APPENDIX

Integration Formulae

E

E.1 Linear triangles

Let i, j, and k be the nodes of a triangular element. Integrating over the triangular area gives

$$A = \int dx_1 dx_2 = \frac{1}{2}\begin{vmatrix} 1 & x_{1i} & x_{2i} \\ 1 & x_{1j} & x_{2j} \\ 1 & x_{1k} & x_{2k} \end{vmatrix} \tag{E.1}$$

where A is the area of the triangle. For a linear triangular element (shape functions are the same as local coordinates), the integration of the shape functions can be written as

$$\int_\Omega N_i^a N_j^b N_k^c \mathrm{d}\Omega = \frac{a!b!c!2A}{(a+b+c+2)!} \tag{E.2}$$

On the boundaries

$$\int_\Gamma N_i^a N_j^b \mathrm{d}\Gamma = \frac{a!b!l}{(a+b+1)!} \tag{E.3}$$

Note that $i - j$ is assumed to be the boundary side. The above equation is identical to the integration formula of a one-dimensional linear element. In the above equation l is the length of a boundary side.

E.2 Linear tetrahedron

Let i, j, k, and m be the nodes of a linear tetrahedron element. Integrating over the volume gives

$$V = \int dx_1 dx_2 dx_3 = \frac{1}{6}\begin{vmatrix} 1 & x_{1i} & x_{2i} & x_{3i} \\ 1 & x_{1j} & x_{2j} & x_{3j} \\ 1 & x_{1k} & x_{2k} & x_{3k} \\ 1 & x_{1m} & x_{2m} & x_{3m} \end{vmatrix} \tag{E.4}$$

where V is the volume of a tetrahedron. For linear shape functions, the integration formula can be written as

$$\int_\Omega N_i^a N_j^b N_k^c N_m^d \mathrm{d}\Omega = \frac{a!b!c!d!6V}{(a+b+c+3)!} \tag{E.5}$$

Finite Element Meth Fluid Dynam, 7E, Volume 3. http://dx.doi.org/10.1016/B978-1-85617-635-4.00029-7

On the boundaries

$$\int_{\Gamma} N_i^a N_j^b N_k^c \mathrm{d}\Gamma = \frac{a!b!c!2A}{(a+b+c+2)!} \tag{E.6}$$

Note that the above formula is identical to the integration formula of triangular elements within the domain. In the above equation A is the area of a triangular face.