APPENDIX

Multigrid Method

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It is intuitively obvious that whenever iterative techniques are used to solve a finite element or finite difference problem it is useful to start from a coarse mesh solution and then to use this coarse mesh solution as a starting point for iteration in a finer mesh. This process repeated on many meshes has been used frequently and obviously accelerates the total convergence rate. This acceleration is particularly important when a hierarchical formulation of the problem is used. We have indeed discussed such hierarchical formulations in Chapter 6 of Ref. [1] and the advantages are pointed out there.

The simple process which we have just described involves going from coarser meshes to finer ones. However it is not useful if no return to the coarser mesh is done. In hierarchical solutions such returning is possible as the coarser mesh matrix is embedded in the finer one with the same variables and indeed the iteration process can be described entirely in terms of the fine mesh solution. The same idea is applied to the multigrid form of iteration in which the coarse and fine mesh solution are suitably linked and use is made of the fact that the fine mesh iteration converges very rapidly in eliminating the higher frequencies of error while the coarse mesh solution is important in eliminating the low frequencies.

To describe the process let us consider the problem of

$$\mathcal{L}\phi = f \quad \text{in} \quad \Omega \tag{H.1}$$

which we discretize incorporating suitable boundary conditions. On a coarse mesh the discretization results in

$$\mathbf{K}^c \tilde{\boldsymbol{\phi}}^c = \mathbf{f}^c \tag{H.2}$$

which can be solved directly or iteratively and generally will converge quite rapidly if $\tilde{\boldsymbol{\phi}}^c$ is not a big vector. The fine mesh discretization is written in the form

$$\mathbf{K}^f \tilde{\boldsymbol{\phi}}^f = \mathbf{f}^f \tag{H.3}$$

and we shall start the iteration after the solution has been obtained on the coarse mesh. Here we generally use a *prolongation* operator which is generally an interpolation from which the fine mesh values at all nodal points are described in terms of the coarse mesh values. Thus

$$\boldsymbol{\phi}_i^f = \mathbf{P}\boldsymbol{\phi}_{i-1}^c + \Delta\boldsymbol{\phi}_i^f \tag{H.4}$$

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where $\Delta\boldsymbol{\phi}_i^f$ is the increment obtained in direct iteration. If the meshes are nested then of course the matter of obtaining $\mathbf{P}$ is fairly simple but this can be done quite generally by interpolating from a coarser to a finer mesh even if the points are not coincident. Obviously the values of the matrices $\mathbf{P}$ will be close to unity whenever the fine mesh points lie close to the coarse mesh ones. This leads to an almost hierarchical form. Once the prolongation to $\boldsymbol{\phi}^f$ has been established at a particular iteration i the fine mesh solutions can be attempted by solving

$$\mathbf{K}^f \Delta\tilde{\boldsymbol{\phi}}^f = \mathbf{f}^f - \mathbf{R}_i^f \tag{H.5}$$

where the residual $\mathbf{R}$ is easily evaluated from the actual equations. We note that the solution need not be complete and can well proceed for a limited number of cycles after which a return to the coarse mesh is again made to cancel out major low frequency errors. At this stage it is necessary to introduce a matrix $\mathbf{Q}$ which transforms values from the fine mesh to the coarse mesh. We now write for instance

$$\tilde{\boldsymbol{\phi}}_i^c = \mathbf{Q}\tilde{\boldsymbol{\phi}}_i^f \tag{H.6}$$

where one choice for $\mathbf{Q}$ is, of course, $\mathbf{P}^{\mathrm{T}}$. In a similar way we can also write

$$\mathbf{R}_i^c = \mathbf{Q}\mathbf{R}_i^f \tag{H.7}$$

where $\mathbf{R}_i$ are residuals. The above interpolation of residuals is by no means obvious but is intuitively at least correct and the process is self-checking as now we shall start a coarse mesh solution written as

$$\mathbf{K}^c(\tilde{\boldsymbol{\phi}}_{i+1}^c - \tilde{\boldsymbol{\phi}}_i^c) = \mathbf{R}_i^c \tag{H.8}$$

At this stage we solve for $\tilde{\boldsymbol{\phi}}_{i+1}^c$ using the values of previous iterations of $\tilde{\boldsymbol{\phi}}_i^c$ and putting the collected residuals on the right-hand side. This way of transferring residuals is by no means unique but has established itself well and the process is rapidly convergent.

In general more than two mesh subdivisions will be used and suitable operators $\mathbf{P}$ and $\mathbf{Q}$ have to be established for transition between each of the stages. The total process of solution is vastly accelerated and proceeds well as shown by the many papers cited in Chapter 7.

Reference

[1] O.C. Zienkiewicz, R.L. Taylor, J.Z. Zhu, The Finite Element Method: Its Basis and Fundamentals, seventh ed., Elsevier, Oxford, 2013.