WI4011-17
Computational Fluid Dynamics
Assignment 1.2
Deadline - 23:59, April 14, 2024

Instructions and assessment criteria to keep in mind:

- A submission for Assignment 1.2 is **necessary for passing** WI4011-17.
- Group submission is accepted but not mandated for this assignment. Recommended group size is 3.
- The report should be **typed in LaTeX or Word** and should be submitted in PDF format. It must also contain the code to any computer program that you used for numerical computations.
- The **deadline** for uploading your solutions to Brightspace is 23:59, April 14, 2024.
- Late submissions will NOT be accepted.
- Provide **clear and motivated answers** to the questions. No/reduced points will be awarded if your solutions are unaccompanied by explanations.

Upwind and Petrov Galerkin finite elements: 1D (6 points)

We consider a weak form of the scalar advection-diffusion equation on domain $\Omega = (0, L)$, which you derived in Assignment 1.1, i.e., find $\phi \in \mathcal{V}$ such that

$$\forall w \in \mathcal{W}, \quad \int_\Omega w u \phi_{,x} + w_{,x} \epsilon \phi_{,x} \, dx = \int_\Omega w f \, dx \ .$$

and whose Galerkin discretization using a basis of hat functions is given as

$$\sum_{k=i-1}^{i} \int_{x_k}^{x_{k+1}} B_i u \left(\sum_{j=0}^{N+1} \phi_j B_{j,x} \right) + B_{i,x} \epsilon \left(\sum_{j=0}^{N+1} \phi_j B_{j,x} \right) \, dx = \sum_{k=i-1}^{i} \int_{x_k}^{x_{k+1}} B_i f \, dx, \quad i = 1, \ldots, N. \tag{1}$$

(a) (2 points) For this part, assume that $f = 0$.

- (1.5 points) Design a *one-point quadrature scheme* for evaluating the integrals in the Galerkin discretization of the above weak form such that the discrete equations (1) are equivalent to the nodally-exact finite-difference scheme derived in Assignment 1.1, i.e.,

$$\begin{aligned} &u \frac{\phi_{i+1} - \phi_{i-1}}{2h} - (\epsilon + \bar{\epsilon}) \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2} = 0 \ , \\ &\bar{\epsilon} := \frac{uh}{2} \left[\coth(\mathrm{Pe}_h) - \frac{1}{\mathrm{Pe}_h} \right] \ , \quad \mathrm{Pe}_h := \frac{uh}{2\epsilon} \ . \end{aligned} \tag{2}$$

In other words, find the quadrature point $\xi \in (0, h)$ such that the optimal scheme (2) is reproduced by approximating all integrals in eq. (1) as,

$$\int_{x_k}^{x_{k+1}} g(x)\ dx \approx \mathcal{I}\left(\int_{x_k}^{x_{k+1}} g(x)\ dx\right) := hg(x_k + \xi)\ . \tag{3}$$

- (0.5 points) Examine the dependence of ξ on Pe_h, and interpret it in the finite element context, specifically in relation to upwinding.

Now, we consider the streamline upwind Petrov Galerkin (SUPG) method for the same problem, i.e., find $\phi_h \in \mathcal{V}_h$ such that

$$\forall w_h \in \mathcal{W}_h, \quad \int_\Omega w_h u\phi_{h,x} + w_{h,x}\epsilon\phi_{h,x}\ dx + \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} p_h\left(u\phi_{h,x} - \epsilon\phi_{h,11} - f\right) = \int_\Omega w_h f\ dx\ , \tag{4}$$

where $p_h := \tau u w_{h,x}$; τ is the SUPG stabilisation parameter and dimensional analysis shows that it should possess the units of time.

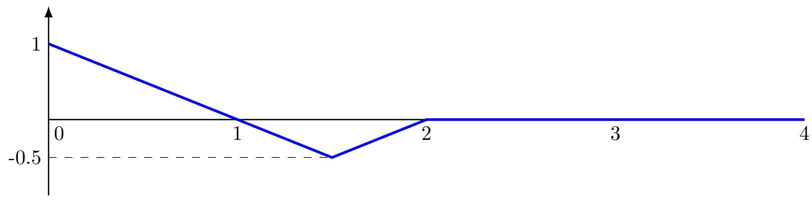
(b) (1.5 points) For $f = 0$, derive an expression for τ such that the SUPG method is equivalent to the optimal scheme (2). Plot how τ varies with Pe_h and, discuss the SUPG method for the limiting values of τ as $\mathrm{Pe}_h \to 0$ and $\mathrm{Pe}_h \to \infty$.

(c) (1 point) Now, we examine the one-point quadrature scheme you designed in (a) for $f \neq 0$. Assuming that f is a piecewise-constant function defined over the mesh, find out any additional restrictions that f must satisfy to ensure that the one-point approximation of the Galerkin is equivalent to the exact SUPG scheme. Specifically, find the conditions that f must satisfy so that

$$\mathcal{I}\left(\int_{x_{i-1}}^{x_{i+1}} B_i f\ dx\right) = \int_{x_{i-1}}^{x_{i+1}} (B_i + \tau u B_{i,x}) f\ dx\ .$$

(d) (1.5 points) Using the same definition of τ as derived in problem (b), implement the SUPG method in a computer code where you can modify the forcing f. For $N = 19$; present three plots for each subitem below, where each plot compares the solutions obtained using the SUPG, standard Galerkin, one-point quadrature approximation to Galerkin derived in (a), and the "exact solution" (obtained using the standard Galerkin method for a very fine mesh).

- $\epsilon \in \{0.01, 0.1, 1\}$ and $f = 0$;
- $\epsilon \in \{0.01, 0.1, 1\}$ and f defined as below;



Discuss your results.

Streamline Upwind/Petrov Galerkin finite elements: 2D (6 points)

We consider the scalar steady convection-diffusion equation on $\Omega = (0, L)^2$ with no source, i.e., $f = 0$ and with the following boundary conditions.

$$\begin{aligned}
\varphi(\mathbf{x}) &= 1, \quad \mathbf{x} \in \Gamma_1 := \{(x, 0) : x \in [0, L]\} \cup \{(0, y) : y \in [0, L/5]\}, \\
\varphi(\mathbf{x}) &= 0, \quad \mathbf{x} \in \partial\Omega \backslash \Gamma 1.
\end{aligned}$$

Similarly to Assignment 1.1, the finite element space will be based on a uniform cartesian mesh with $(n \times n)$ elements of size $h = \frac{L}{n}$ in each direction and we use the functions B_{ij} as our basis for the finite element space.

Recall the SUPG approximation to the steady, convection diffusion equation, i.e., find $\varphi_h \in \mathcal{V}^h$ such that

$$\begin{aligned}
\forall w_h \in \mathcal{W}_h, \quad \int_\Omega w_h \nabla \cdot (\mathbf{u}\varphi_h) + \nabla w_h \cdot (\varepsilon \nabla \varphi_h)\, \mathrm{d}\mathbf{x} + \sum_e \int_{\Omega_e} \tau \mathcal{P}(w_h) \nabla \cdot (\mathbf{u}\varphi_h - \varepsilon \nabla \varphi_h)\, \mathrm{d}\mathbf{x} = \\
\int_\Omega w_h f \mathrm{d}\mathbf{x} + \sum_e \int_{\Omega_e} \tau \mathcal{P}(w_h) f\, \mathrm{d}\mathbf{x},
\end{aligned} \tag{5}$$

where $\mathcal{P}(w_h) = \mathbf{u} \cdot \nabla w_h$, and we consider the following ad-hoc generalization of the stabilization parameter τ from the 1D case

$$\begin{aligned}
\tau &= \frac{\bar{\varepsilon}}{\|\mathbf{u}\|^2}, \\
\bar{\varepsilon} &= (\beta_1 u_1 + \beta_2 u_2) h/2, \\
\beta_1 &= \coth \mathrm{Pe}_{h1} - \frac{1}{\mathrm{Pe}_{h1}}, \quad \beta_2 = \coth \mathrm{Pe}_{h2} - \frac{1}{\mathrm{Pe}_{h2}}, \\
\mathrm{Pe}_{h1} &= \frac{u_1 h}{2\varepsilon}, \quad \mathrm{Pe}_{h2} = \frac{u_2 h}{2\varepsilon}.
\end{aligned}$$

(a) (1 point) Similarly to Assignment 1.1, derive the linear system corresponding to the SUPG approximation (5) without evaluating any integrals.

(b) (5 points) Approximate the integrals in the above system using a *one-point Gaussian quadrature* and implement this discrete SUPG scheme in a computer code. Present the SUPG results for the following cases in comparison with the corresponding Bubnov-Galerkin approximation obtained by setting $\tau = 0$ in your code.

Choose $L = 1$, $\mathbf{u} = [1, 0]$.

- For $\varepsilon = 0.01$, compare the results of SUPG ($n = 10$), Bubnov-Galerkin ($n = 10$) and the "exact" solution (obtained from Bubnov-Galerkin by setting $n = 100$).
- For $\varepsilon = 10^{-6}$, compare the results of SUPG and Bubnov-Galerkin for $n = 10$. What happens to the Bubnov-Galerkin approximation upon refining the mesh? Which solution would you trust? Discuss.