Aula prática nº 10

3.5 Exercícios (Pg - 148)

1) Calcule os integrais de linha seguintes:

Obs: $f = (f_1, \dots, f_m) : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$

$\alpha = (\alpha_1, \dots, \alpha_m) : [a, b] \to \mathbb{R}^m$

$$\int_C f\,d\alpha = \int_C f_1\,dx_1 + \dots + f_m\,dx_m = \int_a^b f(\alpha(t)) \mid \alpha'(t)\,dt =$$
$$= \int_a^b f_1(\alpha(t))\alpha_1'(t) + \dots + f_m(\alpha(t))\alpha_m'(t)\,dt.$$

1.a) de $f(x,y) = (x^2 - 2xy)\vec{e_1} + (y^2 - 2xy)\vec{e_2}$ ao longo da parábola $y = x^2$ entre os pontos $(-1,1)$ e $(1,1)$.

$\alpha(t) = (t, t^2)$, $t \in [-1, 1]$ e $\alpha'(t) = (1, 2t)$.

$$\int_C f\,d\alpha = \int_a^b f(\alpha(t)) \mid \alpha'(t)\,dt = \int_{-1}^1 (t^2 - 2t^3, t^4 - 2t^3) \mid (1, 2t)\,dt =$$

$$= \int_{-1}^1 2t^5 - 4t^4 - 2t^3 + t^2\,dt =$$

$$= \frac{1}{3}\underbrace{\left[t^6\right]_{-1}^1}_{=0} - \frac{4}{5}\left[t^5\right]_{-1}^1 - \frac{4}{2}\underbrace{\left[t^4\right]_{-1}^1}_{=0} + \frac{1}{3}\left[t^3\right]_{-1}^1 =$$

$$= -\frac{8}{5} + \frac{2}{3} = -\frac{24}{15} + \frac{10}{15} = -\frac{14}{15}$$

~~1.b)~~ De $g(x,y)=(2a-y,x)$, ao longo de

$\alpha(t)=(a(t-\sin t),\ a(1-\cos t))$, $0\leq t\leq 2\pi$ e $a>0$.

$\alpha'(t)=(a(1-\cos t),\ a\sin t)$

$$\int_c g\,d\alpha=\int_0^{2\pi}(2a-a+a\cos t)(a-a\cos t)+(at-a\sin t)(a\sin t)\,dt=$$

$$=\int_0^{2\pi}a^2-a^2\cancel{\cos^2 t}+a^2t\sin t-a^2\cancel{\sin^2 t}\,dt=$$

$$=a^2\int_0^{2\pi}t\sin t\,dt=a^2\left(\left[-t\cos t\right]_0^{2\pi}+\int_0^{2\pi}\cos t\,dt\right)=$$

$\begin{cases}u'=\sin t\\ v=t\end{cases}\Rightarrow\begin{cases}u=-\cos t\\ v'=1\end{cases}$

$$=a^2\Big(-2\pi+\underbrace{\left[\sin t\right]_0^{2\pi}}_{=0}\Big)=-2\pi a^2.$$

10-3

~~1.e)~~ De $h(x,y,z) = (y^2-z^2)\vec{e}_1 + 2yz\,\vec{e}_2 - x^2\vec{e}_3$

ao longo de $\alpha(t) = (t, t^2, t^3)$ com $0 \leq t \leq 1$.

$\alpha'(t) = (1, 2t, 3t^2)$

$$\int_C h\,dx = \int_C (y^2 - z^2)dx + 2yz\,dy - x^2 dz =$$

$$\boxed{= \int_0^1 (t^4 - t^6)(1) + 2t^5(2t) - t^2(3t^2)\,dt =}$$

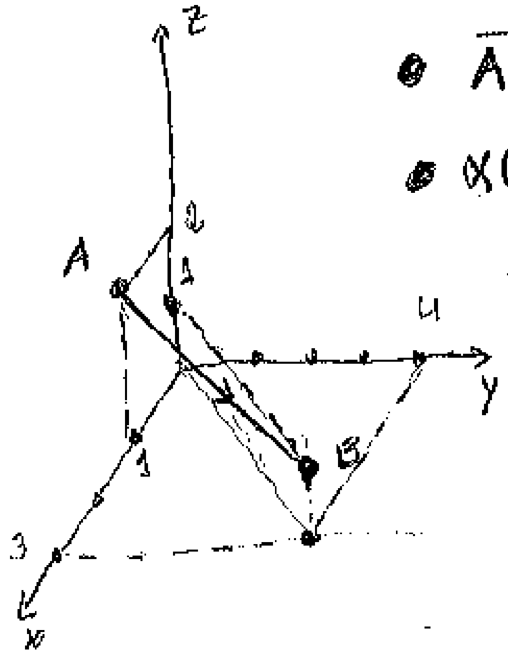
$$= \int_0^1 t^4 - t^6 + 4t^6 - 3t^4\,dt =$$

$$\boxed{= \int_0^1 3t^6 - 2t^4\,dt =}$$

$$\boxed{= \frac{3}{7}\Big[t^7\Big]_0^1 - \frac{2}{5}\Big[t^5\Big]_0^1} =$$

$$= \frac{3}{7} - \frac{2}{5} = \frac{15}{35} - \frac{14}{35} = \boxed{\frac{1}{35}}$$

1.d) De $f(x,y,z) = (2xy, x^2+z, y)$, ao longo do segmento de reta que une os pontos $A(1,0,2)$ e $B(3,4,1)$.



- $\overrightarrow{AB} = B - A = (3,4,1) - (1,0,2) = (2,4,-1)$
- $\alpha(t) = A + t\overrightarrow{AB}$, $t \in [0,1]$
- $\alpha(t) = (1,0,2) + t(2,4,-1) =$
  $= (1+2t, 4t, 2-t)$
- $\alpha'(t) = (2,4,-1)$

- $2xy = 2(1+2t)(4t) = 16t^2 + 8t$
- $x^2 + z = (1+2t)^2 + (2-t) = 1 + 4t + 4t^2 + 2 - t = 4t^2 + 3t + 3$

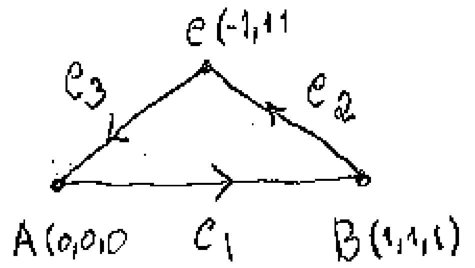
- $\int_C f\,d\alpha = \int_C 2xy\,dx + (x^2+z)\,dy + y\,dz =$

$= \int_0^1 (16t^2+8t)(2) + (4t^2+3t+3)(4) + 4t(-1)\,dt =$

$= \int_0^1 48t^2 + 24t + 12\,dt = \frac{48}{3}\left[t^3\right]_0^1 + \left[12t^2\right]_0^1 + 12\left[t\right]_0^1 =$

$= 16 + 12 + 12 = 40.$

~~2)~~ Considere a força $f(x,y,z)=(yz,\, xz,\, xy+x)$.
Calcular o trabalho realizado por f ao longo do contorno do triângulo de vertices $A(0,0,0)$, $B(1,1,1)$ e $C(-1,1,1)$, por esta ordem.



$C = C_1 \cup C_2 \cup C_3$

O trabalho realizado por f ao longo de C é

$$T = \int_C f\, d\alpha = \int_{C_1} f\, d\alpha + \int_{C_2} f\, d\alpha + \int_{C_3} f\, d\alpha =$$

- Cálculo ao longo de C_1
  - $\overrightarrow{AB} = B - A = (1,1,1) - (0,0,0) = (1,1,1)$
  - $\alpha_1(t) = A + t\overrightarrow{AB} = (t,t,t),\ t\in[0,1]$; $\alpha_1'(t) = (1,1,1)$

$$\int_C f\, d\alpha = \int_0^1 t^2(1) + t^2(1) + (t^2+t)(1)\, dt =$$

$$= \int_0^1 3t^2 + t\, dt = \left[t^3\right]_0^1 + \frac{1}{2}\left[t^2\right]_0^1 = 1 + \frac{1}{2} = \frac{3}{2}.$$

- $f(x,y,z) = (yz, xz, xy+x)$

- Cálculo ao longo de C_2

- $\overrightarrow{BC} = C - B = (-1,1,1) - (1,1,1) = (-2,0,0)$

- $\alpha_2(t) = B + t\overrightarrow{BC} = (1,1,1) + t(-2,0,0) = (1-2t, 1, 1), t \in [0,1]$

$\alpha_2'(t) = (-2,0,0)$

$$\int_{C_2} f\, d\alpha = \int_0^1 1(-2) + (1-2t)(1)(0) + [(1-2t)+(1-2t)](0)\, dt =$$

$$= \int_0^1 -2\, dt = -2$$

- Cálculo ao longo de C_3

- $\overrightarrow{CA} = A - C = (1,-1,-1)$

- $\alpha_3(t) = C + t\overrightarrow{CA} = (-1,1,1) + t(1,-1,-1) = (t-1, 1-t, 1-t), t \in [0,1]$

- $\alpha_3'(t) = (1,-1,-1)$

$$\int_{C_3} f\, d\alpha = \int_0^1 (1-t)^2(1) + (t-1)(1-t)(-1) + (t-1)(1-t)(-1) + (t-1)(-1)\, dt$$

$$= \int_0^1 3(1-t)^2 + 1 - t\, dt = \int_0^1 3(1-2t+t^2) + 1 - t\, dt =$$

$$= \int_0^1 3 - 6t + 3t^2 + 1 - t\, dt = \int_0^1 3t^2 - 7t + 4\, dt =$$

$$= [t^3]_0^1 - \frac{7}{2}[t^2]_0^1 + 4[t]_0^1 = 5 - \frac{7}{2} = \frac{3}{2}$$

$$* \int_C f\, d\alpha = \frac{3}{2} - 2 + \frac{3}{2} = 1 \blacksquare$$

3) Calcule os integrais de linha, com respeito ao comprimento de arco: 10-7

~~3a)~~ Observação:

Se $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ e $\alpha : [a,b] \to \mathbb{R}^n$, então:

$$\int_C f\,dS = \int_a^b f(\alpha(t))\,\|\alpha'(t)\|\,dt.$$

$\int_C \overbrace{XY}^{=f(X,Y)}\,dS$ sobre a curva dada por $y = \frac{X^2}{2}$ da origem ao ponto $(1, 1/2)$.

- $\alpha(t) = \left(t, \frac{t^2}{2}\right)$, $t \in [0,1]$
- $\alpha'(t) = (1, t)$, $\|\alpha'(t)\| = \|(1,t)\| = \sqrt{1+t^2}$.
- $\int_C XY\,dS = \int_a^b f(\alpha(t))\,\|\alpha'(t)\|\,dt =$

$$= \frac{1}{2}\int_0^1 t^3\sqrt{1+t^2}\,dt = \frac{1}{2}\frac{1}{2}\int_1^2 \underbrace{\cancel{\sqrt{u-1}}}_{=t}\,\underbrace{(u-1)}_{=t^2}\sqrt{u}\,\underbrace{\frac{1}{\cancel{\sqrt{u-1}}}}_{\varphi'(u)}\,du =$$

$$\boxed{\begin{array}{l} u = 1+t^2 = \varphi^{-1}(t) \\ t = \sqrt{u-1} = \varphi(u) \\ \varphi'(u) = \frac{1}{2\sqrt{u-1}} \\ \varphi^{-1}(0) = 1,\ \varphi^{-1}(1) = 2 \end{array}}$$

$$= \frac{1}{4}\int_1^2 u^{\frac{3}{2}} - u^{1/2}\,du =$$

$$= \frac{1}{4}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^2 = \dots$$

$$= \frac{1}{4}\left(\frac{2}{5}\sqrt{2^5} - \frac{2}{5} - \frac{2}{3}\sqrt{2^3} + \frac{2}{3}\right) =$$

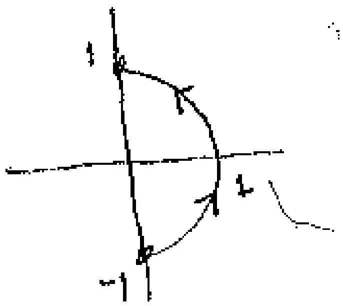
$$= \frac{1}{4}\left(\frac{8}{5}\sqrt{2} - \frac{4}{3}\sqrt{2} + \frac{2}{3} - \frac{2}{5}\right) =$$

$$= \frac{1}{4}\left(\frac{24\sqrt{2} - 20\sqrt{2} + 10 - 6}{15}\right) =$$

$$= \frac{1}{4}\left(\frac{4\sqrt{2}+4}{15}\right) \dots \boxed{= \frac{\sqrt{2}+1}{15}}\,.$$

3. b) Sebenta pg - 137

~~3. c)~~ $\int_C (2+x^2y)\, dS$, onde C é a parte da circunferência $x^2+y^2=1$, para $x \geq 0$, percorrida no sentido anti-horário (sentido direto).



$$\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

$$\alpha'(\theta) = (-\sin\theta, \cos\theta)$$

$$\|\alpha'(\theta)\| = \sqrt{(-\sin\theta)^2 + \cos^2\theta} = 1$$

$$\int_C (2+x^2y)\, dS = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2 + \cos^2\theta \sin\theta \, d\theta =$$

$$= 2\left[\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{- \frac{1}{3}\left[\cos^3\theta\right]_{-\frac{\pi}{2}}^{\pi/2}}_{=0} = 2\pi$$

3. d) $\int_{\mathcal{C}} x^2+y^2-z\,dS$, em que $\mathcal{C}$ é a hélice dada pelas equações paramétricas:

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases} \quad (t \in [0, 2\pi])$$

do ponto $P(1,0,0)$ até $Q(1,0,2\pi)$.

- $\alpha'(t) = (-\sin t, \cos t, 1)$
- $\|\alpha'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} = \sqrt{2}$.
- $\int_{\mathcal{C}} x^2+y^2-z\,dS = \int_0^{2\pi} (\underbrace{\cos^2 t + \sin^2 t}_{=1} - t)\sqrt{2}\,dt =$

$$= \sqrt{2}\left([t]_0^{2\pi} - \frac{1}{2}[t^2]_0^{2\pi} \right) = \sqrt{2}(2\pi - 2\pi^2) =$$

$$= 2\sqrt{2}(\pi - \pi^2).$$

5) Obs: Para um fio com a forma do gráfico de $\alpha:[a,b] \to \mathbb{R}^n$ com densidade $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$, a massa total é:

$$M = \int_C f \, dS \,;$$

as coordenadas $(\bar{x}, \bar{y})$ do centro de massa são dadas por

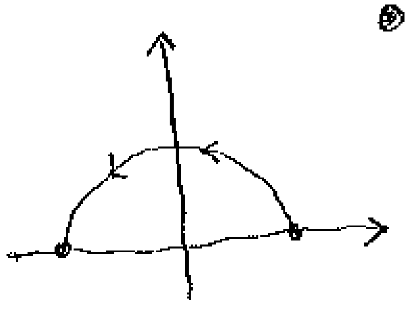
$$\bar{x} = \frac{1}{M} \int_C x f \, dS \quad \text{e} \quad \bar{y} = \frac{1}{M} \int_C y f \, dS \,,$$

e os momentos de inércia são:

$$I_x = \int_C y^2 f \, dS \,, \quad I_y = \int_C x^2 f \, dS \,.$$

— // —

- Considere um fio homogêneo de forma semi-circular de raio r.



- $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \ \theta \in [0, \pi]$

- $f(x,y) = K$, com K constante.

- $\|\alpha'(\theta)\| = r$.

~~5a)~~ Prove que o centro de massa está situado sobre o eixo de simetria, a uma distância de $\frac{2r}{\pi}$ do centro (origem).

- $M = \int_C f\,dS = \int_a^b f(\alpha(\theta))\,\|\alpha'(\theta)\|\,d\theta =$

$$= \int_0^{\pi} k r \, d\theta = \pi k r$$

- $\bar{X} = \frac{1}{M}\int_C X f\,dS = \frac{1}{\pi \cancel{k} r}\int_0^{\pi} r\cos(\theta)\,\cancel{k}\, r\, d\theta =$

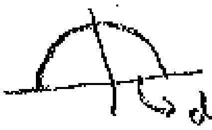
$$= \frac{r}{\pi}\left[\operatorname{sen}\theta\right]_0^{\pi} = 0$$

- $\bar{Y} = \frac{1}{M}\int_C Y f\,dS = \frac{1}{\pi \cancel{k} \cancel{r}}\int_0^{\pi} r\,\operatorname{sen}(\theta)\,\cancel{k}\, r\, d\theta =$

$$= \frac{r}{\pi}\left[-\cos\theta\right]_0^{\pi} = \frac{2r}{\pi};$$

$\therefore\ (\bar{X}, \bar{Y}) = \left(0, \frac{2r}{\pi}\right)$

5.b) Mostre que o momento de inércia em relação ao diâmetro que passa pelas extremidades do fio é $\frac{M}{2} r^2$, onde M é a massa do fio.



- $M = \pi k r$
- $I_x = \int_C y^2 \rho \, dS = \int_0^{\pi} r^2 \sin^2(\theta) k r \, d\theta =$

$$= r^3 k \left[\frac{\theta - \cos\theta \sin\theta}{2} \right]_0^{\pi} =$$

$$= \frac{\pi}{2} k r^3 = \frac{M}{2} r^2.$$

- $\int u' v = u v - \int u v'$

$\int \sin^2\theta = -\cos\theta \sin\theta + \int \cos^2\theta =$

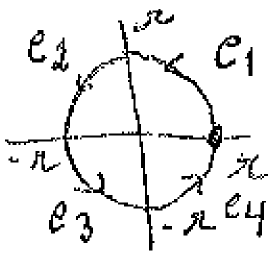
$\begin{cases} u' = \sin\theta \\ v = \sin\theta \end{cases} \Rightarrow \begin{cases} u = -\cos\theta \\ v' = \cos\theta \end{cases}$

$= -\cos\theta \sin\theta + \int (1 - \sin^2\theta) =$

$= \dfrac{-\cos\theta \sin\theta + \theta}{2}.$

~~6)~~ Um fio tem a forma circular de raio r.

Determine a sua massa e o momento de inércia em relação ao diâmetro, se a densidade é $f(x,y) = |x|+|y|$.



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad \theta \in [0, 2\pi]$$

$$\|\alpha'(\theta)\| = r$$

$$f(x,y) = \begin{cases} x+y, & x,y \geq 0 \\ -x+y, & x \leq 0, y \geq 0 \\ -x-y, & x,y \leq 0 \\ x-y, & x \geq 0, y \leq 0 \end{cases}.$$

Usar a simetria da densidade e da circunferência

$$M = \int_C f\,dS = \int_{C_1} x+y\,dS + \int_{C_2} y-x\,dS + \int_{C_3} -x-y\,dS + \int_{C_4} x-y\,dS =$$

$$= 4\left(\int_0^{\frac{\pi}{2}} r(\cos\theta+\sin\theta)r\,d\theta\right) + \int_{\frac{\pi}{2}}^{\pi} r^2(\cos\theta-\sin\theta)\,d\theta + \int_{\pi}^{\frac{3}{4}\pi} r^2(-\cos\theta-\sin\theta)\,d\theta +$$

$$+ \int_{\frac{3}{4}\pi}^{2\pi} r^2(\cos\theta-\sin\theta)\,d\theta = \cdots$$

$$= 4r^2\left(\left[\sin\theta-\cos\theta\right]_0^{\frac{\pi}{2}}\right) + \left[-\cos\theta-\sin\theta\right]_{\frac{\pi}{2}}^{\pi} + \left[-\sin\theta+\cos\theta\right]_{\pi}^{\frac{3}{4}\pi} + \left[\sin\theta+\cos\theta\right]_{\frac{3}{4}\pi}^{2\pi}\Big) =$$

$$= 4r^2(2) + 2 + 2 + 2) = \boxed{8r^2.}$$

$$I_x = \int_C y^2 f\,ds = \int_C y^2(|x|+|y|)\,ds =$$

$$= 4\int_{C_1} y^2x + y^3\,ds = 4\int_0^{\frac{\pi}{2}} (r^2\sin^2\theta\, r\cos\theta + r^3\sin^3\theta)\, r\,d\theta =$$

$$= 4r^4 \int_0^{\frac{\pi}{2}} \sin^2\theta\cos\theta + \sin^3\theta\,d\theta =$$

$$= 4r^4\left[\frac{\sin^3\theta}{3} - \cos\theta\sin^2\theta - \frac{2}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}} =$$

$$= 4r^4\left(\frac{1}{3}+\frac{2}{3}\right) = 4r^4 .$$

$$\int u'v = uv - \int uv'$$

$$\int \sin^3\theta = -\cos\theta\sin^2\theta + 2\int\cos^2\theta\sin\theta =$$

$$\begin{cases} u' = \sin\theta \\ v = \sin^2\theta \end{cases} \Rightarrow \begin{cases} u = -\cos\theta \\ v' = 2\sin\theta\cos\theta \end{cases}$$

$$= -\cos\theta\sin^2\theta - \frac{2}{3}\cos^3\theta$$

7) Verifique se as funções indicadas são, ou não, um gradiente e, em caso afirmativo, determine a função potencial.

Obs: Se $f=(f_1,\dots,f_n)$ é um gradiente, então

$$\frac{\partial f_i}{\partial x_j}=\frac{\partial f_j}{\partial x_i}\ ,\quad i,j\in\{1,\dots,n\}.$$

- $\boxed{\text{Em } \mathbb{R}^2:\ f=(f_1,f_2),\ \dfrac{\partial f_1}{\partial y}=\dfrac{\partial f_2}{\partial x}}$

- Em $\mathbb{R}^3$: $f=(f_1,f_2,f_3)$:

$$\begin{cases}\dfrac{\partial f_1}{\partial y}=\dfrac{\partial f_2}{\partial x}\\[2ex]\dfrac{\partial f_1}{\partial z}=\dfrac{\partial f_3}{\partial x}\\[2ex]\dfrac{\partial f_2}{\partial z}=\dfrac{\partial f_3}{\partial y}\end{cases}$$

7.a) $f(x,y) = x\vec{e}_1 + y\vec{e}_2 = (x,y)$

$\bullet \ \frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \iff 0 = 0$ ✓

Queremos $\varphi : \mathbb{R}^2 \to \mathbb{R}$ tal que $\nabla\varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) = (x,y)$:

$\bullet \ \frac{\partial \varphi}{\partial x} = x \Rightarrow \varphi(x,y) = \frac{x^2}{2} + g(y)$

$\bullet \ \frac{\partial \varphi}{\partial y} = g'(y) = y \Rightarrow g'(y) = \frac{y^2}{2} + k, \ k \in \mathbb{R}$

$\therefore \ \varphi(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + k, \ k \in \mathbb{R}.$

~~7.b)~~ $g(x,y) = (3x^2y, x^3)$

$\bullet \ \frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \iff 3x^2 = 3x^2$ ✓

$\bullet \ \frac{\partial \varphi}{\partial x} = 3x^2y \Rightarrow \varphi = x^3y + g(y)$

$\bullet \ \frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y}\left(x^3y + g(y)\right) = \cancel{x^3} + g'(y) = \cancel{x^3} \Rightarrow g'(y) = 0 \Rightarrow g(y) = k$

$\therefore \ \varphi(x,y) = x^3y + k, \ k \in \mathbb{R}.$

$\cancel{7.c)}$ $h(x,y,z) = (x+z, -y-z, x-y)$

$\dfrac{\partial f_1}{\partial y} = \dfrac{\partial f_2}{\partial x} \Leftrightarrow 0 = 0$ ✓

$\dfrac{\partial f_1}{\partial z} = \dfrac{\partial f_3}{\partial x} \Leftrightarrow 1 = 1$ ✓

$\dfrac{\partial f_2}{\partial z} = \dfrac{\partial f_3}{\partial y} \Leftrightarrow -1 - 1$ ✓

- $\dfrac{\partial \varphi}{\partial x} = x + z \Rightarrow \varphi = \dfrac{x^2}{2} + xz + g(y,z)$

- $\dfrac{\partial \varphi}{\partial y} = \dfrac{\partial g}{\partial y}(y,z) = -y - z \Rightarrow g(y,z) = \dfrac{-y^2}{2} - yz + h(z)$

$\therefore \varphi(x,y,z) = \dfrac{x^2}{2} + xz - \dfrac{y^2}{2} - yz + h(z)$

- $\dfrac{\partial \varphi}{\partial z} = x - y + h'(z) = x - y \Rightarrow h'(z) = 0 \Rightarrow h(z) = K, K \in \mathbb{R}.$

$\therefore \varphi(x,y,z) = \dfrac{x^2}{2} + xz - \dfrac{y^2}{2} - yz + K, K \in \mathbb{R}.$

7.d) $f = (4xy - 3x^2z^2 + 1,\ 2x^2 + 2,\ -2x^3z - 3z^2)$

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \Leftrightarrow 4x = 4x \checkmark$$

$$\frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x} \Leftrightarrow -6x^2z = -6x^2z \checkmark$$

$$\frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} \Leftrightarrow 0 = 0 \checkmark$$

- $\frac{\partial \varphi}{\partial x} = 4\cancel{x}y - 3x^2z^2 + 1 \Rightarrow \varphi = 2x^2y - x^3z^2 + x + g(y,z)$

- $\frac{\partial \varphi}{\partial y} = 2x^2 + \frac{\partial g}{\partial y} = 2x^2 + 2 \Rightarrow \frac{\partial g}{\partial y} = 2 \Rightarrow g = 2y + h(z)$

- $\varphi(x,y,z) = 2x^2y - x^3z^2 + x + 2y + h(z)$

- $\frac{\partial \varphi}{\partial z} = \cancel{-2x^3z} + h'(z) = \cancel{-2x^3z} - 3z^2 \Rightarrow h(z) = -z^3 + k,\ k \in \mathbb{R}$

$\therefore\ \varphi(x,y,z) = 2x^2y - x^3z^2 + x + 2y - z^3 + k,\ k \in \mathbb{R}.$