Mathmatical Game Theory:

Slides for Part I on Combinatorial Games

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These slides were created by Prof. von Stengel of the LSE for a course taught jointly with me. I thank him for letting me use them. They go with his notes, hopefully to become a book (see Game Theory Basics).

Nim – game positions and moves

Game position in Nim =

heaps (rows) of **chips** (matchsticks, cards) of certain **sizes**,
for example three heaps with sizes $\mathbf{1, 2, 3}$.

A **move** is to remove **some chips** (at least one, possibly all) from **one heap**.

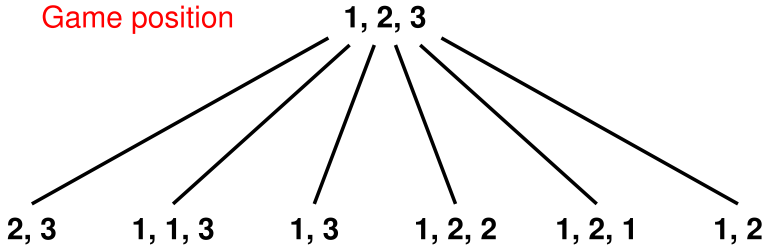
Example: The possible moves from $\mathbf{1, 2, 3}$ are to the positions

$\mathbf{0, 2, 3}$	$= \mathbf{2, 3}$	(remove 1 chip from heap of size **1**)
$\mathbf{1, 1, 3}$		(remove 1 chip from heap of size **2**)
$\mathbf{1, 0, 3}$	$= \mathbf{1, 3}$	(remove 2 chips from heap of size **2**)
$\mathbf{1, 2, 2}$		(remove 1 chip from heap of size **3**)
$\mathbf{1, 2, 1}$		(remove 2 chips from heap of size **3**)
$\mathbf{1, 2, 0}$	$= \mathbf{1, 2}$	(remove 3 chips from heap of size **3**)

Options

The reachable positions from a game position are called its **options**.

In a **game tree** (drawn top-down) shown like this:



its six options (next game positions)

Nim – winning rule

2 players which **alternate** in moving

(player I moves from the starting position, then player II, then again player I, then player II, and so on)

Normal play rule: The player who makes the last move **wins.**

Misère play rule: The player who makes the last move **loses.**

(We look almost exclusively at **normal** play;
misère play is much harder to analyse for general games.)

Nim with two heaps

Consider the position **3, 3** and its options, with player I to move.

Claim:

player II will win because she can always
copy the move of player I as follows:

from **2, 3** to **2, 2** (losing position, see next slide)

from **1, 3** to **1, 1** (losing position, see next slide)

from **0, 3** to **0, 0** (definitely a losing position)

Two equal-sized heaps = losing position

Lemma 1.1

A two-heap position $\boldsymbol{n}, \boldsymbol{n}$ in Nim is **losing**.

That is, the player to move loses if the other plays optimally.

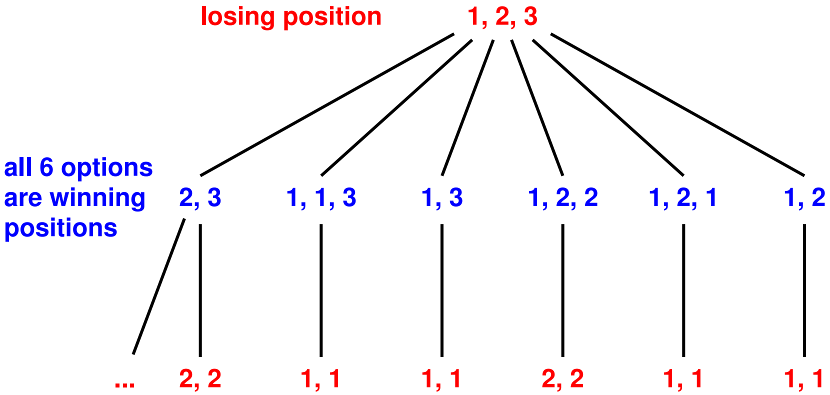
In contrast, a two-heap position $\boldsymbol{n}, \boldsymbol{m}$ with $\boldsymbol{n} < \boldsymbol{m}$ is **winning**. The **winning move** makes the two heaps equal by reducing the larger heap of size $\boldsymbol{m}$ to size $\boldsymbol{n}$, creating the losing position $\boldsymbol{n}, \boldsymbol{n}$.

Winning and losing positions

Consider 2 alternating players, where every game ends after a finite number of moves with a win for one player and a loss for the other (no “draws” due to infinite play or “ties”).

A position is **losing** if **every** move leads to a winning position (for the other player).

1, 2, 3 is a losing position



Recall: **one** winning move suffices for a position to be winning.

Optimal play

“Winning” and “losing” positions only refer to **optimal** play of the player in the winning position.

The art movie Last Year at Marienbad shows people playing Misère Nim, sometimes badly. At

`http://www.maths.lse.ac.uk/Personal/stengel/marienbad.avi`

(a 30 MB video file) we show three such scenes from the movie.

Combinatorial games

Combinatorial games, like Nim, are played as follows:

- 2 players, moving alternately throughout the game
- clear **rules** that define the possible moves for each position
- each play ends after finitely many moves with a win or loss
- normal play convention: a player unable to move loses
- perfect information
- no chance moves such as rolling dice or shuffling cards

Simpler games

Let $\boldsymbol{H}$ and $\boldsymbol{G}$ be two game positions (or just "games").

Then $\boldsymbol{H}$ is called **simpler** than $\boldsymbol{G}$, written $\boldsymbol{H} < \boldsymbol{G}$,
if $\boldsymbol{H}$ can be reached by a nonempty sequence of moves from $\boldsymbol{G}$.

In particular, the **options** of $\boldsymbol{G}$ are simpler games than $\boldsymbol{G}$.

Then $<$ defines a partial order on games: for all games $\boldsymbol{G}, \boldsymbol{H}, \boldsymbol{K}$

if $\boldsymbol{K} < \boldsymbol{H}$ and $\boldsymbol{H} < \boldsymbol{G}$ then $\boldsymbol{K} < \boldsymbol{G}$ ($<$ is transitive), and

not $\boldsymbol{G} < \boldsymbol{G}$ ($<$ is irreflexive) .

Top-down induction

Consider a set $\boldsymbol{S}$ with a partial order $<$ so that
every nonempty subset of $\boldsymbol{S}$ has at least one minimal element.

($\boldsymbol{x}$ is **minimal** in a set $\boldsymbol{T}$ if there is no $\boldsymbol{y}$ in $\boldsymbol{T}$ with $\boldsymbol{y} < \boldsymbol{x}$.)

The Top-Down Induction Principle:

Consider a property $\boldsymbol{P}(\boldsymbol{x})$ that holds for $\boldsymbol{x}$ in $\boldsymbol{S}$
if $\boldsymbol{P}(\boldsymbol{y})$ holds for all $\boldsymbol{y}$ in $\boldsymbol{S}$ with $\boldsymbol{y} < \boldsymbol{x}$.

Then $\boldsymbol{P}(\boldsymbol{x})$ holds for all $\boldsymbol{x}$ in $\boldsymbol{S}$.

Proof of the top-down induction principle

Consider a property $P(x)$ that holds for x in S
if $P(y)$ holds for all y in S with $y < x$.

Then $P(x)$ holds for all x in S.

Proof

Let $T = \{y \in S \mid P(y) \text{ is false }\}$, and suppose T is not empty.

Then T has a minimal element x.

So all y in S with $y < x$ do not belong to T, that is, $P(y)$ is true.

But then, by assumption, $P(x)$ is true.

This contradicts $x \in T$. So T is empty. $\square$

Winning strategy

A **strategy** (of a player) defines a move for every game position where that player moves.

A **winning strategy** forces a win for that player no matter what the other player does.

Lemma 1.2

In a combinatorial game, exactly one player has a winning strategy.

If that is player I, then all game positions of player I are winning and all game positions of player II are losing.

(Otherwise the same holds with the players exchanged.)

Proof by top-down induction

Claim (which implies **Lemma 1.2**)

Every position is either losing (all moves lead to a winning position) or winning (at least one move leads to a losing position).

Proof

Consider a game position G.

Suppose the claim holds for all simpler games than G, in particular the options of G.

If all options of G are winning positions, then G is losing.

Otherwise, at least one option H of G is a losing position. So G is a winning position, where moving to H is a winning move.

By top-down induction, the claim holds for all games G.

The game without moves, called $*\mathbf{0}$

Consider the game position without any options, that is, without any moves.

This is, by the normal play convention, a **losing** position.

Does it fit the description ‘all its options are winning’?

Yes, because it has no options, so all these options are winning.

The game without moves is like an empty Nim heap of size **0**.

We denote it by $*\mathbf{0}$.

(It is also the minimal game for the ‘simpler than’ order $<$.)

Impartial games

Nim is a combinatorial game, and except for the possibility of draws by an unlimited number of moves (or by stalemate), so is Chess.

Chess is vastly more complex than Nim.

Another key difference is that the two players have **different options** in each position: White can only move the white pieces, Black can only move the black pieces.

A game is **impartial** if in any game position, both players have always the **same options**. Nim is impartial.

(Games like Chess that are not impartial are called **partizan**.)

What we plan to do

In the first two weeks (= 4 lecture hours) of this course, we will develop the theory of impartial games.

In this beautiful theory, the game of Nim plays a central role. Essentially, we will show (but not in that order) that

- every impartial game is **equivalent** to some Nim heap
- in particular, every Nim position (with many heaps) is equivalent to a single Nim heap
- that position is losing if and only if the Nim heap has size zero
- the size of each Nim heap carries additional information, used when playing **sums** of games

Sums of games

Consider a Nim position with sizes $\mathbf{2, 6, 2, 6}$.

Is it winning or losing?

This can also be seen as two Nim games $\boldsymbol{G}$, each of which has 2 heaps of sizes $\mathbf{2, 6}$, and player I can move in **one** of these two games. (We will write this combined game as $\boldsymbol{G} + \boldsymbol{G}$.)

This is a **losing** position: any move of player I in one part $\boldsymbol{G}$ can be **copied** by player II in the other part.

Example:

$\mathbf{2, 6,\ 2, 6}$ player I $\rightarrow$ $\mathbf{2, 4,\ 2, 6}$ player II $\rightarrow$ $\mathbf{2, 4,\ 2, 4}$

and so player II has always a move left and will win.

Sum of games – definition

Given two combinatorial games G and H, the game **sum** $G + H$ is the game where player I moves in either G or H and leaves the position of the other game unchanged.

Formally, if

$G_1, \ldots, G_m$ are the m options of G and

$H_1, \ldots, H_n$ are the n options of H, then

the $m + n$ options of $G + H$ are

$$G_1 + H, \ldots, G_m + H, G + H_1, \ldots, G + H_n \;.$$

The copycat principle for impartial games

Lemma 1.6

$G + G$ is a losing game for any impartial game G.

Proof

By top-down induction:

If H is an option of G and player I moves to the option $G + H$ of $G + G$, then player II responds by moving to $H + H$.

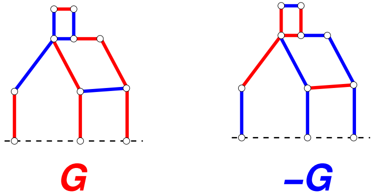
By induction, $H + H$ is losing, so player II has always a winning counter-move (the 'copycat' move), which shows that $G + G$ is losing. $\square$

Side note: copycat for partizan games

Even for combinatorial games that are not impartial, every game $\boldsymbol{G}$ has a 'negative' game $-\boldsymbol{G}$ so that $\boldsymbol{G} + (-\boldsymbol{G})$ is a losing game (for the player who starts).

Example: red-blue Hackenbush

(where player I can remove red edges and player II can remove blue edges, and all remaining edges that are no longer connected to the ground disappear)



A zero element for game-sums

Can we add a game Z to any game G so that $G + Z = G$?

Yes, $Z = *\mathbf{0}$ with the no-move game $*\mathbf{0}$ does this, because then Z has no options, so $G + Z$ is the same game as G.

Crucial observation

Let Z be a **losing** game and G be any (impartial) game.

Then $G + Z$ is essentially the **same** game as G in the sense that $G + Z$ is winning if and only if G is winning.

A more general zero for game-sums

Z losing $\Rightarrow$ $G + Z$ is winning if and only if G is winning.

Proof

If player I has a winning move in G, for example to option H, then we show that moving to $H + Z$ is a winning move in $G + Z$.

Why? For that, $H + Z$ has to be a losing position. Now, all moves in H lead to winning positions K, and all moves in Z lead to winning positions Y, say (because Z **is losing**).

So no matter whether player II moves from $H + Z$ to $K + Z$ or to $H + Y$, player I can move back to a losing position $H' + Z$ or $H + Z'$. By induction, these are losing positions, and therefore is $H + Z$ is a losing position as well. $\square$

Equivalent games

Definition 1.4

Two games G, H are called **equivalent**, written $G \equiv H$, if for any other game J

$$G + J \text{ is winning } \Leftrightarrow H + J \text{ is winning} .$$

With $J = *0$, this means G and H must be both winning or both losing.

In addition, this property is **preserved** in any game sum with another game J.

Adding a losing game gives an equivalent game

Crucial observation (again, with the same reasoning)

$\boldsymbol{Z}$ losing $\Rightarrow$ $\boldsymbol{G} + \boldsymbol{Z} \equiv \boldsymbol{G}$.

Will consider this later in more detail.

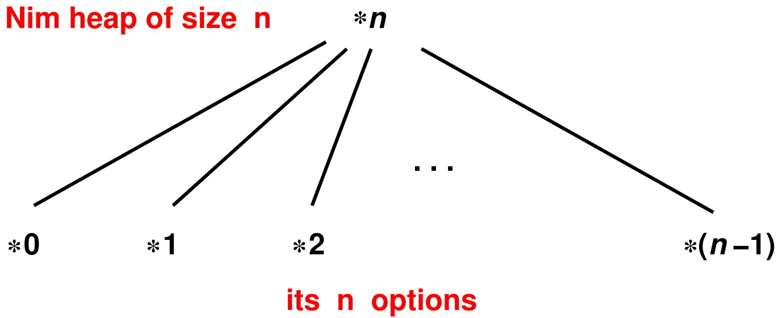
Next goal: equivalent Nim games.

Notation for Nim heaps

We have written a Nim position with three heaps as e.g. **1, 2, 3** .

This is a game-sum of three Nim heaps of sizes **1**, **2**, **3**.

Instead of $\mathbf{1+2+3}$, we write this game-sum as $*\mathbf{1} + *\mathbf{2} + *\mathbf{3}$
where $*n$ is the **game** which is the single Nim heap of size n.



Binary representation of numbers

Lemma

Every nonnegative integer n can be written uniquely as the sum of **distinct** powers of **2**:

$$n = 2^a + 2^b + 2^c + \cdots$$

where $a > b > c > \cdots \geq 0$. (This sum is empty if $n = 0$.)

Examples:

$5 = 4 + 1 = 2^2 + 2^0$

$11 = 8 + 2 + 1$

etc.

Proof of the binary representation

Want to show $n = 2^a + 2^b + 2^c + \cdots$.

If $n = 0$, we are done.

Otherwise, let a be the largest integer ($a \geq 0$) so that

$n = 2^a + q$ for some $q \geq 0$.

Then $q < n$, so we can assume by induction that there are unique exponents $b > c > \cdots \geq 0$ (possibly none if $q = 0$) with

$$q = 2^b + 2^c + \cdots$$

so if $a > b$, then the claim holds for n. We have $a > b$ because $q < 2^a$, because otherwise $q = 2^a + r$ for some $r \geq 0$ and thus

$$n = 2^a + q = 2^a + 2^a + r = 2^{a+1} + r$$

and a would not be maximal. $\square$

The main theorem for Nim heaps

Theorem 1.10

Let $n \geq 1$, and $n = 2^a + 2^b + 2^c + \cdots$, where $a > b > c > \cdots \geq 0$. Then

$$*n \equiv *(2^a) + *(2^b) + *(2^c) + \cdots .$$

Here, the left side shows a single Nim heap $*n$.

The right side shows a game-sum of several Nim heaps, and all their sizes are distinct powers of **2**.

Examples of the Theorem

$*\mathbf{5}$ is equivalent to $*\mathbf{4} + *\mathbf{1}$, that is, to the Nim position with two heaps of sizes **4, 1** .

$*\mathbf{11}$ is equivalent to $*\mathbf{8} + *\mathbf{2} + *\mathbf{1}$, that is, to the Nim position with three heaps of sizes **8, 2, 1** .

$*\mathbf{14}$ is equivalent to $*\mathbf{8} + *\mathbf{4} + *\mathbf{2}$, that is, to the Nim position with three heaps of sizes **8, 4, 2** .

Now, what about three Nim heaps of sizes **5, 11, 14** ?

This position is equivalent to

$$(*\mathbf{4} + *\mathbf{1}) \;+\; (*\mathbf{8} + *\mathbf{2} + *\mathbf{1}) \;+\; (*\mathbf{8} + *\mathbf{4} + *\mathbf{2})$$

which are eight Nim heaps, two each of sizes **8, 4, 2, 1.**

Equal powers of 2 cancel in pairs

Now the Nim position

$$*\mathbf{4} + *\mathbf{1} \ + \ *\mathbf{8} + *\mathbf{2} + *\mathbf{1} \ + \ *\mathbf{8} + *\mathbf{4} + *\mathbf{2}$$

is losing!

Why? The order of the Nim heaps does not matter, so we can write it as

$$*\mathbf{1} + *\mathbf{1} \ + \ *\mathbf{2} + *\mathbf{2} \ + \ *\mathbf{4} + *\mathbf{4} \ + \ *\mathbf{8} + *\mathbf{8}$$

but $*\mathbf{1} + *\mathbf{1}$ is a losing game and can be omitted from the game sum (remember $\boldsymbol{G} + \boldsymbol{Z} \equiv \boldsymbol{G}$ for losing games $\boldsymbol{Z}$) and so is $*\mathbf{2} + *\mathbf{2}$, etc., so this eventually reduces to $*\mathbf{0}$ which is losing.

So three Nim heaps of sizes $\mathbf{5}, \mathbf{11}, \mathbf{14}$ are a losing position:

$$*\mathbf{5} + *\mathbf{11} + *\mathbf{14} \ \equiv \ *\mathbf{0} \,.$$

Moving in winning positions

Consider three Nim heaps of sizes **5, 11, 8** (which can be reached from the losing position **5, 11, 14**, so it must be a winnning position).

It is equivalent to $*4 + *1 \; + \; *8 + *2 + *1 \; + \; *8$,

that is, to $*4 \; + \; *2$ (which simply means that the powers of two **4** and **2** occur an odd number of times). This is equivalent to a single Nim heap of size **6**, which is indeed winning.

What is the winning move from **5, 11, 8** ?

We have to make **4** and **2** appear an even number of times.

This is obtained simply by reducing **4** to **2**, that is, by taking two chips from the heap of size **5**. The resulting position is **3, 11, 8** and indeed

$$*3 + *11 + *8 \;\equiv\; *2 + *1 \; + \; *8 + *2 + *1 \; + \; *8 \;\equiv\; *0\,.$$

Plan

- Properties of equivalent games
- Understand and use the main theorem about Nim heaps
- Nim sums and binary representations
- Proof of the main theorem

Losing games are equivalent to $*0$

Definition 1.4 (Equivalent games) $\quad G \equiv H \;\Leftrightarrow$

for any game J : $\;G + J$ is losing $\Leftrightarrow$ $H + J$ is losing .

Lemma 1.5 $\quad G$ is a losing game $\iff$ $G \equiv *0$

Proof of '$\Rightarrow$': Suppose G is losing and J is any game. Then

$$\begin{aligned}
& G + J \text{ is losing} \\
\Leftrightarrow \quad & J + G \text{ is losing} \\
\Leftrightarrow \quad & J \text{ is losing} \quad \text{(shown earlier, with } Z \text{ instead of } G\text{)} \\
\Leftrightarrow \quad & J + *0 \text{ is losing} \\
\Leftrightarrow \quad & *0 + J \text{ is losing .}
\end{aligned}$$

This shows that $G \equiv *0$. $\quad\square$

Losing games are equivalent to $*0$

Definition 1.4 (Equivalent games) $\quad G \equiv H \quad \Leftrightarrow$

for any game J : $\quad G + J$ is losing $\Leftrightarrow$ $H + J$ is losing .

Lemma 1.5 $\quad G$ is a losing game $\Longleftrightarrow$ $G \equiv *0$

Corollary: All **losing** games are equivalent.

This is not true for **winning** games:

Lemma 1.9 Only Nim heaps of **equal** size are equivalent.

Example: $*4 \not\equiv *6$ because (with $J = *4$)

$*4 + *4$ is losing, but $*6 + *4$ is winning.

Equivalent games sum to $*0$

Lemma 1.7

Two impartial games G and H are equivalent if and only if $G + H \equiv *0$.

Proof

If $G \equiv H$, then (exercise!) $\quad G + H \equiv H + H \equiv *0.$

If $G + H \equiv *0$ then

$$G \equiv G + *0 \equiv G + (H + H) \equiv G + H + H \equiv *0 + H \equiv H. \qquad \square$$

Equivalent options suffice for equivalence

Lemma 1.8

If for every option of $\boldsymbol{G}$ there is an equivalent option of $\boldsymbol{H}$ and vice versa, then $\boldsymbol{G} \equiv \boldsymbol{H}$.

(We will prove a similar statement later.)

Using the main theorem for Nim heaps

Let $n \geq 1$, and $n = 2^a + 2^b + 2^c + \cdots$, where $a > b > c > \cdots \geq 0$. Then

$$*n \;\equiv\; *(2^a) + *(2^b) + *(2^c) + \cdots .$$

Use of this Theorem 1.10 for multiple Nim heaps:

(1) Convert each Nim heap to distinct power-of-two-sized heaps.

(2) Cancel resulting equal powers of two in pairs, using $G + G \equiv *0$. Only **distinct** powers of two remain.

(3) Take the sum s of these distinct powers of two. If $s = 0$, the position is losing, so assume $s > 0$.

(4) Take the game sum $*n + *s$ for any of the original Nim heaps $*n$. When the resulting Nim heap is smaller (which occurs for an odd number of heaps), this is a winning move.

Example

Three Nim heaps $*5$, $*11$, $*8$.

(1) $*5 \equiv *4 + *1$, $*11 \equiv *8 + *2 + *1$, $*8 \equiv *8$.

(2) $*8$ and $*1$ occur an even number of times and cancel.
$*4$ and $*2$ occur an odd number of times and remain.

(3) $s = 6 = 4 + 2$. We have $*5 + *11 + *8 \equiv *s$.

(4) $*5 + *s \equiv *4 + *1 + *4 + *2 \equiv *1 + *2 \equiv *3$.
The move from $*5$ to $*3$ is winning, creating the position
$*3 + *11 + *8 \equiv *5 + *s + *11 + *8 \equiv *0$.

(4') $*11 + *s \equiv *8 + *2 + *1 + *4 + *2 \equiv *8 + *4 + *1 \equiv *13$.
Moving from $*11$ to $*13$ would be winning but is not allowed.

(4'') $*8 + *s \equiv *8 + *4 + *2 \equiv *14$.
Moving from $*8$ to $*14$ would be winning but is not allowed.

Using the binary system

heap	8	4	2	1
5	0	1	0	1
11	1	0	1	1
8	1	0	0	0
$s = 6$	0	1	1	0

The winning move: flip the ‘**1**’ bits of the Nim sum s

	heap	8	4	2	1
$\rightarrow$	3	0	0	1	1
	11	1	0	1	1
	8	1	0	0	0
	$s = 0$	0	0	0	0

Another example

heap	8	4	2	1
6	0	1	1	0
11	1	0	1	1
14	1	1	1	0
$s = 3$	0	0	1	1

3 winning moves: flip the '**1**' bits of the Nim sum s

heap	8	4	2	1
$\rightarrow$ 5	0	1	0	1
11	1	0	1	1
14	1	1	1	0
$s = 0$	0	0	0	0

Another example

heap	8	4	2	1
6	0	1	1	0
11	1	0	1	1
14	1	1	1	0
$s = 3$	0	0	1	1

3 winning moves: flip the '**1**' bits of the Nim sum s

	heap	8	4	2	1
	6	0	1	1	0
$\rightarrow$	8	1	0	0	0
	14	1	1	1	0
	$s = 0$	0	0	0	0

Another example

heap	8	4	2	1
6	0	1	1	0
11	1	0	1	1
14	1	1	1	0
$s = 3$	0	0	1	1

3 winning moves: flip the ‘**1**’ bits of the Nim sum s

	heap	8	4	2	1
	6	0	1	1	0
	11	1	0	1	1
$\rightarrow$	13	1	1	0	1
	$s = 0$	0	0	0	0

Binary system: why Nim sum $s = 0$ is losing

heap	8	4	2	1
6	0	1	1	0
11	1	0	1	1
13	1	1	0	1
$s = 0$	0	0	0	0

Changing one Nim heap **changes at least one bit** in the corresponding row of the binary representation. The affected columns will get an **odd** number of **1**'s.

So it creates a **non-zero** Nim sum, which is a winning position.

This was discovered and published by Charles Bouton in 1902.

Note: this system works well on paper, but not in your head. Instead, visualize splitting Nim heaps into heaps of powers of two.

Nim sums

The **Nim sum** $n \oplus m$ of two **integers** n and m is the size of the equivalent Nim heap of the **game** $*n + *m$, that is,

$$*(n \oplus m) \equiv *n + *m\,.$$

The operation $\oplus$ is like binary "addition without carry" in the binary system, where only the **parity** (**0** for even, **1** for odd) of each power of **2** is recorded.
This is the same as "equal powers of **2** cancel in pairs".

Examples:

$$\mathbf{1 \oplus 3 = 1 \oplus 2 \oplus 1 = 2}$$

$$\mathbf{5 \oplus 11 = 4 \oplus 1 \oplus 8 \oplus 2 \oplus 1 = 8 \oplus 4 \oplus 2 = 14}$$

Theorem 1.10 implies Lemma 1.11

Theorem 1.10

Let $n \geq 1$, and $n = 2^a + 2^b + 2^c + \cdots$, where $a > b > c > \cdots \geq 0$. Then

$$*n \equiv *(2^a) + *(2^b) + *(2^c) + \cdots .$$

Will prove Theorem 1.10 by induction together with the following:

Lemma 1.11

Let $0 \leq p, q < 2^a$. Then $p \oplus q < 2^a$. That is,

$$*p + *q \equiv *r \quad \text{where } 0 \leq r < 2^a .$$

Proof The lower powers of 2 to represent p and q might cancel with $\oplus$ but never go above 2^a. $\square$

Proof by induction

We will prove Theorem 1.10 and its immediate consequence Lemma 1.11 by induction, assuming they hold for all smaller values of n. They hold trivially if $n = 2^a$.

Let $n = 2^a + q$ for some q where $0 < q < 2^a$. By induction, $q = 2^b + 2^c + \cdots$ where $a > b > c > \cdots \geq 0$ and $*q \equiv *(2^b) + *(2^c) + \cdots$, so all we need to show is

$$*n \equiv *(2^a) + *q\,.$$

We will show that $*n$ and the game sum $*(2^a) + *q$ are equivalent by showing that they have equivalent **options**.

Two types of options

Example: $n = 7 = 4 + 3 = 2^a + q$.

Options of $*7$:		Options of $*4 + *3$:		
$*0$		$*0 + *3$	$\equiv$	$*3$
$*1$		$*1 + *3$	$\equiv$	$*2$
$*2$		$*2 + *3$	$\equiv$	$*1$
$*3$		$*3 + *3$	$\equiv$	$*0$
$*4$	$\equiv$	$*4 + *0$	(by induction)	
$*5$	$\equiv$	$*4 + *1$	(by induction)	
$*6$	$\equiv$	$*4 + *2$	(by induction)	

Two types of options

General case: $n = 2^a + q$.

Two types of options

General case: $n = 2^a + q$.

Options of $*n$:

Two types of options

General case: $n = 2^a + q$.

Options of $*n$:		Options of $*(2^a) + *q$:	
$*0$		$*0 + *q$	$\equiv$?
$*1$		$*1 + *q$	$\equiv$?
$\vdots$		$\vdots$	
$*(2^a - 1)$		$*(2^a - 1) + *q$	$\equiv$?
$*(2^a)$	$\equiv$	$*(2^a) + *0$	(by induction)
$*(2^a + 1)$	$\equiv$	$*(2^a) + *1$	(by induction)
$\vdots$		$\vdots$	
$*(2^a + q - 1)$	$\equiv$	$*(2^a) + *(q - 1)$	(by induction)

Equivalence of the first 2^a options

Consider two options of $(2^a) + *q$ of the form

$$*p + *q \quad \text{and} \quad *r + *q \qquad (0 \le p, r < 2^a).$$

If they are **equivalent**, then

$$*p \equiv *p + *q + *q \equiv *r + *q + *q \equiv *r$$

and so they are **equal**, $p = r$.

So all 2^a options $*p + *q$ for $p = 0, 1, \ldots, 2^a - 1$ are equivalent to **distinct** Nim heaps of size $< 2^a$ (by Lemma 1.11, because $q < 2^a$), so together they must be **all** the Nim heaps $*0, *1, \ldots, *(2^a - 1)$ which are the first 2^a options of $*n$.

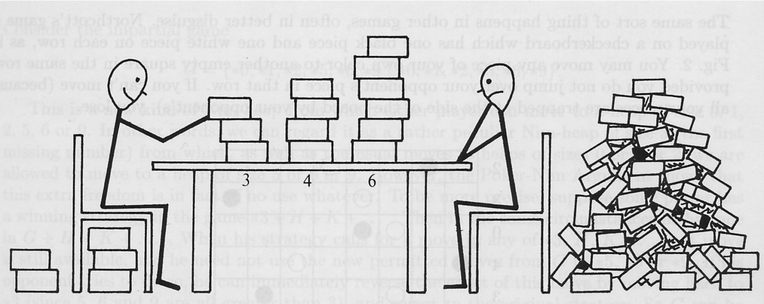
This completes the induction. $\square$

Plan

- Poker Nim and the mex rule
- Every impartial game is equivalent to a Nim heap
- Using the mex rule
- Apply the mex rule to other impartial games

Rules of Poker Nim – a variant of Nim

- A player remove some chips from a heap as in ordinary Nim
- Alternatively, the player may **add** some chips that he collected earlier to an existing heap, even to the empty heap $*\mathbf{0}$.



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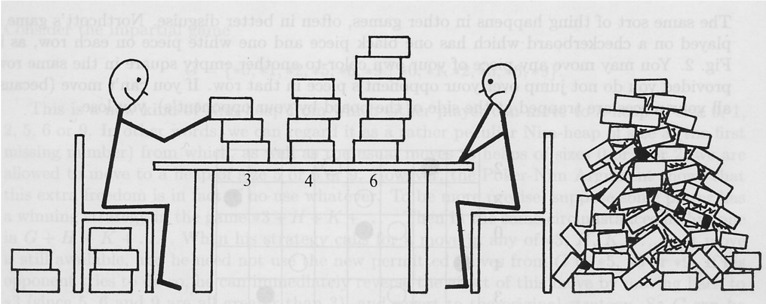


A well advanced game of Poker Nim (from *"Winning Ways"*)

How different is Poker Nim from Nim?

Losing positions in Poker Nim are the **same** as in ordinary Nim:

- any **added** chips can just be removed by the winning opponent.

A player in a winning position never needs to add chips to a heap and thus can make sure the game terminates.

This does not change if there is a **restriction** on **how many** chips can be added to a particular heap, for example, if player I is only allowed to add **1**, **3**, or **5** chips to an existing heap: Then player II will just remove these chips in her counter-move.

Example

Consider the losing position $*\mathbf{1} + *\mathbf{4} + *\mathbf{5}$.

Suppose player I is allowed to add **1**, **3**, or **4** chips to the heap $*\mathbf{5}$ in restricted Poker Nim. Then his options for that heap are

$$*\mathbf{0}, *\mathbf{1}, *\mathbf{2}, *\mathbf{3}, *\mathbf{4}, \boxed{}, *\mathbf{6}, *\mathbf{8}, *\mathbf{9}.$$

Among all the Nim heaps that player I can move to, $*\mathbf{5}$ is the $\boxed{\text{first}}$ that is $\boxed{\text{absent}}$. (All smaller sizes $*\mathbf{0}, *\mathbf{1}, *\mathbf{2}, *\mathbf{3}, *\mathbf{4}$ can be reached because $*\mathbf{5}$ is the given heap.)

- Will show: the larger options $*\mathbf{6}, *\mathbf{8}, *\mathbf{9}$ do not matter.

We write $\boxed{\mathbf{mex}(\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{8}, \mathbf{9}) = \mathbf{5}}$ because **5** is the **m**inimum **ex**cluded number in $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{8}, \mathbf{9}\}$.

The mex function

Given a set S of nonnegative integers, the **m**inimum **ex**cluded number from S is

$$\mathbf{mex}(S) = \mathbf{min}\{k \geq 0 \mid k \notin S\}$$

Examples

$$\mathbf{mex}(0, 2, 3) = 1$$

$$\mathbf{mex}(0, 1, 2, 5, 8, 100) = 3$$

$$\mathbf{mex}(0, 1, 2, 3, 4) = 5$$

$$\mathbf{mex}(2, 3, 7) = 0$$

$$\mathbf{mex}(\emptyset) = 0$$

Proof

Options of G are $\{*s \mid s \in S\}$. Let $m = \textbf{mex}(S)$.

Show $G + *m \equiv *0$ (losing game), implies $G \equiv *m$.

Example: $S = \{0, 1, 2, 5, 8, 100\}$, $m = 3$.

Options of $*m$ are $*0, *1, *2$.

If player I moves to $*p + *m$ or to $G + *p$ for $p < m$,

then player II has the winning counter-move to $*p + *p$.

If player I moves to $*q + *m$ for $q > m$,

then player II has the winning counter-move to $*m + *m$.

(just like removing the added chips in Poker Nim). $\square$

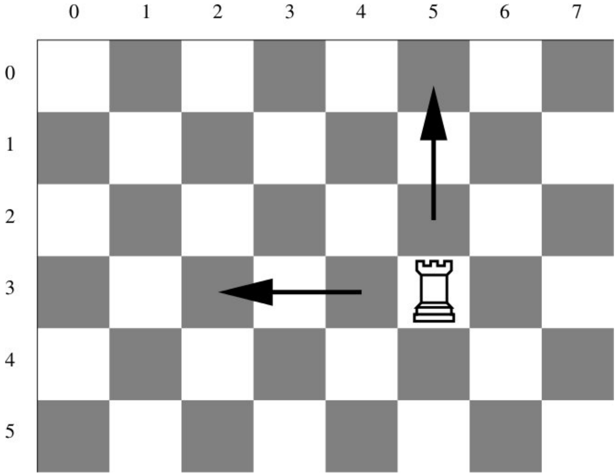
Corollary

Every impartial game $\boldsymbol{G}$ is equivalent to some Nim heap.

(By the mex rule, since this holds by induction for all options of $\boldsymbol{G}$.)

The rook move game

Move the rook like in Chess but only up or left:

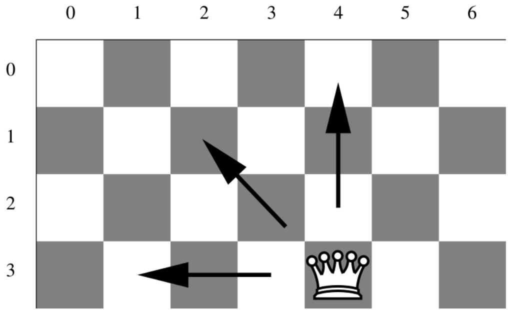


Nim values for the rook move game

	0	1	2	3	4	5	6	7
0	$*0$	$*1$	$*2$	$*3$	$*4$	$*5$		
1	$*1$	$*0$	$*3$	$*2$	$*5$	$*4$		
2	$*2$	$*3$	$*0$	$*1$	$*6$	$*7$		
3	$*3$	$*2$	$*1$	$*0$	$*7$	$*6$		
4	$*4$	$*5$	$*6$	$*7$	$*0$	$*1$		
5	$*5$	$*4$	$*7$	$*6$	$*1$	$*0$		

The queen move game

Move the queen like in Chess but only up, left, or diagonally up and left:



Nim values for the queen move game

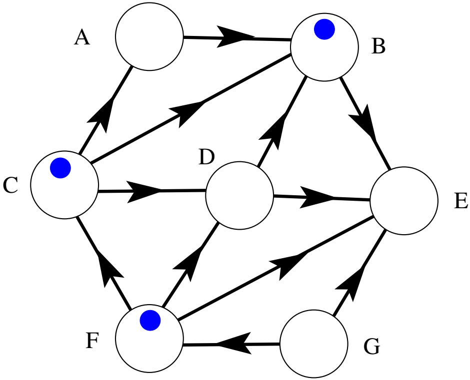
	0	1	2	3	4	5	6
0	$*\mathbf{0}$						
1							
2							
3							

The counter game on a directed graph

Given: an acyclic directed graph.

Each node can contain any number of **counters**.

Move **one counter** in the arrow direction to an adjacent node.



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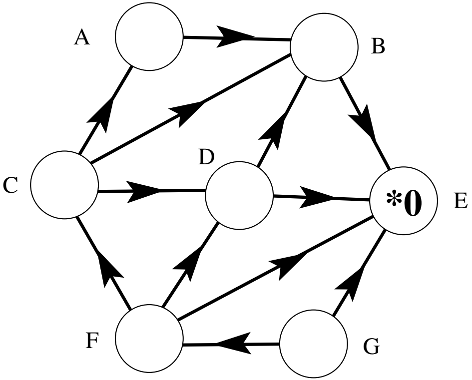

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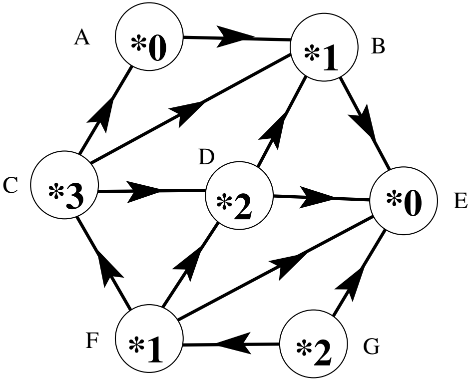

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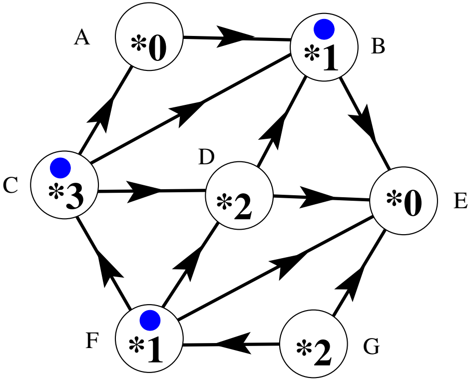

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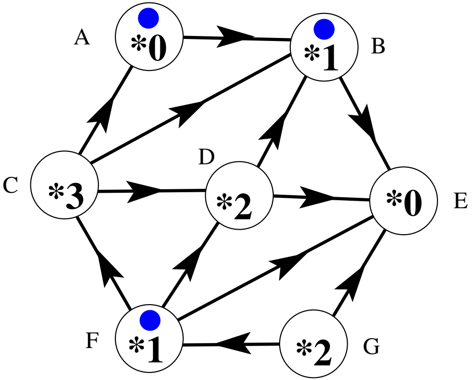

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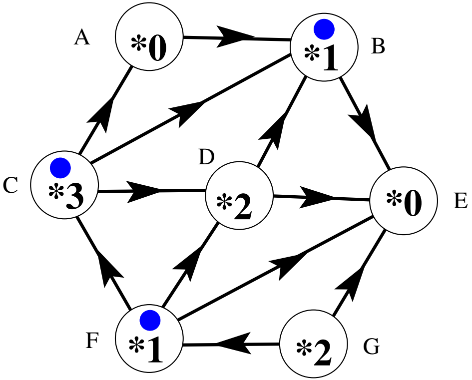

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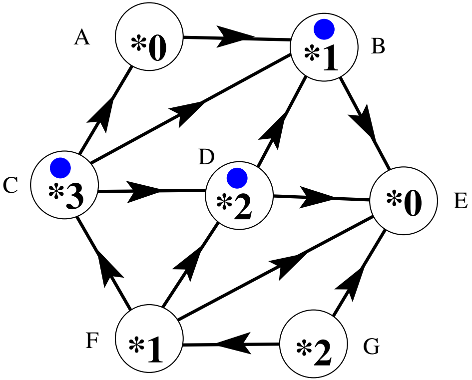
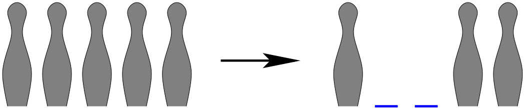


Kayles

Rule: Given a row of $\boldsymbol{n}$ bowling pins, a move knocks out one or two **consecutive** pins.



If this game is called K_n, then knocking out pin p creates the game sum $K_{p-1} + K_{n-p}$, and knocking out pins p and $p+1$ (in the picture, $n = 5$, $p = 2$) creates the game sum $K_{p-1} + K_{n-p-1}$ (above, $K_1 + K_2$). The options of K_n are these game sums for all possible p, where $p > n/2$ need not be considered by symmetry.

Note: In this game, options happen to be **sums** of games. For the mex rule, only their Nim value is important.

Nim values for Kayles

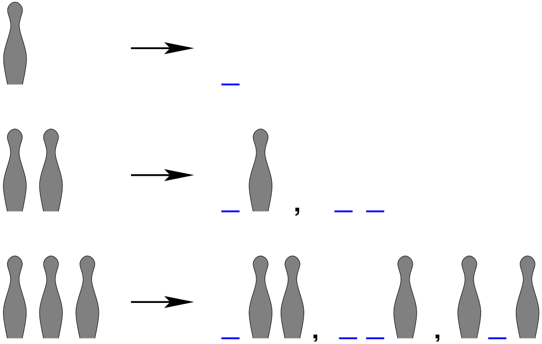
Nim value (= size of equivalent Nim heap) of K_n

$$\begin{aligned} K_1 &\equiv *\mathbf{mex}(0) \\ &= *1 \end{aligned}$$

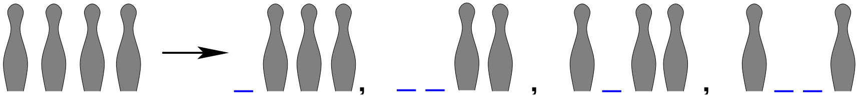
$$\begin{aligned} K_2 &\equiv *\mathbf{mex}(1,0) \\ &= *2 \end{aligned}$$

$$\begin{aligned} K_3 &\equiv *\mathbf{mex}(2,1,0), \\ &= *3 \end{aligned}$$

because $0 = 1 \oplus 1$

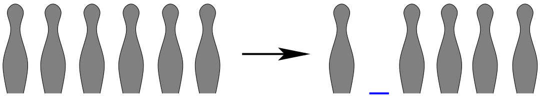


Kayles with 4 pins and 6 pins



$$K_4 \equiv *\mathbf{mex}(3, 2, 1 \oplus 2, 1 \oplus 1) = *\mathbf{mex}(3, 2, 3, 0) = *1$$

So this is an unexpected winning move for 6 pins:



Show directly that $K_1 + K_4$ is a losing position by finding all possible countermoves!