Hilbert Functions of Chopped Ideals

Fulvio Gesmundo[1] Leonie Kayser[2] Simon Telen[2]

[1]University of Saarland [2]Max Planck Institute for Mathematics in the Sciences

Chopped ideals

Let $S = \Bbbk[x_0, \dots, x_n]$ be a polynomial ring over an algebraically closed field of characteristic 0. Let $0 \neq I \subseteq S$ be a homogeneous ideal.

Definition. The *chopped ideal* of I is the ideal $I_{\langle d \rangle} \coloneqq \langle I_d \rangle_S$ generated by the nonzero component of lowest degree d.

Now let $I = I(Z)$ be the ideal of a finite set of points $Z = \{z_1, \dots, z_r\} \subseteq \mathbb{P}^n(\Bbbk)$. In chopping the ideal, we consider only the equations on Z of lowest degree. The goal is to recover Z from $I_{\langle d \rangle}$. In most cases $I_{\langle d \rangle} \subsetneq I$, but one may hope that

$$I(Z) \stackrel{?}{=} (I_{\langle d \rangle})^{\text{sat}} \coloneqq (I_{\langle d \rangle} : \mathfrak{m}^\infty), \qquad \mathfrak{m} = \langle x_0, \dots, x_n \rangle_S.$$

This is the case if and only if Z is scheme-theoretically cut out by $I(Z)_d$.

Theorem. *Let $Z \subseteq \mathbb{P}^n$ be a sufficiently general collection of r points and let d be the chop degree of $I = I(Z)$.*

- *If $r > \binom{n+d}{n} - n$, then $\mathrm{V}(I_{\langle d \rangle})$ is a positive-dimensional complete intersection.*
- *If $r = \binom{n+d}{n} - n$, then $\mathrm{V}(I_{\langle d \rangle})$ is a complete intersection of d^n points.*
- *If $r < \binom{n+d}{n} - n$, then $I_{\langle d \rangle}$ cuts out Z scheme-theoretically.* □

In particular, $(I_{\langle d \rangle})^{\text{sat}} = I$ if and only if $r < \binom{n+d}{n} - n$ or $(n, r) = (2, 4)$.

Hilbert functions

The *Hilbert function* of a finite graded S-module M is $h_M(j) \coloneqq \dim_\Bbbk M_j$. The Hilbert function of a set of r points $h_Z \coloneqq h_{S/I(Z)}$ is non-decreasing towards r. If Z is in general position, then

$$h_Z(j) = \min\{h_S(j), r\}, \qquad h_S(j) = \max\{\tbinom{n+j}{n}, 0\}.$$

As noted in the previous section, for general Z one has

$$(I_{\langle d \rangle})_d = I_d, \qquad (I_{\langle d \rangle})_j = I_j, \quad j \gg 0,$$

since saturation alters only finitely many components.

Question. *What are the values of $h_{S/I_{\langle d \rangle}}$ in between? How large is the saturation gap?*

The components of $I_{\langle d \rangle}$ are the images under the multiplication maps

$$\mu_e \colon S_e \otimes I_d \to I_{d+e}.$$

Assuming this linear map has the largest rank possible, one expects:

Conjecture (Expected syzygy conjecture). *For Z general and $j \geq d$ one has*

$$h_{S/I_{\langle d \rangle}}(j) = \max\left\{ \sum_{k \geq 0} (-1)^k \cdot h_S(j - kd) \cdot \binom{h_S(d) - r}{k}, r \right\}. \qquad (1)$$

The Hilbert function of $S/I_{\langle d \rangle}$ returns to r as soon as this sum is less than or equal to r.

This expression is always a lower bound due to a famous theorem of Fröberg [1]. The case $j = d + 1$ is equivalent to the *Ideal Generation Conjecture* (IGC) [2]; our conjecture is a natural generalization thereof.

Example: 18 points in the plane

The smallest interesting example occurs for $r = 18$ general points Z in the plane (see also fig. 4). The Hilbert function h_Z becomes constant in degree 5, which is also its chop degree.

The component I_5 is generated by $h_S(5) - 18 = 3$ quintics q_1, q_2, q_3, but these do *not* generate I; indeed $h_I(6) = 10 > 3 \cdot 3 = h_I(5) \cdot h_S(1)$. In this case the multiplication map μ_e has maximal rank, and the Hilbert function of $S/\langle q_1, q_2, q_3 \rangle$ is as follows:

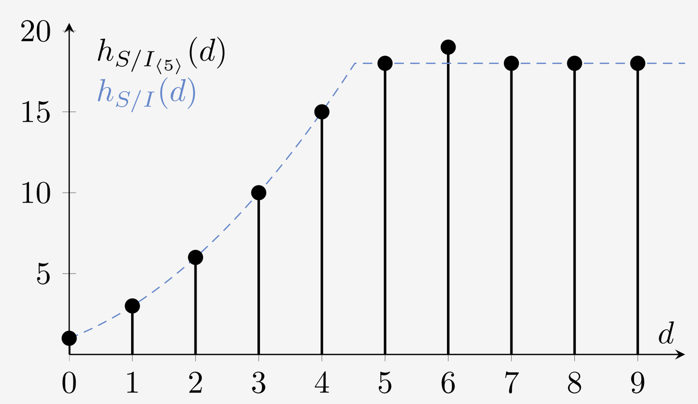

Figure 1. The Hilbert function of the chopped ideal of 18 points.

Which generator in I_6 is missing? Splitting the points in groups of $9 + 9$, there is a unique cubic through each group. Their product spans a complement of $(I_{\langle 5 \rangle})_6$ in I_6.

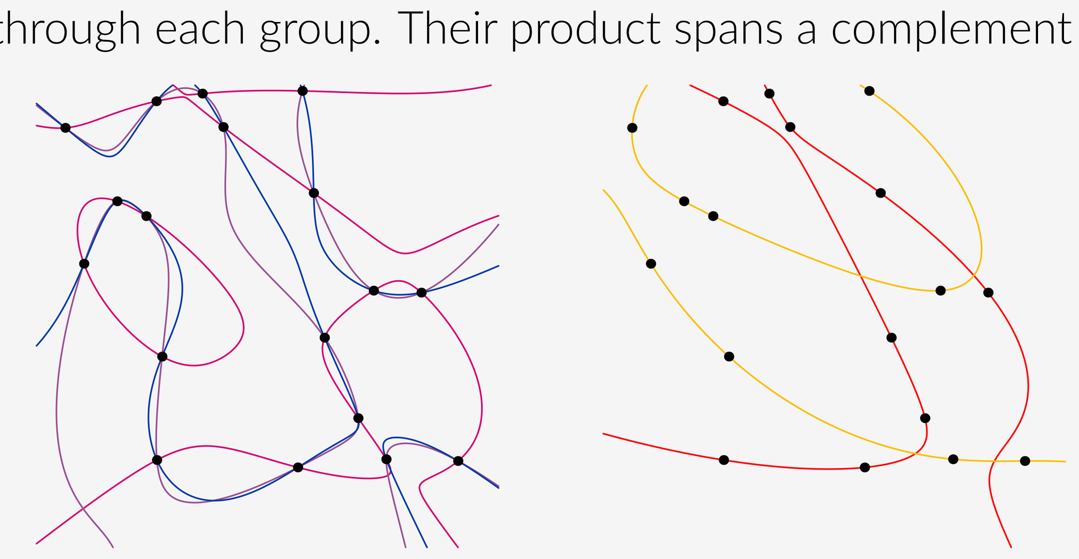

Figure 2. Three quintics (left) and a split sextic (right) through 18 points.

Main results

Theorem. *Conjecture (1) is true in the following cases:*

- *$r_{\max} \coloneqq h_S(d) - (n+1)$ for all d in all dimensions n.*
- *In the plane for $r_{\min} = \frac{1}{2}(d+1)^2$ when d is odd.*
- *$r \leq \frac{1}{n}\big((n+1)h_S(d) - h_S(d+1)\big)$ and $n \leq 4$, or generally whenever (IGC) holds.*
- *In a large number of individual cases in low dimension, see next section.*

Furthermore, the length of the saturation gap is bounded above by

$$\min\{\, e > 0 \mid (I_{\langle d \rangle})_{d+e} = I_{d+e} \,\} \leq (n-1)d - (n+1). \qquad □$$

The alternating sum in (1) is really necessary, as witnessed by fig. 3 ($e > d$).

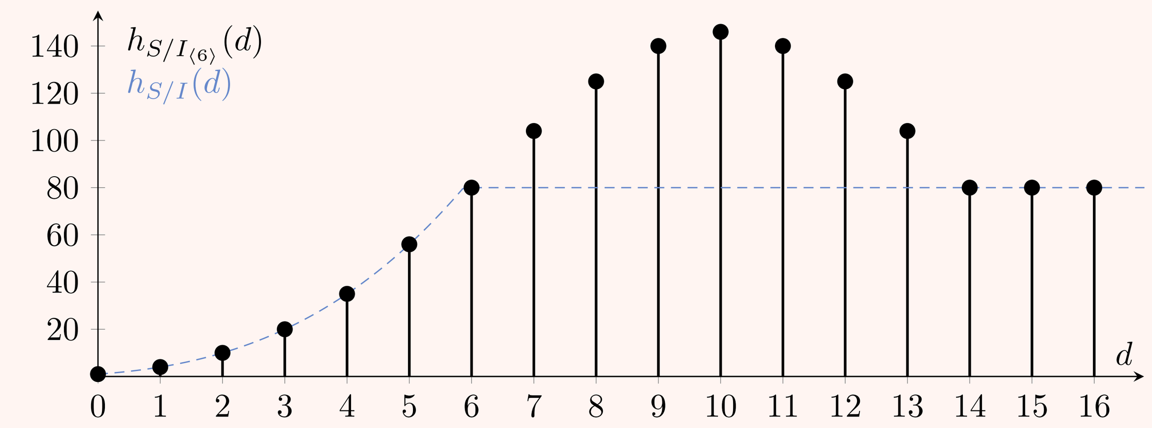

Figure 3. The Hilbert function of the chopped ideal of 80 points in $\mathbb{P}^3$.

Theorem. *Whenever $I_{\langle d \rangle}$ is non-saturated, one has*

$$\operatorname{reg}_{\mathrm{CM}} S/I_{\langle d \rangle} = \operatorname{reg}_{\mathrm{H}} S/I_{\langle d \rangle} - 1. \qquad □$$

Here $\operatorname{reg}_{\mathrm{CM}}$ is the *Castelnuovo-Mumford regularity* and $\operatorname{reg}_{\mathrm{H}}$ is the degree from which on the Hilbert function becomes constant.

The methods used in the proofs include syzygies, liaison (in higher codimension), the mapping cone construction, Hilbert-Burch theory and local cohomology.

Verification using computer algebra

To test the conjecture, one can randomly sample r points from $\mathbb{P}^n(\mathbb{Q})$ and calculate the Hilbert function using computer algebra. If the sample satisfies (1), then by the following theorem the conjecture holds true for this r.

Theorem. *The map $Z \mapsto h_{S/I(Z)_{\langle d \rangle}}(j)$ is upper semicontinuous on the set $U \subseteq (\mathbb{P}^n)^r$ of points with generic Hilbert function.* □

To further speed up the computation, it suffices to verify the conjecture over a finite field $\mathbb{F}_p$. Using `Macaulay2` we verified the conjecture in $\mathbb{P}^2$ for $r \leq 430$ points, in $\mathbb{P}^3$ for $r \leq 157$ points and in $\mathbb{P}^4$ for $r \leq 65$ points.

The following figure shows the distributions of the gap lengths for various r in the plane.

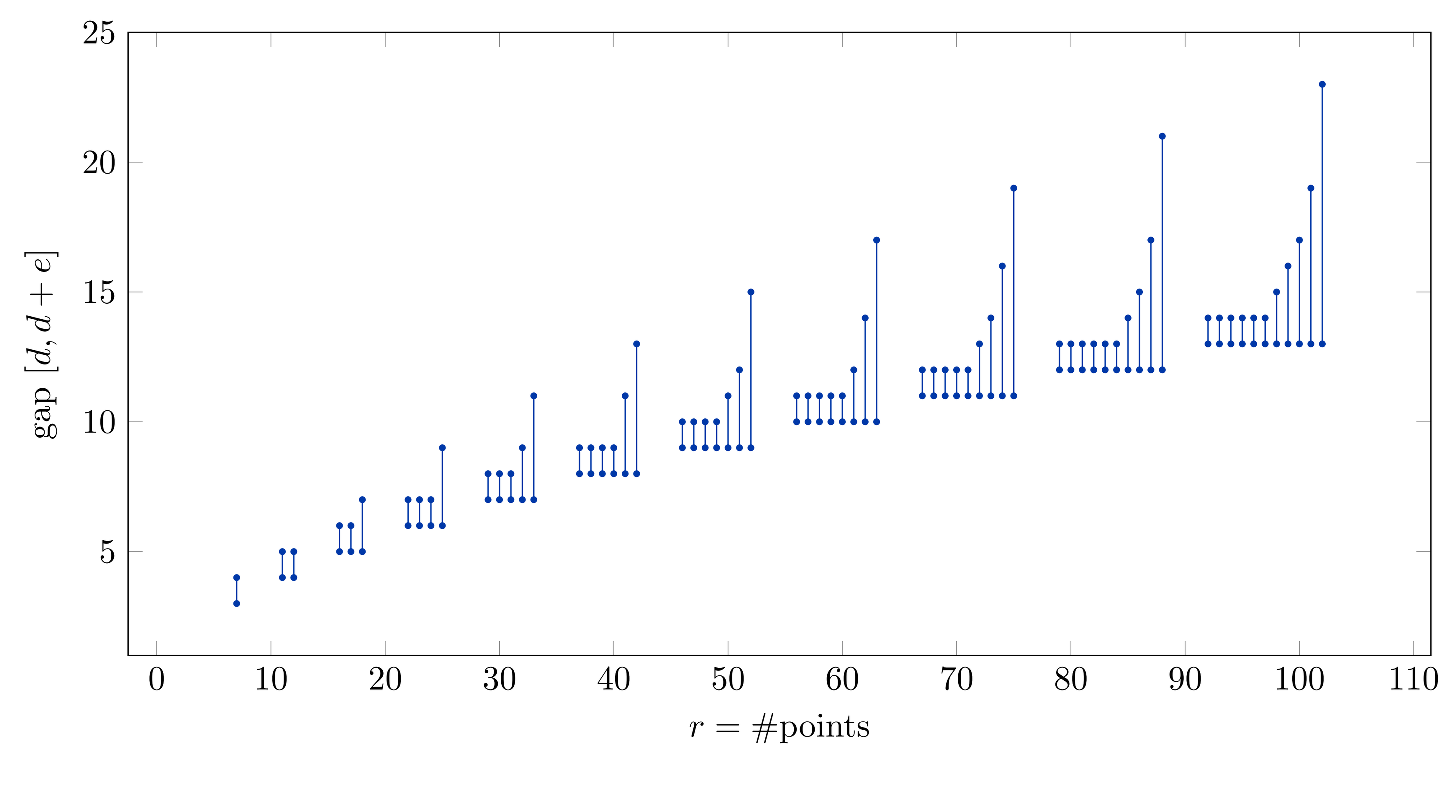

Figure 4. The saturation gaps for all values of $r \leq 102$ in $\mathbb{P}^2$.

Motivation: Tensor decomposition using eigenvalue methods

In symmetric tensor decomposition the goal is to express a form $F \in T \coloneqq \Bbbk[X_0, \dots, X_n]$ of degree D as a minimal sum of powers of linear forms $F = L_1^D + \cdots + L_r^D$. Consider the low rank case $r < h_S(\lfloor D/2 \rfloor) - n$, then this decomposition is generically unique.

The points $Z = \{[L_1], \dots, [L_r]\} \subseteq \mathbb{P}(T_1)$ are solutions to an over-determined system of polynomial equations arising from the *catalecticant matrix* of F. These equations generate the chopped ideal of Z! The case of $F \in \Bbbk[X_0, X_1, X_2]_{10}$ of rank 18 corresponds to the example on the left; the catalecticant method produces the three quintics.

In order to numerically solve these systems, one can employ *numerical normal form methods* [3]. These transform zero-dimensional systems of polynomial equations into eigenvalue problems. The complexity of this algorithms is governed by the Hilbert regularity of the equations $I_{\langle d \rangle}$ at hand, predicted by the conjecture. This procedure has been implemented in `Julia` and routinely outperforms homotopy continuation methods.

References

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