Introduction

Text [1].

2 FEM

The simulation domain is seen in Fig.

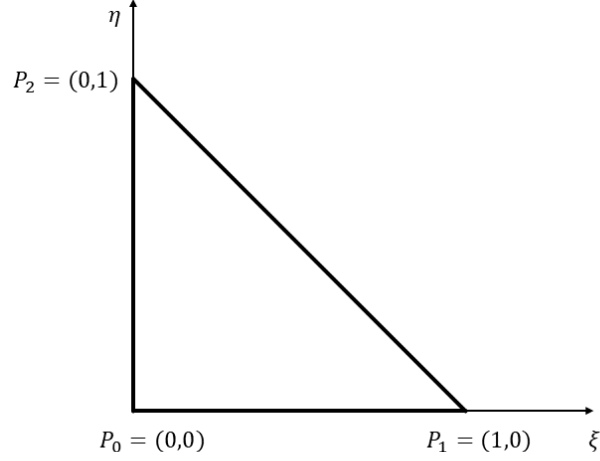


Figure 2.1: Radiation.

Using the definitions

$$x_i \equiv P_{i,x} \quad (2.1)$$

$$y_i \equiv P_{i,y} \quad (2.2)$$

and

$$\Delta x_k = x_0 - x_k \quad (2.3)$$

$$\Delta y_k = y_0 - y_k \quad (2.4)$$

the cartesian coordinates can be expressed in terms of the new coordinates as

$$x(\xi, \eta) = x_0 + \Delta x_1 \xi + \Delta x_2 \eta \quad (2.5)$$

$$y(\xi, \eta) = y_0 + \Delta y_1 \xi + \Delta y_2 \eta \quad (2.6)$$

and the new coordinates as

$$\xi(x, y) = \frac{1}{\det(J(x, y))} (\Delta y_2(x - x_0) - \Delta x_2(y - y_0)) \quad (2.7)$$

$$\eta(x, y) = \frac{1}{\det(J(x, y))} (-\Delta y_1(x - x_0) + \Delta x_1(y - y_0)) \quad (2.8)$$

with the Jacobian matrix

$$J(x, y) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} = \begin{bmatrix} \Delta x_1 & \Delta x_2 \\ \Delta y_1 & \Delta y_2 \end{bmatrix} \quad (2.9)$$

and its determinant

$$\det(J(x, y)) = \Delta x_1 \Delta y_2 - \Delta x_2 \Delta y_1. \quad (2.10)$$

The Jacobian for the backward transformation can be expressed as

$$J(\xi, \eta) = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{\det(J(x, y))} \begin{bmatrix} \Delta y_2 & -\Delta x_2 \\ -\Delta y_1 & \Delta x_1 \end{bmatrix} \quad (2.11)$$

and its determinant

$$\det(J(\xi, \eta)) = \frac{\Delta x_1 \Delta y_2 + \Delta x_2 \Delta y_1}{\Delta x_1 \Delta y_2 - \Delta x_2 \Delta y_1}. \quad (2.12)$$

Required derivatives can be transformed as

$$\frac{\partial u(\xi, \eta)}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} \quad (2.13)$$

$$\frac{\partial u(\xi, \eta)}{\partial y} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y}. \quad (2.14)$$

With use tetrahedral ansatz functions $\phi(\xi, \eta)$ with the definition

$$\phi_i \equiv \phi(P_i) \quad (2.15)$$

which can be explicitly given as for the new coordinate system as

$$\phi_0(\xi, \eta) = -\xi - \eta + 1 \quad (2.16)$$

$$\phi_1(\xi, \eta) = \xi \quad (2.17)$$

$$\phi_2(\xi, \eta) = \eta. \quad (2.18)$$

Using this ansatz function the temperature is defined in the support space as

$$T(\xi, \theta) = \sum_i^3 T_i \phi_i \quad (2.19)$$

and its gradient

$$\nabla T(\xi, \eta) = \begin{pmatrix} \frac{\partial T}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial T}{\partial \eta} \frac{\partial \eta}{\partial x} \\ \frac{\partial T}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial T}{\partial \eta} \frac{\partial \eta}{\partial y} \end{pmatrix} \quad (2.20)$$

$$= \frac{1}{\det(J(x, y))} \begin{pmatrix} (-T_0 + T_1)\Delta y_2 - (-T_0 + T_2)\Delta y_1 \\ -(-T_0 + T_1)\Delta x_2 + (-T_0 + T_2)\Delta x_1 \end{pmatrix} \quad (2.21)$$

$$= \frac{1}{\det(J(x, y))} \begin{pmatrix} T_0(\Delta y_1 - \Delta y_2) + T_1\Delta y_2 - T_2\Delta y_1 \\ T_0(\Delta x_2 - \Delta x_1) - T_1\Delta x_2 + T_2\Delta x_1 \end{pmatrix}. \quad (2.22)$$

3 Boundary Conditions

3.1 Heat Flux

Given a boundary segment we can prescribe the a heat flux normal to that boundary as

$$\lambda (\vec{n} \cdot \nabla T) = \dot{q}_\perp \quad (3.1)$$

where λ is the thermal conductivity, \vec{n} the normal vector of the boundary, ∇T the temperature gradient at a given point in the cell and \dot{q}_\perp the differential heat flux perpendicular to the boundary.

We use linear element as our ansatz function for T and triangles as cells which means that \vec{n} and ∇T are constants. If we now assume a per cell constant thermal conductivity λ this means that the differential heat flux \dot{q}_\perp must be constant over a given cell side.

Using the FEM ansatz with linear tetrahedral elements we can write

$$\lambda \int_0^1 \int_0^{1-\eta} (\vec{n} \cdot \nabla T) \phi_0 \det(J(x, y)) d\xi d\eta = \dot{q}_\perp \int_0^1 \int_0^{1-\eta} \phi_0 \det(J(x, y)) d\xi d\eta. \quad (3.2)$$

Assuming the boundary is the bottom one the normal vector is simply

$$\vec{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (3.3)$$

which leads to

$$\vec{n} \cdot \nabla T = -\frac{\partial T}{\partial y} = \frac{T_0(\Delta x_1 - \Delta x_2) + T_1\Delta x_2 - T_2\Delta x_1}{\det(J(x, y))} \quad (3.4)$$

which leads to

$$(T_0(\Delta x_1 - \Delta x_2) + T_1\Delta x_2 - T_2\Delta x_1) \iint \phi_0 d\xi d\eta = \quad (3.5)$$

$$\frac{\dot{q}_\perp \det(J(x, y))}{\lambda} \iint \phi_0 d\xi d\eta \quad (3.6)$$

$$\boxed{T_0(\Delta x_1 - \Delta x_2) + T_1\Delta x_2 - T_2\Delta x_1 = \frac{\dot{q}_\perp}{\lambda} \det(J(x, y))}. \quad (3.7)$$

Assuming the boundary is on the left we get to a similar argumentation to

$$\boxed{T_0(\Delta y_2 - \Delta y_1) - T_1\Delta y_2 + T_2\Delta y_1 = \frac{\dot{q}_\perp}{\lambda} \det(J(x, y))}. \quad (3.8)$$

3.2 Radiation

The net radiation heat flux from surface 1 to surface 2 using grey body radiation can be calculated as

$$\dot{Q}_{1 \rightarrow 2} = A_1 F_{1 \rightarrow 2} E_1 - A_2 F_{2 \rightarrow 1} E_2. \quad (3.9)$$

using the formula for emission of grey bodies

$$E_i = \epsilon_i \sigma T_i^4 \quad (3.10)$$

and the reciprocity rule for configuration factors $A_1 F_{1 \rightarrow 2} = A_2 F_{2 \rightarrow 1}$ we can write

$$\dot{Q}_{1 \rightarrow 2} = \sigma A_1 F_{1 \rightarrow 2} (\epsilon_1 T_1^4 - \epsilon_2 T_2^4) \quad (3.11)$$

and $\dot{Q}_{1 \rightarrow 2} = -\dot{Q}_{2 \rightarrow 1}$. Given two line segments as seen in Fig. 3.1

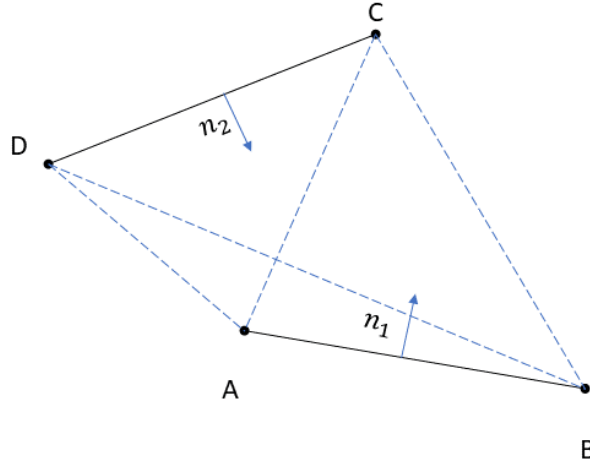


Figure 3.1: Radiation.

the configuration factor from surface \overline{AB} to surface \overline{CD} can be calculated as

$$F_{\overline{AB} \rightarrow \overline{CD}} = \frac{\overline{AC} + \overline{BD} - \overline{AD} - \overline{BC}}{2\overline{AB}} \quad (3.12)$$

where \overline{XY} is the distance from X to Y. The total heat flux from or to a single surface is the sum of the heat fluxes to other surfaces plus the heat flux to the background

$$\dot{Q}_{tot} = \sigma A_1 \left\{ \epsilon_1 T_1^4 \sum_i F_{1 \rightarrow i} - \sum_i \epsilon_i F_{1 \rightarrow i} T_i^4 \right\} + \dot{Q}_{backgr} \quad (3.13)$$

$$\dot{Q}_{tot} = \sigma A_1 \left\{ \epsilon_1 T_1^4 \sum_i F_{1 \rightarrow i} - \sum_i \epsilon_i F_{1 \rightarrow i} T_i^4 + F_{1 \rightarrow bg} (\epsilon_1 T_1^4 - \epsilon_{bg} T_{bg}^4) \right\} \quad (3.14)$$

Since $F_{1 \rightarrow bg} = 1 - \sum_i F_{1 \rightarrow i}$ we get

$$\dot{Q}_{tot} = \sigma A_1 \left\{ \epsilon_1 T_1^4 - \sum_i \epsilon_i F_{1 \rightarrow i} T_i^4 - F_{1 \rightarrow bg} \epsilon_{bg} T_{bg}^4 \right\} \quad (3.15)$$

In terms of boundary conditions, using $\dot{q}_{1 \rightarrow 2} = \dot{Q}_{1 \rightarrow 2}/A_1$, we can write

$$\lambda (\vec{n} \cdot \nabla T) = \dot{q}_{1 \rightarrow 2} \quad (3.16)$$

Bibliography

- [1] Hans Dieter Baehr and Karl Stephan. *Wärme- und Stoffübertragung*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2019. ISBN: 978-3-662-58440-8 978-3-662-58441-5. DOI: 10.1007/978-3-662-58441-5. URL: <http://link.springer.com/10.1007/978-3-662-58441-5> (visited on 10/20/2024).