

Modeling and Nonlinear Control of a Single-link Flexible Joint Manipulator

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1 Introduction

In this experiment, we consider the nonlinear control of a robotic arm with joint flexibility. The specific mechanism under investigation is an attachment to rotary Quanser experimental setup. Flexibility is attained by attaching the link to the motor shaft using a pair of springs. There is some discretion in choosing the joint stiffness as there are different locations that the springs can be attached on the body as well as several spring choices.

The goal of this project is to model and control the flexible joint for the case in which the arm is oriented vertically. This introduces a non-linearity into the system in the form of a sinusoid, as a result of the potential energy due to gravity. The output to be controlled is the tip angle, which is given by the sum of the motor angle θ and the joint deflection α with respect to the motor shaft. The non-linear model of the dynamics will be linearized by means of state feedback in such a way that the input-output response between the voltage applied to the motor and the tip angle becomes that of a linear system. After the system has been rendered linear by feedback, it can be controlled using any linear control system design.

2 Feedback Linearization of Nonlinear Systems

In this section, we give the fundamentals of feedback linearization for single-input, single-output (SISO) nonlinear control systems which are *affine* in the control input, that is, systems of the form

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}\tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$. We assume that the vector fields $f : \mathcal{D} \rightarrow \mathbb{R}^n$, $g : \mathcal{D} \rightarrow \mathbb{R}^n$ and the readout map $h : \mathcal{D} \rightarrow \mathbb{R}$ are *smooth* in the domain $\mathcal{D} \subset \mathbb{R}^n$, that is that their partial derivatives with respect to x of *any order* exist and are continuous in \mathcal{D} . Our objective is to find a smooth control law of the form

$$u = \alpha(x) + \beta(x)v\tag{2}$$

with $v \in \mathbb{R}$ is an overall control input, such that the closed-loop system

$$\begin{aligned}\dot{x} &= f(x) + g(x)\alpha(x) + g(x)\beta(x)v \\ y &= h(x)\end{aligned}\tag{3}$$

depicted in Figure 1 possesses a *linear* input/output map $y = G(s)v$ between the new input v and the system output y . This problem is known as the *input/output feedback linearization problem*, and it is sometimes referred to as *partial linearization problem* (1). If the objective is that of finding a control law of the kind (2) and a set of coordinates under which the closed loop system (3) is

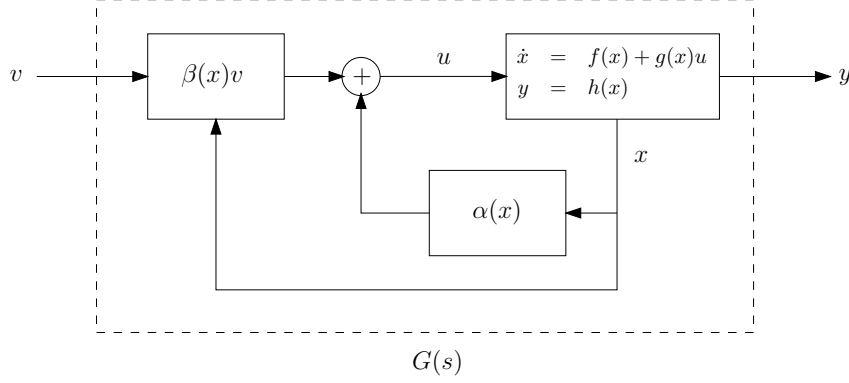


Figure 1: Linearization by feedback

altogether a *realization* of the map u/y , then the problem in question is referred to as the *exact feedback linearization problem*. It is worth noting that, although an output $y = h(x)$ is specified for the system (1), the entire state vector $x \in \mathbb{R}^n$ is assumed available for feedback.

2.1 Notation

We will adopt the following notation:

- A smooth vector field $f(x)$ is a mapping assigning a point $x \in \mathbb{R}^n$ to a *column vector* whose components are smooth functions of $x \in \mathbb{R}^n$

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}.$$

- Given a vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a function $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$, both defined and smooth over a domain $\mathcal{D} \subset \mathbb{R}^n$, we denote by

$$L_f \lambda : \mathbb{R}^n \rightarrow \mathbb{R}, \quad x \mapsto L_f \lambda(x) \doteq \frac{\partial \lambda}{\partial x} f(x)$$

the derivative of $\lambda(x)$ along the vector field $f(x)$. Note that $L_f \lambda(x)$ is again a smooth function. Therefore, higher order derivatives of $\lambda(x)$ along $f(x)$ are easily defined using the recursive relation

$$L_f^k \lambda(x) \doteq \frac{\partial L_f^{k-1} \lambda(x)}{\partial x} f(x),$$

for any integer $k > 1$.

2.2 Coordinate transformations in nonlinear systems

A function $\Phi : \mathcal{D} \rightarrow \mathbb{R}^n$ defined on a domain $\mathcal{D} \subset \mathbb{R}^n$ of a point $x^\circ \in \mathbb{R}^n$, is a *local diffeomorphism* of \mathbb{R}^n in a neighborhood of x° if

1. Φ is one-to-one, that is, the inverse function $\Phi^{-1} : \Phi(\mathcal{D}) \rightarrow \mathcal{D}$ such that

$$\Phi^{-1} \circ \Phi(x) = x$$

is well defined for all $x \in \mathcal{D}$.

2. Φ is smooth in \mathcal{D} and Φ^{-1} is smooth in $\Phi(\mathcal{D})$ ¹.

If the above properties hold for $\mathcal{D} = \Phi(\mathcal{D}) = \mathbb{R}^n$, then Φ is said to be a , *global diffeomorphism*. One easy way to recognize a local diffeomorphism is the following. Suppose Φ is a smooth function over a domain \mathcal{D} . If the jacobian matrix $\partial\Phi/\partial x$ is non singular at a point $x^\circ \in \mathcal{D}$, then Φ is a local diffeomorphism in a neighborhood of x° ².

The importance of a local diffeomorphism for nonlinear systems lies in the fact that it represents a nonlinear coordinate transformation, in the same way that a nonsingular matrix defines a change of basis in a linear system. Suppose we are given a system of the kind (1) over a domain $\mathcal{D} \subset \mathbb{R}^n$, and a local diffeomorphism Φ defined in \mathcal{D} . Letting $z = \Phi(x)$, we obtain

$$\dot{z}(t) = \frac{d}{dt}\Phi(x(t)) = \frac{\partial\Phi}{\partial x}\dot{x}(t)$$

and thus, *in the new coordinate z* the system (1) has the expression

$$\begin{aligned} \dot{z} &= \bar{f}(z) + \bar{g}(z)u \\ y &= \bar{h}(z) \end{aligned} \tag{4}$$

where

$$\bar{f}(z) = \left[\frac{\partial\Phi}{\partial x} f(x) \right]_{x=\Phi^{-1}(z)}, \quad \bar{g}(z) = \left[\frac{\partial\Phi}{\partial x} g(x) \right]_{x=\Phi^{-1}(z)}, \quad \bar{h}(z) = [h(x)]_{x=\Phi^{-1}(z)}.$$

Note that (4) is defined in the domain $\Phi(\mathcal{D})$.³ Therefore, the coordinate transformation

$$z = \Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_n(x) \end{pmatrix},$$

yields a transformed system (4) in which

$$\bar{f}(z) = \begin{pmatrix} L_f \phi_1(x) \\ L_f \phi_2(x) \\ \vdots \\ L_f \phi_n(x) \end{pmatrix}_{x=\Phi^{-1}(z)} \quad \bar{g}(z) = \begin{pmatrix} L_g \phi_1(x) \\ L_g \phi_2(x) \\ \vdots \\ L_g \phi_n(x) \end{pmatrix}_{x=\Phi^{-1}(z)} \quad \bar{h}(z) = h(\Phi^{-1}(z)).$$

2.3 Relative degree and normal forms

Consider again a smooth affine system of the form (1). The following definition is of fundamental importance.

Definition 2.1 (Relative degree) *We say that the system (1) has relative degree $r \leq n$ at the point x° it:*

1. $L_g h(x) = 0$, for any x in a neighborhood of x° ;
2. $L_g L_f^i h(x) = 0$, for any x in a neighborhood of x° , for all $i = 1, \dots, r-2$;
3. $L_g L_f^{r-1} h(x^\circ) \neq 0$.

Remark 2.1 *It is important to point out that the relative degree of a systems is defined for a specific output map $h(x)$, and it can not exceed the dimension of the state space n .*

¹The set $\Phi(\mathcal{D})$ is the set of all points $y \in \mathbb{R}^n$ such that $y = \Phi(x)$ for some $x \in \mathcal{D}$

²This can be verified using the Implicit Function Theorem

³It is easy to verify that, as a result of the continuity of Φ^{-1} , the set $\Phi(\mathcal{D})$ is open and connected.

2.3.1 Example

In a linear SISO systems

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

the relative degree is defined at each point in the state space. In this case, since $f(x) = Ax$, $g(x) = B$ and $h(x) = Cx$, we have

$$L_f^k h(x) = CA^k x, \quad \text{and} \quad L_g L_f^k h(x) = CA^k B.$$

Therefore, the integer r is characterized by the conditions (independent of x)

$$\begin{aligned}CA^k B &= 0 \quad \text{for all } 0 \leq k < r-1 \\ CA^{r-1} B &\neq 0.\end{aligned}$$

It can be shown that the integer satisfying these conditions is exactly equal to the difference between the degree of the denominator polynomial and the degree of the numerator polynomial of the transfer function of the system

$$G(s) = C(sI - A)^{-1}B.$$

Let us discuss a simpler and more intuitive characterization of the relative degree of a system from the point of view of its trajectories. Suppose that $x(t)$ is the trajectory of (1) originating from a point $x(t_0) = x^\circ$ at which a relative degree $r < n$ is defined. For a sufficiently small interval $[t_0, t_1]$ the trajectory $x(t)$ with $t \in [t_0, t_1]$ remains close to x° and thus we can assume that $x(t)$ remains in the neighborhood of x° in which conditions 1-3 in definition 2.1 hold. Taking derivative of the output $y(t)$ of the system, we obtain

$$y^{(1)}(t) = \frac{\partial h}{\partial x} \dot{x} = L_f h(x(t)) + L_g h(x(t))u(t).$$

If the relative degree is greater than 1, for all $t \in [t_0, t_1]$ we have $L_g h(x(t)) = 0$, and therefore

$$y^{(1)}(t) = L_f h(x(t)).$$

If this is the case, then the second derivative of the output is given by

$$y^{(2)}(t) = L_f^2 h(x(t)) + L_g L_f h(x(t))u(t)$$

which, if the relative degree is greater than 2, yields

$$y^{(2)}(t) = L_f^2 h(x(t)), \quad t \in [t_0, t_1].$$

Continuing along these lines, it is easy to see that the relative degree at x° being r means that the derivatives of the output $y(t)$ in a sufficiently small interval of time are given by

$$\begin{aligned}y^{(k)}(t) &= L_f^k h(x(t)) \quad \forall k < r \\ y^{(r)}(t_0) &= L_f^r h(x^\circ) + L_g L_f^{r-1} h(x^\circ)u(t_0).\end{aligned}$$

The relative degree r is exactly equal to the number of times one needs to differentiate the output $y(t)$ at time $t = t_0$ in order to have the input $u(t_0)$ appear explicitly.

The following result is of crucial importance:

Lemma 2.1 Suppose the system (1) has relative degree r at x° . Then, the $r \times n$ matrix

$$\begin{pmatrix} \frac{\partial h(x)}{\partial x} \\ \frac{\partial L_f h(x)}{\partial x} \\ \vdots \\ \frac{\partial L_f^{r-1} h(x)}{\partial x} \end{pmatrix}$$

has rank equal to r for all x in a neighborhood of x° .

This results suggests that the r smooth functions $h(x)$, $L_f h(x)$, \dots , $L_f^{r-1} h(x)$ are good candidates to define a local coordinate transformation in a neighborhood of x° . Since the dimension of the state space is n and $r \leq n$, we need $n - r$ more functions to complete the construction of a local diffeomorphism. At this regard, we can invoke the following result, which constitutes a nonlinear analogue of the theorem of the completion of basis in linear algebra:

Theorem 2.2 Suppose the system (1) has relative degree r at x° . Let

$$\begin{aligned} \phi_1(x) &\doteq h(x) \\ \phi_2(x) &\doteq L_f h(x) \\ &\dots \\ \phi_r(x) &\doteq L_f^{r-1} h(x). \end{aligned}$$

If $r < n$ there exist $n - r$ smooth functions $\phi_{r+1}(x), \dots, \phi_n(x)$ such that the mapping

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \dots \\ \phi_n(x) \end{pmatrix}$$

has a nonsingular jacobian matrix $\partial\Phi/\partial x$ in x° , and thus $\Phi(x)$ defines a local diffeomorphism in a neighborhood of x° . Furthermore, it is always possible to choose $\phi_{r+1}(x), \dots, \phi_n(x)$ in such a way that

$$L_g \phi_i(x) = 0, \quad i = r + 1, \dots, n$$

for all x in a neighborhood of x° .

At this point, we know that we can construct a coordinate transformation $z = \Phi(x)$ in such a way that the first r components are exactly $z_1 = h(x)$, $z_2 = L_f h(x)$, \dots , $z_r = L_f^{r-1} h(x)$. In terms of trajectories, this means that

$$z_1(t) = y(t), \quad z_2(t) = y^{(1)}(t), \quad \dots \quad z_r(t) = y^{(r-1)}(t)$$

for all t in a sufficiently small interval $[t_0, t_1]$. In view of this, and due to the choice of the other $n - r$ components of the diffeomorphism $\Phi(x)$, the system (1) in the new coordinates has the form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= a(z) + b(z)u \\ \dot{z}_{r+1} &= q_{r+1}(z) \\ &\vdots \\ \dot{z}_n &= q_n(z) \\ y &= z_1 \end{aligned} \tag{5}$$

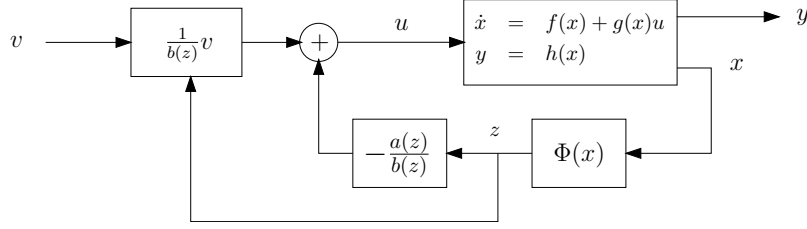


Figure 2: Feedback linearization control

where

$$a(z) = L_f^r h(x)|_{x=\Phi^{-1}(z)}, \quad b(z) = L_g L_f^{r-1} h(x)|_{x=\Phi^{-1}(z)},$$

$$q_i(z) = L_f^i h(x)|_{x=\Phi^{-1}(z)}, \quad i = r+1, \dots, n.$$

The system written in the form (5) is said to be in *normal form*. It is worth noting that only the derivative of the r -th component of z depends on the input u . This is a consequence of both a suitable choice of $\phi_{r+1}(x), \dots, \phi_n(x)$, as suggested by Theorem 3.2, and the definition of relative degree.

2.4 Input/output linearization

Let us finally turn our attention to the problem of finding a feedback control law of the kind (2) such that the closed loop system (3) has a linear input/output mapping between v and y . If the system (1) has relative degree r in x° then it can be put into the normal form (5) by means of the coordinate transformation we have previously discussed. Note that the assumption of relative degree r implies that, at the point $z^\circ = \Phi(x^\circ)$ we have

$$b(z^\circ) = L_g L_f^{r-1} h(x^\circ) \neq 0.$$

As $b(z)$ is a smooth function, $b(z) \neq 0$ in a neighborhood z° . Consider the feedback law (see Figure 2)

$$u = \frac{1}{b(z)} [-a(z) + v], \quad (6)$$

which in the original coordinates x reads as

$$u = \frac{1}{L_g L_f^{r-1} h(x)} [-L_f^r h(x) + v].$$

The feedback law (6), which is well defined in a neighborhood of z° (x° in the original coordinates) yields the closed loop system

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= v \\ \dot{z}_{r+1} &= q_{r+1}(z) \\ &\vdots \\ \dot{z}_n &= q_n(z) \\ y &= z_1. \end{aligned} \quad (7)$$

It is easy to see that the control law has acted in such a way that the closed loop system has been split into a linear and a nonlinear part. More precisely, let $z = \text{col}(\xi, \eta)$, with $\dim \xi = r$ and $\dim \eta = n - r$. System (7) can be rewritten as

$$\begin{aligned}\dot{\xi} &= A\xi + Bv \\ y &= C\xi \\ \dot{\eta} &= q(\eta, \xi)\end{aligned}\tag{8}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad C = (1 \quad 0 \quad \cdots \quad 0).$$

The interconnection between the two subsystem is depicted in Figure 3. For the closed loop system in Figure 2, the I/O map between v and y is that of the linear system (A, B, C) with transfer function

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s^r}$$

which is the transfer function of a chain of r integrators. It is easy to see that r is precisely the difference between the degree of the denominator polynomial of $G(s)$ and the degree of its numerator polynomial.

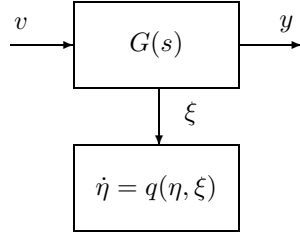


Figure 3: The closed-loop system

It is worth noting that the control law (6) has rendered the partial state η *unobservable* from the output y . Also, keep in mind that the control law (6), and consequently the closed loop system (8), remain valid only in the neighborhood of the point x° at which the relative degree is defined.

If the system has relative degree n , then the diffeomorphism $\Phi(x)$ is found immediately as

$$\Phi(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{pmatrix},$$

and the linearizing feedback

$$u = \frac{1}{L_g L_f^{n-1} h(x)} [v - L_f^n h(x)]$$

yields the n dimensional *linear* closed loop system

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= v \\ y &= z_1.\end{aligned}\tag{9}$$

It is thus clear that our ability to find a change of coordinates and a feedback law that at the same time produces a linear I/O map and transform the original nonlinear system into a linear system depends exclusively on *an intrinsic property* of the nonlinear system, that is, its relative degree at a point. Note that the relative degree of a system remains the same independently of the chosen set of coordinates.

2.4.1 Example [1]

Consider the controlled Van der Pol oscillator

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + (1 - \epsilon x_1^2)x_2 + u \\ y &= x_1.\end{aligned}$$

The system has global ⁴ relative degree 2 with respect to the given output. It is easy to see, in fact, that the system is already in normal form, with

$$a(x) = -x_1 + (1 - \epsilon x_1^2)x_2, \quad b(x) = 1.$$

The linearizing feedback is therefore given by

$$u = x_1 - (1 - \epsilon x_1^2)x_2 + v$$

which produces the closed loop system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= v \\ y &= x_1.\end{aligned}$$

Assume, on the other hand, that the output for the system is given by $y = x_1^2 + x_2$. It is easy to say that in this case the system has global relative degree 1, since

$$L_g h(x) = 1, \quad \forall x \in \mathbb{R}^2.$$

The choice for the first function of the coordinate transformation is obviously $\phi_1(x) = h(x) = x_1^2 + x_2$. For the second function $\phi_2(x)$, we are free to choose any function which form a local diffeomorphism with $\phi_1(x)$ and such that $L_g \phi_2(x) = 0$. For instance, $z_2 = x_1$ is a good choice, since

$$\Phi(x) = \begin{pmatrix} x_1^2 + x_2 \\ x_1 \end{pmatrix}$$

yields the jacobian matrix

$$\frac{\partial \Phi}{\partial x} = \begin{pmatrix} 2x_1 & 1 \\ 1 & 0 \end{pmatrix}$$

⁴That is, for each $x \in \mathbb{R}^2$.

which is nonsingular for any $x \in \mathbb{R}^2$. The inverse transformation is readily seen to be

$$x = \Phi^{-1}(z) = \begin{pmatrix} z_2 \\ z_1 - z_2^2 \end{pmatrix}.$$

In the new coordinates (z_1, z_2) the system has the form

$$\begin{aligned} \dot{z}_1 &= 2z_1z_2 - 2z_2^3 - z_2 + z_1 - z_2^2 - \epsilon z_1z_2^2 + \epsilon z_2^4 + u \\ \dot{z}_2 &= z_1 - z_2^2 \\ y &= z_1 \end{aligned}$$

and the linearizing feedback is

$$u = v - (2z_1z_2 - 2z_2^3 - z_2 + z_1 - z_2^2 - \epsilon z_1z_2^2 + \epsilon z_2^4).$$

The closed loop system is given by

$$\begin{aligned} \dot{z}_1 &= v \\ \dot{z}_2 &= z_1 - z_2^2 \\ y &= z_1 \end{aligned}$$

which is clearly decomposed into a linear subsystem (actually, an integrator) and a nonlinear, unobservable subsystem.

2.5 Inner-loop / outer-loop control

Feedback linearization is an effective method to implement a multi-loop controller for nonlinear systems. Basically, the idea is to render the input/output map of a given nonlinear system that of a linear system, and then apply a linear feedback to achieve the control objectives. To clarify this idea, assume that we want to regulate the output y of the system (1) to a constant setpoint y_d . For the sake of simplicity, assume also that the system (1) has relative degree n for all $x \in \mathbb{R}^n$. Then, there exists a change of coordinates $z = \Phi(x)$ and a feedback control of the form (6) such that the closed loop system reduces to the chain of integrators

$$y^{(n)}(t) = v(t).$$

At this point, define the setpoint error $e_1(t) = y(t) - y_d$, and, since y_d is constant, note that $e(t)$ satisfies

$$e_1^{(n)}(t) = v(t).$$

The above equation has state-space realization

$$\begin{aligned} \dot{e} &= Ae + Bv \\ e_1 &= Ce \end{aligned}$$

where $e = \text{col}(e_1, e_2, \dots, e_n)$ and, as before,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad C = (1 \quad 0 \quad \cdots \quad 0).$$

In order to drive the error to zero asymptotically it suffices to compute a state feedback matrix $K \in \mathbb{R}^{1 \times n}$ such that $A + BK$ has all eigenvalues with $\text{Re}[s] < 0$, which exists since the pair (A, B) is controllable, and apply the outer/loop control $v = Ke$. The resulting control system is shown in Figure 4.

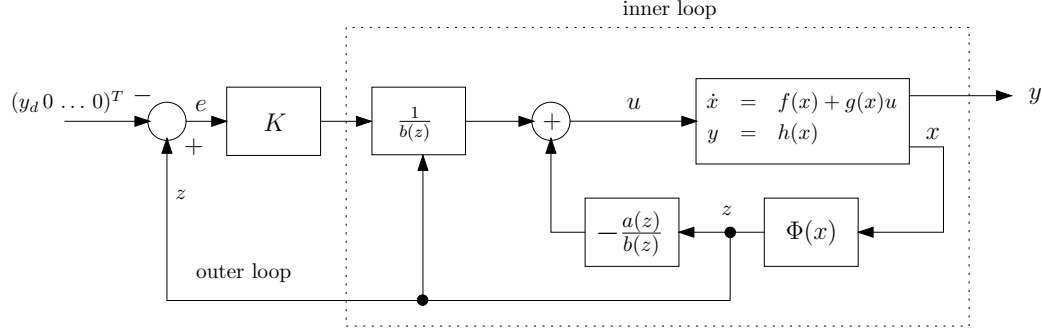


Figure 4: Multiloop nonlinear controller

3 Mathematical Model of the Flexible Joint

In this section, we consider the single-link manipulator with flexible joint operating on a vertical plane. The main difference from the model of the flexible joint manipulator on a horizontal plane considered in the previous experiment is the presence of gravity, which introduces a nonlinearity in the equations of the dynamics. The mathematical model of the dynamics of the link with flexible joint in vertical position is easily obtained from Lagrange's equations of motion. It is clear that the systems has two degrees of freedom, corresponding to the rotation of the motor shaft with respect to a coordinate frame fixed to the base, and the rotation of the flexible joint to which the link is attached with respect to the motor. The generalized coordinates are therefore the angular position of the motor θ and the angular displacement of the flexible joint α (see Figure 5). The potential and kinetic energy of the system are given respectively by

$$\mathcal{K} = \mathcal{K}_h + \mathcal{K}_l, \quad \mathcal{V} = \mathcal{V}_g + \mathcal{V}_s$$

being

$$\begin{aligned} \mathcal{V}_g &= mgh \cos(\theta + \alpha) \\ \mathcal{V}_s &= \frac{1}{2} K_s \alpha^2 \\ \mathcal{K}_h &= \frac{1}{2} J_h \dot{\theta}^2 \\ \mathcal{K}_l &= \frac{1}{2} J_l (\dot{\theta} + \dot{\alpha})^2, \end{aligned}$$

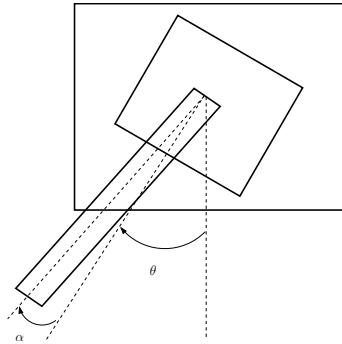


Figure 5: Definition of generalized coordinates for the link

where \mathcal{V}_g is the potential energy due to gravity, \mathcal{V}_s is the potential energy due to the springs, \mathcal{K}_h is the kinetic energy of the hub, and \mathcal{K}_l is the kinetic energy of the load. Also, m is the mass of the shaft, g is the gravity constant, h is the height of the center of gravity of the link with respect to the rest position, J_h is the inertia at the motor output, J_l is the inertia of the arm, and K_s is the spring stiffness. The value of the constant K_s depends on the way the springs are attached to the link. Following the Quanser manual, the expression for K_s is

$$\begin{aligned} K_s &= \frac{2R}{D^{\frac{3}{2}}} \left((Dd - Rr^2)F_r + (D^{\frac{3}{2}}d - DLd + Rr^2L)K \right) \\ D &= r^2 + (R - d)^2, \end{aligned}$$

where R is the arm anchor point, d is the body “Y” anchor point, r is the body “X” anchor point, K is the spring stiffness, F_r is the spring restoring force, and L is the spring length at rest. The values of the constants can be found in Table 1.

The Lagrangian is given by $\mathcal{L} = \mathcal{K} - \mathcal{V}$, and the Lagrange equations of motion read as

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} - \frac{\partial \mathcal{L}}{\partial \alpha} &= 0 \\ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} &= \tau, \end{aligned}$$

which yield, after easy manipulations,

$$\begin{aligned} J_l \ddot{\theta} + J_l \ddot{\alpha} + K_s \alpha - mgh \sin(\theta + \alpha) &= 0 \\ (J_h + J_l) \ddot{\theta} + J_l \ddot{\alpha} - mgh \sin(\theta + \alpha) &= \tau \end{aligned} \tag{10}$$

In the above equations, τ is the torque produced by the motor. The torque is commanded by the voltage v applied to the armature, which represent the control input to the system. The relationship between the torque and the applied voltage is given in the Quanser lab manual as

$$v = iR_m + K_m K_g \omega$$

which gives

$$i = \frac{v}{R_m} - \frac{K_m K_g}{R_m} \omega,$$

being ω the angular velocity of the motor, i the armature current, and R_m the motor resistance, and K_m and K_g constant parameters. Since

$$\begin{aligned} i &= \frac{\tau}{K_g K_m} \\ \dot{\theta} &= \omega, \end{aligned}$$

we obtain the desired relationship as

$$\tau = \frac{K_m K_g}{R_m} v - \frac{K_m^2 K_g^2}{R_m} \dot{\theta}.$$

Choosing

$$\begin{aligned} \theta &= x_1 \\ \alpha &= x_2 \\ \dot{\theta} &= x_3 \\ \dot{\alpha} &= x_4, \end{aligned}$$

Parameter	Symbol	Value
Spring Stiffness	K_s	1.61 [N/m]
Inertia of hub	J_h	0.0021 [Kg m ²]
Link Mass	m	0.403 [Kg]
Grav. Const.	g	-9.81 [N/m]
Height of C.M.	h	0.06 [m]
Motor Const.	K_m	0.00767 [N/rad/s]
Gear Ratio	K_g	70
Load Inertia	J_l	0.0059 [Kg m ²]
Motor Resist.	R_m	2.6 [Ω]

Table 1: Plant parameters

the system is written as

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= \frac{K_s}{J_h}x_2 - \frac{K_m^2 K_g^2}{R_m J_h}x_3 + \frac{K_m K_g}{R_m J_h}v \\
\dot{x}_4 &= -\frac{K_s}{J_h}x_2 + \frac{K_m^2 K_g^2}{R_m J_h}x_3 - \frac{K_m K_g}{R_m J_h}v - \frac{K_s}{J_l}x_2 + \frac{mgh}{J_l}\sin(x_1 + x_2).
\end{aligned}$$

Letting $u = v$, and choosing as output the tip position $y = x_1 + x_2$, the system can now be given the usual state-space form

$$\begin{aligned}
\dot{x} &= f(x) + g(x)u \\
y &= h(x),
\end{aligned} \tag{11}$$

being

$$\begin{aligned}
f(x) &= \begin{pmatrix} x_3 \\ x_4 \\ \frac{K_s}{J_h}x_2 - \frac{K_m^2 K_g^2}{R_m J_h}x_3 \\ -(\frac{K_s}{J_h} + \frac{K_s}{J_l})x_2 + \frac{mgh}{J_l}\sin(x_1 + x_2) + \frac{K_m^2 K_g^2}{R_m J_h}x_3 \end{pmatrix} \\
g(x) &= \begin{pmatrix} 0 \\ 0 \\ \frac{K_m K_g}{R_m J_h} \\ -\frac{K_m K_g}{R_m J_h} \end{pmatrix}, \quad h(x) = x_1 + x_2.
\end{aligned}$$

4 Problems

In preparation for the laboratory, you are asked to solve the following problems:

- **Problem 1** Determine the relative degree of the flexible joint manipulator model (11) with respect to the given input and the given output. Is the relative degree of the system (11) dependent on the state x ?
- **Problem 2** Compute a change of coordinates $z = \Phi(x)$ that puts the given system in normal form.

- **Problem 3** Find a feedback linearizing control $u = \alpha(x) + \beta(x)v$. What is the transfer function $G(s)$ of the closed loop system between the new input v and the output y ?
- **Problem 4** Implement the original plant model and the linearizing controller in Simulink.
- **Problem 5** Compute an outer loop state-feedback linear controller $v = Kz$ that regulates the tip position $y(t)$ to a constant setpoint y_d . You can use either LQR or pole placement.
- **Problem 6** Implement the resulting multiloop controller in Simulink. Perform simulations to test your design.