(Aula-6)

$\boxed{3\text{-}13}$ Considere as funções

$f(x,y,z) = (z, -x^2, -y^2)$ e $g(x,y,z) = x+y+z$

e sejam $u = (2, 3, \frac{1}{2})$ e $v = (1, 2, 3)$.

a) Calcule as matrizes jacobianas de $f$, $g$ e $g \circ f$.

$$J_f(x,y,z) = \begin{bmatrix} 0 & 0 & 1 \\ -2x & 0 & 0 \\ 0 & -2y & 0 \end{bmatrix},$$

$$J_g(x,y,z) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \quad \text{e}$$

$$J_{g\circ f}(x,y,z) = J_g[f(x,y,z)]\; J_f(x,y,z) =$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -2x & 0 & 0 \\ 0 & -2y & 0 \end{bmatrix} = \begin{bmatrix} -2x & -2y & 1 \end{bmatrix}$$

b) Calcule as seguintes derivadas:

$$\frac{\partial f}{\partial v}(1,1,1),\ \frac{\partial f}{\partial u}(0,0,1),\ \frac{\partial g}{\partial v}(0,1,0) \text{ e } \frac{\partial (g \circ f)}{\partial u}(2,0,1).$$

- $\frac{\partial f}{\partial v}(1,1,1) = f'_v(1,1,1) = J_f(1,1,1)\,V =$

$$= \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}.$$

- $f'_u(0,0,1) = J_f(0,0,1)\,U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}.$

- $g'_v(0,1,0) = \nabla g(0,1,0) \mid v = J_g(0,1,0)\,V = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 6$

- $(g \circ f)'_u(2,0,1) = J_{g\circ f}(2,0,1)\,U =$

$$= \begin{bmatrix} -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1/2 \end{bmatrix} = -\frac{15}{2}$$

$\boxed{3\text{-}15}$ Seja $f: D \subseteq \mathbb{R}^3 \to \mathbb{R}$ uma função de classe $C^1$ tal que:

$Df(0,0,e) = [1\ 2\ 3]$ e seja

$h(x,y,z) = f(xy^2z^3, \sin x, z e^{5-y^2})$.

Calcule $\frac{\partial h}{\partial x}(0,-2,1)$.

Observações:

i) $(f \circ g)'(a) = f'[g(a)]\, g'(a)$

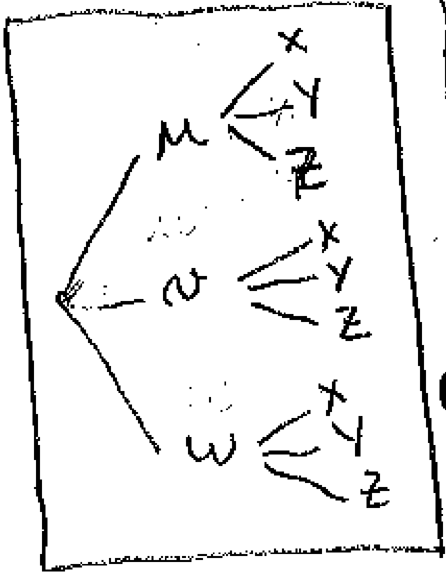
ii) $Df(0,0,e) = f'(0,0,e) =$

$$= \left[ \frac{\partial f}{\partial x}(0,0,e) \quad \frac{\partial f}{\partial y}(0,0,e) \quad \frac{\partial f}{\partial z}(0,0,e) \right] =$$

$$= [1 \quad 2 \quad 3].$$

iii) $h = f(u, v, v)$ onde

$$g(x,y,z) = \begin{cases} u = x y^2 z^2 \\ v = \sin x \\ w = z\, e^{5-y^2} \end{cases} ; \quad \boxed{g(0,-2,1) = \begin{cases} u = 0 \\ v = 0 \\ w = e \end{cases}}$$




$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}$$

$$= \frac{\partial f}{\partial u}\, y^2 z^3 + \frac{\partial f}{\partial v} \cos x.$$

Assim:

$$\frac{\partial h}{\partial x}(0,-2,1) = \frac{\partial f}{\partial u}\left[g(0,-2,1)\right](4) + \frac{\partial f}{\partial v}\left[g(0,-2,1)\right] =$$

$$= \frac{\partial f}{\partial u}(0,0,e)(4) + \frac{\partial f}{\partial v}(0,0,e) =$$

$$= 4 + 2 = 6.$$

$\boxed{3\text{-}20}$ Considere a função $f: \mathbb{R}^3 \to \mathbb{R}^3$ definida por:

$$f(x,y,z) = (\sin(xy), \cos(xy), xz).$$

Calcule o diferencial de $f$ no ponto $P = (0,2,1)$ segundo o vetor $u = (-1,2,1)$.

- O diferencial de $f$ num ponto arbitrário $(x,y,z)$ é

$$(df)(x,y,z) = \begin{bmatrix} y\cos(xy) & x\cos(xy) & 0 \\ -y\sin(xy) & -x\sin(xy) & 0 \\ z & 0 & x \end{bmatrix}$$

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$$(df)(0,2,1) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

e

$$(df)_{(-1,2,1)}^{(0,2,1)} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}.$$

$\boxed{3-28}$ Determine uma equação da reta normal e do plano tangente, no ponto $P=(3,4,-2)$, ao cone

$$C=\{(x,y,z)\in\mathbb{R}^3 : z=3-\sqrt{x^2+y^2}\}$$

Uma superfície é definida por uma equação da forma

$$\varphi(x,y,z)=0$$

neste caso:

$$S:\ z+\sqrt{x^2+y^2}-3=0$$

Convém verificar que o ponto $P=(x_0,y_0,z_0)=(3,4,-2)\in S$:

$$-2+\sqrt{3^2+4^2}-3=0 \Leftrightarrow -5+5=0 \Leftrightarrow$$

$$\Leftrightarrow 0=0 \checkmark$$

Calculem-se as derivadas parciais de $\varphi(x, y, z)$ em P.

$$\frac{\partial \varphi}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \quad , \quad \frac{\partial \varphi}{\partial x}(P) = \frac{3}{5}$$

$$\frac{\partial \varphi}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \quad , \quad \frac{\partial \varphi}{\partial y}(P) = \frac{4}{5}$$

$$\frac{\partial \varphi}{\partial z} = 1 \quad , \quad \frac{\partial \varphi}{\partial z}(P) = 1.$$

Pelo que a equação do plano tangente à superfície S no ponto P é:

$$\Pi_t : \nabla \varphi(P) \mid (x - x_0, y - y_0, z - z_0) = 0$$

$$\Leftrightarrow \left(\frac{3}{5}, \frac{4}{5}, 1\right) \mid (x-3, x-4, z+2) = 0$$

$$\Leftrightarrow \frac{3}{5}(x-3) + \frac{4}{5}(x-4) + z + 2 = 0$$

A equação da reta normal no ponto P pode ser apresentada usando qualquer uma das suas equações por exemplo:

1) equação vetorial:

$$r_n: (x, y, z) = P + t\,\nabla\varphi(t) \quad , \quad t \in \mathbb{R}$$

$$r_n: (x, y, z) = (3, 4, -2) + t\left(\frac{3}{5}, \frac{4}{5}, 1\right), \quad t \in \mathbb{R}.$$

2) equações paramétricas:

$$r_n: \begin{cases} x = x_0 + t\,\dfrac{\partial\varphi}{\partial x}(P) \\ y = y_0 + t\,\dfrac{\partial\varphi}{\partial y}(P) \\ z = z_0 + t\,\dfrac{\partial\varphi}{\partial z}(P) \end{cases} \quad , \quad t \in \mathbb{R};$$

$$r_n = \begin{cases} x = 3 + \dfrac{3}{5}t \\ y = 4 + \dfrac{4}{5}t \\ z = t - 2 \end{cases} \quad , \quad t \in \mathbb{R};$$

3) equação Cartesiana

$$r_n: \frac{x-x_0}{\frac{\partial \varphi}{\partial x}(P)} = \frac{y-y_0}{\frac{\partial \varphi}{\partial y}(P)} = \frac{z-z_0}{\frac{\partial \varphi}{\partial z}(P)},$$

$$r_n: \frac{x-3}{\frac{3}{5}} = \frac{y-4}{\frac{4}{5}} = \frac{z+2}{1} \Leftrightarrow$$

$$\Leftrightarrow \frac{5}{3}(x-3) = \frac{5}{4}(y-4) = z+2.$$

$\boxed{3-34}$ Determine a equação dos planos tangentes à superfície

$$x^2+2y^2+z^2=1$$

de modo que sejam paralelos ao plano

$$y=2z.$$

A superfície é definida por

$$S:\ \varphi(x,y,z)=0 \iff x^2+2y^2+z^2-1=0.$$

Para os planos serem paralelos

$$y-2z=0$$

temos de ter:

$$\nabla\varphi(P)=\nabla\varphi(x_0,y_0,z_0)=\alpha(0,1,-2).$$

Assim:

$$\frac{\partial \varphi}{\partial x}(P) = 0 \Leftrightarrow 2x_0 = 0 \Leftrightarrow x_0 = 0$$

$$\frac{\partial \varphi}{\partial y}(P) = \alpha \Leftrightarrow 4y_0 = \alpha \Leftrightarrow y_0 = \frac{\alpha}{4}$$

$$\frac{\partial \varphi}{\partial z}(P) = -2\alpha \Leftrightarrow 2z_0 = -2\alpha \Leftrightarrow z_0 = -\alpha.$$

Pelo que os pontos pretendidos são da forma:

$$P = \left(0, \frac{\alpha}{4}, -\alpha\right);$$

além disso, P tem de pertencer a S:

$$2y_0^2 + z_0^2 = 1 \Leftrightarrow 2\frac{\alpha^2}{16} + \alpha^2 = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{9}{8}\alpha^2 = 1 \Leftrightarrow \alpha^2 = \frac{8}{9} \Leftrightarrow \alpha = \pm\frac{2\sqrt{2}}{3}.$$

Pelo que temos dois pontos

$$P_1 = \left(0, \frac{\sqrt{2}}{6}, -\frac{2\sqrt{2}}{3}\right) \text{ e}$$

$$P_2 = \left(0, -\frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right).$$

Recorde que $\nabla\varphi = (2x, 4y, 2z)$,

pelo que: $\nabla\varphi(P_1) = \left(0, \frac{2\sqrt{2}}{3}, -\frac{4\sqrt{2}}{3}\right)$

assim:

$$\Pi_t^1 : \nabla\varphi(P_1) \mid (x - x_0, y - y_0, z - z_0) = 0$$

$$\Leftrightarrow \left(0, \frac{2\sqrt{2}}{3}, -\frac{4\sqrt{2}}{3}\right) \mid \left(x, y - \frac{\sqrt{2}}{6}, z + \frac{2\sqrt{2}}{3}\right) = 0$$

$$\Leftrightarrow \frac{2\sqrt{2}}{3}\left(y - \frac{\sqrt{2}}{6}\right) - \frac{4\sqrt{2}}{3}\left(z + \frac{2\sqrt{2}}{3}\right) = 0$$

A equação de $\Pi_t^2$ fica a cargo do aluno.