WI4011-17 - 2023/24
Computational Fluid Dynamics
Assignment 2.1
Deadline - 23:59, May 28, 2024

Instructions and assessment criteria to keep in mind:

- A submission for Assignment 2.1 is **necessary for passing** WI4011-17.
- Group submission is accepted but not mandated for this assignment. Recommended group size is 3.
- The reports need to be **typed in LaTeX or Word** and should be submitted in PDF format. It must also contain the code to any computer program that you used for numerical computations.
- The **deadline** for uploading your solutions to Brightspace is 23:59, May 28, 2024.
- Late submissions will NOT be accepted.
- Provide **clear and motivated answers** to the questions. No/reduced points will be awarded if your solutions are unaccompanied by explanations.

Part A (6 points)

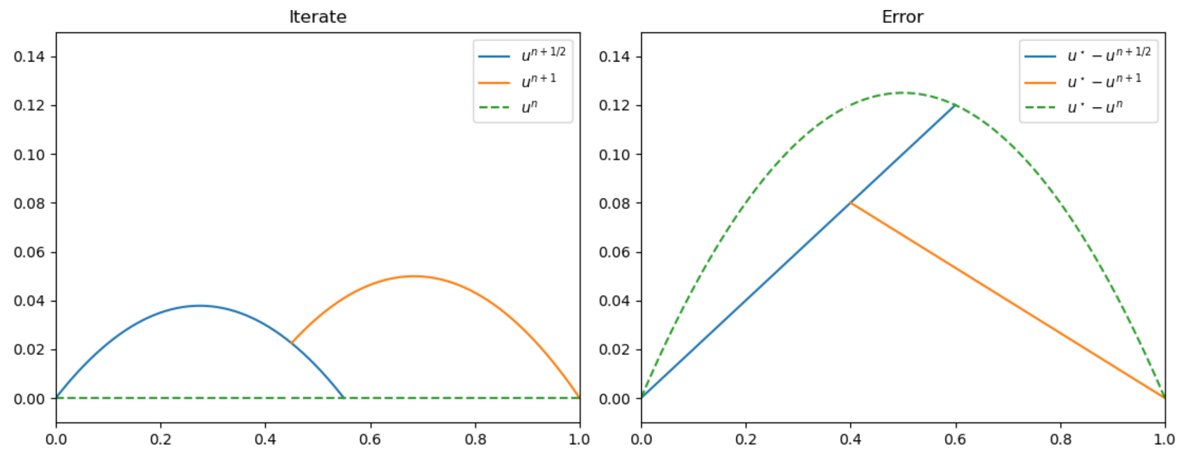

Figure 1: Solution and error of the alternating Schwarz method in the first iteration.

(a) Consider the alternating Schwarz algorithm introduced in the lectures for the one-dimensional model problem, **(3 pt.)**

$$-u'' = 1, \text{ in } [0,1], \quad u(0) = u(1) = 0.$$

with initial guess $u^0 \neq u^\star$ in Ω and subdomains $\Omega_1 = \left(0, \frac{1}{2} + \delta\right)$ and $\Omega_2 = \left(\frac{1}{2} - \delta, 1\right)$; cf. Figure 1. Here $u^\star$ is the analytical solution. Derive a formula for the error reduction factor $C(\delta)$:

$$\|u^\star - u^{n+1}\|_{L^\infty} = C(\delta)\|u^\star - u^n\|_{L^\infty}.$$

As indicated in the formula, the factor *may* depend on the width of the overlap δ.

Reminder:

$$\|u^\star - u^n\|_{L^\infty} = \sup_{x\in\Omega} |u^\star - u^n|.$$

(b) Recall the definition of the Schwarz operators $P_0, \ldots, P_N$ from the lectures: Let $a(\cdot,\cdot)$ and $a_i(\cdot,\cdot)$ be symmetric positive definite bilinear forms on V and V_i, and let

$$a(u,v) = v^\top K u \quad \text{and} \quad a_i(u_i,v_i) = v_i^\top K_i u_i,$$

for all $u, v \in V$ and $u_i, v_i \in V_i$, $i = 0, \ldots, N$. Moreover, the case where

$$a_i(u_i, v_i) = a(R_i^\top u_i, R_i^\top v_i) \qquad \text{for } u_i, v_i \in V_i$$

is denoted as the **use of exact solvers**. The Schwarz operators are defined as

$$P_i = R_i^\top \tilde{P}_i : V \to R_i^\top V_i \subset V \quad \text{for } i = 0, ..., N,$$

where $\tilde{P}_i : V \to V_i$ is given by

$$a_i(\tilde{P}_i u, v_i) = a(u, R_i^\top v_i), \tag{1}$$

for all $v_i \in V_i$.

Prove the following Lemma from the lectures: **(3 pt.)**

LEMMA 11.1

The Schwarz operator P_i, for $i = 0, \ldots, N$, has the following properties:

1. Matrix representation: $P_i = R_i^\top K_i^{-1} R_i K$.
2. P_i is self-adjoint, i.e., $a(P_i u, v) = a(u, P_i v) \quad u, v \in V$.
3. P_i is positive semi-definite, i.e., $a(P_i u, u) \geq 0 \quad u \in V$.
4. For exact solvers, the Schwarz operator P_i is a projection, i.e.,

$$P_i^2 = P_i.$$

Solution

(a) Firstly, we show that the difference $w = u - v$ between two solutions u and v of the PDE is a linear function:

$$-w'' = -(u-v)'' = -u'' - v'' = 1 - 1 = 0,$$

and therefore,

$$w(x) = ax + b$$

with some $a, b \in \mathbb{R}$. Now, let

$$E := u^\star\left(\frac{1}{2} + \delta\right) - u^n\left(\frac{1}{2} + \delta\right)$$

Then, since the error is

- continuous in Ω,
- linear everywhere except for the point $\frac{1}{2} + \delta$, and
- 0 at $x = 0$ and $x = 1$,

we know that the maximum absolute error is attained at $\frac{1}{2} + \delta$:

$$\|u^\star - u^n\|_{L^\infty} = \sup_{x \in \Omega} |u^\star - u^n| = |E|.$$

By simple derivations using

- $u^\star(0) = u^{n+1/2}(0) = 0$ and
- $u^\star - u^{n+1/2}$ is linear in Ω_1,

we can show that

$$u^\star(x) - u^{n+1/2}(x) = \frac{E}{\frac{1}{2} + \delta} x \quad \text{in } \Omega_1.$$

Then, using

$$u^\star\left(\frac{1}{2} - \delta\right) - u^{n+1}\left(\frac{1}{2} - \delta\right) \overset{u^{n+1}(\frac{1}{2}-\delta) = u^{n+1/2}(\frac{1}{2}-\delta)}{=} u^\star\left(\frac{1}{2} - \delta\right) - u^{n+1/2}\left(\frac{1}{2} - \delta\right) = E \frac{\frac{1}{2} - \delta}{\frac{1}{2} + \delta}.$$

as well as

- $u^\star(1) = u^{n+1}(1) = 0$ and
- $u^\star - u^{n+1}$ is linear in Ω_2,

and by some simple calculations, we obtain that

$$u^\star(x) - u^{n+1}(x) = E\frac{\left(\frac{1}{2}-\delta\right)}{\left(\frac{1}{2}+\delta\right)^2}(1-x) \quad \text{in } \Omega_2.$$

Finally,

$$\|u^\star - u^n\|_{L^\infty} = \left|u^\star\left(\frac{1}{2}+\delta\right) - u^{n+1}\left(\frac{1}{2}+\delta\right)\right| = |E|\frac{\left(\frac{1}{2}-\delta\right)^2}{\left(\frac{1}{2}+\delta\right)^2},$$

which gives an error reduction of

$$\|u^\star - u^{n+1}\|_{L^\infty} = \frac{\left(\frac{1}{2}-\delta\right)^2}{\left(\frac{1}{2}+\delta\right)^2}\|u^\star - u^n\|_{L^\infty}.$$

(b) 1. Consider the operator $\tilde{P}_i$ defined by (1) and derive the matrix form

$$\begin{aligned} & a_i\left(\tilde{P}_i u, v_i\right) = a\left(u, R_i^\top v_i\right) \\ \Leftrightarrow \quad & v_i^\top K_i \tilde{P}_i u = \left(R_i^\top v_i\right)^\top K u, \end{aligned}$$

for all $u \in V$ and $v_i \in V_i$. Hence, we have that

$$K_i \tilde{P}_i = R_i K.$$

From $K_i \tilde{P}_i = R_i K$ and K_i invertible, we then obtain

$$\begin{aligned} & \tilde{P}_i = K_i^{-1} R_i K \\ \Leftrightarrow \quad & \underbrace{R_i^\top \tilde{P}_i}_{=P_i} = R_i^\top K_i^{-1} R_i K \end{aligned}$$

2. To prove that P_i is self-adjoint, let $u, v \in V$. Note that, since $a(\cdot,\cdot)$ and $a_i(\cdot,\cdot)$ are symmetric, K and K_i are symmetric as well. Using the matrix form above and the fact that K and K_i are symmetric, we easily derive

$$a\left(P_i u, v\right) = v^\top K\left(R_i^\top K_i^{-1} R_i K u\right) = \left(R_i^\top K_i^{-1} R_i K v\right)^\top K u = a\left(u, P_i v\right).$$

3. The positive semi-definiteness of P_i follows from the fact that K_i is symmetric positive definite. In particular, this implies that K_i^{-1} is symmetric positive definite as well. Then,

$$a\left(P_i u, u\right) = u^\top K P_i u = u^\top K R_i^\top K_i^{-1} R_i K u \overset{v_i := R_i K u}{=} v_i^\top K_i^{-1} v_i \geq 0$$

4. Using exact solvers means that

$$K_i = R_i K R_i^\top.$$

We obtain

$$\begin{aligned} P_i^2 &= \left(R_i^\top K_i^{-1} R_i K\right)^2 = R_i^\top K_i^{-1} \underbrace{R_i K R_i^\top}_{=K_i} K_i^{-1} R_i K \\ &= R_i^\top K_i^{-1} R_i K = P_i \end{aligned}$$

Part B (7 points)

Consider the boundary value problem on a rectangular domain $\Omega = [0,2] \times [0,1]$: find u such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

with $f = 1$. In this part of the assignment, you will solve this problem using the alternating and parallel Schwarz methods. As the discretization, you can use piece-wise bilinear (Q1) finite elements on a structured quadrilateral mesh; see Figure 2.

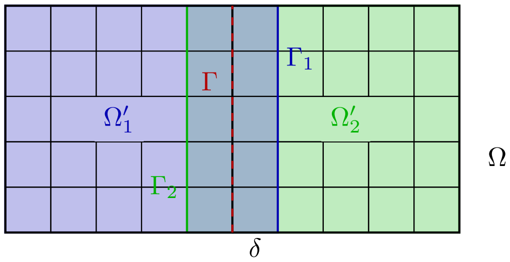

Figure 2: Overlapping domain decomposition of the rectangular domain $\Omega = [0,2] \times [0,1]$ with a structured quadrilateral mesh into two overlapping subdomains.

> ***Remark:*** *In Project 1, you already implemented piece-wise bilinear (Q1) finite elements for discretizing the Stokes problem. Therefore, you should (be able to) re-use large parts of the implementation from Project 1.*

(a) Implement the classical alternating Schwarz iteration from the lectures, starting with some initial guess u^0 that satisfies the boundary conditions of eq. (2). The fixed-point iteration reads: **(3 pt.)**

$$
\begin{aligned}
(D_1)&\begin{cases} -\Delta u^{n+1/2} = f & \text{in } \Omega_1', \\ u^{n+1/2} = u^n & \text{on } \partial\Omega_1' \\ u^{n+1/2} = u^n & \text{on } \Omega_2 := \Omega_2' \setminus \overline{\Omega_1'} \end{cases} \\
(D_2)&\begin{cases} -\Delta u^{n+1} = f & \text{in } \Omega_2, \\ u^{n+1} = u^{n+1/2} & \text{on } \partial\Omega_2' \\ u^{n+1} = u^{n+1/2} & \text{on } \Omega_1 := \Omega_1' \setminus \overline{\Omega_2'} \end{cases}
\end{aligned}
\tag{3}
$$

Implement a reasonable stopping criterion and motivate your decision.

(b) Modify the alternating Schwarz iteration in order to obtain the parallel Schwarz iteration: **(2 pt.)**

$$
\begin{aligned}
(D_1)&\begin{cases} -\Delta u_1^{n+1} = f_1 & \text{in } \Omega_1', \\ u_1^{n+1} = u_2^n & \text{on } \partial\Omega_1' \end{cases} \\
(D_2)&\begin{cases} -\Delta u_2^{n+1} = f_2 & \text{in } \Omega_2, \\ u_2^{n+1} = u_1^n & \text{on } \partial\Omega_2' \end{cases}
\end{aligned}
\tag{4}
$$

Note that, as discussed in the lectures, the iterates u_1^n and u_2^n are only defined inside Ω_1' and Ω_2'. Again, implement a reasonable stopping criterion and motivate your decision.

(c) Compare the performance of two methods for different mesh refinement levels and different widths of overlap. In your investigations, consider both the convergence in terms of iterations as well as the required computing times. Discuss your results. **(2 pt.)**