APPENDIX

Nonconservative Form of Navier-Stokes Equations

B

To derive the Navier-Stokes equations in their nonconservative form, we start with the conservative form.

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = \frac{\partial \rho}{\partial t} + \rho \frac{\partial u_i}{\partial x_i} + u_i \frac{\partial \rho}{\partial x_i} = 0 \tag{B.1}$$

Conservation of momentum:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(u_j \rho u_i)}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial p}{\partial x_i} = 0 \tag{B.2}$$

Conservation of energy:

$$\frac{\partial(\rho E)}{\partial t} + \frac{\partial(u_j \rho E)}{\partial x_j} - \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) + \frac{\partial(u_j p)}{\partial x_j} - \frac{\partial(\tau_{ij} u_j)}{\partial x_j} = 0 \tag{B.3}$$

Rewriting the momentum equation with terms differentiated as

$$\rho \frac{\partial u_i}{\partial t} + u_i\left(\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_j}{\partial x_j} + u_j \frac{\partial \rho}{\partial x_j}\right) + \rho u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial p}{\partial x_i} = 0 \tag{B.4}$$

and substituting the equation of mass conservation [Eq. (B.1)] into the above equation gives the reduced momentum equation

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} - \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = 0 \tag{B.5}$$

Similarly as above, the energy equation [Eq. (B.3)] can be written with differentiated terms as

$$E\left(\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_j}{\partial x_j} + u_j \frac{\partial \rho}{\partial x_j}\right) + \rho \frac{\partial E}{\partial t} + \rho u_j \frac{\partial E}{\partial x_j} - \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) + \frac{\partial(u_i p)}{\partial x_i} - \frac{\partial(\tau_{ij} u_j)}{\partial x_i} = 0 \tag{B.6}$$

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Again substituting the continuity equation into the above equation, we have the reduced form of the energy equation

$$\frac{\partial E}{\partial t} + u_j \frac{\partial E}{\partial x_j} - \frac{1}{\rho} \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) + \frac{1}{\rho} \frac{\partial (u_i p)}{\partial x_i} - \frac{1}{\rho} \frac{\partial (\tau_{ij} u_j)}{\partial x_i} = 0 \tag{B.7}$$

Some authors use Eqs. (B.1), (B.5), and (B.7) to study compressible flow problems. However these nonconservative equations can result in multiple or incorrect solutions in certain cases. This is true especially for high-speed compressible flow problems with shocks. The reader should note that such conservative equations are not suitable for simulation of compressible flow problems.