Solutions 2 Game Theory MA300 / 301 / 402

Solution to Exercise 1.4

(a) The completed table for columns 5 and 6 is

	0	1	2	3	4	5	6
0	$*0$	$*1$	$*2$	$*3$	$*4$	$*5$	$*6$
1	$*1$	$*2$	$*0$	$*4$	$*5$	$*3$	$*7$
2	$*2$	$*0$	$*1$	$*5$	$*3$	$*4$	$*8$
3	$*3$	$*4$	$*5$	$*6$	$*2$	$*0$	$*1$

(b) The queen is on a square with nim value 2, so one winning move is to reduce the nim heap to size 2 to make this a losing position. However, there are also two squares with nim value 4 that the queen can reach, on row 3 column 1 and on row 0 column 4. These are the only such squares. Moving the queen to either of these also gives a losing position because it is equivalent to $*4 + *4$. Any other move would produce a sum of two different nim heaps, which is not a losing position, so there are no other winning moves.

Solution to Exercise 1.5

(a) Putting the domino anywhere on the board produces two independent boards of size $1 \times k$ and $1 \times (n-k-2)$. The two board lengths add up to $n-2$ because two squares are taken away by the domino. The resulting position is a sum of two games because the player can only move in one of them, which is equivalent to the sum of nim heaps $*D_k + *D_{n-k-2}$. Hence,

$$D_n = \text{mex}(\{D_k \oplus D_{n-k-2} \mid 0 \le k \le n-2\}).$$

(b) We use the result in (a). By symmetry, we only have to consider $k \le n/2 - 1$. With the simpler notation $\text{mex}(\ldots)$ instead of $\text{mex}(\{\ldots\})$,

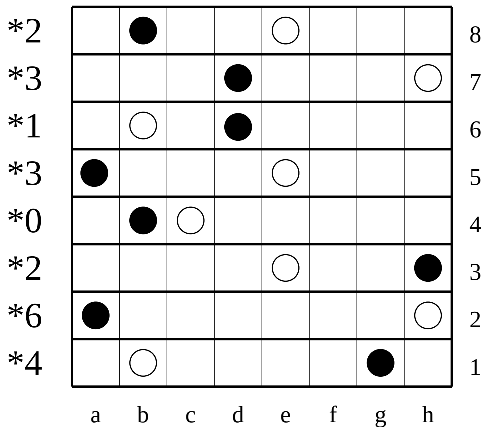
$$
\begin{aligned}
D_0 &= D_1 = 0,\\
D_2 &= \text{mex}(D_0 \oplus D_0) = \text{mex}(0) = 1,\\
D_3 &= \text{mex}(D_0 \oplus D_1) = \text{mex}(0) = 1,\\
D_4 &= \text{mex}(D_0 \oplus D_2, D_1 \oplus D_1) = \text{mex}(1,0) = 2,\\
D_5 &= \text{mex}(D_0 \oplus D_3, D_1 \oplus D_2) = \text{mex}(1,1) = 0,\\
D_6 &= \text{mex}(D_0 \oplus D_4, D_1 \oplus D_3, D_2 \oplus D_2) = \text{mex}(2,1,0) = 3,\\
D_7 &= \text{mex}(D_0 \oplus D_5, D_1 \oplus D_4, D_2 \oplus D_3) = \text{mex}(0,2,0) = 1,\\
D_8 &= \text{mex}(D_0 \oplus D_6, D_1 \oplus D_5, D_2 \oplus D_4, D_3 \oplus D_3) = \text{mex}(3,0,3,0) = 1,\\
D_9 &= \text{mex}(D_0 \oplus D_7, D_1 \oplus D_6, D_2 \oplus D_5, D_3 \oplus D_4) = \text{mex}(1,3,1,3) = 0,\\
D_{10} &= \text{mex}(D_0 \oplus D_8, D_1 \oplus D_7, D_2 \oplus D_6, D_3 \oplus D_5, D_4 \oplus D_4) = \text{mex}(1,1,2,1,0) = 3.
\end{aligned}
$$

For your possible interest, the sequence $D_0, D_1, \ldots$ is $0,0,1,1,2,0,3,1,1,0,3,3,2,2,4,0,5,2,2,3,3,0,1,1,3,0,\ldots$ and eventually repeats with period 34, the highest occurring value being 9 (which requires that 8 occurs – explain! But no 6 occurs – how can that be?). Recognised from those n where $D_n = 0$, dominos on a $1 \times n$ board is a losing game for $n = 0,1,5,9,15$, and others.

Solution to Exercise 1.6

(a) Here, black will win, by simply closing the gap to the white counter whenever white makes a move. So black always has a move left and wins.

(b) The close relationship, hopefully revealed by (a), is to poker nim, where a move that widens the gap between the two counters amounts to adding chips to the heap, whereas narrowing the gap is to reduce the heap size. That is, the gap between the two counters is the size of the nim heap corresponding to that row. The equivalent nim heaps are shown on the left:



As an experienced nim player, you may spot that the top three rows $8,7,6$, equivalent to $*2$, $*3$, $*1$, sum to a losing position, and so do the bottom three rows $3,2,1$ that are equivalent to $*2$, $*6$, $*4$, and of course row 4 which is equivalent to $*0$. So one winning move is to turn any of the rows equivalent to $*3$ to $*0$, like by moving e5 to b5, or h7 to e7. Alternatively, you may spot that the equal-sized nim heaps in rows 8 and 3 (equivalent to $*2$) and in rows 7 and 5 (equivalent to $*3$) cancel out, leaving you with rows 6, 2, and 1 and equivalent nim heaps $*1$, $*6$, $*4$. Another winning move is then to reduce $*6$ to $*5$ because $*1 + *5 + *4$ is losing (see exercise 1.1(a)), with the counter moved from h2 to g2.

This game is known as *Northcott's game*. You can play this game interactively on the internet, see `http://www.cut-the-knot.org/recurrence/Northcott.shtml`, which also gives the explanation in terms of nim heaps. Now you should win whenever you can.