Geometric Invariant Theory
and Nonabelian Hodge Correspondence
Reading Seminar

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Seminar Outline

The nonabelian Hodge correspondence is a deep and beautiful theory that identifies three moduli spaces: the **Betti moduli space** $\mathcal{M}_{\mathrm{B}}$ of surface group representations, the **de Rham moduli space** $\mathcal{M}_{\mathrm{dR}}$ of flat connections and the **Dolbeault moduli space** $\mathcal{M}_{\mathrm{Dol}}$ of Higgs bundles, whose objects on the surface appear to arise from very different contexts. The moduli spaces themselves are constructed as GIT quotients based on the **geometric invariant theory** pioneered by Mumford. The correspondence between these moduli spaces is the cumulative work of Corlette, Donaldson, Hitchin and Simpson built on that of many mathematicians and the machinery involved to prove these results lies in the intersection of algebraic geometry, complex geometry, geometric analysis and gauge theory.

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{B}} \arrow[r, "\cong"] & \mathcal{M}_{\mathrm{dR}} \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & \mathcal{M}_{\mathrm{Dol}} \\
& & \{\text{Harmonic bundles}\} \arrow[ur, "\cong"'] &
\end{tikzcd}$$

The goal of this reading seminar is two-fold:

- Understand the workings of geometric invariant theory and how they are applied in the construction of the moduli spaces involved in the nonabelian Hodge correspondence.
- Understand the geometric properties of objects in the moduli spaces and without going into too much nasty detail, sketch the correspondence between the moduli spaces for $G = \mathrm{GL}(n, \mathbb{C})$.

This topic lies in the intersection of many aspects of mathematics and may yield something useful for everyone in spite of different background and interests. For those more geometrically minded who are working with the moduli spaces, we provide the algebro-geometric foundation of their construction, while for those more algebraically minded, these are excellent examples of GIT and testing ground for your understanding. We aim to divide the topics in a way that strikes a balance between algebra and geometry, thereby bringing forth interaction and collaboration.

Format and Schedule

For our first introductory meeting, we meet on 16.11. at 11:00 in the common area of the library. There we will decide a most convenient time and place to meet.

- Place: A3 02
- Time: On the listed dates (usually Thursdays) 11:00–12:30
- Organisers: Please email us for questions and comments!
  - Enya Hsiao: `enya.hsiao@mis.mpg.de`
  - Leonie Kayser: `leo.kayser@mis.mpg.de`

Each week, a participant will give a 45-60 minute talk summarizing our reading and provide additional information (details in the list of talks below). We will then spend 30 minutes to discuss one problem/exercise/example chosen by the speaker. Depending on whether participants find it useful, we can hold discussion sessions roughly every four meetings to clear up any accumulated confusion.

Tentative dates

The following is a possible plan of the seminar assuming that we hold regular discussion sessions and that we meet weekly on Thursdays with minor modifications.

Date	Topic	Reading
16.11.	0. Introduction and Overview	This document
23.11.	1. Affine and projective varieties	[Gat21, 1–7]
30.11.	2. Algebraic groups and invariant theory	[Hos15, 3.1–3.2, 4.1–4.4]
07.12.	3. Affine GIT	[Hos15, 3.3–3.5, 4.5–4.6]
14.12.	*Discussion: Gauge Theory*	
11.01.	4. The Betti Moduli space: Character variety	[Tho23, 2]
18.01.	5. The de Rham Moduli space: Flat connections	[Tho23, 3, 4, 5]
25.01.	6. The Riemann-Hilbert correspondence	[Tho23, 3]
01.02.	*Discussion: Ample line bundles*	
08.02.	7. Projective GIT	[Hos15, 5.1–5.2, 6.1–6.2]
15.02.	8. Hilbert-Mumford Criterion	[Hos15, 5.3–5.5, 6.3–6.4]
22.02.	9. The moduli problem	[Hos15, 2, 3.6]
28.02.	*Discussion: Complex geometry*	
07.03.	10. The Dolbeault Moduli space: Higgs bundles	[Tho23, 7, 8]
14.03.	11. Harmonic bundles and Donaldson-Corlette correspondence	[Tho23, 9]
21.03.	12. The Hitchin-Simpson correspondence	[Tho23, 9]
28.03.	*Discussion*	

List of Talks

The schedule above is built upon the following list of preliminary talks, interweaving the two main topics.

Topic A: Geometric Invariant Theory

For geometric invariant theory, we will have six talks following the set of lecture notes by Victoria Hoskins [Hos15]. Exercise problems complementary to her lectures can be found on her website: `https://www.math.ru.nl/~vhoskins/moduli_and_GIT.html`. For the relevant background on Algebraic Geometry we recommend Andreas Gathmann's lecture notes [Gat21], available at `https://agag-gathmann.math.rptu.de/de/alggeom.php`. Additional resources for classical algebraic geometry are Klaus Hulek's *Elementary Algebraic Geometry* and Igor R. Shafarevich's *Basic Algebraic Geometry 1*. For affine algebraic groups: James E. Humphreys' *Linear Algebraic Groups*, Armand Borel's *Linear Algebraic Groups* and James S. Milne's *Algebraic Groups* (the latter assumes familiarity with scheme theory).

1. Affine and projective varieties (Leonie)

This is a very condensed intro to the theory of quasi-projective varieties. We emphasize the correspondence of varieties and reduced finite-type $\mathbb{C}$-algebras, which is crucial to understand the GIT quotient construction.

- Affine algebraic sets $V(\mathfrak{a})$, vanishing ideals $I(X)$, Zariski topology, Nullstellensatz
- The structure sheaf $\mathcal{O}_X$ of an affine variety $X \subseteq \mathbb{C}^n$
- Morphisms of affine varieties, the correspondence $\{$affine varieties$\} \leftrightarrow \{$f.t. reduced $\mathbb{C}$-algebras$\}$
- (Abstract varieties and their morphisms, not too important, as we only need quasiprojective varieties)
- Projective varieties, homogeneous ideals, projective Nullstellensatz, quasiprojective varieties
- Products of affine and (quasi)projective varieties

2. Algebraic groups and invariant theory (Barbara)

In this talk we learn about affine algebraic groups $G \subseteq \mathrm{GL}(n, \mathbb{C})$ and invariant rings A^G. As we only care about groups over $\mathbb{C}$, the many caveats in defining subgroups and quotients (group object in schemes, reducedness/smoothness, flatness) do not matter to us at all. In this regard, this part of [Hos15] is written in too much generality for us, so the main goal of these talks will be to translate the relevant results into "classical" language

- (Affine/linear) algebraic group, homomorphism of algebraic groups, normal subgroups
- Linear representations, group actions, equivariant maps
- Algebraic tori, maximal tori, weight space decompositions
- Group action on $\mathbb{C}$-algebras, the invariant ring A^G, Hilbert's 14th problem and its solutions
- (Linearly/geometrically) reductive groups, the main theorems over $\mathbb{C}$

3. Affine GIT (Anaelle)

7. Projective GIT (Bernhard)

8. Hilbert-Mumford criterion (Max)

9. The moduli problem (Leonie)

Topic B: Nonabelian Hodge Correspondence

For nonabelian Hodge correspondence we will have six talks focusing on a subset of the following references: [Tho23; Hos13; Li19; WG11; Wen16; Got14]. As there is a plethora of sources on this subject without a unifying standard text, we will take the more accessible lecture notes [Tho23] as our base and build up the details from here.

4. The Betti Moduli space: Character variety (Fernando)

Let S be a surface of $g \geq 2$. In this talk we discuss the character variety as the affine GIT of the representation variety $\mathrm{Hom}(\pi_1 S, G)$ by the conjugation action of a complex reductive group. Start by introducing the representation variety with its a natural affine algebraic variety structure and give a description of the invariant ring for some nice group, e.g. $\mathrm{GL}(n, \mathbb{C})$. In the spirit of affine GIT, identify the polystable orbits as those arising from completely reducible representations and the stable points from irreducible representations. Translate these definitions using parabolic, levi subgroups into the familiar definition for representations.

Additional references: [Mar], [Sik10].

5. The de Rham Moduli space: Flat connections (Jiajun)

The goal of this talk is to construct the moduli space of flat connections over a surface via symplectic reduction and state (!) that this corresponds to some GIT quotient via the Kempf-Ness theorem. Begin by defining a flat connection on a principal bundle and observe that the space of connections has a natural symplectic structure preserved by the gauge group action. Define the moment map of a Hamiltonian action and its corresponding symplectic reduction. If the resulting space is a manifold, it inherits a symplectic structure. Show that the gauge group action is Hamiltonian and the moment map is given by curvature.

Additional references: [Hos13], [MS17].

6. The Riemann-Hilbert correspondence (Pengfei)

Exhibit the equivalence of categories between flat connections on a G-principal bundle, G-local systems and surface group representations into G. To do so, one defines the holonomy of a connection and show that when the connection is flat this induces a surface group representation up to conjugation. This correspondence gives a complex analytic isomorphism of the Betti and de Rham moduli spaces. Could you explain why the isomorphism is not algebraic?

Additional references: [Sim94].

10. The Dolbeault Moduli space: Higgs bundles (Tim)

11. Harmonic bundles and Donaldson-Corlette correspondence (Christian)

12. The Hitchin-Simpson correspondence (Enya)

References

[Gat21] Andreas Gathmann. *Algebraic Geometry.* Class Notes TU Kaiserslautern 2021/22. 2021. URL: `https://agag-gathmann.math.rptu.de/class/alggeom-2021/alggeom-2021.pdf`.

[Got14] Peter B. Gothen. "Representations of surface groups and higgs bundles". In: *Moduli Spaces* (2014), 151–178. DOI: `10.1017/cbo9781107279544.004`.

[Hos13] Victoria Hoskins. ON ALGEBRAIC ASPECTS OF THE MODULI SPACE OF FLAT CONNECTIONS. 2013. URL: `https://www.math.ru.nl/~vhoskins/talk_connections.pdf`.

[Hos15] Victoria Hoskins. *Moduli problems and geometric invariant theory - fu-berlin.de.* Algebraic Geometry II: Moduli and GIT, Winter Semester 2015/2016. 2015. URL: `https://userpage.fu-berlin.de/hoskins/M15_Lecture_notes.pdf`.

[Li19] Qiongling Li. "An introduction to higgs bundles via harmonic maps". In: *Symmetry, Integrability and Geometry: Methods and Applications* (2019). DOI: `10.3842/sigma.2019.035`.

[Mar] Arnaud Maret. URL: `https://arnaudmaret.com/files/character-varieties.pdf`.

[MS17] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology.* Oxford University Press, 2017.

[Sik10] Adam S. Sikora. *Character Varieties.* 2010. arXiv: `0902.2589 [math.RT]`.

[Sim94] Carlos T. Simpson. "Moduli of representations of the fundamental group of a smooth projective variety. II". In: *Publications mathématiques de l'IHÉS* 80.1 (1994), 5–79. DOI: `10.1007/bf02698895`.

[Tho23] Alexander Thomas. "A gentle introduction to the non-abelian hodge correspondence". In: *arXiv.org* (2023). URL: `https://arxiv.org/abs/2208.05940`.

[WG11] R. O. Wells and O. Garcia-Prada. "Appendix - Moduli spaces and geometric structures". In: *Differential analysis on complex manifolds.* Springer, 2011.

[Wen16] Richard Wentworth. "Higgs bundles and local systems on Riemann surfaces". In: *Advanced Courses in Mathematics - CRM Barcelona* (2016), 165–219. DOI: `10.1007/978-3-319-33578-0_4`.