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Secure State Reconstruction in Differentially Flat Systems Under Sensor Attacks Using Satisfiability Modulo Theory Solving

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Abstract— **We address the problem of reconstructing the state of a differentially flat system from measurements that may be corrupted by an adversarial attack. In cyber-physical systems, malicious attacks can directly compromise the system's sensors or manipulate the communication between sensors and controllers. We consider attacks that only corrupt a subset of sensor measurements. We show that the possibility of reconstructing the state under such attacks is characterized by a suitable generalization of the notion of s-sparse observability, previously introduced by some of the authors in the linear case. We also extend our previous work on the use of Satisfiability Modulo Theory solvers to reconstruct the state under sensor attacks to the context of differentially flat systems. The effectiveness of our approach is illustrated on the problem of controlling a quadrotor under sensor attacks.**

I. Introduction

A broad range of today's critical infrastructures is increasingly relying on sensors and cyber components (e.g., digital processors and networks) to closely monitor and control physical processes. While helping improve the overall system efficiency and performance, such a tight interaction is often at the outset of unprecedented vulnerabilities and malicious attacks. A striking example of such attacks is the Stuxnet virus targeting SCADA systems [1]. In this attack, sensor measurements are replaced by previously recorded data, which, once they are fed to the controller, can lead to catastrophic situations. Other examples include the injection of false data in "smart" systems [2], and the non-invasive sensor spoofing attacks in the automotive domain [3].

To secure these cyber-physical systems, a possible strategy is to exploit an accurate mathematical model of the dynamics of the physical system under control, and analyze any discrepancy between the actual sensor measurements and the ones predicted by the model, to decide about the existence of an adversarial attack [4], [5]. Once the malicious sensors, if any, are detected and identified, it is then possible to reconstruct the actual system state by using the data collected from the attack-free sensors. In the following, we refer to this approach as *secure state reconstruction*.

The problem of state reconstruction in the presence of disturbances, in its general form, has attracted considerable attention from the control community over the years. Previous work addresses the problem in terms of robust filter (estimator) design against outliers [6], [7], [8]. However, the lack of a priori knowledge about the attack signals tends to limit the applicability of robust estimation techniques to security problems. In secure state reconstruction, no assumptions are usually made about the attacks, e.g., their stochastic properties, time evolution, or energy bounds.

A game theoretic formulation for the secure state reconstruction problem has been proposed in the literature, when the physical system is scalar [9]. An alternative reconstruction technique, still in the context of a scalar system equipped with one sensor, has also been derived based on the analysis of the performance degradation of a Kalman filter when the sensor is under attack [10]. Finally, the general case of a multidimensional system equipped with multiple sensors has been tackled [11], [5], [12], [13], [14], [15], [16], [17], [18] when the attackers are restricted to corrupt an unknown subset of the system sensors. However, all of the above contributions focus on problems for which the underlying dynamics can be described by a linear system.

Unlike previous work, we focus in this paper on the problem of secure state reconstruction for a class of nonlinear systems. Specifically, we consider physical systems whose dynamics can be described by a differentially flat system [19]. Differentially flat systems represent an important class of nonlinear systems, in that they encompass a wide range of mechanical systems, including several examples of ground and aerial vehicles.

While differentially flat systems can be converted into linear systems using dynamic feedback linearization and a change of coordinates, this technique would, however, require the knowledge of the system state. Since this is clearly our ultimate goal, it is not possible to directly apply the results from linear secure state reconstruction to differentially flat systems. We follow instead a different approach, by building on our previous work on sound and complete secure state reconstruction for linear systems [15], [16] to develop an algorithm that can efficiently identify the corrupted sensors by leveraging Satisfiability Modulo Theory (SMT) solving [20] to tackle the combinatorial aspects of the problem. For the sake of brevity, we omit

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all the proofs in this paper. For further details, we refer the reader to our extended report [21], where we discuss a more general notion of s-sparse observability for nonlinear systems and its connections with the solution of the secure state reconstruction problem.

II. The Secure State Reconstruction Problem

A. Notation

The symbols $\mathbb{N}, \mathbb{R}$ and $\mathbb{B}$ denote the sets of natural, real, and Boolean numbers, respectively. If S is a set, we denote by $|S|$ its cardinality. The support of a vector $x \in \mathbb{R}^n$, denoted by $\text{supp}(x)$, is the set of indices of the non-zero components of x. Similarly, the complement of the support of a vector x is denoted by $\overline{\text{supp}(x)} = \{1, \dots, n\} \setminus \text{supp}(x)$. We call a vector $x \in \mathbb{R}^n$ s-sparse, if x has s nonzero elements, i.e., if $|\text{supp}(x)| = s$.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function given by $f(x) = (f_1(x), \dots, f_m(x))$, where $f_i : \mathbb{R}^n \to \mathbb{R}$ is the ith component of f. Then, for the set $\Gamma \subseteq \{1, \dots, m\}$, we denote by f_Γ the vector function obtained from f by removing all the components except those indexed by Γ. Similarly, $f_{\overline{\Gamma}}$ is obtained from f by removing the components indexed by Γ. Finally, we use the notation $\nabla_x f$ to denote the Jacobian matrix of f evaluated at x.

B. Dynamics and Attack Model

We consider a system of the form:

$$\Sigma_a \quad \begin{cases} x^{(t+1)} & = f\left(x^{(t)}, u^{(t)}\right), \\ y^{(t)} & = h\left(x^{(t)}\right) + a^{(t)} \end{cases} \tag{II.1}$$

where $x^{(t)} \in \mathcal{X} \subseteq \mathbb{R}^n$ is the system state, $u^{(t)} \in \mathcal{U} \subseteq \mathbb{R}^m$ is the system input, and $y^{(t)} \in \mathbb{R}^p$ is the observed output, all at time $t \in \mathbb{N}$. The map $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ represents the system dynamics. We will use the notation $f_u(x) = f(x, u)$ in the remainder of this paper. We also use the notation $f_{u^{(k)} \dots u^{(1)}}(x)$ to denote the k-fold composition of f, i.e.,

$$f_{u^{(k)} u^{(k-1)} \dots u^{(1)}}(x) = \\ f\left(f\left(f\left(f\left(u^{(1)}, x\right), \dots\right), u^{(k-1)}\right), u^{(k)}\right).$$

An attacker corrupts the sensor measurements y by either spoofing the sensor outputs, or by manipulating the data transmitted from the sensors to the controller. Independently of how the attack is implemented, its effect can be described by the s-sparse vector $a^{(t)} \in \mathbb{R}^p$. If sensor $i \in \{1, \dots, p\}$ is attacked then the ith component of $a^{(t)}$ is non-zero; otherwise the ith sensor is not attacked. Hence, s describes the number of attacked sensors. We make no assumptions on the vector $a^{(t)}$ other than being s-sparse. In particular, we do not assume bounds, statistical properties, or restrictions on the time evolution of the elements in $a^{(t)}$. While the value of s is not known, we assume the knowledge of an upper bound $\overline{s}$ on it.

C. Problem Formulation

Solving the secure state reconstruction problem implies estimating the state x from a set of measurements collected over a window of length $\tau \in \mathbb{N}$. Hence, we start by grouping the measurements from the ith sensor as:

$$Y_i^{(t)} = H_{u,i}\left(x^{(t-\tau+1)}\right) + E_i^{(t)} \tag{II.2}$$

where:

$$Y_i^{(t)} = \begin{bmatrix} y_i^{(t-\tau+1)} \\ y_i^{(t-\tau+2)} \\ \vdots \\ y_i^{(t)} \end{bmatrix}, E_i^{(t)} = \begin{bmatrix} a_i^{(t-\tau+1)} \\ a_i^{(t-\tau+2)} \\ \vdots \\ a_i^{(t)} \end{bmatrix},$$

$$H_{u,i}\left(x^{(t-\tau+1)}\right) = \begin{bmatrix} h_i\left(x^{(t-\tau+1)}\right) \\ h_i\left(f_{u^{(t-\tau+1)}}\left(x^{(t-\tau+1)}\right)\right) \\ \vdots \\ h_i\left(f_{u^{(t)} \dots u^{(t-\tau+1)}}\left(x^{(t-\tau+1)}\right)\right) \end{bmatrix}.$$

We then define:

$$Y^{(t)} = \begin{bmatrix} Y_1^{(t)} \\ \vdots \\ Y_p^{(t)} \end{bmatrix}, \quad E^{(t)} = \begin{bmatrix} E_1^{(t)} \\ \vdots \\ E_p^{(t)} \end{bmatrix}, \quad H_u = \begin{bmatrix} H_{u,1} \\ \vdots \\ H_{u,p} \end{bmatrix}$$

where, with some abuse of notation, Y_i, E_i and $H_{u,i}$ are used to denote the ith block of $Y^{(t)}, E^{(t)}$ and H_u, respectively. Using the same notation, we denote by Y_Γ, E_Γ, and $H_{u,\Gamma}$ the blocks indexed by the elements in the set Γ. Moreover, for simplicity, we drop the time t argument in the following, since we assume that the secure state reconstruction problem is to be solved at every time instance.

Let (x^*, E^*) denote the actual state of the system and the actual attack vector. Let also $b^* \in \mathbb{B}^p$ be a vector of binary indicator variables such that $b_i^* = 0$ when the ith sensor is attack-free and $b_i = 1$ otherwise. It follows from (II.2) that:

$$Y_i = \begin{cases} H_{u,i}(x^*) & \text{if } b_i^* = 0 \\ H_{u,i}(x^*) + E_i^* & \text{if } b_i^* = 1. \end{cases}$$

Therefore, we are interested in a state estimate x and a vector of binary indicator variables $b = (b_1, \dots, b_p)$ such that the discrepancy between the collected measurements Y_i and the expected outputs $H_{u,i}(x)$ is zero for all the sensors that are labeled as attack-free sensors ($b_i^* = 0$). Furthermore, the estimated state x should be equal to x^*. These requests can be formalized as follows.

Problem 2.1: **(Secure State Reconstruction)** For the control system under attack Σ_a (defined in (II.1)), construct the estimate $\eta = (x, b) \in \mathbb{R}^n \times \mathbb{B}^p$ such that $\eta \models \phi$ (i.e., η satisfies the formula ϕ), where:

$$\phi ::= \bigwedge_{i=1}^{p} \left(\neg b_i \Rightarrow \|Y_i - H_{u,i}(x)\|_2^2 = 0\right) \bigwedge \left(\sum_i^p b_i \leq \overline{s}\right), \tag{II.3}$$

subject to $(x^* = x) \wedge (\text{supp}(b^*) \subseteq \text{supp}(b))$.

The second clause in the formula ϕ rules out the trivial solution in which all sensors are labelled as attacked, by enforcing a cardinality constraint on the number of attacked sensors, which is required to be bounded by $\overline{s}$.

As in the case of linear systems [16], the secure state reconstruction problem formulation in Problem 2.1 does not ask for a solution with the minimal number of attacked sensors. However, it is possible to obtain the minimal set of sensors under attack by invoking a solver for Problem 2.1 multiple times [16].

D. Differential Flatness and s-Sparse Flatness

For linear systems, the notion of s-sparse observability plays a key role in determining the existence of a solution for Problem 2.1 [12], [13]. In this subsection, we generalize this notion for a special class of nonlinear systems known as *differentially flat*[1] systems. A system is differentially flat if the state and the input can be reconstructed from current and previous outputs. More formally,

Definition 2.2 (Differential Flatness): System is differentially flat if there exist an integer $k \in \mathbb{N}$, and functions α and β, such that the state and the input can be reconstructed from the outputs $y^{(t)}$ as follows:

$$x^{(t)} = \alpha\left(y^{(t)}, y^{(t-1)}, \ldots, y^{(t-k+1)}\right) \qquad \text{(II.4)}$$

$$u^{(t)} = \beta\left(y^{(t)}, y^{(t-1)}, \ldots, y^{(t-k+1)}\right). \qquad \text{(II.5)}$$

In such case, the output $y^{(t)}$ is called a flat output.
In the remainder of this paper, we assume that the window length τ in (II.2) is chosen such that $\tau = k$.

*Definition 2.3 (**s-Sparse Flat System**):* The nonlinear control system Σ_a, defined by (II.1), is said to be s-sparse flat if for every set $\Gamma \subseteq \{1, \ldots, p\}$ with $|\Gamma| = s$, the system $\Sigma_{\overline{\Gamma}}$:

$$\Sigma_{\overline{\Gamma}} \quad \begin{cases} x^{(t+1)} &= f\left(x^{(t)}, u^{(t)}\right), \\ y^{(t)} &= h_{\overline{\Gamma}}\left(x^{(t)}\right) \end{cases} \qquad \text{(II.6)}$$

is differentially flat.
In other words, the system is s-sparse flat if any choice of $p - s$ sensors is a flat output. Using the notion of s-sparse flatness, we can then establish the following result on the correctness of the solutions of Problem 2.1.

Theorem 2.4: If the dynamical system Σ_a defined by (II.1) is $2\overline{s}$-sparse flat, then for any pair $\eta = (x, b)$ that satisfy ϕ in (II.3) the following holds:

$$x^* = x \qquad \wedge \qquad \text{supp}(b^*) \subseteq \text{supp}(b).$$

III. Secure State Reconstruction Using SMT Solving

The secure state reconstruction problem is combinatorial, since a direct solution would require constructing the state from all different combinations of $p - \overline{s}$ sensors to determine which sensors are under attack. In this section, we show how using SMT solving can dramatically reduce the complexity of the reconstruction algorithm.

To decide whether the combination of Boolean and nonlinear constraints in (II.3) is satisfiable, we develop the detection algorithm IMHOTEP-SMT using the *lazy* SMT paradigm [20]. By building upon the IMHOTEP-SMT solver [16], [22], our decision procedure combines a SAT solver (SAT-SOLVE) and a theory solver ($\mathcal{T}$-SOLVE). However, differently than [16], [22], the theory solver in this paper can also reason about the nonlinear constraints in (II.3), as generated from a differentially flat system. The SAT solver efficiently reasons about combinations of Boolean and pseudo-Boolean constraints[2], using a modern instantiation of the David-Putnam-Logemann-Loveland (DPLL) algorithm [23] to suggest possible assignments for the nonlinear constraints. The theory solver checks the consistency of the given assignments, and provides the reason for the conflict, a *certificate*, or a counterexample, whenever inconsistencies are found. Each certificate results in learning new constraints which will be used by the SAT solver to prune the search space. The complex decision task is thus broken into two simpler tasks, respectively, over the Boolean and nonlinear domains.

A. Overall Architecture

As illustrated in Algorithm 1, we start by mapping each nonlinear constraint to an auxiliary Boolean variable c_i to obtain the following (pseudo-)Boolean satisfiability problem:

$$\phi_B := \bigwedge_{i=1}^{p} \left(\neg b_i \Rightarrow c_i\right) \bigwedge \left(\sum_{i=1}^{p} b_i \le \overline{s}\right)$$

where $c_i = 1$ if $\|Y_i - H_{u,i}(x)\|_2 = 0$ is satisfied, and zero otherwise. By only relying on the Boolean structure of the problem, SAT-SOLVE(ϕ_B) returns an assignment for the variables b_i and c_i (for $i = 1, \ldots, p$), thus hypothesizing which sensors are attack-free, hence which nonlinear constraints should be jointly satisfied. This Boolean assignment is then used by $\mathcal{T}$-SOLVE to determine whether there exists a state $x \in \mathbb{R}^n$ which satisfies all the nonlinear constraints related to the unattacked sensors, i.e. $\{\|Y_i - H_{u,i}(x)\|_2 = 0 | i \in \overline{\text{supp}(b)}\}$ is the set of constraints sent to $\mathcal{T}$-SOLVE. If x is found, IMHOTEP-SMT terminates with `SAT` and provides the solution (x, b). Otherwise, the `UNSAT` certificate ϕ_{cert} is generated in terms of new Boolean constraints, explaining which sensor measurements are conflicting and may be under attack. A naïve certificate can always be generated in the form of:

$$\phi_{\text{triv-cert}} = \sum_{i \in \overline{\text{supp}(b)}} b_i \ge 1, \qquad \text{(III.1)}$$

which encodes the fact that at least one of the sensors in the set $\overline{\text{supp}(b)}$ (i.e. for which $b_i = 0$ in the current iteration) is actually under attack, and must be set to one in the next assignment of the SAT solver. The augmented Boolean problem is then fed back to SAT-SOLVE to produce a new assignment, and the sequence of new SAT queries repeats until $\mathcal{T}$-SOLVE terminates with `SAT`.

[1] Although the term *difference flatness* is sometimes used in the literature for systems governed by difference equations, we choose to employ the widely accepted term *differential flatness*.

[2] A pseudo-Boolean constraint is a linear constraint over Boolean variables with integer coefficients.

Algorithm 1 IMHOTEP-SMT

```
status := UNSAT;
$\phi_B := \bigwedge_{i=1}^{p} (\neg b_i \Rightarrow c_i) \bigwedge (\sum_{i=1}^{p} b_i \leq \overline{s})$;
while status == UNSAT do
    $(b, c) :=$ SAT-SOLVE($\phi_B$);
    (status, $x$) := $\mathcal{T}$-SOLVE.CHECK($\overline{\text{supp}(b)}$);
    if status == UNSAT then
        $\phi_{\text{cert}} :=$ $\mathcal{T}$-SOLVE.CERTIFICATE($b, x$);
        $\phi_B := \phi_B \wedge \phi_{\text{cert}}$;
    end if
end while
return $\eta = (x, b)$;
```

By assuming that the system is $2\overline{s}$-sparse flat, it follows from Theorem 2.4 that there always exists a solution to Problem 2.1, hence Algorithm 1 will always terminate. However, to help the SAT solver quickly converge towards the correct assignment, a central problem in lazy SMT solving is to generate succinct explanations whenever conjunctions of nonlinear constraints are infeasible.

The rest of this section will then focus on the implementation of the two main tasks of $\mathcal{T}$-SOLVE, namely, (i) checking the satisfiability of a given assignment ($\mathcal{T}$-SOLVE.CHECK), and (ii) generating succinct UNSAT certificates ($\mathcal{T}$-SOLVE.CERTIFICATE).

B. Satisfiability Checking

It follows from the $2\overline{s}$-sparse flatness property discussed in Section II, that for a given assignment of the Boolean variables b, with $|\text{supp}(b)| \leq \overline{s}$, the remaining $p - \overline{s}$ sensors define a flat output as:

$$y_{\mathcal{I}}^{(t)}, \quad \ldots, \quad y_{\mathcal{I}}^{(t-\tau+1)}$$

where $\mathcal{I} = \overline{\text{supp}(b)}$. The next step is to use the flat output in order to calculate the estimate $x = \alpha\left(y_{\mathcal{I}}^{(t)}, \ldots, y_{\mathcal{I}}^{(t-\tau+1)}\right)$. Finally, we evaluate if the condition:

$$\left\| Y_{\overline{\text{supp}(b)}} - H_{u,\overline{\text{supp}(b)}}(x) \right\|_2^2 = 0 \qquad \text{(III.2)}$$

is satisfied. This procedure is summarized in Algorithm 2.

Algorithm 2 $\mathcal{T}$-SOLVE.CHECK($\mathcal{I}$)

```
Construct the state estimate:
        $x := \alpha\left(y_{\mathcal{I}}^{(t)}, \ldots, y_{\mathcal{I}}^{(t-\tau+1)}\right)$
if $\left\| Y_{\mathcal{I}} - H_{u,\mathcal{I}(x)} \right\|_2 == 0$ then
    status := SAT;
else
    status := UNSAT;
end if
return (status, $x$)
```

C. Generating Succinct UNSAT Certificates

Whenever $\mathcal{T}$-SOLVE.CHECK provides UNSAT, the naïve certificate can always be generated as in (III.1). However, such trivial certificate does not provide much information, since it only excludes the current assignment from the search space, and can lead to exponential execution time, as reflected by the following proposition.

Proposition 3.1: Let the linear dynamical system Σ_a defined in (II.1) be $2\overline{s}$-sparse observable. Then, Algorithm 1 which uses the trivial UNSAT certificate $\phi_{\text{triv-cert}}$ in (III.1) returns $\eta = (x, b)$ such that:

$$x^* = x \quad \wedge \quad \text{supp}(b^*) \subseteq \text{supp}(b),$$

where x^* and b^* are the actual system state and attack indicator vector, as defined in Section II-C. Moreover, the upper bound on the number of iterations of Algorithm 1 is $\sum_{s=0}^{\overline{s}} \binom{p}{s}$.

The generated UNSAT certificate heavily affects the overall execution time of Algorithm 1: the smaller the certificate, the more information is learnt and the faster is the convergence of the SAT solver to the correct assignment. For example, a certificate with $b_i = 1$ would identify exactly one attacked sensor at each step. Therefore, our objective is to design an algorithm that can lead to more *compact certificates* to enhance the execution time of IMHOTEP-SMT. To do so, we exploit the specific structure of the secure state reconstruction problem and generate customized, yet stronger, UNSAT certificates. The existence of a compact Boolean constraint that explains a conflict is guaranteed by the following Lemma.

Lemma 3.2: Let the nonlinear dynamical system Σ_a defined in (II.1) be $2\overline{s}$-sparse flat. If $\mathcal{T}$-SOLVE.CHECK($\mathcal{I}$) is UNSAT for a set $\mathcal{I}$, with $|\mathcal{I}| > p - 2\overline{s}$, then there exists a subset $\mathcal{I}_{temp} \subset \mathcal{I}$ with $|\mathcal{I}_{temp}| \leq p - 2\overline{s} + 1$ such that $\mathcal{T}$-SOLVE.CHECK($\mathcal{I}_{temp}$) is also UNSAT.

Based on Lemma 3.2 and the intuition provided by its proof [21], our algorithm works as follows. First, we construct the set of indices $\mathcal{I}'$ by picking any random set of $p - 2\overline{s}$ sensors. We then search for one additional sensor i' which can lead to a conflict with the sensors indexed by $\mathcal{I}'$. To do this, we call $\mathcal{T}$-SOLVE.CHECK by passing the set $\mathcal{I}_{temp} := \mathcal{I}' \cup i'$ as an argument. If the check returned SAT, then we label these sensors as "non-conflicting" and we repeat the same process by replacing the sensor indexed by i' with another sensor until we reach a conflicting set. It then follows from Lemma 3.2 that this process terminates revealing a collection of $p - 2\overline{s} + 1$ conflicting sets. Once this collection is discovered, we stop by generating the following, more compact, certificate:

$$\phi_{\text{cert}} := \sum_{i \in \mathcal{I}_{temp}} b_i \geq 1.$$

Although the prescribed process will always terminate regardless of the selection of the initial set $\mathcal{I}'$ or the order followed to select i', the execution time may change. In Algorithm 3, we implement a heuristic for the selection of the initial set $\mathcal{I}'$ and the succeeding indices, inspired by the strategy we have adopted in the context of linear systems [16]. We are now ready to state the main result of this section.

Theorem 3.3: Let the nonlinear dynamical system Σ_a defined in (II.1) be $2\overline{s}$-sparse flat. Then, Algorithm 1 using

Algorithm 3 $\mathcal{T}$-SOLVE.CERTIFICATE$(\mathcal{I}, x)$

```
Compute normalized residuals
    $r := \bigcup_{i \in \mathcal{I}} \{r_i\}$,
        $r_i := \|Y_i - H_{u,i}(x)\|_2^2 \,/\, \|\nabla_x H_{u,i}\|_2^2\,,\ i \in \mathcal{I}$;
Sort the residual variables
    $r\_sorted :=$ sortAscendingly$(r)$;
Pick the index corresponding to the maximum residual
    $\mathcal{I}\_max\_r :=$ Index$(r\_sorted_{\{|\mathcal{I}|, |\mathcal{I}|-1, \ldots, p-2\overline{s}+1\}})$;
    $\mathcal{I}\_min\_r :=$ Index$(r\_sorted_{\{1, \ldots, p-2\overline{s}\}})$;
Search linearly for the UNSAT certificate
    status = SAT;    counter $= 1$;
    $\mathcal{I}\_temp := \mathcal{I}\_min\_r \cup \mathcal{I}\_max\_r_{counter}$;
while status == SAT do
    (status, $x$) := $\mathcal{T}$-SOLVE.CHECK($\mathcal{I}\_temp$);
    if status == UNSAT then
        $\phi_{\text{cert}} := \sum_{i \in \mathcal{I}\_temp} b_i \geq 1$;
    else
        counter := counter + 1;
        $\mathcal{I}\_temp := \mathcal{I}\_min\_r \cup \mathcal{I}\_max\_r_{counter}$;
    end if
end while
return $\phi_{\text{cert}}$
```

the conflicting UNSAT certificate ϕ_{cert} in Algorithm 3 returns $\eta = (x, b)$ such that:

$$x^* = x \quad \wedge \quad \text{supp}(b^*) \subseteq \text{supp}(b).$$

IV. CASE STUDY: SECURING A QUADROTOR MISSION

We implemented a new version of the IMHOTEP-SMT solver, which extends our previous work [22] by replacing the theory solver originally designed for linear systems with Algorithms 2 and 3. We then demonstrate the effectiveness of our detection algorithm by applying it to a waypoint navigation mission for a quadrotor unmanned aerial vehicle (UAV), in which the UAV needs to cross a workspace from a starting point to a desired goal. The dynamical models of the quadrotor and its controller are based on the models in [24], [25].

The quadrotor is equipped with a GPS measuring the position vector and two inertial measurement units (IMUs), whose outputs are fused to generate an estimate for the body angular and linear velocities. We numerically simulate the model of the quadrotor and the controller. In our scenario, the quadrotor goal is to takeoff vertically and then move along a square trajectory. However, one of the IMU's output, the vertical velocity sensor, is attacked by injecting a sinusoidal signal on top of the actual sensor readings. As shown in Fig. 1 (bottom), the attack is injected after the quadrotor has completed the takeoff maneuver, and only along two parallel sides of the whole square trajectory.

To implement the secure state reconstruction algorithm, IMHOTEP-SMT uses an approximate discretized model of the plant along with the sensor measurements. To discretize the model we use the same sampling time ($T_s = 20$ ms) of the controller. Moreover, we adopt a first-order forward Euler approximation scheme which preserves the differential flatness of the original system. To accommodate the model mismatch due to the discrete approximation, as well as round-off errors, we replace the condition in line 2 of Algorithm 2 with $\|Y_{\mathcal{I}} - H_{u,\mathcal{I}}(x)\|_2 \leq \epsilon$, where we set ϵ to 0.1 in our experiments.

Fig. 1(a) (top) shows the effect of the attack when the quadrotor operates without secure state reconstruction algorithm. As evident from the two corners of the square trajectory corresponding to coordinates $(2.5, 0, 1)$ and $(0, 2.5, 1)$, the injected attack harmfully impairs the stability of the quadrotor, due to incorrect state reconstruction, as shown in Fig. 1(a) (middle). Fig. 1(b) (top) shows instead the trajectory of the quadrotor when operated using IMHOTEP-SMT to perform secure state reconstruction. The estimation error on v_z produced by IMHOTEP-SMT is in the order of 10^{-2} m/s, and always bounded, where the bound depends on the error due to mismatch between the model used for estimation and the actual quadrotor dynamics (the controller is designed to be robust against bounded perturbations). The state and the support of the attack are correctly estimated also in the presence of model mismatch: the quadrotor is able to follow the required trajectory and achieve its goal. Finally, the average execution time of 16.1 ms (smaller than the 20-ms sampling time of the position controller) on an Intel Core i7 3.4-GHz processor with 8 GB of memory, is compatible with several real-time applications.

V. CONCLUSIONS

We have investigated, for the first time, the state reconstruction problem from a set of adversarially attacked sensors for a class of nonlinear systems, namely differentially flat systems. Given an upper bound $\overline{s}$ on the number of attacked sensors, we showed that $2\overline{s}$-sparse flatness is a sufficient condition for reconstructing the state in spite of the attack. We have then proposed a Satisfiability Modulo Theory based detection algorithm for differentially flat systems, by extending our previous results, reported in [15], [16], to differentially flat systems. Numerical results show that secure state estimation in complex nonlinear systems, such as in waypoint navigation of a quadrotor under sensor attacks, can indeed be performed with our algorithm in an accurate and efficient way.

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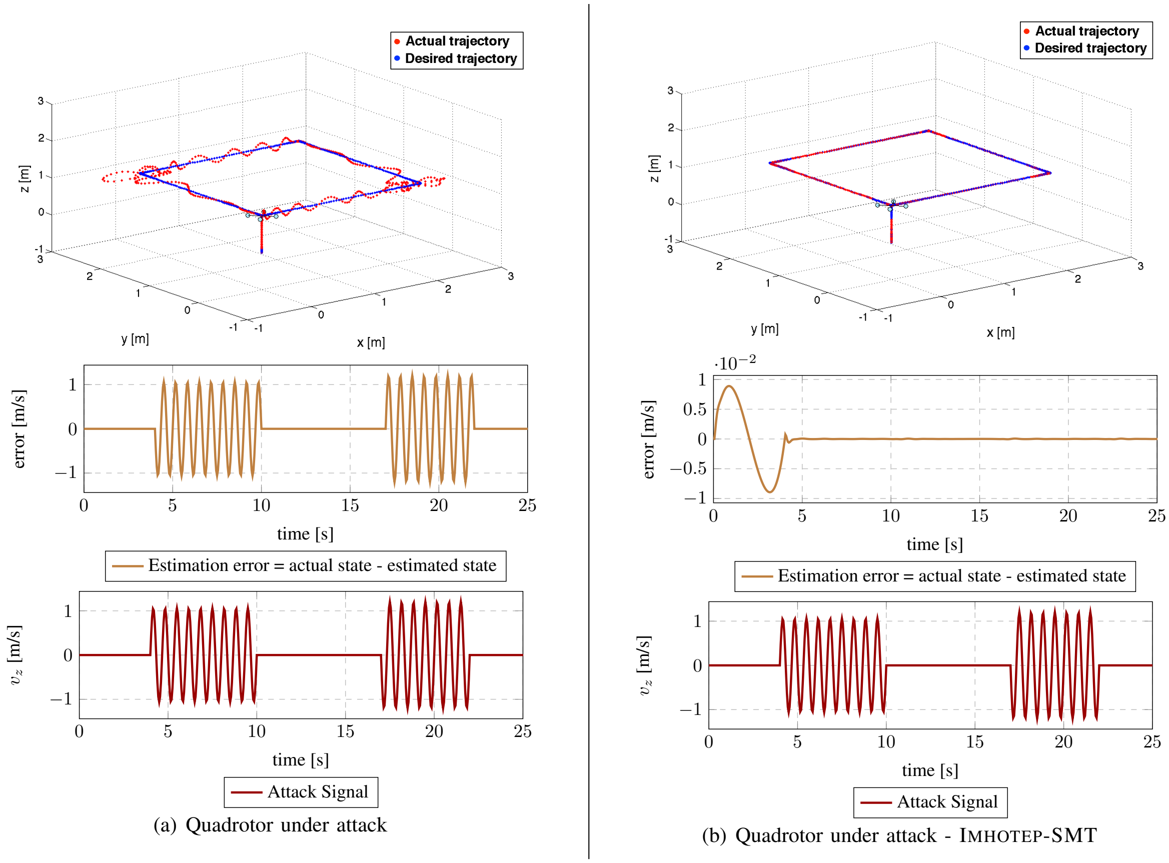

(a) Quadrotor under attack

(b) Quadrotor under attack - IMHOTEP-SMT

Fig. 1. Performance of the quadrotor under sensor attack: (a) without secure state reconstruction algorithm, and (b) with IMHOTEP-SMT. For each scenario, we show the quadrotor trajectory (top), the estimation error on the quadrotor vertical velocity (middle), and the attack signal (bottom).

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