APPENDIX

F Edge-Based Finite Element Formulation

Edge-based data structures have been used in many finite element formulations for flow problems. As mentioned in Section 7.9, Chapter 7, this formulation has many advantages such as smaller storage, etc. To explain the formulation we shall consider the Euler equations and a few assembled linear triangular elements on a two-dimensional finite element mesh as shown in Fig. F.1. From Eq. (1.25) we rewrite the following Euler equations:

$$\frac{\partial \mathbf{\Phi}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} = \mathbf{0} \tag{F.1}$$

where $\mathbf{\Phi}$ are the conservation variables. If the element-based formulation for the above equation omits the stabilization terms, the weak form can be written we get

$$\int_\Omega \mathbf{N}^{\mathrm{T}} \frac{\Delta \mathbf{\Phi}}{\Delta t} \mathrm{d}\Omega = -\int_\Omega \mathbf{N}^{\mathrm{T}} \frac{\partial \mathbf{F}_i}{\partial x_i} \mathrm{d}\Omega \tag{F.2}$$

In a fully explicit form of solution procedure, the left-hand side becomes $\mathbf{M}(\Delta\mathbf{\Phi}/\Delta t)$ and here $\mathbf{M}$ is the consistent mass matrix (see Chapter 3). We can write the RHS of the above equation for an interior node I (Fig. F.1a) by interpolating $\mathbf{F}_i$ in each element and after applying Green's theorem we get

$$\sum_{E \in I} \int_{A_E} \frac{\partial N_I}{\partial x_i} \sum_k (N_k \mathbf{F}_i^k) \mathrm{d}\Omega = \sum_{E \in I} \left[\frac{A_E}{3} \frac{\partial N_I}{\partial x_i} \right]_E (\mathbf{F}_i^I + \mathbf{F}_i^J + \mathbf{F}_i^K) \tag{F.3}$$

where A_E is the area and I, J, and K are the three nodes of the element (triangle) E. This is an acceptable added approximation which is frequently used in the Taylor-Galerkin method (see Chapter 2). In another form, the above RHS can be written as (Fig. F.1a)

$$\frac{A_1}{3} \frac{\partial N_I}{\partial x_i} (\mathbf{F}_i^I + \mathbf{F}_i^1 + \mathbf{F}_i^2) + \frac{A_2}{3} \frac{\partial N_I}{\partial x_i} (\mathbf{F}_i^I + \mathbf{F}_i^2 + \mathbf{F}_i^3) + \frac{A_3}{3} \frac{\partial N_I}{\partial x_i} (\mathbf{F}_i^I + \mathbf{F}_i^3 + \mathbf{F}_i^1) \tag{F.4}$$

where A_1, A_2, and A_3 are the areas of elements 1, 2, and 3, respectively. For integration over the boundary on the RHS, we can write the following in the element formulation:

Finite Element Meth Fluid Dynam, 7E, Volume 3. http://dx.doi.org/10.1016/B978-1-85617-635-4.00030-3

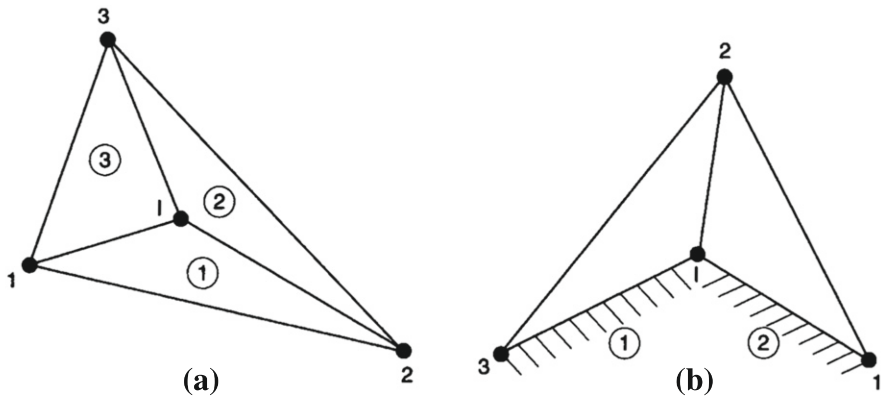

FIGURE F.1

Typical patch of linear triangular elements: (a) inside node; (b) boundary node.

$$\sum_{B\in I}\int_{\Gamma_B} N_I(N_k\mathbf{F}_i^k)\mathrm{d}\Gamma\mathbf{n}_B = \sum_{B\in I}\left[\frac{\Gamma_B}{6}(2\mathbf{F}_i^I+\mathbf{F}_i^J)\mathbf{n}\right]_B \tag{F.5}$$

where $\mathbf{n}$ is the boundary normal. The above equation can be rewritten for the node I in Fig. F.1b as

$$\frac{\Gamma_{B1}}{6}(2\mathbf{F}_i^I+\mathbf{F}_i^3)\mathbf{n}_1+\frac{\Gamma_{B2}}{6}(2\mathbf{F}_i^I+\mathbf{F}_i^1)\mathbf{n}_2 \tag{F.6}$$

where Γ_{B1} and Γ_{B2} are appropriate edge lengths.

The above Eqs. (F.3) and (F.5) can be reformulated for an edge-based data structure. In such a procedure, (Eq. F.3) can be rewritten as (for an interior node I)

$$\sum_{E\in I}\int_{\Omega_E}\frac{\partial N_I}{\partial x_i}(N_k\mathbf{F}_i^k)\mathrm{d}\Omega = \sum_{S=1}^{m_s}\left\{\sum_{E\in II_s}\left[\frac{A_E}{3}\frac{\partial N_I}{\partial x_i}\right]_E(\mathbf{F}_i^I+\mathbf{F}_i^{I_s})\right\} \tag{F.7}$$

where m_s is the number of edges in the mesh which are directly connected to the node I and the summation $\sum_{E\in II_s}$ extends over those elements that contain the edges II_s. The coefficient in Eq. (F.7) is

$$C_i^{II_s} = \sum_{E=II_s}\frac{A_E}{3}\left[\frac{\partial N_I}{\partial x_i}\right]_E \tag{F.8}$$

It can be easily verified that

$$\sum_{S=1}^{m_s} C_i^{II_s} = 0 \tag{F.9}$$

for all i. The user can now readily verify that the above equation is identically equal to the standard element formulation of Eq. (F.4) if we consider the node I in Fig. F.1(a). For the boundary nodes, however, Eq. (F.9) is not satisfied and thus the element formulation is not reproduced. For the boundary edges, in addition to Eqs. (F.6) and (F.7) the following addition is necessary

$$-\left[\frac{\Gamma_{B1}}{6}\mathbf{n_1}+\frac{\Gamma_{B2}}{6}\mathbf{n_2}\right]\mathbf{F}_i^I \tag{F.10}$$