Opinion Dynamics over Cellular Sheaves

Enhancing the expressivity of time diffusion processes over graphs

Social Networks and Online Markets

Leonardo Di Nino (1919479)

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Sapienza Università di Roma

The purpose of my final project in the *"Social Networks and Online Marketds"* course was to address opinion dynamics modeling from an algebraic topological perspective. The main reference is the following paper:

- *Opinion Dynamics on Discourse Sheaves*, Jacob Hansen, Robert Ghrist, 2020

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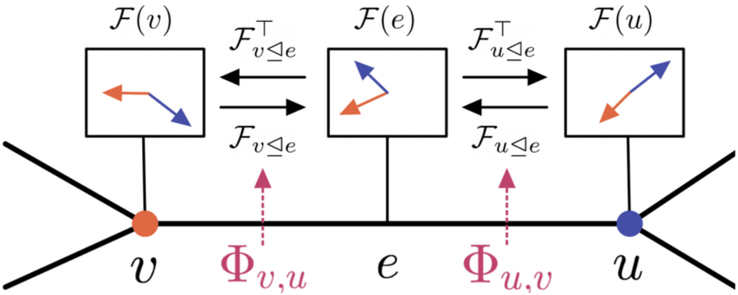
Why should we incorporate sheaves in networking?



Cellular sheaves introduce **geometric structures** over a certain combinatorial topology, enhancing the expressivity and the possibilities of **learning and representation problems**

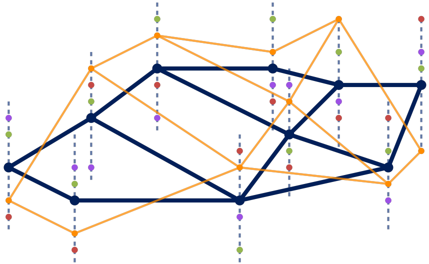
Definition (Graph cellular sheaf)

Considering a graph $G(V, E)$, we define a **graph cellular sheaf** $\mathcal{F}$ specificied by:

- A vector space $\mathcal{F}(v)$ for each node $v \in V$ (**stalk** over a vertex);
- A vector space $\mathcal{F}(e)$ for each edge $e \in E$ (**stalk** over a edge);
- A linear map $\mathcal{F}_{v \trianglelefteq e} : \mathcal{F}(v) \to \mathcal{F}(e)$ for each incidency relation $v \trianglelefteq e$ for each edge (**restriction maps**).

The direct sum of the stalks are called **spaces of cochains**. In particular:

- 0-cochains x belong to $C^0(G,\mathcal{F}) = \bigoplus_{v\in V}\mathcal{F}_v$;
- 1-cochains ξ belong to $C^1(G,\mathcal{F}) = \bigoplus_{e\in E}\mathcal{F}_e$.



The **coboundary map** is the linear operator $\delta : C^1(G, \mathcal{F}) \to C^0(G, \mathcal{F})$ defined as follows, up to an arbitrary orientation of the restriction maps:

$$(\delta^T x)_e = \mathcal{F}_{v \trianglelefteq e} x_v - \mathcal{F}_{u \trianglelefteq e} x_u \tag{1}$$

The **sheaf Laplacian** is the operator defined as follows:

$$L_{\mathcal{F}} = \delta \delta^T : C^0(G, \mathcal{F}) \to C^0(G, \mathcal{F}) \tag{2}$$

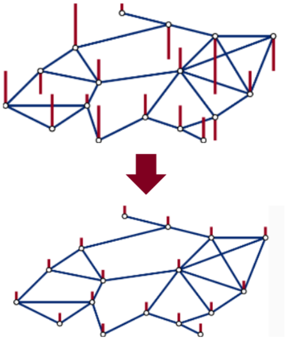
The key concepts - Dirichlet Energy over edges

A classic result in network science is that we can quantify the network ”disagreement” using the quadratic form in the Laplacian:

$$x^T L x = \sum_{i,j \in V} A[i,j](x_i - x_j)^2 \tag{3}$$

The Laplacian of a graph cellular sheaf supports a generalization of such a concept:

$$x^T L_{\mathcal{F}} x = \sum_{e \in E} ||\mathcal{F}_{v \trianglelefteq e} x_v - \mathcal{F}_{u \trianglelefteq e} x_u||^2 \tag{4}$$



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The reference model - $\mathbb{R}^n$ constant sheaf

In a $\mathbb{R}^n$ constant sheaf every stalk is $\mathbb{R}^n$ and every restriction maps is the identity map. The related discourse sheaf is such that:

- Each agent residing on each node has a *private opinion* being a vector in $\mathbb{R}^n$: each component of this vector represents an opinion on a certain topic among n topics;
- Each edge represent a pairwise communication, leading to an *expressed opinion* over the same n topics;
- The restriction maps are all set to the identity maps $\mathbb{I}_n$: this reconnects to classic approaches where private opinions are expressed without modifications.

For a sheaf $\mathcal{F}$ defined over a graph and a 0-cochain x, the dynamic

$$\frac{dx}{dt} = -\alpha L_{\mathcal{F}} x \tag{5}$$

has solutions equal to

$$x(t) = e^{-\alpha L_{\mathcal{F}} t} x_0 \tag{6}$$

and converges to the projection of the initial state $x_0 \in C^0(G, \mathcal{F})$ over the null space of the Laplacian L, in a way that is completely analogous to the classic graph consensus dynamic. $\ker(L_{\mathcal{F}})$ is called **0th-cohomology**, or **space of global sections**, i.e. 0-cochains satisfying $\mathcal{F}_{v \trianglelefteq e} x_v = \mathcal{F}_{u \trianglelefteq e} x_u$ for each edge.

Laplacian heat equation: from graphs to sheaves

2 Opinion dynamics over discourse sheaves

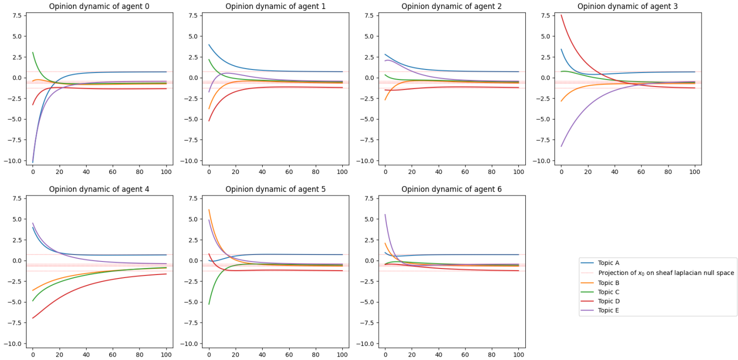


Figure: Opinion trajectories for a constant $\mathbb{R}^n$ sheaf defined over a network of 7 agents and a $n = 5$ topics basis and Gaussian distributed initial opinions

Laplacian heat equation: from graphs to sheaves

2 Opinion dynamics over discourse sheaves

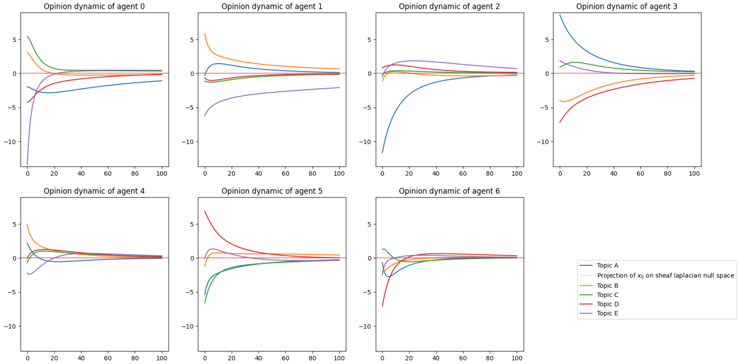


Figure: Opinion trajectories for a random sheaf defined over a network of 7 agents and a $n = 5$ topics basis: the consensus is trivial even if the underlying graph is connected

Modeling stubborn agents

A first variation we can introduce is to define U-restricted dynamics, i.e. considering a set U of agents which do not modify their opinion:

$$\frac{dx}{dt} = -\alpha \Pi_U (L_{\mathcal{F}} x) \tag{7}$$

being Π_U a proper function setting to zero the dynamic of stubborn agents. In this case the dynamic is proved to converge to the *harmonic extension* of $x_0|_U$ being the closest to x_0: the harmonic extension $\widetilde{x}_0|_U$ of $x_0|_U$ satisfies:

$$\begin{cases} [\widetilde{x}_0|_U]_u = [x_0|_U]_u & \forall u \in U \\ [L_{\mathcal{F}} \widetilde{x}_0|_U]_v = 0 & \forall v \notin U \end{cases}$$

Modeling stubborn agents

3 Advanced opinion dynamics modeling

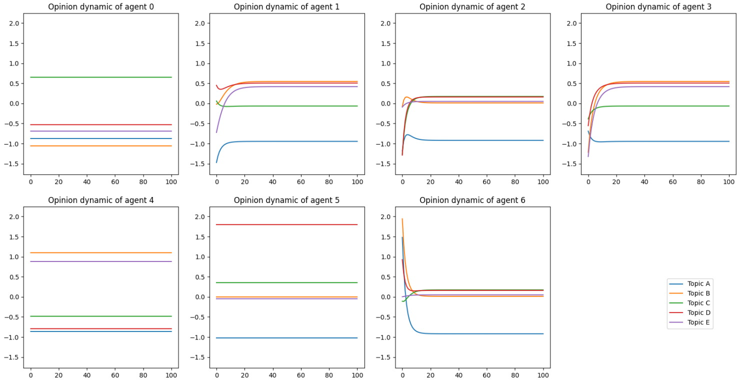


Figure: Opinion trajectories for a constant $\mathbb{R}^n$ sheaf defined over a network of 7 agents and a $n = 5$ topics basis and Gaussian distributed initial opinions with $|U| = 3$ stubborn agents

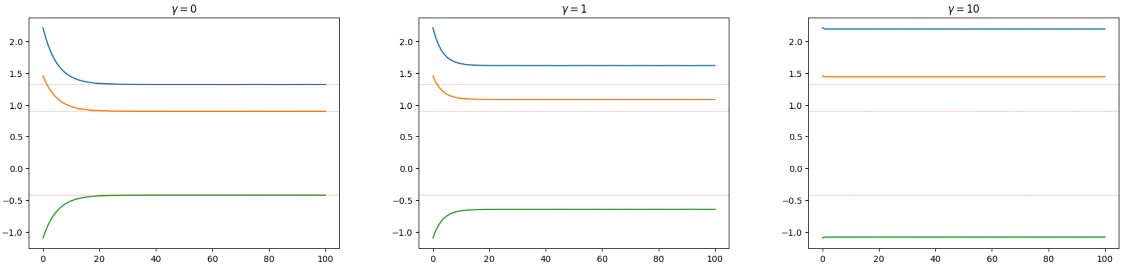
Weighted reluctance

This model requires a specific augmentation construction:

1. Given $G(V, E)$, define a copy of V and attach each $v \in V$ with its copy $v' \in V'$ via e';
2. Extend the sheaf $\mathcal{F}$ to $\mathcal{F}'$ by letting for each $e \sim (v, v')$:
   - $\mathcal{F}'(v) = \mathcal{F}'(v') = \mathcal{F}'(e'), ;$
   - $\mathcal{F}'_{v \trianglelefteq e'} = \mathcal{F}'_{v' \trianglelefteq e'} = \sqrt{\gamma}\mathbb{I}$
3. Given an initial condition $x(0) \in C^0(G, \mathcal{F})$, extend it to $x'(0) \in C^0(G', \mathcal{F}')$ by setting $x_0(v) = x'_0(v')$ for each $e \sim (v, v')$;
4. Apply the stubborn-agents model to $\mathcal{F}'$ defining the set of stubborn agents as V'.

Weighted reluctance

3 Advanced opinion dynamics modeling



Opinion trajectory of one agent on a constant $\mathbb{R}^n$ $(n = 3)$ discourse sheaf over a graph with $|V| = 4$ agents modifying the reluctance weight

Opinion dynamics up to now are all based on private-opinions distributions, i.e. time functionals of the 0-cochains. Using sheaf formalism, we can also model dynamics of the restriction maps using the coboundary matrix:

$$\frac{d\delta^T}{dt} = -\beta \Pi_\delta(\delta^T x x^T) \tag{8}$$

where x is a fixed distribution of private opinions and Π_δ is a projector to preserve the block structure of δ. This equation can be decomposed within each restriction map as:

$$\frac{d\mathcal{F}_{v \trianglelefteq e}}{dt} = -\beta(\mathcal{F}_{v \trianglelefteq e} x_v - \mathcal{F}_{u \trianglelefteq e} x_u) x_v^T \tag{9}$$

The main result is that this dynamic converges to the nearest sheaf in terms of l_2 distance between the respective coboundary maps such that the fixed private opinion distributions is a global section.

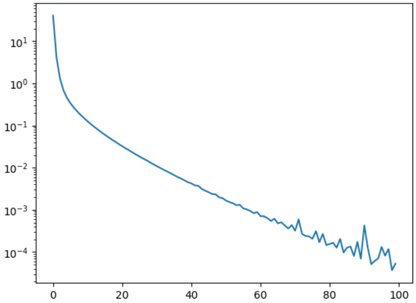


Figure: Network disagreement through time while learning to lie

Joint dynamic of private opinion and expression

The culmination of this modeling approach is to combine in a single system the dynamics of the private opinions and the expression:

$$\begin{cases} \frac{dx}{dt} = -\alpha \Pi_U(\delta\delta^T x) + Bu \\ \frac{d\delta}{dt} = -\beta \Pi_G(\delta^T x x^T) \\ \frac{dy}{dt} = Cx \\ x(0) = x_0 \\ \delta(0) = \delta_0 \end{cases}$$

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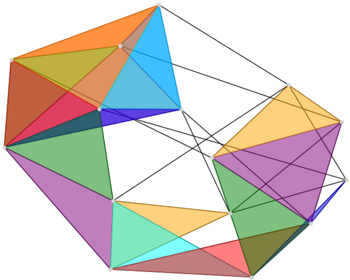
4 Beyond graph modeling: group opinion dynamics over simplicial complexes

Why simplicial complexes?

Complex systems are well modeled by graphs and learning methods on them has gained fortune in exploring *pairwise relations*.
Unfortunately, this well established frameworks are not capable to catch higher order informations residing in more complex relations: further topological structures provides new perspective on this problem, and **simplicial complexes** represent the first step in this direction.



Given a finite set of vertices $\mathcal{V} = \{v_0, ...v_{N-1}\}$, a *k-simplex* σ_i^k is an unordered set of $k+1$ points. An abstract simplicial complex $\mathcal{X}$ is a finite set of simplices that is **closed under the inclusion of faces**.

The fundamental operator when defining such structures is the *boundary operator*, which is well represented algebraically by the set of adjacency matrices B_k, encoding for **inclusions of subsets** and **coherence of orientation**:

$$B_k(i,j) = \begin{cases} 0 & \text{if } \sigma_i^{k-1} \not\subset \sigma_j^k; \\ 1 & \text{if } \sigma_i^{k-1} \subset \sigma_j^k \text{ and } \sigma_i^{k-1} \sim \sigma_j^k; \\ -1 & \text{if } \sigma_i^{k-1} \subset \sigma_j^k \text{ and } \sigma_i^{k-1} \not\sim \sigma_j^k \end{cases} \tag{10}$$

The fundamental property for simplicial complexes is the relation between boundaries of subsequent order:

$$B_k B_{k+1} = 0 \tag{11}$$

Simplicial cellular sheaves

Definition (2-Simplicial cellular sheaf)

Considering a simplicial complex $S(V, E, T)$, we define a **simplicial cellular sheaf** $\mathcal{F}$ specified by:

- A vector space $\mathcal{F}(v)$ for each node $v \in V$;
- A vector space $\mathcal{F}(e)$ for each edge $e \in E$;
- A vector space $\mathcal{F}(t)$ for each triangle $t \in T$;
- A linear map $\mathcal{F}_{v \trianglelefteq e} : \mathcal{F}(v) \to \mathcal{F}(e)$ for each boundary relation $v \trianglelefteq e$ for each edge;
- A linear map $\mathcal{F}_{e \trianglelefteq t} : \mathcal{F}(e) \to \mathcal{F}(t)$ for each boundary relation $e \trianglelefteq t$ for each triangle;

The structure of a K-simplicial complex is well described by the set of its *combinatorial Laplacians*, encoding for the lower and the upper adjacencies:

$$L_k = B_k^T B_k + B_{k+1} B_{k+1}^T \tag{12}$$

Similarly, for a simplicial cellular sheaf we generalize this idea using the k-th order coboundary map δ_k:

$$L_{\mathcal{F}_k} = \delta_k^T \delta_k + \delta_{k+1} \delta_{k+1}^T \tag{13}$$

$\mathbb{R}^n$ constant simplicial discourse sheaf

We shift our initial model to our brand new higher order topology. The related simplicial discourse sheaf is such that:

- Each agent resides on one edge and has a *private opinion* as a vector in $\mathbb{R}^n$;
- Each triangle represent a triple-wise communication, leading to a *tri-expressed opinion* over the same n topics;

This implies that now:

- The stack of all *private opinions* is a 1-cochain $\xi \in C^1(S, \mathcal{F})$;
- The stack of all *tri-shared opinions* is a 2-cochain $\tau \in C^2(S, \mathcal{F})$;

We used the quadratic form $\langle x, L_{\mathcal{F}} x\rangle$ when working with 0-cochains flow as a **measure of agreement** among agents: let's derive a similar expression for the 1-cochains opinion flow.

$$\begin{aligned} \xi^T L_{\mathcal{F}_1}\xi = \xi^T(\delta_1^T\delta_1 + \delta_2\delta_2^T)\xi = \xi^T\delta_1^T\delta_1\xi + \xi^T\delta_2\delta_2^T\xi = \\ \sum_{v\in V}||(\delta_1)_v\xi||_2^2 + \sum_{t\in T}||\mathcal{F}_{e_1\trianglelefteq t}\xi_{e_1} + \mathcal{F}_{e_2\trianglelefteq t}\xi_{e_2} - \mathcal{F}_{e_3\trianglelefteq t}\xi_{e_3}||_2^2 \end{aligned} \qquad (14)$$

This makes us question the tractability of lower order dynamics from higher order ones. If we just consider the entanglement of 1-cochains and 0-cochains to be $\xi = \delta_0^T x$ (*agents on edges as **mediators** between agents on nodes*), this would yield:

$$\langle \xi, L_{\mathcal{F}_1}\xi\rangle = \langle \delta_1^T x, (\delta_1^T\delta_1 + \delta_2\delta_2^T)x\rangle = x^T\delta_1\delta_1^T\delta_1\delta_1^T x + \cancel{x^T\delta_1\delta_2\delta_2^T\delta_1^T x} = ||L_{\mathcal{F}_0}x||_2^2 \qquad (15)$$

1-cochains opinion dynamics

We'll consider the following dynamic over the 1-cochains:

$$\frac{d\xi}{dt} = -\gamma L_{\mathcal{F}_1}\xi = -\gamma(\delta_1^T\delta_1 + \delta_2\delta_2^T)\xi \tag{16}$$

Given an initial state $\xi(0)$, the trajectory has the form:

$$\xi(t) = \mathrm{e}^{-\gamma(\delta_1^T\delta_1+\delta_2\delta_2^T)t}\xi(0) \tag{17}$$

which converges to the projection of the initial state over $\ker(\delta_1^T\delta_1 + \delta_2\delta_2^T)$.

Numerical experiments

4 Beyond graph modeling: group opinion dynamics over simplicial complexes

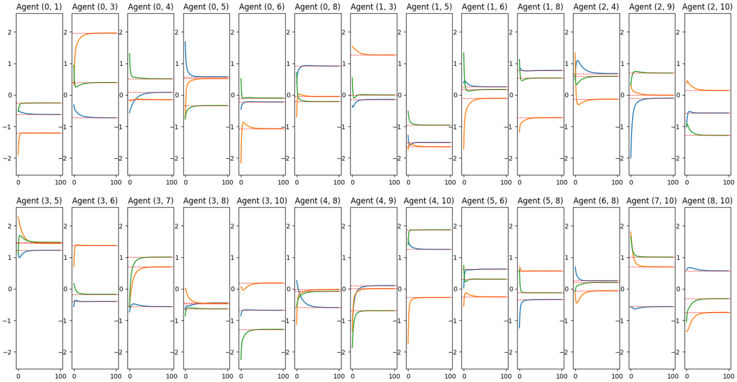


Figure: Opinion trajectories for a constant $\mathbb{R}^n$ sheaf defined for a network of 26 agents placed over the edges of an ER graph of 11 nodes, expanded to a 2-SC with 3 triangles (2, 4, 10), (0, 5, 8), (1, 3, 6)

Numerical experiments

4 Beyond graph modeling: group opinion dynamics over simplicial complexes

As we can see, the disagreement between agents is a decreasing function of time depending on the superpositon of the effect of both the lower and the upper adjacencies. Further tools might be used to investigate the nature of $\ker(L_{\mathcal{F}_1})$ in relation with $\ker(\delta_1^T\delta_1)$ and $\ker(\delta_2\delta_2^T)$.

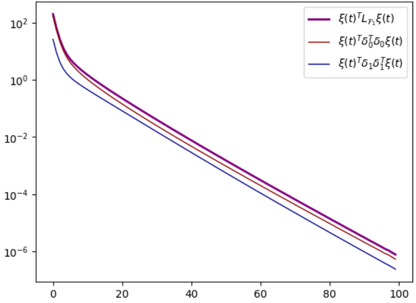


Figure: Disagreement through time as agents modify their opinions aiming at consensus

Similar to what we did on graph cellular sheaves we can design dynamics for the restriction maps: in this case we want the agent to be capable of modifying the way they express their opinion.
The dynamic system we designed to generalize the *learning to lie* takes into account both lower and upper adjacencies:

$$\begin{cases} \frac{d\delta_1}{dt} = -\beta \Pi_{\delta_1}(\delta_1 \xi \xi^T) \\ \frac{d\delta_2^T}{dt} = -\gamma \Pi_{\delta_2^T}(\delta_2^T \xi \xi^T) \end{cases}$$

Learning to lie over simplicial complexes

Via similar arguments it can be proven that this dynamic converges to the nearest sheaf in terms of l_2 distance between the respective coboundary maps such that the fixed private opinion distributions satisfies agreement among networks.

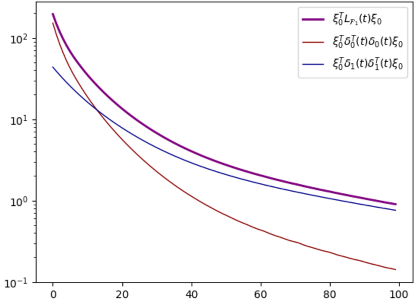


Figure: Disagreement through time as agents learn how to lie on a simplicial discourse sheaf