11-17-a

4.13 Exercícios (pg 179) (continuação).

5) Calcule o momento de inércia de uma placa delgada S no plano XoY, limitada pelas curvas definidas pelas equações abaixo, representando por f(x, y) a densidade de S num ponto arbitrário (x, y):

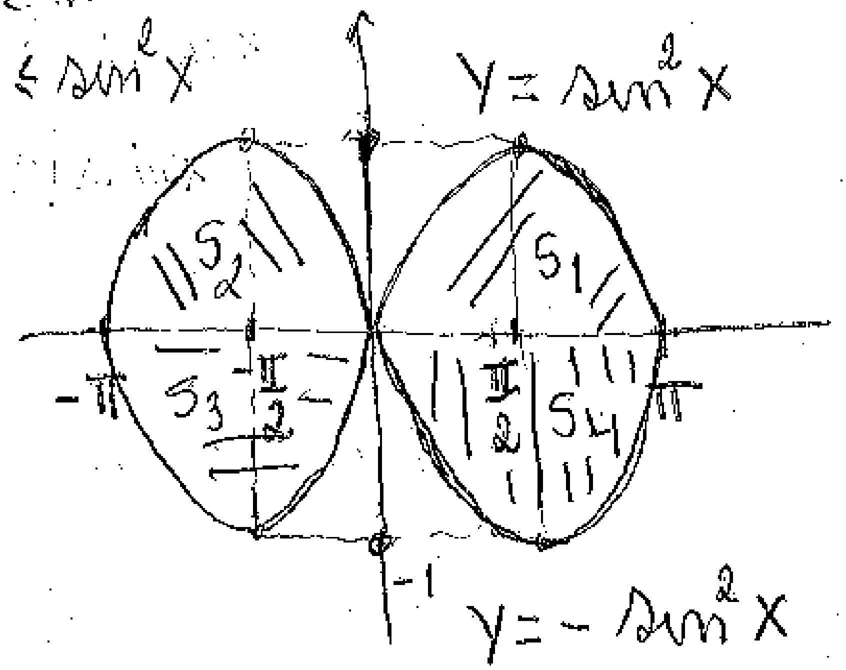
Obs: O momento polar de inércia é:

$$I_0 = I_x + I_y = \iint_S (x^2+y^2) f \, dA.$$

~~5.~~ a) $y = \sin^2 x$, $y = -\sin^2 x$, $-\pi \leq x \leq \pi$

e a densidade é $f(x,y) \equiv 1$.

$S_1 : \begin{cases} 0 \leq x \leq \pi \\ 0 \leq y \leq \sin^2 x \end{cases}$



Vamos usar a simetria da placa e a da densidade, assim:

$$I_0 = 4 \iint_{S_1} (x^2 + y^2)\, f\, dA = 4 \int_0^{\pi} \int_0^{\sin^2 x} x^2 + y^2 \, dy\, dx =$$

$$= 4 \int_0^{\pi} \underbrace{x^2 \sin^2 x}_{\text{Por partes}} + \underbrace{\frac{\sin^6 x}{3}}_{\text{Complicado}} \, dx = \dots = \frac{8\pi^3 - 7\pi}{48}$$

11-18-b

Problem:

$$\boxed{\int x^2 \sin^2(x)\,dx}$$

Apply product-to-sum formulas:

$$\sin(x)\sin(y) = \tfrac{1}{2}(\cos(y-x) - \cos(y+x)),\ \boxed{\sin^2(x) = \tfrac{1}{2}(1 - \cos(2x)),}$$
$$\cos(x)\cos(y) = \tfrac{1}{2}(\cos(y+x) + \cos(y-x)),\ \cos^2(x) = \tfrac{1}{2}(\cos(2x) + 1),$$
$$\sin(x)\cos(y) = \tfrac{1}{2}(\sin(y+x) - \sin(y-x)),\ \cos(x)\sin(x) = \tfrac{1}{2}\sin(2x)$$

$$= \int x^2 \left(\frac{1}{2} - \frac{\cos(2x)}{2}\right) dx$$

Expand:

$$= \int \left(\frac{x^2}{2} - \frac{x^2\cos(2x)}{2}\right) dx$$

Apply linearity:

$$\frac{1}{2}\int x^2\,dx - \frac{1}{2}\int x^2\cos(2x)\,dx$$

Now solving:

$$\int x^2\,dx$$

Apply power rule:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \text{ with } n = 2:$$

$$= \frac{x^3}{3}$$

Now solving:

$$\int x^2\cos(2x)\,dx$$

11-18-e

Integrate by parts: $\int fg' = fg - \int f'g$

$$f = x^2, \quad g' = \cos(2x)$$
$$\downarrow \underline{\text{steps}} \qquad \downarrow \underline{\text{steps}}$$
$$f' = 2x, \quad g = \frac{\sin(2x)}{2}:$$

$$= \frac{x^2 \sin(2x)}{2} - \int x \sin(2x)\,dx$$

Now solving:

$$\int x \sin(2x)\,dx$$

Integrate by parts: $\int fg' = fg - \int f'g$

$$f = x, \quad g' = \sin(2x)$$
$$\downarrow \underline{\text{steps}} \qquad \downarrow \underline{\text{steps}}$$
$$f' = 1, \quad g = -\frac{\cos(2x)}{2}:$$

$$= -\frac{x\cos(2x)}{2} - \int -\frac{\cos(2x)}{2}\,dx$$

Now solving:

$$\int -\frac{\cos(2x)}{2}\,dx$$

Substitute $u = 2x \longrightarrow \frac{du}{dx} = 2$ (steps) $\longrightarrow dx = \frac{1}{2}\,du$:

$$= -\frac{1}{4}\int \cos(u)\,du$$

Now solving:

$$\int \cos(u)\,du$$

This is a standard integral:

$$= \sin(u)$$

11-18-d

Plug in solved integrals:

$$-\frac{1}{4}\int \cos(u)\,du$$

$$= -\frac{\sin(u)}{4}$$

Undo substitution $u = 2x$:

$$= -\frac{\sin(2x)}{4}$$

Plug in solved integrals:

$$-\frac{x\cos(2x)}{2} - \int -\frac{\cos(2x)}{2}\,dx$$

$$= \frac{\sin(2x)}{4} - \frac{x\cos(2x)}{2}$$

Plug in solved integrals:

$$\frac{x^2\sin(2x)}{2} - \int x\sin(2x)\,dx$$

$$= \frac{x^2\sin(2x)}{2} - \frac{\sin(2x)}{4} + \frac{x\cos(2x)}{2}$$

Plug in solved integrals:

$$\frac{1}{2}\int x^2\,dx - \frac{1}{2}\int x^2\cos(2x)\,dx$$

$$= -\frac{x^2\sin(2x)}{4} + \frac{\sin(2x)}{8} - \frac{x\cos(2x)}{4} + \frac{x^3}{6}$$

The problem is solved:

$$\int x^2 \sin^2(x)\,dx$$

11-18-l

$$= -\frac{x^2 \sin(2x)}{4} + \frac{\sin(2x)}{8} - \frac{x\cos(2x)}{4} + \frac{x^3}{6} + C$$

Rewrite/simplify:

$$= -\frac{(6x^2-3)\sin(2x) + 6x\cos(2x) - 4x^3}{24} + C$$

ANTIDERIVATIVE COMPUTED BY MAXIMA:

$\int f(x)\,dx \quad F(x) =$

$$-\frac{(6x^2-3)\sin(2x) + 6x\cos(2x) - 4x^3}{24} + C$$

No further simplification found!

DEFINITE INTEGRAL:

$\int_0^{\pi} f(x)\,dx$

$$\frac{2\pi^3 - 3\pi}{12}$$

Simplify/rewrite:

$$\frac{\pi^3}{6} - \frac{\pi}{4}$$

11-18-B

Problem:

$$\boxed{\int \sin^6(x)\,dx}$$

Pode se feito por partes, ver página 11-29.

Apply reduction formula:

$$\int \sin^n(x)\,dx = \frac{n-1}{n}\int \sin^{n-2}(x)\,dx - \frac{\cos(x)\sin^{n-1}(x)}{n}$$

with $n = 6$:

$$= -\frac{\cos(x)\sin^5(x)}{6} + \frac{5}{6}\int \sin^4(x)\,dx$$

... or choose an alternative:

Apply product-to-sum formulas

Now solving:

$$\int \sin^4(x)\,dx$$

Apply the last reduction formula again with $n = 4$:

$$= -\frac{\cos(x)\sin^3(x)}{4} + \frac{3}{4}\int \sin^2(x)\,dx$$

... or choose an alternative:

Apply product-to-sum formulas

Now solving:

$$\int \sin^2(x)\,dx$$ pode ser feito por partes.

Apply the last reduction formula again with $n = 2$:

11-18-9

$$= -\frac{\cos(x)\sin(x)}{2} + \frac{1}{2}\int 1\,dx$$

... or choose an alternative:

Apply product-to-sum formulas

Now solving:

$$\int 1\,dx$$

Apply constant rule:

$$= x$$

Plug in solved integrals:

$$-\frac{\cos(x)\sin(x)}{2} + \frac{1}{2}\int 1\,dx$$

$$= \frac{x}{2} - \frac{\cos(x)\sin(x)}{2}$$

Plug in solved integrals:

$$-\frac{\cos(x)\sin^3(x)}{4} + \frac{3}{4}\int \sin^2(x)\,dx$$

$$= -\frac{\cos(x)\sin^3(x)}{4} - \frac{3\cos(x)\sin(x)}{8} + \frac{3x}{8}$$

Plug in solved integrals:

$$-\frac{\cos(x)\sin^5(x)}{6} + \frac{5}{6}\int \sin^4(x)\,dx$$

11-18-21

$$= -\frac{\cos(x)\sin^5(x)}{6} - \frac{5\cos(x)\sin^3(x)}{24} - \frac{5\cos(x)\sin(x)}{16} + \frac{5x}{16}$$

The problem is solved:

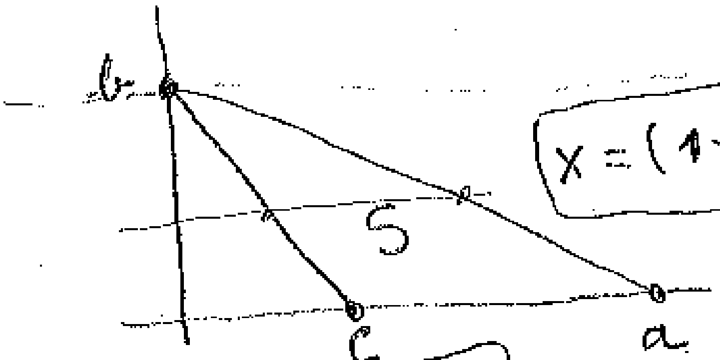
$$\int \sin^6(x)\,dx$$

$$= -\frac{\cos(x)\sin^5(x)}{6} - \frac{5\cos(x)\sin^3(x)}{24} - \frac{5\cos(x)\sin(x)}{16} + \frac{5x}{16} + C$$

Rewrite/simplify:

$$= -\frac{\sin(6x) - 9\sin(4x) + 45\sin(2x) - 60x}{192} + C$$

$\cancel{5\text{-}b)}$ $\frac{x}{a}+\frac{y}{b}=1$, $\frac{x}{c}+\frac{y}{b}=1$, $y=0$, $0<c<a$, $b>0$, $f(x,y)=1$ $\boxed{11\text{-}19}$



$\boxed{x=\left(1-\frac{y}{b}\right)a}$

$\boxed{x=\left(1-\frac{y}{b}\right)c}$

$S=\left\{(x,y) : 0\le y\le b ,\ \left(1-\frac{y}{b}\right)c\le x\le\left(1-\frac{y}{b}\right)a\right\}$

- Vamos fazer $\boxed{a=2,\ b,c=1}$

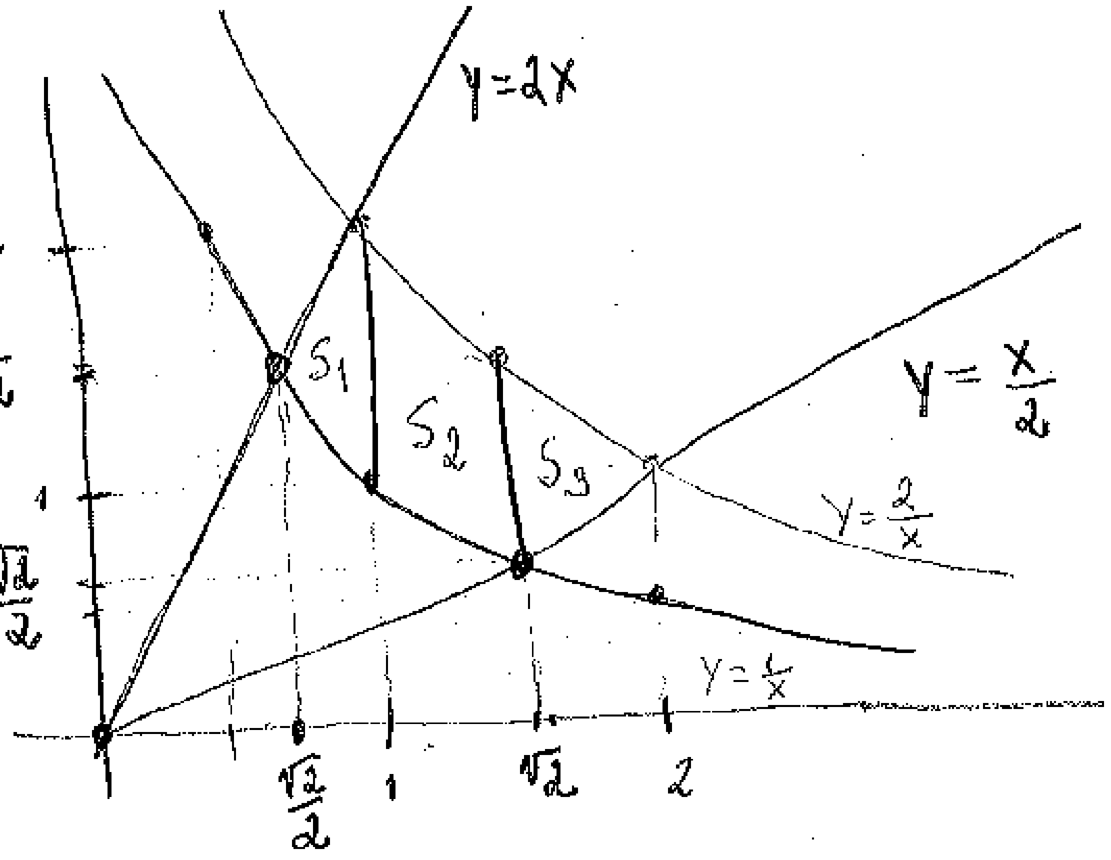
$S=\{(x,y) : 0\le y\le 1 ,\ 1-y\le x\le 2-2y\}$

$$I_0=\int_0^1\int_{1-y}^{2-2y} x^2+y^2\,dx\,dy=\frac{1}{3}\int_0^1 -10y^3+24y^2-21y+7\,dy=$$

$$=\frac{1}{3}\left[-\frac{5}{2}\left[y^4\right]_0^1+8\left[y^3\right]_0^1-\frac{21}{2}\left[y^2\right]_0^1+7\left[y\right]_0^1\right]=$$

$$=\frac{1}{3}\left(-\frac{5}{2}+8-\frac{21}{2}+7\right)=\frac{1}{3}(15-13)=\frac{2}{3}$$

5\. c) $xy=1$, $xy=2$, $x=2y$, $y=2x$, $x,y\geq 0$, $f(x,y)=1$



$$\cap\begin{cases} xy=1 \\ y=2x \end{cases} \Leftrightarrow \begin{cases} 2x^2=1 \\ - \end{cases} \Leftrightarrow \begin{cases} x=\frac{\sqrt{2}}{2} \\ y=\sqrt{2} \end{cases}$$

$$\cap\begin{cases} xy=1 \\ 2y=x \end{cases} \Leftrightarrow \begin{cases} 2y^2=1 \\ \end{cases} \Leftrightarrow \begin{cases} y=\frac{\sqrt{2}}{2} \\ x=\sqrt{2} \end{cases}$$

$$I_0=\underbrace{\int_{\sqrt{2}}^{1}\int_{\frac{1}{x}}^{2x} x^2+y^2\,dy\,dx}_{I\ 1}+\underbrace{\int_{1}^{\sqrt{2}}\int_{\frac{1}{x}}^{\frac{2}{x}} x^2+y^2\,dy\,dx}_{I\ 2}+\underbrace{\int_{\sqrt{2}}^{2}\int_{\frac{x}{2}}^{\frac{2}{x}} x^2+y^2\,dy\,dx}_{I\ 3}=\dots$$

contas (s.c) $\boxed{11\text{-}21}$

$$I_{1} = \int_{\sqrt{2}}^{1} \left[y x^2 \right]_{y=\frac{1}{x}}^{y=2x} + \left[\frac{y^3}{3} \right]_{y=\frac{1}{x}}^{y=2x} dx =$$

$$= \int_{\sqrt{2}}^{1} \frac{14}{3}x^3 - \frac{1}{3x^3} - x \, dx = \frac{1}{6}\left[7x^4 - 3x^2 + \frac{1}{x^2} \right]_{x=\sqrt{2}}^{x=1}$$

$$= \boxed{-\frac{35}{12}}.$$

$$I_{2} = \int_{1}^{\sqrt{2}} \left[x^2 y + \frac{y^3}{3} \right]_{y=\frac{1}{x}}^{y=\frac{2}{x}} dx = \int_{1}^{\sqrt{2}} x + \frac{7}{3x^3} \, dx =$$

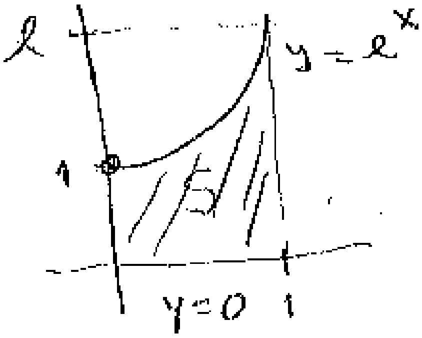
$$= \left[\frac{x^2}{2} - \frac{7}{6x^2} \right]_{x=1}^{x=\sqrt{2}} = \boxed{\frac{13}{12}}$$

$$I_{3} = \int_{\sqrt{2}}^{2} \left[x^2 y + \frac{y^3}{3} \right]_{y=\frac{x}{2}}^{y=\frac{2}{3}} dx = \int_{\sqrt{2}}^{2} -\frac{13}{24}x^3 + 2x + \frac{8}{3x^3} dx$$

$$= \left[-\frac{13}{96}x^4 + x^2 - \frac{4}{3}x^2 \right]_{x=\sqrt{2}}^{x=2} = \boxed{\frac{17}{24}}$$

$$I_0 = -\frac{70}{24} + \frac{26}{24} + \frac{17}{24} = \boxed{-\frac{29}{24}}$$

$\boxed{5.d}$ $y=e^x,\ y=0,\ 0\le x\le 1$ e $f(x,y)=xy$ $\boxed{11\text{-}22}$



$S=\{(x,y): 0\le x\le 1,\ 0\le y\le e^x\}$

$$I_0=\int_0^1\int_0^{e^x}(x^2+y^2)xy\,dy\,dx=\int_0^1\int_0^{e^x}x^3y+xy^3\,dy\,dx=$$

$$=\int_0^1 \frac{x^3}{2}\Big[y^2\Big]_{y=0}^{y=e^x}+\frac{x}{4}\Big[y^4\Big]_{y=0}^{y=e^x}\,dx=$$

$$=\underbrace{\frac{1}{2}\int_0^1 x^3e^{2x}\,dx}_{(**)}+\underbrace{\frac{1}{4}\int_0^1 x e^{4x}\,dx}_{(*)}=\frac{3e^4+4e^2+13}{4}$$

$$(*)\ \frac{1}{4}\int_0^1 x e^{4x}\,dx=\frac{1}{4}\ =\frac{1}{4}\left(\Big[\frac{x}{4}e^{4x}\Big]_0^1-\frac{1}{4}\int_0^1 e^{4x}\,dx\right)=$$

$$\boxed{\begin{cases}u'=e^{4x}\\ v=x\end{cases}\Rightarrow\begin{cases}u=\frac{e^{4x}}{4}\\ v'=1\end{cases}}$$

$$=\frac{1}{4}\left(\frac{e^4}{4}-\frac{1}{16}\Big[e^{4x}\Big]_0^1\right)=\frac{1}{4}\left(\frac{4e^4}{16}-\frac{e^4}{16}+\frac{1}{16}\right)=$$

$$=\frac{3e^4+1}{64}$$

44

$\boxed{11\text{-}23}$

$$P x^3 e^{2x} = \frac{x^3}{2} e^{2x} - \frac{3}{2} P x^2 e^{2x} =$$

$$\begin{cases} u' = e^{2x} \\ v = x^3 \end{cases} \Rightarrow \begin{cases} u = \frac{e^{2x}}{2} \\ v' = 3x^2 \end{cases} , \quad \begin{cases} u' = e^{2x} \\ v = x^2 \end{cases} \Rightarrow \begin{cases} u = \frac{e^{2x}}{2} \\ v = 2x \end{cases}$$

$$= \frac{x^3}{2} e^{2x} - \frac{3}{2} \left(\frac{x^2}{2} e^{2x} - P x e^{2x} \right) =$$

$$= \frac{x^3}{2} e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{3}{2} P x e^{2x} =$$

$$\begin{cases} u' = e^{2x} \\ v = x \end{cases} \Rightarrow \begin{cases} u = \frac{e^{2x}}{2} \\ v = 1 \end{cases}$$

$$= \frac{x^3}{2} e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{3}{4} x e^{2x} - \frac{3}{4} P e^{2x} =$$

$$= \frac{x^3}{2} e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{3}{4} x e^{2x} - \frac{3}{8} e^{2x}$$

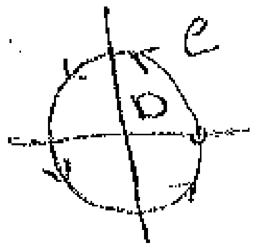
$$\frac{1}{2} \int_0^1 x^3 e^{2x} \, dx = \frac{1}{2} \left[\frac{4x^3 e^{2x} - 6x^2 e^{2x} + 6x e^{2x} - 3 e^{2x}}{8} \right]_0^1 =$$

$$= \frac{1}{2} \left(\frac{4e^2 - \cancel{6e^2} + \cancel{6e^2} - 3e^2 + 3}{8} \right) = \frac{e^2 + 3}{16} .$$

$$\therefore \frac{4e^2 + 12}{64} + \frac{3e^4 + 1}{64} = \frac{3e^4 + 4e^2 + 13}{64}$$

6) Aplicar o teorema de Green para calcular o integral de linha

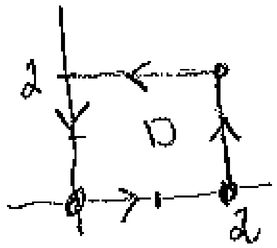
$$\oint_C y^2\,dx + x\,dy$$



quando:

Obs: $\oint_C P\,dx + \varphi\,dy = \iint_D \frac{\partial \varphi}{\partial x} - \frac{\partial P}{\partial y}\,dA$

6.a) C é o quadrado com vértices (0,0), (2,0), (2,2) e (0,2);



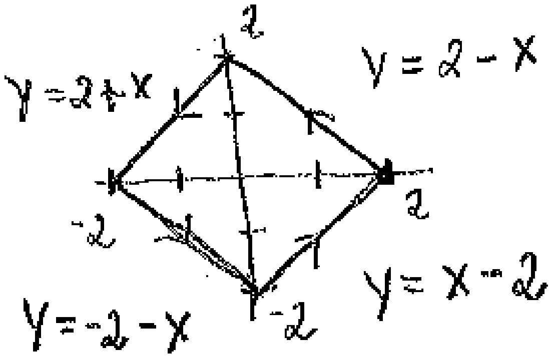
$P = y^2 \ , \ \frac{\partial P}{\partial y} = 2y$; $\boxed{\frac{\partial \varphi}{\partial x} - \frac{\partial P}{\partial y} = 1 - 2y}$

$\varphi = x \ , \ \frac{\partial \varphi}{\partial x} = 1$

$$\oint_C y^2\,dx + x\,dy = \int_0^2 \int_0^2 1 - 2y\,dx\,dy = \int_0^2 (1-2y)\left[x\right]_{x=0}^{x=2} dy =$$

$$= 2\int_0^2 1 - 2y\,dy = 2\left[y\right]_0^2 - 2\left[y^2\right]_0^2 = 4 - 8 = -4.$$

6.b) C é o quadrado ~~da~~ com vértices $(\pm 2, 0)$ e $(0, \pm 2)$.



$$\boxed{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1-2y}$$

$$\oint_C y^2dx + x\,dy = \int_{-2}^{0}\int_{-2-x}^{2+x} 1-2y\,dy\,dx + \int_{0}^{2}\int_{x-2}^{2-x} 1-2y\,dy\,dx =$$

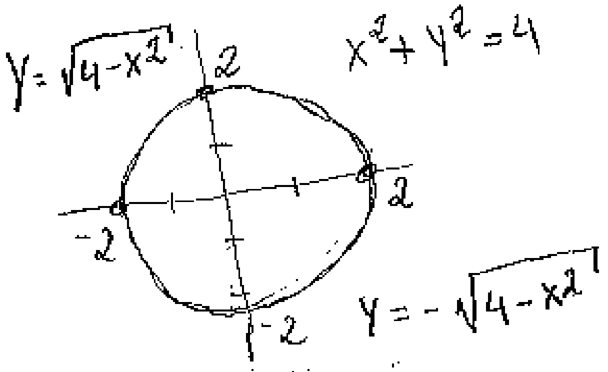
$$= \int_{-2}^{0} [y]_{y=-2-x}^{y=2+x} - \underbrace{[y^2]_{y=-2-x}^{y=2+x}}_{=0} dx + \int_{0}^{2} [y]_{y=x-2}^{y=2-x} - \underbrace{[y^2]_{y=x-2}^{y=2-x}}_{=0} dx =$$

$$= \int_{-2}^{0} 4+2x\,dx + \int_{0}^{2} 4-2x\,dx =$$

$$= 4[x]_{-2}^{0} + [x^2]_{-2}^{0} + 4[x]_{0}^{2} - [x^2]_{0}^{2} =$$

$$= 8-4+8-4 = 8$$

~~6.c)~~ C é a circunferência de raio 2 e centro na origem.



$$\boxed{\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=1-2y}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\oint_C y^2\,dx + x\,dy = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1-2y\;dy\,dx =$$

$$=\int_0^2\int_0^{2\pi}(1-2r\sin\theta)\,r\,d\theta\,dr = \int_0^2\int_0^{2\pi} r-2r^2\sin\theta\,d\theta\,dr =$$

$$=\int_0^2\Big[r\theta+\underbrace{2r\cos\theta}_{=0}\Big]_{\theta=0}^{\theta=2\pi}dr = \pi\int_0^2 2r\,dr =$$

$$=\pi\left[r^2\right]_0^2 = 4\pi$$

7) Calcular os integrais, passando para coordenadas polares.

7.a) Obs: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \text{arctg}\left(\frac{y}{x}\right) \end{cases}$.

$$\iint_{R_{xy}} f(x,y)\, dx\,dy = \iint_{R_{r\theta}} f(r\cos\theta, r\sin\theta)\, \underset{\sim}{r}\, dr\, d\theta.$$

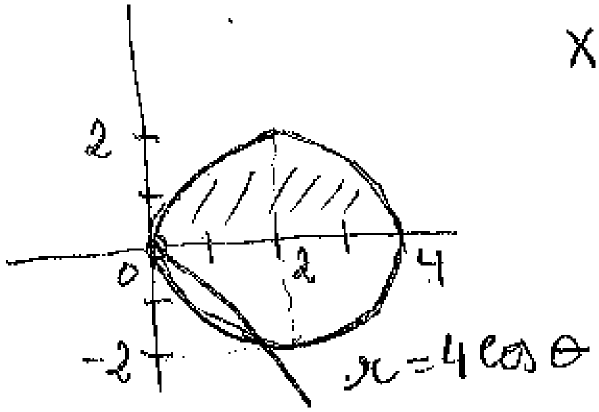
importante

~~7.a)~~ $\displaystyle\int_0^4 \int_0^{\sqrt{4x-x^2}} x^2+y^2\, dy\, dx. = *$

Ver primeiro pag 11-31

$y = \sqrt{4x-x^2} \Rightarrow y^2 = 4x - x^2 \Leftrightarrow x^2-4x+4+y^2 = 4 \Leftrightarrow$

$\Leftrightarrow (x-2)^2 + y^2 = 4$



$x^2+y^2 = 4x \Leftrightarrow r^2 = 4r\cos\theta \Leftrightarrow$

$\Leftrightarrow r = 4\cos\theta$

$\begin{cases} \cancel{-\frac{\pi}{2}}\ 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 4\cos\theta \end{cases}$

$$* = \int_{\cancel{-\frac{\pi}{2}}\ 0}^{\frac{\pi}{2}} \int_0^{4\cos\theta} r^2 \cdot r\, dr\, d\theta = 64 \int_{\cancel{-\frac{\pi}{2}}\ 0}^{\frac{\pi}{2}} \cos^4\theta\, d\theta = 24\pi.$$

$$\circledast \; \int \cos^4\theta = \cos^3\theta \sin\theta + 3\int \cos^2\theta \sin^2\theta =$$

$$\begin{cases} u' = \cos\theta \\ v = \cos^3\theta \end{cases} \Rightarrow \begin{cases} u = \sin\theta \\ v' = -3\cos^2\theta \sin\theta \end{cases}$$

$$= \cos^3\theta \sin\theta + 3\int \cos^2\theta (1 - \overset{\cos^2\theta}{\cancel{\sin^2\theta}}) =$$

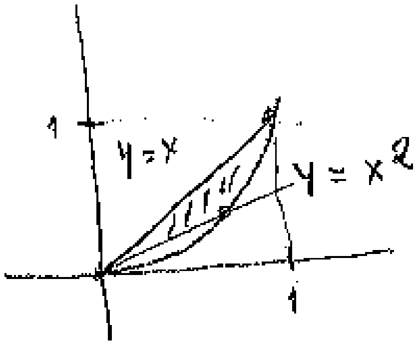
$$= \cos^3\theta \sin\theta + 3\int \cos^2\theta - 3\int \cos^4\theta =$$

$$= \cos^3\theta \sin\theta + \frac{3}{2}(\theta + \cos\theta \sin\theta) - 3\int \cos^4\theta =$$

$$= \frac{2\cos^3\theta \sin\theta + 3\theta + 3\cos\theta \sin\theta}{8}$$

$\boxed{11-30}$

~~Ex 2)~~ $\displaystyle\int_0^1 \int_{x^2}^{x} \sqrt{x^2+y^2}\, dy\, dx = *$



$y = x^2 \Leftrightarrow r \sin\theta = r^2 \cos^2\theta \Leftrightarrow$

$\Leftrightarrow r = \dfrac{\sin\theta}{\cos^2\theta} = \sin\theta \sec^2\theta$

$$\begin{cases} 0 \leq \theta \leq \frac{\pi}{4} \\ 0 \leq r \leq \sin\theta \sec^2\theta \end{cases}$$

$$* = \int_0^{\frac{\pi}{4}} \int_0^{\sin\theta\sec^2\theta} r^2\, dr\, d\theta = \frac{1}{3}\int_0^{\frac{\pi}{4}} \left[r^3\right]_{r=0}^{r=\frac{\sin\theta}{\cos^2\theta}} d\theta =$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{\sin^3\theta}{\cos^6\theta}\, d\theta = \frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{\sin^3\theta}{\cos^3\theta} \cdot \frac{1}{\cos^3\theta}\, d\theta =$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{\tan^3\theta}{\cos^3\theta}\, d\theta = \underbrace{\ldots}_{\text{Complicado}} =$$

11-30-a

Problem:

$$\int \frac{\tan^3(o)}{3\cos^3(o)}\,do$$

Apply linearity:

$$= \frac{1}{3}\int \frac{\tan^3(o)}{\cos^3(o)}\,do$$

Now solving:

$$\int \frac{\tan^3(o)}{\cos^3(o)}\,do$$

Prepare for substitution, use:

$$\cos(o) = \frac{1}{\sec(o)},$$

$$\tan^2(o) = \sec^2(o) - 1$$

$$= \int \sec^2(o)\left(\sec^2(o) - 1\right) \cdot \sec(o)\tan(o)\,do$$

Substitute $u = \sec(o) \longrightarrow \frac{du}{do} = \sec(o)\tan(o)$ (steps) $\longrightarrow do = \frac{1}{\sec(o)\tan(o)}\,du$:

$$= \int u^2\left(u^2 - 1\right)\,du$$

... or choose an alternative:

Substitute $\cos(o)$

Expand:

$$= \int \left(u^4 - u^2\right)\,du$$

Apply linearity:

$$= \int u^4\,du - \int u^2\,du$$

11-30-b

Now solving:

$$\int u^4 \, du$$

Apply power rule:

$$\int u^n \, du = \frac{u^{n+1}}{n+1} \quad \text{with } n = 4:$$

$$= \frac{u^5}{5}$$

Now solving:

$$\int u^2 \, du$$

Apply power rule with $n = 2$:

$$= \frac{u^3}{3}$$

Plug in solved integrals:

$$\int u^4 \, du - \int u^2 \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3}$$

Undo substitution $u = \sec(\theta)$:

$$= \frac{\sec^5(\theta)}{5} - \frac{\sec^3(\theta)}{3}$$

Plug in solved integrals:

$$\frac{1}{3} \int \frac{\tan^3(\theta)}{\cos^3(\theta)} \, d\theta$$

$$= \frac{\sec^5(\theta)}{15} - \frac{\sec^3(\theta)}{9}$$

11-30-C

The problem is solved:

$$\int \frac{\tan^3(o)}{3\cos^3(o)}\,do$$

$$= \frac{\sec^5(o)}{15} - \frac{\sec^3(o)}{9} + C$$

ANTIDERIVATIVE COMPUTED BY MAXIMA:

$\int f(o)\,do = F(o) =$

$$-\frac{5\cos^2(o) - 3}{45\cos^5(o)} + C$$

Simplify/rewrite:

$$\frac{5\sin^2(o) - 2}{45\cos^5(o)} + C$$

DEFINITE INTEGRAL:

$\int_0^{\frac{\pi}{4}} f(o)\,do =$

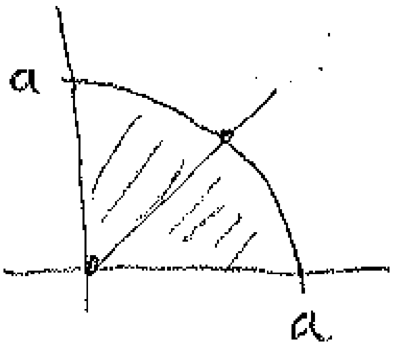
$$\frac{\frac{2^{\frac{5}{2}}}{5} - \frac{2^{\frac{3}{2}}}{3} + \frac{2}{15}}{3}$$

Simplify/rewrite:

$$\frac{2^{\frac{3}{2}} + 2}{45}$$

$\boxed{11-31}$

~~Fue)~~ $\displaystyle\int_0^a \int_0^{\sqrt{a-y^2}} (x^2+y^2)\,dx\,dy =$



$$\begin{cases} 0 \leq \theta \leq \frac{\pi}{2} \\ 0 \leq r \leq a \end{cases}$$

$$\int_0^a \int_0^{\sqrt{a-y^2}} (x^2+y^2)\,dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^a r^2 \cdot r \, dr \, d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4} \right]_{r=0}^{r=a} d\theta = \frac{a^4}{4} \int_0^{\frac{\pi}{2}} d\theta = \frac{a^4}{8} \pi .$$

~~8)~~ Considere a aplicação definida por:

$$x = \underbrace{u+v}_{4} \quad e \quad y = \underbrace{v-u}_{4} .$$

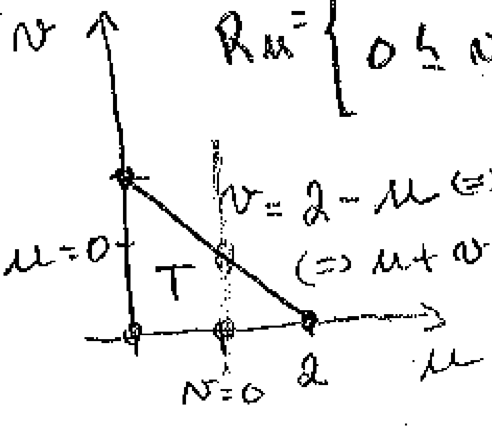
8.a) Calcule o jacobiano ~~da~~ $J(u,v)$

$$\left|\begin{matrix}\nabla\varphi \\ \nabla\psi\end{matrix}\right| = \left|\begin{matrix}1 & 1 \\ -1 & 1\end{matrix}\right| = 1+1 = 2$$

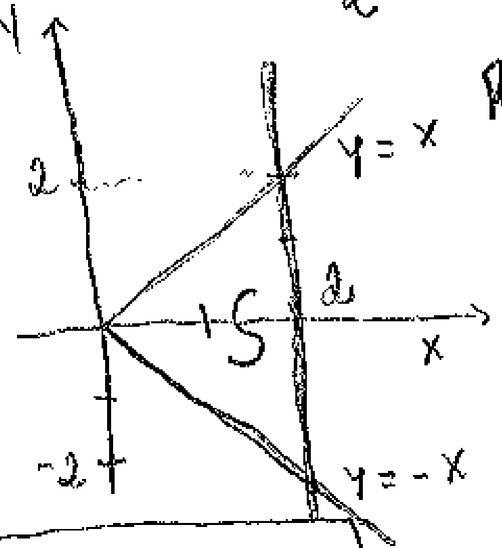
~~8.b)~~ Um triângulo T no plano UoV tem vértices $(0,0)$, $(2,0)$ e $(0,2)$. Desenhe a sua imagem S no plano XoY.

$$\begin{cases} x = u+v \\ y = v-u \end{cases} \Leftrightarrow \begin{cases} u = x-v \\ y = v-x+v \end{cases} \Leftrightarrow \begin{cases} u = \dfrac{x-y}{2} \\ v = \dfrac{x+y}{2} \end{cases}$$

$$R_u = \begin{cases} 0 \le u \le 2 \\ 0 \le v \le 2-u \end{cases}$$



$$R_x \begin{cases} 0 \le x \le 2 \\ -x \le y \le x \end{cases}$$



$$\left| \begin{array}{l} v = au+b = 2-u \\ (2,0):\ 0 = 2a+b \\ (0,2):\ 2 = b \end{array} \right. \Leftrightarrow \begin{cases} a = -1 \\ b = 2 \end{cases}$$

$\boxed{u+v = 2 \Leftrightarrow x = 2}$

$\boxed{v = 0 \Leftrightarrow \dfrac{x+y}{2} = 0 \Leftrightarrow y = -x}$

$\boxed{u = 0 \Leftrightarrow \dfrac{x-y}{2} = 0 \Leftrightarrow y = x}$

8(c) Calcular a área de S por intermédio de um integral duplo estendido a S e por outro estendido a T.

Obs: $\begin{cases} x = \varphi(u,v) \\ y = \psi(u,v) \end{cases}$

$$\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R_{uv}} f(\varphi(u,v), \psi(u,v))\,|J(u,v)|\,du\,dv.$$

- $A(S) = \int_0^2 \int_{-x}^{x} dy\,dx = \int_0^2 2x\,dx = \left[x^2\right]_0^2 = 4$

- $|J(u,v)| = |2| = 2$

- $A(S) = \int_0^2 \int_0^{2-u} 2\,dv\,du = 2\int_0^2 2-u\,du =$

$$= 2\left[2u - \frac{u^2}{2}\right]_0^2 = 2\left[4-2\right] = 4.$$

~~8.d~~ Calcule

(i) $\iint_S (x-y+1)^{-2}\,dx\,dy = \int_0^2 \left[\int_{-x}^{x} (x-y+1)^{-2}\,dy\right] dx =$

$= \int_0^2 \left[\frac{1}{x-y+1}\right]_{y=-x}^{y=x} dx = \int_0^2 1 - \frac{1}{2x+1}\,dx =$

$= \left[x - \frac{\ln(|2x+1|)}{2}\right]_{x=0}^{x=2} = -\frac{\ln(5)-4}{2}.$

(ii) $\iint_S (x-y+1)^{-2}\,dx\,dy = \iint_T (2u+1)^{-2}(2)\,dA$

$\begin{cases} x = u+v \\ y = v-u \end{cases}$

$= \int_0^2 \int_0^{2-u} 2(2u+1)^{-2}\,dv\,du = \int_0^2 \left[\frac{2v}{(2u+1)^2}\right]_{v=0}^{v=2-u} du =$

$= \int_0^2 \frac{4-2u}{(2u+1)^2}\,du = \left[-\frac{\ln(|2u+1|)}{2} - \frac{5}{4u+2}\right]_0^2 =$

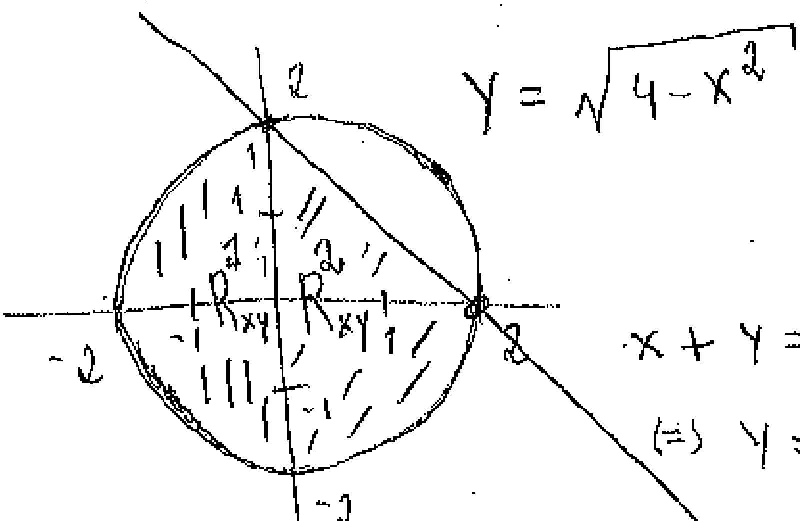
$= -\frac{\ln(5)-4}{2} \simeq 1.195.$

Pode ser calculado das 2 formas.

9) Calcule o volume do sólido limitado pelas superfícies:

9.a) $X^2+Y^2=4$, $X+Y+Z=2$ e $Z=0$.

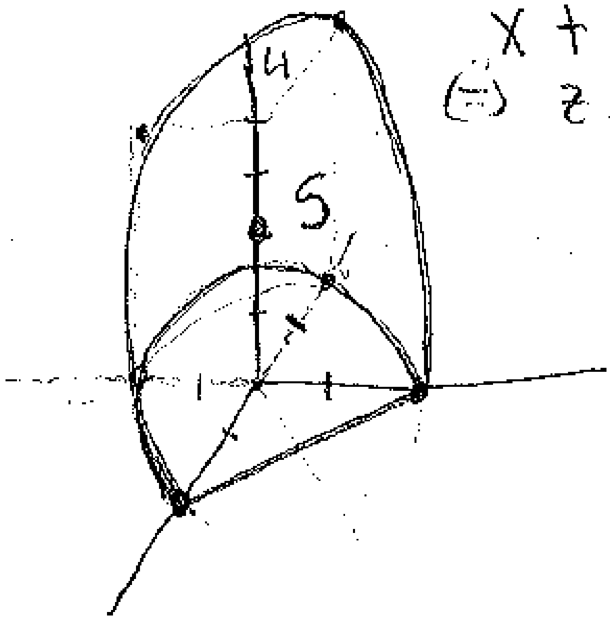
• Projeção do sólido em XoY, isto é, quando $Z=0$:



$X+Y=2 \Leftrightarrow$
$\Leftrightarrow Y=2-X$

$$R^1_{xy}: \begin{cases} -2 \leq X \leq 0 \\ -\sqrt{4-X^2} \leq Y \leq \sqrt{4-X^2} \end{cases}, \quad R^2_{xy}: \begin{cases} 0 \leq X \leq 2 \\ -\sqrt{4-X^2} \leq Y \leq 2-X \end{cases}$$

- Representação do sólido em $\mathbb{R}^3$:



$x + y + z = 2 \Leftrightarrow$
$\Leftrightarrow z = 2 - x - y = f(x,y) \geq 0$

S é uma espécie de uma cunha.

$$V = \iint_{R_{xy}} f(x,y)\, dA =$$

$$= \int_{-2}^{0}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2-x-y\, dy\, dx + \int_{0}^{2}\int_{-\sqrt{4-x^2}}^{2-x} 2-x-y\, dy\, dx =$$

$$= \int_{-2}^{0}\left[2y - xy - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx + \int_{0}^{2}\left[2y - xy - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{2-x} dx$$

$$= 4\int_{-2}^{0} \sqrt{4-x^2}\, dx + \int_{-2}^{0} -2x(4-x^2)^{\frac{1}{2}}\, dx =$$

$$= \frac{1}{2}\int_{0}^{2} -2x(4-x)^{\frac{1}{2}}\, dx + 2\int_{0}^{2} \sqrt{4-x^2}\, dx + \int_{0}^{2} 4-2x\, dx =$$

$$= 4\pi + \frac{2}{3}\left[(4-x^2)^{\frac{3}{2}}\right]_{-2}^{0} + \frac{1}{3}\left[(4-x^2)^{\frac{3}{2}}\right]_{0}^{2} +$$

$$+ 2\pi + 4[x]_0^2 - [x^2]_0^2 =$$

$$= 4\pi + \frac{16}{3} - \frac{8}{3} + 2\pi + \frac{12}{3} = 6\pi + \frac{20}{3}.$$

Problem:

$$\int 4\sqrt{4-x^2}\,dx$$

Apply linearity:

$$= 4\int \sqrt{4-x^2}\,dx$$

Now solving:

$$\int \sqrt{4-x^2}\,dx$$

Perform trigonometric substitution:

Substitute $x = 2\sin(u) \longrightarrow u = \arcsin(\frac{x}{2})$, $dx = 2\cos(u)\,du$ (steps):

$$= \int 2\cos(u)\sqrt{4-4\sin^2(u)}\,du$$

Simplify using $4-4\sin^2(u) = 4\cos^2(u)$:

$$= 4\int \cos^2(u)\,du$$

... or choose an alternative:

Perform hyperbolic substitution

Now solving:

$$\int \cos^2(u)\,du$$

Apply reduction formula:

$$\int \cos^n(u)\,du = \frac{n-1}{n}\int \cos^{n-2}(u)\,du + \frac{\cos^{n-1}(u)\sin(u)}{n}$$

with $n = 2$:

$$= \frac{\cos(u)\sin(u)}{2} + \frac{1}{2}\int 1\,du$$

11-39

... or choose an alternative:

Apply product-to-sum formulas

Now solving:

$$\int 1\,du$$

Apply constant rule:

$$= u$$

Plug in solved integrals:

$$\frac{\cos(u)\sin(u)}{2} + \frac{1}{2}\int 1\,du$$

$$= \frac{\cos(u)\sin(u)}{2} + \frac{u}{2}$$

Plug in solved integrals:

$$4\int \cos^2(u)\,du$$

$$= 2\cos(u)\sin(u) + 2u$$

Undo substitution $u = \arcsin(\frac{x}{2})$, use:

$$\sin(\arcsin(\frac{x}{2})) = \frac{x}{2}$$

$$\cos(\arcsin(\frac{x}{2})) = \sqrt{1 - \frac{x^2}{4}}$$

$$= x\sqrt{1 - \frac{x^2}{4}} + 2\arcsin(\frac{x}{2})$$

Plug in solved integrals:

$$4\int \sqrt{4-x^2}\,dx$$

11-40

$$= 4x\sqrt{1-\frac{x^2}{4}} + 8\arcsin\left(\frac{x}{2}\right)$$

The problem is solved:

$$\int 4\sqrt{4-x^2}\,dx$$

$$= 4x\sqrt{1-\frac{x^2}{4}} + 8\arcsin\left(\frac{x}{2}\right) + C$$

Rewrite/simplify:

$$= 2x\sqrt{4-x^2} + 8\arcsin\left(\frac{x}{2}\right) + C$$

ANTIDERIVATIVE COMPUTED BY MAXIMA:
$\int f(x)\,dx = F(x)$ —

$$4\left(\frac{x\sqrt{4-x^2}}{2} + 2\arcsin\left(\frac{x}{2}\right)\right) + C$$

Simplify/rewrite:

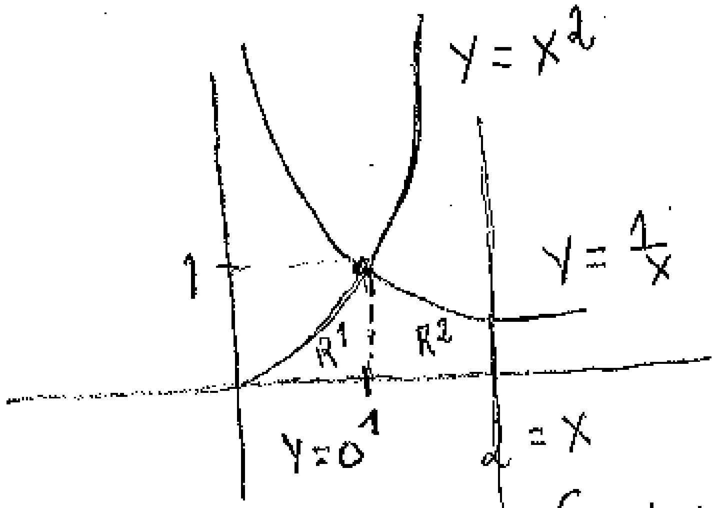
$$2x\sqrt{4-x^2} + 8\arcsin\left(\frac{x}{2}\right) + C$$

DEFINITE INTEGRAL:
$\int_{-2}^{0} f(x)\,dx =$

$$4\pi$$

g-b) $z = x^2+y^2$, $y=x^2$, $xy=1$, $x=2$, $y=0$ e $z=0$.

• Projeção do sólido em XoY ($z=0$).



$$R^1_{xy}: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le x^2 \end{cases} \quad , \quad R^2_{xy}: \begin{cases} 1 \le x \le 2 \\ 0 \le y \le \frac{1}{x} \end{cases}$$

$$V = \underbrace{\int_0^1 \int_0^{x^2} x^2+y^2 \, dy\, dx}_{I_1} + \underbrace{\int_1^2 \int_0^{\frac{1}{x}} x^2+y^2 \, dy\, dx}_{I_2} =$$

$$\dots = \frac{26}{105} + \frac{13}{8} = \frac{1573}{840}$$

$$\bullet\ I_{①} = \int_0^1 x^2 \left[y\right]_{y=0}^{y=x^2} + \frac{1}{3}\left[y^3\right]_{y=2}^{y=x^2} dx =$$

$$= \int_0^1 x^4 + \frac{x^6}{6}\, dx = \left[\frac{x^5}{5}\right]_0^1 + \left[\frac{x^7}{21}\right]_0^1 =$$

$$= \frac{1}{5} + \frac{1}{21} = \boxed{\frac{26}{105}}$$

$$\bullet\ I_{②} = \int_1^2 x^2 \left[y\right]_{y=0}^{y=\frac{1}{x}} + \left[\frac{x^3}{3}\right]_{y=0}^{y=\frac{1}{x}} dx =$$

$$= \int_1^2 x + \frac{1}{3x^3}\, dx = \left[\frac{x^2}{2}\right]_1^2 - \left[\frac{1}{6x^2}\right]_1^2 =$$

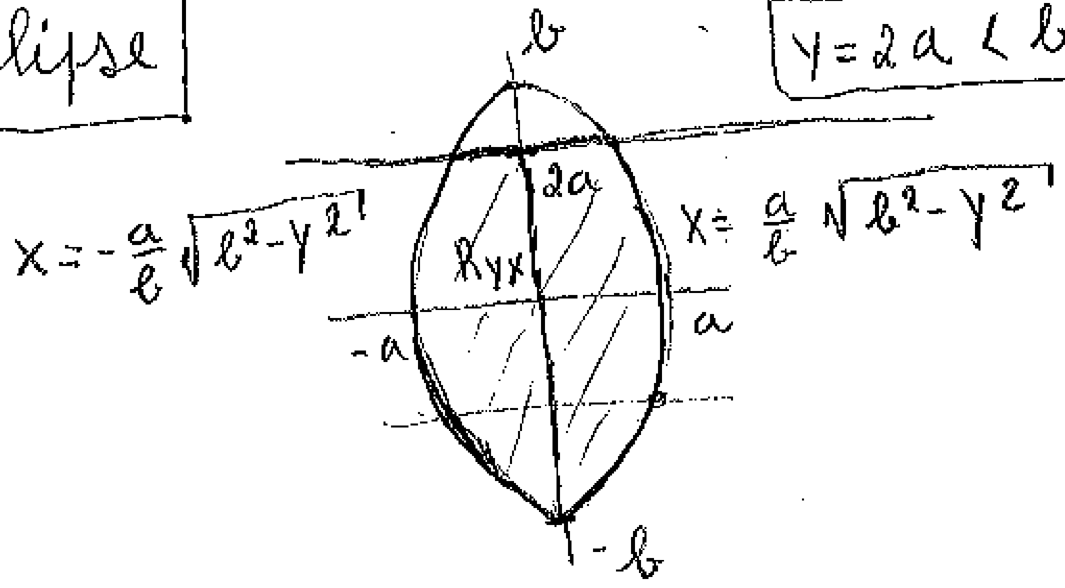
$$= \frac{1}{2}(4-1) - \frac{1}{6}\left(\frac{1}{4} - 1\right) = \frac{3}{2} - \frac{1}{6}\left(-\frac{3}{4}\right) =$$

$$= \frac{3}{2} + \frac{3}{24} = \frac{12}{8} + \frac{1}{8} = \frac{13}{8}.$$

9.c) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=2a-y$ $(2a<b)$.

Projecção do sólido em XoY :

$\boxed{\text{elipse}}$ $\boxed{y=2a<b}$

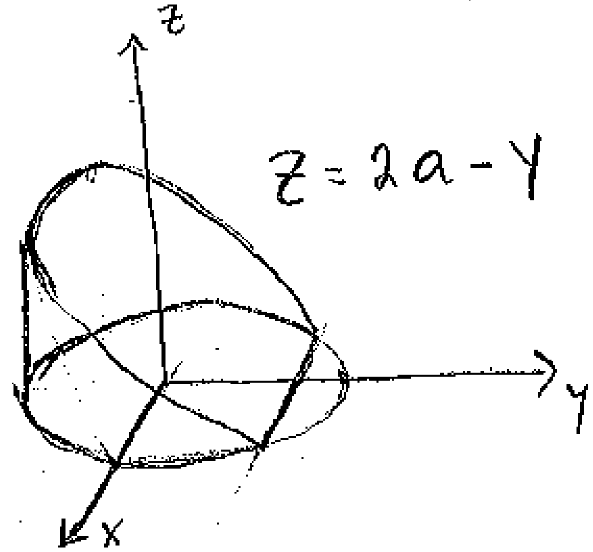


$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Leftrightarrow \frac{x^2}{a^2}=1-\frac{y^2}{b^2} \Leftrightarrow$$

$$\Leftrightarrow x^2=\frac{a^2}{b^2}\left(b^2-y^2\right) \Leftrightarrow x=\pm\frac{a}{b}\sqrt{b^2-y^2}$$

$$R_{yx}=\begin{cases} -b\leq y\leq 2a \\ -\frac{a}{b}\sqrt{b^2-y^2}\leq x\leq\frac{a}{b}\sqrt{b^2-y^2}. \end{cases}$$

b sólido :



$$V = \int_{-b}^{2a} \int_{-\frac{a}{b}\sqrt{b^2-y^2}}^{\frac{a}{b}\sqrt{b^2-y^2}} 2a - y \, dx \, dy =$$

$$= \int_{-b}^{2a} 2a \left[x\right]_{-\frac{a}{b}\sqrt{b^2-y^2}}^{\frac{a}{b}\sqrt{b^2-y^2}} - y \left[x\right]_{-\frac{a}{b}\sqrt{b^2-y^2}}^{\frac{a}{b}\sqrt{b^2-y^2}} \, dy =$$

$$= \frac{4a^2}{b} \underbrace{\int_{-b}^{2a} \sqrt{b^2-y^2} \, dy}_{I_1 \ \text{(substituição)} \ (y = b\sin(u))} + \frac{a}{b} \underbrace{\int_{-b}^{2a} -2y\left(b^2-y^2\right)^{\frac{1}{2}} dy}_{\text{Integral imediato.}} =$$

$= \dots$