CHAPTER

1 Introduction to the Equations of Fluid Dynamics and the Finite Element Approximation

1.1 General remarks and classification of fluid dynamics problems discussed in this book

The problems of solid and fluid behavior are in many respects similar. In both media stresses occur and in both the material is displaced. There is however one major difference. Fluids cannot support any deviatoric stresses when at rest, where only a pressure or a mean compressive stress can be carried. As we know, in solids general forms of stresses can exist and a solid material can support both deviatoric and pressure stresses.

In addition to pressure, deviatoric stresses can develop when the fluid is in motion and such motion of the fluid will always be of primary interest in *fluid dynamics*. We shall therefore concentrate on problems in which displacement is continuously changing and in which velocity is the main characteristic of the flow. The deviatoric stresses which can now occur will be characterized by a quantity that has great resemblance to the shear modulus of solid mechanics and which is known as *dynamic viscosity* or molecular viscosity.

Up to this point the equations governing fluid flow and solid mechanics appear to be similar with the velocity vector **u** replacing the displacement, which often uses the same symbol. However, there is one further difference. Even when the flow has a constant velocity (steady-state), *convective acceleration* effects add terms which make the fluid dynamics equations *non-self-adjoint* (see Appendix A). Therefore, in most cases, unless the velocities are very small so that the convective acceleration is negligible, the treatment has to be somewhat different from that of solid mechanics. The reader should note that for self-adjoint forms, approximating the equations by the Galerkin method gives the minimum error in the energy norm and thus such approximations are in a sense optimal. In general, this is no longer true in fluid mechanics, though for slow flows such as creeping flows where the convective acceleration terms are negligible the situation is somewhat similar.

With a fluid which is in motion, conservation of mass is always essential and, unless the fluid is highly compressible, we require that the divergence of the velocity vector be zero. Similar problems are encountered in the context of incompressible elasticity and the *incompressibility constraint* can introduce difficulties in the formulation

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(viz. Ref. [1]). In fluid dynamics the same difficulty again arises and all fluid dynamics approximations have to be such that, even if compressibility is possible, the limit of incompressibility can be modeled. This precludes the use of many elements which are otherwise acceptable.

In this book we shall introduce the reader to finite element treatment of the equations of motion for various problems of fluid dynamics. Much of the earlier activity in fluid dynamics has however pursued a *finite difference* formulation and later a derivative of this known as the *finite volume* technique. Competition between finite element methods and techniques of finite differences has appeared and led to a much slower adoption of the finite element process in fluid dynamics than in structures. The reasons for this are perhaps simple. In solid mechanics or structural problems, the treatment of continua often arises in combination with other structural forms, e.g., trusses, beams, plates, and shells. The engineer often dealing with structures composed of structural elements does not need to solve continuum problems. In addition when continuum problems are encountered, the system can lead to use of many different material models which are easily treated using a finite element formulation. In fluid mechanics, practically all situations of flow require a two- or three-dimensional treatment and here approximation is required. This accounts for the early use of finite differences in the 1950s before the finite element process was made available. However, as pointed out in Ref. [1], there are many advantages of using the finite element process. This not only allows a fully unstructured and arbitrary domain subdivision to be used but also provides an approximation which in self-adjoint problems is always superior to or at least equal to that provided by finite differences.

A methodology which appears to have gained an intermediate position is that of finite volumes, which were initially derived as a subclass of finite difference methods. As shown later in this chapter, these are simply another kind of finite element form in which subdomain collocation is used. We do not see much advantage in using this form of approximation; however, there is one point which seems to appeal to some investigators. That is the fact the finite volume approximation explicitly satisfies conservation conditions for each finite volume [2]. In full finite element analysis, however, satisfaction of conservation conditions is implicitly achieved [3–10]. Explicit satisfaction of the conservation conditions on an individual element is not an advantage if the general finite element approximation gives results which are superior.

In this book we will discuss various classes of problems, each of which has a certain behavior in the numerical solution. Here we start with incompressible flows or flows where the only change of volume is elastic and associated with transient changes of pressure (Chapters 4 and 5). For such flows full incompressible constraints must be available.

Further, with very slow speeds, convective acceleration effects are often negligible and the solution can on occasion be reached using identical methods to those derived for linear incompressible elasticity. This indeed was the first venture of finite element developers into the field of fluid mechanics, thus transferring the direct knowledge from solid mechanics to fluids. In particular the so-called linear Stokes flow is the case where fully incompressible but elastic behavior occurs. A particular variant of Stokes

flow is that used in metal forming where the material can no longer be described by a constant viscosity but possesses a viscosity which is non-Newtonian and depends on the strain rates and temperature. Chapter 5 is devoted to such problems. Here the fluid (flow formulation) can be applied directly to problems such as the forming of metals or plastics and we shall discuss this extreme situation in Chapter 5. However, even in incompressible flows, when the speed increases convective acceleration terms become important. Here often steady-state solutions do not exist or at least are extremely unstable. This leads us to such problems as vortex shedding. Vortex shedding indicates the start of instability which becomes very irregular and indeed random when high-speed flow occurs in viscous fluids. This introduces the subject of turbulence, which occurs frequently in fluid dynamics. In turbulent flows random fluctuation of velocity occurs at all points and the problem is highly time dependent. With such turbulent motion, it is possible to obtain an averaged solution using time-averaged equations. Details for some available time-averaged models are summarized in Chapter 8.

Chapter 6 deals with incompressible flow in which free surface and other gravity controlled effects occur. In particular we show three different approaches for dealing with free surface flows and explain the necessary modifications to the general formulation.

The next area of fluid dynamics to which much practical interest is devoted is of course that of flow of gases for which the compressibility effects are much larger. Here compressibility is problem dependent and generally obeys gas laws which relate the pressure to temperature and density. It is now often necessary to add the energy conservation equation to the system governing the motion so that the temperature can be evaluated. Such an energy equation can of course be written for incompressible flows but this shows only a weak or no coupling with the dynamics of the flow. This is not the case in compressible flows where coupling between all equations is very strong. In such compressible flows the flow speed may exceed the speed of sound and this may lead to shock development. This subject is of major importance in the field of aerodynamics and we shall devote Chapter 7 to this particular problem.

In a real fluid, viscosity is always present but at high speeds such viscous effects are confined to a narrow zone in the vicinity of solid boundaries (the so-called *boundary layer*). In such cases, the remainder of the fluid can be considered to be inviscid. There we can return to the fiction of an ideal fluid in which viscosity is not present and here various simplifications are again possible. Such simplifications have been used since the early days of aerodynamics and date back to the work of Prandtl and Schlichting [11]. One simplification is the introduction of potential flow and we shall mention this later in this chapter. Potential flows are indeed the topic of many finite element investigators, but unfortunately such solutions are not easily extendible to realistic problems.

A particular form of viscous flow problem occurs in the modeling of flow in porous media. This important field is discussed in Chapter 9. In the topic of flow through porous media, two extreme situations are often encountered. In the first, the porous medium is stationary and the fluid flow occurs only in the narrow passages between solid grains. Such an extreme is the basis of porous medium flow modeling in

applications such as geo-fluid dynamics where the flow of water or oil through porous soils or rocks occurs. The other extreme of porous media flow is the one in which the solid occupies only a small part of the total volume (for example, representing thermal insulation systems, heat exchangers, etc.). In such problems flow is almost the same as that occurring in fluids without the solid phase which only applies an added, distributed, resistance to flow. Both extremes are discussed in Chapter 9.

Another major field of fluid mechanics of interest to us is that of shallow-water flows that occur in coastal estuaries or elsewhere. In this class of problems the depth dimension of flow is very much less than the horizontal ones. Chapter 10 deals with such problems in which essentially the distribution of pressure in the vertical direction is almost hydrostatic. For such shallow-water problems a free surface also occurs. This dominates the flow characteristics and here we note that shallow-water flow problems result in a formulation which is closely related to gas flow.

Whenever a free surface occurs it is possible for transient phenomena to happen, generating waves such as those occurring in oceans or other bodies of water. Chapters 11 and 12 deal with this particular aspect of fluid dynamics. Such wave phenomena are also typical of some other physical problems. For instance, acoustic and electromagnetic waves can be solved using similar approaches. Indeed, one can show that the treatment for this class of problems is very similar to that of surface wave problems.

The interaction of two or more distinctive disciplines is often an integral part of many engineering problems. One such example is the subject of fluid and structure interaction. This topic is the primary focus in many offshore and biomedical engineering problems. In Chapter 13, we have introduced some fundamental approaches to modeling the fluid-solid interaction using the finite element method.

The treatments used for fluid dynamics and fluid-structure interaction problems have a dramatic influence on the biofluid dynamics problems. Although biofluid dynamics is a topic of applied fluid dynamics, there are many situations that require additional knowledge and treatment. Chapter 14 describes some treatments that are important in the solution of biofluid dynamics problems.

In what remains of this chapter we shall introduce the general equations of fluid dynamics valid for most compressible and incompressible flows, showing how a particular simplification occurs in some categories of problems mentioned above. However, before proceeding with the recommended discretization procedures, which we present in Chapter 3, we must introduce the treatment of problems in which convection and diffusion occur simultaneously. This we shall do in Chapter 2 using the scalar convection-diffusion equation. Based on concepts given in Chapter 2, Chapter 3 will introduce a general algorithm capable of solving most of the fluid mechanics problems encountered in this book. There are many possible algorithms and very often specialized ones are used in different areas of applications. However the general algorithm of Chapter 3 produces results which are at least as good as others achieved by more specialized means. We feel that this will give a certain unification to the whole subject of fluid dynamics and, without apology, we will omit reference to many other methods or discuss them only in passing.

For completeness we shall show in the present chapter some detail of the finite element process to avoid the repetition of basic finite element presentations which we assume are known to the reader either from Ref. [1] or from any of the numerous texts available.

1.2 The governing equations of fluid dynamics [2, 11–17]

1.2.1 Velocity, strain rates, and stresses in fluids

As noted above, the essential characteristic of a fluid is its inability to sustain deviatoric stresses when at rest. Here only hydrostatic "stress" or pressure is possible. Any analysis must therefore concentrate on the motion, and the essential independent variable is the velocity $\mathbf{u}$ or, if we adopt indicial notation (with the coordinate axes referred to as x_i, $i = 1, 2, 3$),

$$u_i, i = 1, 2, 3 \quad \text{or} \quad \mathbf{u} = \begin{bmatrix} u_1, & u_2, & u_3 \end{bmatrix}^T \tag{1.1}$$

This replaces the displacement variable which is of primary importance in solid mechanics. The rates of strain are the primary cause of the general stresses, σ_{ij}, and these are defined in a manner analogous to that of infinitesimal strain in solid mechanics as

$$\dot{\varepsilon}_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.2}$$

This is a well-known tensorial definition of strain rates but for use later in variational forms is written as a vector, which is more convenient in finite element analysis. Details of such matrix forms are given fully in Ref. [1] but for completeness we summarize them here. Thus, the strain rate is written as a vector ($\dot{\boldsymbol{\varepsilon}}$) and is given by the form

$$\dot{\boldsymbol{\varepsilon}} = \begin{bmatrix} \dot{\varepsilon}_{11}, \dot{\varepsilon}_{22}, \dot{\varepsilon}_{33}, 2\dot{\varepsilon}_{12}, 2\dot{\varepsilon}_{23}, 2\dot{\varepsilon}_{31} \end{bmatrix}^T \tag{1.3a}$$

in three dimensions with the form in two dimensions given by*

$$\dot{\boldsymbol{\varepsilon}} = \begin{bmatrix} \dot{\varepsilon}_{11}, \dot{\varepsilon}_{22}, 2\dot{\varepsilon}_{12} \end{bmatrix}^T \tag{1.3b}$$

In the above $2\dot{\varepsilon}_{ij}, i \neq j$ is an engineering shear strain rate which we will write as

$$\dot{\varepsilon}_{ij} = \dot{\gamma}_{ij} \quad \text{for} \quad i \neq j \tag{1.3c}$$

When such vector forms are used we can write the strain rate vector in the form

$$2\dot{\boldsymbol{\varepsilon}} = \boldsymbol{\mathcal{S}}\mathbf{u} \tag{1.4}$$

where $\boldsymbol{\mathcal{S}}$ is known as the strain rate operator and $\mathbf{u}$ is the velocity given in Eq. (1.1).

Similarly, we will write the tensor components of stress as σ_{ij} and express this in matrix form as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{31} \end{bmatrix}^T \tag{1.5}$$

*In plane flows $\dot{\varepsilon}_{33} = 0$ and we solve in the x_1, x_2 plane.

1.2.2 Constitutive relations for fluids

The stress-strain rate relations for a linear (Newtonian) isotropic fluid require the definition of two constants. The first of these links the *deviatoric stresses* τ_{ij} to the *deviatoric strain rates*:

$$\tau_{ij} \equiv \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\,\sigma_{kk} = 2\mu\left(\dot{\varepsilon}_{ij} - \tfrac{1}{3}\delta_{ij}\dot{\varepsilon}_{kk}\right) \tag{1.6}$$

In the above equation the quantity in brackets is known as the deviatoric strain rate, δ_{ij} is the Kronecker delta,

$$\delta_{ij} = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$

and a repeated index implies summation over the range of the index; thus

$$\sigma_{kk} \equiv \sigma_{11} + \sigma_{22} + \sigma_{33} \qquad \text{and} \qquad \dot{\varepsilon}_{kk} \equiv \dot{\varepsilon}_{11} + \dot{\varepsilon}_{22} + \dot{\varepsilon}_{33} \tag{1.7}$$

The coefficient μ is known as the dynamic (shear) viscosity or simply viscosity and is analogous to the shear modulus G in linear elasticity.

The second relation is that between the mean stress changes and the volumetric strain rate. This defines the pressure as

$$p = -\tfrac{1}{3}\,\sigma_{kk} = -\kappa\,\dot{\varepsilon}_{kk} + p_0 \tag{1.8}$$

where κ is a *volumetric viscosity* coefficient analogous to the bulk modulus K in linear elasticity and p_0 is the initial hydrostatic pressure independent of the strain rate (note that p and p_0 are invariably defined as positive when compressive).

We can immediately write the "constitutive" relation for fluids from Eqs. (1.6) and (1.8) as

$$\begin{aligned} \sigma_{ij} &= \tau_{ij} - \delta_{ij}\,p \\ &= 2\mu\left(\dot{\varepsilon}_{ij} - \tfrac{1}{3}\delta_{ij}\dot{\varepsilon}_{kk}\right) + \kappa\,\delta_{ij}\,\dot{\varepsilon}_{kk} - \delta_{ij}\,p_0 \end{aligned} \tag{1.9a}$$

or

$$\sigma_{ij} = 2\mu\dot{\varepsilon}_{ij} + \delta_{ij}(\kappa - \tfrac{2}{3}\mu)\dot{\varepsilon}_{kk} - \delta_{ij}\,p_0 \tag{1.9b}$$

There is little evidence about the existence of volumetric viscosity and, in what follows, we shall take

$$\kappa\,\dot{\varepsilon}_{kk} \equiv 0 \tag{1.10}$$

giving the essential constitutive relation as (now dropping the suffix on p_0)

$$\sigma_{ij} = 2\mu\left(\dot{\varepsilon}_{ij} - \tfrac{1}{3}\delta_{ij}\dot{\varepsilon}_{kk}\right) - \delta_{ij}\,p \equiv \tau_{ij} - \delta_{ij}\,p \tag{1.11a}$$

without necessarily implying incompressibility $\dot{\varepsilon}_{kk} = 0$.

In the above,

$$\tau_{ij} = 2\mu\left(\dot{\varepsilon}_{ij} - \tfrac{1}{3}\delta_{ij}\,\dot{\varepsilon}_{kk}\right) = \mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \tfrac{2}{3}\,\delta_{ij}\,\frac{\partial u_k}{\partial x_k}\right] \tag{1.11b}$$

The above relationships are identical to those of isotropic linear elasticity as we will note again later for incompressible flow. However, in solid mechanics we often consider anisotropic materials where a larger number of parameters (i.e., more than 2) are required to define the stress-strain relations. In fluid mechanics use of such anisotropy is rare and in this book we will limit ourselves to purely isotropic behavior.

Nonlinearity of some fluid flows is observed with a coefficient μ depending on strain rates and/or other variables such as temperature. We shall term such flows "non-Newtonian."

We now consider the basic conservation principles used to write the equations of fluid dynamics. These are *mass conservation*, *momentum conservation*, and *energy conservation*.

1.2.3 Mass conservation

If ρ is the fluid density then the balance of mass flow ρu_i entering and leaving an infinitesimal control volume (Fig. 1.1) is equal to the rate of change in density as expressed by the relation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho u_i\right) \equiv \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}^{\mathrm{T}}\left(\rho \mathbf{u}\right) = 0 \tag{1.12}$$

where $\boldsymbol{\nabla}^{\mathrm{T}} = \left[\partial/\partial x_1, \quad \partial/\partial x_2, \quad \partial/\partial x_3\right]$ is known as the gradient operator.

It should be noted that in this section, and indeed a majority of subsequent ones, the control volume remains fixed in space. This is known as the "Eulerian form" and displacements of a particle are ignored. This is in contrast to the usual treatment in solid mechanics where displacement is a primary dependent variable.

It is possible to recast the above equations in relation to a moving frame of reference and, if the motion follows the particle, the equations will be named "Lagrangian." Such Lagrangian frame of reference is occasionally used in fluid dynamics and briefly discussed in Chapter 6.

1.2.4 Momentum conservation: Dynamic equilibrium

In the jth direction the balance of linear momentum leaving and entering the control volume (Fig. 1.1) is to be in dynamic equilibrium with the stresses σ_{ij} and body forces ρg_j. This gives a typical component equation

$$\frac{\partial(\rho u_j)}{\partial t} + \frac{\partial}{\partial x_i}[(\rho u_j)u_i] - \frac{\partial}{\partial x_i}(\sigma_{ij}) - \rho g_j = 0, \quad j = 1, 2, 3 \tag{1.13}$$

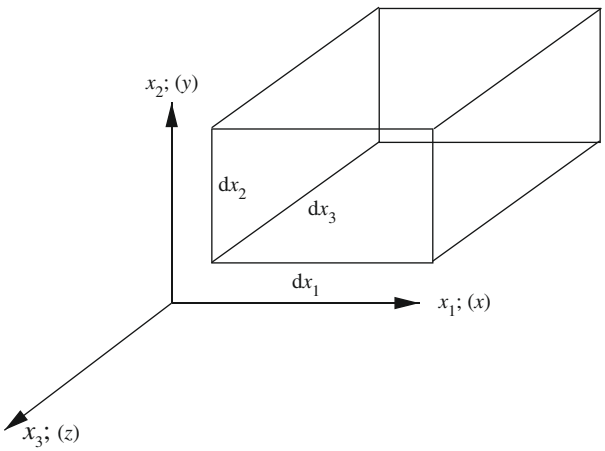

FIGURE 1.1
Coordinate direction and the infinitesimal control volume.

or using (1.11a),

$$\frac{\partial(\rho u_j)}{\partial t} + \frac{\partial}{\partial x_i}[(\rho u_j)u_i] - \frac{\partial \tau_{ij}}{\partial x_i} + \frac{\partial p}{\partial x_j} - \rho g_j = 0, \quad j = 1, 2, 3 \tag{1.14}$$

with (1.11b) implied.

The conservation of angular momentum merely requires the stress to be symmetric, i.e.,

$$\sigma_{ij} = \sigma_{ji} \quad \text{or} \quad \tau_{ij} = \tau_{ji}$$

In the sequel we will use the term *momentum conservation* to imply both linear and angular forms.

1.2.5 Energy conservation and equation of state

We note that in the equations of Sections 1.2.3 and 1.2.4 the dependent variables are u_i (the velocity components), p (the pressure), and ρ (the density). The deviatoric stresses, of course, are defined by Eq. (1.11b) in terms of velocities and hence are dependent variables.

Obviously, there is one variable too many for this equation system to be capable of solution. However, if the density is assumed constant (as in incompressible fluids) or if a single relationship linking pressure and density can be established (as in isothermal flow with small compressibility) the system becomes complete and solvable.

More generally, the pressure (p), density (ρ), and absolute temperature (T) are related by an *equation of state* of the form

$$\rho = \rho(p, T) \tag{1.15a}$$

For an *ideal gas* this takes the form

$$\rho = \frac{p}{RT} \tag{1.15b}$$

where R is the universal gas constant.

In such a general case, it is necessary to supplement the governing equation system by the equation of *energy conservation*. This equation is of interest even if it is not coupled with the mass and momentum conservation, as it provides additional information about the behavior of the system.

Before proceeding with the derivation of the energy conservation equation we must define some further quantities. Thus we introduce e, the *intrinsic energy* per unit mass. This is dependent on the state of the fluid, i.e., its pressure p and temperature T or

$$e = e(T, p) \tag{1.16}$$

The total energy per unit mass, E, includes of course the kinetic energy per unit mass and thus

$$E = e + \tfrac{1}{2} u_i u_i \tag{1.17}$$

Finally, we can define the *enthalpy* as

$$h = e + \frac{p}{\rho} \qquad \text{or} \qquad H = h + \tfrac{1}{2} u_i u_i = E + \frac{p}{\rho} \tag{1.18}$$

and these variables are found to be convenient to express the conservation of energy relation.

Energy transfer can take place by convection and by conduction (radiation generally being confined to boundaries). The conductive heat flux q_i for an isotropic material is defined as

$$q_i = -k \frac{\partial T}{\partial x_i} \tag{1.19}$$

where k is thermal conductivity.

To complete the relationship it is necessary to determine heat source terms. These can be specified per unit volume as q_H due to chemical reaction (if any) and must include the energy dissipation due to internal stresses, i.e., using Eq. (1.11a),

$$\frac{\partial}{\partial x_i}(\sigma_{ij} u_j) = \frac{\partial}{\partial x_i}(\tau_{ij} u_j) - \frac{\partial}{\partial x_j}(p u_j) \tag{1.20}$$

The balance of energy in an infinitesimal control volume can now be written as

$$\frac{\partial(\rho E)}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i E) - \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) + \frac{\partial}{\partial x_i}(p u_i) - \frac{\partial}{\partial x_i}(\tau_{ij} u_j) - \rho g_i u_i - q_H = 0 \tag{1.21a}$$

or using Eq. (1.18) more simply

$$\frac{\partial(\rho E)}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i H) - \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) - \frac{\partial}{\partial x_i}(\tau_{ij} u_j) - \rho g_i u_i - q_H = 0 \tag{1.21b}$$

Here, the penultimate term represents the rate of work done by body forces.

1.2.6 Boundary conditions

On the boundary of a typical fluid dynamics problem *boundary conditions* need to be specified to make the solution possible. These are given simply as follows:

a. The velocities can be described as

$$u_i = \bar{u}_i \quad \text{on } \Gamma_u \tag{1.22a}$$

or traction as

$$t_i = n_j \sigma_{ij} = \bar{t}_i \quad \text{on } \Gamma_t \tag{1.22b}$$

where $\Gamma_u \bigcup \Gamma_t = \Gamma$ and $\Gamma_u \bigcap \Gamma_t = 0$. Generally traction is resolved into normal and tangential components to the boundary.

b. In problems for which consideration of energy is important the temperature on the boundary is expressed as

$$T = \bar{T} \quad \text{on } \Gamma_T \tag{1.23a}$$

or thermal flux

$$q_n = -n_i k \frac{\partial T}{\partial x_i} = -k \frac{\partial T}{\partial n} = \bar{q}_n \quad \text{on } \Gamma_q \tag{1.23b}$$

where $\Gamma_T \bigcup \Gamma_q = \Gamma$ and $\Gamma_T \bigcap \Gamma_q = 0$.

c. For problems of compressible flow the input density is specified as

$$\rho = \bar{\rho} \quad \text{on } \Gamma_\rho \tag{1.24}$$

1.2.7 Navier-Stokes and Euler equations

The governing equations derived in the preceding sections can be written in a general conservative form as

$$\frac{\partial \mathbf{\Phi}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} + \frac{\partial \mathbf{G}_i}{\partial x_i} + \mathbf{Q} = \mathbf{0} \tag{1.25}$$

in which Eqs. (1.12), (1.14), or (1.21b) provide the particular entries to the vectors.

Thus, using indicial notation the vector of independent unknowns is

$$\mathbf{\Phi} = \begin{Bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho E \end{Bmatrix} \tag{1.26a}$$

The convective flux is expressed as

$$\mathbf{F}_i = \begin{Bmatrix} \rho u_i \\ \rho u_1 u_i + p\delta_{1i} \\ \rho u_2 u_i + p\delta_{2i} \\ \rho u_3 u_i + p\delta_{3i} \\ \rho H u_i \end{Bmatrix} \tag{1.26b}$$

Similarly, the diffusive flux is expressed as

$$\mathbf{G}_i = \begin{Bmatrix} 0 \\ -\tau_{1i} \\ -\tau_{2i} \\ -\tau_{3i} \\ -(\tau_{ij}u_j) - k\dfrac{\partial T}{\partial x_i} \end{Bmatrix} \tag{1.26c}$$

and the source terms as

$$\mathbf{Q} = \begin{Bmatrix} 0 \\ -\rho g_1 \\ -\rho g_2 \\ -\rho g_3 \\ -\rho g_i u_i - q_H \end{Bmatrix} \tag{1.26d}$$

with

$$\tau_{ij} = \mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \tfrac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right]$$

The complete set of (1.25) is known as the *Navier-Stokes equation*. A particular case when viscosity is assumed to be zero and no heat conduction exists is known as the "Euler equation" (where $\tau_{ij} = 0$ and $q_i = 0$).

The above equations are the basis from which all fluid dynamics studies start and it is not surprising that many alternative forms are given in the literature obtained by combinations of the various equations [13].

The above set is, however, convenient and physically meaningful, defining the conservation of important quantities. It should be noted that only equations written in conservation form will yield the correct, physically meaningful, results in problems where shock or other forms of discontinuities are present.

In Appendix B, we show a particular set of nonconservative equations which are frequently used. The reader is cautioned not to extend the use of nonconservative equations to problems of high-speed flow.

In many actual situations one or another feature of the flow is predominant. For instance, frequently the viscosity is only of importance close to the boundaries at which velocities are specified. In such cases the problem can be considered separately in two parts: one as a *boundary layer* near such boundaries and another as *inviscid flow* outside the boundary layer.

Further, in many cases a steady-state solution is not available with the fluid exhibiting *turbulence*, i.e., a random fluctuation of velocity. Here it is still possible to use the general Navier-Stokes equations now written in terms of the mean flow with an additional *Reynolds stress* term. Turbulent instability is inherent in the Navier-Stokes equations. It is in principle always possible to obtain the transient, turbulent, solution modeling of the flow, provided the mesh size is capable of reproducing the small eddies which develop in the problem. Such computations, although possible, are extremely costly and hence the Reynolds averaging approach is of practical importance.

Two further points have to be made concerning *inviscid flow* (ideal fluid flow as it is sometimes known). Firstly, the Euler equations are of a purely convective form

$$\frac{\partial \mathbf{\Phi}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} + \mathbf{Q} = \mathbf{0} \qquad \mathbf{F}_i = \mathbf{F}_i(\mathbf{\Phi}) \tag{1.27}$$

and hence very special methods for their solutions will be necessary. These methods are applicable and useful mainly in *compressible flow*, as we shall discuss in Chapter 7. Secondly, for incompressible (or nearly incompressible) flows it is of interest to introduce a *potential* that converts the Euler equations to a simple self-adjoint form (see Appendix A). We shall discuss this potential approximation in Section1.3. Although potential forms are applicable also to compressible flows we shall not use them as they fail in complex situations.

1.3 Inviscid, incompressible flow

In the absence of viscosity and compressibility, ρ is constant and Eq. (1.12) can be written as

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{1.28}$$

and (1.14) as

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_j u_i) + \frac{1}{\rho}\frac{\partial p}{\partial x_i} - g_i = 0 \tag{1.29}$$

1.3.1 Velocity potential solution

The Euler equations given above may be simplified by introducing a potential, ϕ, and defining velocities as

$$u_1 = -\frac{\partial \phi}{\partial x_1} \qquad u_2 = -\frac{\partial \phi}{\partial x_2} \qquad u_3 = -\frac{\partial \phi}{\partial x_3}$$

or

$$u_i = -\frac{\partial \phi}{\partial x_i} \tag{1.30}$$

If such a potential exists then insertion of Eq. (1.30) into (1.28) gives a single governing equation

$$\frac{\partial^2 \phi}{\partial x_i \partial x_i} \equiv \nabla^2 \phi = 0 \tag{1.31}$$

which, with appropriate boundary conditions, can be readily solved. Equation (1.31) is a classical Laplacian equation. For contained flow we can of course impose the normal velocity u_n on the boundaries:

$$u_n = -\frac{\partial \phi}{\partial n} = \bar{u}_n \tag{1.32}$$

As we shall see later, this provides a *natural* boundary condition for a weighted residual or finite element solution.

Of course we must be assured that the potential function ϕ exists, and indeed determine what conditions are necessary for its existence. Here we observe that so far in the definition of the problem we have not used the momentum conservation equation (1.29), to which we shall now return. However, we first note that a single-valued potential function implies that

$$\frac{\partial^2 \phi}{\partial x_j \, \partial x_i} = \frac{\partial^2 \phi}{\partial x_i \, \partial x_j} \tag{1.33}$$

Defining vorticity as rotation rate per unit area

$$\omega_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{1.34}$$

we note that the use of the velocity potential in (1.34) gives

$$\omega_{ij} = 0 \tag{1.35}$$

and the flow is therefore named *irrotational*.

Inserting the definition of potential into the first term of Eq. (1.29) and using Eqs. (1.28) and (1.34) we can rewrite Eq. (1.29) as

$$-\frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial t}\right) + \frac{\partial}{\partial x_i}\left[\tfrac{1}{2}u_j u_j + \frac{p}{\rho} + P\right] = 0 \tag{1.36}$$

in which P is a potential of the body forces given by

$$g_i = -\frac{\partial P}{\partial x_i} \tag{1.37}$$

In problems involving constant gravity forces in the x_3 direction the body force potential is simply

$$P = g\, x_3 \tag{1.38}$$

Equation (1.36) is alternatively written as

$$\nabla\left(-\frac{\partial \phi}{\partial t} + H + P\right) = 0 \tag{1.39}$$

where H is the *enthalpy*, given by (1.18).

If isothermal conditions pertain, the specific energy is constant and (1.39) implies that

$$-\frac{\partial \phi}{\partial t} + \tfrac{1}{2}u_i u_i + \frac{p}{\rho} + P = \text{constant} \tag{1.40}$$

over the whole domain. This can be taken as a corollary of the existence of the potential and indeed is a condition for its existence. In steady-state flows it provides the well-known Bernoulli equation that allows the pressures to be determined throughout the whole potential field once the value of the constant is established.

We note that the governing potential equation (1.31) is self-adjoint (see Appendix A) and that the introduction of the potential has sidestepped the difficulties of dealing with convective terms. It is also of interest to note that the Laplacian equation, which is obeyed by the velocity potential, occurs in other contexts. For instance in two-dimensional flow it is convenient to introduce a *stream function* the contours of which lie along the streamlines. The stream function, ψ, defines the velocities as

$$u_1 = \frac{\partial \psi}{\partial x_2} \quad \text{and} \quad u_2 = -\frac{\partial \psi}{\partial x_1} \tag{1.41}$$

which satisfy the incompressibility condition (1.28):

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_1}\left(\frac{\partial \psi}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(-\frac{\partial \psi}{\partial x_1}\right) = 0 \tag{1.42}$$

For an existence of a unique potential for irrotational flow we note that $\omega_{12} = 0$ gives a Laplacian equation

$$\frac{\partial^2 \psi}{\partial x_i \partial x_i} = \nabla^2 \psi = 0 \tag{1.43}$$

The stream function is very useful in getting a pictorial representation of flow. In Appendix C we show how the stream function can be readily computed from a known distribution of velocities.

1.4 Incompressible (or nearly incompressible) flows

We observed earlier that the Navier-Stokes equations are completed by the existence of a state relationship giving [Eq. (1.15a)]

$$\rho = \rho(p, T)$$

In (nearly) incompressible relations we shall frequently assume that

a. The problem is isothermal.
b. The variation of ρ with p is very small, i.e., such that in product terms of velocity and density the latter can be assumed constant.

The first assumption will be relaxed, as we shall see later, allowing some thermal coupling via the dependence of the fluid properties on temperature. In such cases we shall introduce the coupling iteratively. For such cases the problem of density-induced currents or temperature-dependent viscosity will be typical (see Chapters 5 and 6).

If the assumptions introduced above are used we can still allow for small compressibility, noting that density changes are, as a consequence of elastic deformability, related to pressure changes. Thus we can write

$$\mathrm{d}\rho = \frac{\rho}{K}\mathrm{d}p \tag{1.44a}$$

where K is the elastic bulk modulus. This also can be written as

$$\mathrm{d}\rho = \frac{1}{c^2}\mathrm{d}p \tag{1.44b}$$

or

$$\frac{\partial \rho}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} \tag{1.44c}$$

with $c = \sqrt{K/\rho}$ being the acoustic wave velocity.

Equations (1.25) and (1.26a)–(1.26d) can now be rewritten omitting the energy transport equation (and condensing the general form) as

$$\frac{1}{c^2}\frac{\partial p}{\partial t} + \rho\frac{\partial u_i}{\partial x_i} = 0 \tag{1.45a}$$

$$\frac{\partial u_j}{\partial t} + \frac{\partial}{\partial x_i}(u_j u_i) + \frac{1}{\rho}\frac{\partial p}{\partial x_j} - \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_i} - g_j = 0 \tag{1.45b}$$

In three dimensions $j = 1, 2, 3$ and the above represents a system of four equations in which the variables are u_j and p. Here

$$\frac{1}{\rho}\tau_{ij} = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \tfrac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

The reader will note that the above equations, with the exception of the convective acceleration terms, are *identical to those governing the problem of incompressible* (*or slightly compressible*) *elasticity* (e.g., see Chapter 10 of Ref. [1]).

1.5 Numerical solutions: Weak forms, weighted residual, and finite element approximation

1.5.1 Strong and weak forms

We assume the reader is already familiar with basic ideas of finite element and finite difference methods. However, to avoid a constant cross reference to other texts (e.g., [1]), we provide here a brief introduction to *weighted residual* (or Galerkin) and *finite element methods*.

The Laplace equation, which we introduced in Section 1.3, is a very convenient example for the start of numerical approximations. We shall generalize slightly and discuss in some detail the quasi-harmonic (Poisson) equation

$$-\frac{\partial}{\partial x_i}\left(k\frac{\partial \phi}{\partial x_i}\right)+Q=0 \tag{1.46}$$

where k and Q are specified functions. These equations together with appropriate boundary conditions define the problem uniquely. The boundary conditions can be Dirichlet type,

$$\phi=\bar{\phi} \qquad \text{on } \Gamma_\phi \tag{1.47a}$$

or Neumann type,

$$q_n=-k\frac{\partial \phi}{\partial n}=\bar{q}_n \qquad \text{on } \Gamma_q \tag{1.47b}$$

where a bar denotes a specified quantity.

Equations (1.46) to (1.47b) are defined as the *strong form* of the problem.

1.5.1.1 Weak form of equations

We note that direct use of Eq. (1.46) requires computation of second derivatives to solve a problem using approximate techniques. This requirement may be weakened by considering an integral expression for Eq. (1.46) written as

$$\int_\Omega v\left[-\frac{\partial}{\partial x_i}\left(k\frac{\partial \phi}{\partial x_i}\right)+Q\right]\mathrm{d}\Omega=0 \tag{1.48}$$

in which v is an arbitrary function. A proof that Eq. (1.48) is equivalent to Eq. (1.46) is simple.

If we assume Eq. (1.46) is not zero at some point x_i in Ω then we can also let v be a positive parameter times the same value resulting in a positive result for the integral of Eq. (1.48). Since this violates the equality we conclude that Eq. (1.46) must be zero for every x_i in Ω hence proving its equality with Eq. (1.48).

We may integrate by parts the second derivative terms in Eq. (1.48) to obtain

$$\int_\Omega \frac{\partial v}{\partial x_i}\left(k\frac{\partial \phi}{\partial x_i}\right)\mathrm{d}\Omega+\int_\Omega v\,Q\mathrm{d}\Omega-\int_\Gamma v\,n_i\left(k\frac{\partial \phi}{\partial x_i}\right)\mathrm{d}\Gamma=0 \tag{1.49}$$

We now split the boundary into two parts, Γ_ϕ and Γ_q, with $\Gamma=\Gamma_\phi\bigcup\Gamma_q$, and $\Gamma_\phi\bigcap\Gamma_q=0$, and use (1.47b) in (1.49) to give

$$\int_\Omega \frac{\partial v}{\partial x_i}\left(k\frac{\partial \phi}{\partial x_i}\right)\mathrm{d}\Omega+\int_\Omega v\,Q\,\mathrm{d}\Omega+\int_{\Gamma_q} v\,\bar{q}_n\,\mathrm{d}\Gamma=0 \tag{1.50}$$

which is valid only if v vanishes on Γ_ϕ. Hence we must impose Eq. (1.47a) for equivalence.

Equation (1.50) is known as the *weak form* of the problem since only first derivatives are necessary in constructing a solution. Such forms are the basis for the finite element solutions we use throughout this book.

1.5.2 Weighted residual approximation

In a weighted residual scheme an approximation to the independent variable ϕ is written as a sum of known *trial functions* (basis functions) $N_a(x_i)$ and unknown parameters $\tilde{\phi}^a$. Thus we can always write

$$\begin{aligned}\phi \approx \hat{\phi} &= N_1(x_i)\tilde{\phi}^1 + N_2(x_i)\tilde{\phi}^2 + \cdots \\ &= \sum_{a=1}^{n} N_a(x_i)\tilde{\phi}^a = \mathbf{N}(x_i)\tilde{\boldsymbol{\phi}}\end{aligned} \tag{1.51}$$

where

$$\mathbf{N} = \begin{bmatrix} N_1, & N_2, \ldots, & N_n \end{bmatrix} \tag{1.52a}$$

and

$$\tilde{\boldsymbol{\phi}} = \begin{bmatrix} \tilde{\phi}^1, & \tilde{\phi}^2, \ldots, & \tilde{\phi}^n \end{bmatrix}^T \tag{1.52b}$$

In a similar way we can express the arbitrary variable v as

$$\begin{aligned} v \approx \hat{v} &= W_1(x_i)\tilde{v}^1 + W_2(x_i)\tilde{v}^2 + \cdots \\ &= \sum_{a=1}^{n} W_a(x_i)\tilde{v}^a = \mathbf{W}(x_i)\tilde{\mathbf{v}} \end{aligned} \tag{1.53}$$

in which W_a are *test functions* and $\tilde{v}^a$ arbitrary parameters. Using this form of approximation will convert (1.50) to a set of algebraic equations.

In the finite element method and indeed in all other numerical procedures for which a computer-based solution can be used, the test and trial functions will generally be defined in a local manner. It is convenient to consider each of the test and basis functions to be defined in partitions Ω_e of the total domain Ω. This division is denoted by

$$\Omega \approx \Omega_h = \bigcup \Omega_e \tag{1.54}$$

and in a finite element method Ω_e are known as *elements*. The very simplest uses lines in one dimension, triangles in two dimensions, and tetrahedra in three dimensions in which the basis functions are usually linear polynomials in each element and the unknown parameters are *nodal values* of ϕ. In Fig. 1.2 we show a typical set of such linear functions defined in two dimensions.

In a weighted residual procedure we first insert the approximate function $\hat{\phi}$ into the governing differential equation creating a residual, $R(x_i)$, which of course should be zero at the exact solution. In the present case for the quasi-harmonic equation we obtain

$$R = -\frac{\partial}{\partial x_i}\left(k \sum_a \frac{\partial N_a}{\partial x_i}\tilde{\phi}^a\right) + Q \tag{1.55}$$

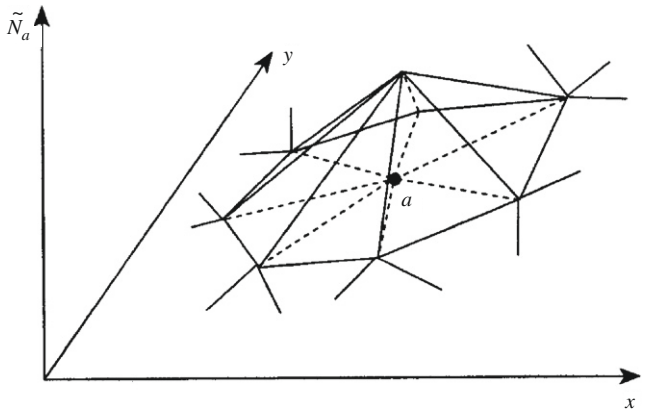

FIGURE 1.2
Basis function in linear polynomials for a patch of triangular elements.

and we now seek the best values of the parameter set $\tilde{\phi}^a$ which ensures that

$$\int_\Omega W_b \, R \mathrm{d}\Omega = 0, \quad b = 1, 2, \ldots, n \tag{1.56}$$

Note that this is the term multiplying the arbitrary parameter $\tilde{v}^b$. As noted previously, integration by parts is used to avoid higher-order derivatives (i.e., those greater than or equal to two) and therefore reduce the constraints on choosing the basis functions to permit integration over individual elements using Eq. (1.54). In the present case, for instance, the weighted residual after integration by parts and introducing the natural boundary condition becomes

$$\int_\Omega \frac{\partial W_b}{\partial x_i} \left(k \sum_a \frac{\partial N_a}{\partial x_i} \tilde{\phi} \right) \mathrm{d}\Omega + \int_\Omega W_b \, Q \mathrm{d}\Omega + \int_{\Gamma_q} W_b \, \bar{q}_n \, \mathrm{d}\Gamma = 0 \tag{1.57}$$

where b denotes a specific node where the residual is evaluated.

1.5.3 The Galerkin finite element method

In the Galerkin method we simply take $W_b = N_b$, which gives the assembled system of equations

$$\sum_{a=1}^{n} K_{ba} \tilde{\phi}^a + f_b = 0, \quad b = 1, 2, \ldots, n - r \tag{1.58}$$

where r is the number of nodes appearing in the approximation to the Dirichlet boundary condition [i.e., Eq. (1.47a)] and K_{ba} is assembled from element contributions K^e_{ba} with

$$K^e_{ba} = \int_{\Omega_e} \frac{\partial N_b}{\partial x_i} \, k \, \frac{\partial N_a}{\partial x_i} \mathrm{d}\Omega \tag{1.59}$$

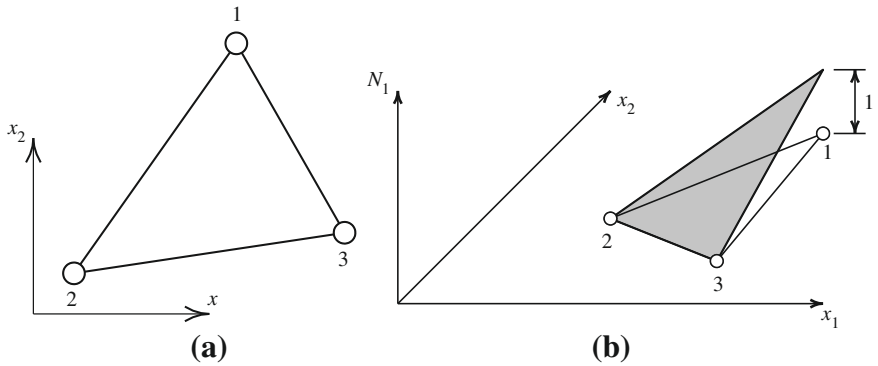

FIGURE 1.3
(a) Three-node triangular element and (b) shape function for N_1.

Similarly, f_b is computed from the element as

$$f_b^e = \int_{\Omega_e} N_b \, Q \, \mathrm{d}\Omega + \int_{\Gamma_{eq}} N_b \, \bar{q}_n \, \mathrm{d}\Gamma \tag{1.60}$$

To impose the Dirichlet boundary condition we set

$$\tilde{\phi}^b = \bar{\phi}^b \text{ at each boundary node} \tag{1.61}$$

and replace $\tilde{\phi}^a$ by given $\bar{\phi}^a$ for the r boundary nodes.

It is evident in this example that the Galerkin method results in a symmetric set of algebraic equations (e.g., $K_{ba} = K_{ab}$). However this only happens if the differential equations are *self-adjoint* (see Appendix A). Indeed the existence of symmetry provides a test for self-adjointness and for steady-state problems existence of a variational principle whose stationarity is sought [1].

It is necessary to remark here that if we were considering a pure convection equation

$$u_i \frac{\partial \phi}{\partial x_i} + Q = 0 \tag{1.62}$$

with u_i being the velocity components, symmetry would not exist and the equations can often become unstable if the Galerkin method is used. We will discuss this matter further in the next chapter.

Example 1.1. Shape functions for triangle with three nodes

A typical finite element with a triangular shape is defined by the local nodes 1, 2, 3 and straight line boundaries between nodes as shown in Fig. 1.3a and will yield the shape of N_a of the form shown in Fig. 1.3b. Writing a scalar variable as

$$\phi = \alpha_1 + \alpha_2 \, x_1 + \alpha_3 \, x_2 \tag{1.63}$$

we may evaluate the three constants by solving a set of three simultaneous equations which arise if the nodal coordinates are inserted and the scalar variable equated to the appropriate nodal values. For example, nodal values may be written as

$$\begin{aligned}\tilde{\phi}^1 &= \alpha_1 + \alpha_2 x_1^1 + \alpha_3 x_2^1 \\ \tilde{\phi}^2 &= \alpha_1 + \alpha_2 x_1^2 + \alpha_3 x_2^2 \\ \tilde{\phi}^3 &= \alpha_1 + \alpha_2 x_1^3 + \alpha_3 x_2^3\end{aligned} \tag{1.64}$$

We can easily solve for α_1, α_2, and α_3 in terms of the nodal values $\tilde{\phi}^1$, $\tilde{\phi}^2$, and $\tilde{\phi}^3$ and substituting into Eq. 1.63 gives

$$\phi = \frac{1}{2\Delta}\left[\left(a_1 + b_1 x_1 + c_1 x_2\right)\tilde{\phi}^1 + \left(a_2 + b_2 x_1 + c_2 x_2\right)\tilde{\phi}^2 + \left(a_3 + b_3 x_1 + c_3 x_2\right)\tilde{\phi}^3\right] \tag{1.65}$$

in which

$$\begin{aligned}a_1 &= x_1^2 x_2^3 - x_1^3 x_2^2 \\ b_1 &= x_2^2 - x_2^3 \\ c_1 &= x_1^3 - x_1^2\end{aligned} \tag{1.66}$$

where x_i^a is the i-direction coordinate of node a and other coefficients are obtained by cyclic permutation of the subscripts in the order 1, 2, 3, and

$$2\Delta = \det \begin{vmatrix} 1 & x_1^1 & x_2^1 \\ 1 & x_1^2 & x_2^2 \\ 1 & x_1^3 & x_2^3 \end{vmatrix} = 2 \cdot (\text{area of triangle } 123) \tag{1.67}$$

From Eq. (1.65) we see that the shape functions are given by

$$N_a = (a_a + b_a x_1 + c_a x_2)/(2\Delta), \quad a = 1, 2, 3 \tag{1.68}$$

Since the unknown nodal quantities defined by these shape functions vary linearly along any side of a triangle the interpolation equation (1.65) guarantees continuity between adjacent elements and, with identical nodal values imposed, the same scalar variable value will clearly exist along an interface between elements. We note, however, that in general the derivatives will not be continuous between element assemblies [1].

Example 1.2. Poisson equation in two dimensions: Galerkin formulation with triangular elements

The relations for a Galerkin finite element solution have been given in (1.58)–(1.60). The components of K_{ba} and f_b can be evaluated for a typical element or subdomain and the system of equations built by standard methods.

For instance, considering the set of nodes and elements shown shaded in Fig. 1.4a, to compute the equation for node 1 in the assembled patch, it is only necessary to

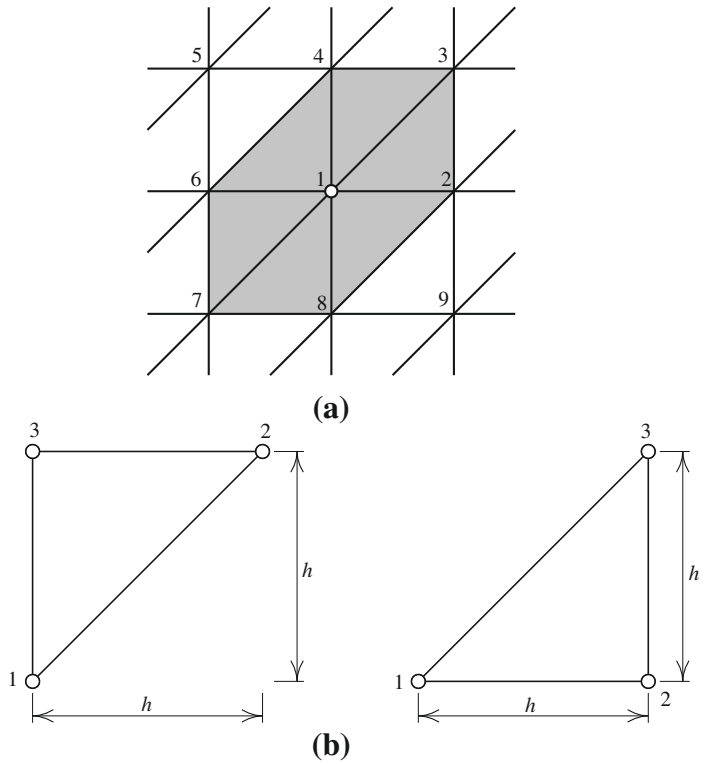

FIGURE 1.4

Linear triangular elements for Poisson equation example: (a) "connected" equations for node 1; (b) Type 1 and Type 2 element shapes in mesh.

compute the K^e_{ba} for two element shapes as indicated in Fig. 1.4b. For the Type 1 element (left element in Fig. 1.4b) the shape functions evaluated from (1.68) using (1.66) and (1.67) and placing the origin of coordinates at node 1 give

$$N_1 = 1 - \frac{x_2}{h}, \quad N_2 = \frac{x_1}{h}, \quad N_3 = \frac{x_2 - x_1}{h}$$

Thus, the derivatives are given by

$$\frac{\partial \mathbf{N}}{\partial x_1} = \begin{Bmatrix} \dfrac{\partial N_1}{\partial x_1} \\ \dfrac{\partial N_2}{\partial x_1} \\ \dfrac{\partial N_3}{\partial x_1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dfrac{1}{h} \\ -\dfrac{1}{h} \end{Bmatrix} \quad \text{and} \quad \frac{\partial \mathbf{N}}{\partial x_2} = \begin{Bmatrix} \dfrac{\partial N_1}{\partial x_2} \\ \dfrac{\partial N_2}{\partial x_2} \\ \dfrac{\partial N_3}{\partial x_2} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{1}{h} \\ 0 \\ \dfrac{1}{h} \end{Bmatrix}$$

Similarly, for the Type 2 element the shape functions are expressed by

$$N_1 = 1 - \frac{x_1}{h}, \quad N_2 = \frac{x_1 - x_2}{h}, \quad N_3 = \frac{x_2}{h}$$

and their derivatives by

$$\frac{\partial \mathbf{N}}{\partial x_1} = \begin{Bmatrix} \dfrac{\partial N_1}{\partial x_1} \\ \dfrac{\partial N_2}{\partial x_1} \\ \dfrac{\partial N_3}{\partial x_1} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{1}{h} \\ \dfrac{1}{h} \\ 0 \end{Bmatrix} \quad \text{and} \quad \frac{\partial \mathbf{N}}{\partial x_2} = \begin{Bmatrix} \dfrac{\partial N_1}{\partial x_2} \\ \dfrac{\partial N_2}{\partial x_2} \\ \dfrac{\partial N_3}{\partial x_2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\dfrac{1}{h} \\ \dfrac{1}{h} \end{Bmatrix}$$

Evaluation of the matrix K^e_{ba} and f^e_b for Type 1 and Type 2 elements gives (refer to Appendix F for integration formulae)

$$\mathbf{K}^e\tilde{\boldsymbol{\phi}}^e = \frac{1}{2}k\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{Bmatrix} \tilde{\phi}^{1e} \\ \tilde{\phi}^{2e} \\ \tilde{\phi}^{3e} \end{Bmatrix} \quad \text{and} \quad \mathbf{K}^e\tilde{\boldsymbol{\phi}}^e = \frac{1}{2}k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \tilde{\phi}^{1e} \\ \tilde{\phi}^{2e} \\ \tilde{\phi}^{3e} \end{Bmatrix}$$

respectively. The force vector for a constant Q over each element is given by

$$\mathbf{f}^e = \frac{1}{6}Q\,h^2\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

for both types of elements. Assembling the patch of elements shown in Fig. 1.4a gives the equation with nonzero coefficients for node 1 as (refer to [1] and [18] for assembly procedure)

$$k\begin{bmatrix} 4 & -1 & -1 & -1 & -1 \end{bmatrix}\begin{Bmatrix} \tilde{\phi}^1 \\ \tilde{\phi}^2 \\ \tilde{\phi}^4 \\ \tilde{\phi}^6 \\ \tilde{\phi}^8 \end{Bmatrix} + Q\,h^2 = 0$$

Using a central finite difference approximation directly in the differential equation (1.46) gives the approximation

$$\frac{k}{h^2}\begin{bmatrix} 4 & -1 & -1 & -1 & -1 \end{bmatrix}\begin{Bmatrix} \tilde{\phi}^1 \\ \tilde{\phi}^2 \\ \tilde{\phi}^4 \\ \tilde{\phi}^6 \\ \tilde{\phi}^8 \end{Bmatrix} + Q = 0$$

and we note that the assembled node is identical to the finite difference approximation though presented slightly differently. If all the boundary conditions are forced (i.e., $\phi = \bar{\phi}$) no differences arise between a finite element and a finite difference solution for the regular mesh assumed. However, if any boundary conditions are of natural type or the mesh is irregular, differences will arise, with the finite element solution

generally giving superior answers. Indeed, no restrictions on shape of elements or assembly type are imposed by the finite element approach.

Example 1.3. Local and global conservation of the finite element method

As mentioned before the finite element method can be locally and globally conservative. The local conservation is achieved by considering a patch of elements surrounding a node a as shown in Fig. 1.2. Rewriting a conservation equation in terms of flux gives

$$-\frac{\partial \mathbf{F_i}}{\partial x_i} + Q = 0 \tag{1.69}$$

Here $\mathbf{F_i}$ contains all fluxes including the diffusive and convective fluxes normally encountered in fluid dynamics. Subjecting the above to the Galerkin finite element process explained in the previous section results in

$$\int_\Omega \frac{\partial N_b}{\partial x_i}\hat{\mathbf{F}}_\mathbf{i}\mathrm{d}\Omega + \int_\Omega N_b Q\mathrm{d}\Omega - \int_{\Gamma_q} N_b\hat{\mathbf{F}}_\mathbf{i}n_i\mathrm{d}\Gamma = 0 \tag{1.70}$$

where $\hat{\mathbf{F}}_\mathbf{i}$ is the approximation for the flux. Since the sum of the trial function derivatives for an inside node surrounded by an element patch is zero, the equation for an inside patch becomes

$$\int_{\Omega^p} N_b\, Q\mathrm{d}\Omega - \int_{\Gamma_q^p} N_b\hat{\mathbf{F}}_\mathbf{i}n_i\,\mathrm{d}\Gamma = 0 \tag{1.71}$$

where Ω^p and Γ_q^p respectively indicate the volume and surface of an inner patch.

The above equation clearly shows that net production (source) is equal to net flux leaving the patch, demonstrating the local conservation of quantities. Now, assuming this is true for all inner patches and assuming a uniform source, the above equation is valid for the entire domain of the problem as long as Neumann boundaries surround the domain. If Dirichlet boundary conditions are applied, the global conservation is not directly demonstrable as these boundary conditions require the weighting functions to vanish on the boundaries. To make the finite element globally conservative, an extra flux boundary condition may be applied along the boundaries.

Example 1.4. In Fig. 1.5 an example of a typical potential solution as described in Section 1.3 is given. Here we show the finite element mesh and streamlines for a domain of flow around a symmetric aerofoil.

Example 1.5. Some problems of specific interest are those of flow with a free surface [19–21]. Here the governing Laplace equation for the potential remains identical, but the free surface position has to be found iteratively. In Fig. 1.6 an example of such a free surface flow solution is given [20].

For gravity forces given by (1.38) the free surface condition in two dimensions (x_1, x_2) requires

$$\tfrac{1}{2}(u_1^2 + u_3^2) - gx_2 = 0$$

Solution of such conditions involves an iterative, nonlinear algorithm, as illustrated by examples of overflows in Ref. [19].

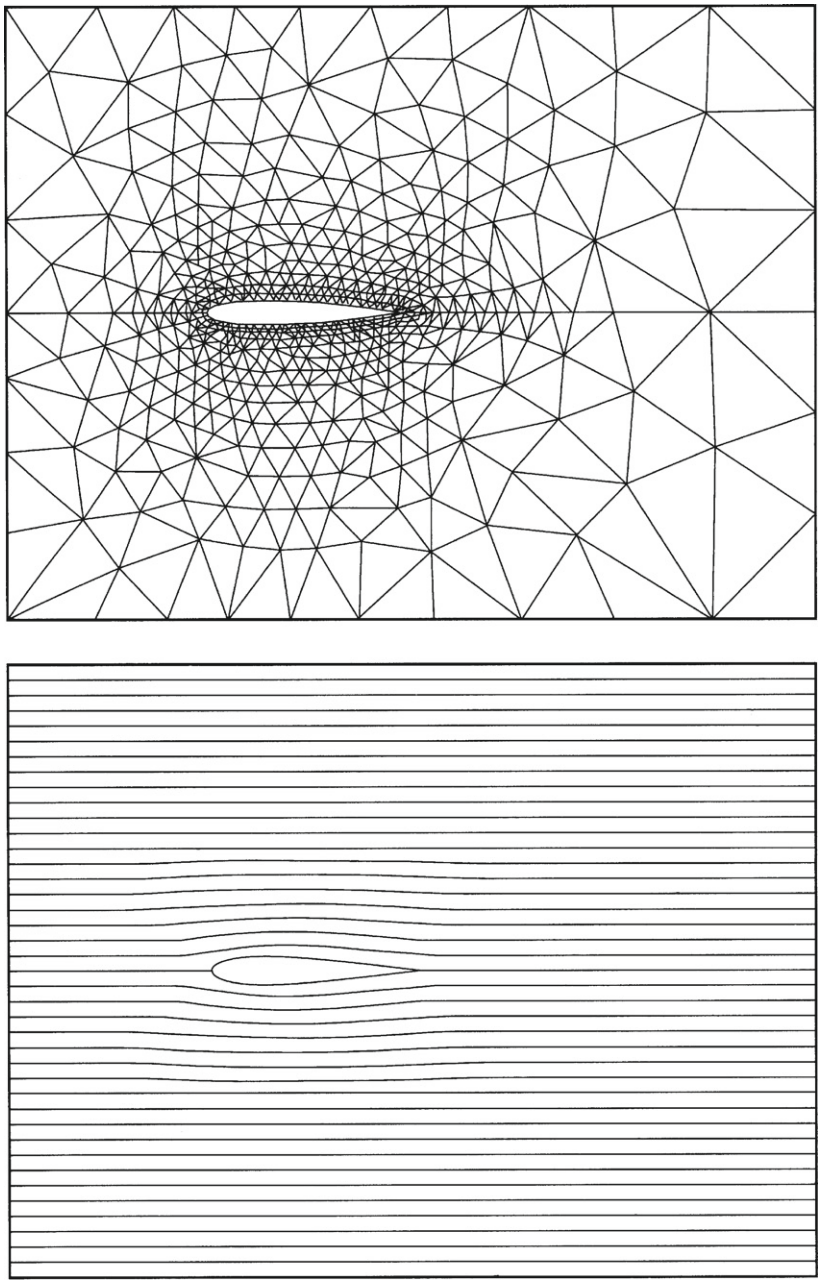

FIGURE 1.5
Potential flow solution around an aerofoil. Mesh and streamline plots.

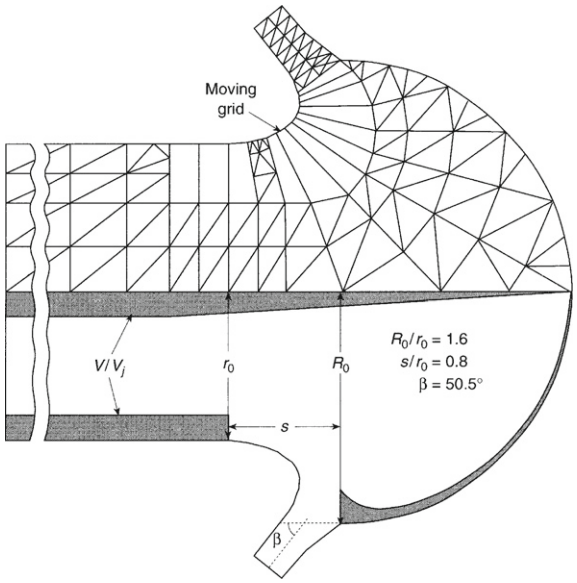

FIGURE 1.6
Free surface potential flow, illustrating an axisymmetric jet impinging on a hemispherical thrust reverser (from Sarpkaya and Hiriart [20]).

1.5.4 A finite volume approximation

Many choices of basis and weight functions are available. A large number of procedures are discussed in Ref. [1]. An approximation which is frequently used in fluid dynamics is the *finite volume* process which many consider to be a generalized finite difference form. Here the weighting function is often taken as unity over a specified subdomain Ω_b and two variants are used: (a) an element (cell) centered approach and (b) a node (vertex) centered approach. Here we will consider only a node centered approach with basis functions as given in Eq. (1.51) for each triangular subdomain and the specified integration cell (dual cell) for each node as shown in Fig. 1.7. For a solution of the Poisson equation discussed above, integration by parts of (1.56) for a unit W_b gives

$$\int_{\Omega_b} Q \mathrm{d}\Omega - \int_{\Gamma_b} n_i \frac{\partial \phi}{\partial x_i} \mathrm{d}\Gamma = 0 \tag{1.72}$$

for each subdomain Ω_b with boundary Γ_b. In this form the integral of the first term gives

$$\int_{\Omega_b} Q \mathrm{d}\Omega = Q\, \Omega_b \tag{1.73}$$

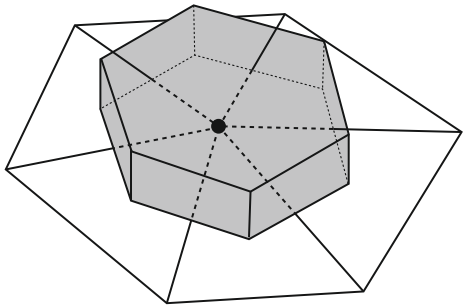

FIGURE 1.7
Finite volume weighting. Vertex centered method.

when Q is constant in the domain. Introduction of the basis functions into the second term gives

$$\int_{\Gamma_b} n_i \frac{\partial \phi}{\partial x_i}\, \mathrm{d}\Gamma \approx \int_{\Gamma_b} n_i \frac{\partial \hat{\phi}}{\partial x_i}\, \mathrm{d}\Gamma = \int_{\Gamma_b} n_i \frac{\partial N_a}{\partial x_i}\, \mathrm{d}\Gamma\, \tilde{\phi}^a \tag{1.74}$$

requiring now only boundary integrals of the shape functions. In order to make the process clearer we again consider the case for the patch of elements shown in Fig. 1.4a.

Example 1.6. Poisson equation in two dimensions: Finite volume formulation with triangular elements

The subdomain for the determination of the equation for node 1 using the finite volume method is shown in Fig. 1.8a. The shape functions and their derivatives for the Type 1 and Type 2 elements shown in Fig. 1.8b are given in Example 1.1. We note especially that the derivatives of the shape functions in each element type are constant. Thus, the boundary integral terms in Eq. (1.74) become

$$\int_{\Gamma_b} n_i \frac{\partial N_a}{\partial x_i} \mathrm{d}\Gamma = \sum_e \int_{\Gamma_e} n_i^e \,\mathrm{d}\Gamma \frac{\partial N_a^e}{\partial x_i}$$

where e denotes the elements surrounding node a. The integrals on Γ_e may be evaluated by computing I_1, I_2 and I_3 as shown for Type 1 and Type 2 elements. It is simple to show for the x_1 derivative we obtain

$$I_1^1 = \int_{\Gamma_e} n_1^e \,\mathrm{d}\Gamma = \int_4^c n_1^e \,\mathrm{d}\Gamma + \int_c^6 n_1^e \,\mathrm{d}\Gamma = x_2^6 - x_2^4$$

where x_2^4, x_2^6 are mid-edge coordinates of the triangle as shown in Fig. 1.8b. Similarly, for the x_2 derivative we obtain

$$I_1^2 = \int_{\Gamma_e} n_2^e \,\mathrm{d}\Gamma = \int_4^c n_2^e \,\mathrm{d}\Gamma + \int_c^6 n_2^e \,\mathrm{d}\Gamma = x_1^4 - x_1^6$$

Thus, for the integral I_1 we obtain

$$I_1 = \int_{\Gamma_e} n_i \frac{\partial N_a}{\partial x_i} \mathrm{d}\Gamma = \frac{\partial N_a}{\partial x_1}(x_2^6 - x_2^4) + \frac{\partial N_a}{\partial x_2}(x_1^4 - x_1^6)$$

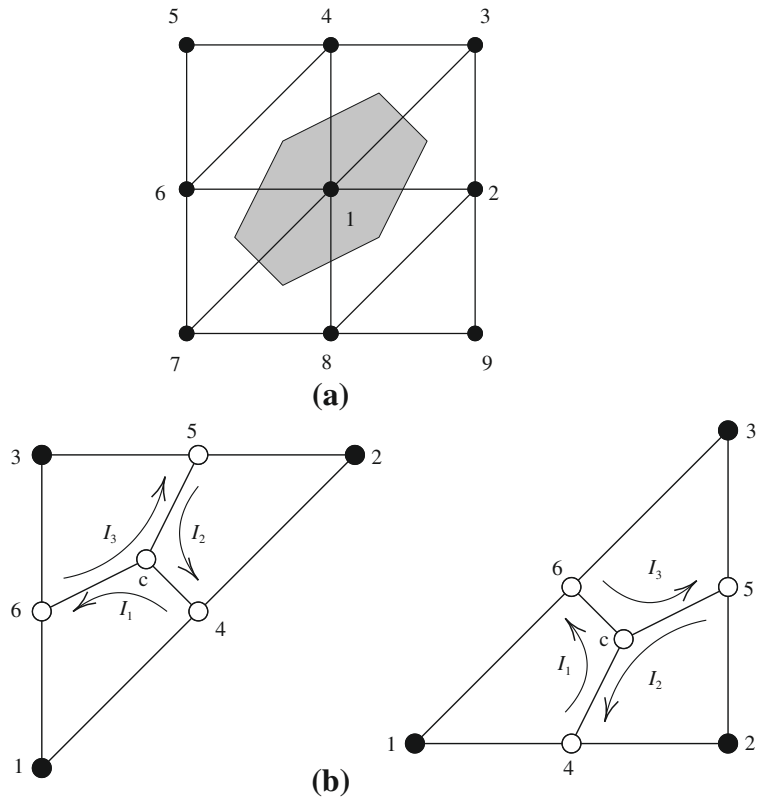

FIGURE 1.8

Finite volume domain and integrations for vertex centered method: "connected" equations for node 1; (b) Type 1 and Type 2 element boundary integrals.

The results for I_2 and I_3 are likewise obtained as

$$I_2 = \frac{\partial N_a}{\partial x_1}(x_2^4 - x_2^5) + \frac{\partial N_a}{\partial x_2}(x_1^5 - x_1^4)$$

$$I_3 = \frac{\partial N_a}{\partial x_1}(x_2^5 - x_2^6) + \frac{\partial N_a}{\partial x_2}(x_1^6 - x_1^5)$$

Using the above we may write the finite volume result for the subdomain shown in Fig. 1.8a as

$$k\begin{bmatrix}4 & -1 & -1 & -1 & -1\end{bmatrix}\begin{Bmatrix}\tilde{\phi}^1\\ \tilde{\phi}^2\\ \tilde{\phi}^4\\ \tilde{\phi}^6\\ \tilde{\phi}^8\end{Bmatrix} + Q\,h^2 = 0$$

We note that for the regular mesh the result is identical to that obtained using the standard Galerkin approximation. This identity does not generally hold when irregular meshes are considered and we find that the result from the finite volume approach applied to the Poisson equation will not yield a symmetric coefficient matrix [due to the fact that first derivative of shape functions appears in Eq. (1.72)]. As we know, the Galerkin method is optimal in terms of energy error and, thus, has more desirable properties than either the finite difference or the finite volume approaches.

Using the integrals defined on "elements," as shown in Fig. 1.8b, it is possible to implement the finite volume method directly in a standard finite element program. The assembled matrix is computed elementwise by assembly for each node on an element. The unit weight will be "discontinuous" in each element, but otherwise all steps are standard.

1.6 Concluding remarks

We have observed in this chapter that a full set of Navier-Stokes equations can be written incorporating both compressible and incompressible behavior. At this stage it is worth remarking that

1. More specialized sets of equations such as those which govern shallow-water flow or surface wave behavior (Chapters 10–12) will be of similar forms and need not be repeated here.
2. The essential difference from solid mechanics equations involves the non-self-adjoint convective terms.
3. For simplified problems on structured meshes, the finite element, finite volume, and finite difference methods give identical solutions.

Before proceeding with discretization and indeed the finite element solution of the full fluid equations, it is important to discuss in more detail the finite element procedures which are necessary to deal with such convective transport terms.

We shall do this in the next chapter where a standard scalar convective-diffusive-reactive equation is discussed.

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