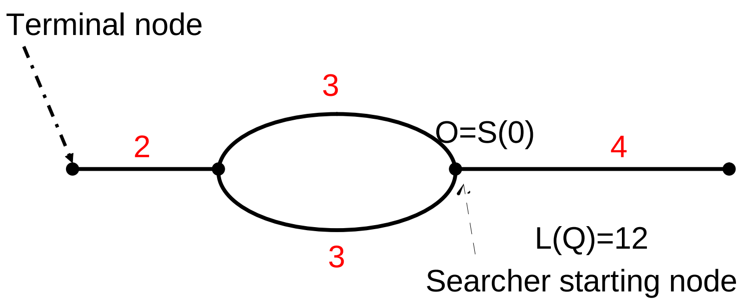
Search for Immobile Hider on a Network

Every edge e of Q has a length $L(e)$ and the total length is denoted by $L(Q) = \mu$.
I will use 'vertex' and 'node' interchangeably.



The distance function d on Q is the 'shortest path' distance.

The Game G=G(Q,O)

The Hider simply picks (as a pure strategy) any point H in Q (not necessarily a node). The $\mathcal{H} = Q$. A mixed strategy h is a probability distribution on Q, for $A \subset Q$ we write $h(A)$ as the probability that H is in A. Thus $h(Q) = 1$.

The Searcher picks a unit speed path $S = S(t)$ which covers Q. A mixed strategy is a probability distribution s over $\mathcal{S}$ (such paths).

The payoff is the time

$$T = T(S, H) = \min\{t : S(t) = H\}.$$

Note that T is only *lower semicontinuous*, but the game has a Value and an optimal mixed searcher strategy and an $\varepsilon-$optimal mixed Hider strategy anyway. We shall not worry about these matters, as we don't use this theorem. [Appendix A].

Calculating Expected Meeting Time

Suppose we fix search space the Q, and Searcher and Hider strategies (at least one of them mixed). Let $F(t)$ denote the probability that $T \leq t$. Then the expected value of T, denoted $E(T)$ can be calculated in either of the following ways (the first is the definition) [Prop. 2.4]:

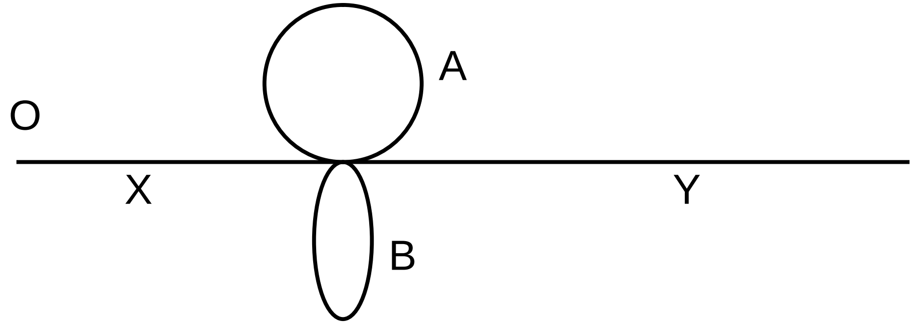
$$E(T) = \int_0^\infty t \; dF(t) = \int_0^\infty \mathsf{Pr}[T > t] \; d\cancel{F}(t).$$

Note: for continuous distributions with $F'(t) = f(t)$, the left integral is

$$\int_0^\infty t \; f(t) \; dt.$$

Search Higher Density Region First

For fixed Q (below) and Hider distribution h,
which has lower expected time E,
X,A,B,Y or X,B,A,Y?



Suppose density B > density A, where density(C) is
h(C)/(time in C), recall h(C) is probability Hider in C
(recall answer from ‘alternating search’)

Also recall that constant density searches can be carried out without interruption.

Search Higher Density Region First

Theorem: Fix network and Hider distribution. Suppose S (with T distribution F) searches disjoint regions A and B in time intervals $[a, b]$ and $[b, c]$, while S' search in other order.

$$\text{If } \frac{F(c) - F(b)}{c - b} \geq \frac{F(b) - F(a)}{b - a}, \text{ then } E(S) \geq E\left(S'\right).$$

Proof: $E(S) - E(S')$ is

$$\begin{aligned}
& \left[\int_a^b t\ dF(t) + \int_b^c t\ dF(t)\right] - \\
& \left[\int_a^b (t + (c-b))\ \ dF(t) + \int_b^c (t - (b-a))\ \ dF(t)\right] \\
= \quad & -\int_a^b (c-b)\, dF(t) + \int_b^c (b-a)\ \ dF(t) \\
= \quad & -(b-c)(F(b) - F(a)) + (c-b)(F(c) - F(b)) \\
= \quad & \frac{F(c) - F(b)}{c-b} - \frac{F(b) - F(a)}{b-a} \geq 0.
\end{aligned}$$

Note: For continuous distributions with $F'(t) = f(t)$, $E(S) - E(S')$ is

$$
\begin{aligned}
& \left[\int_a^b t\ f(t)\ dt + \int_b^c t\ f(t)\ dt\right] - \\
& \left[\int_a^b (t + (c-b))\ f(t)\ dt + \int_b^c (t - (b-a))\ f(t)\ dt\right] \\
= \ & -\int_a^b (c-b)\, f(t)\ dt + \int_b^c (b-a)\ f(t)\ dt \\
= \ & -(b-c)(F(b) - F(a)) + (c-b)(F(c) - F(b)) \\
= \ & \frac{F(c) - F(b)}{c-b} - \frac{F(b) - F(a)}{b-a} \leq 0.
\end{aligned}
$$

Uniform Hider Strategy

A mixed strategy always available to the Hider is the *uniform strategy* $h = u$ which hides in any interval J of any edge with probability proportional to it's length, that is $h(J) = L(J)/\mu$.

Theorem [Thm 3.3]: For any (Q, O) and any $S, T(S, u) \geq \mu/2$. Hence $V \geq \mu/2$.

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Proof: Since S has unit speed, the region $S_t = S[0, t]$ searched by time t has total length satisfying $L_t \leq t$ and hence

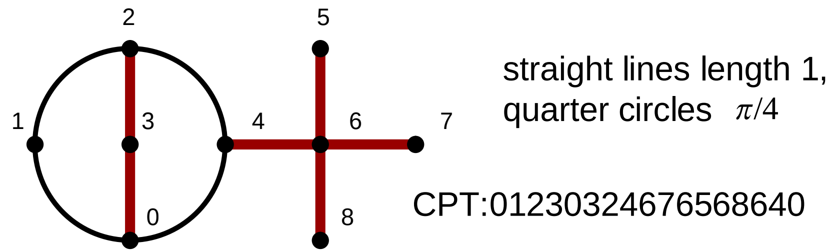
$$\Pr[T \leq t] = F(t) = u(S_t) \leq L_t/\mu \leq t/\mu. \text{ So}$$

$$\begin{aligned} T(S, u) &= \int_0^\infty (1 - F(t))\, dt \geq \int_0^\infty \max[1 - t/\mu, 0]\ dt \\ &= \int_0^\mu (1 - t/\mu)\, dt = \mu/2. \end{aligned}$$

Note: equality holds iff $S([0, \mu]) = Q$, Eulerian path from O

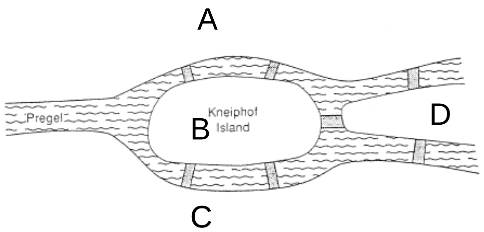
Chinese Postman Tours

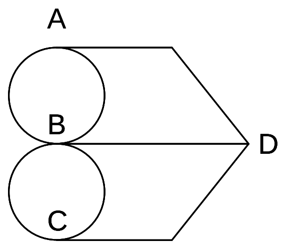
A covering path S is called a tour if it ends back at O. If it has minimum length, denoted $\bar{\mu}$, it is called a *Chinese Postman Tour*. A tour is called *Eulerian* if it has length μ (traverses each edge once). An Eulerian tour exists iff all nodes have even degree (number of incident edges), in which case Q is called *Eulerian*.

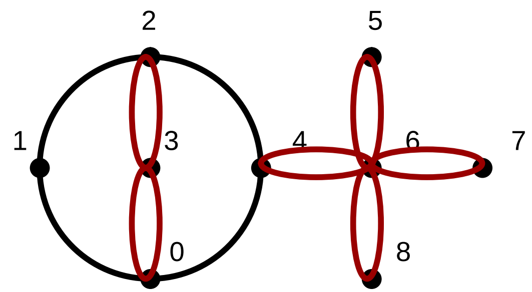


Thick red arcs Q_2 traversed twice, thin black ones Q_1 once. $\bar{\mu} = L(Q_1) + 2\,L(Q_2)$.
Q_1 is union of Eulerian tours

Eulerian Networks







Lemma [3.15]: *Any minimal tour of* Q *satisfies* $\bar{\mu} \leq 2\mu$, *with equality only for trees.*

Proof: Define Q' by doubling every edge of Q (adding another edge with same length). All nodes of Q' have even degree so there is an Eulerian tour of Q' of length 2μ, which also constitutes a covering tour of Q. If Q is not a tree it contains a circuit (closed path of distinct edges), in which case we don't have to double the edges in the circuit.

Random Chinese Postman Tours

Suppose that $S : [0, \bar{\mu}] \to Q$ is a CPT. Let S^r denote it's reverse path defined by $S^r(t) = S(\bar{\mu} - t)$. A Random Chinese Postman Tour (RCPT) $\bar{s}$ is an equiprobable mixture of S and S^r.

Lemma[3.18]: Let $\bar{s}$ be a RCPT on any network Q, O. Then for any $H \in Q$, $T(\bar{s}, H) \leq \bar{\mu}/2$. Hence $V \leq \bar{\mu}/2$.

Proof: For some least $t \leq \bar{\mu}$ we have $S(t) = H$, and so $T(S, H) = t$. Since $S^r(\bar{\mu} - t) = S(t) = H$, we have $T(S^r, H) \leq \bar{\mu} - t$. Consequently

$$T(\bar{s}, H) = \frac{1}{2}T(S, H) + \frac{1}{2}T(S^r, H) \leq \frac{1}{2}(t + (\bar{\mu} - t)) = \frac{\bar{\mu}}{2}.$$

Bounds on V=V(Q,O) for a General Network

Summarizing our previous results, we have shown.

Theorem[3.19]: For any network (Q, O), the value V of the search game for an immobile hider satisfies

$$\frac{\mu}{2} \leq v \leq \frac{\bar{\mu}}{2} \leq \mu.$$

The lower bound holds iff Q is Eulerian (has Eulerian Tour). The upper is only possible if Q is a tree.

Search Game on an Interval

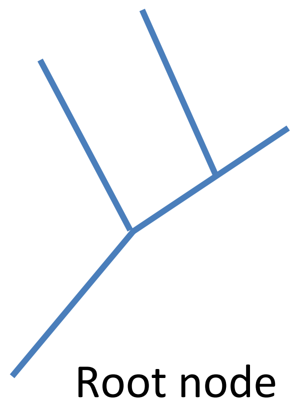
Let Q be an interval of length $\mu = a_1 + a_2$ as shown below

Only the terminal nodes v_1, v_2 are undominated by for the Hider. So assume his mixed strategy has $h\left(v_1\right) = p_1$ and $h\left(v_2\right) = p_2$. The Searcher has two 'searches' which can be done in either order, $A_1 = Ov_1O$ and $A_2 = Ov_2O$. We showed earlier that he should adopt the higher density (probability found/time) one first. That is, A_1A_2 if $p_1/\left(2a_1\right)$ gt$p_{2/}\left(2a_2\right)$. If the hider makes both densities equal (so $p_i = a_i/\left(a_1 + a_2\right)$, then A_1A_2 and A_2A_1 are equally good and the expected payoff is (using A_1A_2)

$$T = \frac{a_1}{a_1 + a_2}\left(a_1\right) + \frac{a_2}{a_1 + a_2}\left(2a_1 + a_2\right) = a_1 + a_2 = \mu = V$$

Depth-First Search on a Tree

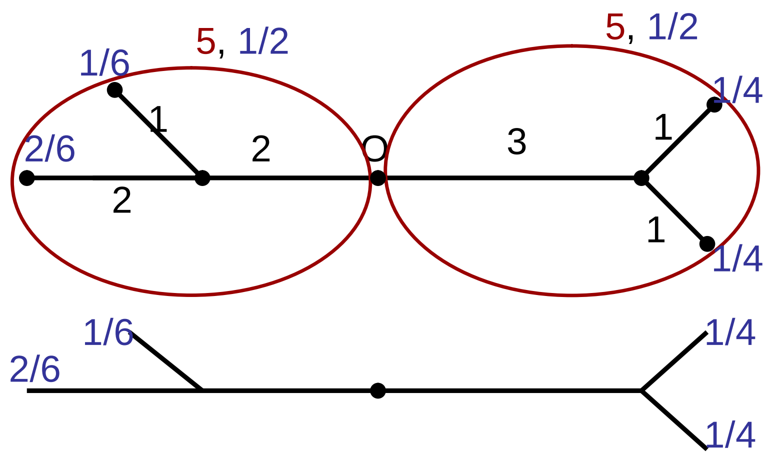
A path on a rooted tree is called **depth-first** if whenever reaching a branch node, it searches all the branches fully before going back towards the root.



A tour of a tree, starting from the root, has minimal length (CPT) if and only if it is a depth-first search path followed by the direct path back to the root. Such a tour traverses every edge exactly twice.

Equal Branch Density (EBD) Hider Distributions

So for the interval the optimal hider strategy is *concentrated on the terminal nodes and at every nonterminal node the search density (total probability/twice total length) of all branches are equal*. Such a distribution on any tree will be called an *Equal Branch Density (EBD)* distribution. The same argument used for the interval shows that an EBD distribution on a tree guarantees that $T \geq \mu$, and hence $V = \mu$.



Consecutive Search of Adjacent Terminal Nodes

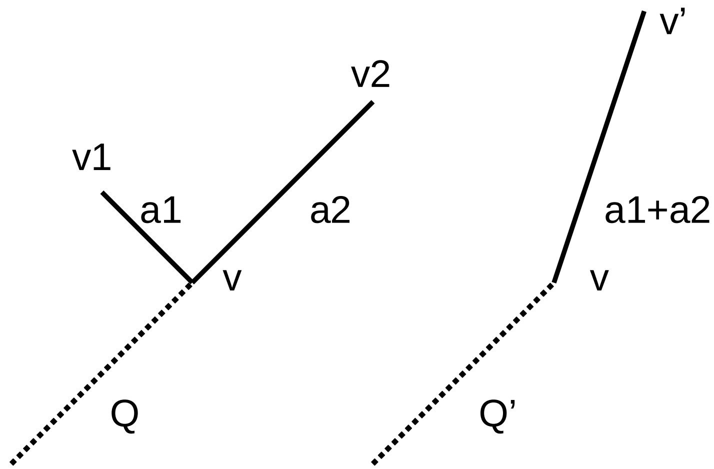
EBD Lemma: There is an optimal search strategy $S(t)$ for a tree with an EBD Hider distribution h, which searches two terminal vertices v_1 and v_2 adjacent to a common vertex v consecutively (no other terminal vertices in between).

Proof: Suppose S is optimal against h and adopts the searches $A_1 = v, v_1, v$, $A_2 = v, v_2, v$ with some search A (starting and ending at v) in between. The 'higher density first' property of optimal searches implies that $\rho(A_1) \leq \rho(A) \leq \rho(A_2)$, where ρ denotes density (probability discovered/time taken). But since h is EBD, we have $\rho(A_1) = \rho(A_2) = \rho(A)$. Consequently the search which changes the order to A_1A_2A has the same expected time, and has the required property.

EBD Theorem: If h is an EBD distribution on a tree Q, then $T(S,h) \geq \mu$ for any search strategy S. So $V \geq \mu$.

Proof (by induction on the number m of edges): If $m=1$, Q is an interval with O at one end, so it takes time μ to reach the other end. So assume the Theorem is true for trees with m edges. Let Q be a tree with $m+1$ edges, and let v_1 and v_2 be vertices at distances a_1 and a_2 from a common vertex v. Define an m edge tree Q' by replacing the edges vv_1 and vv_2 by a single edge va of length a_1+a_2. Define h' on Q' as h except that $h'(a) = h(v_1) + h(v_2)$. Since the density of the edge va is the same as that of vv_1 and vv_2, h' is also EBD. So $T(S', h') \geq \mu$ for any search S' on Q', by the induction hypothesis. But if S^* is optimal on Q, h, searching v_1, v_2 consecutively, the S' on Q' which searches vav when S^* searches vv_1vv_2v has $T_Q(S^*, h) = T_{Q'}(S', h') \geq \mu$.

Induction Step in EBD Theorem



Converting Q to Q' by adding
two adjacent terminal edges

$V = \mu = \bar{\mu}/2$ for Trees

Tree Theorem: Let Q be a tree of length μ with any starting vertex 0. Then $V = \mu$

Proof: We showed (using a random CPT for the Searcher) that for any network Q we have $V \leq \bar{\mu}/2$, where $\bar{\mu}$ is the length of a CPT. For a tree we have $\bar{\mu} = 2\mu$, so this implies $V \leq \mu$. For a tree, the EBD Theorem showed (using an EBD hider distribution) that $V \geq \mu$. Thus for a tree we have $V = \mu$.

Arc-Adding Lemma: Get Q' from a Q by adding edge e of length $l \geq 0$ between points $x, y \in Q$. Then

1. $V(Q') \leq V(Q) + 2l$, so $V(Q') \leq V(Q)$ if we identify v_1, v_2 $(l = 0)$

2. If $l \geq d_Q(v_1, v_2)$, then $V(Q') \geq V(Q)$. Any hiding strategy on Q does as well on Q'.

Arc-Adding Lemma: Get Q' from a Q by adding edge e of length $l \geq 0$ between points $x, y \in Q$. Then

1. $V(Q') \leq V(Q) + 2l$, so $V(Q') \leq V(Q)$ if we identify v_1, v_2 $(l = 0)$

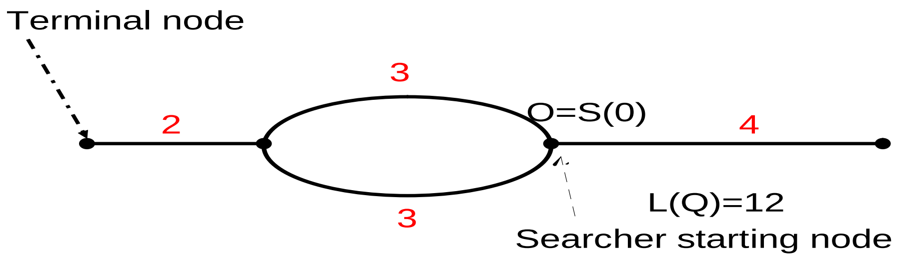
Proof. [1] Replace every pure S in an optimal strategy s by S' which follows S until it reaches x, then tours e, then follows S again. $T(s, z) \leq V(Q) + l$ for $z \in e$, $T(s, z) \leq V(Q) + 2l$ for $z \in Q$. ■

Arc-Adding Lemma: Get Q' from a Q by adding edge e of length $l \geq 0$ between points $x, y \in Q$. Then

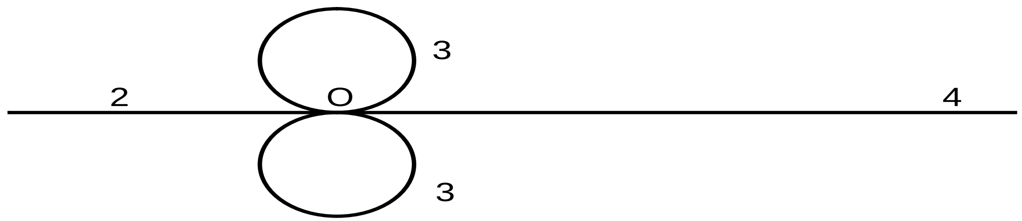
2 If $l \geq d_Q(v_1, v_2)$, then $V(Q') \geq V(Q)$. Any hiding strategy on Q does as well on Q'.

Proof. [2] Let h' on Q' be same as h (don't hide in e). Note that for $H \in Q$, $T_{Q'}(S', H) \geq T_Q(S, H)$, where S like S' but replaces e by x to y in Q. ∎

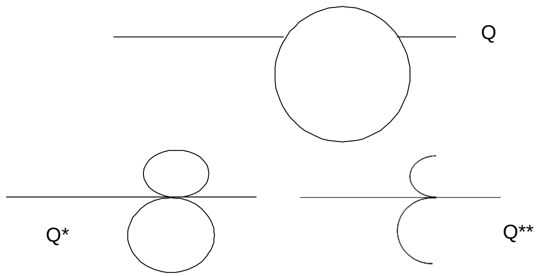
Recall Problem 1 from Homework 3



Identify the two nodes of the circuit, to produce network Q^* below, same $\bar{\mu} = 18$, so $V(Q) \geq V(Q^*) = \bar{\mu}/2 = 9$. But always $V(Q) \leq \bar{\mu}/2$, so $V(Q) = \bar{\mu}/2$.



Proposition: If Q is a weakly Eulerian network then $V = \bar{\mu}/2$.



All three networks have same $\bar{\mu}$. $V(Q^*) \leq V(Q)$ by (?).

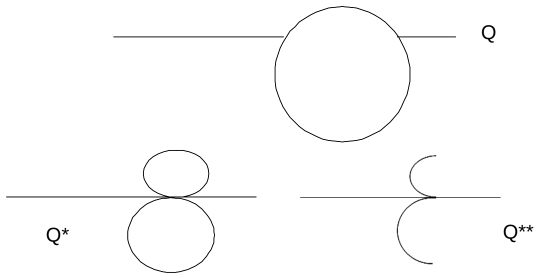
$V(Q^*) \geq V(Q^{**})$ by (?). $V(Q^{**}) = \bar{\mu}$ by (?). So

$$\bar{\mu} \leq V(Q) \leq \bar{\mu}.$$

Weakly Eulerian Networks

Definition: A network is weakly Eulerian if it contains a set of disjoint Eulerian networks such that shrinking each to a point transforms the network into a tree. (In top network on previous page, the 'shrunk' network is tree with edges of length 2 and 4.) Equivalently a network is WE if removing all disconnecting edges leaves network of all even degrees (Eulerian). (In top network of previous page, this leaves a figure eight network and two isolated nodes of degree zero.)

Proposition: If Q is a weakly Eulerian network then $V = \bar{\mu}/2$.



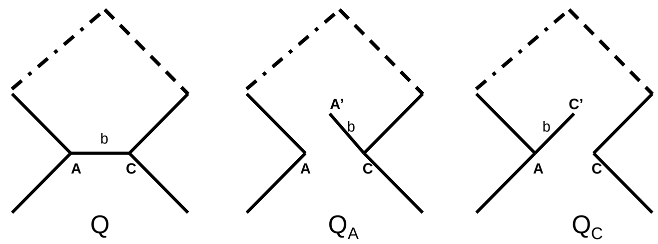
All three networks have same $\bar{\mu}$. $V(Q^*) \leq V(Q)$ by (?).

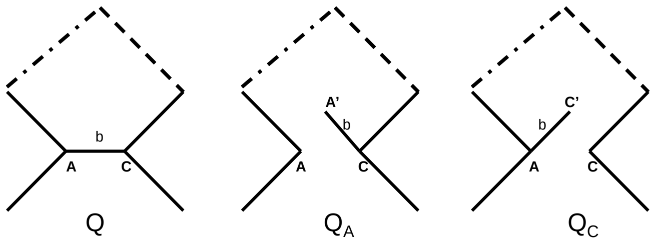
$V(Q^*) \geq V(Q^{**})$ by (?). $V(Q^{**}) = \bar{\mu}$ /2 . So

$$\bar{\mu}/2 \leq V(Q) \leq \bar{\mu}/2$$

Proposition: *If h is a mixed hiding strategy on Q with $T(S,h) = \bar{\mu}/2$ for all S, then Q is weakly Eulerian.*

Proof: Fix any CP Tour of Q and let Q_1 and Q_2 denote the arcs traversed once and twice by it. We know that $Q - Q_2$ is the union of disjoint Eulerian tours and hence that $Q - a$ is connected if a is in Q_1. So by the (alternate) definition of WE, we must show that arcs in Q_2 disconnect Q. So suppose, on the contrary, that there is an arc $b = AC$ of Q_2 such that $Q - b$ is connected. Let Q_A be the network resulting from disconnecting b from A and adding a new terminal node A', with Q_C the same for C. All three networks Q, Q_A and Q_B have the same value of $\bar{\mu} = L(Q_1) + 2L(Q_2)$.





Since the Searcher has strategies guaranteeing $T \leq \bar{\mu}/2$ for both Q_A and Q_B it must be that h is optimal for both. In $G(Q_A)$ points $b - A'$ are dominated and in $G(Q_C)$ those in $b - C'$ are dominated. So no point of b is undominated in both games, and hence h gives zero probility to hiding in b. But for such an h, a randomized CPT of $Q - b$ guarantees an expected capture time $T \leq (\bar{\mu} - 2\ L(b))/2 < \bar{\mu}/2$, contradicting the definition of h.

Gal's Theorem

Theorem [3.26]: For any network Q, if $V = \bar{\mu}/2$ then Q is weakly Eulerian.

Note that our previous result proves this for the case that the Hider has an optimal strategy. If he has only $\varepsilon-$optimal strategies, then the proof becomes more involved, but the idea is largely the same. See [p.30]. Instead of concluding that hiding in b is impossible, we can only say the hiding in b must have very low probability. In that case searching b last finds the low probability hiders in b.

Details of ε-argument: Suppose h guarantees that $T(S,h) \geq \bar{\mu}/2 - \varepsilon$. Let $p = p_\varepsilon = h(b)$ and $l = L(b)$ denote the length of b. Without loss of generality we may assume that the center of gravity of h on b is at least as close to C as to A. Let s be a RCP Tour of Q_A, so that $T(s,H) \leq \bar{\mu}/2$ for all H. For H in b we have $T(s,H) = T(s,A) - d(A,H)$ and the mean on $H \in b$ of $d(A,H)$ is at least $l/2$. Hence

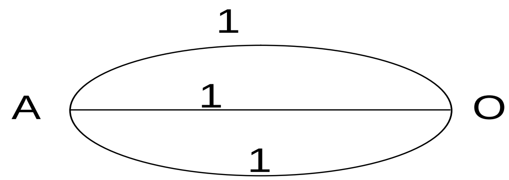
$$T(s,h) \leq (1-p)\frac{\bar{\mu}}{2} + p\left(\frac{\bar{\mu}}{2} - \frac{l}{2}\right) = \frac{\bar{\mu}}{2} - \frac{pl}{2}, \text{ so } p \leq \frac{2\varepsilon}{l}.$$

Let s' be a RCP tour of $Q - b$ followed by a path which reaches A' at some finite time M ($\leq 3\bar{\mu}$). Then

$$\begin{aligned} T\left(s', h\right) &\leq (1-p)\frac{\bar{\mu} - 2l}{2} + pM = \frac{\bar{\mu}}{2} - l + p\left(M - \frac{\bar{\mu} - 2l}{2}\right) \\ &\leq \frac{\bar{\mu}}{2} - l + p_\varepsilon\left(M - \frac{\bar{\mu} - 2l}{2}\right) \to \frac{\bar{\mu}}{2} - l \end{aligned}$$

which leads to a contradiction for ε sufficiently small.

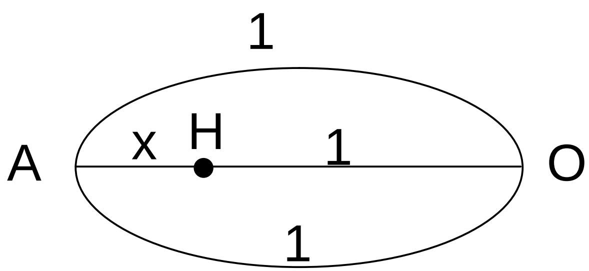
The 'Three Arc' Network

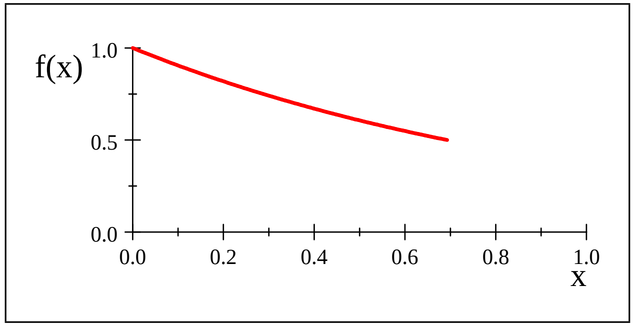


$$\mu = 3,\ \bar{\mu} = 4$$

If Searcher follows minimal search path (or tour), and $d\,(H, A) = x$, $0 < x < 1$,

$$T = \frac{2}{3}\left(\frac{1}{2}\,(1 - x) + \frac{1}{2}\,(1 + x)\right) + \frac{1}{3}\,(3 - x) = \frac{5 - x}{3} \le \frac{5}{3}.$$





Best to hide near A. Pick x (on random arc) with probability density $f(x) = e^{-x}$ $0 < x < \ln 2 \approx .693$. Searcher goes to A, back a bit on another arc, back to A, back to O, back towards A. (S. Gal, L. Pavlovic). $V = (4 + \ln 2)/3 \approx 1.56 < \bar{\mu}/2$.