Support Material for: Effective sample size for a mixture prior

Leonardo Egidi*

Department of Economics, Business, Mathematics and Statistics, University of Trieste, Via Tigor 22, 34124, Trieste, Italy.

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1. Proof of Theorem 1

Proof 1. *For simplicity of notation we denote with q, p and π the noninformative, the informative and the mixture prior $q(\theta)$, $p(\theta)$ and $\pi(\theta) = \psi q(\theta) + (1 - \psi)p(\theta)$, respectively. Unless otherwise stated, the dependence of the quantities introduced in Section 2 of the paper on the parameter $\theta \in \mathbb{R}$ is here implicit. We compute the negative second log-derivative for the mixture prior in general terms as*

$$
\begin{aligned}
D_\pi = -\frac{d^2 \log\{\pi(\theta)\}}{d\theta^2} &= -\frac{d^2 \log\{\psi q(\theta) + (1-\psi)p(\theta)\}}{d\theta^2} = -\frac{d}{d\theta}\left[\frac{\psi q' + (1-\psi)p'}{\psi q + (1-\psi)p}\right] \\
&= \frac{(\psi q' + (1-\psi)p')^2 - (\psi q'' + (1-\psi)p'')(\psi q + (1-\psi)p)}{(\psi q + (1-\psi)p)^2}.
\end{aligned}
\tag{1}
$$

After some simple expansions we can rewrite (1) *and apply some minorations:*

$$
\begin{aligned}
D_\pi =& \frac{\psi^2[(q')^2 - q''q] + (1-\psi)^2(p')^2 + 2\psi(1-\psi)p'q' - \psi(1-\psi)q''p - \psi(1-\psi)qp'' - (1-\psi)^2 pp''}{(\psi q + (1-\psi)p)^2} \\
\leq& \left[\frac{(q')^2 - q''q}{q^2}\right] + \frac{(1-\psi)^2(p')^2 - (1-\psi)^2 pp''}{(1-\psi)^2 p^2} + \frac{2\psi(1-\psi)p'q' - \psi(1-\psi)q''p - \psi(1-\psi)qp''}{\psi^2 q^2} \\
=& D_q + K_1,
\end{aligned}
\tag{2}
$$

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*Leonardo Egidi

Email address: `legidi@units.it` (Leonardo Egidi)

where K_1 collects all the terms which do not enter in D_q. Analogously, we can find another minoration:

$$
\begin{aligned}
D_\pi &\leq \left[\frac{(p')^2 - p''p}{p^2}\right] + \frac{\psi^2(q')^2 - \psi^2 qq'' + 2\psi(1-\psi)p'q' - \psi(1-\psi)q''p - \psi(1-\psi)qp''}{\psi^2 q^2} \\
&= D_p + K_2.
\end{aligned}
\tag{3}
$$

From (2) and (3) it stems that

$$
K_1 - K_2 = \left[\frac{(p')^2 - p''p}{p^2}\right] - \left[\frac{(q')^2 - q''q}{q^2}\right] = D_p - D_q, \tag{4}
$$

with $D_p - D_q > 0$ for assumption (see Table (1) in the paper). Hence we have found the following conditions hold:

$$
\begin{cases}
\mathbf{A}\ D_\pi \leq D_q + K_1 \\
\mathbf{B}\ D_\pi \leq D_p + K_2
\end{cases}
\tag{5}
$$

Condition $\mathbf{B}$ *implies* $D_\pi \leq D_p + K_2 + (K_1 - K_2) = D_p + K_1$ *and yields the further condition*

$$
\mathbf{C}\ D_\pi \leq D_p + K_1.
$$

Thus, we may collect the three conditions already found

$$
\begin{cases}
\mathbf{A}\ D_\pi \leq D_q + K_1 \\
\mathbf{B}\ D_\pi \leq D_p + K_2 \\
\mathbf{C}\ D_\pi \leq D_p + K_1
\end{cases}
\tag{6}
$$

Now we may distinguish three separate cases which satisfy the condition $K_1 - K_2 > 0$:

(a) $K_1 > K_2 > 0$

We use conditions $\mathbf{B}, \mathbf{C}$

$$\begin{cases} \mathbf{B}\ D_\pi \leq D_p + K_2 \\ \mathbf{C}\ D_\pi \leq D_p + K_1 \end{cases} \rightarrow \begin{cases} 2D_\pi \leq 2D_p + 2K_2 \\ D_\pi \leq D_p + 2K_1 \end{cases} \rightarrow \begin{cases} D_\pi \leq D_p + 2(K_2 - K_1) \leq D_p \\ - \end{cases} \tag{7}$$

and we conclude that $D_\pi \leq D_p$.

(b) $K_1 > 0,\ K_2 < 0$

By applying condition $\mathbf{B}$ *, it follows* $D_\pi \leq D_p$.

(c) $K_1 < 0,\ K_2 < 0$

By applying condition $\mathbf{B}$ *or* $\mathbf{C}$ *, it follows* $D_\pi \leq D_p$.

We have proved that for any possible sign of $K_1,\ K_2$, $D_\pi \leq D_p$. *By definition of effective sample size we know that*

$$ESS_\pi = \underset{n \in \mathbb{N}}{Argmin}\{\delta(n, \bar{\theta}, \pi, q_n)\} = \underset{n \in \mathbb{N}}{Argmin}\{|D_\pi(\bar{\theta}) - D_{q_n}(\bar{\theta})|\},$$

evaluated in the plug-in estimate $\bar{\theta} = E_\pi(\theta)$. *From Table (1) we also know that the observed information of the posterior* D_{q_n} *is a linear function of the sample size* n *and is increasing,* $dD_{q_n}/dn > 0,\ \forall n \in \mathbb{N}$. *Thus we may conclude that from* $D_\pi \leq D_p$ *it follows:*

$$ESS_\pi = \underset{n \in \mathbb{N}}{Argmin}\{|D_\pi(\bar{\theta}) - D_{q_n}(\bar{\theta})|\} \leq \underset{n \in \mathbb{N}}{Argmin}\{|D_p(\bar{\theta}) - D_{q_n}(\bar{\theta})|\} = ESS_p\ .\ \square$$