AM II (Aula 7)

3-35 Calcule divergência e rotacional da função $f(x,y,z) = (xy, yz, xz)$

$$\operatorname{div} f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = y + z + x$$

$$\operatorname{rot} f = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{array}{ccc} + & - & + \end{array} \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix} =$$

$$= (-y, -z, -x).$$

3- 40 Para $h(x,y,z) = 5xy + 3x^2y + 2xz + 5yz - z^3$

Calcule:

a) $\nabla h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}\right) =$

$= \left(5y + 6xy + 2z,\ 5x + 3x^2 + 5z,\ 2x + 5y - 3z^2\right)$

b) $|H| = \begin{vmatrix} \nabla\left(\frac{\partial h}{\partial x}\right) \\ \nabla\left(\frac{\partial h}{\partial y}\right) \\ \nabla\left(\frac{\partial h}{\partial z}\right) \end{vmatrix} = \begin{vmatrix} 6y & (5+6x) & 2 \\ (5+6x) & 0 & 5 \\ 2 & 5 & -6z \end{vmatrix}$

$|H((0,b,1)| = \begin{vmatrix} 6b & 5 & 2 \\ 5 & 0 & 5 \\ 2 & 5 & -6 \end{vmatrix} =$

$= -5\begin{vmatrix} 5 & 2 \\ 5 & -6 \end{vmatrix} - 5\begin{vmatrix} 6b & 5 \\ 2 & 5 \end{vmatrix} = -5(-30-10) - 5(30b-10) =$

$= 200 - 150b + 50 = 250 - 150b$

c) $\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 6y - 6z.$

$\boxed{4\text{-}g}$ $g(x,y,z) = \begin{cases} u = x + y + z \\ v = 2y + z \\ w = -x + 2z \end{cases}$

a) Determine o jacobiano de g.

$$|J| = \begin{vmatrix} \nabla g_1 \\ \nabla g_2 \\ \nabla g_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = 6 - 1 = 5$$

b) g é invertível?

Sim, porque $|J| = 5 \neq 0$

c) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -1 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{4}{5}u - \frac{2}{5}v - \frac{w}{5} \\ -\frac{u}{5} + \frac{3}{5}v - \frac{w}{5} \\ \frac{2}{5}u - \frac{v}{5} + \frac{2}{5}w \end{bmatrix}$$

$\boxed{4\text{-}10}$ $f(x,y,z) = (y^2+z^2,\ x^2+z^2,\ x^2+y^2)$

a) Estude a existência da função inversa localmente.

São os pontos onde o determinante da matriz jacobiana é diferente de zero.

$$|J| = \begin{vmatrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \end{vmatrix} = \begin{vmatrix} 0 & 2y & 2z \\ 2x & 0 & 2z \\ 2x & 2y & 0 \end{vmatrix} = -2y(-4xz) + 2z(4xy) = 16xyz.$$

Temos de ter $xyz \neq 0$.

b)...

c) $(f^{-1})'(f(x)) = [f'(x)]^{-1}$ $\quad ((2,2,2) = f(1,1,1))$

$$(f^{-1})'(2,2,2) = [f'(1,1,1)]^{-1} = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

(4.10 Extra) Cálculo da matriz inversa.

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 2 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right]^{L_1 \leftrightarrow L_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right]^{\frac{L_1}{2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right]^{L_3 - 2L_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 2 & -2 & 0 & -1 & 1 \end{array}\right]^{\frac{L_2}{2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 1/2 & 0 & 0 \\ 0 & 2 & -2 & 0 & -1 & 1 \end{array}\right]^{L_3 - 2L_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -4 & -1 & -1 & 1 \end{array}\right]^{-\frac{L_3}{4}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/4 & 1/4 & -1/4 \end{array}\right]^{L_2 - L_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1/4 & -1/4 & 1/4 \\ 0 & 0 & 1 & 1/4 & 1/4 & -1/4 \end{array}\right]^{L_1 - L_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/4 & \frac{1}{4} & 1/4 \\ 0 & 1 & 0 & 1/4 & -1/4 & 1/4 \\ 0 & 0 & 1 & 1/4 & 1/4 & -1/4 \end{array}\right]$$

$$\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}^{-1} = \frac{1}{4} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$\boxed{4-12}$ Mostre que $\overbrace{x^2 y + \sin y - x}^{\varphi(x,y)} = 0$ define implicitamente uma função diferenciável $y = f(x)$ e determine $\frac{dy}{dx}$, numa vizinhança do ponto $(0,0)$.

- $\varphi(0,0) = 0$ ✓
- $\frac{\partial \varphi}{\partial x}(x,y) = 2xy - 1$
- $\frac{\partial \varphi}{\partial y}(x,y) = x^2 + \cos y$

Como estas derivadas parciais são contínuas φ é de classe C^1

- $\frac{\partial \varphi}{\partial y}(0,0) = \cos(0) = 1 \neq 0$

logo a equação define y implicitamente como função de x numa vizinhança de $(0,0)$ e

$$\frac{dy}{dx}(x) = -\frac{\frac{\partial \varphi}{\partial x}(x,y)}{\frac{\partial \varphi}{\partial y}(x,y)} = -\frac{2xy-1}{x^2+\cos y}$$

$\boxed{4-16}$ $\underbrace{x+2xy+3z^2+2x^2z-1}_{=\varphi(x,y,z)}=0$

a) Para que valores de z a equação define, implicitamente, z como função de x e y numa vizinhança de $(1,0,z)$.

• $\varphi(1,0,z)=0 \Leftrightarrow 1+3z^2+2z-1=0 \Leftrightarrow$

$\Leftrightarrow z(3z+2)=0 \Leftrightarrow z=0 \vee z=-\frac{2}{3}$.

• $\begin{cases} \frac{\partial\varphi}{\partial z}=6z+2x^2 \\ \frac{\partial\varphi}{\partial x}=1+2y+4xz \\ \frac{\partial\varphi}{\partial y}=2x \end{cases}$, como todas estas derivadas parciais são contínuas φ é C^1

• $\begin{cases} \frac{\partial\varphi}{\partial z}(1,0,0)=2\neq 0 \\ \frac{\partial\varphi}{\partial z}(1,0,-\frac{2}{3})=-2\neq 0 \end{cases}$ logo $\varphi=0$ define z, implicitamente, como função de x e y em vizinhanças dos pontos $(1,0,0)$ e $(1,0,-\frac{2}{3})$

b) $\frac{\partial z}{\partial x}=-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial z}}=-\frac{1+2y+4xz}{6z+2x^2}$

$\frac{\partial z}{\partial y}=-\frac{\frac{\partial\varphi}{\partial y}}{\frac{\partial\varphi}{\partial z}}=-\frac{2x}{6z+2x^2}=\frac{-x}{3z+x^2}$

c) $\frac{\partial^2 z}{\partial x\partial y}=\ldots$

fica a cargo do aluno.

$\boxed{4-17}$ $\begin{cases} \underbrace{e^{u} + x\cos v}_{\varphi_1(x,y,u,v)} = 0 \\ \underbrace{e^{u} + y\sin v - 1}_{\varphi_2(x,y,u,v)} = 0 \end{cases}$

(i) Verifique que o sistema define u e v, implicitamente, como funções de x e y numa vizinhança de $(-1,1,0,0)$.

- $\begin{cases} \varphi_1(-1,1,0,0) = 0 \checkmark \\ \varphi_2(-1,1,0,0) = 0 \checkmark \end{cases}$

- $\begin{cases} \nabla\varphi_1 = (\cos v, 0, e^{u}, -x\sin v), \ \varphi_1 \text{ é } C^1 \\ \nabla\varphi_2 = (0, \sin v, e^{u}, y\cos v), \ \varphi_2 \text{ é } C^1 \end{cases}$

- $\dfrac{\partial(\varphi_1,\varphi_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial\varphi_1}{\partial u} & \frac{\partial\varphi_1}{\partial v} \\ \frac{\partial\varphi_2}{\partial u} & \frac{\partial\varphi_2}{\partial v} \end{vmatrix} = \begin{vmatrix} e^{u} & -x\sin v \\ e^{u} & y\cos v \end{vmatrix}$

$\dfrac{\partial(\varphi_1,\varphi_2)}{\partial(u,v)}(-1,1,0,0) = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0$, logo o sistema define u, v, implicitamente, como funções de x e y numa vizinhança de $(-1,1,0,0)$.

(1c) Determine $\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}_{(-1,1)}$

- Pelo teorema da derivada da função implícita:

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}_{(-1,1)} = - \begin{bmatrix} \frac{\partial \varphi_1}{\partial u} & \frac{\partial \varphi_1}{\partial v} \\ \frac{\partial \varphi_2}{\partial u} & \frac{\partial \varphi_2}{\partial v} \end{bmatrix}^{-1}_{(-1,1,0,0)} \begin{bmatrix} \frac{\partial \varphi_1}{\partial x} & \frac{\partial \varphi_1}{\partial y} \\ \frac{\partial \varphi_2}{\partial x} & \frac{\partial \varphi_2}{\partial y} \end{bmatrix}_{(-1,1,0,0)} =$$

$$= - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} =$$

$$= - \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$\boxed{4-22}$ Escreva a fórmula de Taylor, de ordem 2, da função $f(x,y,z) = xyz$ no ponto $(1,1,1)$.

$$f\left[(1,1,1)+h\right] = f(1,1,1) + f'_h(1,1,1) + \frac{1}{2} f''_h(1,1,1) + \frac{1}{3!} f'''_h\left((1,1,1)+\theta h\right)$$

com $0 < \theta < 1$.

- $f(x,y,z) = xyz \quad , \quad f(1,1,1) = 1$
- $\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (yz, xz, xy)$
- $\nabla f(1,1,1) = (1,1,1)$
- $f'_h(1,1,1) = \nabla f(1,1,1) \mid h = (1,1,1) \mid (h_1, h_2, h_3) = h_1 + h_2 + h_3$.
- $$H(x,y,z) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix} = \begin{bmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix}$$

$$H(1,1,1) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\left(H = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}\right)$$

$$f''_h(1,1,1) = H^T \mathcal{H}(1,1,1)\, H =$$

$$= \begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} =$$

$$= \begin{bmatrix} (h_2+h_3) & (h_1+h_3) & (h_1+h_2) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} =$$

$$= (h_2+h_3)h_1 + (h_1+h_3)h_2 + (h_1+h_2)h_3 =$$

$$= 2(h_1 h_2 + h_1 h_3 + h_2 h_3).$$

Pelo que:

$$f[(1,1,1)+h] = 1 + h_1 + h_2 + h_3 + (h_1 h_2 + h_1 h_3 + h_2 h_3) +$$

$$+ \frac{1}{3!} f'''_h((1,1,1)+\theta h), \text{ com } 0 < \theta < 1.$$