FOSA CFD – Leo Rauschenberger

- *1:30h; 3 Exercises -> 30min per Exercise!*
- *No calculator!*

Basics:

$\boldsymbol{\nu} = \frac{\boldsymbol{\eta}}{\boldsymbol{\rho}}$

$\boldsymbol{\nabla} = \begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix} \rightarrow \boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$

$\vec{\boldsymbol{v}} = \begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}$

Substantial derivative (non-conservative):

$\frac{D\vec{\boldsymbol{\phi}}}{Dt} = \frac{\partial\vec{\boldsymbol{\phi}}}{\partial t} + (\vec{\boldsymbol{v}}\cdot\boldsymbol{\nabla})\vec{\boldsymbol{\phi}}$

for $\phi = v$: $= \begin{bmatrix} u_t + uu_x + vu_y \\ u_t + uv_x + vv_y \end{bmatrix}$

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

$2\cos\theta = e^{I\theta} + e^{-I\theta}$

$2\sin\theta = e^{I\theta} - e^{-I\theta}$

$e^{I\theta_x} = \cos\theta_x + I\sin\theta_y$

$e^{-I\theta_x} = \cos\theta_x - I\sin\theta_y$

$\cos^2\theta_x + \sin^2\theta_y = 1$

Scalar product:

$\nabla\vec{v} = \begin{cases}u_x\\ v_y\end{cases}$ VS. $\vec{v}\cdot\nabla = \begin{pmatrix}u\\ v\end{pmatrix}\cdot\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$

Foudre: $Fr = \frac{\sqrt{u^2+v^2}}{\sqrt{gh}}$

$\rho = \frac{p}{RT}$

$\bar{\bar{\sigma}} = pI + \sigma$

$\vec{q} = \begin{pmatrix}q_x\\ q_y\end{pmatrix} = -\lambda\begin{pmatrix}T_x\\ T_y\end{pmatrix}$

General conservation equations:

Integral form:

$$\boxed{\int_\tau \frac{\partial\vec{\boldsymbol{U}}}{\partial t}d\tau + \oint_A \vec{\boldsymbol{H}}\cdot\vec{\boldsymbol{n}}dA = \int_\tau \vec{\boldsymbol{F}}d\tau}$$

Temporal change of the **conservation quantities $\vec{\boldsymbol{U}}$** in the **volume $\boldsymbol{\tau}$**
+ generalized **flux $\vec{\boldsymbol{H}}$** normal to the **surface $\boldsymbol{A}$**
= Effect of the **volume forces $\vec{\boldsymbol{F}}$**

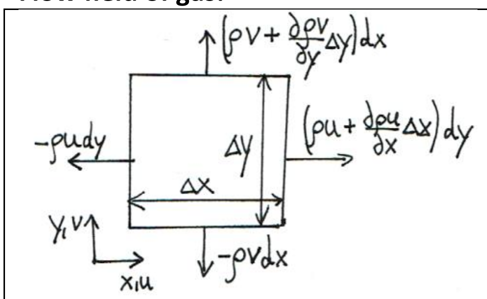
Divergence form:

$$\boxed{\frac{\partial\vec{\boldsymbol{U}}}{\partial t} + \boldsymbol{\nabla}\cdot\vec{\boldsymbol{H}} = \vec{\boldsymbol{F}}}$$

	integral	differential conservative	diff. non-conservative
Mass	$\int_\tau \frac{\partial\rho}{\partial t}d\tau + \oint_A \rho\vec{v}\cdot\vec{n}dA = 0$	$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{v} = 0$	$\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{v} = 0$
Momentum	$\int_\tau \frac{\partial\rho\vec{v}}{\partial t}d\tau + \oint_A(\rho\vec{v}\vec{v} + \bar{\bar{\sigma}})\cdot\vec{n}dA = \int_\tau \vec{f}_{vol}d\tau$	$\frac{\partial\rho\vec{v}}{\partial t} + \nabla\cdot(\rho\vec{v}\vec{v} + \bar{\bar{\sigma}}) = \vec{f}_{vol}$	$\frac{D\vec{v}}{Dt} + \frac{1}{\rho}\nabla\cdot\bar{\bar{\sigma}} = \frac{1}{\rho}\vec{f}_{vol}$
Energy	$\int_\tau \frac{\partial\rho E}{\partial t}d\tau + \oint_A(\rho E\vec{v} + \bar{\bar{\sigma}}\vec{v} + \vec{q})\cdot\vec{n}dA = \int_\tau \vec{f}_{vol}\vec{v}d\tau$	$\frac{\partial\rho E}{\partial t} + \nabla\cdot(\rho E\vec{v} + \bar{\bar{\sigma}}\vec{v} + \vec{q}) = \vec{f}_{vol}\cdot\vec{v}$	$\frac{DE}{Dt} + \frac{1}{\rho}\nabla\cdot(\bar{\bar{\sigma}}\vec{v} + \vec{q}) = \frac{1}{\rho}\vec{f}_{vol}\cdot\vec{v}$

- The non-conservative form is basically the form that is already broken up into its components
- Oftentimes, several different forms of the non-conservative form are possible!

Flow field of gas:



$\lim\limits_{\Delta\to d}$

Equations:

Nav.-Stokes

- heat conduction
- continuum flow
- unsteady
- viscous!

Incompressible

The incompressible forms are always non-conservative

$\vec{\boldsymbol{v}}, (\boldsymbol{p})$	$u_x + v_y = 0$ $u_t + uu_x + vu_y + \frac{1}{\rho} p_x = \nu(u_{xx} + u_{yy})$ (x-momentum) $v_t + uv_x + vv_y + \frac{1}{\rho} p_y = \nu(v_{xx} + v_{yy})$ (x-momentum)	$\nabla \cdot \vec{v} = 0$ $\frac{D\vec{v}}{Dt} + \frac{1}{\rho}\nabla p = \nu\nabla^2\vec{v}$
$\boldsymbol{\psi}, \boldsymbol{\omega}$	$\psi_{xx} + \psi_{yy} = -\omega$ poisson eq. of streamline ft. $\omega_t + u\omega_x + v\omega_y = \nu(\omega_{xx} + \omega_{yy})$ vorticity transport equation Vorticity transp. Eq. from curl of momentum equations $\nabla \times \begin{pmatrix} x\ momentum\ eq. \\ y\ momentum\ eq. \end{pmatrix}$ Subtract and use: $\omega = v_x - u_y$ & $u_x + v_y = 0$ Introduction of stream function ψ eliminates conti., because stream-function satisfies conti.-eq.	$\nabla^2\psi = -\omega$ $\frac{D\omega}{Dt} = \nu\nabla^2\omega$
$\boldsymbol{p}, (\vec{\boldsymbol{v}})$	$\nabla^2 p = -\rho[u_x^2 + v_y^2 + 2v_x u_y]$ poisson eq. for pressure Divergence of momentum eq. with $u_x + v_y = 0$ $\nabla \cdot \begin{pmatrix} x\ momentum\ eq. \\ y\ momentum\ eq. \end{pmatrix}$	

Compressible

int.	$\int_\tau \frac{\partial \vec{U}}{\partial t} d\tau + \oint_A (E_{inv} + E_{vis}) dy = \oint_A (F_{inv} + F_{vis}) dx$
diff cons.	$\rho_t + (\rho u)_x + (\rho v)_y = 0$ $(\rho u)_t + (\rho u^2 + p + \sigma_{xx})_x + \left(\rho uv + \sigma_{xy}\right)_y = 0$ (x-momentum) $(\rho u)_t + \left(\rho uv + \sigma_{yx}\right)_x + \left(\rho v^2 + p + \sigma_{yy}\right)_y = 0$ (y-momentum) $(\rho E)_t + \left(\rho uE - up + u\sigma_{xx} + v\sigma_{xy} + q_x\right)_x + \left(\rho vE + vp + v\sigma_{yy} + u\sigma_{xy} + q_y\right)_y = 0$ Or deduced from the incompressible form: $\rho_t + (\rho u)_x + (\rho v)_y = 0$ $(\rho u)_t + (\rho u^2)_x + (\rho uv)_y + p_x = \left(\nu u_{xx} + \nu u_{yy}\right)$
non-cons.	$\frac{D\rho}{Dt} + \rho u_x + \rho v_x = 0$ $\frac{Du}{Dt} + \frac{1}{\rho}(p + \sigma_{xx})_x + \frac{1}{\rho}\sigma_{xyy} = 0$ $\frac{Dv}{Dt} + \frac{1}{\rho}\sigma_{xxy} + \frac{1}{\rho}(p + \sigma_{yy}) = 0$ $\frac{DE}{Dt} + \frac{1}{\rho}\left(up + u\sigma_{xx} + v\sigma_{xy} + q_x\right)_x + \frac{1}{\rho}\left(vp + v\sigma_{yy} + u\sigma_{xy} + q_y\right)_y = 0$

Deduce type:

Steady:	**Unsteady:**
from $\boldsymbol{\psi}, \boldsymbol{\omega}$ formulation: $\begin{vmatrix} \Omega_x^2 + \Omega_y^2 & 0 \\ 0 & -\nu(\Omega_x^2 + \Omega_y^2) \end{vmatrix} = 0 \rightarrow$ elliptic Solve with	Assumption! Uncoupled $\boldsymbol{\psi}, \boldsymbol{\omega}$ formulation: $\psi_{xx} + \psi_{yy} = -\omega \rightarrow$ elliptic $\omega_t + u\omega_x + v\omega_y = \nu\left(\omega_{xx} + \omega_{yy}\right) \rightarrow$ parabolic! ➔ Solve with combination of iteration scheme for the elliptic part and marching scheme for parabolic part.

Euler

- inviscid $\nu = 0$
- unsteady

Incompressible

$\vec{\boldsymbol{v}}, \boldsymbol{p}$	$u_x + v_y = 0$ $u_t + uu_x + vu_y + \frac{1}{\rho} p_x = 0$ $v_t + uv_x + vv_y + \frac{1}{\rho} p_y = 0$	$\nabla \cdot \vec{v} = 0$ $\frac{D\vec{v}}{Dt} + \frac{1}{\rho} \nabla p = 0$
$\boldsymbol{\psi}, \boldsymbol{\omega}$	$\psi_{xx} + \psi_{yy} = -\omega$ $\omega_t + u\omega_x + v\omega_y = 0$	$\nabla^2 \psi = -\omega$ $\frac{D\omega}{Dt} = 0$
p	$\nabla^2 p = -\rho[u_x^2 + v_y^2 + 2v_x u_y]$	

Compressible

Integral	$\int_\tau \frac{\partial \vec{U}}{\partial t} d\tau + \oint_A E_{inv} dy = \oint_A F_{inv} dx$
diff cons.	$\rho_t + (\rho u)_x + (\rho v)_y = 0$ $(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0$ (x-momentum) $(\rho u)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0$ (y-momentum) $(\rho E)_t + (\rho uE + up)_x + (\rho vE + vp)_y = 0$
non-cons.	$\frac{D\rho}{Dt} + \rho u_x + \rho v_y = 0$ $\frac{Du}{Dt} + \frac{1}{\rho} p_x = 0$ $\frac{Dv}{Dt} + \frac{1}{\rho} p_y = 0$ $\frac{DE}{Dt} + \frac{1}{\rho}(up)_x + \frac{1}{\rho}(vp)_y = 0$

Deduce type:

Steady & Unsteady: from $\boldsymbol{\psi}, \boldsymbol{\omega}$ formulation: $\begin{vmatrix} \Omega_x^2 + \Omega_y^2 & 0 \\ 0 & (\Omega_t) + u\Omega_x + v\Omega_y \end{vmatrix} = 0 \rightarrow$ elliptic & hyperbolic mixed

Boundary Layer

- attached flows
- heat conduction
- high Reynolds numbers $Re \gg 1$
- **steady**

➔ parabolic

Incompressible

$\vec{\boldsymbol{v}}, \boldsymbol{p}$	$u_x + v_y = 0$ (conti) $uu_x + vu_y + \frac{1}{\rho} p_x = \left(\nu u_y\right)_y$ (x-momentum) $p_y = 0$ (y-momentum – all derivatives of v=0)	$\nabla \cdot \vec{v} = 0$ $(\vec{v} \cdot \nabla)u + \frac{1}{\rho} p_x = \left(\nu u_y\right)_y$ $p_y = 0$

Compressible

int.	-
diff cons.	$(\rho u)_x + (\rho v)_y = 0$ $\rho u u_x + \rho v u_y + p_x = \left(\eta u_y\right)_y$ $p_y = 0$ $\rho u h_x + \rho v h_y - u p_x = \left(\lambda T_y\right)_y + \eta u_y^2$ $\quad h = c_p T$ Or $\rho c_p \left(u T_x + v T_y\right) - u p_x = \left(\lambda T_y\right)_y + \eta u_y^2$
non-cons.	**-**

Potential

- *Can be obtained from the Euler equations!*

- irrotational flow
- steady
- isoenergetic
- isentropic
- inviscid

➔ elliptic!

Incompressible

$\vec{v}$	$u_x + v_y = 0$ (conti) $v_x - u_y = 0$ (condition vorticity $\omega = 0$ because irrotational!)	$\nabla \cdot \vec{v} = 0$ $\nabla \times \vec{v} = 0$
ψ, ω	Conti satisfied $\psi_{xx} + \psi_{yy} = 0$	$\nabla^2 \psi = 0$
ϕ (potential)	$\phi_{xx} + \phi_{yy} = 0$ (condition for irrotational satisfied!)	$\nabla^2 \phi = 0$
p	Bernoulli: $p_0 = p + \frac{\rho}{2}(u^2 + v^2)$	

Compressible

- supersonic flow

int.	-
diff cons.	Continuity & $\phi_x = u$; $\phi_y = v$ $(\rho\phi_x)_x + \left(\rho\phi_y\right)_y = 0$
non-cons.	From steady, isoenergetic 2D Euler equations with $dp = a^2 d\rho$ $(\rho u)_x + (\rho v)_y = 0$ (conti) $(\rho u^2)_x + (\rho uv)_y + p_x = 0$ (x-momentum) $(\rho uv)_x + (\rho v^2)_y + p_y = 0$ (y-momentum) Split momentum eq. into parts and: x-momentum * u y-momentum * v With: $\phi_x = u$; $\phi_y = v$ $(u^2 - a^2)\phi_{xx} + 2uv\phi_{xy} + (v^2 - a^2)\phi_{yy} = 0$ Divide by a^2: $(1 - M_x^2)\phi_{xx} - 2M_x M_y \phi_{xy} + \left(1 - M_y^2\right)\phi_{yy} = 0$ With: $M_x = \frac{u}{a}$, $M_y = \frac{v}{a}$, $M^2 = M_x^2 + M_y^2$ Supersonic -> hyperbolic M>1 Sonic -> parabolic M=1 Subsonic -> elliptic M<1 N.B.: This scalar equation is easily solvable.

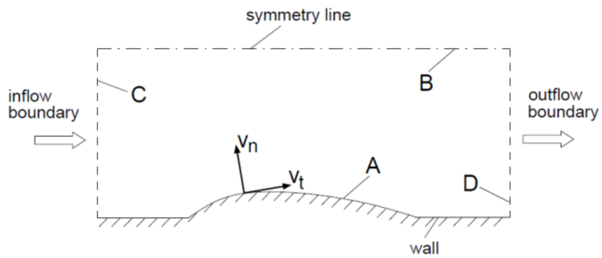
Further equations:

vorticity ω (or ζ)	$\omega = v_x - u_y$	$\omega = \nabla \times \vec{v}$
Potential ϕ	$\phi_x = u$ $\phi_y = v$	$\nabla\phi = \vec{v}$
streamline function ψ	$\psi_x = -v$ $\psi_y = u$	
Cauchy-Riemann diff. eq.	$u_x + v_y = 0$ $v_x - u_y = 0$	
Poisson eq. of the streamline function.	$\psi_{xx} + \psi_{yy} = -\omega$	
Vorticity transport eq. / Eddy transport eq.	$\omega_t + u\omega_x + v\omega_y = \nu\nabla^2\omega$	
Convection-diffusion eq. / Heat conduction eq.	$u_t + a\, u_x = \nu(u_{xx} + u_{yy})$	$a\, u_x$ convection (called advection for molecules) $\nu(u_{xx} + u_{yy})$ diffusion
Heat equation	$T_t - \lambda T_{xx} = cT$	$c > 0$ heat source $c < 0$ heat sink $-\lambda T_{xx}$ thermal diffusion = Wärmeleitung T_t rate of change
wave eq.	$u_{tt} - c^2 u_{xx} = 0$	
perturbation potential eq.	$(Ma^2 - 1)\phi_{xx} - \phi_{yy} = 0$	
Burger's eq.	$u_t + uu_x = 0$	nonlinear

Characterize equation systems:

2D	x, y
steady	$\frac{\partial}{\partial t} = 0$
incompressible	$\rho = ₵$ note: if contains e.g. $\frac{\partial}{\partial y}\left(\eta \frac{\partial u}{\partial y}\right) \rightarrow$ compressible
Viscous/viscid	$\nu = \frac{\eta}{\rho} \neq 0$
low Reynolds number	nonlinear convective terms can be neglected e.g. Nav-S.: $v_t + \frac{1}{\rho} p_y = 0$
linear / nonlinear	nonlinear if $uv,\ u^2, v^2$
isoenergetic	$h = 0$
irrotational	$\nabla \times \vec{v} = 0$ 2D: $v_x - u_y = 0$
isotropic	
conservative	Dependant variables (=variables in derivatives) are cons. quantities $\vec{U} = \vec{U}(\rho, \vec{v}, E)$ in practice: $\frac{Dx}{\mathrm{Dt}} + \nabla x$ …
non-conservative	Dependent variables are for example $\rho, \vec{v}, E$ In practice: $\frac{Dx}{\mathrm{Dt}} + \nabla y$ …
Boundary layer	$Re \gg 1$
Irrotational	$\omega = 0$

Boundary conditions in fluid dynamics



No-slip on wall	$u = u_{wing}\ or\ 0$ $v = v_{wing}\ or\ 0$	
Isothermal wall	$T = T_w$	
adiabatic	$\frac{\partial T}{\partial n} = \frac{\partial T}{\partial x} n_x + \frac{\partial T}{\partial y} n_y$	
pressure	$\frac{\partial p}{\partial n} = \frac{\partial p}{\partial x} n_x + \frac{\partial p}{\partial y} n_y$	

<u>Types of boundary conditions:</u>

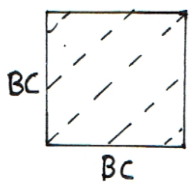
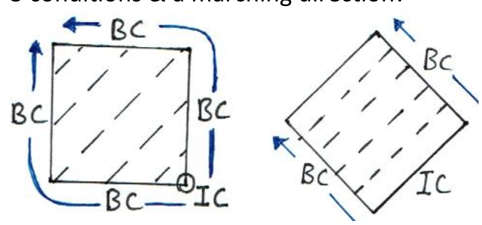
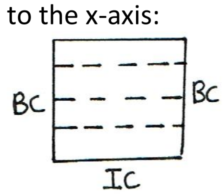
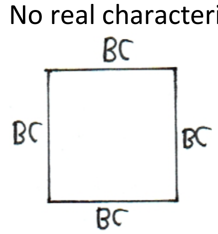
1) Dirichlet
2) Von Neuman
3) Mixed
4) Periodic

Classification

Determine the Type of equation:

1st order PDE ①	**e.g.** $\boldsymbol{au_x + bu_y = 0}$ **Slope of char. line**: $\boxed{-\frac{\Omega_x}{\Omega_y} = \frac{dy}{dx}\Big\vert_{Co}} = \frac{b}{a}$ or $\boxed{-\frac{\Omega_t}{\Omega_x} = \frac{dx}{dt}\Big\vert_{Co}}$ integrate $dy = \frac{b}{a}dx$ to obtain eq. of the **<u>char. base / line</u>** curve: $\int_y^{y_0} dy = \frac{b}{a}\int_x^{x_0} dx$ $\rightarrow y - y_0 = \frac{b}{a}(x - x_0)$
mixed	Only the highest order terms are relevant. e.g. $\boldsymbol{au_x + bu_{yy} = 0}$ $\Omega_x^2 = 0$
2nd order PDE ①	The equations can be written as e.g. Euler eq. $\begin{pmatrix} \frac{\partial}{\partial_x} & \frac{\partial}{\partial_y} & 0 \\ u\frac{\partial}{\partial_x} + v\frac{\partial}{\partial_y} & 0 & \frac{1}{\rho}\frac{\partial}{\partial_x} \\ 0 & u\frac{\partial}{\partial_x} + v\frac{\partial}{\partial_y} & \frac{1}{\rho}\frac{\partial}{\partial_y} \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = 0$ With: $\boxed{\frac{\partial}{\partial_x} = \Omega_x}$ or $\partial_x \rightarrow \Omega_x, \quad \partial_{xy} \rightarrow \Omega_x\Omega_y, \quad \partial_{xx} \rightarrow \Omega_x^2$ $0 = det \begin{vmatrix} \Omega_x & \Omega_y & 0 \\ u\Omega_x + v\Omega_y & 0 & \frac{1}{\rho}\Omega \\ 0 & u\Omega_x + v\Omega_y & \frac{1}{\rho}\Omega \end{vmatrix} = \cdots$ With: $\boxed{-\frac{\Omega_x}{\Omega_y} = \frac{dy}{dx}}$ e.g.: $-\frac{\Omega_x}{\Omega_x} = \frac{dy}{dx}\Big\vert_{1,2} = \pm\sqrt{-1} = \pm I$ $-\frac{\Omega_x}{\Omega_x} = \frac{dy}{dx}\Big\vert_{3} = \frac{u}{v}$ $x_{1,2} = \frac{-b\pm\sqrt{\Delta}}{2a}$ with the discriminant $\Delta = b^2 - 4ac$

- *If the region of influence is constrained by boundaries, initial value problems need BCs as well (= initial-boundary value problem)*
- Hyperbolic: Specify ICs, BCs on all inflow boundaries! (Determined by sign of u,v)

	1st order	**2nd order**	**Sketch IC & BC , Solution scheme**	**2nd order normal form**
Hyperbolic *Initial value problem*	real char. 1st order PDE always hyperbolic!  e.g. $u_t + au_x = 0$	$\Delta > 0$ real char. $\left.\frac{dy}{dx}\right\|_1 \neq \left.\frac{dy}{dx}\right\|_2$	There needs to be an I.C. or B.C. at the origin of the slopes (characteristics). That means that each slope has 2 conditions which may or may not coincide with those for other slopes. e.g. $u_{tt} - c^2 u_{xx} = 0$	$u_{\xi\xi} - u_{\eta\eta} = 0$ or $u_{\xi\eta} = 0$
Parabolic *Initial-value problem*		$\Delta = 0$ real double char. $\left.\frac{dy}{dx}\right\|_1 = \left.\frac{dy}{dx}\right\|_2$	3 conditions & a marching direction!  Parabolic problems are initial value problems -> IC required! If $\left.\frac{dy}{dx}\right\|_{1,2} = 0$ the characteristcs are paralell to the x-axis:  e.g. $u_t - \lambda u_{xx} = 0$	
Elliptic *Boundary value problem*		$\Delta < 0$ complex char. $\left.\frac{dy}{dx}\right\|_1 \neq \left.\frac{dy}{dx}\right\|_2$	No real characteristics to be drawn!  Solution scheme must provide coupling in all 4 directions -> sweeps along all 4 edges! e.g. $u_{xx} + u_{yy} = 0$ $\phi_{xx} + \phi_{yy} = 0$	$u_{\xi\xi} + u_{\eta\eta} = 0$
ODE		$\frac{dy}{dx} \to \infty$	-	
Mixed			Create 2 graphs!	

Canonical or normal form:
$d\xi_1 = \alpha dx - dy$
$d\eta_1 = \beta dx$

Characteristic solution:

= solution along characteristic base curve

Task: Find the char. sol. of a given PDE:

$u_t = \xi_t u_\xi + \tau_t u_\tau$ $u_x = \xi_x u_\xi + \tau_x u_\tau$ $u_y = \xi_y u_\xi + \tau_y u_\tau$
$\begin{pmatrix} d\xi \\ d\tau \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \tau_x & \tau_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \cdots$ or $= \begin{pmatrix} \xi_x & \xi_t \\ \tau_x & \tau_t \end{pmatrix} \begin{pmatrix} dx \\ dt \end{pmatrix} = \cdots$

$$d\xi = \left.\frac{dy}{dx}\right|_{C_0} dx - dy$$

PDE: $\boldsymbol{a u_x} + \boldsymbol{b u_y} = \boldsymbol{c}$

1. $(\boldsymbol{x}, \boldsymbol{y}) \to (\boldsymbol{\tau}, \boldsymbol{\xi})$
   $\boldsymbol{a}(\xi_x u_\xi + \tau_x u_\tau) + \boldsymbol{b}(\xi_y u_\xi + \tau_y u_\tau) = \boldsymbol{c}$

2. **Find ξ_x etc by comparison:**
   $\begin{cases} d\xi = \frac{b}{a} dx - dy \\ d\tau = dx \end{cases} \to \begin{cases} d\xi = \xi_x dx + \xi_y dy \\ d\tau = \tau_x dx + \tau_y dy \end{cases}$ **OR** $\begin{cases} d\xi = \xi_x dx + \xi_t dt \\ d\tau = \tau_x dx + \tau_t dt \end{cases}$

General form	if c=0:
normal form (characteristic form) $a\left(\frac{b}{a} u_\xi + u_\tau\right) - b(-u_\xi) = c$ $\boxed{\to u_\tau = \left.\frac{\partial u}{\partial \tau}\right\|_{\xi = ¢} = \frac{c}{a}}$	$u_\tau = \left.\frac{\partial u}{\partial \tau}\right\|_{\xi = ¢} = 0$
characteristic solution $\partial u = \frac{c}{a} \partial \tau$ $\boxed{\to u(\tau, \xi) = \frac{c}{a} \tau + k(\xi)}$	$u(\tau, \xi) = k(\xi)$
introduce I.C. & $(\boldsymbol{\tau}, \boldsymbol{\xi}) \to (\boldsymbol{x}, \boldsymbol{y})$ $u(x, y) = \frac{c}{a}(x - x_0) + u_0(x_0, y_0)$	$u(x, y) = u_0(x_0, y_0)$ on $\xi = \frac{b}{a} x - y = \frac{b}{a} x_0 - y_0 = const.$ (1)
$u(\tau, \xi) = \frac{c}{a} \tau + k(\xi)$	
With: $\xi = \frac{b}{a} x - y \to y = -\xi$ $x = \tau$ $u(\tau = 0, \xi) = \boldsymbol{k}(\boldsymbol{\xi}) = -\xi$	With: Provided solution e.g.: $u(x = 0, y) = y$

General solution: $u(\tau,\xi) = \frac{c}{a}\tau - \xi$ With: $\xi = \frac{b}{a}x - y$ $\tau = x$ $u(x,y) = \frac{c}{a}x - \left(\frac{b}{a}x - y\right)$	
Particular solution for u(2,1): $u(2,1) = \frac{c}{a}2 - \left(\frac{b}{a}2 - 1\right) = \cdots$	

<u>Task: Find the PDE of a given char. sol.:</u>

new coord and u in new coord.

$$\boldsymbol{u_\xi = x_\xi u_x + \cdots}$$

old

Char sol.:

$$\begin{aligned} u_{\xi\eta} &= (u_\xi)_\eta \\ &= (x_\xi u_x + t_\xi u_t)_\eta \\ &= x_{\xi\eta} u_x + x_\xi(x_\eta u_{xx} + t_\eta u_{xt}) + \cdots \end{aligned}$$

$\to f(x_\xi, t_\xi, \ldots)$

$$\begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_t \\ \eta_x & \eta_t \end{pmatrix} \begin{pmatrix} dx \\ dt \end{pmatrix} = \bar{\bar{J}} \begin{pmatrix} dx \\ dt \end{pmatrix}$$

$$\begin{pmatrix} dx \\ dt \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta \\ t_\xi & t_\eta \end{pmatrix} \begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = \bar{\bar{J}}^{-1} \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}$$

$x_\xi = \cdots$ etc

Task: Sketch

1. Sketch characteristic lines in (x,t)-Diagram using provided sample solution
2. Plot horizontal line at desired time t in (x,t)-Diagram
3. Plot to (u, t) diagram

Discretization

1. Discretization on cartesian grids

- *Establish the Taylor series for the discretization points you need.*
- *For consistency, you must discretize <u>around the same discretization point</u> in a given PDE! Meaning: Discretizations around u_i and $u_{i+1/2}$ cannot be mixed!*
- *The discretization point should always be in the middle to reduce the error!*

Taylor series expansions

Around $\boldsymbol{u_i}$ in 1D & 2D:

$$u_{i\pm1} = u_i \pm \Delta x\, u_x|_i + \frac{\Delta x^2}{2} u_{xx}|_i \pm \frac{\Delta x^3}{6} u_{xxx}\Big|_i + \frac{\Delta x^4}{24} u_{xxxx}\Big|_i \pm \frac{\Delta x^5}{120} u_{(5x)}\Big|_i + \frac{\Delta x^6}{720} u_{(6x)}\Big|_i + \cdots$$

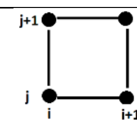
$$u_{i\pm2} = u_i \pm 2\Delta x\, u_x|_i + \frac{4\Delta x^2}{2} u_{xx}|_i \pm \frac{8\Delta x^3}{6} u_{xxx}\Big|_i + \frac{16\Delta x^4}{24} u_{xxxx}\Big|_i \pm \frac{32\Delta x^5}{120} u_{(5x)}\Big|_i + \frac{64\Delta x^6}{720} u_{(6x)}\Big|_i + \cdots$$

$$u_{i\pm\frac{1}{2}} = u_i \pm \frac{\Delta x}{2}\, u_x\Big|_i + \frac{\Delta x^2}{8} u_{xx}|_i \pm \frac{\Delta x^3}{48} u_{xxx}\Big|_i + \frac{\Delta x^4}{384} u_{xxxx}\Big|_i + \cdots$$

$$u_{i+1,j+1} = u_{i,j} + \Delta x u_x + \Delta y u_y + \frac{\Delta x^2}{2} u_{xx} + \Delta x \Delta y u_{xy} + \frac{\Delta y^2}{2} u_{yy} + \cdots$$
$$\ldots + \frac{\Delta x^3}{6} u_{xxx} + \frac{\Delta x^2 \Delta y}{2} u_{xxy} + \frac{\Delta x \Delta y^2}{2} u_{xyy} + \frac{\Delta y^3}{6} u_{yyy} + O(\Delta x^4, \Delta y^4)$$

Around $\boldsymbol{u_{i+\frac{1}{2}}}$ in 1D & 2D:

$$\boldsymbol{u}_{\ \boldsymbol{n}}^{\boldsymbol{n+1}} = u^{n+1/2} \pm \frac{\Delta t}{2}\, u_t\Big|^{n+1/2} + \frac{\Delta t^2}{8} u_{tt}\Big|^{n+1/2} \pm \frac{\Delta t^3}{48} u_{ttt}\Big|^{n+1/2} + \frac{\Delta t^4}{384} u_{tttt}\Big|^{n+1/2} + \cdots$$



$$c := i + \tfrac{1}{2}, j + \tfrac{1}{2}$$

$$u_{i+1,j+1} = u_c + \frac{\Delta x}{2} u_x + \frac{\Delta y}{2} u_y + \frac{\Delta x^2}{8} u_{xx} + \frac{\Delta x \Delta y}{4} u_{xy} + \frac{\Delta y^2}{8} u_{yy} + \cdots$$
$$\ldots + \frac{\Delta x^3}{48} u_{xxx} + \frac{\Delta x^2 \Delta y}{16} u_{xxy} + \frac{\Delta x \Delta y^2}{16} u_{xyy} + \frac{\Delta y^3}{48} u_{yyy} + O(\Delta x^4, \Delta y^4)$$

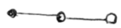
Discretize $\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{y}$, …:

$u \to u_{i,j}$
$x \to i\Delta x$
$y \to j\Delta y$
$t^n \to n\Delta t$

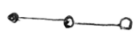
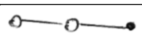
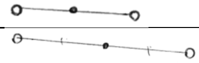
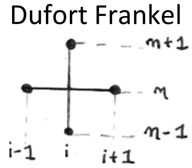
Discretize u:

Nbr points	Schematic	Result (truncated)	$= u_i + ERROR$	Order
center		$\frac{u_{i+1/2} + u_{i-1/2}}{2}$	$\frac{\Delta x^2}{8} u_{xx}$	$O(\Delta x^2)$
center		$\frac{u_{i+1} + u_{i-1}}{2}$	$\frac{\Delta x^2}{2} u_{xx}$	$O(\Delta x^2)$

Discretize u_x:

Nbr points	Schematic	Result (truncated)	$= u_x + ERROR$	Order
forward		$\frac{u_{i+1} - u_i}{\Delta x}$	$\frac{\Delta x}{2} u_{xx}$	$O(\Delta x)$
backward		$\frac{u_i - u_{i-1}}{\Delta x}$	$-\frac{\Delta x}{2} u_{xx}$	$O(\Delta x)$
central		$\frac{u_{i+1} - u_{i-1}}{2\Delta x}$	$\frac{\Delta x^2}{6} u_{xxx}$	$O(\Delta x^2)$
one-sided		$\frac{-3u_i + 4u_{i+1} - u_{i+2}}{2\Delta x}$ $\frac{3u_i - 4u_{i-1} + u_{i-2}}{2\Delta x}$	$-\frac{\Delta x^2}{3} u_{xxx}$ $-\frac{\Delta x^2}{3} u_{xxx}$	$O(\Delta x^2)$
Central, 4 points		$\frac{8(u_{i+1} - u_{i-1}) - (u_{i-2} - u_{i+2})}{12\Delta x}$	$-\frac{\Delta x^4}{30} u_{5x}$	$O(\Delta x^4)$

Discretize u_{xx}:

	Schematic	Result (truncated)	$= u_{xx} + ERROR$	Order
one sided		$\frac{u_{i+2} - 2u_{i+1} + u_i}{\Delta x^2}$	$\Delta x\, u_{xxx}$	$O(\Delta x)$
one sided		$\frac{u_{i-2} - 2u_{i-1} - u_i}{\Delta x^2}$	$-\Delta x\, u_{xxx}$	$O(\Delta x)$
central		$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$	$\frac{\Delta x^2}{12} u_{xxxx}$	$O(\Delta x^2)$
central		$\frac{u_{i+2} - 2u_i + u_{i-2}}{4\Delta x^2}$	$\frac{\Delta x^2}{3} u_{xxxx}$	$O(\Delta x^2)$
Central, 5 points		$\frac{16(u_{i+1} + u_{i-1}) - 30u_i - (u_{i+2} + u_{i-2})}{12\Delta x^2}$	$-\frac{\Delta x^4}{90} u_{6\cdot x}$	$O(\Delta x^4)$
Central, with 3-time levels	Dufort Frankel  	$\frac{u_{i+1}^n - (u_i^{n-1} + u_i^{n+1}) + u_{i-1}^n}{\Delta x^2}$ Procedure: Use $\frac{u_{i+1} - 2u_i^n + u_{i-1}}{\Delta x^2}$ and replace u_i^n!	$-\frac{\Delta x^2}{12} u_{xxxx} + \frac{\Delta x^2}{\Delta t^2} u_{tt}$ Conditionally consistent! $\Delta x \ll \Delta t$	$O\left(\Delta x^2, \frac{\Delta x^2}{\Delta t^2}\right)$

Discretize u_{xxx}:

Central, 4 point		$\frac{u_{i+2} - 2(u_{i+1} - u_{i-1}) - u_{i-2}}{2\Delta x^3}$		$O(\Delta x^2)$

Discretize u_{xxxx}:

Central, 5 point	$(u_{i+2} + u_{i-2}) - 4(u_{i+1} + u_{i-1})$	$\frac{u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i+2}}{\Delta x^4}$	$\frac{\Delta x^2}{6} u_{6\cdot x}$	$O(\Delta x^2)$

Discretize u_{xy}:

<table>
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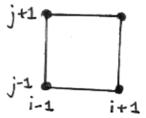
<td>
</td>
<td>$\frac{u_{i+1,j+1}-u_{i-1,j+1}-u_{i+1,j-1}+u_{i-1,j-1}}{4\Delta x\Delta y}$</td>
<td></td>
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</tr>
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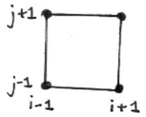
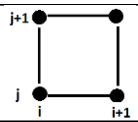
<td>
</td>
<td>$\frac{u_{i+1,j+1}-u_{i-1,j+1}-u_{i+1,j-1}+u_{i-1,j-1}}{4\Delta x\Delta y}$</td>
<td>$+\frac{\Delta x^2}{6}u_{xxxy}+\frac{\Delta y^2}{6}u_{xyyy}$</td>
<td>$O(\Delta x^2,\Delta y^2)$</td>
</tr>
<tr>
<td>
</td>
<td>$\frac{u_{i,j}-u_{i+1,j}-u_{i,j+1}+u_{i+1,j+1}}{\Delta x\Delta y}$</td>
<td>$+\frac{\Delta x^2}{24}u_{xxxy}+\frac{\Delta y^2}{24}u_{xyyy}$</td>
<td>$O(\Delta x^2,\Delta y^2)$</td>
</tr>
</table>

Discretize on non-equidistant grid

Establish equations e.g.

$u_{i-1}=u_i-u_xh_1+\cdots$

$u_{i+1}=u_i+u_xh_2+\cdots$

Discretize on provided stencil

2 procedures:

1) use the discretization along lines as above and then add additional discretizations as needed.
2) Write down all points in stencil in table with Taylor expansions and guesstimate suitable solution

	$u_{i,j}$	$\Delta x u_x$	$\Delta y u_y$	$\frac{\Delta x^2}{2}u_{xx}$	$\Delta x\Delta y u_{xy}$	$\frac{\Delta y^2}{2}u_{yy}$	$\frac{\Delta x^3}{6}u_{xxx}$	$\frac{\Delta x^2\Delta y}{2}u_{xxy}$	$\frac{\Delta x\Delta y^2}{2}u_{xyy}$	$\frac{\Delta y^3}{6}u_{yyy}$
$u_{i+1,j}$										
$u_{i,j+1}$										
$u_{i+1,j+1}$										
SUM										

Additional terms (if needed): $\frac{\Delta x^4}{24}u_{xxxx}+\frac{\Delta x^3\Delta y}{6}u_{xxxy}+\frac{\Delta x^2\Delta y^2}{4}u_{xxyy}+\frac{\Delta x\Delta y^3}{6}u_{xyyy}+\frac{\Delta y^4}{24}u_{yyyy}+\cdots$

	u_i^n	$\Delta x u_x$	$\Delta t u_t$	$\frac{\Delta x^2}{2}u_{xx}$	$\Delta x\Delta t u_{xt}$	$\frac{\Delta t^2}{2}u_{tt}$	$\frac{\Delta x^3}{6}u_{xxx}$	$\frac{\Delta x^2\Delta t}{2}u_{xxt}$	$\frac{\Delta x\Delta t^2}{2}u_{xtt}$	$\frac{\Delta t^3}{6}u_{ttt}$
u_i^{n+1}		+	+	+	+	+	+	+	+	+
u_{i+1}^{n+1}										
u_{i-1}^{n+1}										
SUM										

2. Discretization on curved grids

2.1 Using the Finite Difference Method

Transform equation to general curvilinear coordinates

Jacobi: $\bar{\bar{J}} = \begin{bmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{bmatrix}$ Comparison of coefficients with those found by chain rule: $\begin{pmatrix} u_\xi \\ u_\eta \end{pmatrix} = \begin{bmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{bmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} \rightarrow \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \frac{1}{J} \begin{bmatrix} y_\eta & -y_\xi \\ -x_\eta & x_\xi \end{bmatrix} \begin{pmatrix} u_\xi \\ u_\eta \end{pmatrix} = \begin{cases} u_x = \frac{1}{J}(y_\eta u_\xi - y_\xi u_\eta) \\ u_y = \frac{1}{J}(-x_\eta u_\xi + x_\xi u_\eta) \end{cases}$

Example: $u_t + au_x + bu_y = 0$

1. **Find coefficients :**
   Chain rule:
   $u_x = \xi_x u_\xi + \eta_x u_\eta$
   $u_y = \xi_y u_\xi + \eta_y u_\eta$
   u_t unchanged

   $u_{xx} = (u_x)_x = g_x = \xi_x g_\xi + \eta_x g_\eta$
   $u_{yy} = \left(u_y\right)_y = h_y = \xi_y h_\xi + \eta_y h_\eta$

   $\rightarrow u_t + \cdots u_\xi + \cdots u_\eta = 0$

2. **Discretize using metric terms:**
   Multiply with J

$\xi_x = \frac{y_\eta}{J}$ $\xi_y = -\frac{x_\eta}{J}$	$\eta_x = -\frac{y_\xi}{J}$ $\eta_y = \frac{x_\xi}{J}$

3. Discretize
4. Simplify!

Transform equation to polar coordinates

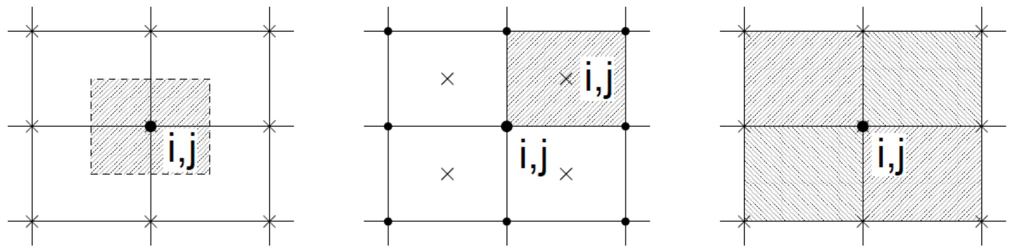
$x = r\cos\Phi$
$y = r\sin\Phi$

$r = \sqrt{x^2 + y^2}$
$\Phi = \arctan\frac{y}{x}$

$\nabla^2 =$

2.2 Using the Finite Volume Method

node centered | cell centered | cell-vertex



$$\nabla^2 = 0$$

Divergence/Gauss theorem:

$$\boxed{\int_\tau \nabla \cdot \vec{f}\, d\tau = \oint \vec{f}\, \vec{n}\, dA} = \sum_{k=1}^{4} \left(\vec{f} \cdot \vec{n}\Delta A\right)_k = 0$$

$$\vec{f} = \nabla u = \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix} \text{ for Laplace eq.}$$

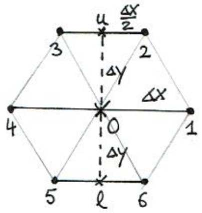
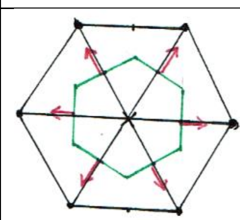
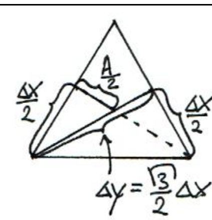
$$\textstyle\sum_{k=1}^{4}\left(\vec{f} \cdot \vec{n}\Delta A\right)_k = \sum_{k=1}^{4}\left(\begin{pmatrix} g \\ h \end{pmatrix}\begin{pmatrix} \Delta y \\ -\Delta x \end{pmatrix}\right)_k = \sum_{k=1}^{4}(g\Delta y - h\Delta x)_k$$

- Variables are stored at the cell centers
- Values on the surface can be reconstructed with a Linear function:
  $u(x, y) = a_0 + a_1 x + a_2 y$

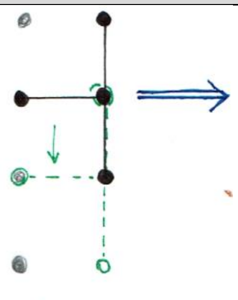
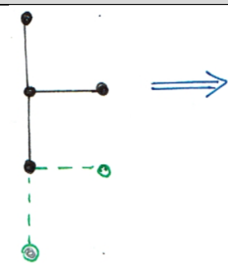
  Also $\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

- Constants with least squares method:
  $$\begin{pmatrix} n & \sum x_i & \sum y_i \\ \sum x_i & \sum x_i^2 & \sum x_i y_i \\ \sum y_i & \sum x_i\, y_i & \sum y_i^2 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \sum u_i \\ \sum u_i x_i \\ \sum u_i y_i \end{pmatrix}$$

  n is the number of points.

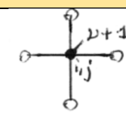
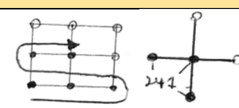
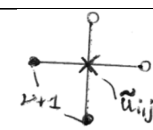
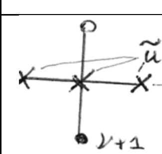
Finite Differences	**Finite Volume**
	 with: 

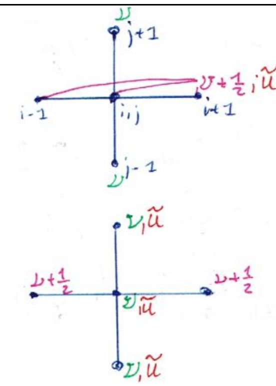
Formulate solution scheme

implicit	explicit
 	
Solve with Thomas algorithm or iteration scheme (see next page). Unconditionally stable!	Can all be solved in parallel Don’t forget to check stability!!
$u_t = cu_{xx}$ $\frac{u_i^{n+1}-u_i^n}{2\Delta t} = c\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{\Delta x^2}$ With: $\sigma = \frac{2\Delta tc}{\Delta x^2}$ ➔ $-\sigma u_{i-1}^{n+1} - (1+2\sigma)u_i^{n+1} - \sigma u_{i-1}^{n+1} = u_i^n$	$u_t = cu_{xx}$ $\frac{u_i^{n+1}-u_i^n}{2\Delta t} = c\frac{u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n}}{\Delta x^2}$ ➔ $u_i^{n+1} = \sigma(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$

Iteration schemes:

- *For **elliptic** equations! N.B.: For hyperbolic and parabolic equations, marching schemes are employed!*
- ***Point Iterative Methods:** At each step the approximate solution is modified at a single point of the domain. Each $u_{i,j}^{n+1}$ is determined explicitly i.e. simultaneous solution of equations not required.*
- ***Block Iterative Methods**: Generally, some level of implicitness leads to increased convergence rates. Here, only simple rows/columns are investigated (=line iterative methods).*
- ***Line iterative methods** can be solved using the **Thomas algorithm** (see 3-5 script)*

Type:	**Point-wise form:**	**Visual:**
Poisson eq.	$u_{xx} + u_{yy} = -f(x,y)$ ↓ discretize using central differences: $u_{i,j} = \theta_x(u_{i-1,j} + u_{i+1,j}) + \theta_y(u_{i,j-1} + u_{i,j+1}) + \delta^2 f_{i,j}$ With: $\theta_x = \frac{\Delta y^2}{2(\Delta x^2+\Delta y^2)}$ & $\theta_x = \frac{\Delta x^2}{2(\Delta x^2+\Delta y^2)}$ & $\delta^2 = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2+\Delta y^2)}$	
Jacobi (-Point)	Does not use updated values from previous! - most simple - bad rate of convergence - straight forward parallel execution (no coupling across domain boundaries) - doesn't require structural mesh	
	$u_{i,j}^{\nu+1} = \theta_x(u_{i-1,j}^{\nu} + u_{i+1,j}^{\nu}) + \theta_y(u_{i,j-1}^{\nu} + u_{i,j+1}^{\nu}) + \delta^2 f_{i,j}$	
Gauss Seidel:	Unlike Jacobi, it uses the updated values from previous lines!!	
G-S Point	$u_{i,j}^{\nu+1} = \theta_x(u_{i-1,j}^{\nu+1} + u_{i+1,j}^{\nu}) + \theta_y(u_{i,j-1}^{\nu+1} + u_{i,j+1}^{\nu}) + \delta^2 f_{i,j}$ - convergence rate 2x better than Jacobi - uses updated values from neighboring points as soon as available -> direction dependent! - NO parallel exec. EXCEPT by dropping coupling across boundaries	
G-S Point (accelerated /over-relaxed)	$u_{i,j}^{\nu+1} = u_{i,j}^{\nu}(1-\omega) + \omega\tilde{u}_{i,j}$ $\tilde{u}_{i,j} = \theta_x(u_{i-1,j}^{\nu+1} + u_{i+1,j}^{\nu}) + \theta_y(u_{i,j-1}^{\nu+1} + u_{i,j+1}^{\nu}) + \delta^2 f_{i,j}$ combined: $u_{i,j}^{\nu+1} = u_{i,j}^{\nu}(1-\omega) + \omega[\theta_x(u_{i-1,j}^{\nu+1} + u_{i+1,j}^{\nu}) + \theta_y(u_{i,j-1}^{\nu+1} + u_{i,j+1}^{\nu}) + \delta^2 f_{i,j}]$ Stability: $0 < \omega \leq 2$	
G-S Point (with red/black ordering)	- Parallel exec. possible (no coupling during the two sub-steps; in between data exchange necessary) - usually improvement of performance	
G-S Line	In x-direction: $-\theta_x u_{i-1,j}^{\nu+1} + u_{i,j}^{\nu+1} - \theta_x u_{i+1,j}^{\nu+1} = \theta_y(u_{i,j-1}^{\nu+1} + u_{i,j+1}^{\nu}) + \delta^2 f_{i,j}$ In y-direction: $-\theta_y u_{i,j+1}^{\nu+1} + u_{i,j}^{\nu+1} - \theta_y u_{i,j+1}^{\nu+1} = \theta_x(u_{i-1,j}^{\nu+1} + u_{i+1,j}^{\nu}) + \delta^2 f_{i,j}$ - NO parallel exec. (requires inversion of tridiagonal matrix in each substep)	
G-S Line (accelerated /over-relaxed)	$\boxed{u_{i,j}^{\nu+1} = u_{i,j}^{\nu}(1-\omega) + \omega\tilde{u}_{i,j}}$ $-\theta_x\tilde{u}_{i-1,j} + \tilde{u}_{i,j} - \theta_x\tilde{u}_{i+1,j} = \theta_y(u_{i,j-1}^{\nu+1} + u_{i,j+1}^{\nu}) + \delta^2 f_{i,j}$	

	for stability check; apply to both equations: $\tilde{u}_{i,j}^{\nu} = \tilde{V}^{\nu} e^{I(i\theta_x + j\theta_y)}$ - convergence rate 2x better than G-S Point - **NOT** applicable to unstructured meshes (no i,j-ordering) Stability: $0 < \omega \leq 2$	
Alternating line …	**(1) x-Line** $u_{i,j}^{\nu+1/2} = u_{i,j}^{\nu}(1-\omega) + \omega \tilde{u}_{i,j}^{\nu+1/2}$ $-\theta_x \tilde{u}_{i-1,j}^{\nu+1/2} + \tilde{u}_{i,j}^{\nu+1/2} - \theta_x \tilde{u}_{i+1,j}^{\nu+1/2} = \theta_y \left(u_{i,j-1}^{\nu} + u_{i,j+1}^{\nu}\right) + \delta^2 f_{i,j}$ **(2) y-line** $u_{i,j}^{\nu+1} = u_{i,j}^{\nu+1/2}(1-\omega) + \omega \tilde{u}_{i,j}^{\nu+1}$ $-\theta_y \tilde{u}_{i-1,j}^{\nu+1} + \tilde{u}_{i,j}^{\nu+1} - \theta_y \tilde{u}_{i+1,j}^{\nu+1} = \theta_x \left(u_{i,j-1}^{\nu+1/2} + u_{i,j+1}^{\nu+1/2}\right) + \delta^2 f_{i,j}$ -avoids performing line iterations only in one direction, which slows the convergence down - stable for all relaxation parameters ω - use optimized parameter for each iteration - comparison with over-relaxed schemes difficult	

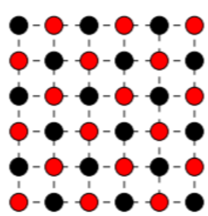
Red Black ordering:

- The mesh-points are split up into red and black points like a checkerboard

1) The values on all red points are computed with Gauss-Seidel method, taking into account the surrounding black points (NO other red points)
2) The values on all black points are computed using the red points from (1)

Advantage: *Vectorization* of the solution procedure as the solution at different points can be computed SIMULTANEOUSLY (NOT recursively)

More complex stencils might need more than 2 stages in each iteration e.g. 4



Red depends only on black, and vice-versa.
Generalization: multi-color orderings

Truncation error

PDE: $L(u)$
FDE: $L_\Delta(u)$

$$\boxed{\tau(u) = L(u) - L_\Delta(u)}$$

Example:
$L(u) = u_{xx}$
$L(u) = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} = u_{xx} + \frac{\Delta x^2}{12} u_{xxxx} + \cdots$
$\tau(u) = L(u) - L_\Delta(u) = -\frac{\Delta x^2}{12} u_{xxxx} + \cdots = O(\Delta x^2)$

- To reduce the truncation error: Introduce an additional point in the FDE.

Convergence

$$\boxed{Convergence = Wellposedness\ +\ Consistency\ +\ Stability}$$

The convergence of a finite difference equation requires **consistency and stability.**

Consistency

$$\boxed{\lim_{\Delta x, \Delta y \to 0} \tau\,(u) = 0}$$

For $\tau = \frac{\Delta x^2}{\Delta t} \rightarrow$ conditionally consistent if $\Delta x^2 \ll\ \Delta t$

Stability

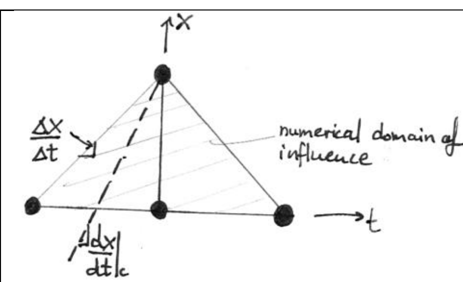
Implicit FDEs are unconditionally stable!

CFL- condition

- Only a necessary condition for stability
- Results from von Neumann stability analysis and CFL might differ!
- Only applicable to **hyperbolic** PDEs

$$\boxed{\frac{\Delta x}{\Delta t} \geq \left.\frac{dx}{dt}\right|_C = \lambda}$$

$$\frac{\Delta x}{\Delta t} \geq \max \left| \left.\frac{dx}{dt}\right|_{1,2,\ldots} \right|$$



Courant-number:

$$C = \frac{\max \left| \left.\frac{dx}{dt}\right|_C \right|}{\frac{\Delta x}{\Delta t}} = \frac{exact\ info\ rate}{computer\ info\ rate} \leq 1$$

Discrete error perturbation theory (3-8)

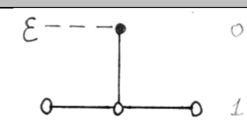
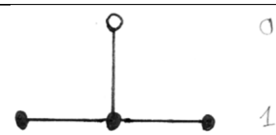
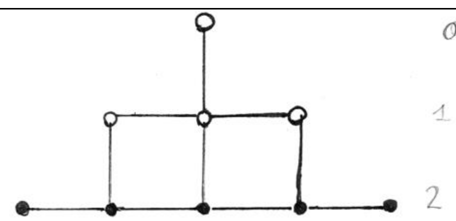
Empirical method for investigation of stability.
A disturbance ϵ overlays the exact solution U.

$$\boldsymbol{W = U + \epsilon}$$

$$\boxed{\boldsymbol{u_i^n = u_{exact}|_i^n + \varepsilon_i^n}}$$

The modulus of the perturbation must decrease for stability!!

Example:

$\boldsymbol{n =}$	$\boldsymbol{i =}$	$\boldsymbol{\varepsilon_i}$	**Scheme**	**The value** $\max\left\|\frac{\varepsilon_i}{\varepsilon}\right\|$ should decrease with every iteration, otherwise perturbations are amplified.
0	is	$\boxed{\boldsymbol{\varepsilon_{is}^0 = \varepsilon}}$		
	$else$	0		
1	is	$\varepsilon_{is}^1 = \cdots$		
	$is + 1$	$\varepsilon_{is+1}^1 = \cdots$		
	$is - 1$	$\varepsilon_{is-1}^1 = \cdots$		$\max\left\|\frac{\varepsilon_i^1}{\varepsilon}\right\| \leq 1$
2	is	$\varepsilon_{is}^2 = \cdots$		
	$is + 1$	$\varepsilon_{is+1}^2 = \cdots$		
	$is - 1$	$\varepsilon_{is-1}^2 = \cdots$		
	$is + 2$	$\varepsilon_{is+2}^2 = \cdots$		
	$is - 2$	$\varepsilon_{is-2}^2 = \cdots$		$\max\left\|\frac{\varepsilon_i^2}{\varepsilon}\right\| \leq 1$
n				$\max\left\|\frac{\varepsilon_i^n}{\varepsilon}\right\| \leq 1$ Must asymptotically approach a stability limit.

Procedure: Always introduce the previous ε_i into the next equation.

$$\max\left|\frac{\varepsilon_i}{\varepsilon}\right| \leq 1$$

Neumann Stability analysis:

$$\boxed{\boldsymbol{u_i^n = u_{exact}|_i^n + V^n e^{I(i\alpha + j\beta)}}}$$

1	Substitute periodic perturbation for $u_{i,j}$ in FDE	$u_{i,j}^{\nu} = V^{\nu} e^{I(i\theta_x + j\theta_y)}$ (Fourier) $I = \sqrt{-1}$ or $u_i^{\nu} = V^{\nu} e^{I\theta i}$
2	Therme zusammenfassen; durch $e^{I(i\theta_x + j\theta_y)}$ teilen	$(\ldots)V^{\nu+1} = (\ldots)V^{\nu}$
3		$G = \frac{V^{\nu+1}}{V^{\nu}} = \frac{V^{\nu}}{V^{\nu-1}}$ or if both $V^{\nu+1}$ and $V^{\nu-1}$ are present: $G^2 = \frac{V^{\nu+1}}{V^{\nu-1}} = \frac{V^{\nu+1}}{V^{\nu}} \frac{V^{\nu}}{V^{\nu-1}}$ Or $G = \frac{V^{\nu+1}}{V^{\nu+\frac{1}{2}}} \frac{V^{\nu+\frac{1}{2}}}{V^{\nu}}$
		$2\cos\theta = e^{I\theta} + e^{-I\theta}$ $2I\sin\theta = e^{I\theta} - e^{-I\theta}$ $e^{I\theta} = \cos\theta + I\sin\theta$ $e^{-I\theta} = \cos\theta - I\sin\theta$
4	Show that the absolute error amplification factor <1	$\|G\|^2 \leq 1$ $\boxed{\boldsymbol{\|G\| \leq 1}}$ If complex: $\|z\| = \sqrt{a^2 + b^2}$ $\left\|\frac{a \pm bI}{c \pm dI}\right\| = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$

Hirt analysis

Idea: return to known form such as:

$u_{tt} = a^2 u_{xx}$

<u>Scenario 1</u>

Given a hyperbolic equation, the goal is to make the equation parabolic, because that allows comparison to the parabolic equation $u_t = \nu_{num}(u_{xx} + u_{yy})$

Hyperbolic PDE	$u_t + au_x = 0$
Discretize terms	$u_t^n + u_{tt}^n \frac{\Delta t}{2} + a\left(u_x^n - u_{xx}^n \frac{\Delta x^2}{6}\right) + \cdots = 0$
Replace u_{tt}	$u_{tt} = -au_{xt} = a^2 u_{xx}$
Find ν_{num}	$\nu_{num} > 0$!!! ➔ unstable if negative numerical viscosity

<u>Scenario 2</u>

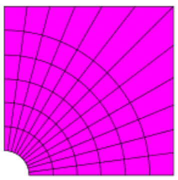
Given a parabolic equation, the goal is to make the equation hyperbolic, because that allows application of the CFL-condition.

Parabolic PDE	$u_t - \nu u_{xx} = 0$
Discretize terms	$u_t^n + u_{tt}^n \frac{\Delta t}{2} - -\nu u_{xx} + \cdots = 0$
Apply CFL	$\frac{\Delta x}{\Delta t} \geq \left.\frac{dx}{dt}\right\|_C$

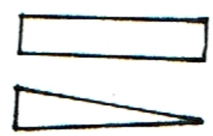
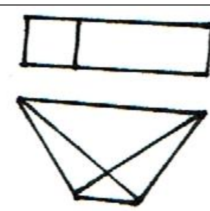
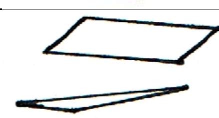
Warning: Sometimes, Hirt is not applicable e.g. when no physical meaning can be found.

The mesh

- *Unstructured if no global i,j ordering can be introduced!*
- *Structured meshes have regular connectivity.*
- *Hybrid grids are also possible.*

Structured	**Unstructured**
	
+ computational efficiency + memory efficiency (neighbor relationship in data arrangement) + higher-order schemes feasible	+ suitable for complex geometries + solution-based adaptation possible + automatic grid generation
- tedious to construct - not suited for complex geometries - no automatic grid generation	- complex algorithms - slower memory access - increased memory consumption (neighbor relationship has to be stored)
• suitable for FV or FD formulation	• suitable for FV formulation

Aspects of the mesh that have influence on the truncation error:

Cell aspect ratio ➔ difference in length of the sides	
Mesh smoothness ➔ difference in size of the cells	
Mesh skewness ➔ difference in angles at the corners	
Size of spatial step	

Task: Change order of equation

$u_{tt} - a u_{xx} = 0$

Set:

$q = u_t$

$p = u_x$

$$\begin{cases} q_t - a p_x = 0 \\ q_x - p_t = 0 \end{cases}$$