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L9.1: An introduction to particle filtering

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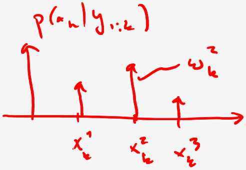
- Gaussian filtering is a useful technique to perform nonlinear filtering.
- Limitations: Gaussian filtering methods do not perform well when
  - the models are highly nonlinear,
  - when the posterior distribution is significantly non-Gaussian, e.g., a multimodal density.
- For such problems we need a different type of approximation to the posterior density!

Basic idea

- Use a non-parametric representation

$$p(\mathbf{x}_k|\mathbf{y}_{1:k}) \approx \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{x}_k - \mathbf{x}_k^{(i)})$$



where $\mathbf{x}_k^{(i)}$ are particles and $w_k^{(i)}$ are associated weights.

- Filtering is (essentially) performed by
  1. propagating $\mathbf{x}_{k-1}^{(i)} \to \mathbf{x}_k^{(i)}$ over time,
  2. updating the weights, $w_k^{(i)}$.
- Basic version: $\mathbf{x}_k^{(i)} \sim p(\mathbf{x}_k|\mathbf{x}_{k-1}^{(i)})$, $w_k^{(i)} \propto w_{k-1}^{(i)} p(\mathbf{y}_k|\mathbf{x}_k^{(i)})$.

After this lecture you should be able to

- explain the concepts of Monte Carlo sampling and importance sampling,
- describe what particle degeneracy is and why resampling is useful,
- and implement a particle filter.

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L9.2: Monte Carlo (MC) approximations and Importance Sampling (IS)

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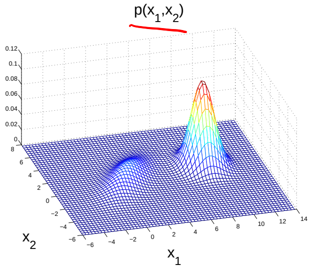
Monte Carlo approximations

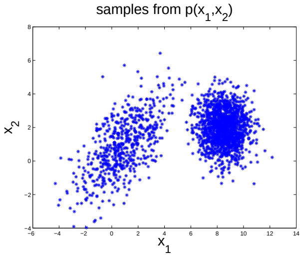
Two perspectives on Monte Carlo approximation

Given independent samples $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(N)} \sim p(\mathbf{x})$ we can approximate

$$\mathbb{E}[\mathbf{g}(\mathbf{x})] \approx \frac{1}{N} \sum_{i=1}^{N} \mathbf{g}(\mathbf{x}^{(i)}) \qquad (1)$$

$$p(\mathbf{x}) \approx \frac{1}{N} \sum_{i=1}^{N} \delta(\mathbf{x} - \mathbf{x}^{(i)}) \qquad (2)$$





Remarks on Monte Carlo approximations:

- non-parametric approximation to $p(\mathbf{x})$.
- approximate all kinds of densities, $p(\mathbf{x})$. Very flexible!
- does not suffer from the *curse of dimensionality*, e.g., $\hat{\mu}$

$$\text{Cov}(\hat{\mu}) = \text{Cov}\Bigg(\underbrace{\frac{1}{N}\sum_{i=1}^{N}\mathbf{x}^{(i)}}_{\hat{\mu}}\Bigg) = \frac{1}{N}\text{Cov}(\mathbf{x})$$

  independently on dim$(\mathbf{x})$!
- Weakness: it is often difficult to generate samples from $p(\mathbf{x})$.

Importance sampling

What can we do when it is difficult to sample from $p(\mathbf{x})$?

Importance sampling

- Generate samples, $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(N)}$, from a proposal density $q(\mathbf{x})$:

$$\mathbb{E}_{p(\mathbf{x})}[\mathbf{g}(\mathbf{x})] = \int \underbrace{g(x) \cdot \frac{p(x)}{q(x)}}_{\tilde{g}(x)} q(x)\,dx \approx \frac{1}{N} \sum_{i=1}^{N} g(x^{(i)}) \frac{p(x^{(i)})}{q(x^{(i)})}$$

$$\propto \sum_{i=1}^{N} \underbrace{\frac{p(x^{(i)})}{q(x^{(i)})}}_{\tilde{w}^{(i)}} \cdot g(x^{(i)})$$

Importance sampling approximation to $p(\mathbf{x})$

- Generate samples, $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(N)}$, from $q(\mathbf{x})$ and set

$$p(\mathbf{x}) \approx \sum_{i=1}^{N} w^{(i)} \delta(\mathbf{x} - \mathbf{x}^{(i)})$$

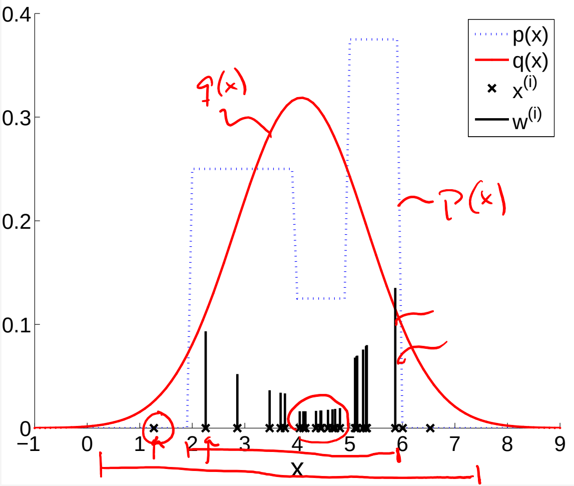
where

$$w^{(i)} = \frac{\tilde{w}^{(i)}}{\sum_{n=1}^{N} \tilde{w}^{(n)}} \quad \text{and} \quad \tilde{w}^{(i)} = \frac{p(\mathbf{x}^{(i)})}{q(\mathbf{x}^{(i)})}.$$

- Importance sampling is a flexible and powerful tool.
- It can perform very well as long as:
  1. it is easy to sample from $q(\mathbf{x})$,
  2. the support of $q(\mathbf{x})$ contains the support of $p(\mathbf{x})$,
  3. $q(\mathbf{x})$ is "similar" to $p(\mathbf{x})$.

Example – Importance sampling

- Approximate $p(x)$ using N independent samples from $q(x) = \mathcal{N}(x; 4, 1.5^2)$.



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L9.3: Sequential Importance Sampling (SIS)

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- Objective: to recursively and accurately approximate the filtering density, $p(\mathbf{x}_k | \mathbf{y}_{1:k})$.
- Assumption: both the motion and measurement models

$$p(\mathbf{x}_k | \mathbf{x}_{k-1}) \qquad \text{and} \qquad p(\mathbf{y}_k | \mathbf{x}_k)$$

  can be easily evaluated point-wise.
- A common example is

$$\begin{aligned} \mathbf{x}_k &= f(\mathbf{x}_{k-1}) + \mathbf{q}_{k-1}, & \mathbf{q}_{k-1} &\sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{k-1}) \\ \mathbf{y}_k &= h(\mathbf{x}_k) + \mathbf{r}_k & \mathbf{r}_k &\sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k), \end{aligned}$$

  where, e.g., $p(\mathbf{x}_k | \mathbf{x}_{k-1}) = \mathcal{N}(\mathbf{x}_k; f(\mathbf{x}_{k-1}), \mathbf{Q}_{k-1})$ is generally easy to evaluate for any values of $\mathbf{x}_k$ and $\mathbf{x}_{k-1}$.

- Particle filters are also known as *sequential importance resampling* or *sequential Monte Carlo*.
- The basis of these methods is an algorithm called sequential importance sampling (SIS).

Standard SIS algorithm

- For $i = 1, \ldots, N$ and at each time k:
  - Draw $\mathbf{x}_k^{(i)} \sim q(\mathbf{x}_k | \mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k)$.
  - Compute weights

$$w_k^{(i)} \propto w_{k-1}^{(i)} \frac{p(\mathbf{y}_k | \mathbf{x}_k^{(i)}) p(\mathbf{x}_k^{(i)} | \mathbf{x}_{k-1}^{(i)})}{q(\mathbf{x}_k^{(i)} | \mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k)}.$$

  - Normalize the weights.

- We then approximate $p(\mathbf{x}_k | \mathbf{y}_{1:k}) \approx \sum_{i=1}^N w_k^{(i)} \delta(\mathbf{x}_k - \mathbf{x}_k^{(i)})$.

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- Assuming that we describe our posterior using the following approximation $p(\mathbf{x}_k|\mathbf{y}_{1:k}) \approx \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{x}_k - \mathbf{x}_k^{(i)})$. What is then the MMSE estimate of $\mathbf{x}_k$?
  - $\hat{\mathbf{x}}_k = \sum_i^N w_k^{(i)} \mathbf{x}_k^{(i)}$
  - $\hat{\mathbf{x}}_k = \mathbf{x}_k^{(j)}$, where $j = \arg\max_i w_k^{(i)}$
  - $\hat{\mathbf{x}}_k = \frac{1}{N} \sum_i^N \mathbf{x}_k^{(i)}$
  - It is not possible to calculate a MMSE estimate from this approximation.

$$\hat{x}_k = E\{x_k | y_k\} = \int x_k \sum_{i=1}^{N} \omega_k^{(i)} \delta(x_k - x_k^{(i)}) dx_k$$

$$= \sum_{i=1}^{N} \omega_k^{(i)} x_k^{(i)}$$

Derivation - Basic strategy

Recursively at time $k = 1, 2, \ldots$

1. Draw particles

$$\mathbf{x}_{0:k}^{(i)} \sim q(\mathbf{x}_{0:k}|\mathbf{y}_{1:k})$$

2. Update weights

$$w_k^{(i)} \propto \frac{p(\mathbf{x}_{0:k}^{(i)}|\mathbf{y}_{1:k})}{q(\mathbf{x}_{0:k}^{(i)}|\mathbf{y}_{1:k})}$$

Comments on drawing particles:

- Let us assume that

$$q(\mathbf{x}_{0:k}|\mathbf{y}_{1:k}) = q(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_k)q(\mathbf{x}_{0:k-1}|\mathbf{y}_{1:k-1}).$$

- we generate $\mathbf{x}_{0:k-1}^{(i)} \sim q(\mathbf{x}_{0:k-1}|\mathbf{y}_{1:k-1})$ at time $k-1$,
- it is sufficient to generate $\mathbf{x}_k^{(i)} \sim q(\mathbf{x}_k|\mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k)$ and append that to $\mathbf{x}_{1:k-1}^{(i)}$!

- It remains to derive the expression for the weights:

$$w_k^{(i)} \propto \frac{p(\mathbf{x}_{0:k}^{(i)}|\mathbf{y}_{1:k})}{q(\mathbf{x}_{0:k}^{(i)}|\mathbf{y}_{1:k})}$$

$\propto p(x_{0:k-1}^{(i)}, x_k^{(i)}, y_k | y_{1:k-1}) = p(y_k | x_k^{(i)}) p(x_k^{(i)} | x_{k-1}^{(i)}) p(x_{0:k-1}^{(i)} | y_{1:k-1})$

$$\propto \frac{p(\mathbf{y}_k|\mathbf{x}_k^{(i)})p(\mathbf{x}_k^{(i)}|\mathbf{x}_{k-1}^{(i)})}{q(\mathbf{x}_k^{(i)}|\mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k)} \underbrace{\frac{p(\mathbf{x}_{0:k-1}^{(i)}|\mathbf{y}_{1:k-1})}{q(\mathbf{x}_{0:k-1}^{(i)}|\mathbf{y}_{1:k-1})}}_{w_{k-1}^{(i)}}$$

$$\propto w_{k-1}^{(i)} \frac{p(\mathbf{y}_k|\mathbf{x}_k^{(i)})p(\mathbf{x}_k^{(i)}|\mathbf{x}_{k-1}^{(i)})}{q(\mathbf{x}_k^{(i)}|\mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k)}$$

- We have thus derived the SIS algorithm:

Standard SIS algorithm

- For $i = 1, \ldots, N$ and at each time k:
  - Draw $\mathbf{x}_k^{(i)} \sim q(\mathbf{x}_k | \mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k)$.
  - Compute weights

$$w_k^{(i)} \propto w_{k-1}^{(i)} \frac{p(\mathbf{y}_k | \mathbf{x}_k^{(i)}) p(\mathbf{x}_k^{(i)} | \mathbf{x}_{k-1}^{(i)})}{q(\mathbf{x}_k^{(i)} | \mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k)}.$$

  - Normalize the weights.

- A simple choice of importance density is

$$q(\mathbf{x}_k | \mathbf{x}_{k-1}, \cancel{\mathbf{y}_k}) = p(\mathbf{x}_k | \mathbf{x}_{k-1})$$

for which $w_k^{(i)} \propto w_{k-1}^{(i)} p(\mathbf{y}_k | \mathbf{x}_k^{(i)})$.

Example - Nonlinear filter benchmark

- The following is a common benchmark for nonlinear filters

$$x_k = \frac{x_{k-1}}{2} + \frac{25 x_{k-1}}{1 + x_{k-1}^2} + 8\cos(1.2k) + q_{k-1}$$

$$y_k = \frac{x_k^2}{20} + r_k$$

where $q_{k-1} \sim \mathcal{N}(0, 10)$ and $r_k \sim \mathcal{N}(0, 1)$.

- Let us see how the above filter performs on this challenging problem!

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L9.4: Sequential Importance Resampling (SIR)

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- One can show that all SIS filters suffer from degeneracy:

  after a few time steps all but one particle will have negligible weight.

- Consequences of degeneracy:
  - the filter believes that it knows $\mathbf{x}_k$ exactly,
  - we obtain very poor state estimates,
  - most of our calculations are wasted on insignificant particles.

  These are very serious drawbacks!

- A key technique to improve performance is resampling.

- Challenge: we have $p(\mathbf{x}_k|\mathbf{y}_{1:k}) \approx \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{x}_k - \mathbf{x}_k^{(i)})$ where most weights $w_k^{(i)}$ are very small.

Idea: use Monte Carlo sampling

- Generate independent samples $\tilde{\mathbf{x}}_k^{(1)}, \ldots, \tilde{\mathbf{x}}_k^{(N)}$ from $p(\mathbf{x}_k|\mathbf{y}_{1:k})$ and set

$$p(\mathbf{x}_k|\mathbf{y}_{1:k}) \approx \sum_{i=1}^{N} \frac{1}{N} \delta(\mathbf{x}_k - \tilde{\mathbf{x}}_k^{(i)}).$$

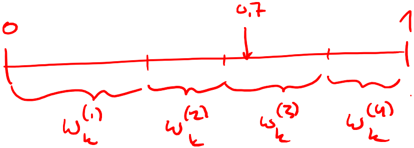
- After resampling we get
  - equal weights (they are all $1/N$),
  - multiple copies of high probability particles.

Resampling algorithm

1) Draw N samples with replacement from $\mathbf{x}_k^{(1)}, \mathbf{x}_k^{(2)}, \ldots, \mathbf{x}_k^{(N)}$, where the probability of selecting $\mathbf{x}_k^{(i)}$ is $w_k^{(i)}$.

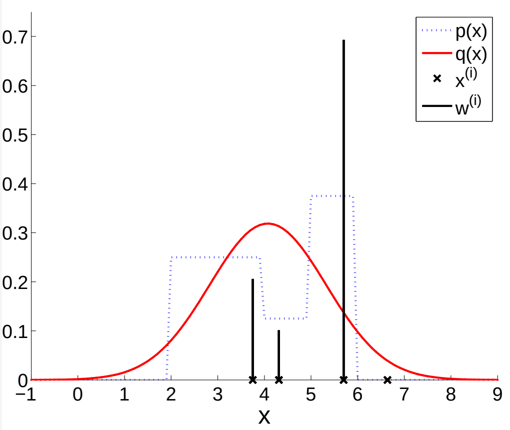
2) Replace the old sample set with the new one and set all weights to $1/N$.

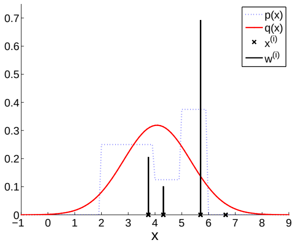
- A few remarks:
  - We use $\mathbf{x}_k^{(i)}$ and $w_k^{(i)}$ to denote the particles and their weights also after resampling.
  - We can use samples from the uniform distribution, unif[0, 1], to draw samples from the discrete distribution $p(\mathbf{x}_k | \mathbf{y}_{1:k})$.



Self-assessment – Resampling

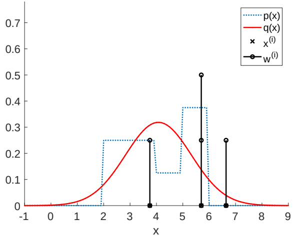
- Perform resampling on the density to the right and illustrate the result.
- Assume that the numbers 0.65, 0.03, 0.84 and 0.93 are drawn from unif[0, 1].

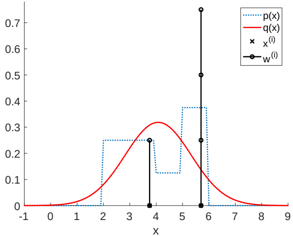


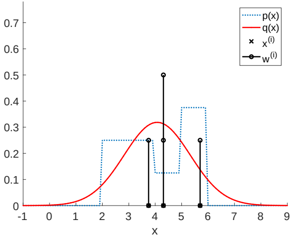


- Perform resampling on the density to the right and illustrate the result.
- Assume that the numbers 0.65, 0.03, 0.84 and 0.93 are drawn from unif[0, 1].

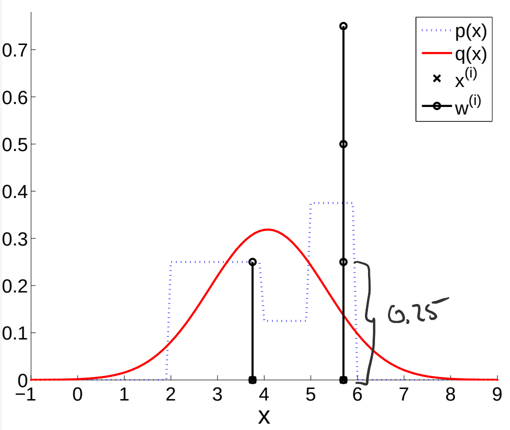
- Choose the figure below that illustrates the resampled particles:



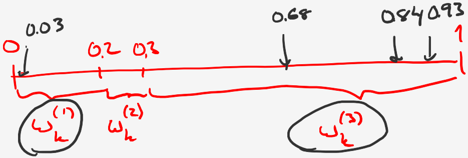




Self-assessment – Solution



- If particles where ordered in ascending order, $x^{(1)} < \cdots < x^{(4)}$, resampling gives $x^{(1)} = 3.8$ and $x^{(2)} = x^{(3)} = x^{(4)} = 5.7$.



- Resampling costs some calculations and introduces some errors, but improves performance immensely over time.
- An estimate for the *effective number of particles* is

$$N_{eff} = \frac{1}{\sum_{i=1}^{N} \left(w_k^{(i)}\right)^2}.$$

- Many algorithms only resample when N_{eff} is below some threshold, e.g., $N/4$.

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L9.5: Choice of importance distribution

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- A carefully selected importance distribution, $q(\mathbf{x}_k | \mathbf{x}_{k-1}, \mathbf{y}_k)$, can slow down the degeneracy and improve performance.

Intuition: if most particles are placed in "high probability regions" there is less need to get rid of useless particles.

Optimal importance density

- The optimal importance density is

$$q(\mathbf{x}_k | \mathbf{x}_{k-1}, \mathbf{y}_k) = p(\mathbf{x}_k | \mathbf{x}_{k-1}, \mathbf{y}_k).$$

- Unfortunately, in most nonlinear settings, $p(\mathbf{x}_k | \mathbf{x}_{k-1}, \mathbf{y}_k)$, is difficult to both draw samples from and to evaluate.

- We can approximate $p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_k)$ using, e.g., linearization.

- The most common choice is still, $q(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_k) = p(\mathbf{x}_k|\mathbf{x}_{k-1})$, and the bootstrap algorithm.

The bootstrap PF

At each time k:

- Draw $\mathbf{x}_k^{(i)} \sim p(\mathbf{x}_k|\mathbf{x}_{k-1}^{(i)})$, for $i = 1, \ldots, N$.
- Calculate $w_k^{(i)} \propto w_{k-1}^{(i)} p(\mathbf{y}_k|\mathbf{x}_k^{(i)})$ and normalize to 1.
- Resample.

- Note: if we resample at every time step, we get $w_k^{(i)} \propto p(\mathbf{y}_k|\mathbf{x}_k^{(i)})$ since $w_{k-1}^{(i)} = 1/N \;\; \forall i$ after resampling.
- Note 2: the Auxiliary PF (APF) is variation of the SIR algorithm that makes use of $\mathbf{y}_k$.

- Particle filters (PFs) can handle highly nonlinear and non-Gaussian systems.
- Particle filters are asymptotically exact as you increase N.
- The complexity is roughly $O(N)$ but the gain in performance flattens out as you increase N.
- Unfortunately, PFs suffer from the curse of dimensionality and are intractable in higher dimensions.

- The output from a PF is an approximation

$$p(\mathbf{x}_k | \mathbf{y}_{1:k}) \approx \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{x}_k - \mathbf{x}_k^{(i)}) \iff \Pr\{x_k = x' | y_{1:k}\} = \begin{cases} \omega_k^{(i)} & \text{if } x' = x_k^{(i)} \\ 0 & \text{otherwise} \end{cases}$$

which implies that

$$\mathbb{E}\left[\mathbf{g}(\mathbf{x}_k) | \mathbf{y}_{1:k}\right] \approx \sum_{i=1}^{N} w_k^{(i)} \mathbf{g}(\mathbf{x}_k^{(i)}).$$

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L9.6: Rao-Blackwellized Particle Filter

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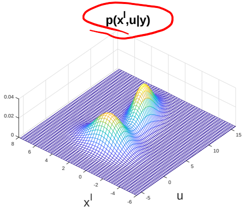
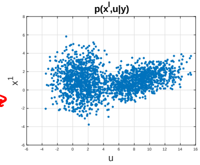
- Background:
  - particle filters are intractable in high dimensions.
  - many systems are linear in some dimensions.

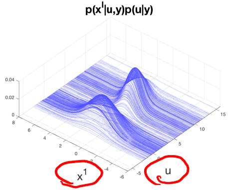
- Idea 1: *"combine a particle filter for the nonlinear states with a Kalman filter for the linear states"*.

- Idea 2: If $\mathbf{x}_k = \begin{bmatrix} \mathbf{x}_k^l \\ \mathbf{u}_k \end{bmatrix}$ where $\mathbf{x}_k^l$ and $\mathbf{u}_k$ are the linear and nonlinear states:

$$p(\mathbf{x}_k^l, \mathbf{u}_{1:k} | \mathbf{y}_{1:k}) = \underbrace{p(\mathbf{x}_k^l | \mathbf{u}_{0:k}, \mathbf{y}_{1:k})}_{\text{Gaussian}} \underbrace{p(\mathbf{u}_{0:k} | \mathbf{y}_{1:k})}_{\text{Particle filter}}$$







- Assuming we have $\mathbf{x}_k = \begin{bmatrix} \mathbf{x}_k^l \\ \mathbf{u}_k \end{bmatrix}$, Rao-Blackwellized particle filters are often used for models on the form

$$\mathbf{x}_k^l = f_{k-1}^l(\mathbf{u}_{k-1}) + \mathbf{A}_{k-1}^l(\mathbf{u}_{k-1})\mathbf{x}_{k-1}^l + \mathbf{q}_{k-1}^l$$

$$\mathbf{u}_k = f_{k-1}^u(\mathbf{u}_{k-1}) + \mathbf{A}_{k-1}^u(\mathbf{u}_{k-1})\mathbf{x}_{k-1}^l + \mathbf{q}_{k-1}^u$$

$$\mathbf{y}_k = h_k(\mathbf{u}_k) + \mathbf{H}_k(\mathbf{u}_k)\mathbf{x}_k^l + \mathbf{r}_k$$

where all the noises are Gaussian.

Bearing only tracking

- Bearing only tracking with a constant velocity motion in 2D.
  What is $\mathbf{x}_k^l$, $\mathbf{u}_k^l$ and $\mathbf{y}_k$ in this example?

- $\mathbf{x}_k^l$: position, $\mathbf{u}_k^l$: velocity, $\mathbf{y}_k$: bearing to target
- $\mathbf{x}_k^l$: velocity, $\mathbf{u}_k^l$: position, $\mathbf{y}_k$: bearing to target
- $\mathbf{x}_k^l$: velocity, $\mathbf{u}_k^l$: bearing to target, $\mathbf{y}_k$: position
- $\mathbf{x}_k^l$: position, $\mathbf{u}_k^l$: bearing to target, $\mathbf{y}_k$: bearing to target

Bearing only tracking – system models

- Let us denote our state vector $\mathbf{x}_k = [\underbrace{x_k^1, x_k^2}_{u_k}, \underbrace{\dot{x}_k^1, \dot{x}_k^2}_{x^l_k}]^T$, the system models can then be written as:

$$x^l_k = \begin{bmatrix} \dot{x}_k^1 \\ \dot{x}_k^2 \end{bmatrix} = \begin{bmatrix} \dot{x}_{k-1}^1 \\ \dot{x}_{k-1}^2 \end{bmatrix} + \mathbf{q}_{k-1}^l = x^l_{k-1} + q^l_{k-1}$$

$$u_k = \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} = \begin{bmatrix} x_{k-1}^1 \\ x_{k-1}^2 \end{bmatrix} + T \begin{bmatrix} \dot{x}_{k-1}^1 \\ \dot{x}_{k-1}^2 \end{bmatrix} + \mathbf{q}_{k-1}^u = u_{k-1} + T \cdot x^l_k + q^u_{k-1}$$

$$\mathbf{y}_k = \operatorname{atan}_2(x_k^2, x_k^1) + \mathbf{r}_k$$

where $\mathbf{r}_k \sim \mathcal{N}(0, \sigma_r^2)$ and $\mathbf{q}_k = \begin{bmatrix} \mathbf{q}_k^u \\ \mathbf{q}_k^l \end{bmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} \frac{T^2}{2}\mathbf{I} \\ T\mathbf{I} \end{bmatrix} \begin{bmatrix} \sigma_q^2 & 0 \\ 0 & \sigma_q^2 \end{bmatrix} \begin{bmatrix} \frac{T^2}{2}\mathbf{I} \\ T\mathbf{I} \end{bmatrix}^T\right)$

- One recursion of the Rao-Blackwellized particle filter contains five steps:

$p(\mathbf{x}^l_{k-1}\|\mathbf{u}_{0:k-1},\mathbf{y}_{1:k-1})p(\mathbf{u}_{0:k-1}\|\mathbf{y}_{1:k-1}) \rightarrow p(\mathbf{x}^l_k\|\mathbf{u}_{0:k},\mathbf{y}_{1:k})p(\mathbf{u}_{0:k}\|\mathbf{y}_{1:k})$	
1) PF-pred:	$p(\mathbf{u}_{1:k-1}\|\mathbf{y}_{1:k-1}) \rightarrow p(\mathbf{u}_{1:k}\|\mathbf{y}_{1:k-1})$
2) KF, dyn. upd.:	$p(\mathbf{x}^l_{k-1}\|\mathbf{u}_{1:k-1},\mathbf{y}_{1:k-1}) \rightarrow p(\mathbf{x}^l_{k-1}\|\mathbf{u}_{1:k},\mathbf{y}_{1:k-1})$
3) KF-pred:	$p(\mathbf{x}^l_{k-1}\|\mathbf{u}_{1:k-1},\mathbf{y}_{1:k-1}) \rightarrow p(\mathbf{x}^l_k\|\mathbf{u}_{1:k-1},\mathbf{y}_{1:k-1})$
4) PF, update:	$p(\mathbf{u}_{1:k}\|\mathbf{y}_{1:k-1}) \rightarrow p(\mathbf{u}_{1:k}\|\mathbf{y}_{1:k})$
5) KF, meas. upd.:	$p(\mathbf{x}^l_k\|\mathbf{u}_{1:k},\mathbf{y}_{1:k-1}) \rightarrow p(\mathbf{x}^l_k\|\mathbf{u}_{1:k},\mathbf{y}_{1:k})$

$u_k^{(i)} = u_{k-1}^{(i)} + T\cdot x_k^l + q_{k-1}^u$

$u_k^{(i)} - u_{k-1}^{(i)} = T x_k^l + q_{k-1}^u$

$x_k^l = x_{k-1}^l + q_{k-1}^l$

$w_k^{(i)} \propto w_{k-1}^{(i)} \cdot p(y_k | u_k)$

- **Note:**
  - Step 2) makes use of the motion model for $\mathbf{u}_k$ to update $\mathbf{x}^l_{k-1}$.
  - The linear states are marginalized from step 1) and 4), similarly to how we normally handle noise.

Bearing only tracking – system models

- Let us denote our state vector $\mathbf{x}_k = [x_k^1, x_k^2, \dot{x}_k^1, \dot{x}_k^2]^T$, the system models can then be written as:

$$\begin{bmatrix} \dot{x}_k^1 \\ \dot{x}_k^2 \end{bmatrix} = \begin{bmatrix} \dot{x}_{k-1}^1 \\ \dot{x}_{k-1}^2 \end{bmatrix} + \mathbf{q}_{k-1}^l$$

$$\begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} = \begin{bmatrix} x_{k-1}^1 \\ x_{k-1}^2 \end{bmatrix} + T \begin{bmatrix} \dot{x}_{k-1}^1 \\ \dot{x}_{k-1}^2 \end{bmatrix} + \mathbf{q}_{k-1}^u$$

$$\mathbf{y}_k = \mathrm{atan}_2(x_k^2, x_k^1) + \mathbf{r}_k$$

where $\mathbf{r}_k \sim \mathcal{N}(0, (\frac{\pi}{180})^2)$ & $\mathbf{q}_k = \begin{bmatrix} \mathbf{q}_k^u \\ \mathbf{q}_k^l \end{bmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} \frac{T^2}{2}\mathbf{I} \\ T\mathbf{I} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{T^2}{2}\mathbf{I} \\ T\mathbf{I} \end{bmatrix}^T\right)$

Concluding remarks:

- Rao-Blackwellized particle filters are useful to reduce the number of particles.
- These filters enable us to handle higher dimensions than normal PFs.
- They are particularly useful if Kalman gains and posterior covariances are independent of the nonlinear states
  $\Rightarrow$ sufficient to compute them one time in each recursion.