Discussion sheet: Complex geometry

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This sheet provides some questions to help remind you of some definitions and concepts in complex geometry that may be useful for the next talks. The answers to most questions can be found in Huybrechts [Huy10], Neitzke [Nei16] and a quick search on the internet.

Complex structures

1. Let's review complex manifolds. Find nontrivial examples of:

   - a Lie group with a complex structure,
   - a compact simply connected manifold with a complex structure,
   - a real even-dimensional smooth manifold that does not admit a complex structure,
   - a matrix Lie group with complex coefficients that does not admit a complex structure,
   - a submanifold of a complex manifold that is not complex.

   Fact: The existence of a complex structure on X is equivalent to the existence of a Dolbeault operator $\overline{\partial} : \Omega^{0,0}(X) \to \Omega^{0,1}(X)$ satisfying the Leibniz rule and $\overline{\partial}^2 = 0$.

2. Let (X, J) be a complex manifold. The $\pm i$-eigenspaces of J decomposes its complex tangent bundle splits into a holomorphic and anti-holomorphic part,

   $$TX \otimes \mathbb{C} =: T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X.$$

   Similarly, the complex cotangent bundle splits into (1,0) and (0,1)-forms

   $$T^*_{\mathbb{C}}X = \Omega^{1,0}(X) \oplus \Omega^{0,1}(X).$$

   What is $\dim_{\mathbb{C}} T^{1,0}X$ in terms of $\dim_{\mathbb{C}} X = n$? Can you recall the canonical complex structure J_0 on $\mathbb{C}^n$ and find a frame for $T^{0,1}\mathbb{C}^n$ and $\Omega^{1,0}(\mathbb{C}^n)$?

3. A **Kähler manifold** is a manifold with three mutually compatible structures: a complex structure J, a Riemannian structure g, and a symplectic structure ω such that for any $u, v \in T_pX$,

   $$g(u, v) = \omega(u, Jv).$$

   The n-torus $\mathbb{T}^n = \mathbb{C}^n/\Lambda$ is a compact Kähler manifold. Assuming that it inherits the canonical complex structure and Euclidean metric from $\mathbb{C}^n$, could you find its Kähler form ω?

   (*)**Fact:** Every closed orientable surface admits a conformal structure, which in turn is equivalent to a complex structure. Fixing a complex structure makes the surface into a 1-dimensional complex manifold called a **Riemann surface**. Every Riemann surface is automatically Kähler, why is that true?

4. A **hyperkähler** manifold (X, g, I, J, K) is a Riemannian manifold endowed with three complex structures that are Kähler with respect to g and such that $IJK = -1$. Show that the Quaternionic space

$$\mathbb{H} = \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}, \quad x_i \in \mathbb{R}\}$$

is a hyperkähler manifold. Convince yourself that the associated symplectic forms are given by:

$$\begin{aligned}
\omega_I &= dx_0 \wedge dx_1 + dx_2 \wedge dx_3 \\
\omega_J &= dx_0 \wedge dx_2 + dx_3 \wedge dx_1 \\
\omega_K &= dx_0 \wedge dx_3 + dx_1 \wedge dx_2.
\end{aligned}$$

5. Let (X, J) be a Riemann surface.The **Hodge star operator** $\star$ by

$$\begin{aligned}
\star : TX^* &\to TX^*, \\
\alpha &\mapsto -\alpha \circ J.
\end{aligned}$$

Note that $\star^2 = -1$. Check that the following map is a complex-linear isomorphism

$$\begin{aligned}
T_p^*X &\to (T_p^{1,0}X)^* = \Omega^{1,0}(X), \\
\alpha &\mapsto \frac{1}{2}(\alpha + i \star \alpha).
\end{aligned}$$

Can you write isomorphism $T^*X \to \Omega^{0,1}(X)$?

Fact: The above definition coincides with the Hodge-star operator defined on an n-dimensional Riemannian manifold (X, g):

$$\star\colon \Omega^k(X) \to \Omega^{n-k}(X), \quad \eta \wedge \star\omega = \langle \eta, \omega \rangle \, \mathrm{dvol}_g.$$

Holomorphic vector bundles

1. Let X be a complex manifold. A complex vector bundle $\pi : E \to X$ is **holomorphic** if one of the following equivalent conditions hold:

   - E admits a complex structure making the projection map π holomorphic,
   - the transition maps $\varphi : U_{ij} \to \mathrm{GL}(n, \mathbb{C})$ are holomorphic,
   - there exists a **holomorphic structure** of E, i.e. a Dolbeault operator

     $$\overline{\partial}_E : \Omega^{0,0}(X, E) \to \Omega^{0,1}(X, E)$$

     satisfying the Leibniz rule and $\overline{\partial}_E^2 = 0$.

   Give distinct examples of a real vector bundle, a complex vector bundle, and a holomorphic vector bundle over $\mathbb{CP}^1$.

2. Let $E \to X$ be a complex vector bundle. A **connection** ∇ is a $\mathbb{C}$-linear map

   $$\nabla : \Omega^0(X, E) \to \Omega^1(X, E),$$

   satisfying the Leibniz rule

   $$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s).$$

We can decompose $\nabla = \nabla^{1,0} + \nabla^{0,1}$, where

$$\nabla^{1,0} \colon \Omega^0(X,E) \to \Omega^{1,0}(X,E),$$
$$\nabla^{0,1} \colon \Omega^0(X,E) \to \Omega^{0,1}(X,E).$$

Using a previous exercise, can you express for $\nabla^{1,0}$ and $\nabla^{0,1}$ in terms of ∇?

Fact: A **hermitian structure** on a E is a fiberwise Hermitian scalar product varying differentiably on x. Suppose $(E, \overline{\partial}_E)$ is a holomorphic vector bundle over X. For every hermitian metric h on E, there exists a unique **unitary** connection ∇, i.e. compatible with h, such that $\nabla^{0,1} = \overline{\partial}_E$. Then ∇ is called the **Chern connection** of $(E, \overline{\partial}_E, h)$.

3. (*) Can you recall the definition of the degree of a line bundle L over a Riemann surface X? In particular, what is the degree of $\mathcal{O}(-n)$ or $\mathcal{O}(1) \oplus \mathcal{O}(1)$? What is the degree of the canonical line bundle $K := T^*X$ over $\mathbb{CP}^1$?

References

[Huy10] Daniel Huybrechts. *Complex geometry: An introduction*. World Publishing Corporation, 2010.

[Nei16] Andrew Neitzke. 2016. URL: `https://gauss.math.yale.edu/~an592/exports/higgs-bundles.pdf`.