Solutions 1 Game Theory MA300 / 301 / 402

Solution to Exercise 1.1

(a) In analysing the games of three nim heaps where one heap has size one, we first look at some examples, and then use *mathematical induction* to prove what we conjecture to be the losing positions. A losing position is one where every move is to a winning position, because then the opponent will win. The point of this exercise is that you formulate precisely the statement to be proved.

Consider three heaps of sizes $1,m,n$, where $n \geq m \geq 1$. We observe the following: $1,1,m$ is winning, by moving to $1,1,0$. Similarly, $1,m,m$ is winning, by moving to $0,m,m$. Next, $1,2,3$ is losing, and hence $1,2,n$ for $n \geq 4$ is winning. $1,3,n$ is winning for any $n \geq 3$ by moving to $1,3,2$. For $1,4,5$, reducing any heap produces a winning position, so this is losing.

The general pattern for the losing positions thus seems to be: $1,m,m+1$, for even numbers m. This includes also the case $m = 0$, which we can take as the base case for an induction. So we prove by induction that $1,m,m+1$, for even numbers m, are the losing positions: The move to $0,m,m+1$ produces a winning position. The move to $1,k,m+1$ for $k < m$ allows the following countermove to a losing position, by inductive hypothesis: If k is even, to $1,k,k+1$; if k is odd, to $1,k,k-1$. This covers all moves in the heap of size m. From the heap of size $m+1$, the move to $1,m,m$ is to a winning position, and to $1,m,k$ for $k < m$ as well, by inductive hypothesis: If k is even, then $k \leq m-2$ because m is even, so one can countermove to $1,k+1,k$, and if k is odd, to $1,k-1,k$. This completes the induction.

Alternatively, one can use the theorem on nim heap sizes represented as sums of powers of two: $*1 + *m + *n$ is losing if and only if, except for 2^0, the powers of two making up m and n come in pairs. So these must be the same powers of two, except for $1 = 2^0$, which occurs in only m or n, where we have assumed that n is the larger number: We have $m = 2^a + 2^b + 2^c + \cdots$ for $a > b > c > \cdots \geq 1$, so m is even, and, with the same $a, b, c, \ldots$, $n = 2^a + 2^b + 2^c + \cdots + 1 = m + 1$. Then $*1 + *m + *n \equiv *0$.

(b) We have $6 = 4+2$, $10 = 8+2$, and $15 = 8+4+2+1$. So $*6 + *10 + *15 = *(4+2) + *(8+2) + *(8+4+2+1) \equiv *4 + *2 + *8 + *2 + *8 + *4 + *2 + *1 \equiv *2 + *1 \equiv *3$ by cancelling repetitions in pairs. So this is a winning position, which we have to change to a losing position by nim-adding $*2 + *1$ to a suitable heap. Here, all three heaps have the largest nim-heap $*2$ of the sum in their representation, so all three heaps can be suitably diminished to obtain the sum $*0$ for the new three heaps. Following steps 1–3 on page 15, we compute: $*6 + (*2 + *1) \equiv *4 + *1$, so the winning move is to reduce the heap of size 6 to size 5. By moving in the heap with 10 chips: $*10 + (*2 + *1) \equiv *8 + *1$, so another winning move is to reduce the heap of size 10 to size 9. Finally, by moving in the heap with 15 chips: $*15 + (*2 + *1) \equiv *8 + *4$, so the winning move is to reduce the heap of size 15 to size 12. (In particular, not all heaps are reduced by the same amount.)

In what amounts to the same computation, we get these moves by converting the heap sizes to binary:

$$
\begin{array}{rcl}
6 & = & 0\,1\,1\,0 \\
10 & = & 1\,0\,1\,0 \\
15 & = & 1\,1\,1\,1 \\
\hline
 & & 0\,0\,1\,1
\end{array}
$$

Because all three rows have a 1 in the leftmost odd "sum" column, each row represents a heap where a winning move can be made. Hence, player I has three winning moves, namely: converting 0110 to a binary number where the binary digits in the last two columns (the 1's in 0011) are changed, giving the binary representation 0101 which is 5 in decimal. The second move is, similarly, obtained by converting 1010 to 1001, which means removing 1 chip from the heap of size 10 to give a heap of size 9. The third move is to convert 1111 to 1100, which means removing 3 chips from the heap of size 15, giving size 12.

Solution to Exercise 1.2

(a) In 3×3 dominos, there are, up to symmetry, only two moves for player I, namely placing the domino such that it occupies a corner square or such that it occupies the centre square. In either case, player II can respond by placing her domino alongside the first domino, such that a 2×2 square in one corner is occupied, leaving the L-shaped remaining 5 squares. Then no matter what player I does, II will still be able to place the last domino, and player I loses. So 3×3 dominos is a win for II.

(b) When both m and n are even, player II will win in $m \times n$ dominos by playing "copycat", using the *central symmetry* of the board. That is, whenever player I places his domino, player II responds by placing her domino on the square obtained by point-reflection on the centre of the board. Then the domino pattern of the board will have that central symmetry whenever player I makes his move, so that the response move of player II will always be possible. That is to say, the two adjacent squares in question are always empty because their symmetric counterparts have been empty when player I made his move.

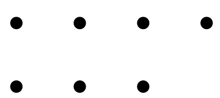
Be careful: Any "copycat" strategy requires that the underlying symmetry of the situation is preserved. This fails when using the symmetry where the board is reflected along a line: here player I could place a domino on the line itself and II would not have a counter-move. So "by copycat" is not a satisfactory answer.

(c) In $m \times n$ dominos when m is odd and n is even, player I will win by placing his domino on the centre pair of squares (in the middle row), and then playing copycat as described before for player II in (b).

Solution to Exercise 1.3

(a) Player I always wins in these situations. We describe directly the winning move in the games with two rows of dots:

For $2 \times m$ chomp, the winning move is to remove the bottom right dot $(2,m)$, leaving a pattern like the following when $m = 4$:



Afterwards, player I can always re-create this pattern by removing the dot that is diagonally adjacent to the dot that player II removed. That is, any move of player II of the form $(1,i)$ for $i > 1$ can be countered by $(2,i-1)$, and any move $(2,i)$ by $(1,i+1)$. So player I has always a move left and wins.

For square games of size $m \times m$, $m \geq 2$, the winning move is $(2,2)$. Then player I can respond to a move of type $(i,1)$ by removing the dot $(1,i)$ and vice versa, until player II is forced to take $(1,1)$ and loses.

(b) Remove the "poisoned cookie", that is, the dot on $(1,1)$. Then the last player loses by not being able to move any more, exactly when before she would have had to take the poisoned cookie.

(c) [A beautiful argument; note that in the case $m = n$, (a) provides an explicit winning move.]

If removing the bottom right dot (m,n) is a winning move, then we are done. If not, there is a counter-move (i, j) by player II that would create a losing situation for player I. But then player I could make (i, j) to start with, creating the same losing situation for player II. So this game is a win for player I, even though we don't know the winning move.