BIG DATA COMPUTING

Homework 3
Dimensionality Reduction

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Assignment 2

Given a square symmetric matrix $A \in \mathbb{R}^{n\times n}$ and its spectral decomposition $A = \sum_{i=1}^{n} \lambda_i \underline{u}_i \underline{u}_i^T$ we can derive more compact formulas for some operations like power of a matrix or matrix exponential.

Question 1

For simplicity we'll consider each of the eigenvector to be L_2 normalized: $\underline{u}_i^T \underline{u}_i = 1$. So we can write:

$$
\begin{gathered}
A^k = \prod_{j=1}^{k} A = \\
\prod_{j=1}^{k} \sum_{i=1}^{n} \lambda_i \underline{u}_i \underline{u}_i^T = \sum_{i=1}^{n} \prod_{j=1}^{k} \lambda_i \underline{u}_i \underline{u}_i^T = \\
\sum_{i=1}^{n} \lambda_i^k \prod_{j=1}^{k} \underline{u}_i \underline{u}_i^T = \sum_{i=1}^{n} \lambda_i^k \underline{u}_i [\prod_{j=1}^{k-1} (\underline{u}_i^T \underline{u}_i)] \underline{u}_i^T = \\
\sum_{i=1}^{n} \lambda_i^k \underline{u}_i \underline{u}_i^T
\end{gathered} \tag{1}
$$

We could also prove it by induction. Using the matrix notation we rewrite the spectral decomposition as $A = U\Lambda U^T$. We want to prove that $A^k = U\Lambda^k U^T$. The base induction is clearly satisfied for $k = 1$, giving us back the spectral decomposition. The inductive step is based on proving that $A^{n+1} = U\Lambda^{n+1}U^T$ stating that $A^n = U\Lambda^n U^T$:

$$
A^{n+1} = AA^n = (U\Lambda U^T)(U\Lambda^n U^T) = U\Lambda(U^T U)\Lambda^n U^T = U\Lambda\Lambda^n U^T = U\Lambda^{n+1}U^T \tag{2}
$$

Question 2

If A is invertible then $\lambda_i \neq 0, i = 1, ..., n$: this clearly implies that also the diagonal matrix of the eigenvalues is invertible, so that we can guess using the previous result that

$$
A^{-1} = \sum_{i=1}^{n} \frac{1}{\lambda_i} \underline{u}_i \underline{u}_i^T = U\Lambda^{-1}U^T \tag{3}
$$

This last claim can be easily proved considering that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ and that since $UU^T = \mathbb{1}$, $U^T = U^{-1}$, so that:

$$
A^{-1} = (U\Lambda U^T)^{-1} = (U^T)^{-1}\Lambda^{-1}U^{-1} = U\Lambda^{-1}U^T \tag{4}
$$

Assignment 3

Now we want to generalize some of the previous results to non-square matrices using singular value decomposition. In particular we are given $A \in \mathbb{R}^{n\times m}$ and its SVD $A = \sum_{i=1}^{r} \sigma_i \underline{u}_i \underline{v}_i^T$

Question 1

We want to derive an expression for $(AA^T)^k$ in terms of its singular values decomposition. First of all, using the matrix notation, we have:

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V \Sigma^T U^T = U\Sigma^2 U^T \tag{5}$$

here we used the fact that both U and V are orthogonal projector assuming each of the left and right singular vector is L_2 normalized, and that Σ is a diagonal matrix, so that $\Sigma\Sigma^T = \Sigma\Sigma = \Sigma^2$.

We can now claim that, using previous results, that

$$(AA^T)^k = U\Sigma^{2k}U^T \tag{6}$$

We can proceed again by induction as we did previously, since the base induction is again satisfied:

$$(AA^T)^k \Bigg|_{k=1} = AA^T = U\Sigma^2 U^T \tag{7}$$

The inductive step must prove that $(AA^T)^{k+1} = U\Sigma^{2(k+1)}U^T$ given that $(AA^T)^k = U\Sigma^{2k}U^T$:

$$\begin{aligned}(AA^T)^{k+1} = (AA^T)^k(AA^T) = (U\Sigma^{2k}U^T)(U\Sigma^2U^T) = \\ U\Sigma^{2k}(U^TU)\Sigma^2U^T = U\Sigma^{2k}\Sigma^2U^T = U\Sigma^{2(k+1)}U^T\end{aligned} \tag{8}$$

In alternative notation, $(AA^T)^k = \sum_{i=1}^{r} \sigma_i^{2k} \underline{u}_i \underline{u}_i^T$.

Question 2

If Q is a square matrix whose columns form an orthonormal basis for $\mathbb{R}^n$ then it means that:

$$Q^TQ = \mathbb{1}_n \tag{9}$$

We want to prove that this implies that the rows of Q also form an orthonormal basis for $\mathbb{R}^n$, i.e.:

$$QQ^T = \mathbb{1}_n \tag{10}$$

This can be easily demonstrated considering the definition of existence of the inverse of a matrix: given a non-singular square matrix A, its inverse A^{-1} satisfies $AA^{-1} = A^{-1}A = \mathbb{1}$. Being in this case $Q^T = Q^{-1}$, our claim is proved.

Question 3

We want to prove that given $A \in \mathbb{R}^{n\times n}$ and its SVD $A = \sum_{i=1}^{n} \sigma_i u_i v_i^T = U\Sigma V^T$, its inverse is $B = \sum_{i=1} \frac{1}{\sigma_i} v_i u_i^T = V\Sigma^{-1}U^T$. We know U and V are orthonormal matrices being A a squared matrix , and from previous result we have:

$$U^TU = UU^T = \mathbb{1}_n \tag{11}$$

$$V^TV = VV^T = \mathbb{1}_n \tag{12}$$

So now we should compute AB and verify it is equal to the identity matrix:

$$AB = (U\Sigma V^T)(V\Sigma^{-1}U^T) = U\Sigma(V^TV)\Sigma^{-1}U^T = U(\Sigma\Sigma^{-1})U^T = UU^T = \mathbb{1}_n \tag{13}$$

Assignment 4

Question 1

For any real valued matrix $A \in \mathbb{R}^{n\times m}$ it's easy to show that AA^T is square and symmetric, and we want to prove that it is positive semi definite. We can reason similarly on the first steps in the constructive demonstration of the existence of SVD, so we now should consider the vector $A^T x$ and its squared norm that clearly is non-negative:

$$0 \leq ||A^T x||^2 = (A^T x)^T A^T x = x^T AA^T x \tag{14}$$

Since we didn't make any assumption on x we can conclude that $x^T AA^T x \geq 0 \ \forall x \in \mathbb{R}^n$ that is the very definition of positive semi-definiteness for AA^T.

Question 2

We are given a square symmetric matrix $A \in \mathbb{R}^{n\times n}$ and we want to prove that it positive semidefinite if and only if all its eigenvalues are non negative.

Firstly let's prove that non-negativity of eigenvalues is a necessary condition for a matrix to be positive semidefinite.

If we consider a positive semidefinite matrix and we consider a generic eigenvalue-eigenvector pair $\{\lambda_i, x_i\}$ we have the following because of definition of positive semidefiniteness $x_i^T Ax_i \geq 0$ that implies $x_i^T \lambda_i x_i \geq 0$. From this last we have that $\lambda_i x_i^T x_i \geq 0$ so $\lambda_i ||x_i||^2 \geq 0$ and finally $\lambda_i \geq 0$.

Now we can prove that the non-negativity of eigenvalues is a sufficient condition for a matrix to be positive semidefinite.

Let's consider again the quadratic form in A for a generic vector and its expansion through its spectral decomposition $A = U\Lambda U^T$:

$$x^T Ax = x^T U\Lambda U^T x = x^T U\Lambda(x^T U)^T \tag{15}$$

So we have a quadratic form in Λ: changing notation we have:

$$x^T U\Lambda(x^T U)^T = \sum_{i=1}^{n} \lambda_i (x^T U)_i^2 \geq 0 \tag{16}$$

where the last inequality is ensured by the hypotesis of non-negativity for the eigenvalues and therefore $x^T Ax \geq 0$: this ends our proof.