Supporting Information for "The thermospheric auroral red line Angle of Linear Polarisation"

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1. Introduction

Computing the apparent angle of the magnetic field as seen by the polarimeter in any direction is a classical spherical geometry problem. However, it is difficult to find the development in geophysical articles or textbooks. Therefore, we chose to provide it here. It is not something new, reason to write it as an appendix.

2. Geographical coordinates

The geography is plotted in Figure S1. The SPP is located at a point A characterized by a latitude Φ_A and a longitude Ψ_A. It looks at the polarisation with an elevation ϵ and an azimuth α. The elevation is the angle with the plan which is tangent to the sphere in A, $\epsilon = 0°$ when pointing at the horizon and$\epsilon = 90°$ when pointing at the zenith. The azimuth is the angle with the meridian in A. It is taken equal to zero pointing North, to 90° East, 180° South and 270° West.

In (*Lilensten, Bommier, Barthélemy, Bernard, Lamy, Moen, Johnsen, Løvhaug, and Pitout*, 2015), we have shown that the theoretical polarisation equals the observations at the altitude at which the red line emission maximizes. Let us call h this altitude. In the above comparison, following (*Lilensten et al.*, 2015), $h = 220\ km$. We call H the point where the line of sight is at an altitude h. If O is the center of the Earth, the axis $\bar{OH}$ intersects the surface of the Earth at a point P of latitude Φ_H and longitude Ψ_H. The line of sight $\overrightarrow{AH}$ has a norm $|AH|$ and is carried by the unit vector $\overrightarrow{u}$.

The frame of reference $\mathcal{R}_O$ at O is as follows: Axis x is at zeroth longitude and latitude. Axis y is horizontal and points positively toward longitude 90. Axis z is vertical and points positively toward the north pole. The frame of reference $\mathcal{R}_A$ at A is as follows: Axis x is the altitude, positive upward. Axis y points positively toward the east in the plan tangent to the surface in A. Axis z points positively toward the north in the same plan. Let us compute Φ_H and Ψ_H. In the following, R_T^A is the radius of the Earth in A.

3. First step: *A* at the equator

We now consider that A is located at the equator such as shown in Figure S2, along the zeroth meridian. The reference frame in A is the same as the reference frame in O, the center of the Earth, translated by a distance R_t^A along the x axis. We call {X,Y,Z} the absolute frame centered around O. A magnification of the geometry around a is plotted in figure S3.

$$\left\{ \begin{array}{lcl} \Phi_A & = & 0 \\ \Psi_A & = & 0 \end{array} \right. \tag{1}$$

Expressed in $\mathcal{R}_A$, the projection of $\overrightarrow{u}$ in the horizontal plane is (Figure S2):

$$\left\{ \begin{array}{lcl} u_y & = & \sin(\alpha_{AH}) \\ u_z & = & \cos(\alpha_{AH}) \end{array} \right. \tag{2}$$

With the choice of the azimuth oriented positively clockwise. Taking the elevation into account, one gets in $\mathcal{R}_A$:

$$\left\{ \begin{array}{lcll} u_x & = & \sin(\epsilon_{AH}) & \text{altitude} \\ u_y & = & \sin(\alpha_{AH})\cos(\epsilon_{AH}) & \text{estward} \\ u_z & = & \cos(\alpha_{AH})\cos(\epsilon_{AH}) & \text{northward} \end{array} \right. \tag{3}$$

The vector that links the center of the Earth O to the observed point H is $\overrightarrow{OH} = \overrightarrow{OA} + |AH|\ \overrightarrow{u}$. We can develop on the three axis {X, Y, Z} of $\mathcal{R}_O$:

$$\overrightarrow{OH} = \begin{cases} R_T^A + |AH|\sin(\epsilon_{AH}) \\ -|AH|\sin(\alpha_{AH})\cos(\epsilon_{AH}) \\ -|AH|\cos(\alpha_{AH})\cos(\epsilon_{AH}) \end{cases} \tag{4}$$

The norm of $\overrightarrow{OH}$ is $|OH| = \sqrt{OH_x^2 + OH_y^2 + OH_z^2}$, which gives:

$$|OH| = \sqrt{(R_T^A)^2 + |AH|^2 + 2R_T^A|AH|\sin(\epsilon_{AH})} \tag{5}$$

The height of the observation point H above the Earth is $h = |PH|$, which we take equal to 220 km. This value is the difference $|OH| - |OP|$. We suppose that the radius of the Earth is the same for $|OA|$ and $|OP|$ so that:

$$|OH| = h + R_T^A \tag{6}$$

from equations 5 and 6 we easily get the value of $|AH|$ through:

$$|AH|^2 + 2|AH|\ R_T^A \sin(\epsilon_{AH}) - 2R_T^A h - h^2 = 0 \tag{7}$$

which gives:

$$|AH| = {R_T^A}^2 \sin(\epsilon)^2 + \sqrt{{R_T^A}^2 \sin(\epsilon)^2 + 2hR_T^A + h^2} \tag{8}$$

From equations 3 and 8, one deduces the vector $\overrightarrow{AH}$:

$$\overrightarrow{AH} = |AH|\ \overrightarrow{u}. \tag{9}$$

4. Second step: At any location

The geometry of the problem is displayed in Figure S1. The SPP may be positioned anywhere on Earth, and this complicate significantly the geometry. The first step is to find a way to express every vector in the same reference frame.

Let us first express $\overrightarrow{u_A}$, the unit vector of our line of sight in $\mathcal{R}_O$. The transformation matrix to pass from the reference frame $\mathcal{R}_A$ at point A to $\mathcal{R}_O$ at point O is:

$$R_{A\to O} = \begin{bmatrix} \cos(\Phi_A)cos(\psi_A) & -\sin(\psi_A) & -\sin(\Phi_A)\cos(\psi_A) \\ \cos(\Phi_A)sin(\psi_A) & \cos(\psi_A) & -\sin(\Phi_A)\sin(\psi_A) \\ \sin(\Phi_A) & 0 & \cos(\Phi_A) \end{bmatrix} \tag{10}$$

And its inverse, $R_{O\to A}$, to pass from $\mathcal{R}_O$ to $\mathcal{R}_A$ is the transposed matrix of $R_{A\to O}$. This will allow us to easily express all our vectors in one reference frame. For example, $\overrightarrow{u_A}$ can be expressed as follows in $\mathcal{R}_0$:

$$\overrightarrow{u_O} = R_{A\to O}.\overrightarrow{u_A} \tag{11}$$

Using again the fact that $\overrightarrow{OH} = \overrightarrow{OA} + |AH|\ \overrightarrow{u}$, we can develop on the three axis {X, Y, Z} of $\mathcal{R}_O$:

$$\overrightarrow{OH} = R_T^A \begin{pmatrix} \cos(\Phi_A)\cos(\Psi_A) \\ \cos(\Phi_A)\sin(\Psi_A) \\ \sin(\Phi_A) \end{pmatrix} + |AH|\overrightarrow{u_O} \tag{12}$$

From the vector $\overrightarrow{OH}$, we can easily deduce the latitude Φ_H and the longitude Ψ_H of the point H. Its elevation is still taken to be 220 km (see Figure S3).

$$\begin{cases} \sin(\Phi_H) & = \frac{OH_z}{|OH|} \\ \cos(\Phi_H) & = \frac{\sqrt{OH_x^2+OH_y^2}}{|OH|} \end{cases} \tag{13}$$

$$\begin{cases} \sin(\Psi_H) &= \frac{OH_y}{\sqrt{OH_x^2+OH_y^2}} \\ \cos(\Psi_H) &= \frac{OH_x}{\sqrt{OH_x^2+OH_y^2}} \end{cases} \tag{14}$$

The results above Ny-Ålesund are shown in Figure S4.

5. Apparent angle of observation

We use IGRF (*Olsen, Sabaka, and Tøffner-Clausen*, 2000) to get the magnetic field $B_H(H)$ at the observed point of coordinates ($latitude\ = \Phi_H\ longitude\ = \Psi_H\ heigh =$ $220\ km$). The configuration is shown in Figure S5. The next question to solve is that of the apparent angle of the magnetic field in H seen from A.

IGRF provides the three coordinates of the magnetic field $\{B_x,B_y,B_y\}$ in the frame of H, characterized by three axis $\{x_H,y_H,z_H\}$. It is necessary to rotate these values in the reference frame of A, characterized by $\{x_A,y_A,z_A\}$. This is illustrated in Figure S6.

The first rotation expresses the magnetic field B_H in $\mathcal{R}_O$ using equation 10:

$$\overrightarrow{B_O} = R_{H\to O}.\overrightarrow{B_H} \tag{15}$$

The second rotation brings the field in $\mathcal{R}_A$:

$$\begin{aligned} \overrightarrow{B_A} &= R_{O\to A}.\overrightarrow{B_O} \\ &= R_{O\to A}.R_{H\to O}.\overrightarrow{B_H} \end{aligned} \tag{16}$$

Computing the magnetic angle seen by the SPP is performed through a last set of rotations that brings the magnetic field $\overrightarrow{B_A(H)}$ in the frame of the polarimeter $\mathcal{R}_{SPP} =$ $\{x_{SPP}, y_{SPP}, z_{SPP}\}$. The rotations are illustrated in Figure S7. The rotation matrix is:

$$R_{SPP\to A} = \begin{pmatrix} \sin(\epsilon) & 0 & \cos(\epsilon) \\ \cos(\epsilon)\sin(\alpha) & -\cos(\alpha) & -\sin(\epsilon)\sin(\alpha) \\ \cos(\epsilon)\cos(\alpha) & \sin(\alpha) & -\sin(\epsilon)\cos(\alpha) \end{pmatrix} \tag{17}$$

And again, its inverse matrix to pass from the reference frame of A to the reference frame of SPP is its transposed matrix $R_{A\to SPP}$.

So that we can now write the vector of the magnetic field $\overrightarrow{B_{SPP}(H)}$ as seen by the polarimeter:

$$\begin{aligned} \overrightarrow{B_{SPP}} &= R_{A\to SPP}\overrightarrow{B_A(H)} \\ &= R_{A\to SPP}.R_{O\to A}.R_{H\to O}.\overrightarrow{B_H(H)} \end{aligned} \tag{18}$$

The apparent angle of the magnetic field is η defined simply by:

$$\begin{cases} \cos(\eta) &= \frac{B_y(SPP)}{\sqrt{B_z^2(SPP)+B_y^2(SPP)}} \\ \sin(\eta) &= \frac{B_z(SPP)}{\sqrt{B_z^2(SPP)+B_y^2(SPP)}} \end{cases} \tag{19}$$

Figure S8 shows this angle as a function of the elevation in four geographic directions. Obviously, at an elevation of 90°, all the values converge since this is a zenith pointing. The wriggle at 85° in the south and north azimuth is due to the shape of the magnetic field, and more exactly to the eastward component of the magnetic field. As seen in Figure S9, this component turns from positive to negative for the northern azimuth above 86° while it turns from negative to positive for the southern azimuth.

6. Angle between the line of sight and the magnetic field in P versus the SPP elevation.

In *Lilensten et al.* (2015), we computed the theoretical Degree of Linear Polarisation. This value depends on the the angle Θ between the line of sight $|AH|$ (equation 9) and the local magnetic field line in H $\overrightarrow{B_H(H)}$ (equation **??**). It is simply computed through:

$$\cos(\Theta) = \frac{\overrightarrow{AH}.\overrightarrow{B_A(H)}}{|AH||B_A(H)|} \tag{20}$$

In Figure S10, we show this angle. Because we are still relatively low at 220 km, the magnetic field lines are almost vertical straight lines. At an elevation of 30° as in the present study, this angle is 67.2° in the west azimuth and 52° in the Eastward direction. North and south directions give roughly the same angle of 60°. This contradicts our first articles (*Lilensten, Moen, Barthélemy, Thissen, Simon, Lorentzen, Dutuit, Amblard, and Sigernes*, 2008) where we mentioned that our north pointing was close to a right angle. The error was due to the fact that we had used a simplified geometry. This has a effect on the published values of the DoLP (out of the scope of this article). The measured $DoLP_{obs}$ is a projection of the absolute value $DoLP_{abs}$ on the line of sight. We have therefore:

$$DoLP_{abs} = DoLP_{obs} \sin\Theta \tag{21}$$

In our case, $\sin\Theta = 0.92$, introducing an error of 8% when a right angle is considered.