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# Integrability for axisymmetric and stationary spacetimes in General Relativity with a conformally coupled scalar field


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# Contents

# Chapter 1

# Introduction

## 1.1 Introduction

General Relativity (GR), as developed by Albert Einstein in 1915, is the universally accepted theory of gravity thanks to the variety of experimental tests that confirm its prediction. The theory owes part of its success to the remarkable interpretation of the field that mediates gravitational interaction. Drawing on new mathematical tools such as Riemannian geometry, Einstein reinterpreted the motion of masses in space not as governed by mutual forces, but rather as motion along straight paths in a space curved by the presence of mass - the gravitational field itself. This wonderful interpretation of the gravitational dynamics is synthesized by Einstein's field equations[1].

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{1.1}$$

Nevertheless, many cosmological and theoretical phenomena, such as dark energy and cosmological inflation, remain unexplained. It thus appears necessary to extend classical General Relativity to account for these phenomena. One of the most straightforward approaches to achieving this is through the use of a Scalar-Tensor theory. Scalar-Tensor theories are those generalizations of classical GR that describe gravity using the classical metric tensor $g_{\mu\nu}$ and a scalar field $\phi$ directly coupled with the curvature scalar. The addition of a scalar field introduces new solutions and symmetries to the equations of classical GR, potentially leading to new gravitational phenomena.

Another challenge of the classical theory lies in solving the field equations (1.1), as they constitute a system of 10 second-order nonlinear partial differential equations. Since the birth of the theory, various techniques for generating solutions to Einstein's equations have emerged, including the method developed by F. J. Ernst [1] for axisymmetric and stationary spacetimes. In particular, Ernst rearranged equations (1.1) defining a complex potential $\mathcal{E}$ that brings out new symmetries into the system. These new symmetries are redefinitions or Gauge transformations of the Ernst's potential $\mathcal{E}$ except for a couple of them that lead to an inequivalent spacetime.

[1]Later modified with the addition of the cosmological constant term $\Lambda g_{\mu\nu}$

Effectively, Ernst's method is a technique that allows one to generate a solution, from a "seed" one, that sometimes posses a different physical interpretation.
In this thesis we export Ernst's method to a Scalar-Tensor theory where the new field $\phi$ is conformally coupled to the action (so that the equation derived from the field posses conformal invariance) and we explore the possible emergence of new symmetries with respect to classical GR. We are particularly interested in conformal invariance because Maxwell's equations, which we will couple to the Scalar-Tensor theory in Section 3.2, exhibit this property (see the appendix of [2]). So the additional scalar field that we introduce is intended to be similar to the electromagnetic field, whose coupling with General Relativity is well-established. Furthermore, there is a historical significance in considering the conformal frame: it was within this framework that Bekenstein [3] discovered the first counterexample to the "No-hair" theorem[2] (commonly referred to as the BBMB solution). In chapter 2 we show how the action of a Scalar-Tensor theory appears, we present the case of interest for us (where the scalar field is conformally coupled to the action) and we introduce the notions of conformal transformation and conformal invariance. In chapter 3 we recast Ernst's method and Ernst's equations for our theory following step by step the original derivation in the vacuum case and in the electrovacuum case, so that also Maxwell's electromagnetism is included into the theory. Then in chapter 4 we find the correlated Lie point symmetries and investigate how those present in the classical theory are modified or whether the coupling in the conformal frame reveals new ones. Finally in chapter 5-6 we first test the symmetries of Ernst's equations building some existing solution starting from the BBMB solution as seed and then we produce some new solutions of the conformal theory. In general we will use the signature $(-,+,+,+)$ and $G=c=\mu_0=1$.

[2]The famous physicist John Wheeler coined this expression stating that "Black holes have no hair" [4]

# Chapter 2

# Scalar-Tensor theories

Scalar-Tensor theories are generalizations of classical General Relativity with a scalar field that directly couples to the curvature scalar $R$. There exist different kinds of Scalar-Tensor theories[1], here we present the action in the most general case. We notice that it is also possible to couple a scalar field such that it slightly affect the ordinary GR equations, then we indicate this frame as minimally coupled (MC) frame. In section 2.3 we report a remarkable theorem to map the conformal and the MC frame.

## 2.1 Coupling a scalar field to the action

We are interested in new solutions that a scalar field could produce once incorporated into the theory. Therefore the most general way of introducing a scalar field $\phi$, is writing the action as

$$S[g_{\mu\nu}, \phi] = \int d^4x\sqrt{-g}\left[\underbrace{f(\phi)R}_{\mathcal{L}_{\mathcal{R}}} \underbrace{-\frac{1}{2}h(\phi)\nabla^\mu\phi\nabla_\mu\phi - V(\phi)}_{\mathcal{L}_\phi} + \mathcal{L}_M(g_{\mu\nu}, \alpha_i)\right] \tag{2.1}$$

where $\mathcal{L}_\phi$ is the pure scalar term, $f, h$ and $V$ are the fields defining the theory and $\mathcal{L}_M$, the matter lagrangian, depends on some fields $\alpha_i$, on the metric but not on $\phi$. Through a variation of this action it is possible to derive the equations of motion (a more precise procedure can be found in [6]). Starting with

$$g^{\mu\nu} \longrightarrow g^{\mu\nu} + \delta g^{\mu\nu}$$

leads to

$$G_{\mu\nu} = f^{-1}\left(\frac{1}{2}T^M_{\mu\nu} + \frac{1}{2}T^\phi_{\mu\nu} + \nabla_\mu\nabla_\nu f - g_{\mu\nu}\Box f\right) \tag{2.2}$$

[1] For example, in the theory proposed by C. Brans and R. H. Dicke [5] the scalar field introduces a spacetime dependence of Newton's gravitational constant G.

where $T^M_{\mu\nu}$ is obtained from the variation of $\mathcal{L}_M$ with respect to $g_{\mu\nu}$ and

$$T^{\phi}_{\mu\nu} = h\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}h\nabla^\rho\phi\nabla_\rho\phi + V(\phi)\right) \tag{2.3}$$

The next step is to produce a variation in the scalar field

$$\phi \longrightarrow \phi + \delta\phi$$

that leads to

$$h\Box\phi + \frac{1}{2}\frac{dh}{d\phi}\nabla^\mu\phi\nabla_\mu\phi - \frac{dV}{d\phi} + R\frac{df}{d\phi} = 0 \tag{2.4}$$

Finally a variation with respect to the fields $\alpha_i$ produces the associated matter equations.
Notice that if we take $f(\phi) = 1$, $h(\phi) = 1$ and $V(\phi) = \mathcal{L}_M = 0$ in (2.1) we obtain the action of GR with a minimally coupled (MC) scalar field.

## 2.2 Coupling a conformal scalar field to the action

### 2.2.1 Conformal transformations

Looking at (2.2) and (2.3), it's clear that working with a scalar field complicates the equations, so sometimes it could be useful to improve a conformal transformation to recover the appearance of classical GR. Let's begin by stating that given an n-dimensional manifold M with metric $\tilde{g}_{\alpha\beta}$ and a smooth function $\Omega$, we can define a new metric as $g_{\alpha\beta} = \Omega^2\tilde{g}_{\alpha\beta}$, then we say that $\tilde{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ are related by a *conformal transformation*. Now we have a $\tilde{\nabla}$ operator associated to $\tilde{g}_{\alpha\beta}$ and $\nabla$ to $g_{\alpha\beta}$ that defines the relations between differential geometry quantities like the Riemann tensor. In particular it is possible to demonstrate (see appendix G of [6] and D of [2] for a complete derivation) that, for $n = 4$ dimensions, the scalar curvature respect

$$R = \Omega^{-2}\tilde{R} - 6\tilde{g}^{\alpha\beta}\Omega^{-3}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\Omega \tag{2.5}$$

with $\nabla_\alpha\phi = \tilde{\nabla}_\alpha\phi = \partial_\alpha\phi$ for scalar fields. We can use this last relation, obviously with $g^{\alpha\beta} = \Omega^{-2}\tilde{g}^{\alpha\beta}$, to recast the $\mathcal{L}_R$ term of the action (2.1) in the classical GR form. In particular taking (2.1) with $h = V = \mathcal{L}_M = 0$ in the $\tilde{g}_{\alpha\beta}$ frame

$$S = \int d^4x\sqrt{-\tilde{g}}f(\phi)\tilde{R}. \tag{2.6}$$

Following this last action we define a conformal transformation such that $g_{\alpha\beta} = f(\phi)\tilde{g}_{\alpha\beta}$ (so $\Omega^2 = f(\phi)$) in order to replace $\tilde{R}$ from 2.5 and $\sqrt{-\tilde{g}} = f^{-2}\sqrt{-g}$ obtaining

$$S = \int d^4x\sqrt{-g}(R + 6g^{\alpha\beta}f^{-1/2}\nabla_\alpha\nabla_\beta f^{1/2}). \tag{2.7}$$

The last term of the action can be integrated by part discarding the surface term, in particular

$$\int d^4x\sqrt{-g}f^{-1/2}\nabla^\alpha\nabla_\alpha f^{1/2} = \cancel{\nabla^\alpha(f^{-1/2}\nabla_\alpha f^{1/2})} - \int d^4x\sqrt{-g}\nabla^\alpha f^{-1/2}\nabla_\alpha f^{1/2} \tag{2.8}$$

resulting in

$$S = \int d^4x\sqrt{-g}\left[R + \frac{3}{2}f^{-2}\left(\frac{df}{d\phi}\right)^2\nabla^\alpha\phi\nabla_\alpha\phi\right] \tag{2.9}$$

that represent the action of a Scalar-Tensor theory with the single term R, exactly as in classical GR. For this similarity in appearance this frame is called *Einstein frame* while the other frame, with $\tilde{R}f(\phi)$ in the action, is called *Jordan* or *string frame.*

### 2.2.2 Conformal invariance

Another interesting aspect of working with conformal transformation is *conformal* or *scale invariance* of equations. An equation for a field $\phi$ is conformally invariant if there exist a number $s \in \mathbb{R}$, called the conformal weight of the field, such that $\phi$ is a solution with metric $\tilde{g}_{\alpha\beta}$ *if and only if* $\psi = \Omega^s\phi$ is a solution with metric $g_{\alpha\beta} = \Omega^2\tilde{g}_{\alpha\beta}$. So when we say that we want to conformally couple a scalar field to our action, we mean that the equation of motion for the field derived from that action must be conformally invariant.

In general, in Jordan frame, we can ensure a conformal invariace for the scalar field equation taking the action (2.1) with (changing notation $\phi \to \psi$) : $f(\psi) = (1 - \frac{k}{6}\psi^2)$, $h(\psi) = 2k$, $V(\psi) = \mathcal{L}_M = 0$ and $k = 8\pi$ (G=1)

$$S[g_{\mu\nu}, \psi] = \frac{1}{2k}\int d^4x\sqrt{-g}\left[R - k\left(\nabla^\mu\psi\nabla_\mu\psi + \frac{R}{6}\psi^2\right)\right] \tag{2.10}$$

that produces the following equations of motion

$$G_{\mu\nu} = k\left[\partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}\partial^\sigma\psi\partial_\sigma\psi + \frac{1}{6}\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}\right)\psi^2\right] \tag{2.11}$$

$$\Box\psi - \frac{R}{6}\psi = 0 \tag{2.12}$$

and this last equation, contrary to the classical one $\Box\psi = 0$, is conformally invariant (see [2]). We indicate this frame as the conformally coupled frame (CC), to distinguish it from the MC frame where the equations of motion are not conformally invariant.

## 2.3 Mapping MC and CC frames

As seen before we can minimally couple a scalar field to the action of general relativity that slightly affect the ordinary equations, in particular we can write from 2.1 a theory with

$$S[\hat{g}_{\mu\nu}, \phi] = \int d^4x \sqrt{-\hat{g}} \left[\hat{R} - k\nabla^\alpha\phi\nabla_\alpha\phi\right] \tag{2.13}$$

that produces the following equations of motion

$$G_{\mu\nu} = k\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}\hat{g}_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi\right) \tag{2.14}$$

$$\Box\phi = 0 \tag{2.15}$$

This equation is not conformally invariant, but what is extremely interesting to know is that Bekenstein found in [7] how to map the solutions of MC theory (2.13) into those of CC theory (2.10). The result is stated with the following theorem

**Theorem 1.** *if $\hat{g}_{\mu\nu}, \bar{F}_{\mu\nu}, \phi, \bar{\rho}$ and $\bar{u}_v$ form a solution of Einstein's equations for a spacetime containing an ordinary scalar field $\phi$, an electromagnetic field $\bar{F}_{\mu\nu}$ and radiation of density $\bar{\rho}$ with 4-velocity $\bar{u}_v$, then $g_{\mu\nu} = \Omega^2\hat{g}_{\mu\nu}$, $\psi = \sqrt{\frac{6}{k}}\tanh\sqrt{\frac{k}{6}}\phi$, $F_{\mu\nu} = \bar{F}_{\mu\nu}$, $\rho = \Omega^{-4}\bar{\rho}$, $u_v = \Omega\bar{u}_v$ is the corresponding solution for a conformal scalar field $\psi$, where $\Omega = \cosh\sqrt{\frac{k}{6}}\phi$.*

We can summarize the result stating that given a solution $(\phi, \hat{g}_{\mu\nu})$ of the MC theory (2.13) we can find the corresponding solution $(\psi, g_{\mu\nu})$ of the CC theory (2.10) by applying the map

$$\phi \longrightarrow \psi = \sqrt{\frac{6}{k}}\tanh\left(\sqrt{\frac{k}{6}}\phi\right),$$

$$\hat{g}_{\mu\nu} \longrightarrow g_{\mu\nu} = \left[1 - \frac{k}{6}\psi^2\right]^{-1}\hat{g}_{\mu\nu}. \tag{2.16}$$

# Chapter 3

# Ernst's technique in the conformal frame

In 1968 F. J. Ernst proposed [1] an extremely elegant method to treat stationary and axisymmetric gravitational field problems. He found that, in terms of the LWP metric, Einstein's equations for the two coupled fields $f, \omega$ can be rewritten in a single equation, named Ernst's equation, of a complex field $\mathcal{E}$, named Ernst's potential. The power of this approach emerges looking at the symmetries of Ernst's equation, not present in the standard ones, that, sometimes, leads to a completely new physical insight. In [8] it is shown how to extend this method with the presence of an electromagnetic field.
An interesting question arises regarding the potential extension of Ernst's method to Scalar-Tensor theories. Specifically, whether Ernst's equation in the conformal frame retains the same symmetries and if this coupling uncovers new ones. Due to the minimal coupling of the scalar field, Ernst's method remains unchanged in the MC frame. Then, as discussed in section 2.3, the most natural approach to this problem might be applying the existing symmetries to a solution in the MC frame and then using Bekenstein's map (2.16) to map them into conformal frame.
Since this is the standard procedure, a different approach will be presented here. We show how to extend Ernst's method directly in the CC frame, without relying to Bekenstein's map. The theory (2.10) defined by the equations of motion (2.11)-(2.12) will be used. The aim of this chapter is to produce an effective action used in chapter 4 to study the correlated symmetries.

## 3.1 Ernst's technique in the CC frame

### 3.1.1 Conformal LWP ansatz

First of all, we need to define a stationary and axisymmetric ansatz in a cylindrical coordinates system $\{-t, \rho, \varphi, z\}$. We identify a spacetime as stationary if it presents a timelike Killing vector $\xi_\mu$ whose orbits are complete, while it is said to be axisymmetric if the Killing vector is spacelike with closed orbits. The simplest choice for a spacetime with this properties that we can use in the conformal frame is the Lewis-Weyl-Papapetrou (LWP) ansatz multiplied by a conformal factor $\Omega$

$$ds^2 = \Omega \left(-f(dt - \omega d\varphi)^2 + f^{-1}\left[\rho^2 d\varphi^2 + e^{2\gamma}(d\rho^2 + dz^2)\right]\right). \tag{3.1}$$

We use this metric because it closely resembles the original LWP ansatz employed by Ernst and, since we expect to identify some symmetries that can be mapped onto those of MC frame [9], this ansatz should make it easier to identify them. A complete derivation of LWP metric can be found in [10]-[11]. Notice that thanks to the symmetries specified, all functions depend only on $\rho, z$. A reasonable alternative to treat this kind of spacetimes in the conformal frame, is removing the $\Omega$ factor and substitute the $\rho^2 d\varphi^2$ term with a generic function $\alpha(\rho, z)d\varphi^2$. In appendix A we repeat the same procedure for Ernst's method with a completely different ansatz.

### 3.1.2 Deriving Ernst's equation

In terms of metric (3.1), the five unknown fields $f, \Omega, \gamma, \omega, \psi$ are described by four Einstein's equations (EE) and equation (2.12) (that is a "Klein-Gordon" equation (K.G.))

$$\nabla \cdot \left[\rho^{-2} f^2 \Omega\chi \nabla\omega\right] = 0 \tag{$EE^{\varphi}_{t}$}$$

$$\nabla \cdot \left[\rho \nabla(\Omega\chi)\right] = 0 \tag{$EE^{\rho}_{\rho} + EE^{z}_{z}$}$$

$$\frac{f}{\Omega\chi}\left[\nabla \cdot (\Omega\chi\nabla f) + f\nabla^2(\Omega\chi)\right] = \nabla f^2 - \rho^{-2} f^4 \nabla\omega^2 \tag{$EE^{t}_{t} - EE^{\varphi}_{\varphi}$}$$

$$\begin{aligned}\partial^2_\rho\gamma + \partial^2_z\gamma = \frac{3}{4}\left(\frac{\nabla\Omega^2}{\Omega^2} + \frac{\nabla\chi^2}{\chi(\chi+3)} - \frac{\nabla^2\chi}{\chi} - \frac{\nabla^2\Omega}{\Omega} - \frac{\nabla^2(\Omega\chi)}{\Omega\chi} - f^{-2}\nabla f^2\right) \\ + \frac{1}{4}\rho^{-2} f^2 \nabla\omega^2 + \frac{1}{2} f^{-1}\nabla^2 f\end{aligned} \tag{$EE^{t}_{t} + EE^{\varphi}_{\varphi}$}$$

$$\begin{aligned}12\psi^{-1}\Omega^{-1}\nabla \cdot (\Omega\nabla\psi) - 3\frac{\nabla\Omega^2}{\Omega^2} + 6\frac{\nabla^2\Omega}{\Omega} + 3f^{-2}\nabla f^2 - \rho^{-2} f^2\nabla\omega^2 \\ - 2f^{-1}\nabla^2 f + 4(\partial^2_\rho\gamma + \partial^2_z\gamma) = 0\end{aligned} \tag{K.G.}$$

where $\chi \equiv 4\pi\psi^2 - 3$ and $\nabla\cdot$, $\nabla^2$ are the flat cylindrical operators. Now the purpose is to rewrite this five equations with Ernst's potential and then find the associated effective action. Following step by step Ernst procedure, we notice that if a function $h$ is independent from $\varphi$ and if $\hat{e}_\varphi$ is the unit vector in azimuthal direction, the following equations hold

$$\nabla\cdot[\rho^{-1}\hat{e}_\varphi\times\nabla h]=0 \tag{3.2}$$

$$\hat{e}_\varphi\times(\hat{e}_\varphi\times\nabla h)=-\nabla h$$

$$(\hat{e}_\varphi\times\nabla h)^2=\nabla h^2.$$

From this it is natural to regard $EE_t^\varphi$ as an integrability condition for the existence of $h$

$$\rho^{-1}\hat{e}_\varphi\times\nabla h=\rho^{-2}f^2\Omega\chi\nabla\omega \tag{3.3}$$

that satisfies

$$\nabla\cdot\left[(f^2\Omega\chi)^{-1}\nabla h\right]=0 \tag{3.4}$$

equivalent to $EE_t^\varphi$, and taking the square of (3.3)

$$\nabla\omega^2=\rho^2 f^{-4}(\Omega\chi)^{-2}\nabla h^2. \tag{3.5}$$

At this point, with (3.5) and defining $\tilde{f}\equiv\Omega\chi f$, the $EE_t^t-EE_\varphi^\varphi$ becomes

$$\frac{\tilde{f}}{\Omega\chi}\nabla\cdot[\Omega\chi\nabla\tilde{f}]=\nabla\tilde{f}^2-\nabla h^2 \tag{3.6}$$

therefore it is natural to define the Ernst's potential $\mathcal{E}\equiv\tilde{f}+ih$ so that equations (3.6) and (3.4) (equivalent to $EE_t^t-EE_\varphi^\varphi$ and $EE_t^\varphi$) can be viewed as the real and the imaginary part of a single complex equation, the Ernst's equation

$$\mathrm{Re}\mathcal{E}\nabla^2(\alpha\nabla\mathcal{E})=\alpha\nabla\mathcal{E}\nabla\mathcal{E} \tag{3.7}$$

where $\alpha\equiv\Omega\chi$. Once Ernst's potential has been defined and with the substitutions $\Omega\chi\to\alpha$, $\nabla\omega^2\to\nabla h^2$, $\alpha f\to\tilde{f}$, the $EE_t^t+EE_\varphi^\varphi$ and $EE_\rho^\rho+EE_z^z$ take the form

$$\partial_\rho^2\gamma+\partial_z^2\gamma=-\frac{9\nabla\chi^2}{4\chi^2(\chi+3)}-2\frac{\nabla^2\alpha}{\alpha}-\frac{\nabla\mathcal{E}\nabla\mathcal{E}^*}{(\mathcal{E}+\mathcal{E}^*)^2}+\frac{\nabla\alpha^2}{\alpha^2} \tag{3.8}$$

$$\nabla\cdot[\rho\nabla\alpha]=0 \tag{3.9}$$

and replacing (3.8) in the last term of K.G.

$$6\nabla\cdot(\alpha\nabla\chi)-\frac{9\alpha(\chi+2)}{\chi(\chi+3)}\nabla\chi^2=0. \tag{3.10}$$

### 3.1.3 Effective action

Finally is straightforward to demonstrate that the four equations of motion (3.7)-(3.10) can be derived from the following effective action

$$S = \int \rho d\rho dz \left[ -\frac{3\alpha \nabla \chi^2}{\chi^2(\chi+3)} + \frac{4}{3}\left( \frac{\nabla \alpha^2}{\alpha} - \alpha \frac{\nabla \mathcal{E} \nabla \mathcal{E}^*}{(\mathcal{E}+\mathcal{E}^*)^2} - \alpha(\partial_\rho^2 \gamma + \partial_z^2 \gamma) \right) \right] \tag{3.11}$$

or equivalently without the second-order derivative term

$$S = \int \rho d\rho dz \left[ -\frac{3\alpha \nabla \chi^2}{\chi^2(\chi+3)} + \frac{4}{3}\left( \frac{\nabla \alpha^2}{\alpha} - \alpha \frac{\nabla \mathcal{E} \nabla \mathcal{E}^*}{(\mathcal{E}+\mathcal{E}^*)^2} + \nabla \alpha \nabla \gamma + \alpha \frac{\nabla \gamma \nabla \rho}{\rho} \right) \right]. \tag{3.12}$$

In particular if $\mathcal{L}$ is the term between brackets in the above action, it's easy to verify that

$$\begin{aligned}
\nabla \frac{\delta \mathcal{L}}{\delta \nabla \mathcal{E}} - \frac{\delta \mathcal{L}}{\delta \mathcal{E}} &= 0 \longrightarrow (3.7) \\
\nabla \frac{\delta \mathcal{L}}{\delta \nabla \chi} - \frac{\delta \mathcal{L}}{\delta \chi} &= 0 \longrightarrow (3.10) \\
\nabla \frac{\delta \mathcal{L}}{\delta \nabla \gamma} - \frac{\delta \mathcal{L}}{\delta \gamma} &= 0 \longrightarrow (3.9) \\
\nabla \frac{\delta \mathcal{L}}{\delta \nabla \alpha} - \frac{\delta \mathcal{L}}{\delta \alpha} &= 0 \longrightarrow (3.8)
\end{aligned}$$

## 3.2 The electromagnetic case

To complete the discussion let's consider Ernst's technique within a scalar-tensor theory that incorporates Maxwell's electromagnetism. The theory is defined below, where we simply introduce the electromagnetic Lagrangian density in the action (2.10)

$$S[g_{\mu\nu}, \psi, F_{\mu\nu}] = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - k\left( \nabla^\mu \psi \nabla_\mu \psi + \frac{R}{6}\psi^2 \right) - F^{\mu\nu}F_{\mu\nu} \right]. \tag{3.13}$$

This leads to the following equations of motion

$$\begin{aligned}
&G_{\mu\nu} = 2T^F_{\mu\nu} + kT^\psi_{\mu\nu} \\
&\Box \psi - \frac{R}{6}\psi = 0 \\
&\partial_\mu(\sqrt{-g}F^{\mu\nu}) = \nabla_\mu F^{\mu\nu} = 0
\end{aligned} \tag{3.14}$$

where $T^{\psi}_{\mu\nu}$ is the right-hand side of equation (2.11), while $T^{F}_{\mu\nu}$ is

$$T^{F}_{\mu\nu} = F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{3.15}$$

In this case as well, we will use the conformal LWP ansatz (3.1) with a four-potential defined as $A = A_0(\rho, z)dt + A_3(\rho, z)d\varphi$, so that it satisfies symmetry conditions (and obviously $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$). This time, there are seven unknown fields $A_0, A_3, f, \omega, \gamma, \Omega, \psi$ described respectively by two Maxwell's equations (ME), four Einstein's equations and a Kleing-Gordon equation

$$\nabla \cdot [\rho^{-2} f(\nabla A_3 - \omega\nabla A_0)] = 0 \tag{$ME_\varphi$}$$

$$\nabla \cdot [f^{-1}\nabla A_0 + \rho^{-2} f\omega(\nabla A_3 - \omega\nabla A_0)] = 0 \tag{$ME_t$}$$

$$\nabla \cdot \left[\rho^{-2} f^2 \Omega\chi\nabla\omega + 12 f\rho^{-2} A_0(\nabla A_3 - \omega\nabla A_0)\right] = 0 \tag{$EE^{\varphi}_t$}$$

$$\nabla \cdot [\rho\nabla(\Omega\chi)] = 0 \tag{$EE^{\rho}_\rho + EE^{z}_z$}$$

$$\begin{aligned}\frac{f}{\Omega\chi}\left[\nabla\cdot(\Omega\chi\nabla f) + f\nabla^2(\Omega\chi)\right] =& \nabla f^2 - \rho^{-2}f^4\nabla\omega^2 \\ &- 6(\Omega\chi)^{-1} f[(\rho^{-2}f^2(\nabla A_3^2 - \omega^2\nabla A_0^2) + \nabla A_0^2]\end{aligned} \tag{$EE^{t}_t - EE^{\varphi}_\varphi$}$$

$$\begin{aligned}\partial^2_\rho\gamma + \partial^2_z\gamma = \frac{3}{4}\left(\frac{\nabla\Omega^2}{\Omega^2} + \frac{\nabla\chi^2}{\chi(\chi+3)} - \frac{\nabla^2\chi}{\chi} - \frac{\nabla^2\Omega}{\Omega} - \frac{\nabla^2(\Omega\chi)}{\Omega\chi} - f^{-2}\nabla f^2\right) \\ + \frac{1}{4}\rho^{-2}f^2\nabla\omega^2 + \frac{1}{2}f^{-1}\nabla^2 f\end{aligned} \tag{$EE^{t}_t + EE^{\varphi}_\varphi$}$$

$$\begin{aligned}12\psi^{-1}\Omega^{-1}\nabla\cdot(\Omega\nabla\psi) - 3\frac{\nabla\Omega^2}{\Omega^2} + 6\frac{\nabla^2\Omega}{\Omega} + 3f^{-2}\nabla f^2 - \rho^{-2}f^2\nabla\omega^2 \\ - 2f^{-1}\nabla^2 f + 4(\partial^2_\rho\gamma + \partial^2_z\gamma) = 0.\end{aligned} \tag{K.G.}$$

As done in the previous section, our starting point to build Ernst's potential is looking at which of this seven equations represent an integrability condition. The first one is the equation $ME_\varphi$ that, with property (3.2), we can use to define a field $\tilde{A}_3$ independent on $\varphi$

$$\rho^{-1}\hat{e}_\varphi \times \nabla\tilde{A}_3 = \rho^{-2} f(\nabla A_3 - \omega\nabla A_0) \tag{3.16}$$

such that

$$\nabla \cdot [f^{-1}\nabla \tilde{A}_3 - \rho^{-1}\omega \hat{e}_\varphi \times \nabla A_0] = 0 \tag{3.17}$$

and using definition (3.16) in $ME_t$

$$\nabla \cdot [f^{-1}\nabla A_0 + \rho^{-1}\omega \hat{e}_\varphi \times \nabla \tilde{A}_3] = 0. \tag{3.18}$$

Defining a new complex field as $\mathbf{\Phi} \equiv A_0 + i\tilde{A}_3$ equations (3.17) and (3.18) can be viewed as the imaginary and the real part of a single complex equation

$$\nabla \cdot [f^{-1}\nabla \mathbf{\Phi} - i\rho^{-1}\omega \hat{e}_\varphi \times \nabla \mathbf{\Phi}] = 0. \tag{3.19}$$

With the new fields $\tilde{A}_3$ and $\tilde{f} \equiv \Omega\chi f$ the $EE_t^\varphi$ can be rewritten as

$$\nabla \cdot [(\Omega\chi)^{-1}\tilde{f}^2\rho^{-2}\nabla\omega + 6\rho^{-1}\hat{e}_\varphi \times \mathrm{Im}(\mathbf{\Phi}^*\nabla\mathbf{\Phi})] = 0 \tag{3.20}$$

and this last equation is the second integrability condition for the existence of a field h

$$\rho^{-1}\hat{e}_\varphi \times \nabla h = (\Omega\chi)^{-1}\tilde{f}^2\rho^{-2}\nabla\omega + 6\rho^{-1}\hat{e}_\varphi \times \mathrm{Im}(\mathbf{\Phi}^*\nabla\mathbf{\Phi}) \tag{3.21}$$

such that

$$\nabla \cdot [\Omega\chi\tilde{f}^{-2}(\nabla h - 6\mathrm{Im}(\mathbf{\Phi}^*\nabla\mathbf{\Phi}))] = 0 \tag{3.22}$$

$$\nabla\omega^2 = (\Omega\chi)^2\rho^2\tilde{f}^{-4}(\nabla h - 6\mathrm{Im}(\mathbf{\Phi}^*\nabla\mathbf{\Phi}))^2. \tag{3.23}$$

Finally we have to recast equation $EE_t^t - EE_\varphi^\varphi$ with $h$ and the new potential $\mathbf{\Phi}$, obtaining

$$\frac{\tilde{f}}{\Omega\chi}\nabla \cdot (\Omega\chi\nabla\tilde{f}) = \nabla\tilde{f}^2 - (\nabla h - 6\mathrm{Im}(\mathbf{\Phi}^*\nabla\mathbf{\Phi}))^2 - 6\tilde{f}\nabla\mathbf{\Phi}\nabla\mathbf{\Phi}^* \tag{3.24}$$

so that with the definition $\mathcal{E} \equiv \tilde{f} + 3|\mathbf{\Phi}|^2 + ih$, equations (3.22) and (3.24) are the imaginary and real part of the equation (3.25), while equation (3.19) yields the (3.26).

$$\left(\mathrm{Re}\mathcal{E} - 3|\mathbf{\Phi}|^2\right)\nabla \cdot (\alpha\nabla\mathcal{E}) = \alpha\nabla\mathcal{E}(\nabla\mathcal{E} - 6\mathbf{\Phi}^*\nabla\mathbf{\Phi}) \tag{3.25}$$

$$\left(\mathrm{Re}\mathcal{E} - 3|\mathbf{\Phi}|^2\right)\nabla \cdot (\alpha\nabla\mathbf{\Phi}) = \alpha\nabla\mathbf{\Phi}(\nabla\mathcal{E} - 6\mathbf{\Phi}^*\nabla\mathbf{\Phi}) \tag{3.26}$$

where $\alpha \equiv \Omega\chi$, that are the Ernst's equations for the electromagnetic case. The last step before finding an effective action of the theory is writing $EE_t^t + EE_\varphi^\varphi$ and

$EE^\rho_\rho + EE^z_z$ in terms of the Ernst's potential, simply replacing $f$, $\omega$, $\Omega\chi$ with $\mathcal{E}$, $\alpha$

$$\partial^2_\rho\gamma + \partial^2_z\gamma = \frac{\nabla\alpha^2}{\alpha^2} - 2\frac{\nabla^2\alpha}{\alpha} - \frac{|\nabla\mathcal{E} - 6\mathbf{\Phi}^*\nabla\mathbf{\Phi}|^2}{(\mathcal{E}+\mathcal{E}^* - 6|\mathbf{\Phi}|^2)^2} - 6\frac{\nabla\mathbf{\Phi}\nabla\mathbf{\Phi}^*}{\mathcal{E}+\mathcal{E}^*-6|\mathbf{\Phi}|^2} - \frac{9\nabla\chi^2}{4\chi^2(\chi+3)} \tag{3.27}$$

$$\nabla\cdot[\rho\nabla\alpha] = 0 \tag{3.28}$$

and replacing (3.27) in the last term of K.G.

$$6\nabla\cdot(\alpha\nabla\chi) - \frac{9\alpha(\chi+2)}{\chi(\chi+3)}\nabla\chi^2 = 0. \tag{3.29}$$

Looking at the effective action (3.12) of the uncharged theory, the Lagrangian density can be split in two contributes (3.30)-(3.31), the first $\mathcal{L}_\mathcal{E}$ that generates Ernst's equation and the second $\mathcal{L}_{\alpha\chi\gamma}$ for the remaining ones.

$$\mathcal{L}_\mathcal{E} = \alpha\frac{\nabla\mathcal{E}\nabla\mathcal{E}^*}{(\mathcal{E}+\mathcal{E}^*)^2} \tag{3.30}$$

$$\mathcal{L}_{\alpha\chi\gamma} = \frac{9\alpha\nabla\chi^2}{4\chi^2(\chi+3)} - \frac{\nabla\alpha^2}{\alpha} - \nabla\alpha\nabla\gamma - \alpha\frac{\nabla\gamma\nabla\rho}{\rho} \tag{3.31}$$

With the presence of electromagnetic field, $\mathcal{L}_{\alpha\chi\gamma}$ is unchanged while $\mathcal{L}_\mathcal{E}$ takes the form

$$\mathcal{L}_{\mathcal{E}\mathbf{\Phi}} = \alpha\frac{(\nabla\mathcal{E} - 6\mathbf{\Phi}^*\nabla\mathbf{\Phi})\cdot(\nabla\mathcal{E}^* - 6\mathbf{\Phi}\nabla\mathbf{\Phi}^*)}{(\mathcal{E}+\mathcal{E}^*-6|\mathbf{\Phi}|^2)^2} + 6\alpha\frac{\nabla\mathbf{\Phi}\nabla\mathbf{\Phi}^*}{\mathcal{E}+\mathcal{E}^*-6|\mathbf{\Phi}|^2} \tag{3.32}$$

so the complete effective action of the theory reads as[1]

$$S = \int\rho d\rho dz\left[\mathcal{L}_{\mathcal{E}\mathbf{\Phi}} + \mathcal{L}_{\alpha\chi\gamma}\right]. \tag{3.33}$$

[1] Obviously in the limit of null scalar field $\psi\to 0$ (and then $\alpha\to -3$, $\mathcal{E}\to -3\mathcal{E}_{GR}$) we recover the action that generates Ernst's equations of classical GR.

# Chapter 4

# The symmetry group

Here, we address one of the main topics of this thesis, the transformation symmetries of equations of motion (3.7)-(3.10) of the conformal frame. In particular we take in consideration the continuous transformation group developed by the mathematician Sophus Lie called Lie point symmetries [12]-[13]. Given a system of differential equations, a Lie point symmetry is a change of variables that transform a solution of the system into a new solution. Actually with this method we are mapping an existing solution, called the seed solution, into a, possibly non trivial, new one. We will not discuss Lie point symmetry here; instead, we will directly apply the method as presented in [11]-[14] for our effective action (3.12). As said before, we are not completely blind on some of this symmetries. Indeed since the symmetry group has been applied in the MC frame [9], we expect to recover this MC symmetries mapped by Bekenstein's map (2.16) in the conformal frame.

## 4.1 Symmetries of the effective action

The starting point is considering the Lagrangian density of effective action (3.12)

$$\mathcal{L} = -\frac{3\alpha\nabla\chi^2}{\chi^2(\chi+3)} + \frac{4}{3}\left(\frac{\nabla\alpha^2}{\alpha} - \alpha\frac{\nabla\mathcal{E}\nabla\mathcal{E}^*}{(\mathcal{E}+\mathcal{E}^*)^2} + \nabla\alpha\nabla\gamma + \alpha\frac{\nabla\gamma\nabla\rho}{\rho}\right)$$

from this we had to find the induced bilinear form. In particular we can make the substitution (see [14] for particular)

$$\nabla \longrightarrow d$$

and rename the field $\mathcal{E} \rightarrow x + iy$ such that the Lagrangian above appear as a metric

$$ds^2 = -\frac{3\alpha}{\chi^2(\chi+3)}d\chi^2 + \frac{4}{3}\frac{d\alpha^2}{\alpha} - \frac{\alpha}{3x^2}(dx^2+dy^2) + \frac{4}{3}d\alpha d\gamma + \frac{4}{3}\alpha\frac{d\rho d\gamma}{\rho}. \tag{4.1}$$

Notice that in order to find Lie point symmetries, we have to consider the coordinate $\rho$ as an unknown field, just like the others, in the action (3.12). From the Killing vectors of the metric (4.1) we can extract the transformations symmetries of equations of motion.

### 4.1.1 Killing vectors

Let's start remembering that a Killing vector is a vector field that satisfy the Killing equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{4.2}$$

Killing vector are of particular interest in General Relativity as they identify the transformations under which the geometry is invariant. It is always possible to extract a conserved quantity with a killing vector field $\xi$ [1]. Here we are interested on Killing vectors because they represent the infinitesimal generators of the transformations and then, by integration, to the finite ones. The max number of Killing vectors on a N dimensional pseudo-Riemannian manifold is $\frac{1}{2}N(N+1)$, so for our metric (4.1) there are at most 21 Killing vectors. In term of our metric, equation (4.2) becomes a system of 21 coupled partial differential equations reported below.

$$\begin{cases}
\rho\xi_\rho - \alpha\xi_\alpha + 8x\xi_x + 8x^2\partial_x\xi_x = 0 \\
\dfrac{2}{x}\xi_y + \partial_y\xi_x + \partial_x\xi_y = 0 \\
-\dfrac{\xi_x}{\alpha} + \partial_\alpha\xi_x + \partial_x\xi_\alpha = 0 \\
-8x\xi_x + \alpha\xi_\alpha - \rho\xi_\rho + 8x^2\partial_y\xi_y = 0 \\
-\dfrac{\xi_y}{\alpha} + \partial_\alpha\xi_y + \partial_y\xi_\alpha = 0 \\
-\dfrac{\xi_\gamma}{\alpha} + \dfrac{\xi_\alpha}{2} - \dfrac{\rho}{2\alpha}\xi_\rho + \partial_\alpha\xi_\gamma + \partial_\gamma\xi_\alpha = 0 \\
\dfrac{\alpha}{2\rho}\xi_\alpha - \dfrac{\xi_\rho}{2} + \partial_\rho\xi_\gamma + \partial_\gamma\xi_\rho = 0 \\
\dfrac{\xi_\alpha}{\alpha} - \rho\dfrac{\xi_\rho}{\alpha^2} + 2\partial_\alpha\xi_\alpha = 0 \\
-\dfrac{\xi_\chi}{\alpha} + \partial_\chi\xi_\alpha + \partial_\alpha\xi_\chi = 0 \\
9(\alpha\xi_\alpha - \rho\xi_\rho) + 12\chi(\chi+2)\xi_\chi + 8\chi^2(\chi+3)\partial_\chi\xi_\chi = 0 \\
-\dfrac{\xi_\rho}{\alpha} + \partial_\rho\xi_\alpha + \partial_\alpha\xi_\rho = 0
\end{cases}
\qquad
\begin{cases}
\partial_\gamma\xi_x + \partial_x\xi_\gamma = 0 \\
\partial_\chi\xi_x + \partial_x\xi_\chi = 0 \\
\partial_\rho\xi_x + \partial_x\xi_\rho = 0 \\
\partial_\gamma\xi_y + \partial_y\xi_\gamma = 0 \\
\partial_\chi\xi_y + \partial_y\xi_\chi = 0 \\
\partial_\rho\xi_y + \partial_y\xi_\rho = 0 \\
\partial_\chi\xi_\gamma + \partial_\gamma\xi_\chi = 0 \\
\partial_\chi\xi_\rho + \partial_\rho\xi_\chi = 0 \\
\xi_\rho + \rho\partial_\rho\xi_\rho = 0 \\
\partial_\gamma\xi_\gamma = 0
\end{cases}$$

[1] In particular given $\xi$ and a geodesic $\gamma$ with tangent vector $U$, the quantity $U_\mu\xi^\mu$ is constant on $\gamma$.

The solution of this system leads to the contravariant components (4.3) of the killing vector field (where we can rise indices with the inverse of (4.1) ). Notice that there is nine different integration constant $c_{1-9}$ that will identify nine finite transformations.

$$\begin{aligned}
\xi^x &= 3x(2c_3y - c_1) \\
\xi^y &= -(3c_3(x^2 - y^2) + 3c_1y + 3c_2) \\
\xi^\gamma &= \frac{3}{2}\operatorname{arctanh}\left[\sqrt{\frac{\chi+3}{3}}\right]c_6 + \frac{3}{2}(c_5 - \gamma c_9) \\
\xi^\alpha &= 0 \\
\xi^\chi &= -\frac{\chi\sqrt{\chi+3}}{6\sqrt{3}}(2\sqrt{3}c_4 + 3c_6(\log[\alpha] + \log[\rho]) + 3\gamma c_7) \\
\xi^\rho &= \frac{3}{2}\rho\left(c_9(\log[\alpha] + \log[\rho]) + \operatorname{arctanh}\left[\sqrt{\frac{\chi+3}{3}}\right]c_7 + c_8\right)
\end{aligned} \tag{4.3}$$

Setting eight constant to zero and the last to one, nine different killing vectors or infinitesimal generators emerge

$$\begin{aligned}
\xi^1 &= -3(x\partial_x + y\partial_y) \\
\xi^2 &= -3\partial_y \\
\xi^3 &= 6xy\partial_x - 3(x^2 - y^2)\partial_y \\
\xi^4 &= -\frac{\chi\sqrt{\chi+3}}{3}\partial_\chi \\
\xi^5 &= \frac{3}{2}\partial_\gamma \\
\xi^6 &= \frac{3}{2}\operatorname{arctanh}\left[\sqrt{\frac{\chi+3}{3}}\right]\partial_\gamma - \frac{\chi\sqrt{\chi+3}}{2\sqrt{3}}(\log[\alpha] + \log[\rho])\partial_\chi \\
\xi^7 &= -\frac{\chi\sqrt{\chi+3}}{2\sqrt{3}}\gamma\partial_\chi + \frac{3}{2}\rho\operatorname{arctanh}\left[\sqrt{\frac{\chi+3}{3}}\right]\partial_\rho \\
\xi^8 &= \frac{3}{2}\rho\partial_\rho \\
\xi^9 &= -\frac{3}{2}\gamma\partial_\gamma + \frac{3}{2}\rho(\log[\alpha] + \log[\rho])\partial_\rho
\end{aligned} \tag{4.4}$$

From this vectors it is possible to generate the finite transformations.

### 4.1.2 Finite transformations

Following [14], the components of infinitesimal generators $\xi^{1-9}$ represent the infinitesimal transformations of the field associated to the direction of that component. Let's take for example the generator $\xi^1$, the component in $x$ direction is $-3x$, this means that the transformed $x'(x, y, \epsilon)$ has the derivative with respect to the transformation parameter $\epsilon$ equals to $-3x'(x, y, \epsilon)$. Then the finite transformation is obtained by solving a system of differential equation

$$\begin{cases} \dfrac{dx'}{d\epsilon} = -3x' \\ \dfrac{dy'}{d\epsilon} = -3y' \end{cases} \rightarrow \begin{cases} x' = xe^{-3\epsilon} \\ y' = ye^{-3\epsilon} \end{cases}$$

where we must impose the boundary condition $x'(x, y, 0) = x$ and $y'(x, y, 0) = y$, since if the parameter $\epsilon$ goes to zero there must be no transformation. The resulting $\mathcal{E}'$ is $\mathcal{E}' = x' + iy' = e^{-3\epsilon}(x + iy) = e^{-3\epsilon}\mathcal{E}$. The same procedure for the other generators leads to nine different finite transformations reported below[2]

$$\begin{array}{llllll} I) & \mathcal{E} \rightarrow \lambda\lambda^*\mathcal{E} & \alpha \rightarrow \alpha & \chi \rightarrow \chi & \gamma \rightarrow \gamma & \rho \rightarrow \rho \\ II) & \mathcal{E} \rightarrow \mathcal{E} + ib & \alpha \rightarrow \alpha & \chi \rightarrow \chi & \gamma \rightarrow \gamma & \rho \rightarrow \rho \\ III) & \mathcal{E} \rightarrow \dfrac{\mathcal{E}}{1 + ic\mathcal{E}} & \alpha \rightarrow \alpha & \chi \rightarrow \chi & \gamma \rightarrow \gamma & \rho \rightarrow \rho \\ VI) & \mathcal{E} \rightarrow \mathcal{E} & \alpha \rightarrow \alpha & \chi \rightarrow \chi' & \gamma \rightarrow \gamma & \rho \rightarrow \rho \end{array}$$

$$\chi' = 3\left(\tanh^2\left[a \pm \operatorname{arctanh}\left[\sqrt{\frac{\chi+3}{3}}\right]\right] - 1\right)$$

$$\begin{array}{llllll} VII) & \mathcal{E} \rightarrow \mathcal{E} & \alpha \rightarrow \alpha & \chi \rightarrow \chi & \gamma \rightarrow \gamma + d & \rho \rightarrow \rho \\ VIII) & \mathcal{E} \rightarrow \mathcal{E} & \alpha \rightarrow \alpha & \chi \rightarrow \chi' & \gamma \rightarrow \gamma' & \rho \rightarrow \rho \end{array}$$

$$\chi' = 3\left(\tanh^2\left[\frac{l}{2}(\log[\alpha] + \log[\rho]) \pm \operatorname{arctanh}\left[\sqrt{\frac{\chi+3}{3}}\right]\right] - 1\right)$$

$$\gamma' = \gamma + 3l\left(\frac{l}{4}(\log[\alpha] + \log[\rho]) \pm \operatorname{arctanh}\left[\sqrt{\frac{\chi+3}{3}}\right]\right)$$

[2]We have omitted labels IV) and V) because, in general, this are attributed to a couple of transformation with the presence of an electromagnetic field.

$$IX) \qquad \mathcal{E} \to \mathcal{E} \qquad \alpha \to \alpha \qquad \chi \to \chi' \qquad \gamma \to \gamma \qquad \rho \to \rho'$$

$$\chi' = 3\left(\tanh^2\left[\frac{l}{2}\gamma \pm \operatorname{arctanh}\left[\sqrt{\frac{\chi+3}{3}}\right]\right] - 1\right)$$

$$\rho' = \rho \exp\left[3l\left(\frac{l}{4}\gamma \pm \operatorname{arctanh}\left[\sqrt{\frac{\chi+3}{3}}\right]\right)\right]$$

$$X) \qquad \mathcal{E} \to \mathcal{E} \qquad \alpha \to \alpha \qquad \chi \to \chi \qquad \gamma \to \gamma \qquad \rho \to \nu\nu^*\rho$$

$$XI) \qquad \mathcal{E} \to \mathcal{E} \qquad \alpha \to \alpha \qquad \chi \to \chi \qquad \gamma \to \frac{\gamma}{p} \qquad \rho \to \rho^p\alpha^{p-1}$$

where $a, b, c, d, l, p \in \mathbb{R}$ with $p > 0$ and $\lambda, \nu \in \mathbb{C}$. Four of this transformations, the I, II, VII and X are gauge symmetries and can be reabsorbed by a change of coordinates, i.e. they do not lead to a nonequivalent physical solution. Transformation III is known as Ehlers transformation (present in classical GR as well as I and II) and it has the peculiarity of adding a NUT parameter to the solution. The NUT or gravomagnetic parameter can be interpreted as the dual of the mass exactly as the magnetic charge can be the dual of electric charge in electromagnetism [15]. The general Taub-NUT solution (see [16]) does not present curvature singularity. It is known [9] that in the MC frame, the scalar field $\hat{\psi}$ presents the symmetry $\hat{\psi} \to \hat{\psi} + a$. If we apply Bekenstein's map to this symmetry, the result will be exactly the transformation VI (this is the symmetry which we expected to find). The remaining transformations VIII, IX and XI are new symmetries revealed thanks to Ernst's method directly applied in the conformal frame. We are not certain about how to implement symmetries involving the coordinate $\rho$. However, we can certainly observe that transformation IX, once understood how to apply it, may be used as a map between the classical theory and the conformal one. Indeed, it can introduce a non trivial scalar field from a solution of GR where $\psi = 0$ ($\chi = 4\pi\psi^2 - 3 = -3$) thanks to the $\gamma$ term.
Once symmetries have been found in the uncharged case, it is straightforward to repeat the process for the theory (3.13) with electromagnetic field. Here we report the main results. The induced bilinear form of Lagrangian density $\mathcal{L}_{\mathcal{E}\boldsymbol{\Phi}} + \mathcal{L}_{\alpha\chi\gamma}$ of

the action (3.33) is

$$ds^2 = \frac{9\alpha}{4\chi^2(\chi+3)}d\chi^2 - \frac{d\alpha^2}{\alpha} - d\alpha d\gamma - \alpha\frac{d\rho d\gamma}{\rho} + ds^2_{\mathcal{E}\mathbf{\Phi}} \tag{4.5}$$

where if we define $\mathcal{E}$ and $\mathbf{\Phi}$ as $\mathcal{E} = x + iy$, $\mathbf{\Phi} = z + iw$ the last term becomes

$$ds^2_{\mathcal{E}\mathbf{\Phi}} = \alpha\frac{dx^2 + dy^2 - 12dx(zdz + wdw) - 12y(zdw - wdz) + 12x(dz^2 + dw^2)}{4(x - 3(z^2 + w^2))^2}.$$

From metric (4.5) we can derive the following killing vectors

$$\begin{aligned}
\xi^1 &= 2x\partial_x + 2y\partial_y + z\partial_z + w\partial_w \\
\xi^3 &= 2xy\partial_x - (x^2 - y^2)\partial_y + (wx + yz)\partial_z + (wy - xz)\partial_w \\
\xi^{10} &= 2(wy - zx)\partial_x - 2(wx + yz)\partial_y + (2w^2 - 2z^2 + \frac{x}{3})\partial_z - (4wz - \frac{y}{3})\partial_w \\
\xi^{11} &= 2(wx + zy)\partial_x + 2(wy - zx)\partial_y + (4wz + \frac{y}{3})\partial_z + (2w^2 - 2z^2 - \frac{x}{3})\partial_w \\
\xi^{12} &= 2(z\partial_x + w\partial_y) + \frac{1}{3}\partial_z \\
\xi^{13} &= 2(-w\partial_x + z\partial_y) - \frac{1}{3}\partial_w \\
\xi^{14} &= w\partial_z - z\partial_w
\end{aligned} \tag{4.6}$$

the remaining ones $\xi^2$, $\xi^{4-9}$ are the same (up to a multiplicative coefficient) of the uncharged case (4.4). From them, it is clear that transformations II, VI, VII, VIII, IX, X and XI are unchanged with the trivial addition $\mathbf{\Phi} \to \mathbf{\Phi}$. The vectors $\xi^{10}$ and $\xi^{11}$ combine together to produce transformation V and the same for $\xi^{12}$ and $\xi^{13}$ to produce IV. $\xi^1$, $\xi^2$ and $\xi^{14}$ are modifications of transformations I and III as below

$$\begin{array}{llll}
I) & \mathcal{E} \to \lambda\lambda^*\mathcal{E} & \mathbf{\Phi} \to \lambda\mathbf{\Phi} & \alpha \to \alpha \quad \chi \to \chi \quad \gamma \to \gamma \quad \rho \to \rho \\
III) & \mathcal{E} \to \dfrac{\mathcal{E}}{1 + ic\mathcal{E}} & \mathbf{\Phi} \to \dfrac{\mathbf{\Phi}}{1 + ic\mathcal{E}} & \alpha \to \alpha \quad \chi \to \chi \quad \gamma \to \gamma \quad \rho \to \rho \\
IV) & \mathcal{E} \to \mathcal{E} - 6\beta^*\mathbf{\Phi} + 3\beta\beta^* & \mathbf{\Phi} \to \mathbf{\Phi} + \beta & \alpha \to \alpha \quad \chi \to \chi \quad \gamma \to \gamma \quad \rho \to \rho \\
V) & \mathcal{E} \to \dfrac{\mathcal{E}}{1 + 3\tilde{\alpha}\tilde{\alpha}^*\mathcal{E} - 6\tilde{\alpha}^*\mathbf{\Phi}} & \mathbf{\Phi} \to \dfrac{\mathbf{\Phi} - \tilde{\alpha}\mathcal{E}}{1 + 3\tilde{\alpha}\tilde{\alpha}^*\mathcal{E} - 6\tilde{\alpha}^*\mathbf{\Phi}} & \alpha \to \alpha \quad \chi \to \chi \quad \gamma \to \gamma \quad \rho \to \rho
\end{array}$$

with $\lambda, \beta, \tilde{\alpha} \in \mathbb{C}$ and $c \in \mathbb{R}$. Again, we have that transformations I and IV are gauge symmetries while the III is the electromagnetic version of Ehlers transformation. Finally the V is called Harrison transformation and, in classical GR can be used to add electric charge to vacuum solutions.

# Chapter 5

# Testing the solutions generating technique on BBMB black hole

Before using the new symmetry transformations of chapter 4, we can try to build some known solution of the conformal theory, to verify that the symmetries actually work. In classical GR it is known that the transformation III, or Ehlers transformation, can incorporate the Taub-NUT parameter into a considered solution (in [11] it is shown how to apply it to Schwarzschild). At the same time in [17] has been found how to include this parameter in the BBMB solution of the conformal theory. So in section 5.1 we test the solutions generating technique by applying the transformation III at BBMB to obtain BBMB + NUT of [17]. Then it is also known that the Harrison transformation, when applied with a real parameter $\tilde{\alpha} \in \mathbb{R}$, introduces an electric charge into an initial vacuum solution. In section 5.2, we demonstrate that our transformation V from Chapter 4 has the same capability by applying it to BBMB (where $\bar{A} = 0$). In appendix C, through the transformation III, we embed the BBMB black hole in a swirling background, that is the second known property of the Ehelrs transformation.

## 5.1 Adding NUT parameter to BBMB black hole

The BBMB solution was the first counter example of the "No-hair" theorem and at the same time one of the first solution of equations (2.11)-(2.12) of the conformal theory. It is a spherically symmetric and static solution proposed by Bekenstein in [7] that reads in spherical coord. as

$$ds^2 = -\left(1 - \frac{m}{r}\right)^2 dt^2 + \left(1 - \frac{m}{r}\right)^{-2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{5.1}$$

with

$$\psi = \sqrt{\frac{6}{k}} \frac{m}{r - m}.$$

In [18] are listed some critical aspects of this solution. As mentioned, following [17], Taub-NUT parameter can be included in solution (5.1) to produce

$$ds^2 = -\frac{(r-\mu)^2}{r^2+n^2}(dt+2n\cos\theta d\varphi)^2 + \frac{r^2+n^2}{(r-\mu)^2}dr^2 + (r^2+n^2)(d\theta^2+\sin^2\theta d\varphi^2), \quad (5.2)$$

$$\psi = \sqrt{\frac{6}{k}}\frac{\sqrt{\mu^2+n^2}}{r-\mu}$$

where $\mu = \sqrt{m^2-n^2}$ and $n$ is the NUT parameter. Let's specify that this solution is obtained from a symmetry of Ernst's equation in prolate spherical coordinates that correspond (has shown in [15]) to the composition $III \circ II$. Summarizing, the purpose is writing solution (5.2) from the application of $III \circ II$ on (5.1).

We can start by defining the transformation from cylindrical to spherical coordinates $\{\rho,\varphi,z\} \to \{r,\theta,\varphi\}$

$$\begin{cases} \rho = (r-m)\sin\theta \\ z = (r-m)\cos\theta \\ \varphi = \varphi \end{cases} \qquad (c \to s)$$

with

$$d\rho^2 + dz^2 = dr^2 + (r-m)^2 d\theta^2 \quad (5.3)$$

so that the conformal LWP ansatz becomes

$$ds^2 = \Omega\left[-f_0(dt-\omega_0 d\varphi)^2 + f_0^{-1}\left(e^{2\gamma}(dr^2+(r-m)^2d\theta^2) + (r-m)^2\sin^2\theta d\varphi^2\right)\right]. \quad (5.4)$$

Now it's easy to check that with the substitutions below, the metric (5.4) corresponds exactly to BBMB solution (5.1).

$$\begin{cases} f_0 = \dfrac{(r-m)^2}{r^2} \\ \chi = 4\pi\psi^2 - 3 = 3r\dfrac{2m-r}{(r-m)^2} \end{cases} \qquad \begin{cases} \omega_0 = 0 \\ \Omega = 1 \\ \gamma = 0 \end{cases}$$

Remembering that we can set $h_0 = 0$ in the static case $\omega_0 = 0$, let's apply the transformation $II$ to $\mathcal{E}_0 = \tilde{f}_0 = \Omega\chi f_0 = -3\left(1-\frac{2m}{r}\right)$, in particular

$$\mathcal{E}_0 \longrightarrow \mathcal{E} = \mathcal{E}_0 + ib = -3\left(1-\frac{2m}{r}\right) + ib \quad (5.5)$$

followed by the Ehlers transformation to the new $\mathcal{E}$

$$\mathcal{E} \longrightarrow \mathcal{E}' = \frac{\mathcal{E}}{1+ic\mathcal{E}} = \underbrace{\frac{3r(2m-r)}{r^2(1-bc)^2+9c^2(2m-r)^2}}_{\tilde{f}'} + i\underbrace{\frac{r^2b(1-bc)-9c(2m-r)^2}{r^2(1-bc)^2+9c^2(2m-r)^2}}_{h'} \tag{5.6}$$

where obviously $\Omega \to \Omega$, $\gamma \to \gamma$, $\chi \to \chi$. Notice that the transformation adds a rotation term, indeed leads to the appearance of the field $\omega'$ that can be evaluated from $h'$ thanks to the definition (3.3)

$$\omega' = -12mc(bc-1)\cos\theta. \tag{5.7}$$

Replacing the new field $f', \omega'$ in (5.4) we obtain a modified BBMB solution

$$ds^2 = -\frac{(r-m)^2}{\eta}(dt + 12mc(bc-1)\cos\theta d\varphi)^2 + \frac{\eta}{(r-m)^2}dr^2 + \eta(d\theta^2 + \sin^2\theta d\varphi^2), \tag{5.8}$$

$$\psi = \sqrt{\frac{6}{k}}\frac{m}{r-m}$$

where $\eta = r^2(1-bc)^2 + 9c^2(2m-r)^2$. In order to recast this solution to a form that resemble the (5.2), we can operate a change of variables

$$r \longrightarrow R = r\sqrt{\xi} - \frac{18c^2m}{\sqrt{\xi}} \qquad\qquad t \longrightarrow t' = \frac{t}{\sqrt{\xi}} \tag{5.9}$$

with $\xi = (1-bc)^2 + 9c^2$. Finally with the condition $b = (1 \pm \sqrt{1-9c^2})/c$ and defining the parameters

$$n = \frac{6cm(1-bc)}{\sqrt{\xi}} \qquad\qquad \mu = \sqrt{m^2 - n^2} \tag{5.10}$$

the metric becomes exactly solution (5.2) with $n$ the NUT parameter.

## 5.2 Charging BBMB black hole

Starting from the same change of variable ($c \to s$) and the same identifications for $f_0$, $\omega_0$, $\omega$, $\chi$, $\gamma$ of the previous section, the Ernst's potentials reduces to $\mathcal{E}_0 = -3\left(1 - \frac{2m}{r}\right)$ and $\mathbf{\Phi}_0 = 0$. Then, the transformation V takes the form

$$\mathcal{E} \to \frac{\mathcal{E}_0}{1 + 3\tilde{\alpha}^2\mathcal{E}_0 - 6\tilde{\alpha}\mathbf{\Phi}_0} = \frac{-3(r-2m)}{r - 9\tilde{\alpha}^2(r-2m)}$$

$$\mathbf{\Phi} \rightarrow \frac{\mathbf{\Phi}_0 - \tilde{\alpha}\mathcal{E}_0}{1 + 3\tilde{\alpha}^2\mathcal{E}_0 - 6\tilde{\alpha}\mathbf{\Phi}_0} = \frac{3\tilde{\alpha}(r - 2m)}{r - 9\tilde{\alpha}^2(r - 2m)}.$$

From the definition of the Ernst's potential $\mathbf{\Phi} = A_t + i\tilde{A}_\varphi$ we can see the appearance of the four potential $\bar{A} = A_t dt$ with $A_t = \mathbf{\Phi}$, since $\mathbf{\Phi}$ is real. At the same time we can evaluate $f$ inverting $\mathcal{E} = \Omega\chi f + 3|\mathbf{\Phi}|^2 + ih$ with $h = 0$, obtaining

$$f = \frac{(r - m)^2}{(r - 9\tilde{\alpha}^2(r - 2m))^2} \tag{5.11}$$

so that the charged solution is

$$ds^2 = -\frac{(r - m)^2}{\eta^2}dt^2 + \frac{\eta^2}{(r - m)^2}dr^2 + \eta^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{5.12}$$

$$\psi = \sqrt{\frac{6}{k}}\frac{m}{r - m}, \qquad \eta = r - 9\tilde{\alpha}^2(r - 2m), \qquad \bar{A} = \frac{3\tilde{\alpha}(r - 2m)}{\eta}dt.$$

Again, we can propose a change of variable to recover the standard BBMB (5.1), in particular

$$r \longrightarrow R = r(1 - 9\tilde{\alpha}^2) + 18m\tilde{\alpha}^2 \qquad t \longrightarrow t' = \frac{t}{1 - 9\tilde{\alpha}^2} \tag{5.13}$$

so that redefining $m$ as $M = m(1 + 9\tilde{\alpha}^2)$ and imposing

$$\tilde{\alpha}^2 = \frac{1}{9}\frac{M - \sqrt{M^2 - e^2}}{M + \sqrt{M^2 - e^2}} \tag{5.14}$$

the solution becomes

$$ds^2 = -\frac{(R - M)^2}{R^2}dt'^2 + \frac{R^2}{(R - M)^2}dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{5.15}$$

$$\psi = \sqrt{\frac{6}{k}}\frac{\sqrt{M^2 - e^2}}{R - M}, \qquad \bar{A} = \left(-\frac{e}{R} + \sqrt{\frac{M - \sqrt{M^2 - e^2}}{M + \sqrt{M^2 - e^2}}}\right)dt'$$

equivalent to the charged BBMB proposed in [9].

# Chapter 6

# New solutions

Now that the symmetries have been verified, we can build some new solutions of the conformal theory. In section 6.1, starting from a "magnetic" version of the LWP ansatz (the result of a double-Wick rotation on LWP), we embed the Reissner–Nordström solution, equipped with a constant scalar parameter $s$ (so called RNS), in a swirling universe by applying the transformation III on it. Although it is a new solution, the result does not surprise us, as the effect of the Ehlers transformation in producing a swirling background is well known and we have shown in chapter 5 that transformation III is equivalent to the Ehlers in classical gravity. Then we can proceed with the unknown symmetries that involves the scalar field. In section 6.2 and 6.3 we apply the transformation VIII respectively to BBMB and RNS solutions.

## 6.1 RNS black hole in a swirling universe

### 6.1.1 The background

Adding the NUT parameter to a given solution is not the only property of the Ehlers transformation, as adding electric charge is not the only effect of the Harrison transformation. Both can lead to different solutions if the starting metric is not the standard LWP but rather the resulting of a discrete symmetry, called double-Wick rotation, acting on that metric. In particular, from the conformal LWP

$$ds^2 = \Omega\left(-f(dt-\omega d\varphi)^2 + f^{-1}\left[\rho^2 d\varphi^2 + e^{2\gamma}(d\rho^2+dz^2)\right]\right)$$

we make the change of variable

$$\begin{cases} t \to i\phi \\ \varphi \to i\tau \end{cases}$$

so that the metric becomes

$$ds_M^2 = \Omega\left(-f(d\phi-\omega d\tau)^2 + f^{-1}\left[\rho^2 d\tau^2 - e^{2\gamma}(d\rho^2+dz^2)\right]\right) \tag{6.1}$$

where we have reabsorbed a minus sign in $f$. Notice that now the Ernst's potential $\mathbf{\Phi}$ is defined as $\mathbf{\Phi} = A_\tau + iA_\phi$. We will refer to (6.1) as the magnetic version of LWP metric, since if we apply Harrison on a seed of this kind with a magnetic monopole, we will embed it in a Melvin magnetic universe. As well, if we apply the III on a magnetic seed we will embed the solution in a swirling universe. But what is this swirling background? A large analysis of it can be found in [19], basically is a spacetime with two rotating sources with opposite angular velocity so that they generates two whirls in the spacetime. In [19] are drawn the geodesics of this metric where we can clearly see this swirling effect. The background metric is

$$ds^2 = F(-dt^2 + dr^2 + r^2 d\theta^2) + F^{-1} r^2 \sin^2\theta (d\varphi + 4jr\cos\theta dt)^2 \tag{6.2}$$

which is a Petrov type D metric and belongs to Kundt class (see [13]).

### 6.1.2 The solution

In the context of solutions of the conformal theory with a constant scalar field $\psi$, it is possible to write a form of the Reissner–Nordström black hole, equipped with a constant scalar hair $s$, as shown in [18]. The solution is

$$ds^2 = -\frac{Q(r)}{r^2}dt^2 + \frac{r^2}{Q(r)}dr^2 + r^2\left(\frac{dx^2}{1-x^2} + (1-x^2)d\varphi^2\right), \tag{6.3}$$

$$\psi = \sqrt{\frac{6}{k}}\sqrt{\frac{s}{s+e^2}}$$

where $Q(r) \equiv r^2 - 2mr + e^2 + s$ and the four-potential is $\bar{A} = -\frac{e}{r}dt$. We have to be careful about the $s$ parameter, indeed since there is no global symmetry associated to $s$, from Noether's theorem we cannot consider it a proper charge. We can map the magnetic LWP in this solution with the change of variables[1] $\{\tau, \rho, z, \phi\} \to \{t, r, x, \varphi\}$

$$\begin{cases} \rho = \sqrt{Q(r)(1-x^2)} \\ z = (r-m)x \end{cases} \qquad \begin{cases} \tau = t \\ \phi = \varphi \end{cases}$$

[1] We report also the differential operators in term of coord. $\{t, r, x, \varphi\}$

$$\bar{\nabla} g(r,x) = (r^2 - 2mr + m^2(1-x^2) + (e^2+s)x^2)^{-\frac{1}{2}}\left[\sqrt{Q(r)}\partial_r g(r,x)\hat{e}_r - \sqrt{1-x^2}\partial_x g(r,x)\hat{e}_x\right]$$

$$\nabla^2 g(r,x) = (r^2 - 2mr + m^2(1-x^2) + (e^2+s)x^2)^{-\frac{1}{2}}\left[\partial_r[\sqrt{Q(r)}\partial_r g(r,x)]\hat{e}_r + (1-x^2)\partial_x^2 g(r,x)\hat{e}_x\right]$$

and defining the fields as

$$\begin{cases} f_0 = -r^2(1-x^2) \\ e^{2\gamma} = \dfrac{r^4(1-x^2)}{r^2-2mr+m^2(1-x^2)+(e^2+s)x^2} \\ A_{\tau_0} = -\dfrac{e}{r} \end{cases} \qquad \begin{cases} \Omega = 1 \\ \omega = 0 \\ \chi = \alpha = -3\dfrac{e^2}{e^2+s}. \end{cases}$$

By applying the III, this solution can be embedded in the background of the previous section. So the first step is find Ernst's potentials. In term of our differential operators the relation

$$\rho^{-1}\hat{e}_\varphi \times \nabla\tilde{A}_\tau = \rho^{-2} f(\nabla A_\tau + \omega\nabla A_\phi) \tag{6.4}$$

becomes

$$\begin{cases} \partial_r\tilde{A}_\tau = -\dfrac{f}{Q(r)}(\partial_x\tilde{A}_\tau + \omega\partial_x A_\phi) \\ \partial_x\tilde{A}_\tau = \dfrac{f}{1-x^2}(\partial_r\tilde{A}_\tau + \omega\partial_r A_\phi) \end{cases}$$

that leads to $\tilde{A}_{\tau_0} = -ex$ and so $\mathbf{\Phi}_0 = -iex$, $\mathcal{E}_0 = \alpha f_0 + 3|\mathbf{\Phi}_0|^2 = -\alpha r^2(1-x^2) + 3e^2x^2$. Now we can apply the Ehlers transformation[2]

$$\mathcal{E}_0 \longrightarrow \mathcal{E} = \frac{\mathcal{E}_0}{1+ij\mathcal{E}_0} = \frac{3e^2x^2 - \alpha r^2(1-x^2)}{\Lambda(r,x)} + i\underbrace{\frac{-j(3e^2x^2-\alpha r^2(1-x^2))^2}{\Lambda(r,x)}}_{h} \tag{6.5}$$

$$\mathbf{\Phi}_0 \longrightarrow \mathbf{\Phi} = \frac{\mathbf{\Phi}_0}{1+ij\mathcal{E}_0} = \underbrace{-jex\frac{3e^2x^2-\alpha r^2(1-x^2)}{\Lambda(r,x)}}_{A_\phi} + i\underbrace{\frac{-ex}{\Lambda(r,x)}}_{\tilde{A}_\tau} \tag{6.6}$$

where we have defined $\Lambda(r,x) = 1 + j^2\mathcal{E}_0^2$. For the magnetic case as well, the Ehlers transformation adds a rotational term thanks to the presence of the field $h$. We can evaluate $\omega$ from the relation

$$\nabla h = \alpha f^2\rho^{-2}\hat{e}_\varphi \times \nabla\omega + 6\mathrm{Im}(\mathbf{\Phi} * \nabla\mathbf{\Phi}) \tag{6.7}$$

that becomes

$$\begin{cases} \partial_r h = -\dfrac{\alpha f^2}{Q(r)}\partial_x\omega + 6(A_\phi\partial_r\tilde{A}_\tau - \tilde{A}_\tau\partial_r A_\phi) \\ \partial_x h = \dfrac{\alpha f^2}{1-x^2}\partial_r\omega + 6(A_\phi\partial_x\tilde{A}_\tau - \tilde{A}_\tau\partial_x A_\phi). \end{cases}$$

[2]For the magnetic case we have renamed the transformation parameter $c \to j$

The final step is recover the potential $A_\tau$ inverting relation (6.4), so that we are left with the RNS black hole embedded in a swirling universe

$$ds^2 = \Lambda(r,x)\left[-\frac{Q(r)}{r^2}d\tau^2 + \frac{r^2}{Q(r)}dr^2 + \frac{r^2}{1-x^2}dx^2\right] + \frac{r^2(1-x^2)}{\Lambda(r,x)}(d\phi - \omega d\tau)^2, \quad (6.8)$$

$$\psi = \sqrt{\frac{6}{k}}\sqrt{\frac{s}{s+e^2}} \qquad \text{and} \qquad \bar{A} = A_\tau d\tau + A_\phi d\phi$$

where

$$\Lambda(r,x) = 1 + 9j^2e^4\left[\frac{r^2}{s+e^2}(1-x^2) + x^2\right]^2, \quad (6.9)$$

$$\omega = -12j\frac{e^2}{s+e^2}\frac{Q(r)}{r}x, \quad (6.10)$$

$$A_\phi = -3je^3\frac{x^2 + \frac{r^2}{s+e^2}(1-x^2)}{\Lambda(r,x)}x, \qquad A_\tau = \frac{e}{r} - \omega A_\phi. \quad (6.11)$$

Clearly when $j = 0$ we recover the seed solution. The spherical symmetry is lost due to the background distorting the spacetime. Since the metric (6.8) is of the same form of Schwarzschild presented in [19], we expect a similar deformation of the event horizon. The solution (6.8) is a Type I metric in Petrov classification. We can conclude this section with an interesting observation, if we define a new parameter $3\tilde{j}/e^2 = j$ and then set to zero all the energy density in solution (6.8) ($e \to 0$, $m \to 0$) we remain with the metric (6.2) plus a correction term $s/r^2$ because of the the scalar parameter

$$ds^2 = F(-\xi dt^2 + \xi dr^2 + r^2 d\theta^2) + F^{-1}r^2\sin^2\theta\left(d\varphi + 4\tilde{j}\frac{\xi}{s}r\cos\theta dt\right)^2, \quad (6.12)$$

$$\psi = \sqrt{\frac{6}{k}} \qquad \text{and} \qquad \xi = 1 + \frac{s}{r^2}.$$

For this last metric the Kretschmann scalar $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ diverges in $r = 0$, so we can consider this last solution as a scalar monopole embedded in a swirling universe. Finally we can recover the standard swirling background with the substitution $s\tilde{\tilde{j}} = \tilde{j}$ and then setting $s \to 0$.

## 6.2 Applying transformation VIII to BBMB black hole

Here we apply the transformation VIII of section 4.1.2 to BBMB black hole and we show that the result is not a trivial solution or a diffeomorphism of the metric (5.1). Starting from the conformal LWP ansatz we can retrieve BBMB metric with the change of coordinates ($c \to s$) of section 5.1 and and with $f = \frac{(r-m)^2}{r^2}$, $\chi = 3r\frac{2m-r}{(r-m)^2}$, $\Omega = 1$, $\gamma = \omega = 0$. Now the transformation VIII reads as

$$\chi \to \chi' = 3\left(\tanh^2\left[\frac{l}{2}\log\left[3r\frac{2m-r}{r-m}\sin\theta\right] \pm \operatorname{arctanh}\left[\frac{m}{r-m}\right]\right] - 1\right),$$

$$\gamma \to \gamma' = 3l\left(\frac{l}{4}\log\left[3r\frac{2m-r}{r-m}\sin\theta\right] \pm \operatorname{arctanh}\left[\frac{m}{r-m}\right]\right)$$

and since $\alpha \to \alpha' = \alpha$ we have to impose $\Omega' = \chi/\chi'$ so that the final solution is

$$ds^2 = \Omega'\left[-\frac{(r-m)^2}{r^2}dt^2 + r^2\left(d\theta^2 + \frac{dr^2}{(r-m)^2}\right)e^{6l(\xi-\eta)} + r^2\sin^2\theta d\varphi^2\right], \quad (6.13)$$

$$\psi = \sqrt{\frac{6}{k}}\sqrt{1-\operatorname{sech}^2\xi}$$

where

$$\xi = 2\eta \pm \operatorname{arctanh}\left[\frac{m}{r-m}\right], \quad \eta = \frac{l}{4}\log\left[3r\frac{2m-r}{r-m}\sin\theta\right], \quad \Omega' = \frac{r(2m-r)}{(\tanh^2\xi - 1)(r-m)^2}$$

with $l \in \mathbb{R}$. To be sure that solution (6.13) is not a gauge transformation of BBMB metric, we can control the Petrov Type (see [13]) of both. The Petrov classification is based on algebraic symmetries of the Weyl tensor, if BBMB and (6.13) solutions belong to different Petrov types then they can not be diffeomorphic. It is known that BBMB is a Type D metric so let's check this to introduce the method. The first step is choosing a null Newman-Penrose tetrad, a set of four null vectors ($\boldsymbol{k}$,$\boldsymbol{l}$,$\boldsymbol{m}$,$\bar{\boldsymbol{m}}$) such that

- $k^\mu l_\mu = -1$ and $m^\mu \bar{m}_\mu = 1$
- $k^\mu m_\mu = k^\mu \bar{m}_\mu = l^\mu m_\mu = l^\mu \bar{m}_\mu = 0$.

In general, for axially symmetric and stationary spacetimes, it can be choose the tetrad as

$$\boldsymbol{k} = \frac{1}{\sqrt{2}}\left(-g_{tt}^{-1}\partial_t + \partial_r\right),$$

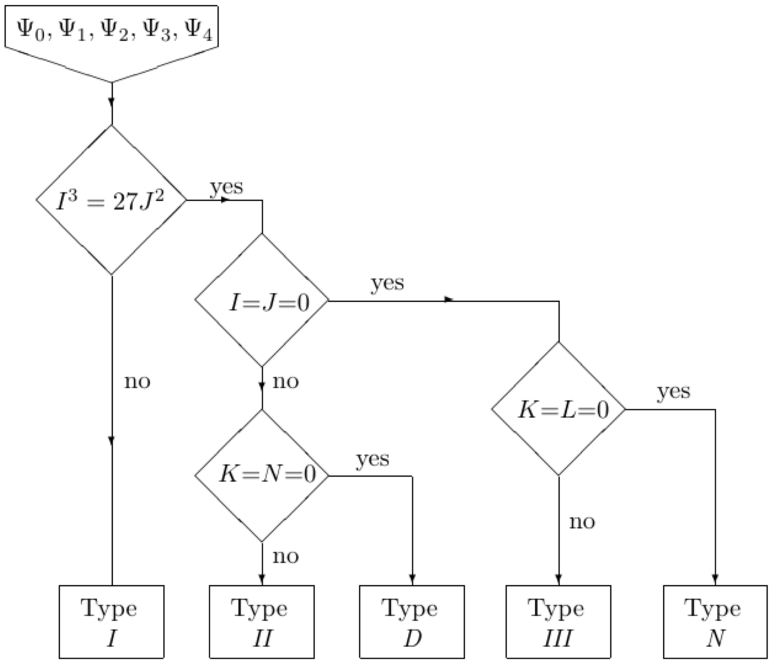


Figure 6.1: Petrov classification based on scalars $\boldsymbol{\psi_{0-4}}$

$$
\begin{aligned}
\boldsymbol{l} &= \frac{1}{\sqrt{2}}\left(\partial_t + g_{tt}\partial_r\right), \qquad (6.14)\\
\boldsymbol{m} &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{g_{\theta\theta}}}\partial_\theta + \frac{i}{\sqrt{g_{\varphi\varphi}}}\partial_\varphi\right),\\
\bar{\boldsymbol{m}} &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{g_{\theta\theta}}}\partial_\theta - \frac{i}{\sqrt{g_{\varphi\varphi}}}\partial_\varphi\right).
\end{aligned}
$$

From that and with the Weyl tensor $C_{\mu\nu\sigma\rho}$ we can construct five scalars ($\boldsymbol{\psi_0}$,$\boldsymbol{\psi_1}$,$\boldsymbol{\psi_2}$,$\boldsymbol{\psi_3}$,$\boldsymbol{\psi_4}$)

$$
\begin{aligned}
\boldsymbol{\psi_0} &= C_{\mu\nu\sigma\rho}k^\mu m^\nu k^\sigma m^\rho,\\
\boldsymbol{\psi_1} &= C_{\mu\nu\sigma\rho}k^\mu l^\nu k^\sigma m^\rho,\\
\boldsymbol{\psi_2} &= C_{\mu\nu\sigma\rho}k^\mu m^\nu \bar{m}^\sigma l^\rho, \qquad (6.15)\\
\boldsymbol{\psi_3} &= C_{\mu\nu\sigma\rho}l^\mu k^\nu l^\sigma \bar{m}^\rho,\\
\boldsymbol{\psi_4} &= C_{\mu\nu\sigma\rho}l^\mu \bar{m}^\nu l^\sigma \bar{m}^\rho.
\end{aligned}
$$

Finally we define some new quantities as

$$
I = \boldsymbol{\psi_0}\boldsymbol{\psi_4} - 4\boldsymbol{\psi_1}\boldsymbol{\psi_3} + 3\boldsymbol{\psi_2}^2, \qquad J = \det\begin{pmatrix} \boldsymbol{\psi_0} & \boldsymbol{\psi_1} & \boldsymbol{\psi_2}\\ \boldsymbol{\psi_1} & \boldsymbol{\psi_2} & \boldsymbol{\psi_3}\\ \boldsymbol{\psi_2} & \boldsymbol{\psi_3} & \boldsymbol{\psi_4} \end{pmatrix} \qquad (6.16)
$$

$$K = \boldsymbol{\psi_1}\boldsymbol{\psi_4^2} - 3\boldsymbol{\psi_4}\boldsymbol{\psi_3}\boldsymbol{\psi_2} + 2\boldsymbol{\psi_3^3}, \quad L = \boldsymbol{\psi_2}\boldsymbol{\psi_4} - \boldsymbol{\psi_3^2}, \quad N = 12L^2 - \boldsymbol{\psi_4^2}I. \tag{6.17}$$

To label a metric with Petrov classification we can easily follow the scheme in figure 6.1 based on quantities (6.16)-(6.17). If we take the BBMB solution with the tetrad (6.14) this leads to $\boldsymbol{\psi_1} = \boldsymbol{\psi_3} = 0$, $\boldsymbol{\psi_0} = \boldsymbol{\psi_4} = 3\boldsymbol{\psi_2}$ and $\boldsymbol{\psi_2} = \frac{m(m-r)}{2r^4}$, so looking at scheme 6.1, BBMB is clearly a Type D solution. Contrary, if we consider solution (6.13) with the same tetrad, after a bit of calculation to evaluate the scalars, we can easily see that $I^3 - 27J^2 \neq 0$, so we have a Type I solution.

## 6.3 Applying transformation VIII to RNS black hole

We can repeat the same procedure for the RNS solution with little effort. The result will be a new solution with constant scalar field since, as seen in the previous section, this transformation is non trivial. The RNS solution in spherical coordinates appear as

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{s+e^2}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{s+e^2}{r^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{6.18}$$
$$\psi = \sqrt{\frac{6}{k}}\sqrt{\frac{s}{s+e^2}}$$

with $\bar{A} = -\frac{e}{r}dt$. We can map the conformal LWP ansatz into (6.18) with the change of coordinates

$$\begin{cases} \rho = \sqrt{Q(r)}\sin\theta \\ z = (r-m)\cos\theta \\ \varphi = \varphi \end{cases}$$

where $Q(r) \equiv r^2 - 2mr + e^2 + s$, so that

$$d\rho^2 + dz^2 = \left(r^2 - 2mr + m^2\sin^2\theta + (e^2+s)\cos^2\theta\right)\left(\frac{dr^2}{Q(r)} + d\theta^2\right)$$

and with the identifications

$$\begin{cases} f = 1 - \dfrac{2m}{r} + \dfrac{e^2+s}{r^2} \\ e^{2\gamma} = \dfrac{Q(r)}{r^2 - 2mr + m^2\sin^2\theta + (e^2+s)\cos^2\theta} \\ Q(r) = r^2 - 2mr + e^2 + s \end{cases} \qquad \begin{cases} \omega = 0 \\ \Omega = 1 \\ \chi = \dfrac{-3e^2}{s+e^2}. \end{cases}$$

From this, the transformation VIII becomes

$$\chi \to \chi' = 3\left(\tanh^2\left[\frac{l}{2}\log\left[\frac{-3e^2\sqrt{Q(r)}}{s+e^2}\sin\theta\right] \pm \operatorname{arctanh}\left[\sqrt{\frac{s}{s+e^2}}\right]\right] - 1\right),$$

$$\gamma \to \gamma' = \gamma + 3l\left(\frac{l}{4}\log\left[\frac{-3e^2\sqrt{Q(r)}}{s+e^2}\sin\theta\right] \pm \operatorname{arctanh}\left[\sqrt{\frac{s}{s+e^2}}\right]\right).$$

In this case as well, we have to impose $\Omega' = \chi/\chi'$. Finally the transformed solution reads as

$$ds^2 = \Omega'\left[-\frac{Q(r)}{r^2}dt^2 + r^2\left(\frac{dr^2}{Q(r)} + d\theta^2\right)e^{6l(\xi-\eta)} + r^2\sin^2\theta d\varphi^2\right],$$

$$\psi = \sqrt{\frac{6}{k}}\sqrt{1-\operatorname{sech}^2\xi} \tag{6.19}$$

where

$$\xi = 2\eta \pm \operatorname{arctanh}\left[\sqrt{\frac{s}{s+e^2}}\right],$$

$$\eta = \frac{l}{4}\log\left[\frac{-3e^2\sqrt{Q(r)}}{s+e^2}\sin\theta\right],$$

$$\Omega' = \frac{-e^2}{(\tanh^2\xi - 1)(s+e^2)}$$

with $l \in \mathbb{R}$ and the electromagnetic potential that remains unchanged $\bar{A} = -\frac{e}{r}dt$. Using the same Newman-Penrose tetrad from the previous section, this solution turns out to be a Petrov Type I.

# Chapter 7

# Conclusions

## 7.1 Conclusions

In this thesis we have examined how a scalar-tensor theory significantly modifies the equations of motion of classical GR, as highlighted in Chapter 2. This led us to employ the Ernst's technique in the conformal frame, where the scalar field is naturally incorporated into the definition of $\mathcal{E}$, but not in the one of $\mathbf{\Phi}$. This suggests that the scalar field does not have significant couplings with the electromagnetic field.

We then explored the Lie point symmetries, observing that in the uncharged case, the number of symmetries triples, increasing from three to nine, with respect to GR. All symmetries present in the classical theory are also present in the conformal frame. At least four of the nine transformations are gauge symmetries. However, the other ones, specifically III, VI and VIII are clearly non-trivial, as they lead to solutions with different Petrov types.

The introduction of the electromagnetic field, as expected, did not produce any surprising new effects, as it involved the introduction of a gauge symmetry and transformation V (a conformal frame version of the Harrison transformation). This further confirmed that the scalar field does not have any particular coupling with the electromagnetic field.

Among the well-known transformations in classical GR, the most significant are those of Ehlers and Harrison, which allow respectively the addition of NUT/electric charge from a standard seed solution, or the immersion into a swirling/Melvin universe from a magnetic seed. As demonstrated in Chapter 5, these properties are shared by transformations III and V in the conformal frame, then we can identify them as a conformal versions of the Ehlers and Harrison transformations.

Building on this identification, we constructed a new solution by immersing a Reissner–Nordström black hole with a scalar parameter into a swirling universe. This approach also revealed another interesting aspect: by turning off all energy densities in the solution, we find a solution that represents a scalar monople embedded in the swirling background. This leads us to propose an interesting future avenue, which involves taking a solution in the conformal frame embedded in Melvin universe and turning off the energy densities to observe how the background is modified. This operation, however, must be performed with a seed different from the BBMB solution, where turning off the mass causes the scalar field to vanish.

The transformation VI corresponds to the conformal version of the symmetry in the MC frame $\hat{\psi} \to \hat{\psi} + d$. In appendix B we confirm it is able to generate a traversable wormhole from the BBMB solution.

In conclusion, about the new non-trivial transformations founded, we cannot say much about them. In Chapter 6, we derived two new solutions based on transformations VIII applied to the BBMB and RNS as seed solutions. However, beyond the classification according to their Petrov type, we have not yet conducted a thorough analysis of their physical implications. Our hope is that, in the future, these new transformations can be physical interpreted, as done for the Ehlers and Harrison transformations, and that they may offer insight about the solutions 6.13 and 6.19. Transformation IX and XI, which appears to be particularly complex, have not yet been analyzed due to the difficulty of them application and remains a subject for future investigation. In addition, we are uncertain about the transformation involving the coordinate $\rho$. The treatement through the symmetries of the effective action might be too naive. It would be better to further investigate those symmetries with the help of the prolongation technique.

In summary, we have laid the groundwork for a deeper understanding of symmetries and solutions in the conformal frame, but much work remains to be done, particularly in assigning a physical interpretation, if there is one, to the new transformations and solutions presented.

# Appendix A

# Ernst's technique in CC frame with a different ansatz

Here we repeat the procedure to derive Ernst's method, and the correlated effective action, starting from a completely different anstaz with respect to the conformal LWP. In this case as well, the new metric must preserve the properties of stationarity and axial symmetry of the spacetime.

## A.1 The ansatz

This ansatz is the same used in [20] to charging Ernst's solution generating technique with cosmological constant and in [21] to produce stationary and axisymmetric solution of Brans-Dicke theory. The ansatz is

$$ds^2 = -\alpha e^{\Omega/2}(dt + \omega d\varphi)^2 + \alpha e^{-\Omega/2} d\varphi^2 + \frac{e^{2\nu}}{\sqrt{\alpha}}(d\rho^2 + dz^2) \tag{A.1}$$

with, thanks to the symmetry conditions, all function depending only on $\rho, z$.

### A.1.1 Deriving Ernst's equation

It is possible to describe the five unknown fields, $\alpha, \Omega, \nu, \omega, \psi$, by four Einstein equations plus a Klein-Gordon equation in term of the ansatz (A.1)

$$\nabla \cdot \left(\alpha \chi e^{\Omega} \nabla \omega\right) = 0 \tag{$EE^{\varphi}_{t}$}$$

$$\nabla \cdot (e^{\Omega} \alpha \chi \omega \nabla \omega) + \frac{1}{2} \nabla \cdot (\alpha \chi \nabla \Omega) = 0 \tag{$EE^{t}_{t} - EE^{\varphi}_{\varphi}$}$$

$$\nabla^2 (\alpha \chi) = 0 \tag{$EE^{\rho}_{\rho} + EE^{z}_{z}$}$$

$$\frac{1}{2} \alpha \chi e^{\Omega} \nabla \omega^2 - \frac{1}{8} \alpha \chi \nabla \Omega^2 - 2 \alpha \chi \nabla^2 \nu + \frac{3}{2} \alpha \frac{\nabla \chi^2}{\chi + 3} - \frac{3}{2} \alpha \nabla^2 \chi = 0 \tag{$EE^{t}_{t} + EE^{\varphi}_{\varphi}$}$$

$$6 \nabla \cdot (\alpha \nabla \psi) + \frac{3}{2} \psi \nabla^2 \alpha = \frac{1}{2} \alpha \psi e^{\Omega} \nabla \omega^2 - 2 \alpha \psi \nabla^2 \nu - \frac{1}{8} \alpha \psi \nabla \Omega^2 \tag{K.G.}$$

where has been defined $\chi \equiv 4\pi\psi^2 - 3$ and we are using $\nabla^2 \equiv \partial_\rho^2 + \partial_z^2$. As done before we want to rewrite this five equation with Ernst potentials and then find the associated effective action. Again the equation $EE_t^\varphi$ is an integrability condition, in particular can be used to defined a new field $h$ such that

$$\hat{e}_\varphi \times \nabla h = \alpha\chi e^{\Omega}\nabla\omega \tag{A.2}$$

that satisfies

$$\nabla \cdot \left[(\alpha\chi e^{\Omega})^{-1}\nabla h\right] = 0 \tag{A.3}$$

equivalent to $EE_t^\varphi$, and taking the square of (A.2)

$$\nabla\omega^2 = (\alpha\chi e^{\Omega})^{-2}\nabla h^2. \tag{A.4}$$

With this last equation and defining $f \equiv \alpha\chi e^{\Omega/2}$ it is possible to rewrite $EE_t^t - EE_\varphi^\varphi$ as

$$f(\alpha\chi)^{-1}\nabla \cdot (\alpha\chi\nabla f) = \nabla f\nabla f - \nabla h^2 \tag{A.5}$$

where has been neglected the $\nabla^2(\alpha\chi)$ term thanks to $EE_\rho^\rho + EE_z^z$. Now it's natural to define a complex potential $\mathcal{E} \equiv f + ih$, named Ernst potential, so that equations (A.5) and (A.3) can be viewed as real and imaginary part of a single complex equation, named Ernst equation

$$\mathrm{Re}\mathcal{E}\tilde{\alpha}^{-1}\nabla^2(\tilde{\alpha}\nabla\mathcal{E}) = \nabla\mathcal{E}\nabla\mathcal{E} \tag{A.6}$$

where $\tilde{\alpha} \equiv \alpha\chi$. Once Ernst potential has been defined and making the substitutions $\alpha\chi \to \tilde{\alpha}$, $\nabla\omega^2 \to \nabla h^2$, $\Omega \to 2\log(f/\tilde{\alpha})$, the $EE_\rho^\rho + EE_z^z$, $EE_t^t + EE_\varphi^\varphi$ and K.G. equations takes the form

$$2\tilde{\alpha}\nabla^2\nu = \frac{3\tilde{\alpha}}{2\chi}\left(\frac{\nabla\chi^2}{\chi+3} - \nabla^2\chi\right) - \frac{\nabla\tilde{\alpha}^2}{2\tilde{\alpha}} - \tilde{\alpha}\frac{\nabla\mathcal{E}^2 + \nabla\mathcal{E}^{*2}}{(\mathcal{E}+\mathcal{E}^*)^2} + \frac{\nabla\mathcal{E} + \nabla\mathcal{E}^*}{\mathcal{E}+\mathcal{E}^*}\nabla\tilde{\alpha} \tag{A.7}$$

$$\nabla^2\tilde{\alpha} = 0 \tag{A.8}$$

$$6\nabla(\tilde{\alpha}\nabla\chi) - \frac{9\tilde{\alpha}(\chi+2)}{\chi(\chi+3)}\nabla\chi^2 = 0. \tag{A.9}$$

### A.1.2 The effective action

Finally is straightforward to demonstrate that the four equation of motion (A.6)-(A.9) can be derived from the following effective action

$$S = \int d\rho dz d\varphi \frac{3\tilde{\alpha}}{\chi^2(\chi+3)} \nabla\chi^2 - \chi^{-1}\nabla\tilde{\alpha}\nabla\chi + \frac{1}{3}\left[4\tilde{\alpha}\frac{\nabla\mathcal{E}\nabla\mathcal{E}^*}{(\mathcal{E}+\mathcal{E}^*)^2} + \frac{\nabla\tilde{\alpha}^2}{\tilde{\alpha}} - 2\frac{\nabla\mathcal{E}+\nabla\mathcal{E}^*}{\mathcal{E}+\mathcal{E}^*}\nabla\tilde{\alpha} - 4\nabla\tilde{\alpha}\nabla\nu\right] \tag{A.10}$$

more precisely

$$\begin{aligned}
\nabla\frac{\delta\mathcal{L}}{\delta\nabla\mathcal{E}} - \frac{\delta\mathcal{L}}{\delta\mathcal{E}} &= 0 \longrightarrow (A.6)\\
\nabla\frac{\delta\mathcal{L}}{\delta\nabla\chi} - \frac{\delta\mathcal{L}}{\delta\chi} &= 0 \longrightarrow (A.9)\\
\nabla\frac{\delta\mathcal{L}}{\delta\nabla\nu} - \frac{\delta\mathcal{L}}{\delta\nu} &= 0 \longrightarrow (A.8)\\
\nabla\frac{\delta\mathcal{L}}{\delta\nabla\tilde{\alpha}} - \frac{\delta\mathcal{L}}{\delta\tilde{\alpha}} &= 0 \longrightarrow (A.7).
\end{aligned}$$

# Appendix B

# Recover a traversable wormhole

Another interesting test that we can do on the solution generating technique, is applying transformation VI to BBMB black hole. As mentioned, this symmetry is the shift transformation on the scalar field, $\hat{\psi} \to \hat{\psi} + a$, of MC frame mapped by Bekenstein's map (2.16). In [22] this symmetry is applied at BBMB black hole, to obtain the solution founded in [23], interpreted as a traversable wormhole. Here we recover the same result by applying directly transformation VI of the CC frame.

## B.1 The transformation

Since the seed solution is the BBMB black hole (5.1), we start again with the change of coordinates ($c \to s$) of section 5.1 and with the same definitions for $f$, $\omega$, $\Omega$, $\gamma$ and in particular $\chi$ that is

$$\chi = 4\pi\psi^2 - 3 = 3r\frac{2m-r}{(r-m)^2}$$

so that the transformation VI, that involves only the scalar field, becomes

$$\chi \to \chi' = 3\left(\tanh^2\left[a \pm \operatorname{arctanh}\left[\frac{m}{r-m}\right]\right] - 1\right)$$

or equivalently

$$\psi \to \psi' = \sqrt{\frac{6}{k}}\tanh\left[a \pm \operatorname{arctanh}\left[\frac{m}{r-m}\right]\right] \tag{B.1}$$

with $a \in \mathbb{R}$. Now defining a new parameter $s$ such that $a = \log[\sqrt{s}]$ and recalling the property of the function tanh

$$\tanh[x+y] = \frac{\tanh[x] + \tanh[y]}{1 + \tanh[x]\tanh[y]}$$

we can rewrite $\psi'$ as

$$\psi' = \sqrt{\frac{6}{k}}\frac{(r-m)(s-1) \pm m(s+1))}{(r-m)(s+1) \pm m(s-1))}. \tag{B.2}$$

The choice of the shift's sign in transformation VI identify two solutions, one with a scalar field $\psi'_+$ and the other with $\psi'_-$

$$\psi'_+ = \sqrt{\frac{6}{k}}\frac{r(s-1)-2ms}{r(s+1)-2ms} \tag{B.3}$$

$$\psi'_- = \sqrt{\frac{6}{k}}\frac{r(s-1)+2m}{r(s+1)-2m} \tag{B.4}$$

## B.2 The solutions

Remembering that $\Omega = 1$ for BBMB metric and $\alpha = \alpha'$ $(\Omega'\chi' = \Omega\chi)$ in transformation VI we have to impose

$$\Omega' = \frac{\chi}{\chi'} = \frac{1-\frac{k}{6}\psi^2}{1-\frac{k}{6}\psi'^2}.$$

Finally we remain with two transformed versions of the Bekenstein's solutions, the first

$$ds^2_+ = \frac{(r(s+1)-2ms)^2}{4s(r-m)^2}ds^2_0 \qquad \psi'_+ = \sqrt{\frac{6}{k}}\frac{r(s-1)-2ms}{r(s+1)-2ms} \tag{B.5}$$

that is exactly the traversable wormhole in the form proposed in [22] and the second

$$ds^2_- = \frac{(r(s+1)-2m)^2}{4s(r-m)^2}ds^2_0 \qquad \psi'_- = \sqrt{\frac{6}{k}}\frac{r(s-1)+2m}{r(s+1)-2m} \tag{B.6}$$

Obviously in both cases, we can recover the standard BBMB solution in the limit $s \to 1$.

# Appendix C

# BBMB black hole in a swirling universe

For the seek of completeness, in this appendix we show that even for magnetic seeds, transformation III behaves exactly as Ehlers transformation of classical GR. In particular we recover the BBMB black hole in the swirling background of section 6.1.1.

## C.1 BBMB in swirling universe

Let's report, for convenience, the BBMB solution (5.1)

$$ds^2 = -\left(1 - \frac{m}{r}\right)^2 dt^2 + \left(1 - \frac{m}{r}\right)^{-2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

$$\psi = \sqrt{\frac{6}{k}}\frac{m}{r-m}$$

We can map the magnetic LWP (6.1) in this solution by the extremal[1] change of coordinates ($c \to s$) of section (5.1) and with the identifications

$$\begin{cases} f_0 = -r^2\sin^2\theta \\ \Omega = 1 \\ \omega_0 = 0 \end{cases} \qquad \begin{cases} e^{2\gamma} = \dfrac{r^4\sin^\theta}{(r-m)^2} \\ \chi = 3r\dfrac{2m-r}{(r-m)^2}. \end{cases}$$

This times the Ernst's potentials are $\mathcal{E}_0 = \alpha f_0 = \Omega\chi f_0$ and $\mathbf{\Phi}_0 = 0$, so that transformation III becomes

$$\mathcal{E}_0 \longrightarrow \mathcal{E} = \frac{\mathcal{E}_0}{1 + ij\mathcal{E}_0} = \underbrace{\frac{-3r^3(2m-r)\sin^2\theta}{(r-m)^2\Lambda(r,\theta)}}_{\tilde{f}} + i\underbrace{\frac{-j9r^6(2m-r)^2\sin^2\theta}{(r-m)^4\Lambda(r,\theta)}}_{h} \tag{C.1}$$

[1] Extremal because this change of coordinates it's the same used for RNS solution in the extremal case of $e^2 + s = m^2$

with $\Lambda(r,\theta) = 1 + j^2\mathcal{E}_0^2$ and $\mathbf{\Phi} = \mathbf{\Phi}_0 = 0$. Again Ehlers transformation adds a rotational term $\omega$ that we can evaluate from the definition of $h$ (6.7). In this coordinates system it becomes

$$\begin{cases} \partial_r\omega = -\sin\theta\alpha^{-1}f^{-2}\partial_\theta h \\ \\ \partial_\theta\omega = (r-m)^2\sin\theta\alpha^{-1}f^{-2}\partial_r h. \end{cases}$$

Combining everything, the BBMB solution in swirling universe is

$$ds^2 = \Lambda(r,\theta)\left[-\frac{(r-m)^2}{r^2}d\tau^2 + \frac{r^2}{(r-m)^2}dr^2 + r^2 d\theta^2\right] + \frac{r^2\sin^2\theta}{\Lambda(r,\theta)}\left(d\phi - \omega d\tau\right)^2,$$

$$\psi = \sqrt{\frac{6}{k}}\frac{m}{r-m} \tag{C.2}$$

where

$$\Lambda(r,\theta) = 1 + 9j^2r^6\frac{(2m-r)^2}{(r-m)^4}\sin^4\theta, \qquad \omega = -\frac{12j}{r-m}(3m^2 - 3mr + r^2)\cos\theta.$$

Notice that setting to zero the energy density ($m \to 0$) we recover the swirling background (6.2) instead of the conformal version (6.12), because in the BBMB solution without the mass the scalar field vanishes.

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