Mukai lifting of self-dual points in $\mathbb{P}^6$

Barbara Betti and Leonie Kayser

Abstract

A set of $2n$ points in $\mathbb{P}^{n-1}$ is self-dual if it is invariant under the Gale transform. Motivated by Mukai's work on canonical curves, Petrakiev showed that a general self-dual set of 14 points in $\mathbb{P}^6$ arises as the intersection of the Grassmannian $\mathrm{Gr}(2,6)$ in its Plücker embedding in $\mathbb{P}^{14}$ with a linear space of dimension 6. In this paper we focus on the inverse problem of recovering such a linear space associated to a general self-dual set of points. We use numerical homotopy continuation to approach the problem and implement an algorithm in `Julia` to solve it. Along the way we also implement the forward problem of slicing Grassmannians and use it to experimentally study the real solutions to this problem.

1   Introduction

In 1988 Mukai proved that the intersection of the Grassmannian $\mathrm{Gr}(2,6)$ in its Plücker embedding in $\mathbb{P}^{14}$ with a general linear space of dimension 7 is a general curve of genus 8 in its canonical embedding [Muk88; Muk92]. This is part of his more general structure results concerning canonical curves and K3 surfaces: Mukai established that the general K3 surface of genus $g \leq 10$ is a linear section of what is now called the *Mukai Grassmannians* X_g. In the case of genus $g = 8$ the Mukai Grassmannian is the classical Grassmannian $X_8 = \mathrm{Gr}(2,6)$ in its Plücker embedding. Prime Fano 3-folds, K3 surfaces, canonical curves and self-dual point configurations arise as the intersection of $\mathrm{Gr}(2,6)$ with a linear subspace of $\mathbb{P}^{14}$ of dimension $9, 8, 7$ and 6, respectively. The *Mukai lifting problem* concerns the inverse task, that is, it deals with recovering a linear section of the Grassmannian that produces a given variety.

In 2009 Petrakiev showed that a general self-dual point configuration of 14 points in $\mathbb{P}^6$ arises as the intersection of $\mathrm{Gr}(2,6)$ with a linear space of dimension 6 in $\mathbb{P}^{14}$ [Pet09, Thm. 3.3, Theorem 2.11 below]. Our paper presents an algorithm that solves the corresponding lifting problem.

Problem 1.1. *Given such a set $\Gamma \subseteq \mathbb{P}^6$, find a linear embedding $L \colon \mathbb{P}^6 \xrightarrow{\sim} \mathbb{L} \subseteq \mathbb{P}^{14}$ such that*

$$L(\Gamma) = \mathbb{L} \cap \mathrm{Gr}(2,6). \tag{1}$$

The existence of a solution is guaranteed by the result of Petrakiev. However, the proof is not constructive. Obtaining a linear section is a challenging problem that involves hard polynomial systems which have not been successfully solved before. Our implementation relies on numerical homotopy continuation to solve equations, as well as exploiting the toric degeneration of the Grassmannian $\mathrm{Gr}(2,6)$. A similar approach is followed in a more general setting in [BSW23], but we avoid to use the defining ideal of the Grassmannian. Instead we directly degenerate its coordinate ring by using a finite Sagbi basis. Such a basis is the analogue of Gröbner bases for subalgebras of polynomial rings.

The name self-dual stems from the defining property of these configurations being invariant under the *Gale transform*. This transform is an involution that takes a tuple Γ of m points in $\mathbb{P}^{n-1}$ into a tuple Γ' of m points in $\mathbb{P}^{m-(n-1)}$, defined up to projective linear transformations. The Gale transform was first studied in the general case of m points in $\mathbb{P}^{n-1}$ by Coble [Cob22], while some specific cases of self-dual points have been studied earlier. A first example is *Pascal's Theorem* stating that the vertices of two triangles circumscribed to the same conics lie on another conic. Hesse gave a similar result for 8 points in 3-space. The transform is named after David Gale, who used it in his studies on convex polyhedra, we refer to [EP00] for a more detailed history of the Gale transform. Applications of Gale transforms are present in coding theory, where the Gale transform of a linear code gives the parity-check matrix, and in solving polynomial systems, where upper bounds for the number of solutions are given by studying the *Gale dual system*, see [Sot11, Chapter 6]. Some more recent results involving Gale duality can be found in [Arz18; BS08; Cam+18].

In this work we turn abstract and classical algebraic geometry into practice using state-of-the-art numerical tools. It is a first step towards solving more challenging lifting problems, for example lifting canonical curves. Our main contribution, other than the algorithmic implementation, regards new normal forms for self-dual points and the parameterization of their moduli space. We also experimentally study the real geometry of the slicing problem.

The paper is organized as follows. In Section 2 we give equivalent characterizations for self-dual points and briefly introduce the moduli spaces of point configurations and the Gale transform. We discuss normal forms of self-dual points and how to parametrize them generically. We finally recall Mukai's results and collect known descriptions of the moduli space of self-dual points $\mathcal{A}_n$ for small n. In Section 3 we introduce the homotopy continuation method and use it to compute the points of in given linear section of the Grassmannian $\mathrm{Gr}(2,6)$, exploiting its Gröbner degeneration. We also experimentally analyze how many real solutions there exist. In Section 4 we finally describe a set of polynomial equations representing the Mukai lifting problem and solve them for a general configuration of points using Homotopy Continuation. Section 5 describes the implementation of our algorithm as well as an extended example and performance results. The full code is implemented in the programming language `Julia` [Bez+12] version 1.9.3 and it is available at the MathRepo page

https://mathrepo.mis.mpg.de/MukaiLiftP6

2   A primer of self-dual points

In this Section we give an elementary introduction to the theory of self-dual points. Our main goal is to define the moduli space $\mathcal{A}_n$ of self-dual points in $\mathbb{P}^n$ and give parametrizations thereof. Our main contribution is the skew normal form of a sufficiently general set of self-dual points, which, together with the orthogonal normal form, give two useful normal forms when working with self-dual points. While the treatment is non self-contained, all details relevant for our algorithmic considerations are given.

2.1   Self-dual point configurations

We consider an algebraically closed field $\mathbb{K}$ of any characteristic, in the later sections we will specialize to $\mathbb{K} = \mathbb{C}$. Let $S = \mathbb{K}[X_1, \ldots, X_n]$ be the homogeneous coordinate ring of $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{K}^n)$ and given a set $X \subseteq \mathbb{P}^{n-1}$ let $I(X)$ be its homogeneous ideal. Recall that a set of points $\Gamma \subseteq \mathbb{P}^{n-1}$ is (linearly) non-degenerate if they are not contained in a hyperplane, namely, $I(\Gamma)_1 = 0$. A set of points is linearly general if any subset of at most n points is linearly independent. Let $\mathrm{Diag}(n) \subseteq \mathrm{GL}(n)$

be the set of invertible diagonal matrices and let $\Gamma \subseteq \mathbb{P}(\mathbb{K}^n)$ be a non-degenerate ordered set of $2n$ distinct points, represented by the columns of a $n \times 2n$ matrix.

Lemma 2.1. *The following are equivalent for Γ:*

(i) *All subsets of $2n-1$ points impose the same number of conditions on quadrics as Γ, in symbols $I(\Gamma)_2 = I(\Gamma \setminus \{\gamma\})_2$ for all $\gamma \in \Gamma$;*

(ii) *There exists an invertible diagonal matrix $\Lambda \in \mathrm{Diag}(2n)$ such that $\Gamma \cdot \Lambda \cdot \Gamma^{\mathsf{T}} = \mathbf{0}$;*

(iii) *There exists a partition $\Gamma = \Gamma_1 \dot\cup \Gamma_2$ and a non-degenerate symmetric bilinear form $Q \in \mathrm{Sym}(n)$ such that both Γ_i are orthogonal bases with respect to Q, i. e. $\Gamma_i^{\mathsf{T}} \cdot Q \cdot \Gamma_i \in \mathrm{Diag}(n)$.*

If Γ is linearly general, these are equivalent to

(iii') *For any partition $\Gamma = \Gamma_1 \dot\cup \Gamma_2$ into $n + n$ points there is a non-degenerate bilinear form $Q \in \mathrm{Sym}(n)$ such that both Γ_i are orthogonal bases with respect to Q.*

Proof. (ii)$\Rightarrow$(i) Let $A = \{\, \alpha \in \mathbb{N}^n \mid |\alpha| = 2 \,\}$, for a point $z \in \mathbb{P}(\mathbb{K}^n)$ define $z^A := (z^\alpha)_{\alpha \in A}$ as the row vector of monomials X^α evaluated in z. Then the linear map $E \colon \mathbb{K}[X_1, \ldots, X_n]_2 \to \mathbb{K}^{2n}$ given by evaluation of quadrics in Γ is represented by

$$E = \left[(\gamma_i)^\alpha\right]_{1 \le i \le 2n, |\alpha|=2} = \begin{bmatrix} -(\gamma_1)^A- \\ \vdots \\ -(\gamma_{2n})^A- \end{bmatrix}, \qquad \Gamma = [\gamma_1|\ldots|\gamma_{2n}].$$

Its kernel $\mathrm{Ker}\, E \subseteq \mathbb{K}[X_1, \ldots, X_n]_2$ consists of quadrics passing through Γ. We need to show that the rank of M does not change when deleting any of the $2n$ rows from M. For a point γ, the vector γ^A consists of the upper half of the matrix $\gamma \cdot \gamma^{\mathsf{T}}$, and by (ii) we have $\sum_i \Lambda_{ii} \gamma_1 \gamma_1^{\mathsf{T}} = \mathbf{0}$. By assumption all Λ_{ii} are nonzero, so any row of M is a linear combination of the other rows, proving the rank condition.

(i)$\Rightarrow$(ii) Reversing the argument, by (i) every $\gamma_j \gamma_j^{\mathsf{T}}$ is a linear combination

$$\gamma_j \gamma_j^{\mathsf{T}} = \sum_{i \ne j} \beta_i^{(j)} \gamma_i \gamma_i^{\mathsf{T}}.$$

Thus the system of equations $\Gamma \Lambda \Gamma^{\mathsf{T}} = \mathbf{0}$, linear in diagonal matrices Λ, has solutions $\mathrm{diag}(\beta^{(j)})$ with $\beta_j^{(j)} = -1 \ne 0$. Since $\mathbb{K}$ is infinite, there exists a solution with all entries nonzero.

(ii)$\Rightarrow$(iii): After rescaling the columns by $\gamma_i \mapsto 1/\sqrt{\Lambda_{ii}}\gamma_i$ we may assume that $\Lambda = I_{2n}$. After reordering the columns we may assume that $\Gamma = [\Gamma_1|\Gamma_2]$ where Γ_1 is invertible. Consider the non-degenerate symmetric matrix $Q := \Gamma_1^{-\mathsf{T}}\Gamma_1^{-1}$, then using

$$\mathbf{0} = \Gamma\Gamma^{\mathsf{T}} = \Gamma_1\Gamma_1^{\mathsf{T}} + \Gamma_2\Gamma_2^{\mathsf{T}}$$

we see that Γ_2 is also invertible and $Q = -\Gamma_2^{-\mathsf{T}}\Gamma_2^{-1}$, and hence $\Gamma_i^{\mathsf{T}} Q \Gamma_i = \pm I_n$.

(iii)$\Rightarrow$(ii): This argument is reversible after rescaling Γ to match $\Gamma_i^{\mathsf{T}} Q \Gamma_i = \pm I_n$.

(ii)$\Leftrightarrow$(iii'): The same argument clearly applies to any partition if Γ is linearly general. $\qquad\square$

Definition 2.2. A set of $2n$ points in $\mathbb{P}^{n-1}$ is *self-dual* or self-associated if it satisfies any of the equivalent conditions (i),(ii),(iii) from the previous lemma.

The name "self-dual" is related to the *Gale transform* on the space of point-configurations, which we will introduce in the next section. Its fixed points are exactly self-dual point configurations. A deeper characterization of self-dual points related to the Gorenstein property of $S/I(\Gamma)$ is given by Eisenbud and Popescu [EP00]. A finite-dimensional $\mathbb{K}$-algebra A is Gorenstein if it is injective as a module over itself. An equivalent characterization is the existence of a $\mathbb{K}$-linear map $e\colon A \to \mathbb{K}$ such that $(f,g) \mapsto e(fg)$ is a perfect pairing. A standard graded $\mathbb{K}$-algebra S_X of Krull dimension n is Gorenstein if there is a regular sequence $\ell_1, \dots, \ell_n \in (S_X)_1$ such that $S_X/\langle \ell_1, \dots, \ell_n \rangle$ is Gorenstein.

Theorem 2.3 ([EP00, Theorem 7.1 & 7.3]). *For a non-degenerate set of $2n$ points $\Gamma \subseteq \mathbb{P}^{n-1}$ the following are equivalent:*

(i) Γ *is self-dual and fails by 1 to impose independent conditions on quadrics;*

(ii) *every subset of $2n - 1$ points imposes independent conditions on quadrics, but Γ does not impose independent conditions;*

(iii) Γ *is arithmetically Gorenstein, i.e. the graded ring $S_\Gamma = S/I(\Gamma)$ is Gorenstein.*

2.2 The space of point configurations

Following [MFK94, Chapter 3] or [DO88, Chapter II], the moduli space $\mathcal{P}_n^m$ of ordered sets of m points in $\mathbb{P}^n$ can be constructed as a GIT quotient as follows:

- The product $(\mathbb{P}^n)^m$ admits group actions of $G = \mathrm{SL}(n+1)$ and $\mathfrak{S}_m$ via

$$g \cdot (x_1, \dots, x_m) = (gx_1, \dots, gx_m), \qquad \sigma \cdot (x_1, \dots, x_m) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(m)}).$$

- Consider the very ample line bundle on $(\mathbb{P}^n)^m$

$$\mathcal{L} := \mathcal{O}_{(\mathbb{P}^n)^m}(1, \dots, 1) = \bigotimes_{i=1}^{m} \pi_i^* \mathcal{O}_{\mathbb{P}^n}(1).$$

  This enables us to embed $(\mathbb{P}^n)^m$ as a projective variety in $\mathbb{P}^{(n+1)^m - 1}$ with homogeneous coordinate ring $\bigoplus_{k \geq 0} H^0((\mathbb{P}^n)^m, \mathcal{L}^{\otimes k})$. The bundle $\mathcal{L}$ linearizes with respect to the action of $\mathrm{SL}(n+1)$ in the sense of [MFK94].

- With respect to this linearized action, a tuple $Z \in (\mathbb{P}^n)^m$ is semi-stable if and only if

$$\dim \mathrm{Span}_{\mathbb{P}^n}(Y) + 1 \geq \frac{(n+1) \cdot \#Y}{m} \qquad \text{for all } Y \subseteq Z.$$

  The tuple Z is stable if and only if this inequality is always strict [DO88, Chapter II Thm. 1].

- Consider the graded ring of invariants $R_n^m := \bigoplus_{k \geq 0} H^0((\mathbb{P}^n)^m, \mathcal{L}^{\otimes k})^G$, then

$$\mathcal{P}_n^m := \mathrm{Proj}\, R_n^m = ((\mathbb{P}^n)^m)^{\mathrm{ss}} /\!/_{\mathcal{L}} G.$$

The moduli space of ordered point configurations $\mathcal{P}_n^m$ is a natural domain for the Gale transformation, which we now introduce (compare to Lemma 2.1(ii), a reference is [DO88, Chapter III]).

Definition 2.4. The *Gale transform* of $\Gamma \in \mathcal{P}_n^m$ is given by any ordered set $G(\Gamma) \in \mathcal{P}_{m-n-2}^m$ of m points in $\mathbb{P}^{m-n-2}$, represented by a matrix of size $(n+1) \times m$ such that there exists an invertible diagonal matrix $\Lambda \in \mathrm{Diag}(m)$ verifying

$$\Gamma \cdot \Lambda \cdot G(\Gamma)^{\mathsf{T}} = 0. \tag{2}$$

In other words, the *Gale transform* is the map between moduli spaces of ordered set of points

$$G \colon \mathcal{P}_n^m \longrightarrow \mathcal{P}_{m-n-2}^m, \qquad \Gamma \longmapsto G(\Gamma).$$

The Gale transform of Γ is given by the transpose of a kernel matrix of Γ. Defining the moduli space $\mathcal{P}_n^m$ as above makes this construction well-defined for any semi-stable configuration. Assuming that $\Gamma = [I_{n+1}|A]$ and taking the diagonal matrix Λ in the definition as $I_m \oplus -I_m$, it follows that the Gale transform of Γ is given by the configuration $G(\Gamma) \in \mathcal{P}_{m-n-2}^m$ represented by the columns of the matrix $[A^{\mathsf{T}}|I_{m-(n+1)}]$.

Proposition 2.5. $\Gamma \in \mathcal{P}_{n-1}^{2n}$ *is self-dual if and only if it is fixed under the Gale transform.*

Proof. Inspecting Equation (2), we see that $\Gamma = G(\Gamma)$ in $\mathcal{P}_{n-1}^{2n}$ is a reformulation to condition (ii) of Lemma 2.1. $\square$

The characterization using the Gale transform finally leads to the definition of the moduli space of self-dual points.

Definition 2.6. The *moduli space of un-ordered self-dual point configurations* in $\mathbb{P}^n$ is the fixed locus of the Gale transform G on $\mathcal{P}_n^{2n+2}/\mathfrak{S}_{2n+2}$. It will be denoted by $\mathcal{A}_n$.

2.3 Normal forms for self-dual points

We consider the problem of parametrizing the space of self-dual points by means of suitable normal forms. Lemma 2.1(iii) suggests a normal form using orthogonal matrices over $\mathbb{K}$

$$\mathrm{SO}(n) := \{ P \in \mathrm{SL}(n) \mid P \cdot P^{\mathsf{T}} = I_n \} .$$

Theorem 2.7 (Orthogonal normal form). *Let $\Gamma = [\gamma_1, \ldots, \gamma_{2n}] \subseteq \mathbb{P}(\mathbb{K}^n)$ be an ordered set of $2n$ points with $\gamma_1, \ldots, \gamma_n$ linearly independent. Then the following hold:*

(i) *Γ is self-dual if and only if there exists a matrix $A \in \mathrm{PGL}(n)$ such that, after rescaling the columns of Γ, we have $A \cdot \Gamma = [I_n|P]$ for some $P \in \mathrm{SO}(n, \mathbb{K})$.*

(ii) *Moreover, the orthogonal matrix P is unique up to multiplying the columns with $\varepsilon \in \{\pm 1\}^n$ such that $\prod_i \varepsilon_i = 1$.*

Proof. (i) If $A \cdot \Gamma = [I_n|P]$, then Γ is self-dual by Lemma 2.1(iii), for the converse we provide a construction of A and P. By Lemma 2.1(ii) there exists an invertible diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{2n})$ such that $\Gamma \cdot \Lambda \cdot \Gamma^{\mathsf{T}} = 0$. Rescaling the columns, i.e. multiplying Γ on the right with $\mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}, \sqrt{-\lambda_{n+1}}, \cdots, \sqrt{-\lambda_{2n}})$, we can assume that $\Lambda = I_n \oplus -I_n$. Since the first n points of $\Gamma = [\Gamma_1|\Gamma_2]$ are linearly independent, the matrix $A := \Gamma_1$ is invertible and $A^{-1} \cdot \Gamma = [I_n|P]$ is self-dual with respect the same diagonal matrix as Γ. Then

$$\begin{bmatrix} I_n & P \end{bmatrix} \cdot \begin{bmatrix} I_n & \\ & -I_n \end{bmatrix} \cdot \begin{bmatrix} I_n \\ P^{\mathsf{T}} \end{bmatrix} = I_n - P \cdot P^{\mathsf{T}} = 0,$$

which shows that P is an orthogonal matrix. If P has determinant -1, we can rescale the last column of Γ and P to obtain a special orthogonal matrix.

5

(ii) The orthogonal matrix P only depends on the choice of the square root of unity in the rescaling with $\sqrt{\pm\lambda_i}$, unique up to sign. $\qquad\square$

Corollary 2.8. *Assume that* $\mathrm{char}(\mathbb{K}) \neq 2$, *we have a surjective morphism* $\mathrm{SO}(n) \twoheadrightarrow \mathcal{A}_{n-1}$ *generically finite of degree* $2^{n-1}(2n)!$.

Proof. The orthogonal normal form $P \mapsto [I_n|P]$ is surjective by the previous theorem and the characterization of semi-stable points. This normal form of an ordered set of self-dual points is unique up to the 2^{n-1} ways of multiplying the columns of $P \in \mathrm{SO}(n)$ with ± 1 (here we use $-1 \neq 1$). Moreover, for a general Γ, all its permutations by $\sigma \in \mathfrak{S}_{2n}$ have distinct normal forms, hence the rational map $\mathrm{SO}(n) \to \mathcal{A}_{n-1}$ is generically finite of degree $2^{n-1}(2n)!$. $\qquad\square$

A parametrization of $\mathcal{A}_{n-1}$ more suitable for computation relies on the parametrization of the orthogonal group via the Cayley transform. Let $U \subseteq \mathbb{K}^{n \times n}$ be the set of matrices A with $I_n + A$ invertible, then the *Cayley transform* is the self-inverse automorphism

$$\mathcal{C} : U \to U, \qquad \mathcal{C}(A) = (I_n - A)(I_n + A)^{-1}.$$

Noting that the product $(I_n - A)(I_n + A)^{-1}$ commutes, a straightforward computation shows that $\mathcal{C}$ induces a birational map between skew-symmetric and special orthogonal matrices

$$\mathcal{C} : \mathrm{Skew}(n) \cap U \leftrightarrow \mathrm{SO}(n) \cap U.$$

Hence the orthogonal normal form of a general set of points can be rewritten as $[I_n \mid \mathcal{C}(S)]$ for some $S \in \mathrm{Skew}(n)$. Since point configurations in $\mathcal{A}_{n-1}$ are only defined up to projective linear transformations, we can apply the transformation $I_n + S$ to obtain

$$[I_n \mid \mathcal{C}(S)] = [I_n \mid (I_n - S)(I_n + S)^{-1}] \quad \rightsquigarrow \quad [I_n + S \mid I_n - S]. \tag{3}$$

This avoids the delicate step of inverting a matrix in the implementation of this parametrization, reducing computation time and numerical error. Since this step was suggested us by Simon Telen, we refer at the representation in (3) as the *skew normal form*.

Theorem 2.9 (Skew normal form). *A general configuration of self-dual points* $\Gamma \in \mathcal{A}_{n-1}$ *admits a skew normal form. Moreover, the dominant rational map*

$$\mathrm{Skew}(n) \dashrightarrow \mathcal{A}_{n-1}, \qquad S \mapsto [I_n + S \mid I_n - S]$$

is generically finite of degree $(2n)! 2^{n-1}$.

Since a skew-symmetric matrix $S \in \mathrm{Skew}(n)$ is uniquely determined by the $\frac{n(n-1)}{2}$ entries in the upper triangular part, a suitable parameter space for $\mathcal{A}_{n-1}$ is $\mathbb{C}^{\frac{n(n-1)}{2}}$:

$$(s_1, \ldots, s_{\frac{n(n-1)}{2}}) \mapsto \left[\begin{array}{cccc|cccc} 1 & s_1 & \cdots & s_{n-1} & 1 & -s_1 & \cdots & -s_{n-1} \\ -s_1 & 1 & \ddots & \vdots & s_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & s_{\frac{n(n-1)}{2}} & \vdots & \ddots & \ddots & -s_{\frac{n(n-1)}{2}} \\ -s_{n-1} & \cdots & -s_{\frac{n(n-1)}{2}} & 1 & s_{n-1} & \cdots & s_{\frac{n(n-1)}{2}} & 1 \end{array} \right]$$

In our situation, 14 general self-dual points in $\mathbb{P}^6$ have exactly

$$2^6 \cdot 14! = 5.579.410.636.800$$

distinct orthogonal or skew normal forms.

2.4 Mukai-type results for $\mathcal{A}_n$

For small values of n, self-dual configurations in $\mathbb{P}^n$ are well-studied. A related landmark result is Mukai's description of polarized K3 surfaces and (canonical) curves of genus g as complete intersections on homogeneous varieties X_g [Muk88; Muk92]. He proved that a general polarized K3 surface $(S, \mathcal{L})$ of genus g, meaning $\mathcal{L}$ is ample and $\mathcal{L}^2 = 2g - 2$, arises as the intersection of homogeneous varieties $X_g \subseteq \mathbb{P}^N$ with general hypersurfaces (in most cases hyperplanes) if and only if $g \leq 10$. Similarly, general canonical curves $C \subseteq \mathbb{P}^{g-1}$, embedded by their very ample canonical bundle ω_C, arise as sections of X_g if $g \leq 9$.

In dimension zero, general self-dual point configurations in $\mathbb{P}^{g-2}$ arise as sections of X_g if $g \leq 8$. This is classical for $g \leq 6$ (see [EP00, Part II] for a modern account) and was proven by Petrakiev for $g = 7, 8$ [Pet09]. In particular, all self-dual point configurations in $\mathbb{P}^n$ for $n \leq 6$ arise as linear sections of canonical curves by Mukai's result. For $g \geq 9$ this is no longer true by a simple moduli count. A summary of these results is given in the following theorem.

Theorem 2.10. (i) *All sets of four points in $\mathbb{P}^1$ are self-dual.*

(ii) *Six points in $\mathbb{P}^2$ are self-dual if and only if they are the intersection of a quadric and cubic.*

(iii) *A general set in $\mathcal{A}_3$ is a complete intersection of three quadric surfaces in $\mathbb{P}^3$.*

(iv) *A general set in $\mathcal{A}_4$ is a quadric section of an elliptic normal curve in $\mathbb{P}^5$; equivalently it is the intersection of $\mathrm{Gr}(2, 5) \subseteq \mathbb{P}^9$ with a quadric and a linear space.*

(v) *A general set in $\mathcal{A}_5$ is a linear section of the Lagrangean Grassmannian $\mathrm{LG}_+(5, 10) \subseteq \mathbb{P}^{15}$.*

(vi) *A general set in $\mathcal{A}_6$ is a linear section of the Grassmannian $X_8 = \mathrm{Gr}(2, 6) \subseteq \mathbb{P}^{14}$.*

In cases (ii)–(vi), an equivalent statement is that the set $\Gamma \subseteq \mathbb{P}^{n-1}$ is a hyperplane section of a canonical curve of genus $n + 1$ in $\mathbb{P}^n$.

Part (iv)–(vi) all correspond to a non-trivial lifting problem, the last one being exactly Problem 1.1. For reference, we record a precise statement of Petrakiev's result on $\mathcal{A}_6$. The group $\mathrm{SL}(6)$ acts on $\mathbb{C}^{15} = \bigwedge^2 \mathbb{C}^6$ and hence induces an action on $\mathrm{Gr}(7, \mathbb{C}^{15})$, leaving $X_8 \subseteq \mathbb{P}(\mathbb{C}^{15})$ invariant.

Theorem 2.11 ([Pet09, Thm. 3.3]). *The slicing map*

$$f \colon \mathrm{Gr}(7, \mathbb{C}^{15}) / \mathrm{SL}(6) \dashrightarrow \mathcal{A}_6, \qquad \mathbb{L} \mapsto \mathbb{L} \cap X_8$$

defines a generically finite dominant morphism to the moduli space of self-dual points $\mathcal{A}_6$.

3 Slicing using homotopy continuation

3.1 Introduction to homotopy continuation

Homotopy continuation is a computational method used to solve a multivariate polynomial system $F(x) = 0$ with a zero-dimensional set of solutions over the complex numbers. It relies on finding a homotopy that deforms $F(x)$ into a system that is easier to solve. HomotopyContinuation.jl can solve square systems without specifying a start system and a homotopy to follow. Usually the system is parameterized introducing some variables $\underline{a}$ giving a family of systems $F(x, \underline{a})$ with the same number of solutions for every $\underline{a}$. The system we want to solve corresponds to the target parameter

$F(x, \underline{a}_{\text{target}}) = F(x)$ and the one that we are able to solve corresponds to the start parameter $F(x, \underline{a}_{\text{start}})$. Defining a homotopy between these two systems, we move the start parameter and, numerically tracking paths, we deform the solution of $F(x, \underline{a}_{\text{start}})$ into the solution of $F(x, \underline{a}_{\text{target}})$. An example of homotopy between two systems G and F is given by a straight line between them:

$$H(x, u) = (1 - u)G(x) + uF(x) \quad u \in [0, 1].$$

This technique has been implemented by Breiding and Timme [BT18] in the `Julia` package `HomotopyContinuation.jl`, as the following snippet of code demonstrates.

```
using HomotopyContinuation as HC
@var x[1:n] a[1:m]
f = [f_1,...,f_l] # Define equations f_i(x,a) appropriately
F = System(f, variables = x[1:n], parameters = a[1:m])
res = HC.solve(F, start_sol; start_parameters = a_start, target_parameters = a_target)
```

For a polynomial system there are infinitely many possible start systems and homotopies. Different choices may have a different impact on the computational time of the `solve()` function. If the start system has the same number of solutions as the $F(x)$, we say that it is *optimal*. Optimal start systems are preferred because they require to track the minimal number of paths possible and still get all solutions of $F(x) = 0$. This choice also shortens significantly the computation time. However, in general there is no way to predict which start system and homotopy will lead to the quickest time to solve. It is also necessary to ensure that the parameterized system $F(x, \underline{a})$ has the same number of solutions. In other words, we need a parameterization $F(x, \underline{a})$ such that we can keep track of all the start solutions along the way. This is a fundamental challenge in the Mukai lifting problem, because it requires to parameterize the space of self-dual points in $\mathbb{P}^6$.

3.2 Computing a linear section of the Grassmannian

In this section we compute a set of self-dual points Γ as a given 6-dimensional linear subspace $\mathbb{L}$ intersecting the Grassmannian $X_8 := \mathrm{Gr}(2, 6) \subseteq \mathbb{P}^{14}$. By construction, the linear space $\mathbb{L} \subseteq \mathbb{P}^{14}$ is a solution to the Mukai lifting problem for Γ. In the next section we set up a homotopy in which Γ serves to get the start parameter and $\underline{a}_{\text{start}}$ and $\mathbb{L}$ serves to get the start solution. We recall that X_8 parameterizes $2-$dimensional subspaces in $\mathbb{C}^6$. We represent an element in an affine chart of X_8 as the row span of the 2×6 matrix:

$$H = \begin{pmatrix} 1 & 0 & t_1 & t_2 & t_3 & t_4 \\ 0 & 1 & t_5 & t_6 & t_7 & t_8 \end{pmatrix}. \tag{4}$$

The Plücker embedding $p \colon \mathrm{Gr}(2, 6) \hookrightarrow \mathbb{P}^{14}$, $p(H) = (p_0 : \cdots : p_{14}) \in \mathbb{P}^{14}$ is given by taking the 2×2 minors of H, these are the 15 Plücker coordinates in $\mathbb{C}[t_1, \ldots, t_8]$. In our affine chart we have $p_0 = 1$.

Lemma 3.1. *Let $A \in \mathbb{C}^{8 \times 15}$ be a general matrix and $\mathbb{L} = \mathbb{P}(\mathrm{Ker}\, A)$. The polynomial system in $\underline{t}$*

$$A \cdot \big(p_0(\underline{t}), \ldots, p_{14}(\underline{t})\big)^{\mathsf{T}} = 0 \tag{5}$$

has exactly 14 solutions $\underline{t}_1, \ldots, \underline{t}_{14}$ and the set $\{\, p(\underline{t}_k) \mid k = 1, \ldots, 14 \,\}$ is the image of a configuration of self-dual points $\Gamma \subseteq \mathbb{P}^6$ under a linear map $L \colon \mathbb{P}^6 \overset{\sim}{\to} \mathbb{L} \subseteq \mathbb{P}^{14}$.

Proof. The Grassmannian X_8 is a $8-$dimensional variety of degree 14 [MS21, Thm 5.13], so its intersection with 8 linear hyperplanes in general position contains exactly 14 points. Let L be a kernel matrix for A. If $\underline{t}_k$ is a solution to (5), then $p(\underline{t}_k)$ is a linear combination of the columns of L and it lies in the image $\mathbb{L} := \text{Im}(L)$ of the linear map $L \colon \mathbb{P}^6 \to \mathbb{P}^{14}$. Since $X_8 \subseteq \mathbb{P}^{14}$ is arithmetically Gorenstein, so is $P \subseteq \mathbb{L}$ and by Theorem 2.3(iii) there exists a self dual set of points $\Gamma \subseteq \mathbb{P}^6$ such that $L(\Gamma) = P$. To compute such a configuration it is enough to compute a solution Γ to:

$$
L \cdot \Gamma = \begin{pmatrix} p_0(\underline{t}_1) & & p_0(\underline{t}_{14}) \\ \vdots & \dots & \vdots \\ p_{14}(\underline{t}_1) & & p_{14}(\underline{t}_{14}) \end{pmatrix}.
$$

$\square$

We point out that the previous Lemma gives an algorithm to construct the self-dual configuration Γ in a linear section $\mathbb{L}$ defined by 8 hyperplanes $\sum_{j=1}^{15} a_{ij} x_j = 0$, $i = 1, \dots, 8$. After fixing a random matrix A, we use a toric degeneration of the Grassmannian to compute solutions $\underline{t}_k$ for (5). We dedicate the following section to describe the construction of the degeneration.

3.3 Toric degeneration of the Grassmannian

The homogeneous coordinate ring $R := S(X) \subseteq \mathbb{K}[t_1, \dots, t_n]$ of a projective variety $X \subseteq \mathbb{P}^{n-1}(\mathbb{K})$ can be studied by looking at the *initial algebra* of R with respect to a term order $\prec$, that is the algebra generated by the initial terms with respect to $\prec$ of every element in R:

$$
\text{in}_\prec(R) := k[\text{in}_\prec(f) : f \in R].
$$

Many geometric properties of the variety X can be understood by looking at the corresponding algebraic properties of the initial algebra $\text{in}_\prec(R)$. For example, if the initial algebra is normal, Cohen-Macaulay or Gorenstein, then so is the coordinate ring R. We refer to [Bru+22] for further details. Although the coordinate ring of a projective variety is a finitely generated algebra, the corresponding initial algebra may not be finitely generated for a given term order. This happens when R does not admit a finite Khovanskii basis for that term order. The existence of universal finite Sagbi basis is an active research problem, see [BC24] for some recent developments.

Definition 3.2. Let $R \subseteq k[t_1, \dots, t_n]$ be a finitely generated polynomial algebra. A set of generators $\mathcal{F}$ of R is a *Khovanskii basis* with respect to a term order $\prec$ if $\text{in}_\prec(R) = k[\text{in}_\prec(f) : f \in \mathcal{F}]$.

Khovanskii bases have been introduced by Robbiano and Sweedler in [RS06] and by Sturmfels in [Stu96], respectively with the name of *Sagbi bases* and *Canonical Subalgebra bases*. Later, Kaveh and Manon in [KM19] gave a more general definition introducing the terminology that we use, even if in the context of this paper is in the Sagbi case. Khovanskii bases are the analogoues of Gröbner bases for ideals to subalgebras and they have been effectively used in computer algebra in [BPT25]. When the coordinate ring R admits a finite Khovanskii basis, the initial algebra is a finitely generated monomial algebra, hence it can be viewed as the coordinate ring of a toric variety. In this case we say that X admits a *toric degeneration* to the projective toric variety $\text{Proj}(\text{in}_\prec(R))$.

Definition 3.3. A toric degeneration of $X \subseteq \mathbb{P}^n$ is a variety $Y \subseteq \mathbb{P}^n \times \mathbb{A}^1$ such that the projection $\pi \colon Y \to \mathbb{A}^1$ is flat and $\pi^{-1}(0) \hookrightarrow \mathbb{P}^n$ is a toric variety and $\pi^{-1}(\lambda) = X$ for every $\lambda \neq 0$.

Degenerations are used to formalize the concept of limit of a family of varieties as the *special fiber* $\pi^{-1}(0)$ can be thought of as the limit of $\pi^{-1}(t)$ for $t \to 0$. Now we illustrate explicitly this degeneration for the Grassmannian X_8. For a *weight vector* ω, we define the ω−initial form $\mathrm{in}_\omega(f)$ of a polynomial $f = \sum c_\alpha t^\alpha$ to be the sum of the terms $c_\alpha t^\alpha$ such that $\alpha \cdot \omega$ is maximal. A weight vector ω represents a term order $\prec$ for a finite set of polynomials $\mathcal{F}$ if $\mathrm{in}_\omega(f) = \mathrm{in}_\prec(f)$ for every $f \in \mathcal{F}$. The Plücker coordinates generate the coordinate ring R of $X_8 \cap \{p_0 = 1\} \subseteq \mathbb{A}^{14}$, that is $R = \mathbb{C}[p_0(\underline{t}), \dots, p_{14}(\underline{t})]$. A *diagonal term order* is a term order for which the initial terms of the Plücker coordinates p_i are the corresponding monomial in a main diagonal of (4).

Proposition 3.4. *[Stu96, Prop. 11.8] The Plücker coordinates are a finite Khovanskii basis for the coordinate ring R of the Grassmannian X_8 with respect to any diagonal term order $\prec_d$, which is represented by the weight vector $\omega = (3, 2, 1, 0, 0, 1, 2, 3)$:*

$$\mathrm{in}_\omega(R) = k[\mathrm{in}_\omega(p_1), \dots, \mathrm{in}_\omega(p_{15})].$$

We introduce a new variable u, that will be the parameter of the degeneration and, for every polynomial $f = \sum_\alpha c_\alpha t^\alpha \in R$, we define $\nu(f) \coloneqq \max\{\omega \cdot \alpha \mid c_\alpha \neq 0\}$ and

$$f_u \coloneqq \sum_\alpha c_\alpha t^\alpha \cdot u^{\nu(f) - \omega \cdot \alpha} \in k[t_1, \dots, t_8, u],$$

$$R_u \coloneqq k[f_u : f \in R] \subseteq k[t_1, \dots, t_8, u]. \tag{6}$$

We observe that $f_0 = \mathrm{in}_\omega(f) = \lim\limits_{u \to 0} t_u$ and $f_1 = f$. In particular we recover the coordinate ring and its initial algebra as $R_0 = \mathrm{in}_\omega(R)$ and $R_1 = R$. By moving the parameter u from 1 to 0 we degenerate the coordinate ring R into a monomial algebra. The latter is the coordinate ring of a toric variety and this corresponds to degenerating X_8 into the toric variety $\mathrm{Spec}(R_0)$.

Remark 3.5. The previous construction is an application of the more general result by Anderson [And13] regarding toric degenerations of projective varieties whose coordinate ring has a full-rank valuation with finitely generated value semigroup. Taking ω as before, the full-rank valuation

$$\nu : R \setminus 0 \to \mathbb{Z}^8, \quad \nu(\sum_\alpha c_\alpha t^\alpha) = \mathrm{argmax}\{\omega \cdot \alpha \mid c_\alpha \neq 0\}.$$

induces a term order represented by the vector ω. The existence of a finite Khovanskii basis for this term order is equivalent to having a finitely generated value semigroup for ν by [KM19].

Going back to Equation (5), we introduce the parameter u to degenerate it to a toric ideal as follows. We denote by $\underline{a}$ the entries of the matrix A from Lemma 3.1 and define the system $F(\underline{t}, u, \underline{a})$ with 8 equations by:

$$F(\underline{t}, u, \underline{a}) = \{A \cdot (p_{0,u}(\underline{t}), \dots, p_{14,u}(\underline{t}))^\mathsf{T} = 0\}. \tag{7}$$

The variables $u, \underline{a}$ are the parameters in the homotopy, the equations in (5) correspond to the target system $F(\underline{t}, 1, \underline{a})$. Fixing random values $\underline{a}_s$ we obtain the start system $F(\underline{t}, 0, \underline{a}_s)$ that involves only linear relations among monomials, i.e. the leading terms of the Plücker coordinates. `HomotopyContinuation.jl` can easily solve this by automatically choosing a polyhedral homotopy. Then we track the solutions to obtain the solutions of $F(\underline{t}, 1, \underline{a}_s)$. Given a matrix A with entries $\underline{a}_t$, we solve $F(\underline{t}, 1, \underline{a}_t)$ as the following snippet of `Julia` code shows.

The equations in (7) are encoded in the vector `F`. The first homotopy compute the solution on the toric degeneration by moving the hyperplanes that intersect it and the parameter $u = 0$ is fixed.

The second homotopy transport the solution from the toric degeneration to the Grassmannian X_8 by increasing the parameter u from 0 to 1 and fixing the parameters $\underline{a}$. The third homotopy finally solves $F(\underline{t}, 1, \underline{a}_t)$ by tracking the solution of $F(\underline{t}, 1, \underline{a}_s)$.

```
C          = System(F, variables = x, parameters = [u;a[:]])

A_rand     = randn(ComplexF64, length(a));
a0_start   = [0;A_rand]
toric_sol  = solutions(HC.solve(C,                  # solutions of F(t,0,a_s)
                    target_parameters = a0_start))

a1_start   = [1;A_rand]
start_sol  = solutions(HC.solve(C, toric_sol; # solutions of F(t,1,a_s)
                    start_parameters = a0_start, target_parameters = a1_start))

a1_target  = [1;A] # given A
X8_cap_A   = solutions(HC.solve(C, start_sol; # solutions of F(t,1,a_t)
                    start_parameters = a1_start, target_parameters = a1_target))
```

Once we have the solutions P to $F(\underline{t}, 1, \underline{a}_t)$, we obtain a configuration of self-dual points Γ that is embedded in $\mathbb{P}^{14}$ by solving the linear system $L(\Gamma) = P$, where $L := \mathrm{Ker}([\underline{a}_t])$ is a kernel basis. This is illustrated in the following snippet of code, the function `plücker_emb` (numerically) evaluates the Plücker embedding on points in the affine chart $H \cong \mathbb{C}^8$.

```
Z = reshape(vcat(plücker_emb.(X8_cap_A)...),15,14) # Embed points in P^14
L = LinearAlgebra.nullspace(A) # Obtain matrix L whose image is Ker(A)
Γ = L\Z # Take the inverse image of Z under L
```

3.4   Real solutions

For a transversal slicing of the Grassmannian X_8 with 8 complex hyperplanes, as described in Equation (5), we get 14 complex solutions. We study how many of these are real for random choices of real hyperplanes. The regions in $\mathrm{Gr}(7, \mathbb{R}^{15})$ corresponding to transversal linear spaces are partitioned into open semi-algebraic cells by the number of real intersection points.

We compute the solutions with `HomotopyContinuation.jl` using $F(\underline{t}, 1, \underline{a}_t)$ as start system, for which we computed a start solution `X8_cap_A` previously. The hyperplanes are chosen as follows: we draw the real coefficients of the defining equations from the standard normal distribution. This gives the uniform distribution on the Grassmannian $\mathrm{Gr}(7, 15)$ with respect to the Haar measure from the compact real Lie group $\mathrm{O}(15)$ [Chi03, Thm. 2.2.1]. In Table 1 we report the result of the sampling after ten million iterations. We observe that the case in which all 14 solutions are real is very rare but it does happen. The columns *succ* and *fail* report the number of times the computations succeeded in computing all the 14 distinct solutions and the number of times it failed.

Real solutions	0	2	4	6	8	10	12	14	succ	fail
Count	178244	2277545	4079864	2547002	790369	116267	10508	192	999991	9
Proportion	1.78%	22.78%	40.80%	25.47%	7.90%	1.16%	0.11%	0.002%	100%	-

Table 1: Slicing X_8 with uniformly sampled real $\mathbb{L} \subseteq \mathbb{P}^{14}(\mathbb{R})$ 10^7 times.

4 Lifting using Homotopy Continuation

In the previous section we computed pairs $(\mathbb{L}, Z)$ with $X_8 \cap \mathbb{L} = Z \subseteq \mathbb{P}^{14}$. In this section these will be turned into the start solution of a parameterized polynomial system solving the lifting problem Problem 1.1. As mentioned, an important step for using Homotopy Continuation is parameterizing the space of self-dual points $\mathcal{A}_6$ in $\mathbb{P}^6$. The main problem is that this space is not convex (in an obvious way) and, more generally, it is crucial to find a path between two self-dual configurations that stays *inside* the space of self-dual points. Using the parametrization given by $\mathbb{C}^{21} \dashrightarrow \mathcal{A}_6$ that exploits the skew normal form in Theorem 2.9, we solve this problem efficiently: after computing normal forms for start and target configuration, we can choose a straight line through the parameter space $\mathbb{C}^{21}$. If s_{start} is chosen sufficiently randomly, this path will avoid the discriminant (branch locus) as it has real codimension 2.

With this structure, the Mukai lifting problem becomes the following: For $S \in \mathrm{Skew}(7) \cong \mathbb{C}^{21}$ we want to recover the embedding $L \colon \mathbb{P}^6 \xrightarrow{\sim} \mathbb{L} \subseteq \mathbb{P}^{14}$ such that

$$L \cdot \{\gamma \in [I_7 + S \mid I_7 - S]\} = \mathbb{L} \cap X_8.$$

Lemma 4.1. *Let γ_i be the i-th column of a set Γ in skew normal form and let $q_1, \ldots q_{15}$ be the Plücker relations of X_8, i.e. the generators of the defining ideal of $X_8 \subseteq \mathbb{P}^{14}$. The embedding $L = [\ell_{ij}] \colon \mathbb{P}^6 \xrightarrow{\sim} \mathbb{L} \subseteq \mathbb{P}^{14}$ is a solution for the Mukai lifting problem for Γ if and only if*

$$q_k(L\gamma_i) = 0 \qquad \text{for all } i = 1, \ldots, 14, \ k = 1, \ldots, 15. \tag{8}$$

Proof. The only condition that L has to satisfy is that the image of Γ lies inside the Grassmannian X_8. This is ensured by the vanishing of the Plücker relations on $L \cdot \gamma_i$ for every $i = 1, \ldots, 14$. Indeed if Equation (8) holds, then $L(\Gamma) \subseteq \mathbb{L} \cap X_8$ and we have equality as the latter is a set of 14 points by Lemma 3.1 and L is injective. $\qquad\square$

We observe that the system in (8) has degree 2 in the 105 variables ℓ_{ij} and has $14 \cdot 15 = 210$ equations, so it is heavily over-determined. Additionally, this system is not zero-dimensional, which is a necessary condition for applying homotopy continuation methods. Indeed, Petrakiev's result [Pet09] implies that the rational map

$$f \colon \mathrm{Gr}(7, \mathbb{C}^{15}) \dashrightarrow \mathcal{A}_6, \qquad \mathbb{L} \mapsto \mathbb{L} \cap X_8$$

generically has 35-dim'l fibers (unions of orbits of the $\mathrm{SL}(6)$ action on $\mathbb{C}^{15} = \bigwedge^2 \mathbb{C}^6$).

Let $\widehat{\mathcal{A}}_6 \subseteq (\mathbb{P}^6)^{14}$ be the locally closed set of semistable self-dual point configuration. In Section 3 we considered the related rational map (note that we considered kernel matrices instead):

$$\hat{f} \colon \mathbb{C}^{7 \times 15} \dashrightarrow \widehat{\mathcal{A}}_6 / \mathfrak{S}_{14}, \qquad A \mapsto A^{-1}(\mathrm{Im}(A) \cap X_8) \subseteq \mathbb{P}^6.$$

The relationship between f, $\hat{f}$ and the skew normal forms can be summarized as follows.

Theorem 4.2. *We have a commutative diagram*

$$
\begin{array}{ccc}
\widehat{\mathcal{A}}_6 & \subseteq & (\mathbb{P}^6)^{14} \\
\downarrow & & \quad\text{SNF} \\
\mathbb{C}^{7\times 15} \overset{\hat{f}}{\dashrightarrow} \widehat{\mathcal{A}}_6/\mathfrak{S}_{14} \subseteq (\mathbb{P}^6)^{14}/\mathfrak{S}_{14} & & \mathrm{Skew}(7) \\
\downarrow & \circlearrowleft & \\
\mathrm{Gr}(7,\mathbb{C}^{15}) & & gen.fin \\
\downarrow & & \\
\mathrm{Gr}(7,\mathbb{C}^{15})/\operatorname{SL}(6) \underset{gen.fin}{\overset{f}{\dashrightarrow}} \mathcal{A}_6 & \subseteq & \mathcal{P}_6^{14}/\mathfrak{S}_{14}
\end{array}
$$

In particular, $\hat{f}$ is dominant with general fiber of dimension $\dim \hat{f}^{-1}(\Gamma) = 36$.

Proof. Commutativity of the diagram follows from the previous discussion. Since f is dominant (Theorem 2.11) and the image of $\hat{f}$ is invariant under $\operatorname{SL}(7)$, $\hat{f}$ is dominant too. The dimension of a general fiber is
$$
\dim \hat{f}^{-1}(\Gamma) = \dim \mathbb{C}^{7\times 15} - \dim \widehat{\mathcal{A}}_6 = 105 - 69 = 36. \qquad \square
$$

To get a system with a finite set of solutions we need to impose a number of conditions on the entries of L equal to the dimension of a general fiber of $\hat{f}$. Hence we can construct L as follows using 69 variables ℓ_i, $i = 1, \ldots, 69$. As described in Section 3 we consider a self-dual start configuration Γ_{start} embedded in the linear space $\mathbb{L}_{\text{start}} := L_{\text{start}}(\mathbb{P}^6) \subseteq \mathbb{P}^{14}$. By adding random linear combinations of the 69 variables in each entry of L_{s}, given by random matrices A_k for $k = 1, \ldots, 69$, we construct L as:
$$
L = L_{\text{start}} + \sum_{k=1}^{69} \ell_k \cdot A_k
$$

With this interpretation, the start solution for `HomotopyContinuation.jl` is a vector of zeros $\ell_{\text{start}} = (0, \ldots, 0)$ and the start parameters are the entries of the skew-symmetric matrix S_{start} in a skew normal form of Γ_{start}.

The number of solutions to this system could be larger than the degree of the slicing map f (which is conjecturally birational), because we are intersecting $\operatorname{SL}(6)$-orbits with linear spaces. Since we only want to construct *some* Mukai lift, we do not investigate this subtlety further, though it would be interesting to study this degree in future work. We finally describe part of the construction of Equation (8) in `Julia`. We concatenate three different systems as follows:

- First we define `L_system` to be the system containing the equations of the image of a linear map $L : \mathbb{P}^6 \to \mathbb{P}^{14}, x \mapsto Lx$, as in Lemma 4.1 and L as described above in the variables $\mathtt{l} \,\hat{=}\, (\ell_1, \ldots, \ell_{69})$. The variables $\mathtt{x}$ are the coordinates of a point in $\mathbb{P}^6$.

- Then we evaluate the Plücker relations in these expressions, obtaining the system `Q_system` of 15 equations in $\mathtt{l}$ and $\mathtt{x}$.

- Finally, we evaluate the equations of `Q_system` in each of the points in $\Gamma_{\mathtt{s}} = [I_7 + S | I_7 - S]$, keeping the variables $\mathtt{s} \,\hat{=}\, S \in \mathrm{Skew}(7)$ as parameters.

A relevant part of the construction is summarized in the following snippet:

```
L_system = System(L*x, variables=[x;l])                                              1
plücker_system = System([oscar_to_HC_Q(plück_oscar[i], q) for i=1:15], variables=q)  2
Q = plücker_system(expressions(L_system));                                           3
Q_system = System(Q, variables=[x;l]);                                               4
equations = vcat([Q_system([Γ[:,i];l]) for i in 1:14]...);                           5
parameterized_system = System(equations, variables=l, parameters=s);                6
```

The full implementation together with the example notebook presented in the next section can be found at

https://mathrepo.mis.mpg.de/MukaiLiftP6

5 Implementation and extended example

In this section we explain the implementation through an instructive example. We consider the self-dual configuration $\Gamma \subseteq \mathbb{P}^6$ given by the columns of the following matrix:

$$
\Gamma = \begin{bmatrix}
7 & -2 & 6 & -1 & -6 & 1 & -9 & 7 & 0 & 6 & 1 & 8 & -3 & 7 \\
-1 & 2 & -5 & -2 & 0 & -4 & 3 & -3 & -4 & -3 & 4 & -2 & 4 & -1 \\
1 & 4 & -1 & -5 & -3 & 6 & 8 & -1 & -8 & -3 & 5 & 1 & -6 & -8 \\
3 & -6 & 4 & -3 & -4 & 6 & 0 & 5 & 8 & 2 & 3 & 2 & -8 & 0 \\
1 & -2 & 1 & 0 & -4 & 2 & 2 & 3 & 4 & -1 & 2 & 2 & -2 & -2 \\
0 & -6 & -5 & 6 & 3 & 7 & -3 & 2 & 8 & -7 & -6 & -3 & -5 & 5 \\
-3 & 3 & -4 & 1 & 4 & 3 & 2 & -3 & -6 & -4 & -3 & -4 & -1 & -2
\end{bmatrix}
$$

To verify that Γ is indeed self-dual we use condition (ii) of Lemma 2.1 and compute an invertible diagonal matrix Λ verifying $\Gamma \cdot \Lambda \cdot \Gamma^{\mathsf{T}} = 0$. This is a linear system in 14 unknowns (the entries of Λ) and $\binom{7+1}{2} = 28$ equations, namely $\sum_{i=1}^{14} \lambda_i \gamma_i \gamma_i^{\mathsf{T}} = 0$. The function `test_self_dual` returns a floating point approximation of the (exact) normed solution

$$
\Lambda = \frac{1}{\sqrt{13.0625}} \operatorname{diag}(-1,-1,-1,-1,-1,-1,-1,1,\tfrac{1}{4},1,1,1,1,1).
$$

```
Lambda = test_self_dual(Gamma)                                               1
norm( Gamma * Lambda * transpose(Gamma), Inf ) #4.085620730620576e-14        2
```

We bring Γ into an orthogonal normal form Γ_{ONF} as in Theorem 2.7, essentially using linear algebra. Applying an appropriate scaling Λ_{scale} and linear map A we have $A \cdot \Gamma_{\mathrm{ONF}} = \Gamma \cdot \Lambda_{\mathrm{scale}}$.

$$
\Gamma_{\mathrm{ONF}} = [I_7 \mid P] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -0.3399 & 0.06 & -0.0924 & 0.235 & 0.7564 & -0.3033 & 0.3911 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -0.2052 & -0.8506 & 0.0197 & -0.1977 & 0.268 & 0.1942 & -0.2922 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0.1008 & 0.2302 & 0.8118 & -0.2422 & 0.3148 & -0.1311 & -0.3208 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0.0948 & 0.319 & -0.4084 & -0.722 & 0.3291 & 0.3032 & -0.0306 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -0.7734 & 0.0615 & 0.1552 & -0.4091 & -0.3747 & -0.1609 & 0.2009 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.2627 & -0.1669 & -0.2427 & -0.2875 & -0.072 & -0.8558 & -0.1552 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -0.3951 & 0.2942 & -0.2868 & 0.2752 & 0.055 & -0.0546 & -0.7703
\end{bmatrix}
$$

$$
A = i \begin{bmatrix}
3.6821 & -1.052 & 3.1561 & -0.526 & -3.1561 & 0.526 & -4.7341 \\
-0.526 & 1.052 & -2.63 & -1.052 & 0.0 & -2.104 & 1.578 \\
0.526 & 2.104 & -0.526 & -2.63 & -1.578 & 3.1561 & 4.2081 \\
1.578 & -3.1561 & 2.104 & -1.578 & -2.104 & 3.1561 & 0.0 \\
0.526 & -1.052 & 0.526 & 0.0 & -2.104 & 1.052 & 1.052 \\
0.0 & -3.1561 & -2.63 & 3.1561 & 1.578 & 3.6821 & -1.578 \\
-1.578 & 1.578 & -2.104 & 0.526 & 2.104 & 1.578 & 1.052
\end{bmatrix} \qquad \Lambda_{\mathrm{scale}} = \sqrt{\Lambda[1{:}7]} \oplus -\sqrt{\Lambda[8{:}14]}
$$

```
Gamma_ONF, A, Lambda_scale = orthogonal_normal_form(Gamma)                         1
norm( A * Gamma_ONF - Gamma * Lambda_scale, Inf ) # 2.6645352591003757e-15         2
```

14

We compute a skew normal form Γ_{SNF} by applying the Cayley transform to P to obtain the skew-symmetric matrix S and we compute $\Gamma_{\mathrm{SNF}} = [I_7 + S | I_7 - S]$. The output is a floating point approximation of the following exact matrix:

$$S = \begin{bmatrix} 0 & -1 & 1 & 0 & -1 & 4 & 2 \\ 1 & 0 & 3 & -2 & -2 & 10 & 12 \\ -1 & -3 & 0 & 2 & 1 & -1 & -2 \\ 0 & 2 & -2 & 0 & -1 & -10 & -6 \\ 1 & 2 & -1 & 1 & 0 & -4 & -4 \\ -4 & -10 & 1 & 10 & 4 & 0 & -6 \\ -2 & -12 & 2 & 6 & 4 & 6 & 0 \end{bmatrix}$$

$$\Gamma_{\mathrm{SNF}} = \begin{bmatrix} 1 & -1 & 1 & 0 & -1 & 4 & 2 & 1 & 1 & -1 & 0 & 1 & -4 & -2 \\ 1 & 1 & 3 & -2 & -2 & 10 & 12 & -1 & 1 & -3 & 2 & 2 & -10 & -12 \\ -1 & -3 & 1 & 2 & 1 & -1 & -2 & 1 & 3 & 1 & -2 & -1 & 1 & 2 \\ 0 & 2 & -2 & 1 & -1 & -10 & -6 & 0 & -2 & 2 & 1 & 1 & 10 & 6 \\ 1 & 2 & -1 & 1 & 1 & -4 & -4 & -1 & -2 & 1 & -1 & 1 & 4 & 4 \\ -4 & -10 & 1 & 10 & 4 & 1 & -6 & 4 & 10 & -1 & -10 & -4 & 1 & 6 \\ -2 & -12 & 2 & 6 & 4 & 6 & 1 & 2 & 12 & -2 & -6 & -4 & -6 & 1 \end{bmatrix}$$

```
P = Gamma_norm[:,8:14]
S = cayley(P)
Gamma_SNF = [I+S I-S]
```

We compute the linear embedding $L \colon \mathbb{P}^6 \xrightarrow{\sim} \mathbb{L} \subseteq \mathbb{P}^{14}$ such that $L(\Gamma_{\mathrm{SNF}}) = \mathbb{L} \cap X_8$. The parametrized polynomial system in Equation (8) has been precomputed as F as described in Section 4 together with start solution l_start = 0 and start parameters S_start. The target parameter of the homotopy are the entries of the matrix S. We executed this computation on a single thread of a 512 GB RAM machine using 2x 12-Core Intel Xeon E5-2680 v3 processor working at 2.50 GHz in 3172 seconds.

```
S_target = skew_to_vector(S) # unrolls the upper right half of S into its 21 entries
@time result = HomotopyContinuation.solve(F, l_start;
            start_parameters=S_start, target_parameters=S_target)  #3172.87531 seconds
sol = solutions(result)[1]
L_tilde = L_start + sum(sol[i]*A_rand[i] for i in eachindex(l))
```

$$L = \begin{bmatrix} -2.399-0.761i & 0.137-0.073i & 0.818-0.374i & -1.274-0.479i & 0.299+0.667i & 1.461+0.897i & -0.055+1.543i \\ 0.12-0.688i & 0.119+0.018i & -0.816-0.003i & 0.586-0.149i & 0.105-0.01i & -0.18-0.6i & 0.61-0.349i \\ -0.875+0.49i & -0.133+0.162i & 0.865-0.349i & -0.468+0.231i & -1.022-0.629i & -0.266+0.524i & -0.484-0.171i \\ 0.09-0.809i & 0.213+0.539i & 0.311-0.255i & -0.134+0.12i & 0.964+0.728i & -0.207+0.732i & -0.337+0.803i \\ -0.661-1.146i & 0.031+0.383i & 0.943-0.325i & -0.774-0.038i & 0.469+0.059i & 0.07+0.714i & -0.554+0.551i \\ 1.919+0.501i & 0.078+0.334i & 1.962+0.777i & 0.14-0.603i & -0.376+1.007i & -0.049+1.538i & -1.346+0.213i \\ -0.267-0.832i & -0.256-0.495i & -0.762+0.311i & -1.382+1.048i & 0.174+0.08i & -0.417-1.807i & -2.217+0.998i \\ -0.225-3.906i & -0.636-0.176i & 0.042+0.974i & 0.293-1.761i & -3.008-1.518i & -0.34+0.89i & -1.235-1.536i \\ -0.409-0.898i & -0.342-0.402i & -0.963+1.551i & 0.086-1.274i & -2.462+0.308i & 0.013-0.069i & -1.701-0.732i \\ 1.336+1.106i & -0.01+0.302i & -0.13-0.816i & 0.907+0.758i & -0.373+0.631i & 0.33+0.598i & 0.557+0.716i \\ -0.113-0.063i & 0.42-0.176i & -0.108-1.307i & -0.154+0.469i & 0.835-0.138i & 0.911-0.294i & 0.508+0.839i \\ -0.01+0.158i & 0.208+0.008i & -0.205-1.466i & 0.093+0.984i & 0.507-0.232i & 0.533-0.285i & 0.734+0.933i \\ -0.188-0.131i & 0.35+0.326i & -0.378+0.453i & 1.402+0.609i & 1.196-0.736i & -0.807+0.391i & 1.627+0.668i \\ -0.559-0.346i & 0.222+0.314i & 0.131+0.08i & 0.644+0.2i & 0.715-0.678i & -0.357+0.44i & 1.039+0.243i \\ -0.79+0.774i & 0.005-0.041i & -0.313-0.484i & -0.329+0.156i & -0.599+0.324i & 0.384-0.152i & -0.418-0.097i \end{bmatrix}$$

To obtain the map $\widehat{L} \colon \mathbb{P}^6 \hookrightarrow \mathbb{P}^{14}$ with $\widehat{L}(\Gamma) \subseteq X_8$, one only needs to compose L with the previous coordinate transformations $\Gamma \rightsquigarrow \Gamma_{\mathrm{SNF}}$:

$$\mathbb{L} \cap X_8 = L(\Gamma_{\mathrm{SNF}}) = L\left((I_7 + S) \cdot \Gamma_{\mathrm{ONF}} \right) = L\left((I_7 + S) \cdot A^{-1} \cdot \Gamma \cdot \Lambda_{\mathrm{scale}} \right).$$

Since the configuration Γ does not change when multiplying on the right by a diagonal matrix, we obtain the solution for the Mukai lifting problem for Γ as:

$$\widehat{L} = L \cdot (I_7 + S) \cdot A^{-1}.$$

We can verify $\widehat{L}(\Gamma) = \mathrm{Im}(\widehat{L}) \cap X_8$ by evaluating $\widehat{L}(\gamma_i)$ in the Plücker relations

```
L = L_tilde*(I+S)*inv(A)
maximum([ norm(q(L*Gamma[:,i]) ,Inf) for i =1:14]) #1.28058016710644476e-11
```

15

Outlook

The last section illustrated the procedure developed in this paper to compute a solution for the Mukai lifting problem of a configuration of self-dual points in $\mathbb{P}^6$. We observe that using our approach, the linear space will *never* be defined over the real numbers, even if Γ is real. It is an interesting question to extend this approach to find real spaces for real self-dual point configurations.

This project focused on $\mathcal{A}_6$, though our methods can also be applied to $\mathcal{A}_5$, replacing $\mathrm{Gr}(2,6)$ with $X_7 = \mathrm{LG}_+(5,10) \subseteq \mathbb{P}^{15}$. It would be an interesting challenge to devise an algorithm for the Mukai lifting of general canonical curves $C \subseteq \mathbb{P}^7$, i.e. for general points $[C] \in \mathcal{M}_8$.

As mentioned in Section 4, Petrakiev's map f (Theorem 2.11) is conjecturally birational modulo $\mathrm{SL}(6)$. One could experimentally study the degree of this map using monodromy methods. Following our approach, this is likely related to the degree of the restriction $\hat{f}_{|W}$ to a general 69-dimensional affine subspace $W \subseteq \mathbb{C}^{7 \times 15}$. This degree is expected to be larger than 1 even if f is birational, as mentioned in Section 4. Such an experimental study would also benefit from a faster evaluation of our polynomial system.

Acknowledgements

References

[And13]   Dave Anderson. "Okounkov bodies and toric degenerations". In: *Mathematische Annalen* 356.3 (2013), pp. 1183–1202. ISSN: 1432-1807. DOI: 10.1007/s00208-012-0880-3.

[Arz18]   Ivan Arzhantsev. "Gale duality and homogeneous toric varieties". In: *Communications in Algebra* 46.8 (2018), pp. 3539–3552. DOI: 10.1080/00927872.2018.1425411.

[BC24]   Winfried Bruns and Aldo Conca. "Sagbi combinatorics of maximal minors and a Sagbi algorithm". In: *Journal of Symbolic Computation* 120 (2024), p. 102237. ISSN: 0747-7171. DOI: 10.1016/j.jsc.2023.102237.

[Bez+12] Jeff Bezanson, Alan Edelman, Stefan Karpinski, and Viral B. Shah. "Julia: A Fast Dynamic Language for Technical Computing". In: (2012). arXiv: 1209.5145 [cs.PL].

[BPT25]   Barbara Betti, Marta Panizzut, and Simon Telen. "Solving equations using Khovanskii bases". In: *Journal of Symbolic Computation* 126 (2025), p. 102340. ISSN: 0747-7171. DOI: https://doi.org/10.1016/j.jsc.2024.102340. URL: https://www.sciencedirect.com/science/article/pii/S0747717124000440.

[Bru+22] Winfred Bruns, Aldo Conca, Claudiu Raicu, and Matteo Varbaro. *Determinants, Gröbner Bases and Cohomology*. Springer Monographs in Mathematics. Springer International Publishing, 2022. ISBN: 9783031054815. DOI: 10.1007/978-3-031-05480-8.

[BS08]   Frédéric Bihan and Frank Sottile. "Gale duality for complete intersections". In: *Annales de l'institut Fourier* 58.3 (2008), pp. 877–891. URL: http://eudml.org/doc/10337.

[BSW23]   Michael Burr, Frank Sottile, and Elise Walker. "Numerical homotopies from Khovanskii bases". In: *Math. Comput.* 92.343 (2023), pp. 2333–2353. DOI: 10.1090/MCOM/3689.

[BT18]   Paul Breiding and Sascha Timme. "HomotopyContinuation.jl: A Package for Homotopy Continuation in Julia". In: *Mathematical Software – ICMS 2018*. Ed. by James H. Davenport, Manuel Kauers, George Labahn, and Josef Urban. Cham: Springer International Publishing, 2018, pp. 458–465.

[Cam+18] Alessio Caminata, Noah Giansiracusa, Han-Bom Moon, and Luca Schaffler. "Equations for point configurations to lie on a rational normal curve". In: *Advances in Mathematics* 340 (2018), pp. 653–683. ISSN: 0001-8708. DOI: 10.1016/j.aim.2018.10.013.

[Chi03]   Yasuko Chikuse. *Statistics on Special Manifolds*. en. 2003rd ed. Lecture notes in statistics. New York, NY: Springer, Feb. 2003. DOI: 10.1007/978-0-387-21540-2.

[Cob22]   Arthur B. Coble. "Associated Sets of Points". In: *Transactions of the American Mathematical Society* 24.1 (1922), pp. 1–20. ISSN: 00029947. URL: http://www.jstor.org/stable/1989044 (visited on 01/31/2024).

[DO88]   Igor Dolgachev and David Ortland. *Point sets in projective spaces and theta functions*. en. Astérisque 165. Société mathématique de France, 1988. URL: http://www.numdam.org/item/AST_1988__165__1_0/.

[EP00]   David Eisenbud and Sorin Popescu. "The Projective Geometry of the Gale Transform". In: *Journal of Algebra* 230.1 (2000), pp. 127–173. ISSN: 0021-8693. DOI: 10.1006/jabr.1999.7940.

[KM19]   Kiumars Kaveh and Christopher Manon. "Khovanskii Bases, Higher Rank Valuations, and Tropical Geometry". In: *SIAM Journal on Applied Algebra and Geometry* 3.2 (2019), pp. 292–336. DOI: 10.1137/17M1160148.

[MFK94]   David Mumford, John Fogarty, and Frances Kirwan. *Geometric Invariant Theory*. Ergebnisse der Mathematik und Ihrer Grenzgebiete, 3 Folge/A Series of Modern Surveys in Mathematics Series. Springer Berlin Heidelberg, 1994. ISBN: 9783540569633.

[MS21]   Mateusz Michalek and Bernd Sturmfels. *Invitation to nonlinear algebra*. Graduate studies in mathematics 211. Providence, Rhode Island: American Mathematical Society, 2021. ISBN: 978-1-4704-5367-1. URL: https://bookstore.ams.org/gsm-211.

[Muk88]   Shigeru Mukai. "Curves, K3 Surfaces and Fano 3-folds of Genus $\leq$ 10". In: *Algebraic Geometry and Commutative Algebra*. Academic Press, 1988, pp. 357–377. ISBN: 978-0-12-348031-6. DOI: 10.1016/B978-0-12-348031-6.50026-7.

[Muk92]   Shigeru Mukai. "Curves and Grassmannians". In: *Algebraic geometry and related topics* (1992), pp. 19–40.

[Pet09]   Ivan Petrakiev. "On Self-Associated Sets of Points in Small Projective Spaces". In: *Communications in Algebra* 37.2 (2009), pp. 397–405. DOI: 10.1080/00927870802451407.

[RS06]   Lorenzo Robbiano and Moss Sweedler. "Subalgebra bases". In: *Commutative Algebra: Proceedings of a Workshop held in Salvador, Brazil, Aug. 8–17, 1988*. Springer. 2006, pp. 61–87. DOI: 10.1007/BFb0085537.

[Sot11]   Frank Sottile. *Real Solutions to Equations from Geometry*. University lecture series. American Mathematical Society, 2011. ISBN: 9780821853313.

[Stu96]   Bernd Sturmfels. *Gröbner Bases and Convex Polytopes*. Memoirs of the American Mathematical Society. American Mathematical Society, 1996. ISBN: 9780821882672.

Authors' addresses:

Barbara Betti, Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany
`betti@mis.mpg.de`

Leonie Kayser, Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany
`kayser@mis.mpg.de`