WI4011-17 - 2023/24
Computational Fluid Dynamics
Assignment 2.2

Deadline - 23:59, June 9, 2024

Instructions and assessment criteria to keep in mind:

- A submission for Assignment 2.2 is **necessary for passing** WI4011-17.
- Group submission is accepted but not mandated for this assignment. Recommended group size is 3.
- The reports need to be **typed in LaTeX or Word** and should be submitted in PDF format. It must also contain the code to any computer program that you used for numerical computations.
- The **deadline** for uploading your solutions to Brightspace is 23:59, June 9, 2024.
- Late submissions will NOT be accepted.
- Provide **clear and motivated answers** to the questions. No/reduced points will be awarded if your solutions are unaccompanied by explanations.

Part A (7 points)

(a) Show that, if K is symmetric and we use exact solvers, the hybrid Schwarz operator **(3 pt.)**

$$P_{\text{hy1}} = I - E_{\text{hy1}}, \quad E_{\text{hy1}} = \big(I - P_0\big)\big(I - \sum_{i=1}^{N} P_i\big)\big(I - P_0\big),$$

yields a symmetric preconditioner of the form

$$M_{\text{hy1}}^{-1} = R_0^\top K_0^{-1} R_0 + \big(I - P_0\big)\big(\sum_{i=1}^{N} R_i^\top K_i^{-1} R_i\big)\big(I - P_0\big)^\top.$$

(b) Recall the **Local Stability** (Assumption 10.3): **(2 pt.)**

There exist an $\omega > 0$, such that

$$a\left(R_i^T u_i, R_i^T u_i\right) \leq \omega a_i\left(u_i, u_i\right), \quad u_i \in \operatorname{range} \tilde{P}_i, \quad 0 \leq i \leq N.$$

Prove the following lemma from the lectures:

> LEMMA 10.3
>
> Let $E_N = (I - P_N) \cdots (I - P_0)$ be the error propagation operator of a multiplicative Schwarz method. By adding another subspace V_{N+1} with a solver satisfying Local Stability (Assumption 10.3) with $\omega \in (0, 2)$, the respective error propagation operator $E_{N+1} = (I - P_{N+1})\, E_N$ satisfies
>
> $$\|E_{N+1}\|_a \leq \|E_N\|_a\,.$$

(c) Recall the **Strengthened Cauchy–Schwarz Inequality** (Assumption 10.2): **(2 pt.)**

There exist constants $0 \leq \epsilon_{ij} \leq 1$, $1 \leq i, j \leq N$, such that

$$\left|a\left(R_i^T u_i, R_j^T u_j\right)\right| \leq \epsilon_{ij}\, a\left(R_i^T u_i, R_i^T u_i\right)^{1/2} a\left(R_j^T u_j, R_j^T u_j\right)^{1/2}$$

for $u_i \in V_i$ and $u_j \in V_j$. We consider $\mathcal{E} = (\epsilon_{ij})$ as a matrix and $\rho(\mathcal{E})$ as its spectral radius.

Prove the following lemma from the lectures:

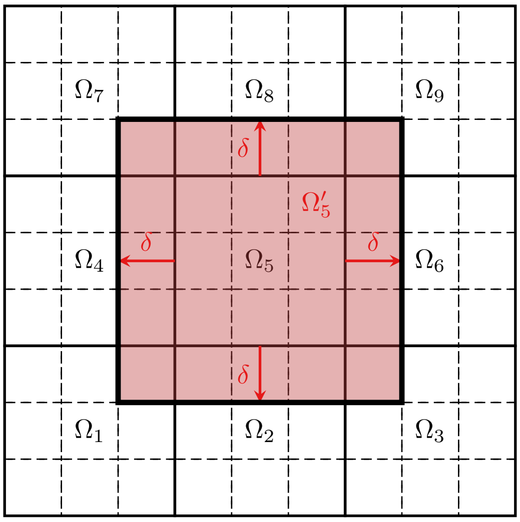

Figure 1: 9×9 Mesh (dashed lines) and 3×3 nonoverlapping subdomains (solid lines) as well as one exemplary overlapping subdomain with one layer of elements of overlap (red); the other eight overlapping subdomains are not shown.

> **LEMMA 10.4**
>
> Let $\mathcal{E} = (\epsilon_{ij})$ be given as in the strengthened Cauchy–Schwarz inequality. Suppose that there are at most N^c nonzeros in each row of $\mathcal{E}$. Then,
>
> $$\rho(\mathcal{E}) \leq N^c.$$

Part B (7 points)

1. **Implementation:** Solve the following boundary value problem using piecewise bilinear finite elements: find u such that **(5 pt.)**

$$-\Delta u = f \quad \text{in } \Omega = [0,1]^2,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

This time, consider the solution of the resulting discrete linear equation system

$$Ku = f$$

using the **preconditioned conjugate gradient (PCG)** method and an additive overlapping Schwarz preconditioner. Therefore, make sure that all requirements for the PCG method are met (discuss this briefly!) and implement the **additive one-level Schwarz preconditioner**

$$M_{1-lvl}^{-1} = \sum\nolimits_{i=1}^{N} R_i^\top K_i^{-1} R_i K \tag{1}$$

with **exact local solvers**. Use a structured domain decomposition into $N = n \times n$ subdomains with overlap δ; see fig. 1.

> **REMARK**
>
> - The goal of these preconditioners is to enable fast convergence of the PCG method. At the same time, they should be implemented efficiently. Some **requirements for the implementation**:
>   - All matrices that are sparse should also be stored in a sparse matrix format.
>   - Avoid unnecessary re-computations of matrices: In a setup phase, construct all necessary components of the preconditioner. In the PCG iteration, **only apply** the preconditioner.
>   - Implement the local and coarse solves K_i^{-1} and K_0^{-1} efficiently: as usual in numerical mathematics, **avoid computing inverse matrices**.

2. Test the **convergence of (P)CG** **(2 pt.)**

   - **without any preconditioner** and
   - using the **one-level Schwarz preconditioner** M_{1-lvl}^{-1}

   varying the subdomain size H, the size of the overlap δ, and the fine mesh size h. Discuss your results.

   > REMARK
   >
   > Among others, consider the following aspects in your discussion:
   >
   > - The theoretical results from the lectures (convergence rate of (P)CG, condition numbers) as well as the computing times.
   > - Which preconditioner settings (H and δ) are most efficient for a given h?
   > - What would be your expectations for a two-level Schwarz preconditioner?