Support Material for: Effective sample size for a mixture prior

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A. ESS for standard Bayesian models

Table A.1: $\theta \in \mathbb{R}$, $c \geq 1$. Suppose $\boldsymbol{y} = (y_1, ..., y_n) \sim f(\boldsymbol{y}|\theta)$. Informative prior $p(\theta)$, non-informative prior $q(\theta)$, negative second log-derivatives and effective sample sizes for the univariate conjugate models: Normal-Normal (NN), Gamma-Poisson (GP), Gamma-Exponential (GExp) and Beta-Binomial (BB). Following [1], we denote $\mathcal{N}(\theta|\mu, \sigma^2)$, $\mathcal{G}\text{a}(\theta|\alpha, \beta)$, $\mathcal{B}\text{e}(\theta|\alpha, \beta)$, $\mathcal{B}\text{in}(n, \theta)$, $\mathcal{P}\text{ois}(\theta)$ and $\mathcal{E}\text{xp}(\theta)$ for the normal, gamma, beta, binomial, Poisson and exponential distributions. Let $\bar{\theta} = E_p(\theta)$ denote the plug-in estimate.

	NN	*GP*	*GExp*	*BB*
$q(\theta)$	$\mathcal{N}(\theta\|\mu, c\tau^2)$	$\mathcal{G}\text{a}(\theta\|\frac{\alpha}{c}, \frac{\beta}{c})$	$\mathcal{G}\text{a}(\theta\|\frac{\alpha}{c}, \frac{\beta}{c})$	$\mathcal{B}\text{e}(\theta\|\frac{\alpha}{c}, \frac{\beta}{c})$
$p(\theta)$	$\mathcal{N}(\theta\|\mu, \tau^2)$	$\mathcal{G}\text{a}(\theta\|\alpha, \beta)$	$\mathcal{G}\text{a}(\theta\|\alpha, \beta)$	$\mathcal{B}\text{e}(\theta\|\alpha, \beta)$
$D_q(\theta)$	$1/c\tau^2$	$\frac{(\alpha/c-1)}{\bar{\theta}^2}$	$\frac{(\alpha/c-1)}{\bar{\theta}^2}$	$(\frac{\alpha}{c}-1)\frac{1}{\bar{\theta}^2} + (\frac{\beta}{c}-1)\frac{1}{(1-\bar{\theta})^2}$
$D_p(\theta)$	$1/\tau^2$	$(\alpha-1)\bar{\theta}^{-2}$	$(\alpha-1)\bar{\theta}^{-2}$	$\frac{(\alpha-1)}{\bar{\theta}^2} + \frac{(\beta-1)}{(1-\bar{\theta})^2}$
$D_q(n, \theta, \boldsymbol{y})$	n/σ^2	$\frac{(\alpha/c+\sum y_i-1)}{\bar{\theta}^2}$	$\frac{(\alpha/c+n-1)}{\bar{\theta}^2}$	$\frac{(\frac{\alpha}{c}+\sum_i y_i-1)}{\bar{\theta}^2} + \frac{(\frac{\beta}{c}+n-\sum_i y_i-1)}{(1-\bar{\theta})^2}$
ESS_q	$\sigma^2/c\tau^2$	0	0	0
ESS_p	σ^2/τ^2	$\frac{\alpha-\alpha/c}{\bar{y}}$	$\alpha-\alpha/c$	$\alpha+\beta$

Table A.1 reports the observed informations and the ESS for some univariate standard (conjugate) Bayesian models, for which the derivation of prior-equivalent sample sizes is straightforward and commonly accepted.

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B. Proof of Theorem 1

Proof 1. *According to the definitions in Equation (6) in the paper,* $ESS_\pi \leq ESS_{p_1} \Leftrightarrow D_{\pi,+}(\bar{\boldsymbol{\theta}}) \leq D_{p_1,+}(\bar{\boldsymbol{\theta}})$*. Thus, denoting* $H_{i,j}(\bar{\boldsymbol{\theta}}) = \frac{\partial^2 p_i(\bar{\boldsymbol{\theta}})}{\partial \theta_j^2}|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}}$, $H_{i,+}(\bar{\boldsymbol{\theta}}) = \sum_{j=1}^{d} H_{i,j}(\bar{\boldsymbol{\theta}})$*, and noting that* $\frac{\partial p_i(\bar{\boldsymbol{\theta}})}{\partial \theta_j}|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}} = 0\ \forall i$ *due to assumption (iii), we have:*

$$
\begin{aligned}
&D_{\pi,+}(\bar{\boldsymbol{\theta}}) \leq D_{p_1,+}(\bar{\boldsymbol{\theta}}) \Leftrightarrow \\
&\qquad -\sum_{j=1}^{d} \frac{\partial^2 \log \pi(\bar{\boldsymbol{\theta}})}{\partial \theta_j^2} \leq -\sum_{j=1}^{d} \frac{\partial^2 \log p_1(\bar{\boldsymbol{\theta}})}{\partial \theta_j^2} \\
&\qquad \Leftrightarrow -\sum_{j=1}^{d} \frac{\partial}{\partial \theta_j} \left[\frac{1}{\sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}})} \sum_{i=1}^{k} \psi_i \left(\frac{\partial p_i(\boldsymbol{\theta})}{\partial \theta_j} \right) \bigg|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}} \right] \leq -\sum_{j=1}^{d} \frac{\partial}{\partial \theta_j} \left[\frac{\frac{\partial p_1(\boldsymbol{\theta})}{\partial \theta_j}|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}}}{p_1(\bar{\boldsymbol{\theta}})} \right] \\
&\qquad \Leftrightarrow -\sum_{j=1}^{d} \frac{\sum_{i=1}^{k} \psi_i H_{i,j}(\bar{\boldsymbol{\theta}}) \sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}}) - (\sum_{i=1}^{k} \psi_i (\frac{\partial p_i(\boldsymbol{\theta})}{\partial \theta_j}|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}}))^2}{(\sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}}))^2} \\
&\qquad\qquad \leq -\sum_{j=1}^{d} \frac{H_{1,j}(\bar{\boldsymbol{\theta}}) p_1(\bar{\boldsymbol{\theta}}) - (\frac{\partial p_1(\boldsymbol{\theta})}{\partial \theta_j}|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}})^2}{(p_1(\bar{\boldsymbol{\theta}}))^2} \\
&\qquad \Leftrightarrow -\sum_{j=1}^{d} \frac{\sum_{i=1}^{k} \psi_i H_{i,j}(\bar{\boldsymbol{\theta}}) \sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}})}{(\sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}}))^2} \leq -\sum_{j=1}^{d} \frac{H_{1,j}(\bar{\boldsymbol{\theta}}) p_1(\bar{\boldsymbol{\theta}})}{(p_1(\bar{\boldsymbol{\theta}}))^2} \\
&\qquad \Leftrightarrow \sum_{j=1}^{d} \left[\frac{H_{1,j}(\bar{\boldsymbol{\theta}}) p_1(\bar{\boldsymbol{\theta}})}{(p_1(\bar{\boldsymbol{\theta}}))^2} - \frac{\sum_{i=1}^{k} \psi_i H_{i,j}(\bar{\boldsymbol{\theta}}) \sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}})}{(\sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}}))^2} \right] \leq 0 \\
&\qquad \Leftrightarrow \sum_{j=1}^{d} H_{1,j}(\bar{\boldsymbol{\theta}}) \sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}}) - p_1(\bar{\boldsymbol{\theta}}) \sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}}) \leq 0 \\
&\qquad \Leftrightarrow \sum_{j=1}^{d} [\psi_1 p_1(\bar{\boldsymbol{\theta}}) H_{1,j}(\bar{\boldsymbol{\theta}}) + H_{1,j}(\bar{\boldsymbol{\theta}}) \sum_{i=2}^{k} \psi p_i(\bar{\boldsymbol{\theta}}) - \\
&\qquad\qquad \psi_1 p_1(\bar{\boldsymbol{\theta}}) H_{1,j}(\bar{\boldsymbol{\theta}}) - p_1(\bar{\boldsymbol{\theta}}) \sum_{i=2}^{k} \psi_i H_{i,j}(\bar{\boldsymbol{\theta}})] \leq 0 \\
&\qquad \Leftrightarrow \sum_{j=1}^{d} H_{1,j}(\bar{\boldsymbol{\theta}}) \sum_{i=2}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}}) \leq \sum_{j=1}^{d} p_1(\bar{\boldsymbol{\theta}}) \sum_{i=2}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}}) \\
&\qquad \Leftrightarrow \sum_{j=1}^{d} H_{1,j}(\bar{\boldsymbol{\theta}}) \leq \sum_{j=1}^{d} p_1(\bar{\boldsymbol{\theta}}) \frac{\sum_{i=2}^{k} \psi_i H_{i,j}(\bar{\boldsymbol{\theta}})}{\sum_{i=2}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}})} \\
&\qquad \Leftrightarrow H_{1,+}(\bar{\boldsymbol{\theta}}) \leq \sum_{j=1}^{d} p_1(\bar{\boldsymbol{\theta}}) \frac{\sum_{i=2}^{k} \psi_i H_{i,j}(\bar{\boldsymbol{\theta}})}{\sum_{i=2}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}})}. \quad \square
\end{aligned}
$$

C. Proof of Theorem 2

Proof 2. *The proof stems directly from the proof of Theorem 1. Since $\boldsymbol{\psi}$ is assigned a hyperprior distribution, we have that the parameter space is of di-*

mension $d+k$ (when $\boldsymbol{\psi}$ is fixed, the parameter space has dimension d). Thus, the observed information for the mixture prior is:

$$
\begin{aligned}
-\sum_{j=1}^{d+k} \frac{\partial^2 \log \pi(\bar{\boldsymbol{\theta}})}{\partial \theta_j^2} &= -\left(\sum_{j=1}^{d} \frac{\partial^2}{\partial \theta_j^2} \log \pi(\bar{\boldsymbol{\theta}}) + \sum_{i=1}^{k} \frac{\partial^2}{\partial \theta_i^2} \log \pi(\bar{\boldsymbol{\theta}})\right) \\
&= -\left(\sum_{j=1}^{d} \frac{\sum_{i=1}^{k} \psi_i H_{i,j}(\bar{\boldsymbol{\theta}})}{\sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}})} + \sum_{i=1}^{k} \frac{\sum_{i=1}^{k} H_{i,i}(\bar{\boldsymbol{\theta}}) p_i(\bar{\boldsymbol{\theta}})}{\sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}})}\right) \\
&= -\left(\sum_{j=1}^{d} \frac{\sum_{i=1}^{k} \psi_i H_{i,j}(\bar{\boldsymbol{\theta}})}{\sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}})} + k \frac{\sum_{i=1}^{k} H_{i,i}(\bar{\boldsymbol{\theta}}) p_i(\bar{\boldsymbol{\theta}})}{\sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}})}\right).
\end{aligned}
$$

We can then proceed as for the proof of Theorem 1:

$$
\begin{aligned}
&D_{\pi,+}(\bar{\boldsymbol{\theta}}) \leq D_{p,+}(\bar{\boldsymbol{\theta}}) \Leftrightarrow \\
&\qquad \Leftrightarrow -\left(\sum_{j=1}^{d} \frac{\sum_{i=1}^{k} \psi_i H_{i,j}(\bar{\boldsymbol{\theta}})}{\sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}})} + k \frac{\sum_{i=1}^{k} H_{i,i}(\bar{\boldsymbol{\theta}}) p_i(\bar{\boldsymbol{\theta}})}{\sum_{i=1}^{k} \psi_i p_i(\bar{\boldsymbol{\theta}})}\right) \leq -\sum_{j=1}^{d} \frac{H_{1,j}(\bar{\boldsymbol{\theta}})}{p_1(\bar{\boldsymbol{\theta}})}.
\end{aligned}
$$

We can then proceed analogously as for the previous proof, and the second addendum in this left term inequality is just summed to the right term inequality of Theorem 1, and the proof is derived. □

D. Second derivatives of some Gaussian priors

Figure D.1 displays the behaviour of the second derivatives from four Gaussian prior distributions: the smaller the variance, the smaller is the second derivative in the point $\bar{\theta}$.

E. Further plots for the multinomial-Dirichlet simulation

Figures E.1 and E.2 report the distances $\delta(n, \bar{\boldsymbol{\theta}}, p_1, q_n)$ and $\delta(n, \bar{\boldsymbol{\theta}}, \pi, q_n)$ for the multinomial-Dirichlet simulation study in Section 3. Figure E.3 shows the ESS for Scenario B.

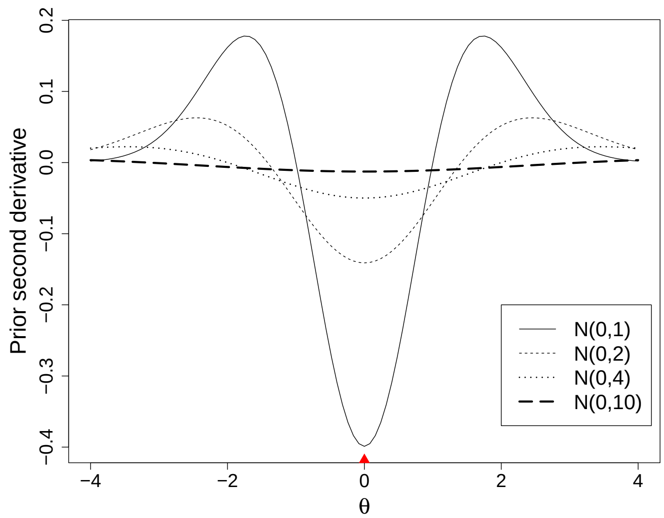


Figure D.1: Second derivative computed in the maximum value $\bar{\theta} = 0$ for four Gaussian priors, with mean equal to zero and variances equal to 1,2,4, and 10, respectively. The value of $p''(\bar{\theta})$ is as low as the variance is small, thus as small as the prior is informative.

F. Jeffreys prior for an exponential model

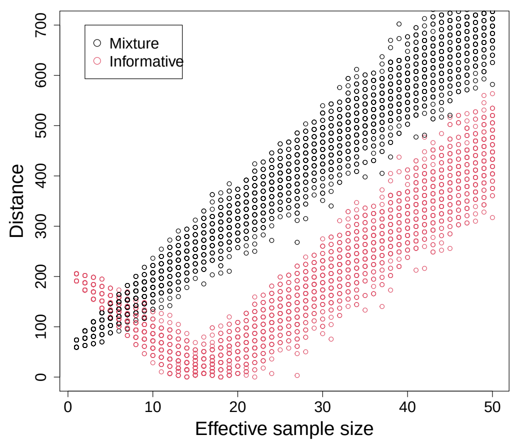
Let $\boldsymbol{y} = (y_1, ..., y_n) \underset{iid}{\sim} \mathcal{E}xp(\boldsymbol{y}|\theta)$, with $p(\theta) = \mathcal{G}a(\theta|\alpha, \beta)$. The model likelihood is then given by:

$$L(\theta; \boldsymbol{y}) = \prod_{i=1}^{n} f(y_i|\theta) = \theta^n \exp(-\theta \sum_i y_i). \tag{1}$$

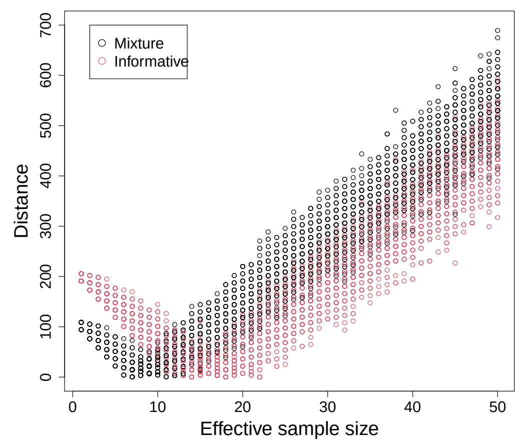
We introduce the Fisher information for the exponential model computed for a single observation, $I_\theta = 1/\theta^2$. Let $q(\theta) = j(\theta)$, where $j(\theta) = \sqrt{I_\theta} = 1/\theta$ is the Jeffreys prior. Thus, the Jeffreys posterior $q_n(\theta|y_1, ..., y_n) = j_n(\theta|y_1, ..., y_n)$ is proportional to $j(\theta)L(\theta; \boldsymbol{y}) = \theta^{-1} \prod_{i=1}^{n} \theta \exp\{-\theta y_i\} = \theta^{n-1} \exp\left\{-\theta \sum_{i=1}^{n} y_i\right\}$, the kernel of a Gamma distribution, $\mathcal{G}a(\theta|n, \sum_i y_i)$. Thus:

$$j_n(\theta|\boldsymbol{y}) = \frac{(\sum_i y_i)^n}{\Gamma(n)} \theta^{n-1} \exp\left\{-\theta \sum_{i=1}^{n} y_i\right\}.$$

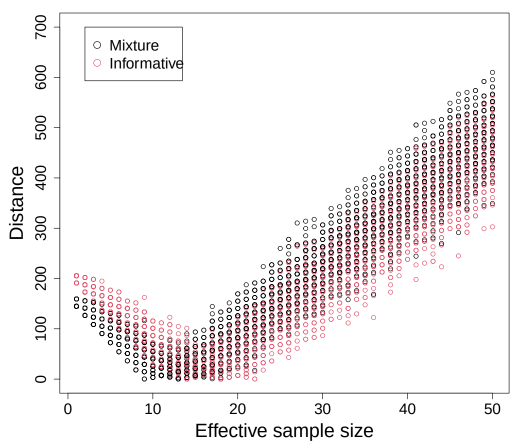
Given that $D_{j_n} = -\frac{d^2}{d\theta^2}\left[j_n(\theta|\boldsymbol{y})\right] = \frac{n-1}{\theta^2}$, by using the plug-in estimate $\bar{\theta} = \alpha/\beta$, we may compute the following quantities: 1) the distance between the



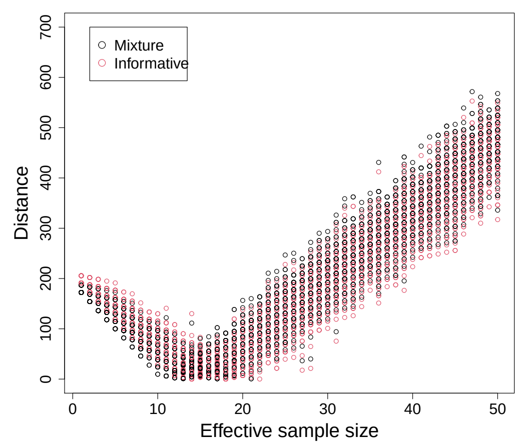
(a) $\boldsymbol{\psi} = (0.1, 0.4, 0.5)$



(b) $\boldsymbol{\psi} = (1/3, 1/3, 1/3)$

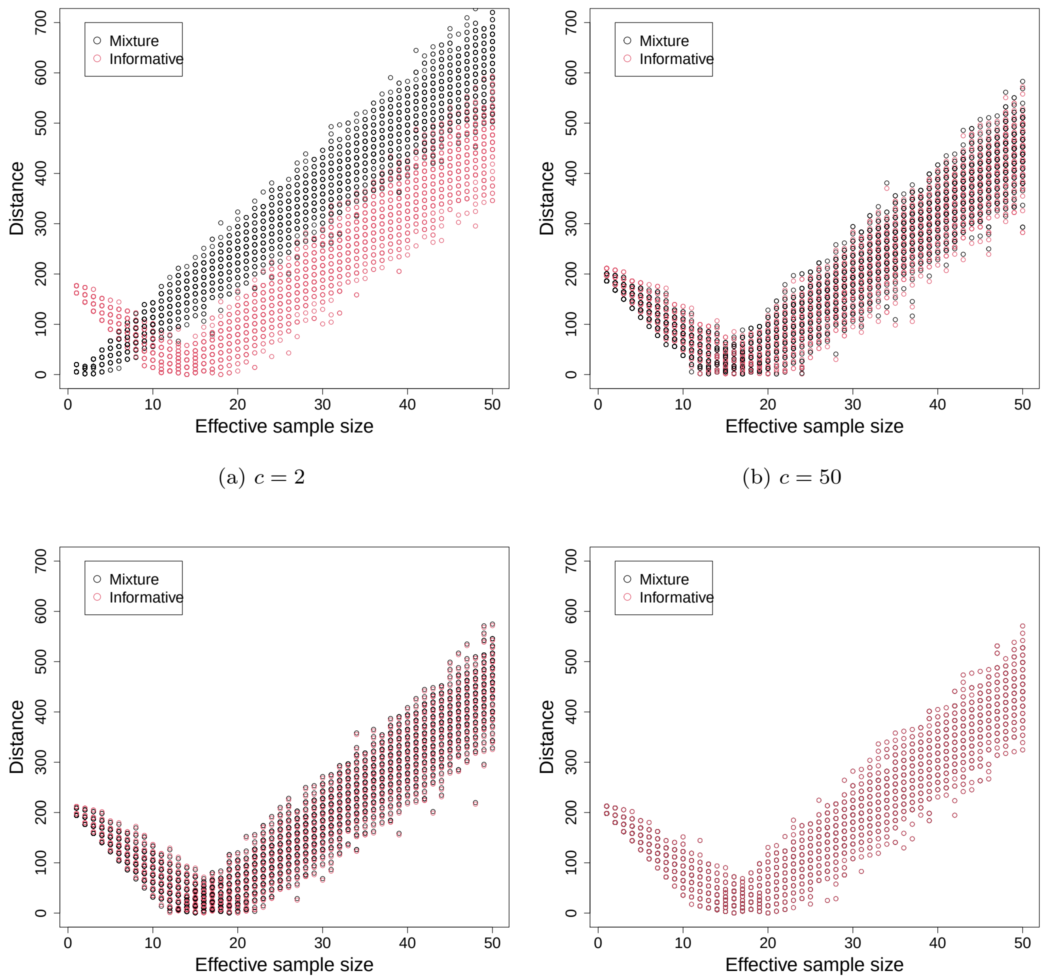


(c) $\boldsymbol{\psi} = (0.5, 0.2, 0.3)$



(d) $\boldsymbol{\psi} = (0.7, 0.1, 0.2)$

Figure E.1: Scenario A: distance $\delta(n, \bar{\boldsymbol{\theta}}, p_1, q_n$ for 100 simulated datasets under the informative $p_1(\boldsymbol{\theta}) \sim \text{Dirichlet}(\boldsymbol{\theta}|\boldsymbol{\alpha})$ and the mixture prior $\sum_{i=1}^{3} \psi_i p_i(\boldsymbol{\theta})$ under different choices for the mixture weights $\boldsymbol{\psi}$, where $p_2(\boldsymbol{\theta}) \sim \text{Dirichlet}(\boldsymbol{\theta}|\boldsymbol{\alpha}/c)$, $p_3(\boldsymbol{\theta}) \sim \text{Dirichlet}(\boldsymbol{\theta}|\boldsymbol{\alpha}/3c)$, $\boldsymbol{\alpha} = (5, 5, 10)$ and $c = 10$.



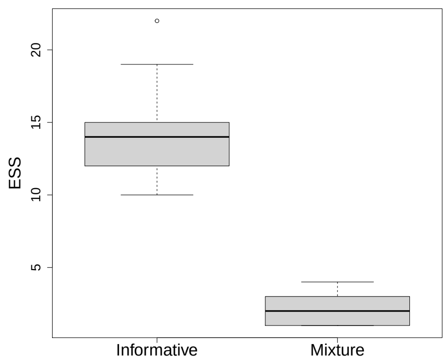
(a) $c = 2$

(b) $c = 50$

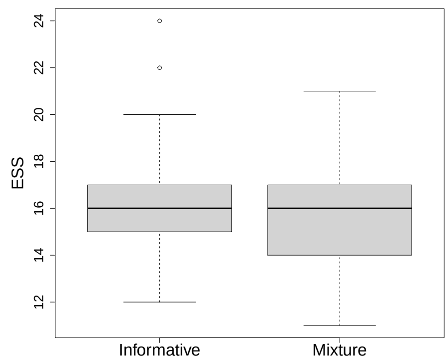
(c) $c = 100$

(d) $c = 1000$

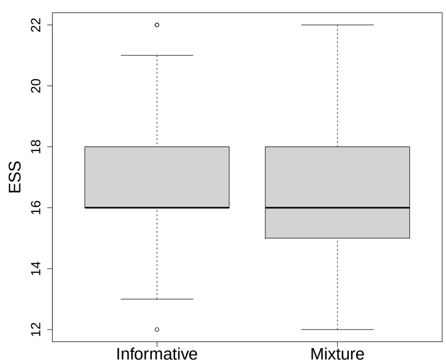
Figure E.2: Scenario B: distance $\delta(n, \bar{\boldsymbol{\theta}}, p_1, q_n)$ for 100 simulated datasets under the informative $p_1(\boldsymbol{\theta}) \sim \text{Dirichlet}(\boldsymbol{\theta}|\boldsymbol{\alpha})$ and the mixture prior $\sum_{i=1}^{3} \psi_i p_i(\boldsymbol{\theta})$ under different choices for the factor c, where $p_2(\boldsymbol{\theta}) \sim \text{Dirichlet}(\boldsymbol{\theta}|\boldsymbol{\alpha}/c)$, $p_3(\boldsymbol{\theta}) \sim \text{Dirichlet}(\boldsymbol{\theta}|\boldsymbol{\alpha}/3c)$, $\boldsymbol{\alpha} = (5, 5, 10)$ and $\boldsymbol{\psi} = (1/3, 1/3, 1/3)$.



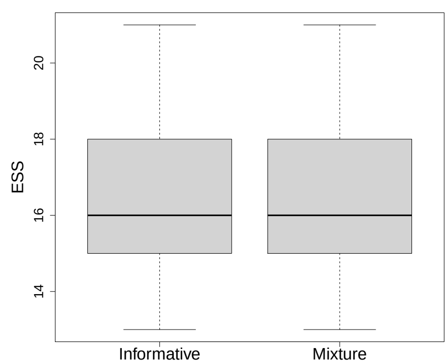
(a) $c = 2$



(b) $c = 50$



(c) $c = 100$



(d) $c = 1000$

Figure E.3: Scenario B: ESS for 100 simulated datasets under the informative $p_1(\boldsymbol{\theta}) \sim \text{Dirichlet}(\boldsymbol{\theta}|\boldsymbol{\alpha})$ and the mixture prior $\sum_{i=1}^{3} \psi_i p_i(\boldsymbol{\theta})$ under different choices for the factor c, where $p_2(\boldsymbol{\theta}) \sim \text{Dirichlet}(\boldsymbol{\theta}|\boldsymbol{\alpha}/c)$, $p_3(\boldsymbol{\theta}) \sim \text{Dirichlet}(\boldsymbol{\theta}|\boldsymbol{\alpha}/3c)$, $\boldsymbol{\alpha} = (5, 5, 10)$ and $\boldsymbol{\psi} = (1/3, 1/3, 1/3)$.

informative prior p and the Jeffreys posterior j_n; 2) the distance between the Jeffreys prior j and the Jeffreys posterior j_n; 3) the distance between the mixture prior π and the Jeffreys posterior j_n.

G. R code for the multinomial-Dirichlet simulation study

For sake of brevity, we report the `R` code for the Scenario A of Section 3 (Scenario B code is pretty identical). The function `DD` allows to compute the second derivative of a probability density function, whereas the main function `Dist` allows to compute the distances between the distinct priors and the non-informative posterior.

```
library(DirichletReg)

# second derivative function
DD <- function(expr, name, order = 1) {
  if(order < 1) stop("'order' must be >= 1")
  if(order == 1) D(expr, name)
  else DD(D(expr, name), name, order - 1)
}

ess_mix = ess_p = c()

## scenario A: varying weights, fixed c

par(mar=c(5,5,2,1))

# possible mixture weights values
aa <- c(0.1, 1/3, 0.5, 0.7)
bb <- c(0.4, 1/3,0.2, 0.1)
cc <- c(0.5, 1/3, 0.3, 0.2)

# 4 simulation for each of the weights' scenarios
for (j in 1:4){
```

```
a <- aa[j]
b <- bb[j]
c <- cc[j]
a1 <- 5   # Dirichlet hyperpar. alpha1
a2 <- 5   # Dirichlet hyperpar. alpha2
a3 <- 10  # Dirichlet hyperpar. alpha3
k <- 3
weights <- c(a1, a2, a3)
w <- 10   # scaling factor
weights_q <- weights/w
theta_hat <- (weights-1)/(sum(weights)-k)  # mode

# 100 simulations, with n=1,2,...,50.
# For each simulation, I draw a multinomial sample Y
for (i in 1:100){

# main function to compute distances between each prior and the posterior
Dist <- function(x){
  Y <- rmultinom(x, 1, prob=c(1/3, 1/3, 1/3)) # multinomial likelihood
  S <- apply(Y,1,sum)


  evaluate_d_prior <- function(theta, weights){ # mixture prior derivative
    theta1 <- theta[1]
    theta2 <- theta[2]
    theta3 <- theta[3]
    weights1 <- weights[1]
    weights2 <- weights[2]
    weights3 <- weights[3]
    D_prior_p1 <- -eval(DD(  expression(log( a*(gamma(weights1+weights2+weights3)/
                             (gamma(weights1)*gamma(weights2)*gamma(weights3)))*
                             (theta1^(weights1-1))*
                              (theta2^(weights2-1))*
```

```
                    (theta3^(weights3-1))+
          b* (gamma(weights1/w+weights2/w+weights3/w)/
                   (gamma(weights1/w)*gamma(weights2/w)*gamma(weights3/w)))*
                   (theta1^(weights1/w-1))*
                    (theta2^(weights2/w-1))*
                    (theta3^(weights3/w-1))+
          c* (gamma(weights1/(3*w)+weights2/(3*w)+weights3/(3*w)))/
                (gamma(weights1/(3*w))*gamma(weights2/(3*w))*gamma(weights3/(3*w)))*
                   (theta1^(weights1/(3*w)-1))*
                    (theta2^(weights2/(3*w)-1))*
                    (theta3^(weights3/(3*w)-1)))), "theta1",2))
D_prior_p2 <- -eval(DD(  expression(log( a*(gamma(weights1+weights2+weights3)/
 (gamma(weights1)*gamma(weights2)*gamma(weights3)))*
                   (theta1^(weights1-1))*
                   (theta2^(weights2-1))*
                    (theta3^(weights3-1))+
          b* (gamma(weights1/w+weights2/w+weights3/w)/
             (gamma(weights1/w)*gamma(weights2/w)*gamma(weights3/w)))*
                   (theta1^(weights1/w-1))*
                   (theta2^(weights2/w-1))*
                   (theta3^(weights3/w-1))+
          c* (gamma(weights1/(3*w)+weights2/(3*w)+weights3/(3*w)))/
             (gamma(weights1/(3*w))*gamma(weights2/(3*w))*gamma(weights3/(3*w)))*
                   (theta1^(weights1/(3*w)-1))*
                    (theta2^(weights2/(3*w)-1))*
                    (theta3^(weights3/(3*w)-1)))), "theta2",2))
D_prior_p3 <- -eval(DD(  expression(log( a*(gamma(weights1+weights2+weights3)/
        (gamma(weights1)*gamma(weights2)*gamma(weights3)))*
        (theta1^(weights1-1))*
        (theta2^(weights2-1))*
        (theta3^(weights3-1))+
          b* (gamma(weights1/w+weights2/w+weights3/w)/
        (gamma(weights1/w)*gamma(weights2/w)*gamma(weights3/w)))*
```

```
          (theta1^(weights1/w-1))*
          (theta2^(weights2/w-1))*
          (theta3^(weights3/w-1))+
            c* (gamma(weights1/(3*w)+weights2/(3*w)+weights3/(3*w)))/
          (gamma(weights1/(3*w))*gamma(weights2/(3*w))*gamma(weights3/(3*w)))*
          (theta1^(weights1/(3*w)-1))*
          (theta2^(weights2/(3*w)-1))*
          (theta3^(weights3/(3*w)-1)) )), "theta3",2))

  D_prior_p <- sum(D_prior_p1, D_prior_p2, D_prior_p3)
  return(D_prior_p)
}

evaluate_d_prior2 <- function(theta, weights){ # informative prior derivative
  theta1 <- theta[1]
  theta2 <- theta[2]
  theta3 <- theta[3]
  weights1 <- weights[1]
  weights2 <- weights[2]
  weights3 <- weights[3]
  D_prior_p1 <- -eval(DD(  expression(log( (gamma(weights1+weights2+weights3)/
  (gamma(weights1)*gamma(weights2)*gamma(weights3)))*(theta1^(weights1-1))*
                                           (theta2^(weights2-1))*
                                           (theta3^(weights3-1)))), "theta1",2))
  D_prior_p2 <- -eval(DD(  expression(log( (gamma(weights1+weights2+weights3)/
  (gamma(weights1)*gamma(weights2)*gamma(weights3)))*(theta1^(weights1-1))*
                                           (theta2^(weights2-1))*
                                           (theta3^(weights3-1)))), "theta2",2))
  D_prior_p3 <- -eval(DD(  expression(log( (gamma(weights1+weights2+weights3)/
  (gamma(weights1)*gamma(weights2)*gamma(weights3)))*(theta1^(weights1-1))*
                                           (theta2^(weights2-1))*
                                           (theta3^(weights3-1)))), "theta3",2))
```

```
  D_prior_p <- sum(D_prior_p1, D_prior_p2, D_prior_p3)
  return(D_prior_p)
}

evaluate_d_posterior <- function(theta, weights, S){
                                # noninformative posterior derivative
  theta1 <- theta[1]
  theta2 <- theta[2]
  theta3 <- theta[3]
  weights1 <- weights[1]+S[1]
  weights2 <- weights[2]+S[2]
  weights3 <- weights[3]+S[3]
  D_post_p1 <- -eval(DD(  expression(log( (gamma(weights1+weights2+weights3)/
  (gamma(weights1)*gamma(weights2)*gamma(weights3)))*(theta1^(weights1-1))*
                                          (theta2^(weights2-1))*
                                          (theta3^(weights3-1)))), "theta1",2))
  D_post_p2 <- -eval(DD(  expression(log( (gamma(weights1+weights2+weights3)/
  (gamma(weights1)*gamma(weights2)*gamma(weights3)))*(theta1^(weights1-1))*
                                          (theta2^(weights2-1))*
                                          (theta3^(weights3-1)))), "theta2",2))
  D_post_p3 <- -eval(DD(  expression(log( (gamma(weights1+weights2+weights3)/
  (gamma(weights1)*gamma(weights2)*gamma(weights3)))*(theta1^(weights1-1))*
                                          (theta2^(weights2-1))*
                                          (theta3^(weights3-1)))), "theta3",2))

  D_post_p <- sum(D_post_p1, D_post_p2, D_post_p3)
  return(D_post_p)
}

# distances
D1 <- abs(evaluate_d_prior(theta_hat, weights)-
           evaluate_d_posterior(theta_hat, weights_q/3, S))
D2 <- abs(evaluate_d_prior2(theta_hat, weights)-
```

```
             evaluate_d_posterior(theta_hat, weights_q/3, S))
  return(c(D1,D2))
}

Dist <- Vectorize(Dist)
result<-Dist(seq(1:50))
par(mfrow=c(1,1))

if (i==1){
plot(1:50, result[1,], ylim=c(0,700),
     xlab = "Effective sample size", ylab = "Distance",
     cex.lab=2, cex.axis =1.6)
  points(1:50, result[2,], col=2)
  legend(2, 700, c("Mixture", "Informative"),col=c(1,2),
         pch=1, cex=1.6)
}else{
  points(1:50, result[1,])
  points(1:50, result[2,], col=2)
}

# ess computation
ess_mix[i] <- which.min(result[1,])
ess_p[i] <- which.min(result[2,])

}
dev.copy2pdf(file=paste("distance", j, ".pdf", sep=""), width=9,
             height=8)


pdf(file=paste("ESS", j, ".pdf", sep=""), width = 9, height = 8)
par(xaxt="n", mar=c(5,5,2,1))
boxplot(ess_p, ess_mix, ylab = "ESS", cex.lab =2, cex.axis =1.6)
par(xaxt="s")
```

```
axis(1,at =c("1", "2"), labels= c("Informative", "Mixture"), cex.lab=2, cex.axis=2)
dev.off()
}
```

H. Logistic regression for phase I trial

[2] proposed a logistic regression to determine the greatest amount of tolerable dose in a phase I trial. In this section we follow the approach of [3], who used the same example aimed at studying the properties of the ESS for different values of the hyperparameters.

The level of dose which each patient may receive is one among 100, 200, 300, 400, 500, 600 mg/m^2, denoted by $x_1, \ldots, x_6$. These values are then standardized on the log scale and denoted with $X_1, ..., X_6$. The response variable is $y_i = 1$ if patient i suffers toxicity, $y_i = 0$ if not. They assume the following logistic model:

$$P(y_i = 1) \equiv p(X_i, \boldsymbol{\theta}) = \text{logit}^{-1}(\mu + \beta X_i),\ i = 1, ..., n, \tag{2}$$

where $\text{logit}^{-1}(x) = e^x/(1 + e^x)$. The dimension of the parameters' space is $d = 2$, $\boldsymbol{\theta} = (\mu, \beta)$, where μ is the intercept of the linear predictor and β is the coefficient associated to the different levels of the doses. To compute the ESS, we need the computational extension to the multivariate case outlined by [3] and not reported here. The likelihood for a sample of n patients $\boldsymbol{y} = (y_1, ..., y_n)$ is

$$f(\boldsymbol{y}|X, \boldsymbol{\theta}) = \prod_{i=1}^{n} p(X_i, \theta)^{y_i}(1 - p(X_i, \theta))^{1-y_i}. \tag{3}$$

[2] elicited two independent informative priors for μ and β based on preliminary sensitivity analysis:

$$\begin{aligned} \mu &\sim p(\mu) = \mathcal{N}(\mu|\tilde{\mu}_\mu, \tilde{\sigma}^2_\mu) = \mathcal{N}(\mu| - 0.11313, 2^2) \\ \beta &\sim p(\beta) = \mathcal{N}(\beta|\tilde{\mu}_\beta, \tilde{\sigma}^2_\beta) = \mathcal{N}(\beta|2.3980, 2^2). \end{aligned} \tag{4}$$

Table H.1: Effective sample sizes $ESS_p^{\mu}, ESS_p^{\beta}$ for the tolerable dose in a phase I trial.

	ESS	ESS_p^{μ}	ESS_p^{β}
$\sigma_\mu^2 = \sigma_\beta^2 = 0.5^2$	37.00	22.8	98.11
$\sigma_\mu^2 = \sigma_\beta^2 = 1^2$	10.00	5.76	25.56
$\sigma_\mu^2 = \sigma_\beta^2 = 2^2$	3.00	1.38	6.53
$\sigma_\mu^2 = \sigma_\beta^2 = 3^2$	2.00	1.03	3.08
$\sigma_\mu^2 = \sigma_\beta^2 = 5^2$	1.00	1.00	1.38

Hence, the noninformative posterior is $q_n(\boldsymbol{\theta}|\boldsymbol{y}) = \mathcal{N}(\boldsymbol{\theta}|\tilde{\mu}_\mu, c\tilde{\sigma}_\mu^2)\mathcal{N}(\boldsymbol{\theta}|\tilde{\mu}_\beta, c\tilde{\sigma}_\beta^2)$, where the hyperparameter $c = 10^4$ allows for the variance inflation. We follow the steps of the algorithm formulated by [3] to determine (i) the ESS of each subvector and (ii) the global effective sample size of the parameter vector $\boldsymbol{\theta}$ as those values minimizing the distances $\delta_1(n_\mu, \bar{\boldsymbol{\theta}}, p_\mu, q_{n_\mu}), \delta_2(n_\beta, \bar{\boldsymbol{\theta}}, p_\beta, q_{n_\beta})$ and $\delta(n, \bar{\boldsymbol{\theta}}, p, q_n)$, respectively, by using the plug-in vector $\bar{\boldsymbol{\theta}} = (\tilde{\mu}_\mu, \tilde{\mu}_\beta)$. In this way, we compute the ESS of each parameter's subvector and then the global ESS of the logistic model. Given the two priors p_μ, p_β in (4), we will denote the first two quantities with $ESS_p^{\mu}, ESS_p^{\beta}$, and the third one simply with ESS.

Table H.1 reports the results, obtained replicating the experiment of [3] and evaluating different values of the priors variances $\sigma_\mu^2, \sigma_\beta^2$. As it is intuitive, the information contained in the prior distributions decreases as the variances increase. In any case, the parameter β, associated to the effect of the doses, yields a greater knowledge than the parameter μ, which represents the average response.

We repeat the same steps above by specifying a mixture prior for the vector parameter $\boldsymbol{\theta}$:

$$\begin{aligned} \mu &\sim \pi(\mu) = \psi\mathcal{N}(\mu|\tilde{\mu}_\mu, c\tilde{\sigma}_\mu^2) + (1-\psi)\mathcal{N}(\mu|\tilde{\mu}_\mu, \tilde{\sigma}_\mu^2) \\ \beta &\sim \pi(\beta) = \psi\mathcal{N}(\beta|\tilde{\beta}_\beta, c\tilde{\sigma}_\beta^2) + (1-\psi)\mathcal{N}(\beta|\tilde{\mu}_\beta, \tilde{\sigma}_\beta^2), \end{aligned} \tag{5}$$

where the hyperparameter c takes four possible values, $(5, 10^2, 10^3, 10^4)$, and ψ is the mixture weight. Usually, in the Bayesian variable selection framework, c

is assumed to take small values: as we'll see, in this framework as well smaller values of c yield more interpretable results in terms of effective sample sizes. For illustration purposes only, we consider three different values for ψ, $\psi = \{0.2, 0.5, 0.8\}$. Then, we compare the so obtained results with those obtained with the above-mentioned prior distributions in Table 2: due to Theorem 1, when $c = 10^4$ the ESS in Table H.1 never exceeds the mixture ESS in Table H.2. As may be noticed from Table H.2, as ψ increases the ESS for the mixture priors (5) slightly decrease. However, this reduction from $\psi = 0.2$ to $\psi = 0.8$ is quite relevant when $c = 5$, wheras it vanishes as c increases, suggesting that the influence of the scaling factor c on reducing the mixture ESS is as much large as c is small: again, we find the paradoxical argument already mentioned on page 8 in the simulation study. One could then tempted to argue that a noninformative Gaussian prior with high variance is not really vague in some practical cases. It would be worth assessing how much varies the information of the mixture priors π by choosing other noninformative priors in place of some flat normal distributions as those proposed by [2].

Let us consider two improper priors, $q(\mu) \propto 1$, $q(\beta) \propto 1$. The resulting mixture priors $\pi(\mu)$, $\pi(\beta)$ are then:

$$\begin{aligned} \mu \sim \pi(\mu) &= \psi + (1-\psi)\mathcal{N}(\mu|\tilde{\mu}_\mu, \tilde{\sigma}^2_\mu) \\ \beta \sim \pi(\beta) &= \psi + (1-\psi)\mathcal{N}(\beta|\tilde{\mu}_\beta, \tilde{\sigma}^2_\beta). \end{aligned} \tag{6}$$

Table H.3 reports the ESS for the priors in (6). In this case, the decrease of the information associated to the mixture priors π is evident: as ψ increases and the improper priors are then favored, the ESS rapidly decreases. This is intuitive, since the improper priors in (6) provide less information if compared to the two normal priors in (5).

The example suggests that inflating the noninformative variances by a great factor c does not affect in a sensible way the amount of information contained in the mixture prior and, thus, does not alter the final posterior conclusions. We may conclude that the best way for reducing an extra amount of information

Table H.2: Effective sample sizes ESS_p^{μ}, ESS_p^{β} for the mixture priors $\pi(\mu) = \psi\mathcal{N}(\mu|\tilde{\mu}_\mu, c\tilde{\sigma}_\mu^2) + (1-\psi)\mathcal{N}(\mu|\tilde{\mu}_\mu, \tilde{\sigma}_\mu^2)$, $\pi(\beta) = \psi\mathcal{N}(\beta|\tilde{\mu}_\beta, c\tilde{\sigma}_\beta^2) + (1-\psi)\mathcal{N}(\beta|\tilde{\mu}_\beta, \tilde{\sigma}_\beta^2)$ according to different values of the mixture weight ψ and the scaling factor c.

		$\psi = 0.2$			$\psi = 0.5$			$\psi = 0.8$		
		ESS	ESS_p^{μ}	ESS_p^{β}	ESS	ESS_p^{μ}	ESS_p^{β}	ESS	ESS_p^{μ}	ESS_p^{β}
$c = 5$	$\sigma_\mu^2 = \sigma_\beta^2 = 0.5^2$	37.00	22.76	98.16	36.00	21.81	95.85	23.00	14.20	63.26
	$\sigma_\mu^2 = \sigma_\beta^2 = 1^2$	10.00	5.75	25.48	9.00	5.28	23.44	5.00	2.77	12.51
	$\sigma_\mu^2 = \sigma_\beta^2 = 2^2$	3.00	1.37	6.53	3.00	1.21	5.60	1.00	1.03	2.55
	$\sigma_\mu^2 = \sigma_\beta^2 = 3^2$	2.00	1.03	3.05	1.00	1.03	2.45	1.00	1.00	1.35
	$\sigma_\mu^2 = \sigma_\beta^2 = 5^2$	1.00	1.00	1.37	1.00	1.00	1.17	1.00	1.00	1.07
$c = 10^2$	$\sigma_\mu^2 = \sigma_\beta^2 = 0.5^2$	38.00	22.76	98.13	37.00	22.49	97.59	31.00	19.02	84.77
	$\sigma_\mu^2 = \sigma_\beta^2 = 1^2$	10.00	5.75	25.52	10.00	5.61	24.91	7.00	4.13	18.38
	$\sigma_\mu^2 = \sigma_\beta^2 = 2^2$	3.00	1.37	6.56	3.00	1.32	6.25	2.00	1.06	3.76
	$\sigma_\mu^2 = \sigma_\beta^2 = 3^2$	2.00	1.03	3.03	2.00	1.03	2.82	1.00	1.00	1.54
	$\sigma_\mu^2 = \sigma_\beta^2 = 5^2$	1.00	1.00	1.36	1.00	1.00	1.35	1.00	1.00	1.06
$c = 10^3$	$\sigma_\mu^2 = \sigma_\beta^2 = 0.5^2$	37.00	22.79	98.12	37.00	22.70	97.96	35.00	21.44	94.59
	$\sigma_\mu^2 = \sigma_\beta^2 = 1^2$	10.00	5.74	25.56	10.00	5.70	25.35	9.00	5.10	22.69
	$\sigma_\mu^2 = \sigma_\beta^2 = 2^2$	3.00	1.37	6.51	3.00	1.37	6.44	2.00	1.20	5.24
	$\sigma_\mu^2 = \sigma_\beta^2 = 3^2$	2.00	1.03	3.06	2.00	1.03	3.02	1.00	1.00	2.21
	$\sigma_\mu^2 = \sigma_\beta^2 = 5^2$	1.00	1.00	1.37	1.00	1.00	1.36	1.00	1.00	1.14
$c = 10^4$	$\sigma_\mu^2 = \sigma_\beta^2 = 0.5^2$	37.00	22.78	98.08	37.00	22.75	98.03	37.00	22.34	97.15
	$\sigma_\mu^2 = \sigma_\beta^2 = 1^2$	10.00	5.74	25.52	10.00	5.73	25.46	9.00	5.51	24.55
	$\sigma_\mu^2 = \sigma_\beta^2 = 2^2$	3.00	1.38	6.53	3.00	1.38	6.50	3.00	1.32	6.05
	$\sigma_\mu^2 = \sigma_\beta^2 = 3^2$	2.00	1.03	3.07	2.00	1.03	3.05	2.00	1.03	2.69
	$\sigma_\mu^2 = \sigma_\beta^2 = 5^2$	1.00	1.00	1.37	1.00	1.00	1.37	1.00	1.00	1.25

Table H.3: Effective sample sizes ESS_π^μ, ESS_π^β for the mixture priors $\pi(\mu) = \psi + (1 - \psi)p_\mu$, $\pi(\beta) = \psi + (1-\psi)p_\beta$ according to different values of the mixture weight ψ.

	$\psi = 0.2$			$\psi = 0.5$			$\psi = 0.8$		
	ESS	ESS_π^μ	ESS_π^β	ESS	ESS_π^μ	ESS_π^β	ESS	ESS_π^μ	ESS_π^β
$\sigma_\mu^2 = \sigma_\beta^2 = 0.5^2$	32.00	19.71	87.65	23.00	14.03	62.43	11.00	6.55	29.06
$\sigma_\mu^2 = \sigma_\beta^2 = 1^2$	6.00	3.58	15.78	3.00	1.68	7.42	1.00	1.03	2.48
$\sigma_\mu^2 = \sigma_\beta^2 = 2^2$	1.00	1.00	1.99	1.00	1.00	1.14	1.00	1.00	1.03
$\sigma_\mu^2 = \sigma_\beta^2 = 3^2$	1.00	1.00	1.10	1.00	1.00	1.03	1.00	1.00	1.03
$\sigma_\mu^2 = \sigma_\beta^2 = 3^2$	1.00	1.00	1.03	1.00	1.00	1.03	1.00	1.00	1.03

is combining an informative prior with a purely noninformative prior as in this section and check the results by varying the weight ψ.

I. `R` code for the phase I trial

We report the `R` code for the logistic regression for phase I trial, Section 4. For technical details about the implementation, see Morita (2008), Section 6, example 7.

```
library(distrEx)

# second derivative function
DD <- function(expr, name, order = 1) {
  if(order < 1) stop("'order' must be >= 1")
  if(order == 1) D(expr, name)
  else DD(D(expr, name), name, order - 1)
}

# initialization
m=m_mu=m_Beta=c()
  m_mix=m_mu_mix=m_Beta_mix=m_Jeffreys=m_mu_Jeffreys=m_Beta_Jeffreys=matrix(NA, 5,5)
  tab<-list()
```

```
# scaling factor values
KK=c(5, 100, 1000,10000)


# simulation
for (jj in 1:4){
K <- KK[jj]
for (k in 1:5){
  sigma2_Beta=c(0.5^2,1,2^2,3^2,5^2)
  sigma2_mu=c(0.5^2,1,2^2,3^2,5^2)
  mu_Beta=2.3980
  mu_mu=-0.1313
  X=c(100,200,300,400,500,600)

  X_stand<-c()

  for (h in 1:6){
    X_stand[h]<-log(X[h])-sum(log(X))/6
  }

Dist_p1=1/sigma2_mu[k]
Dist_p2=1/sigma2_Beta[k]

M=100
T=10000
X_rep<-matrix(NA, M, T)

for (j in 1:M){
  X_rep[j,]<-sample(X_stand, T, prob=rep(1/6,6), replace=T )
}

p<-matrix(NA, M, T)
```

```
for (j in 1:M){
  for (t in 1:T){
  p[j,t]=(exp(mu_mu+mu_Beta*X_rep[j,t]))/(1+exp(mu_mu+mu_Beta*X_rep[j,t])  )
}}

Dist_q1<-matrix(NA, M, T)
for (j in 1:M){
  for (t in 1:T){
Dist_q1[j,t]<-sum(p[1:j,t]*(1-p[1:j,t]))
}}

Dist_q2<-matrix(NA, M, T)

for (j in 1:M){
  for (t in 1:T){
  Dist_q2[j,t]<-sum((X_rep[1:j,t]^(2))*p[1:j,t]*(1-p[1:j,t]))
  }}



# Monte Carlo simulation
theta_rep<-matrix(NA, T,2)  #vector of parameters mu and Beta
y_rep<-matrix(NA, M, T)
p_rep<-matrix(NA, M, T)
Dist_q1_rep<-matrix(NA, M, T)
Dist_q2_rep<-matrix(NA, M, T)

for (t in 1:T){
  theta_rep[t,1]<-rnorm(1, mu_mu, sqrt(sigma2_mu[k]))
  theta_rep[t,2]<-rnorm(1, mu_Beta, sqrt(sigma2_Beta[k]))
    for (j in 1:M){
        p_rep[j,t]<-exp(theta_rep[t,1]+theta_rep[t,2]*X_rep[j,t])/
                    (1+exp(theta_rep[t,1]+theta_rep[t,2]*X_rep[j,t]))
```

```
        y_rep[j,t]<-rbinom(1,1,p_rep[j,t])
        Dist_q1_rep[j,t]<-sum(p_rep[1:j,t]*(1-p_rep[1:j,t]))
        Dist_q2_rep[j,t]<-sum((X_rep[1:j,t]^(2))*p_rep[1:j,t]*(1-p_rep[1:j,t]))
}
}

Dist_MC_mu<-c()
Dist_MC_Beta<-c()
Dist_q<-c()

for (j in 1:M){
  Dist_MC_mu[j]<-(1/T)*sum(Dist_q1_rep[j,])
  Dist_MC_Beta[j]<-(1/T)*sum(Dist_q2_rep[j,])
  Dist_q[j]<-Dist_MC_mu[j]+Dist_MC_Beta[j]
}

min1<-c()
min2<-c()

for (t in 1:T){
    # ess computation
min1[t]<-which.min(abs(Dist_p1-Dist_q1[,t]))
min2[t]<-which.min(abs(Dist_p2-Dist_q2[,t]))
}

m_mu[k]<-mean(min1)
m_Beta[k]<-mean(min2)
m_funct<-function(x){
  return(abs(Dist_p1+Dist_p2-Dist_q))
}
m[k]<-which.min(abs(Dist_p1+Dist_p2-Dist_q))     #valori interi
```

```
for (p in 1:3){
  sigma2_Beta=c(0.5^2,1,2^2,3^2,5^2)
  sigma2_mu=c(0.5^2,1,2^2,3^2,5^2)
H_vec<-c(0.2,0.5,0.8)
H<-H_vec[p]


theta_hat<-mu_mu
sigma2_Beta<-sigma2_Beta[k]
sigma2_mu<-sigma2_mu[k]

# ess for the mixture prior
Dist_p3<- -eval(DD(  expression(log( H*((1/((sqrt(2*pi))*sqrt(K*sigma2_mu)) )*
                     exp(-(0.5/(K*sigma2_mu))*(theta_hat-mu_mu)^2))+
                     (1-H)*((1/((sqrt(2*pi))*sigma2_mu) )*
                     exp(-(0.5/(sigma2_mu))*(theta_hat-mu_mu)^2 )))), "theta_hat",2))
Dist_p3_Jeffreys<- -eval(DD(  expression(log( H+
                          (1-H)*((1/((sqrt(2*pi))*sigma2_mu) )*
                    exp(-(0.5/(sigma2_mu))*(theta_hat-mu_mu)^2 )))), "theta_hat",2))

theta_hat<-mu_Beta

Dist_p4<--eval(DD(  expression(log( H*((1/((sqrt(2*pi))*sqrt(K*sigma2_Beta)) )*
                        exp(-(0.5/(K*sigma2_Beta))*(theta_hat-mu_Beta)^2))+
                        (1-H)*((1/((sqrt(2*pi))*sigma2_Beta) )*
                  exp(-(0.5/(sigma2_Beta))*(theta_hat-mu_Beta)^2 )))), "theta_hat",2))
Dist_p4_Jeffreys<--eval(DD(  expression(log( H+
                  (1-H)*((1/((sqrt(2*pi))*sigma2_Beta) )*
                  exp(-(0.5/(sigma2_Beta))*(theta_hat-mu_Beta)^2 )))), "theta_hat",2))

min3<-c()
min4<-c()
min3_Jeffreys<-c()
```

```r
min4_Jeffreys<-c()

for (t in 1:T){

  min3[t]<-which.min(abs(Dist_p3-Dist_q1[,t]))
  min3_Jeffreys[t]<-which.min(abs(Dist_p3_Jeffreys-Dist_q1[,t]))
  min4[t]<-which.min(abs(Dist_p4-Dist_q2[,t]))
  min4_Jeffreys[t]<-which.min(abs(Dist_p4_Jeffreys-Dist_q2[,t]))
}

m_mu_mix[k,p]<-mean(min3)
m_Beta_mix[k,p]<-mean(min4)
m_mix[k,p]<-which.min(abs(Dist_p3+Dist_p4-Dist_q))

m_mu_Jeffreys[k,p]<-mean(min3_Jeffreys)
m_Beta_Jeffreys[k,p]<-mean(min4_Jeffreys)
m_Jeffreys[k,p]<-which.min(abs(Dist_p3_Jeffreys+Dist_p4_Jeffreys-Dist_q))

}

  vec<-list()
  for (k in 1:5){

    vec[[k]]<-rbind(c(m[k], m_mu[k], m_Beta[k],  m_mix[k,1], m_mu_mix[k,1], m_Beta_mix[k,1],
                      m_mix[k,2], m_mu_mix[k,2], m_Beta_mix[k,2],
                      m_mix[k,3], m_mu_mix[k,3], m_Beta_mix[k,3],
                      m_mix[k,4], m_mu_mix[k,4], m_Beta_mix[k,4],
                      m_mix[k,5], m_mu_mix[k,5], m_Beta_mix[k,5]
                      ))
 } }


# final tab
```

```
tab[[jj]]<-rbind( as.double(vec[[1]]), as.double(vec[[2]]), as.double(vec[[3]]),
            as.double(vec[[4]]), as.double(vec[[5]]))
}

vec_Jeffreys<-list()
for (k in 1:5){

  vec_Jeffreys[[k]]<-rbind(c( m_Jeffreys[k,1], m_mu_Jeffreys[k,1], m_Beta_Jeffreys[k,1],
                    m_Jeffreys[k,2], m_mu_Jeffreys[k,2], m_Beta_Jeffreys[k,2],
                    m_Jeffreys[k,3], m_mu_Jeffreys[k,3], m_Beta_Jeffreys[k,3]))
}

tab2<-rbind( as.double(vec_Jeffreys[[1]]), as.double(vec_Jeffreys[[2]]), as.double(vec_Jeffreys[[3]]),
            as.double(vec_Jeffreys[[4]]),  as.double(vec_Jeffreys[[5]]) )
```

References

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[3] S. Morita, P. F. Thall, P. Müller, Determining the effective sample size of a parametric prior, Biometrics 64 (2) (2008) 595–602.