Tensor Decomposition Using Numerical (Non)Linear Algebra

SIAM Conference on Applied Linear Algebra (LA24)

MAX PLANCK INSTITUTE
FOR MATHEMATICS
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May 14, 2024

Mathematics > Commutative Algebra

[Submitted on 5 Jul 2023]

Hilbert Functions of Chopped Ideals

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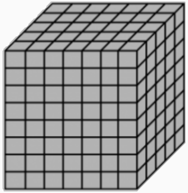
A chopped ideal is obtained from a homogeneous ideal by considering
cases in which the chopped ideal defines the same finite set of points a
computing these points from the chopped ideal is governed by the Hilb
these invariants and prove them in many cases. We show that our conje
decomposition.

What is a tensor?



A tensor...

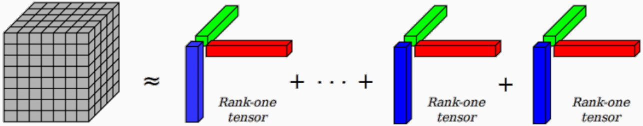
- ...is an object that transforms like a tensor
- ...is an element of a tensor product of vector spaces $U \otimes V \otimes W$
- ...is a **multidimensional array of numbers** $A = (A_{i_1 \dots i_d})_{i_1,\dots,i_d} \in \mathbb{C}^{n_1 \times \cdots \times n_d}$
- ...in $(\mathbb{C}^n)^{\otimes d}$ is symmetric if its entries are invariant under permutations $\sigma \in \mathfrak{S}_d$
- Symmetric tensors can be identified with homogeneous polynomials

$$\mathbb{C}[x_1, \dots, x_n]_d \ni x_{i_1} \cdots x_{i_d} \quad \longleftrightarrow \quad \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(d)}} \in \mathrm{Sym}^d \mathbb{C}^n \subseteq (\mathbb{C}^n)^{\otimes d}$$

Tensor decomposition and rank

- A tensor of the form $(u_i v_j w_k)_{i,j,k} \hat{=} u \otimes v \otimes w$ is simple
- Every tensor is a linear combination of simple tensors

$$A = \sum_{i=1}^{r} \lambda_i u^{(i)} \otimes v^{(i)} \otimes w^{(i)}$$



- The smallest such r is the tensor rank of A
- Generalizes matrix rank: $\mathbb{C}^{m \times n} \ni A = S \cdot \mathrm{diag}(\underbrace{1, \ldots, 1}_{\mathrm{rank}\, A}, 0, \ldots) \cdot T = \sum_{i=1}^{r} S_{*,i} \cdot T_{i,*}$
- If the simple tensors are unique up to scaling, then A is called identifiable
- Symmetric case: Simple tensor $v^{\otimes d} \hat{=} \ell^d$ powers of linear forms, $F = \sum_{i=1}^{r} \lambda_i \ell_i^d$
- Symmetric tensor rank, identifiability, . . .

Forms of small rank often have unique decompositions

Let $T_d = \mathbb{C}[X_0, \ldots, X_n]_d \cong \mathbb{C}^{\binom{n+d}{n}}$ be the vector space of degree d forms

- **(Alexander–Hirschowitz)**
  A general form $F \in T_d$ has rank $\left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil$ except in a few cases
- **(Ballico, Mella, Chiantini–Ottaviani–Vannieuwenhoven, . . .)**
  For $r < \frac{1}{n+1}\binom{n+d}{d}$ a general form of rank r is identifiable except in a few cases

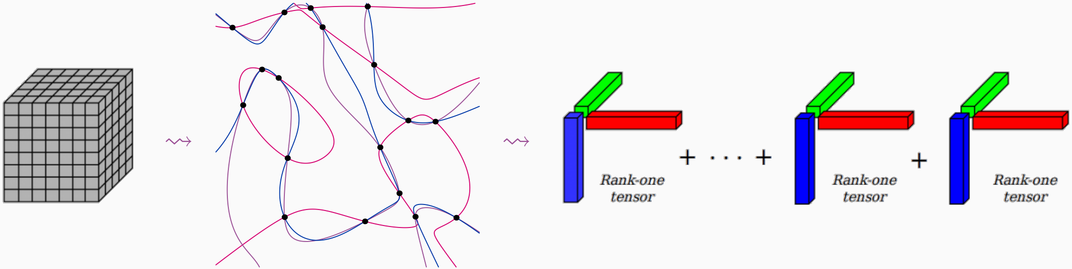
Running example:

A general $F \in \mathbb{C}[X_0, X_1, X_2]_{10}$ has $\operatorname{rk} F = \frac{1}{3}\binom{2+10}{2} = 22$. The set of such forms of rank 18 has dimension 54 in $\mathbb{C}^{66}$. A random such F has a *unique* decomposition

$$F = \ell_1^{10} + \cdots + \ell_{18}^{10}, \qquad \ell_i \in \mathbb{C}[X_0, X_1, X_2]_1.$$

The catalecticant method

- Fix general $F = \sum_{i=1}^{r} \ell_i^d \in T_d$ of rank r
- Linear forms as points in projective space $[\ell_i] \in \mathbb{P}(T_1) \cong \mathbb{P}^n_{\mathbb{C}}$ $\qquad \mathbb{P}(V) = (V \setminus 0)/\mathbb{C}^\times$
- Catalecticant method give polynomials vanishing on $Z = \{[\ell_1], \ldots, [\ell_r]\} \subseteq \mathbb{P}^n$



- In fact: Obtain *all* homog. equations of degree $\leq d/2$ vanishing on Z

$\rightsquigarrow$ Hope: Solutions to equations are exactly the $[\ell_i]$!

The algorithm

- Equations via kernel of catalecticant maps

$$\mathrm{Cat}_j(F)\colon \mathbb{C}[y_0,\dots,y_n]_j \to T_{d-j}, \qquad g \mapsto g(\partial_{X_0},\dots,\partial_{X_n})F(X_0,\dots,X_n)$$

- Algorithmic approach:
  1. Compute kernel basis $\mathcal{F}$ of the *linear* catalecticant map $\mathrm{Cat}_{\lfloor d/2 \rfloor}(F)$
  2. Solve *polynomial* system $\{\mathcal{F}=0\}$ to get $\mathcal{Z}eros(\mathcal{F}) \overset{?}{=} \{[\ell_1],\dots,[\ell_r]\}$,
  3. Solve *linear* equations to get λ_i in $F = \sum_{i=1}^r \lambda_i \ell_i^d$
- (At least) three common approaches:
  - Gröbner bases computation (symbolic)
  - Homotopy continuation (numerical)
  - Eigenvalue/normal form methods (numerical/mixed)

$\rightsquigarrow$ Focus on the eigenvalue method approach here

Eigenvalue methods for polynomial system solving

Task: Given 0-dim'l system $\{\mathcal{F}=0\}$, compute finite set $Z=\{z_1,\dots,z_r\}=\mathcal{Z}eros(\mathcal{F})\subseteq\mathbb{P}^n$

- ▷ Consider ideal $J\coloneqq\langle\mathcal{F}\rangle_S=\bigoplus_{t\geq 0}J_t$, this is a graded subspace of S with

$$J_t=S_{t-\deg f_1}f_1+\cdots+S_{t-\deg f_s}f_s\subseteq S_t$$

- ▷ For t large enough the Hilbert function $h_{S/J}(t)\coloneqq\dim_{\mathbb{C}}(S/J)_t$ is constant r
- ▷ Multiplication map for $g\in S_e$:

$$M_g\colon(S/J)_d\xrightarrow{\cdot g}(S/J)_{d+e}$$

- ▷ Under "suitable conditions" $M_h^{-1}M_g\colon(S/J)_d\to(S/J)_d$ has left eigenpairs

$$\left\{\,(\mathrm{ev}_{z_i},\tfrac{g}{h}(z_i))\mid i=1,\dots,r\,\right\},\qquad \mathrm{ev}_{z_i}(f)=f(z_i)/h(z_i)$$

⇝ Translate problem into large eigenvalue problem, solve numerically

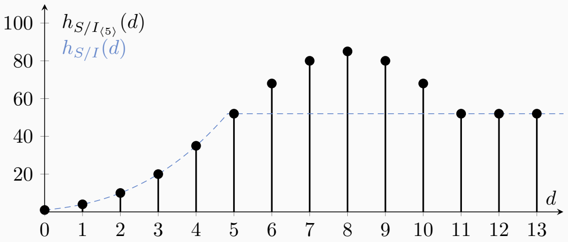
- ▷ For this need $h_{S/J}(d+e)=h_{S/J}(d)=r$, want $d,d+e$ as small as possible

Non-saturated systems are harder to solve

- Z general set of points, $I = \{\, f \in S \mid f(Z) = 0 \,\}$, then $h_{S/I}(t) = \min\{h_S(t), r\}$
  - $\rightsquigarrow$ $d = \min\{\, t \mid h_S(t) \geq r \,\}$ and $e = 1$ work.
- In general, larger saturation gap can be encountered
- Saturation gap governs *algorithmic complexity* of solving J with eigenvalue methods

Bigger example

For a general set $Z \subseteq \mathbb{P}^3$ of 52 points and $J = I_{\langle 5 \rangle}(Z) := \langle \{\, f \in S_5 \mid f(Z) = 0 \,\} \rangle_S$, we have the Hilbert function pictured below. Smallest choice: $d = 5$, $d + e = 11$.



Recap

We are lead to the following setup:

- Given a general form $F = \sum_{i=1}^{r} \ell_i^d \in \mathbb{C}[X_1, \ldots, X_n]_d$ of "small" rank r
- Decomposition is unique, want to find $Z = \{[\ell_1], \ldots, [\ell_r]\} \in \mathbb{P}^n$
- Want to solve Catalecticant polynomial system $\mathcal{F}$ using the eigenvalue method
- Is $\mathcal{Z}eros(\mathcal{F}) = Z$? With(out) multiplicities?
- What is the Hilbert function of the ideal $\langle \mathcal{F} \rangle_S \subseteq S$? When $= r$?

Running example

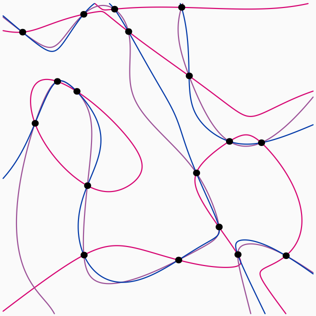
$n = 2$, $d = 10$, $r = 18$, equations $\mathcal{F}$ have degree $d/2 = 5$.

$$F = \sum_{i=1}^{18} \ell_i^{10} \in \mathbb{C}[X_0, X_1, X_2]_{10}, \qquad [\ell_i] \in \mathbb{P}(\mathbb{C}[X_0, X_1, X_2]_1) = \mathbb{P}^2$$

Example: $Z = 18$ points in the plane

t	$\dots$	3	4	5	6	7
$h_S(t)$	$\dots$	10	15	21	28	36
$h_I(t)$	$\dots$	0	0	3	10	18
$h_{I_{\langle 5\rangle}}(t)$	$\dots$	0	0	3	9	18

t	0	1	2	3	4	5	6	7
$h_S(t)$	1	3	6	10	15	21	28	36
$h_{S/I}(t)$	1	3	6	10	15	18	18	18
$h_{S/I_{\langle 5\rangle}}(t)$	1	3	6	10	15	18	19	18



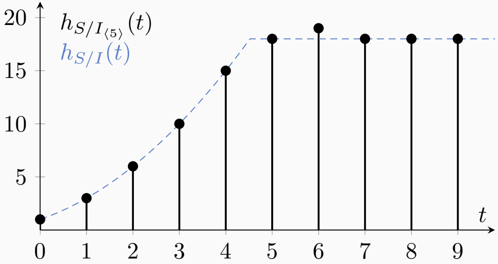


Figure 1: Three quintics $\langle q_1, q_2, q_3\rangle_{\mathbb{C}} = I_5$ passing through 18 general points.

For which forms is our algorithmic approach even possible?

- For a set of points Z consider the vanishing ideal and chopped ideal

$$I = \{\, f \in S \mid f(Z) = 0 \,\}\,, \qquad I_{\langle d \rangle} = \langle \{\, f \in S_d \mid f(Z) = 0 \,\} \rangle_S$$

- Generally $I_{\langle d \rangle} \subsetneq I$, we need $\mathcal{Z}eros(I) \overset{?}{\underset{\text{multiplicities}}{=}} \mathcal{Z}eros(I_{\langle d \rangle}) \subseteq \mathbb{P}^n$

Theorem

Let $Z \subseteq \mathbb{P}^n$ be a general set of r points and $d \in \mathbb{N}$. Then

$$\mathcal{Z}eros(I) = \mathcal{Z}eros(I_{\langle d \rangle}) \iff r < \binom{n+d}{n} - n \text{ or } r = 1 \text{ or } (n, r, d) = (2, 4, 2).$$

The conjectural Hilbert function

Expected syzygy conjecture (ESC)

For a general set of $r < \binom{n+d}{n} - n$ points in $\mathbb{P}^n$ the ideal $I_{\langle d \rangle}$ has Hilbert function

$$h_{S/I_{\langle d \rangle}}(t) = \begin{cases} \sum_{k \geq 0} (-1)^k \cdot \binom{n+t-kd}{n} \cdot \binom{\binom{n+d}{n} - r}{k} & t < t_0, \\ r & t \geq t_0, \end{cases}$$

where t_0 is the first integer $> d$ such that the sum is $\leq r$.

- One can extract the saturation gap length from this formula
- This is always a (lexicographic) lower bound due to Fröberg
- If $W \subseteq S_d$ is a random vector subspace of dim. $\binom{n+d}{n} - r$, then this sum is the expected Hilbert function of $S/\langle W \rangle_S$ (until sum ≤ 0)

Main results

Theorem

Conjecture (ESC) is true in the following cases:

- $r_{\max} := \binom{n+d}{n} - (n+1)$ *for all* d *in all dimensions* n.
- *In the plane for* $r_{\min} = \frac{1}{2}(d+1)^2$ *when* d *is odd.*
- $r \leq \frac{1}{n}\left((n+1)\binom{n+d}{n} - \binom{n+d+1}{n}\right)$ *and [*$n \leq 4$ *or* $d \gg 0$*]*
- *In a large number of individual cases in low dimension (table below).*

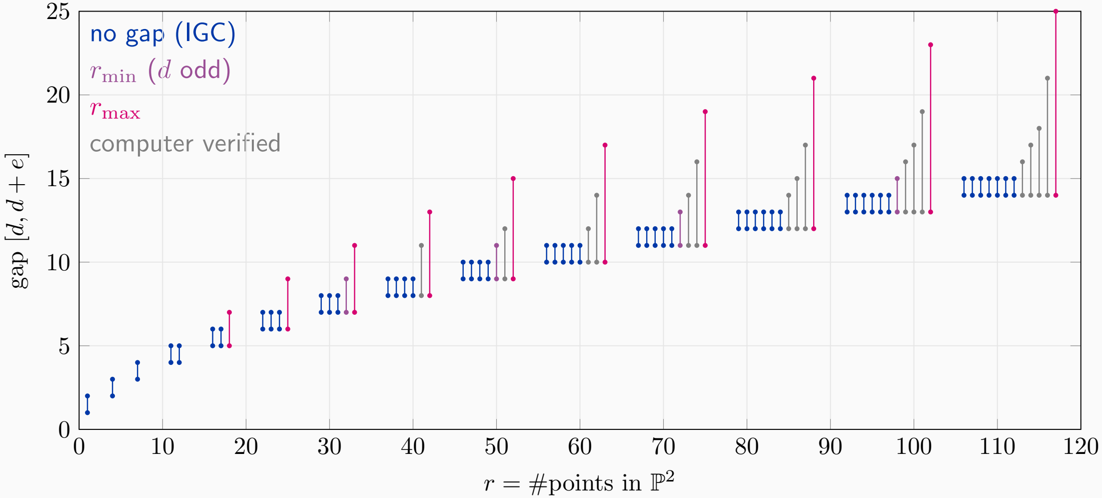
The length of the saturation gap is bounded above by

$$\min\{\, e > 0 \mid (I_{\langle d \rangle})_{d+e} = I_{d+e} \,\} \leq (n-1)d - (n+1).$$

n	2	3	4	5	6	7	8	9	10
r	≤ 1825	≤ 1534	≤ 991	≤ 600	≤ 447	≤ 316	≤ 333	≤ 204	≤ 259
d	≤ 58	≤ 18	≤ 9	≤ 6	≤ 4	≤ 3	≤ 3	≤ 2	≤ 2

Visualization of the saturation gaps in $\mathbb{P}^2$

- ESC predicts exactly how large the difference between I and $I_{\langle d \rangle}$ is



Thank you! Questions?

arXiv:2307.02560

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A chopped ideal is obtained from a homogeneous ideal by considering only the generators of a fixed degree. We investigate cases in which the chopped ideal defines the same finite set of points as the original one-dimensional ideal. The complexity of

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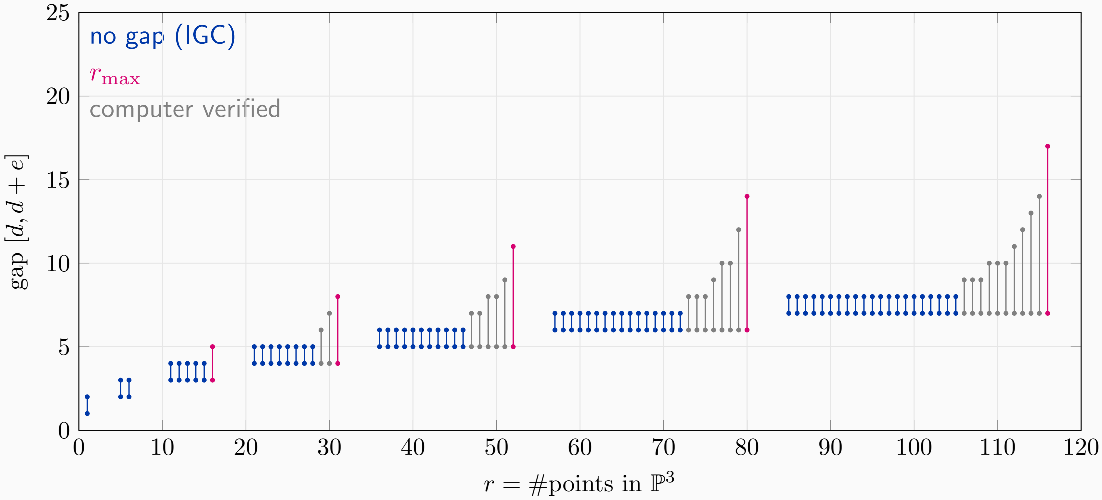
Visualization of the saturation gaps in $\mathbb{P}^3$



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