Aulas 11 e 12

4.13 Exercícios (Pg 179)

1) calcular os seguintes integrais:

1.a) $\int_0^1 \int_0^1 x^2 y + x y^2 \, dx \, dy = \int_0^1 \left[\frac{x^3}{3} y + \frac{x^2}{2} y^2 \right]_{x=0}^{x=1} dy$

$= \int_0^1 \frac{y}{3} + \frac{y^2}{2} \, dy = \left[\frac{y^2}{6} + \frac{y^3}{6} \right]_0^1 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$

~~1.b~~) $\int_0^1 \int_{0=1}^3 \sqrt{y} + x - 3 x y^2 \, dy \, dx = \int_0^1 \left[\frac{2}{3} y^{\frac{2}{3}} + x y - x y^3 \right]_{y=0}^{y=3} dx =$

$= \int_0^1 2\sqrt{3} - 24 x \, dx = \left[2\sqrt{3}\, x - 12 x^2 \right]_0^1 = 2\sqrt{3} - 12.$

1.~~c~~) $\int_0^\pi \int_0^\pi \sin^2 x \sin^2 y \, dx \, dy = \int_0^\pi \sin^2 y \left[\frac{x - \cos x \sin x}{x} \right]_{x=0}^{x=\pi} dy =$

$= \frac{\pi}{2} \int_0^\pi \sin^2 y \, dy = \frac{\pi^2}{4}$

$P \sin^2 x = -\cos x \sin x + P \cos^2 x = -\cos x \sin x + P(1 - \sin^2 x) =$

$\begin{cases} u' = \sin x \\ v = \sin x \end{cases} \Rightarrow \begin{cases} u = -\cos x \\ v' = \cos x \end{cases}$ $\quad = \frac{x - \cos x \sin x}{2}.$

$P u' v = u v - P u v'$

1. d) $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(x+2y)\,dx\,dy = \int_0^{\frac{\pi}{2}} \left[-\cos(x+2y)\right]_{x=0}^{x=\frac{\pi}{2}} dy =$

$$= \int_0^{\frac{\pi}{2}} -\cos\left(\frac{\pi}{2}+2y\right) + \cos(2y)\,dy =$$

$$= \frac{1}{2}\left[-\sin\left(\frac{\pi}{2}+2y\right) + \sin(2y)\right]_0^{\frac{\pi}{2}} =$$

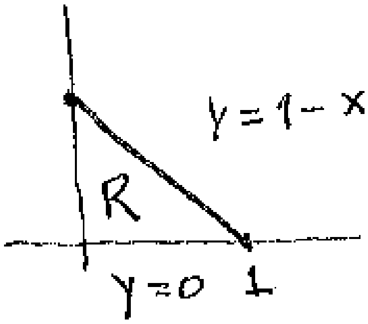
$$= \frac{1}{2}\Big(-\underbrace{\sin\left(\frac{3}{2}\pi\right)}_{=-1} + \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} + \underbrace{\sin(\pi)}_{=0} - \underbrace{\sin(0)}_{=0}\Big) = 1$$

1. e) $\int_0^{t} \int_1^{t} \frac{e^{\frac{tx}{y}}}{y^3}\,dy\,dx = \ldots$ muito complicado.

2) Esboce a região de integração e calcule as integrais

~~a)~~* $\iint_R 1-x-y\,dA$, $R=\{(x,y): \underbrace{x,y\geq 0}, x+y\leq 1\}$

$R_x=\{(x,y): 0\leq x\leq 1, 0\leq y\leq 1-x\}$



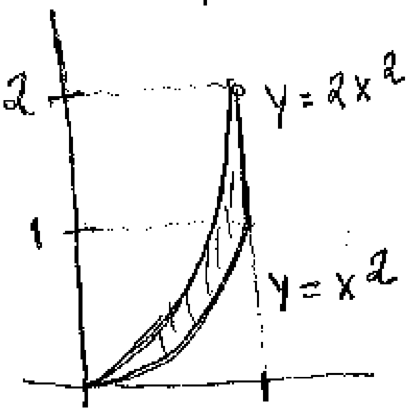
$$\int_0^1\int_0^{1-x} 1-x-y\,dy\,dx = \int_0^1 \left[y-yx-\frac{y^2}{2}\right]_{y=0}^{y=1-x} dx =$$

$$= \int_0^1 1-x-(1-x)x-\frac{(1-x)^2}{2}\,dx =$$

$$= \int_0^1 1-x-x+x^2-\frac{1-2x+x^2}{2}\,dx =$$

$$= \int_0^1 \frac{1}{2}-x+\frac{x^2}{2}\,dx = \left[\frac{x}{2}-\frac{x^2}{2}+\frac{x^3}{6}\right]_0^1 = \frac{1}{6}$$

2. b)* $\iint_R x+y\,dA$, $R = \{(x,y): \underbrace{0 \le x \le 1}, x^2 \le y \le 2x^2\}$



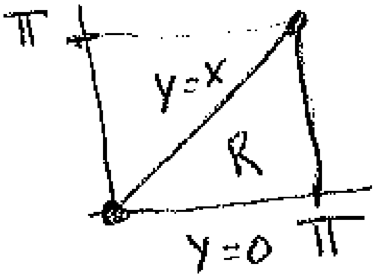
$$\int_0^1 \int_{x^2}^{2x^2} x+y\,dy\,dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_{y=x^2}^{y=2x} dx =$$

$$= \int_0^1 2x^3 - x^3 + \frac{4x^4 - x^4}{2}\,dx = \int_0^1 x^3 + \frac{3}{2}x^4\,dx =$$

$$= \left[\frac{x^4}{4} + \frac{3}{10}x^5\right]_0^1 = \frac{1}{4} + \frac{3}{10} = \frac{5}{20} + \frac{6}{20} = \frac{11}{20}$$

2.c)
2.d) Ver sebenta pg 16 f.

~~2~~ l) $\iint_R x\cos(x+y)\,dA$, onde R é a região triangular de vértices $(0,0)$, $(\pi,0)$ e (π,π).



$R_x = \{(x,y): 0\le x\le\pi,\ 0\le y\le x\}$

$$\int_0^{\pi}\int_0^{x} x\cos(x+y)\,dy\,dx = \int_0^{\pi}\Big[x\sin(x+y)\Big]_{y=0}^{y=x}dx =$$

$$= \int_0^{\pi} x\sin(2x) - x\sin x\,dx =$$

$$\boxed{\int_a^b u'v\,dx = [uv]_a^b - \int_a^b u v'}$$

$$\begin{cases} u' = \sin(2x) \\ v = x \end{cases} \Rightarrow \begin{cases} u = -\dfrac{\cos(2x)}{2} \\ v' = 1 \end{cases}$$

$$\begin{cases} p' = \sin x \\ q = -x \end{cases} \Rightarrow \begin{cases} p = -\cos x \\ q' = -1 \end{cases}$$

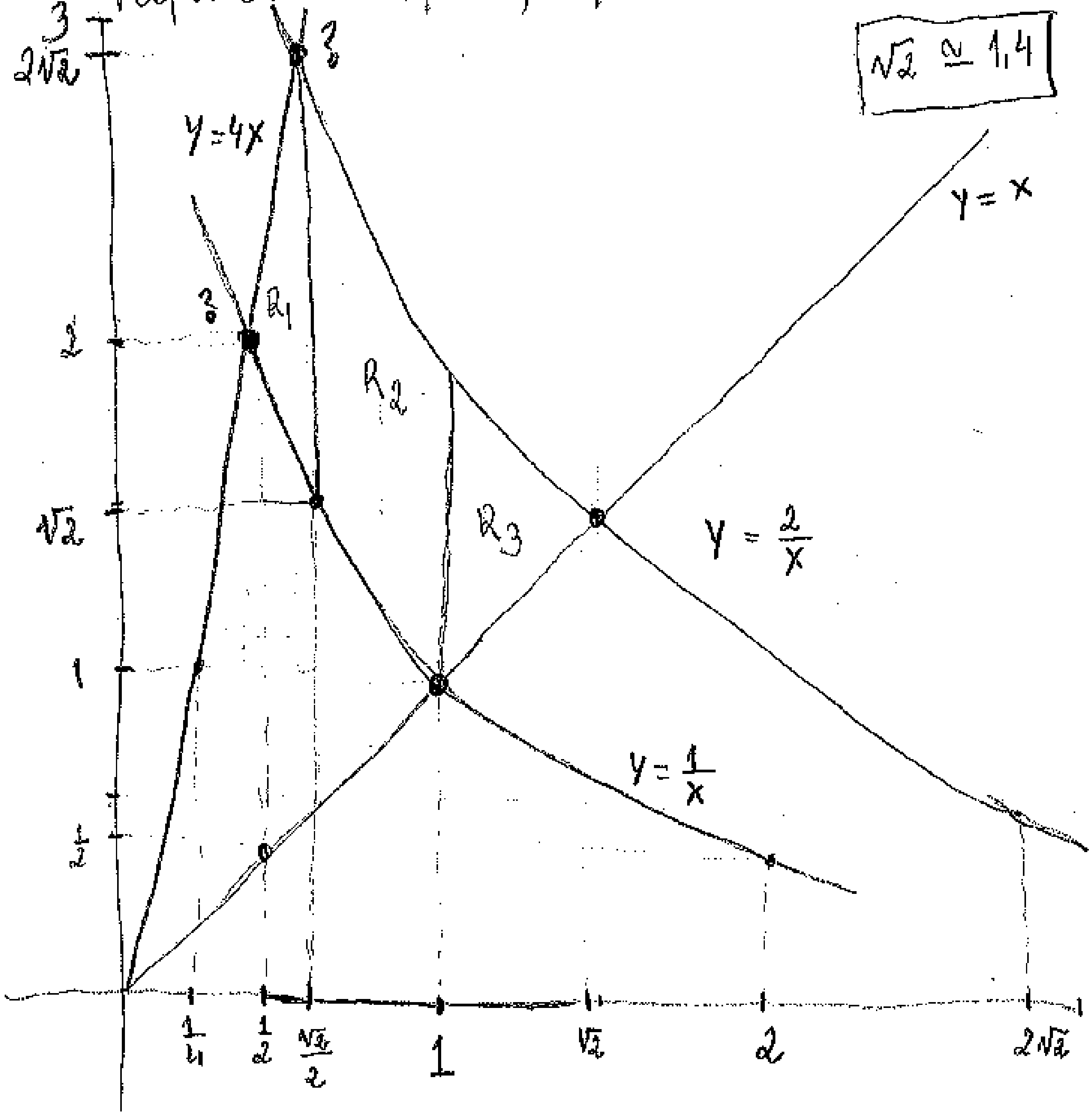
$$= \Big[-\frac{x}{2}\cos(2x)\Big]_0^{\pi} + \frac{1}{2}\int_0^{\pi}\cos(2x)\,dx + \Big[x\cos x\Big]_0^{\pi} - \int_0^{\pi}\cos x\,dx =$$

$$= -\frac{\pi}{2}\cos(2\pi) + \frac{1}{4}\underbrace{\Big[\sin(2x)\Big]_0^{\pi}}_{=0} + \pi\underbrace{\cos(\pi)}_{=-1} - \underbrace{\Big[\sin x\Big]_0^{\pi}}_{=0} = -\frac{3}{2}\pi.$$

$\boxed{11-6}$

2. f) $\iint_R x^2y^2\,dA$, sendo R a região do 1º quadrante entre as hipérboles $xy=1$, $xy=2$ e as retas $y=x$ e $y=4x$.

$\boxed{\sqrt{2} \cong 1{,}4}$



- Intersecção entre as condições

$$\begin{cases} y = 4x \\ xy = 1 \end{cases} \Leftrightarrow \begin{cases} \\ 4x^2 = 1 \end{cases} \Leftrightarrow \begin{cases} y = 2 \\ x = \frac{1}{2} \end{cases}$$

$$\begin{cases} y = 4x \\ xy = 2 \end{cases} \Leftrightarrow \begin{cases} \\ 4x^2 = 2 \end{cases} \Leftrightarrow \begin{cases} y = 2\sqrt{2} \\ x = \frac{\sqrt{2}}{2} \end{cases}$$

$$\iint_R x^2 y^2 \, dx \, dy =$$

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \int_{\frac{1}{x}}^{4x} x^2 y^2 \, dy \, dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_{\frac{1}{x}}^{\frac{2}{x}} x^2 y^2 \, dy \, dx + \int_{1}^{\sqrt{2}} \int_{x}^{\frac{2}{x}} x^2 y^2 \, dy \, dx =$$

$$= \dots = \frac{7}{3} \ln(2)$$

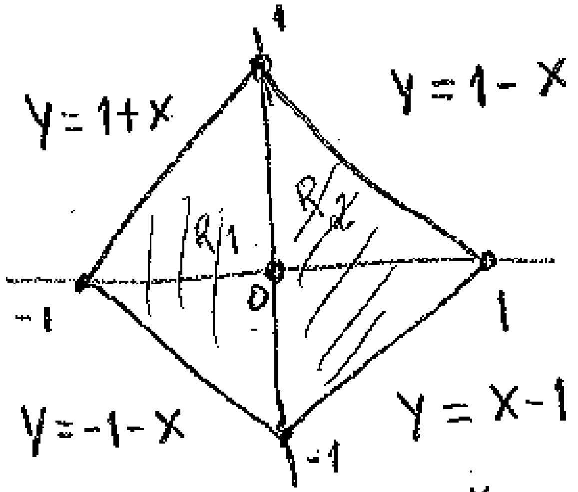
$$* \begin{cases} xy = 1 \\ y = x \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$$

$$* \begin{cases} xy = 2 \\ y = x \end{cases} \Leftrightarrow \begin{cases} x = \sqrt{2} \\ y = \sqrt{2} \end{cases}$$

$\boxed{11-8}$

$\cancel{2g)}\ \iint_R e^{x+y}\, dA$ com $R = \{(x,y) : |x|+|y| \leq 1\}$

$$|x|+|y| = \begin{cases} x+y, & x,y \geq 0 \\ y-x, & x \leq 0, y \geq 0 \\ -x-y, & x,y \leq 0 \\ x-y, & x \geq 0, y \leq 0 \end{cases}$$



$$\int_{-1}^{0}\int_{-1-x}^{1+x} e^{x+y}\, dy\, dx + \int_{0}^{1}\int_{x-1}^{1-x} e^{x+y}\, dy\, dx =$$

$$= \int_{-1}^{0} \left[e^{x+y}\right]_{y=-1-x}^{y=1+x} dx + \int_{0}^{1} \left[e^{x+y}\right]_{y=x-1}^{y=1-x} dx =$$

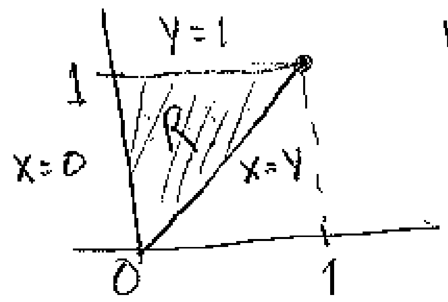
$$= \int_{-1}^{0} e^{2x+1} - e^{-1}\, dx + \int_{0}^{1} e - e^{2x-1}\, dx =$$

$$= \left[\frac{e^{2x+1}}{2} - \frac{x}{e}\right]_{-1}^{0} + \left[ex - \frac{e^{2x+1}}{2}\right]_{0}^{1} = \cancel{\frac{e}{2}} - \cancel{\frac{1}{2e}} - \frac{1}{e} + e - \cancel{\frac{e}{2}} + \cancel{\frac{1}{2e}} =$$

$$= \boxed{e - \frac{1}{e}}$$

3) Admitindo que as integrais em causa existem, esboce a região de integração e permute (inverta) a ordem de integração.

3.a) $\int_0^1 \int_0^y f \, dx \, dy = \int_0^1 \int_x^1 f \, dy \, dx$



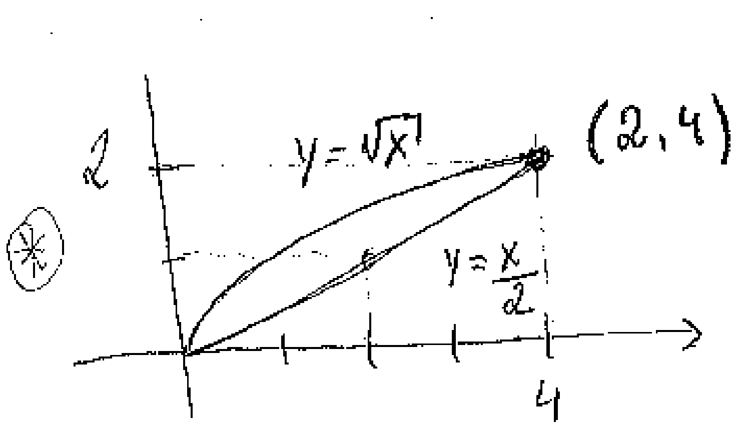
$R_y = \{(x,y) : 0 \le x \le 1, \ x \le y \le 1\}$

$R_x = \{(x,y) : 0 \le y \le 1, \ 0 \le x \le y\}$

3.b) $\int_0^2 \int_{y^2}^{2y} f \, dx \, dy = \int_0^4 \int_{\frac{x}{2}}^{\sqrt{x}} f \, dy \, dx.$

(*)

$\cap \begin{cases} x = 2y \\ x = y^2 \end{cases} \Rightarrow \boxed{\begin{cases} y = \frac{x}{2} \\ y = \sqrt{x} \end{cases}} \Leftrightarrow \begin{cases} - \\ y^2 - 2y = 0 \end{cases} \Leftrightarrow \begin{cases} - \\ y(y-2) = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \boxed{\begin{cases} x = 4 \\ y = 2 \end{cases}}$



$R_y = \{(x,y) : 0 \le x \le 4, \ \frac{x}{2} \le y \le \sqrt{x}\}$

$$\cancel{3}\cancel{c)}\ \int_1^2\int_{2-x}^{\sqrt{2x-x^2}} f\,dy\,dx=\int_0^1\int_{2-y}^{1+\sqrt{1-y^2}} f\,dx\,dy.$$

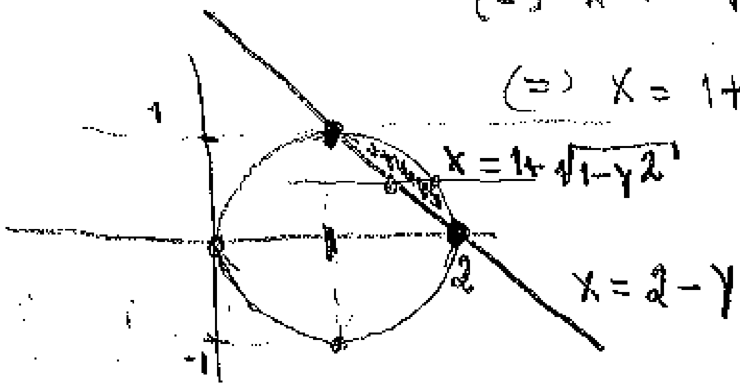
- $y=2-x \iff x=2-y$

$y=\sqrt{2x-x^2} \Rightarrow y^2=2x-x^2 \iff x^2-2x+1+y^2=1 \iff$

$\iff (x-1)^2+y^2=1 \iff (x-1)^2=1-y^2 \iff$

$\iff x-1=\sqrt{1-y^2} \iff$

$\iff x=1+\sqrt{1-y^2}$



$$3.\cancel{d)}\ \int_{-6}^2\int_{\frac{x^2-4}{4}}^{2-x} f\,dy\,dx=$$

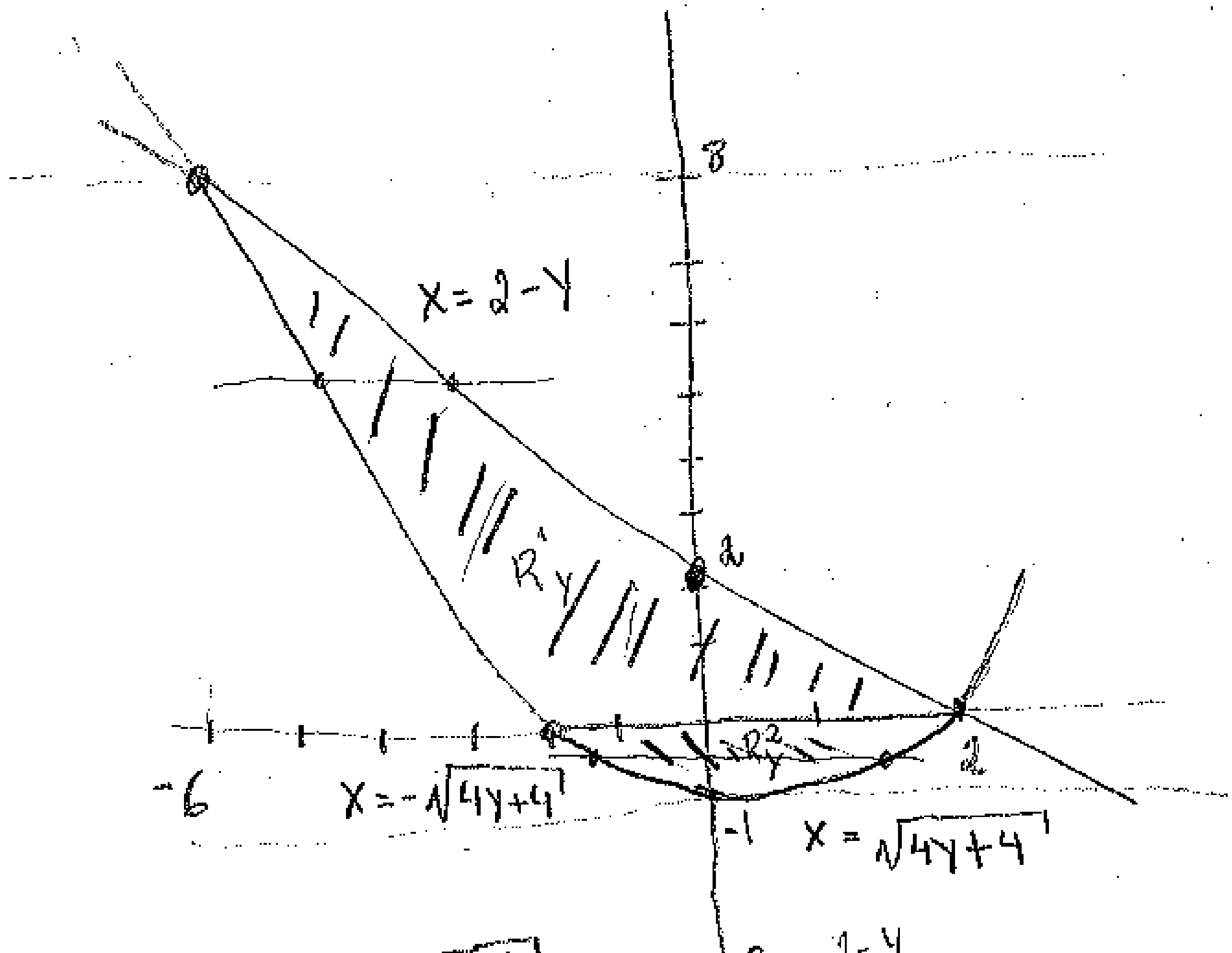
$y=2-x \iff x=2-y$

$y=\dfrac{x^2-4}{4} \iff x^2-4=4y \iff$

$\iff x^2=4y+4 \iff x=\pm\sqrt{4y+4}$

$\begin{cases} y=2-x \\ 4y=x^2-4 \end{cases} \iff \begin{cases} — \\ 8-4x=x^2-4 \end{cases} \iff \begin{cases} — \\ x^2+4x-12=0 \end{cases} \iff$

$\iff \begin{cases} — \\ x=\dfrac{-4\pm\sqrt{16+48}}{2} \end{cases} \iff \begin{cases} y=8 \\ x=-6 \end{cases} \vee \begin{cases} y=0 \\ x=2 \end{cases}$

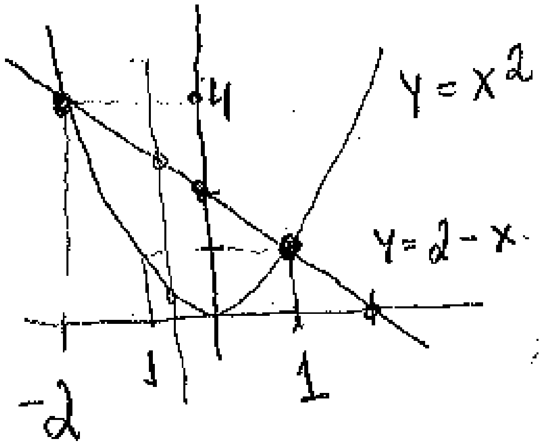


$$\int_{-6}^{2}\int_{\frac{x^2-4}{4}}^{2-x} f\,dy\,dx = \int_{-1}^{0}\int_{-\sqrt{4y+4}}^{\sqrt{4y+4}} f\,dx\,dy + \int_{0}^{8}\int_{-\sqrt{4y+4}}^{2-y} f\,dx\,dy$$

$$R_y = R_y^1 \cup R_y^2$$

4) Considere uma placa homogénea ($f(x,y)=k$, com k constante) com o formato da região S limitada pelas curvas abaixo: Em cada caso represente graficamente S e calcule as coordenadas do centroide:

4.a) $y = x^2$ e $x + y = 2$.



$$\begin{cases} y = x^2 \\ y = 2 - x \end{cases} \Leftrightarrow \begin{cases} x^2 + x - 2 = 0 \\ \end{cases} \Leftrightarrow$$

$$\begin{cases} x = \dfrac{-1 \pm \sqrt{1+8}}{2} \end{cases} \Leftrightarrow \boxed{\begin{cases} x = -2 \\ y = 4 \end{cases} \vee \begin{cases} x = 1 \\ y = 1 \end{cases}}$$

$$S = \{(x,y) : -2 \leq x \leq 1,\ x^2 \leq y \leq 2 - x\}$$

$$M = \iint_S f\, dA = \int_{-2}^{1} \int_{x^2}^{2-x} k\, dy\, dx =$$

$$= k \int_{-2}^{1} 2 - x - x^2\, dx = k \left[2x - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-2}^{1} = \ldots$$

$$\ldots = k\left(2 - \frac{1}{2} - \frac{1}{3} + 4 + \frac{4}{2} - \frac{8}{3}\right) = k\left(6 + \frac{3}{2} - \frac{9}{3}\right) =$$

$$= k\left(3 + \frac{3}{2}\right) = \boxed{\frac{9k}{2}}.$$

$$\bar{X} = \frac{1}{M}\iint_S x f\,dA = \frac{2}{9}\int_{-2}^{1}\int_{x^2}^{2-x} x\,dy\,dx =$$

$$= \frac{2}{9}\int_{-2}^{1} x\Big[y\Big]_{y=x^2}^{y=2-x} dx = \frac{2}{9}\int_{-2}^{1} 2x - x^2 - x^3\,dx =$$

$$= \frac{2}{9}\left[x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_{-2}^{1} = \frac{2}{9}\left(-\frac{9}{4}\right) = \boxed{-\frac{1}{2}}$$

$$\bar{Y} = \frac{1}{M}\iint_S y f\,dA = \frac{2}{9}\int_{-2}^{1}\int_{x^2}^{2-x} y\,dy\,dx =$$

$$= \frac{2}{9}\int_{-2}^{1} \frac{1}{2}\Big[y^2\Big]_{y=x^2}^{y=2-x} dx = \frac{1}{9}\int_{-2}^{1}(2-x)^2 - x^4\,dx =$$

$$= \frac{1}{9}\int_{-2}^{1} 4 - 4x + x^2 - x^4\,dx = \frac{1}{9}\left[4x - 2x^2 + \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \ldots$$

$$= \frac{1}{9}\left(4 - 2 + \frac{1}{3} - \frac{1}{5} + 8 + 8 + \frac{8}{3} - \frac{32}{5}\right) = \ldots$$

$$= \frac{1}{9}\left(2 + \frac{5}{15} - \frac{3}{15} + 16 + \frac{40}{15} - \frac{96}{15}\right) =$$

$$= \frac{1}{9}\left(\frac{30}{15} + \frac{2}{15} + \frac{240}{15} - \frac{56}{15}\right) = \frac{1}{9}\,\frac{216}{15} = \frac{24}{15} = \boxed{\frac{8}{5}}$$

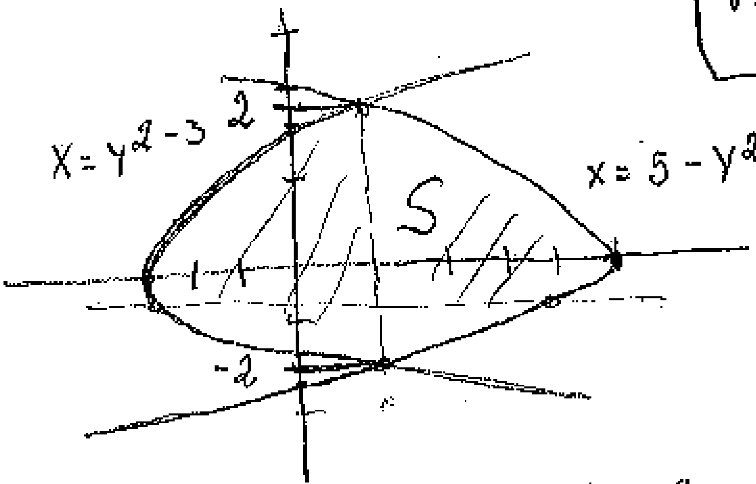
$$\therefore\ (\bar{X}, \bar{Y}) = \left(-\frac{1}{2}, \frac{8}{5}\right)$$

$\boxed{11\text{-}14}$

~~4.b)~~ $y^2 = x+3$ e $y^2 = 5-x$.

$x = y^2 - 3$ e $x = 5 - y^2$

$$\boxed{\begin{array}{l} \sqrt{3} \simeq 1,7 \\ \sqrt{5} \simeq 2,2 \end{array}}$$



$$\cap \begin{cases} x = y^2 - 3 \\ x = 5 - y^2 \end{cases} \Leftrightarrow \begin{cases} 5 - y^2 = y^2 - 3 \\ - \end{cases} \Leftrightarrow \begin{cases} 2y^2 = 8 \\ \end{cases} \Leftrightarrow \boxed{\begin{cases} y = -2 \\ x = 1 \end{cases} \vee \begin{cases} y = 2 \\ x = 1 \end{cases}}$$

$$S = \{(x,y) : -2 \leq y \leq 2,\ y^2 - 3 \leq x \leq 5 - y^2\}$$

$$M = \iint_S f \, dA = \int_{-2}^{2} \int_{y^2-3}^{5-y^2} k \, dx\, dy = k \int_{-2}^{2} 5 - y^2 - y^2 + 3 \, dy =$$

$$= k \int_{-2}^{2} 8 - 2y^2 \, dy = k \left(8 [y]_{-2}^{2} - \frac{2}{3} \left[y^3 \right]_{-2}^{2} \right) =$$

$$= k \left(32 - \frac{2}{3}(8+8) \right) = k \left(\frac{96}{3} - \frac{32}{3} \right) = \boxed{\frac{64}{3} k}$$

$\boxed{11\text{-}15}$

$$\boxed{\bar{x}} = \frac{1}{M}\iint_S x f \, dx = \frac{3}{64}\int_{-2}^{2}\int_{y^2-3}^{5-y^2} x \, dx =$$

$$= \frac{3}{64}\int_{-2}^{2} \frac{1}{2}\left[x^2\right]_{x=y^2-3}^{x=5-y^2} dy = \frac{3}{128}\int_{-2}^{2}(5-y^2)^2-(y^2-3)^2 dy =$$

$$= \frac{3}{128}\int_{-2}^{2} 25-10y^2+\cancel{y^4}-(\cancel{y^4}-6y^2+9)\, dy =$$

$$= \frac{3}{128}\int_{-2}^{2} 16-4y^2 \, dy = \frac{3}{32}\int_{-2}^{2} 4-y^2 \, dy =$$

$$= \frac{3}{32}\left(4\left[y\right]_{-2}^{2} - \frac{1}{3}\left[y^3\right]_{-2}^{2}\right) = \frac{3}{32}\left(\frac{48}{3}-\frac{16}{3}\right) = \boxed{1}$$

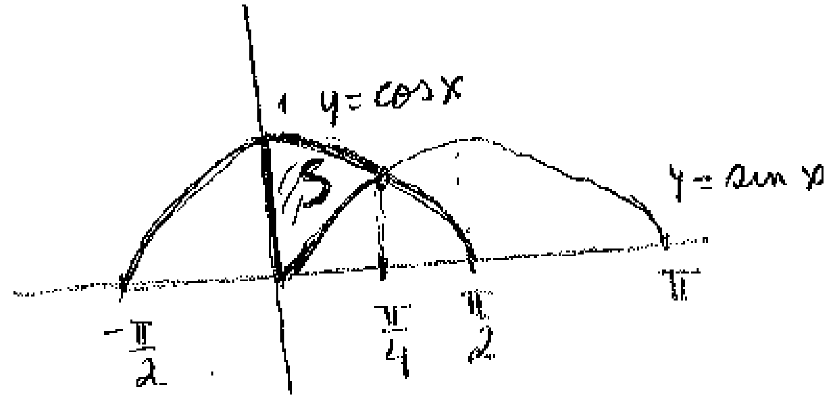
$$\boxed{\bar{y}} = \frac{1}{M}\iint_S y f \, dA = \frac{3}{64}\int_{-2}^{2}\int_{y^2-3}^{5-y^2} y \, dx \, dy =$$

$$= \frac{3}{64}\int_{-2}^{2} y(8-2y^2)\, dy = \frac{3}{64}\int_{-2}^{2} 8y-2y^3 \, dy =$$

$$= \frac{3}{64}\left(4\underbrace{\left[y^2\right]_{-2}^{2}}_{=0} - \frac{1}{2}\underbrace{\left[y^4\right]_{-2}^{2}}_{=0}\right) = \boxed{0}$$

$$\boxed{\therefore\ (\bar{x},\bar{y}) = (1,0)}$$

~~4.c)~~ $y = \sin x,\ y = \cos x,\ 0 \le x \le \frac{\pi}{4}$



$S = \{(x,y) : 0 \le x \le \frac{\pi}{4},\ \sin x \le y \le \cos x\}$

$$M = \iint_S \rho \, dA = \int_0^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} k \, dy\, dx = k \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx =$$

$$= k\left([\sin x]_0^{\frac{\pi}{4}} + [\cos x]_0^{\frac{\pi}{4}} \right) = k\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 \right) = \boxed{k(\sqrt{2}-1)}$$

$$\bar{X} = \frac{1}{M} \iint_S x \rho \, dA = \frac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} \int_0^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} x \, dy\, dx =$$

$$= (\sqrt{2}+1) \int_0^{\frac{\pi}{4}} x\cos x - x \sin x \, dx =$$

$$= (\sqrt{2}+1) \left[x \sin x + \cos x + x \cos x - \sin x \right]_0^{\frac{\pi}{4}} =$$

$$= (\sqrt{2}+1)\left(\frac{\pi}{4}\frac{\sqrt{2}}{2} + \cancel{\frac{\sqrt{2}}{2}} + \frac{\pi}{4}\frac{\sqrt{2}}{2} - \cancel{\frac{\sqrt{2}}{2}} - 1 \right) = (\sqrt{2}+1)\left(\frac{\pi\sqrt{2}}{4} - 1 \right).$$

$$\boxed{P\, x\cos x = x \sin x + \cos x \quad \begin{cases} u' = \cos x \\ v = x \end{cases} \Rightarrow \begin{cases} u = \sin x \\ v' = 1 \end{cases}}\ ; \quad \boxed{-P\, x \sin x = x \cos x - \sin x \quad \begin{cases} u' = \sin x \\ v = -x \end{cases} \Rightarrow \begin{cases} u = -\cos x \\ v' = -1 \end{cases}}$$

11-17

$$\bar{Y} = \frac{1}{M}\iint_S Y f\, dA = (\sqrt{2}+1)\int_0^{\frac{\pi}{4}}\int_{\sin x}^{\cos x} Y\, dY\, dx =$$

$$= \frac{\sqrt{2}+1}{2}\int_0^{\frac{\pi}{4}} \left[Y^2\right]_{Y=\sin x}^{Y=\cos x} dx = \frac{\sqrt{2}+1}{2}\int_0^{\frac{\pi}{4}} \underbrace{\cos^2 x - \sin^2 x\, dx}_{\text{Por partes.}} =$$

$$= \frac{\sqrt{2}+1}{4}\int_0^{\frac{\pi}{4}} 2\cos(2x)\, dx = \frac{\sqrt{2}+1}{4}\left[\sin x\right]_0^{\frac{\pi}{4}} =$$

$$= \frac{\sqrt{2}+1}{4}\Big[\underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{=1} - \underbrace{\sin(0)}_{=0}\Big] = \frac{\sqrt{2}+1}{4}$$

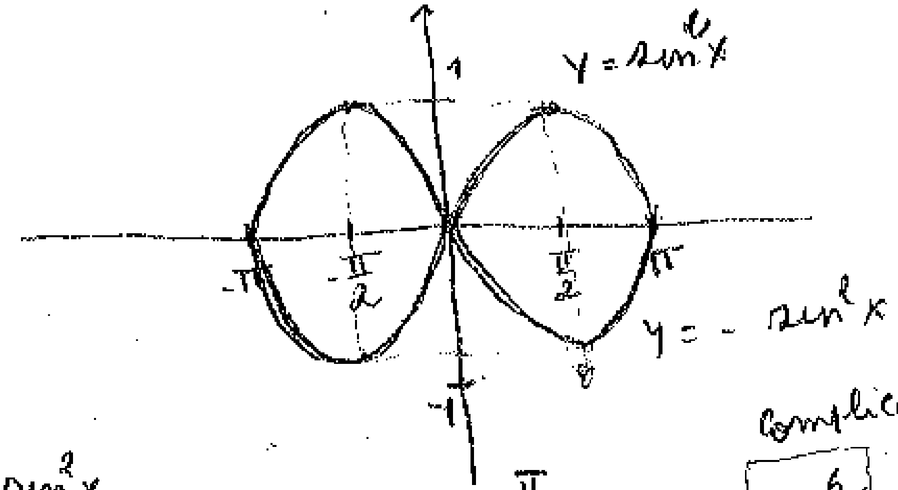
$$\therefore\ (\bar{X},\bar{Y}) = \left((\sqrt{2}+1)\left(\tfrac{\pi}{4}\sqrt{2}-1\right),\ \frac{\sqrt{2}+1}{4}\right).$$

5) Calcular o momento de inércia de uma placa delgada S no plano XOY, limitada pela curvas definidas pelas equações abaixo, representando por $f(x,y)$ a densidade de S num ponto arbitrário (x,y):

5.a) $y = \sin^2 x$, $y = -\sin^2 x$, $-\pi \leq x \leq \pi$, $f(x,y) = 1$.

Obs: Momento polar de inércia é

$$I_0 = I_x + I_y = \iint_S (x^2+y^2)\, f \, dA.$$



complicado

$$I_0 = 4\int_0^{\pi}\int_0^{\sin^2 x} x^2+y^2 \, dy\, dx = 4\int_0^{\pi} x^2 \sin^2 x + \boxed{\frac{\sin^6 x}{3}}\, dx = \frac{8\pi^3 - 7\pi}{48}$$