HAMT

Heat and Mass Transfer

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1 Introduction

Text [1].

2 FEM

The simulation domain is seen in Fig.

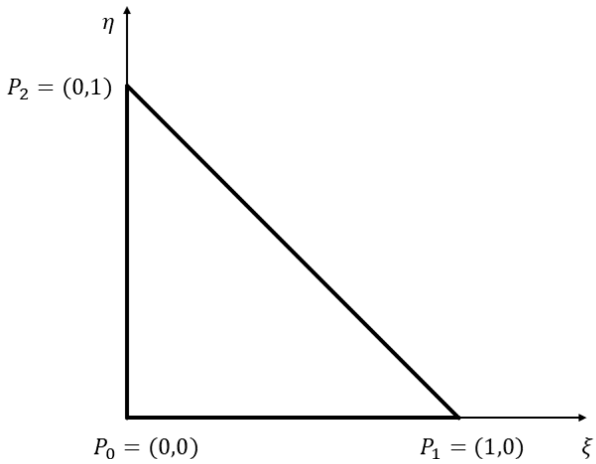

Figure 2.1: Radiation.

Using the definitions

$$x_i \equiv P_{i,x} \tag{2.1}$$
$$y_i \equiv P_{i,y} \tag{2.2}$$

and

$$\Delta x_k = x_0 - x_k \tag{2.3}$$
$$\Delta y_k = y_0 - y_k \tag{2.4}$$

the cartesian coordinates can be expressed in terms of the new coordinates as

$$x(\xi, \eta) = x_0 + \Delta x_1 \xi + \Delta x_2 \eta \tag{2.5}$$
$$y(\xi, \eta) = y_0 + \Delta y_1 \xi + \Delta y_2 \eta \tag{2.6}$$

and the new coordinates as

$$\xi(x, y) = \frac{1}{\det\left(\mathrm{J}(x, y)\right)} \big(\Delta y_2 (x - x_0) - \Delta x_2 (y - y_0)\big) \tag{2.7}$$
$$\eta(x, y) = \frac{1}{\det\left(\mathrm{J}(x, y)\right)} \big(- \Delta y_1 (x - x_0) + \Delta x_1 (y - y_0)\big) \tag{2.8}$$

with the Jacobian matrix

$$\mathrm{J}(x,y) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} = \begin{bmatrix} \Delta x_1 & \Delta x_2 \\ \Delta y_1 & \Delta y_2 \end{bmatrix} \tag{2.9}$$

and its determinant

$$\det\left(\mathrm{J}(x,y)\right) = \Delta x_1 \Delta y_2 - \Delta x_2 \Delta y_1. \tag{2.10}$$

The Jacobian for the backward transformation can be expressed as

$$\mathrm{J}(\xi,\eta) = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{\det\left(\mathrm{J}(x,y)\right)} \begin{bmatrix} \Delta y_2 & -\Delta x_2 \\ -\Delta y_1 & \Delta x_1 \end{bmatrix} \tag{2.11}$$

and its determinant

$$\det\left(\mathrm{J}(\xi,\eta)\right) = \frac{\Delta x_1 \Delta y_2 + \Delta x_2 \Delta y_1}{\Delta x_1 \Delta y_2 - \Delta x_2 \Delta y_1}. \tag{2.12}$$

Required derivatives can be transformed as

$$\frac{\partial u(\xi,\eta)}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \tag{2.13}$$

$$\frac{\partial u(\xi,\eta)}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}. \tag{2.14}$$

With use tetrahedral ansatz functions $\phi(\xi,\eta)$ with the definition

$$\phi_i \equiv \phi(P_i) \tag{2.15}$$

which can be explicitly given as for the new coordinate system as

$$\phi_0(\xi,\eta) = -\xi - \eta + 1 \tag{2.16}$$

$$\phi_1(\xi,\eta) = \xi \tag{2.17}$$

$$\phi_2(\xi,\eta) = \eta. \tag{2.18}$$

Using this ansatz function the temperature is defined in the support space as

$$T(\xi,\theta) = \sum_i^3 T_i \phi_i \tag{2.19}$$

and its gradient

$$\nabla T(\xi,\eta) = \begin{pmatrix} \frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y} \end{pmatrix} \tag{2.20}$$

$$= \frac{1}{\det\left(\mathrm{J}(x,y)\right)} \begin{pmatrix} (-T_0 + T_1)\Delta y_2 - (-T_0 + T_2)\Delta y_1 \\ -(-T_0 + T_1)\Delta x_2 + (-T_0 + T_2)\Delta x_1 \end{pmatrix} \tag{2.21}$$

$$= \frac{1}{\det\left(\mathrm{J}(x,y)\right)} \begin{pmatrix} T_0(\Delta y_1 - \Delta y_2) + T_1 \Delta y_2 - T_2 \Delta y_1 \\ T_0(\Delta x_2 - \Delta x_1) - T_1 \Delta x_2 + T_2 \Delta x_1 \end{pmatrix}. \tag{2.22}$$

3 Boundary Conditions

3.1 Heat Flux

Given a boundary segment we can prescribe the a heat flux normal to that boundary as

$$\lambda\,(\vec{n}\cdot\nabla T) = \dot{q}_\perp \tag{3.1}$$

where λ is the thermal conductivity, $\vec{n}$ the normal vector of the boundary, ∇T the temperature gradient at a given point in the cell and $\dot{q}_\perp$ the differential heat flux perpendicular to the boundary.

We use linear element as our ansatz function for T and triangles as cells which means that $\vec{n}$ and ∇T are constants. If we now assume a per cell constant thermal conductivity λ this means that the differential heat flux $\dot{q}_\perp$ must be constant over a given cell side.

Using the FEM ansatz with linear tetrahedral elements we can write

$$\int_0^1\int_0^{1-\eta}\lambda\,(\vec{n}\cdot\nabla T)\,\phi_i\det\left(\mathrm{J}(x,y)\right)\mathrm{d}\xi\,\mathrm{d}\eta = \dot{q}_\perp\int_0^1\int_0^{1-\eta}\phi_i\det\left(\mathrm{J}(x,y)\right)\mathrm{d}\xi\,\mathrm{d}\eta. \tag{3.2}$$

Assuming the boundary is the button one the normal vector is simply

$$\vec{n} = \begin{pmatrix}0\\1\end{pmatrix} \tag{3.3}$$

which leads to

$$\vec{n}\cdot\nabla T = \frac{\partial T}{\partial y} = \frac{\partial\phi_0}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial\phi_0}{\partial\eta}\frac{\partial\eta}{\partial y} \tag{3.4}$$

which leads to

$$\int_0^1\int_0^{1-\eta} T_i\phi_i\,\mathrm{d}\xi\,\mathrm{d}\eta = \frac{\dot{q}_\perp\det\left(\mathrm{J}(x,y)\right)}{\lambda\,(\Delta x_2-\Delta x_1)}\int_0^1\int_0^{1-\eta}\phi_i\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{3.5}$$

For the individual integrals $[\phi_i]\equiv\int_0^1\int_0^{1-\eta}\phi_i\,\mathrm{d}\xi\,\mathrm{d}\eta$ we get

$$[\phi_0] = [\phi_1] = [\phi_2] = \frac{1}{6} \tag{3.6}$$

$$\sum_i[\phi_i] = \frac{1}{2}. \tag{3.7}$$

In the end we get

$$T_0 + T_1 + T_2 = 3\frac{\dot{q}_\perp}{\lambda}\frac{\Delta x_1\Delta y_2 - \Delta x_2\Delta y_1}{\Delta x_2 - \Delta x_1} \tag{3.8}$$

3.2 Radiation

The net radiation heat flux from surface 1 to surface 2 using grey body radiation can be calculated as

$$\dot{Q}_{1\to 2} = A_1 F_{1\to 2} E_1 - A_2 F_{2\to 1} E_2. \tag{3.9}$$

using the formula for emission of grey bodies

$$E_i = \epsilon_i \sigma T_i^4 \tag{3.10}$$

and the reciprocity rule for configuration factors $A_1 F_{1\to 2} = A_2 F_{2\to 1}$ we can write

$$\dot{Q}_{1\to 2} = \sigma A_1 F_{1\to 2} \left(\epsilon_1 T1^4 - \epsilon_2 T_2^4\right) \tag{3.11}$$

and $\dot{Q}_{1\to 2} = -\dot{Q}_{2\to 1}$. Given two line segments as seen in Fig. 3.1

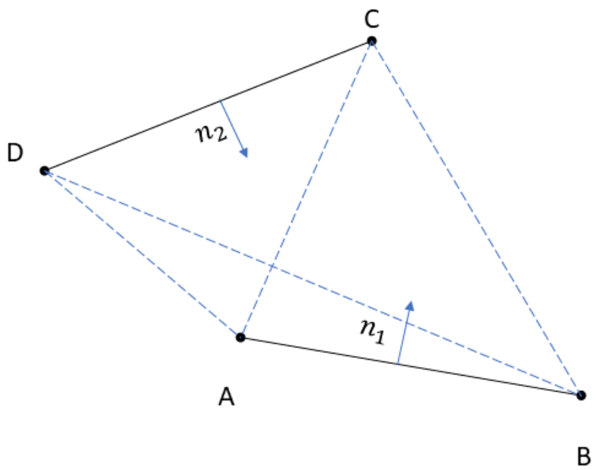

Figure 3.1: Radiation.

the configuration factor from surface $\overline{\mathrm{AB}}$ to surface $\overline{\mathrm{CD}}$ can be calculated as

$$F_{\overline{\mathrm{AB}}\to\overline{\mathrm{CD}}} = \frac{\overline{\mathrm{AC}} + \overline{\mathrm{BD}} - \overline{\mathrm{AD}} - \overline{\mathrm{BC}}}{2\overline{\mathrm{AB}}} \tag{3.12}$$

where $\overline{\mathrm{XY}}$ is the distance from X to Y. The total heat flux from or to a single surface is the sum of the heat fluxes to other surfaces plus the heat flux to the background

$$\dot{Q}_{tot} = \sigma A_1 \left\{ \epsilon_1 T_1^4 \sum_i F_{1\to i} - \sum_i \epsilon_i F_{1\to i} T_i^4 \right\} + \dot{Q}_{backgr} \tag{3.13}$$

$$\dot{Q}_{tot} = \sigma A_1 \left\{ \epsilon_1 T_1^4 \sum_i F_{1\to i} - \sum_i \epsilon_i F_{1\to i} T_i^4 + F_{1\to bg}(\epsilon_i T_1^4 - \epsilon_{bg} T_{bg}^4) \right\} \tag{3.14}$$

Since $F_{1\to bg} = 1 - \sum_i F_{1\to i}$ we get

$$\dot{Q}_{tot} = \sigma A_1 \left\{ \epsilon_1 T_1^4 - \sum_i \epsilon_i F_{1\to i} T_i^4 - F_{1\to bg}) \epsilon_{bg} T_{bg}^4 \right\} \tag{3.15}$$

In terms of boundary conditions, using $\dot{q}_{1\to 2} = \dot{Q}_{1\to 2}/A_1$,we ca write

$$\lambda \left(\vec{n} \cdot \nabla T \right) = \dot{q}_{1\to 2} \tag{3.16}$$

Bibliography

[1] Hans Dieter Baehr and Karl Stephan. *Wärme- und Stoffübertragung*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2019. ISBN: 978-3-662-58440-8 978-3-662-58441-5. DOI: 10.1007/978-3-662-58441-5. URL: http://link.springer.com/10.1007/978-3-662-58441-5 (visited on 10/20/2024).