An introduction to Bayesian statistics

Sensor fusion & nonlinear filtering

Lars Hammarstrand

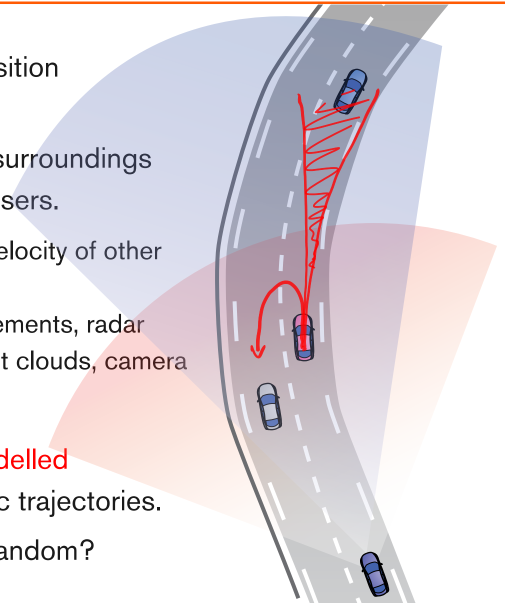
WHAT IS BAYESIAN STATISTICS?

- A statistical inference framework.
- Can be used for estimation, classification, detection, model selection, etc.
- Key characteristic: unknown quantities are described as random.

APPLICATIONS OF BAYESIAN STATISTICS

- A medical application: analyze the disease of a patient.
  - Quantity of interest: the disease, θ.
  - Observations: blood samples, temperature, comments by patient, etc.
- In Bayesian statistics θ is described as random
  $\rightsquigarrow$ we can make statements like: “based on our observations, patient has disease X with 97% probability”.
- Possible concern: is the disease random?

APPLICATIONS OF BAYESIAN STATISTICS



- Self-driving vehicles rely on the ability to position surrounding vehicles.
- This enables the system safely navigate its surroundings without causing accidents with other road users.
  - Quantity of interest: relative position and velocity of other vehicles at the current time.
  - Observations: wheel speeds, INS measurements, radar detections (distance and angle), Lidar point clouds, camera images, etc.
- Bayesian statistics: vehicle motions are modelled statistically $\rightsquigarrow$ helps us to rule out unrealistic trajectories.
- Possible concern: are the vehicle motions random?

COMPARISON: BAYES VS FREQUENTIST

- There are two main strategies to decision making: Bayesian and frequentist statistics.
- In **frequenstist statistics**, the quantities of interest are described as **unknown and deterministic**.

Bayes vs Frequentist

We wish to estimate the height of the Eiffel tower. Is the height random or not?

- **Frequentist perspective:** the tower has a certain height and is therefore not random.
- **Bayesian perspective:** we describe our uncertainties in the height stochastically $\Rightarrow$ height is described as random!

OVERVIEW OF THE BAYESIAN STRATEGY

Suppose we wish to estimate θ given measurements y.

Key steps in a Bayesian method:

1. **Modeling.** Model what we know about θ (using a prior $p(\theta)$) and the how the measurements y relate to θ (using a density $p(y|\theta)$).
2. **Measurement update.** Combine what we knew before (the prior) with our measurement (with $p(y|\theta)$, also called the likelihood) to summarize what we know about θ ($p(\theta|y)$).
3. **Decision making.** Given what we know about θ (described by $p(\theta|y)$) and a loss function, we compute *an optimal decision*.

SELF-ASSESSMENT QUESTIONS

Which of the following statements are correct:

- Bayesian methods can be used to solve many types of decision making problems including estimation, detection and classification.
- We can model the height of the Eiffel tower as random only if we think that there are many similar towers with different heights.
- In Bayesian statistics we describe what we know about θ (the quantity of interest) before observing any measurements.

Check all that apply.

Bayes' rule – a first example

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BAYES' RULE: A FIRST EXAMPLE

Selecting fruit from an urn

- An urn is selected at random (prob. $1/2$, $1/2$).
  From that urn we pick a fruit.

- If fruit is orange, what is probability that we chose the red urn?

PROBABILITY THEORY

- Bayesian statistics is simple! We only need two rules:

Conditional probability (product rule)

$$\Pr\{y, \theta\} = \Pr\{y \,\big|\, \theta\} \Pr\{\theta\}$$

The law of total probability (sum rule)

$$\Pr\{y\} = \sum_{\theta} \Pr\{y, \theta\}$$ discrete variables

$$p(y) = \int_{\theta} p(y, \theta)\, d\theta$$ continuous variables

PROBABILITY THEORY – BAYES' RULE

- Bayes' rule is a consequence of conditional probability,

$$p(y|\theta)p(\theta) = p(\theta|y)p(y).$$

Bayes' rule

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$

- Usage of Bayes' rule: express a relation of interest, $p(\theta|y)$, in terms of the relation that we know $p(y|\theta)$.
- Note that $p(y) = \int_\theta p(y|\theta)p(\theta)\, d\theta$.

BAYES' RULE: A FIRST EXAMPLE

- Let $\theta \in \{r, b\}$ be color of urn, and $y \in \{o, a\}$ be the fruit.
- Question: If fruit is orange, what is probability that we chose the red urn?

$y=o$

$\theta = r$

$$\Pr\{\theta=r \mid y=o\} = \frac{\Pr\{y=o \mid \theta=r\}\Pr\{\theta=r\}}{\Pr\{y=o\}} = \frac{\frac{3}{4}\cdot\frac{1}{2}}{\frac{1}{2}} = \frac{3}{4}$$

$$\Pr\{y=o\} = \Pr\{y=o, \theta=r\} + \Pr\{y=o, \theta=b\} =$$

$$= \frac{3}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{2}$$

$\Pr\{\theta=r\} = \frac{1}{2}$

$\Pr\{y=o \mid \theta=r\} = \frac{3}{4}$

BAYES' RULE: A FIRST EXAMPLE

- Let $\theta \in \{r, b\}$ be color of urn, and $y \in \{o, a\}$ be the fruit.
- Question: If fruit is orange, what is probability that we chose the red urn?
- Bayes' rule gives

$$\Pr\{\theta = r \big| y = o\} = \frac{\Pr\{y = o \big| \theta = r\} \Pr\{\theta = r\}}{\Pr\{y = o\}}$$

where $\Pr\{\theta = r\} = 1/2$, $\Pr\{y = o \big| \theta = r\} = 3/4$ and

$$\begin{aligned} \Pr\{y = o\} &= \Pr\{y = o, \theta = r\} + \Pr\{y = o, \theta = b\} \\ &= \frac{3}{4}\frac{1}{2} + \frac{1}{4}\frac{1}{2} = \frac{1}{2}. \end{aligned}$$

- Thus, $\Pr\{\theta = r \big| y = o\} = \frac{3}{4}$.

Building blocks of Bayesian models – Likelihoods, Priors and Posteriors

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LIKELIHOODS, PRIORS AND POSTERIORS

General problem formulation

- We are interested in an unknown parameter $\theta \in \Theta$ for which we observe some related data y.
- Common problem types are estimation (e.g., $\Theta = \mathbb{R}^n$) and detection problems (e.g., $\Theta = \{-1, 1\}$).

Assumption

- The observed data, y, is distributed as

$$y \sim p(y|\theta),$$

where p is a known distribution.

LIKELIHOODS, PRIORS AND POSTERIORS

Likelihood

- Since y is observed, we often view $p(y|\theta)$ as a function of θ,

$$l(\theta|y) = p(y|\theta),$$

  where $l(\theta|y)$ is called the likelihood function.

- Note: the likelihood function is *not* a density w.r.t. θ.

Prior

- In Bayesian statistics we have a prior distribution on θ, $p(\theta)$.
- Prior means *earlier*, or before, and $p(\theta)$ describes what we know *before* observing y.

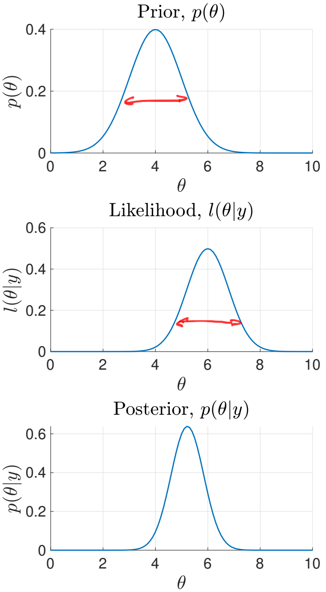
LIKELIHOODS, PRIORS AND POSTERIORS

Posterior

- One objective in Bayesian statistics is to compute the posterior

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} \propto l(\theta|y)p(\theta)$$

- Posterior means *after* and $p(\theta|y)$ describes what we know *after observing y*.

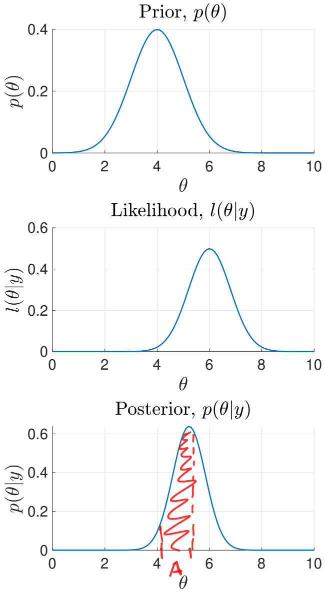


LIKELIHOODS, PRIORS AND POSTERIORS

- We summarize this as

$$\text{posterior} \propto \text{likelihood} \times \text{prior}.$$

- Given the posterior, $p(\theta|y)$ we can answer, e.g.,
  - What is the most probable θ?
  - What is the probability that $\theta \in \mathcal{A}$?
  - What is the posterior mean of θ?
- We can also minimize expected costs in a decision theoretic manner.



EXAMPLE: SCALAR IN GAUSSIAN NOISE

Estimation of scalar in Gaussian noise

- Suppose we observe

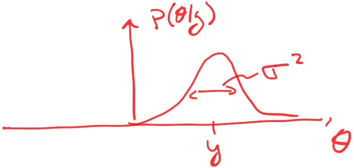
$$y = \theta + v, \qquad v \sim \mathcal{N}(0, \sigma^2)$$

  such that $p(y|\theta) = \mathcal{N}(y; \theta, \sigma^2) \propto \exp\{-(y-\theta)^2/(2\sigma^2)\}$.
- A common non-informative prior on θ is $p(\theta) \propto 1$.

- What is the posterior?

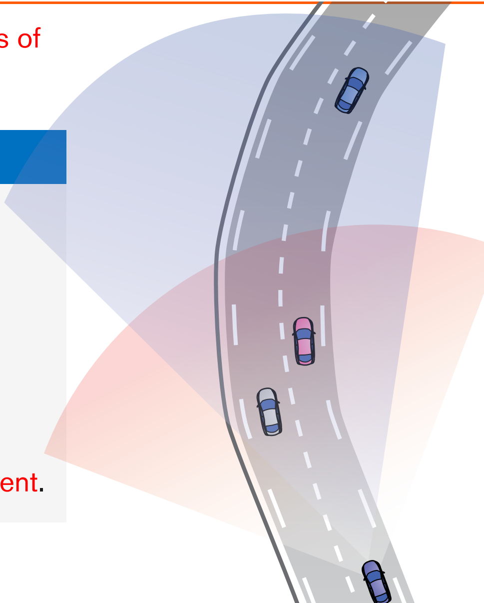
$$p(\theta|y) \propto p(y|\theta) \cdot p(\theta) \propto \exp(-(y-\theta)^2/2\sigma^2) \cdot 1 \propto \mathcal{N}(\theta; y, \sigma^2)$$

$$\Rightarrow p(\theta|y) = \mathcal{N}(\theta; y, \sigma^2)$$



BAYESIAN APPROACH TO SENSOR FUSION

- Suppose we collect measurements from two types of sensors, y_1 and y_2,



Bayesian fusion of independent observations

- We seek the posterior distribution:

$$p(\theta|\underbrace{y_1, y_2}_{y}) \propto p(\theta)p(\underbrace{y_1, y_2}_{y}|\theta).$$

- It is often reasonable to assume that

$$p(y_1, y_2|\theta) \approx p(y_1|\theta)p(y_2|\theta),$$

i.e., that measurements are conditionally independent.

SELF-ASSESSMENT

The posterior distribution is $p(\theta|y) \propto p(y|\theta)p(\theta)$.
It is also true that:

- The normalization factor is not always unique?
- The posterior $p(\theta|y)$ can always be uniquely determined from the fact that $\int p(\theta|y)\, d\theta = 1$?
- The posterior distribution can only be uniquely determined if it is proportional to a well known distribution, e.g., a Gaussian.

Only one statement is correct.

Bayesian Decision Theory

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BAYESIAN DECISION PRINCIPLE

- How can we use $p(\theta|y)$ to make decisions?
- **Examples** of decision problems
  - How to control a self-driving vehicle.
  - How to invest money.
  - Select medicine to give to a patient
  - Estimate a parameter vector (may represent temperature, distance, etc).

Basic principle of Bayesian decision theory

- Minimize expected loss

  or, equivalently,

- Maximize expected utility.

DECISION THEORY – A TOY EXAMPLE

Choosing a course

- A student wants to decide whether to take a course or not.
- Suppose $\theta \in \{\text{good course, fair course, bad course}\}$ and

	good course	fair course	bad course
$\Pr\{\theta \| y\}$	0.3	0.3	0.4

- If the loss function is

	good course	fair course	bad course
Take	0	5	30
Not take	20	5	0

should he/she then take the course?

MINIMUM POSTERIOR EXPECTED LOSS

- We often study loss functions $C(\theta, a)$ instead of utility. (Typically, $C \geq 0$.)
- Let $\hat{\theta}$ denote an estimate of θ.

Optimal Bayesian decisions

Minimize the posterior expected loss

$$\hat{\theta} = \arg\min_a \mathbb{E}\left\{C(\theta, a)\middle|y\right\}$$

where $\mathbb{E}\left\{C(\theta, a)\middle|y\right\} = \int_\Theta C(\theta, a)p(\theta|y)\, d\theta$

- Note: y is given (fixed) and θ is random.

SELF-ASSESSMENT

To make an optimal Bayesian decision it is sufficient to know:

- The prior, $p(\theta)$, the likelihood, $p(y|\theta)$, and a loss function $C(\theta, a)$.
- The likelihood, $p(y|\theta)$, and a loss function $C(\theta, a)$.
- The posterior distribution, $p(\theta|y)$, and a loss function, $C(\theta, a)$.

Check all statements that apply.

COMPARISON: BAYES VS FREQUENTIST

Frequentist	Bayes
θ is fixed and unknown $\Rightarrow$ θ is deterministic	Uncertainties in θ are described stochastically $\Rightarrow$ θ is random
Maximum likelihood (ML) most famous estimator $\hat{\theta}_{ML} = \arg\max_\theta l(\theta\|y)$	Minimum mean square error and maximum a posteriori estimators, e.g., $\hat{\theta}_{MAP} = \arg\max_\theta p(\theta)l(\theta\|y)$
Study performance by averaging over y for fixed θ	Make decisions conditioned on the observation y.

- Note 1: most Bayesians also study frequentist performance.
- Note 2: many frequentists agree that parameters may be random in some situations.

Cost functions in Bayesian decision theory

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BAYESIAN DECISION THEORY – SUMMARY

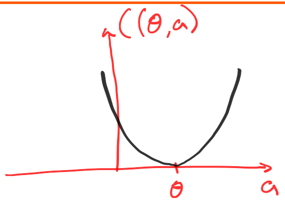
- Bayesian decision theory relies on
  1. Likelihood: $p(y \mid \theta)$
  2. Prior distribution: $p(\theta)$
  3. Loss function: $C(\theta, a)$
- Combining likelihood and prior gives posterior

$$p(\theta \mid y) \propto p(y \mid \theta) p(\theta).$$

Posterior and loss gives decisions

$$\hat{\theta} = \arg\min_a \int_\Theta C(\theta, a) p(\theta \mid y) \, d\theta.$$

THE QUADRATIC LOSS FUNCTION



Minimum mean squared error estimator, MMSE

- Parameter estimation, $\theta \in \Theta = \mathbb{R}^n$
- Most common loss function is the quadratic loss

$$C(\theta, a) = \|\theta - a\|_2^2 = (\theta - a)^T(\theta - a)$$

- Let: $\bar{\theta} = \mathbb{E}\{\theta|y\}$, $\mathbf{P} = \mathrm{Cov}\{\theta|y\} = \mathbb{E}\left\{(\theta - \bar{\theta})(\theta - \bar{\theta})^T|y\right\}$

- Optimal estimator: $E\{C(\theta,a)|y\} = E\{(\theta-a)^T(\theta-a)|y\} = E\{(\underbrace{\theta-\bar\theta}_{\text{zero mean}}+\underbrace{\bar\theta-a}_{\text{Determ.}})^T(\theta-\bar\theta+\bar\theta-a)|y\}$

$$= \underbrace{E\{(\theta-\bar\theta)^T(\theta-\bar\theta)|y\}}_{Tr\{P\}} + \underbrace{E\{(\theta-\bar\theta)^T|y\}}_{=0}(\bar\theta-a) + 0 + (\bar\theta-a)^T(\bar\theta-a)$$

$$\hat\theta_{MMSE} = \min_{a} \arg E\{C(\theta,a)|y\} = \bar\theta$$

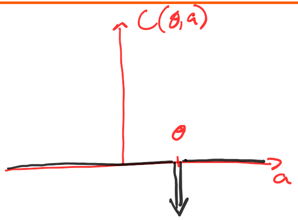
THE $0-1$ LOSS FUNCTION

Maximum a-posteriori estimator, MAP

- Parameter estimation, $\theta \in \Theta = \mathbb{R}^n$.
- Another common choice is the $0-1$ loss function

$$C(\theta, a) = -\delta(\theta - a),$$

$\delta(\cdot)$ is the Dirac's delta function.



- Optimal estimator: $\mathbb{E}\{C(\theta, a)\big|y\} = \mathbb{E}\{-\delta(\theta - a)\big|y\}$
$= -\int p(\theta\big|y)\delta(\theta - a)\,d\theta = -p(\theta\big|y)\Big|_{\theta=a}$

$$\Rightarrow \quad \hat{\theta}_{MAP} = \arg\min_{a} -p(\theta\big|y)\Big|_{\theta=a}$$

$$= \arg\max_{\theta} p(\theta\big|y)$$

SELF-ASSESSMENT

Suppose $p(\theta|y) = \mathcal{N}(\theta; \bar{\theta}, \mathbf{P})$. The MMSE and MAP estimators are, respectively,

- $\bar{\theta} + \mathrm{tr}\{\mathbf{P}\}$ and $\bar{\theta}$.
- $\bar{\theta}$ and $\mathcal{N}(\theta; \bar{\theta}, \mathbf{P})$.
- $\bar{\theta}$ and $\bar{\theta}$.
- $\bar{\theta} + \mathrm{tr}\{\mathbf{P}\}$ and $\mathcal{N}(\theta; \bar{\theta}, \mathbf{P})$.

Only one statement is correct.