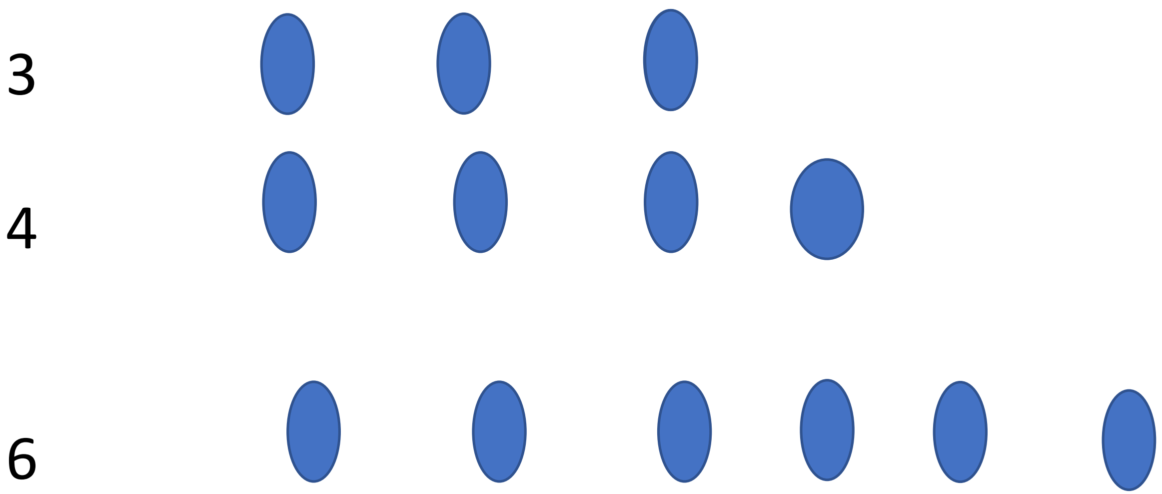
Lecture 1 on NIM - A

In this lecture we define the game of Nim, play a few games, and make a few observations. Then we develop a simple theory based on binary numbers that determines which player should win with best play and how to win in a winning position. Strangely, I will ask you to ‘forget’ this theory at the end of this week.

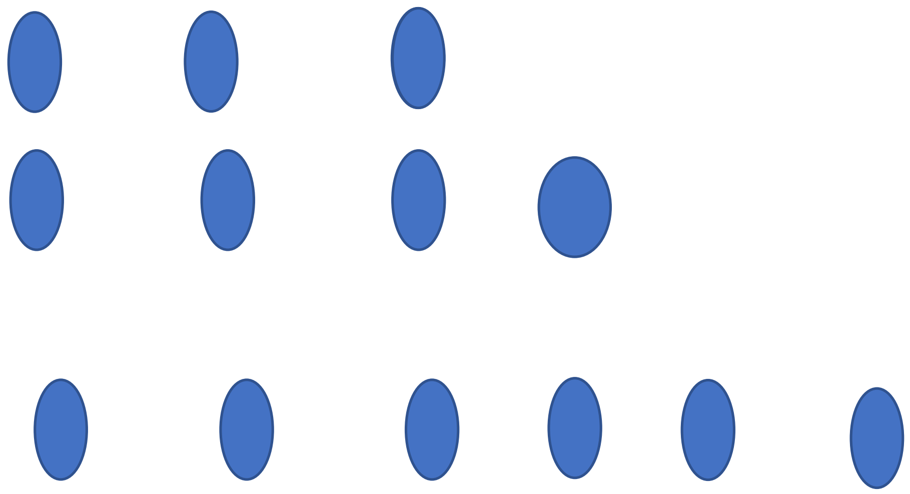
Verbal Description of Game of Nim, one game played out.

We start with say piles of 3, 4 and 6 chips (blue circles). A move takes away Chips from one of the piles. The player who removed the last chip wins. Alternatively, the player who cannot remove any chips (there are none left) Loses.



How Nim is played.

The game of Nim, or say Nim [x,y,z] begins with say three piles of chips, of sizes x,y,z. There are two players, called I and II (I goes first). When moving, a player must remove some chips from a single pile. For example, from [3,4,6] a player can take 2 chips from the second pile, leaving the other player to move in the position [3,2,6]. A player who cannot move, that is faces position [0,0,0], is the loser. Equivalently, the player who takes the last chip wins. It is clear that the game cannot go on forever, so one player wins and the other loses. Of course there can be more than three piles.



Winning Strategies, Winning Positions

A general theorem says that one of the players has a *winning strategy* (a ***strategy*** says how she will move in any position, a winning strategy is one that always ends in a win for the player using it), so a position where I has a winning strategy is called a ***winning position*** and one where II has a winning strategy is called a ***losing position***. For example $[1, 1, 1]$ is winning position and $[1, 1, 1, 1]$ (there can be more than three piles) is a losing position. (In these examples, the players have essentially no choice, they can just remove one of the single-chip piles.)

Q: When is a Nim position with n piles of a single chip each a winning position?

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Q: When is a Nim position with n piles of a single chip each a winning position?

A: If n is even Player II wins; if n is odd Player I wins.

Games to play

1. Try playing [1; 2; 3] with your neighbor (take turns crossing out chips with your pen). The vertical line just separates different copies of the same game, so you get several tries and the same initial position, to hone your skills:

1 2 3

| o oo ooo | o oo ooo | o oo ooo | o oo ooo | o oo ooo |

| o oo ooo | o oo ooo | o oo ooo | o oo ooo | o oo ooo

- How many times did I win ? _____________
- How many times did II win? _____________

Try playing [4,4] with your neighbour (bubble partner)

|oooo   oooo | oooo oooo | oooo oooo | oooo oooo |

Which player should win? What is the winning strategy?

Try playing [4,4] with your neighbour (bubble partner)

|oooo oooo | oooo oooo | oooo oooo | oooo oooo |

Which player should win? What is the winning strategy?

Answer: The second player should win. The winning strategy is the **copycat strategy**: Whatever I does in some pile, II does the same in the other pile.

Try playing [1,4,4] with your neighbor.

$o \quad oooo \; oooo \mid o \; oooo \; oooo \mid o \; oooo \; oooo \mid o \; oooo \; oooo \mid o \; oooo \; oooo$

Q: How can Player I win in this position?

Try playing [1,4,4] with your neighbor.

$o \quad oooo \;\; oooo \mid o \;\; oooo \;\; oooo \mid o \;\; oooo \;\; oooo \mid o \;\; oooo \;\; oooo \mid o \;\; oooo \;\; oooo$

Q: How can Player I win in this position?

A: First move removes the single pile of one. Then play copycat.

Try playing [2,3,2,3] with your neighbor.

oo ooo oo ooo | oo ooo oo ooo | oo ooo oo ooo |
oo ooo oo ooo | oo ooo oo ooo |

Q: Who wins and what is the winning strategy?

Try playing [2,3,2,3] with your neighbor.

oo ooo oo ooo | oo ooo oo ooo | oo ooo oo ooo |
oo ooo oo ooo | oo ooo oo ooo |

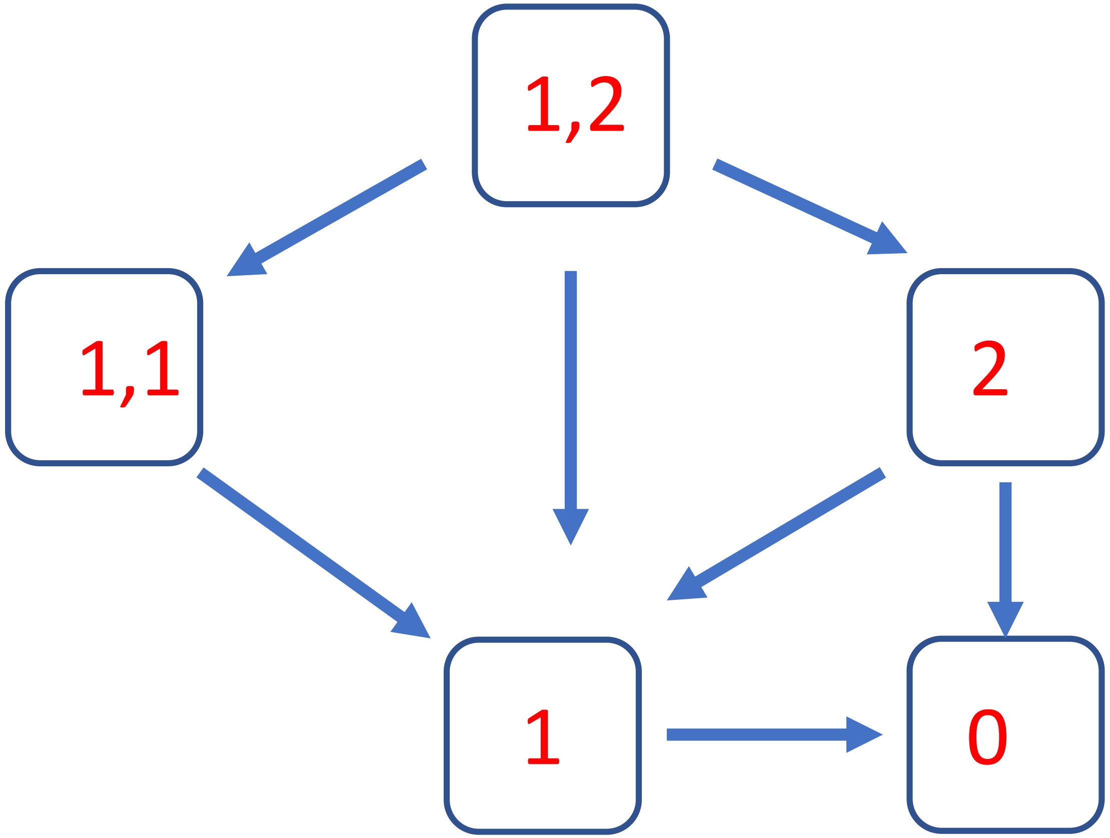
Q: Who wins and what is the winning strategy?

A: Consider this as two games of [2,3]. The second player wins by copying whatever the first does in the other game. For example if I moves to [2,3,2,1] then II moves to [2,1,2,1].

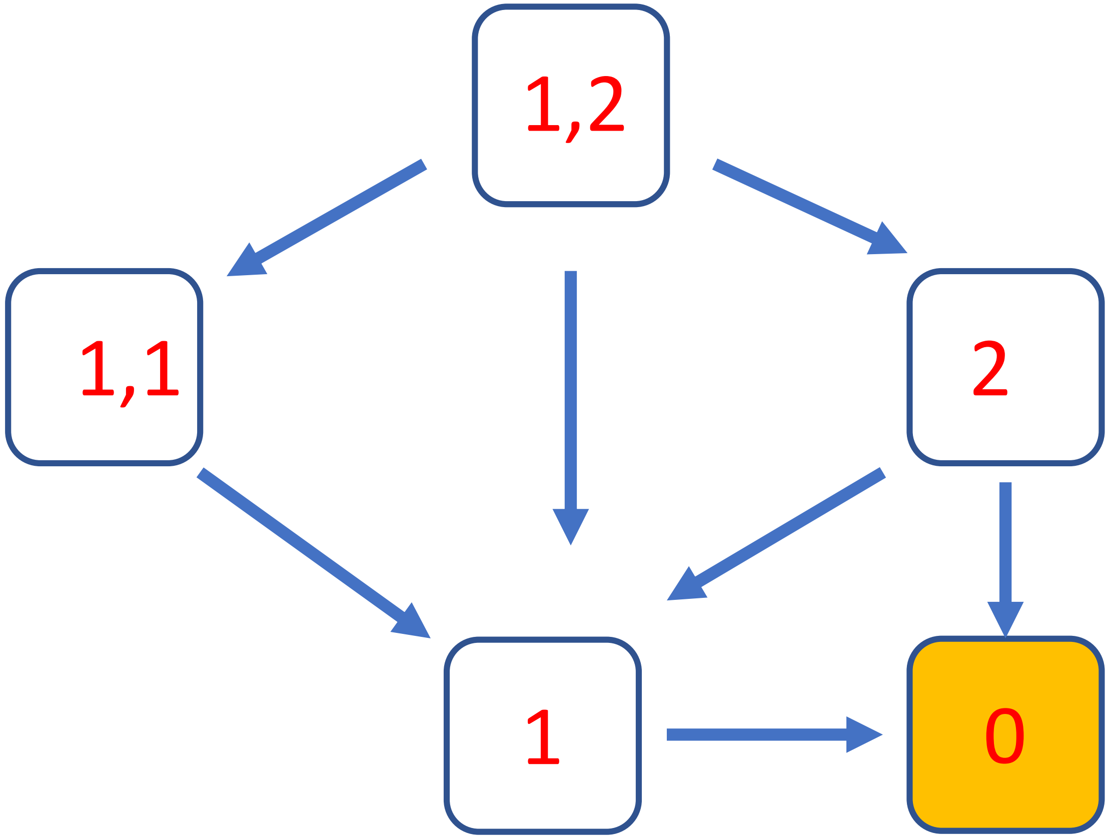
Lecture 1 on Nim – B (Nim as DG Game)

We show how to view a game of Nim in terms of nodes representing positions and directed arcs which represent moves.

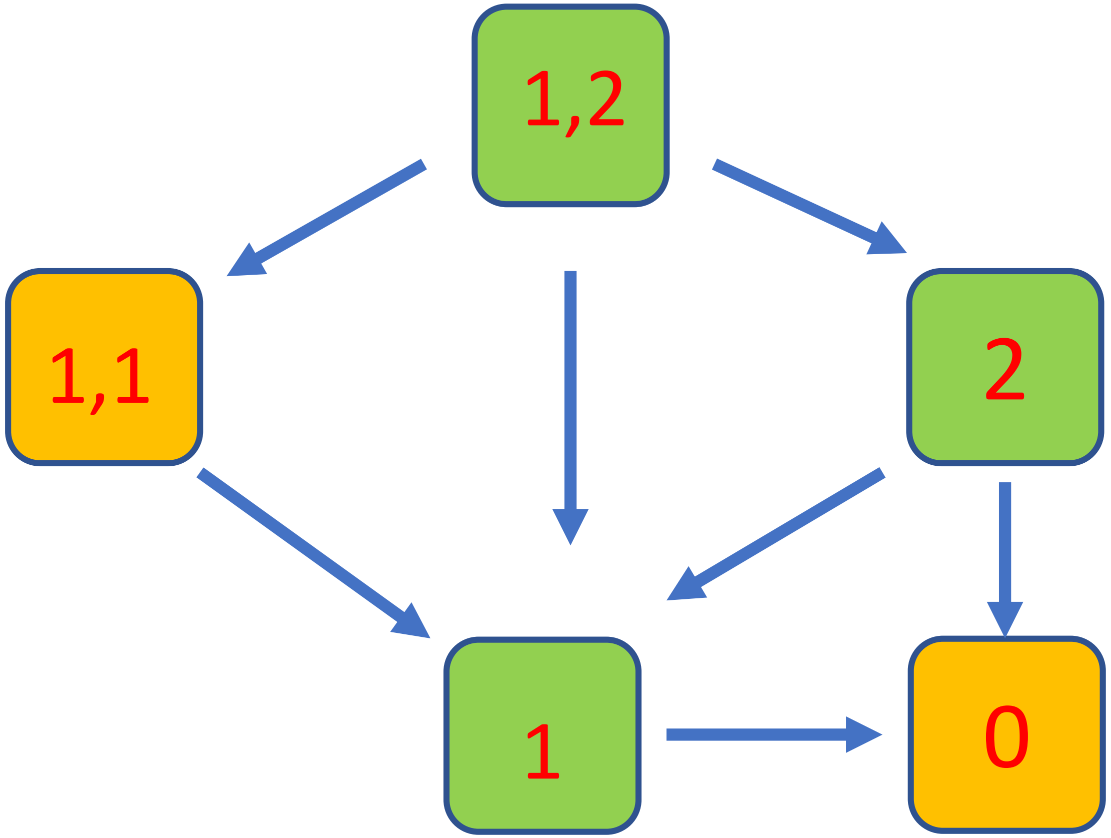
Directed Graph Game



Directed Graph Game

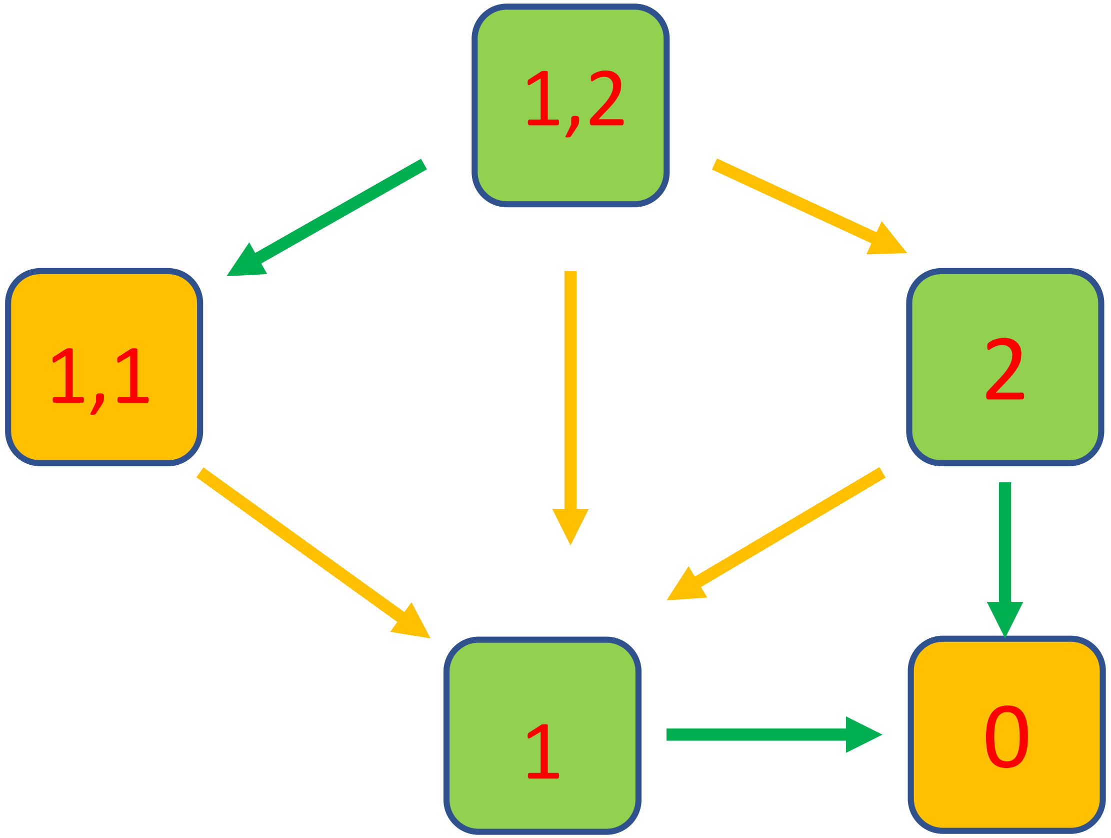


Directed Graph Game



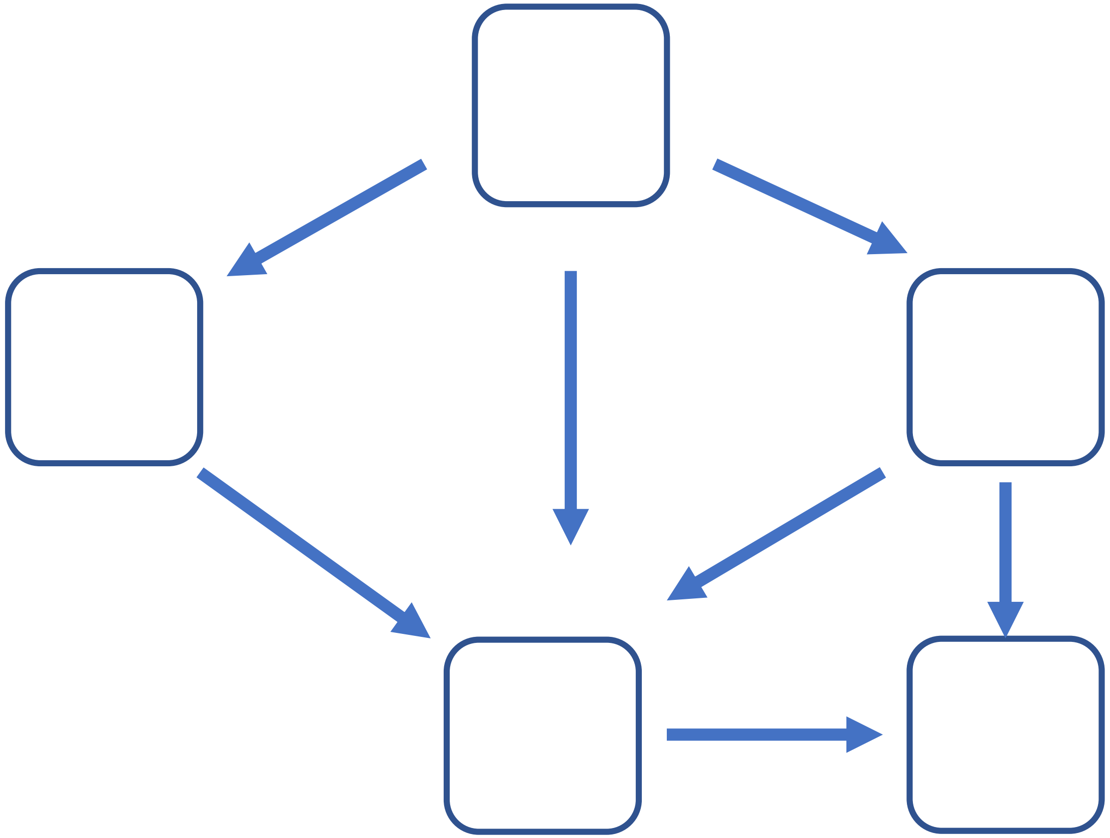
So [1,2] is a winning position, winning move is to go to [1,1]

Directed Graph Game – Winning Arrows (moves) green

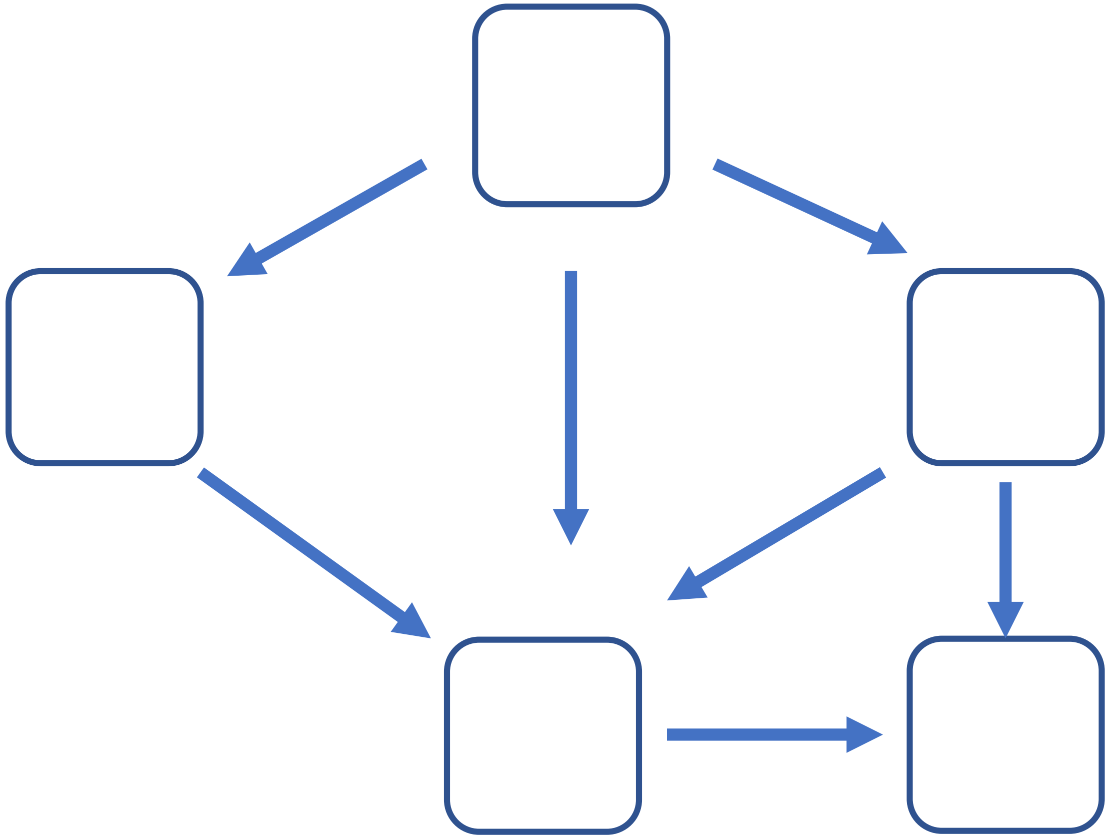


So [1,2] is a winning position, winning move is to go to [1,1]

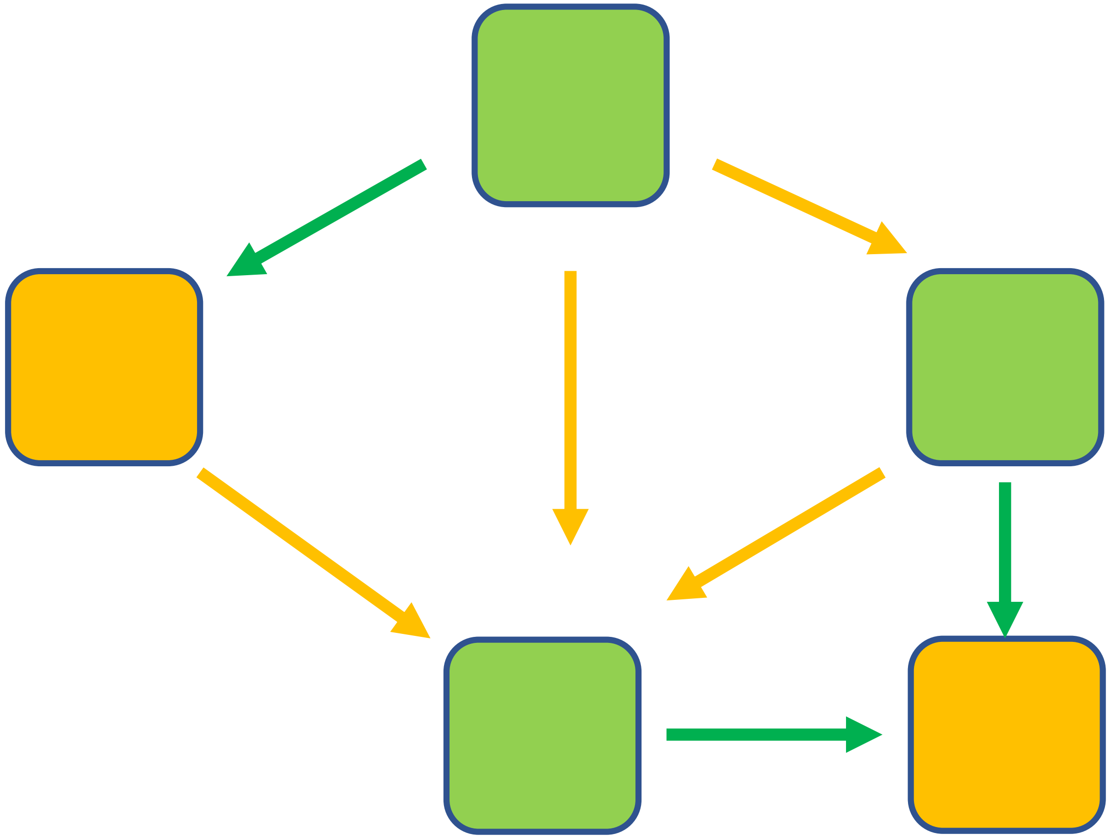
Abstract Directed Graph Game



Abstract Directed Graph Game

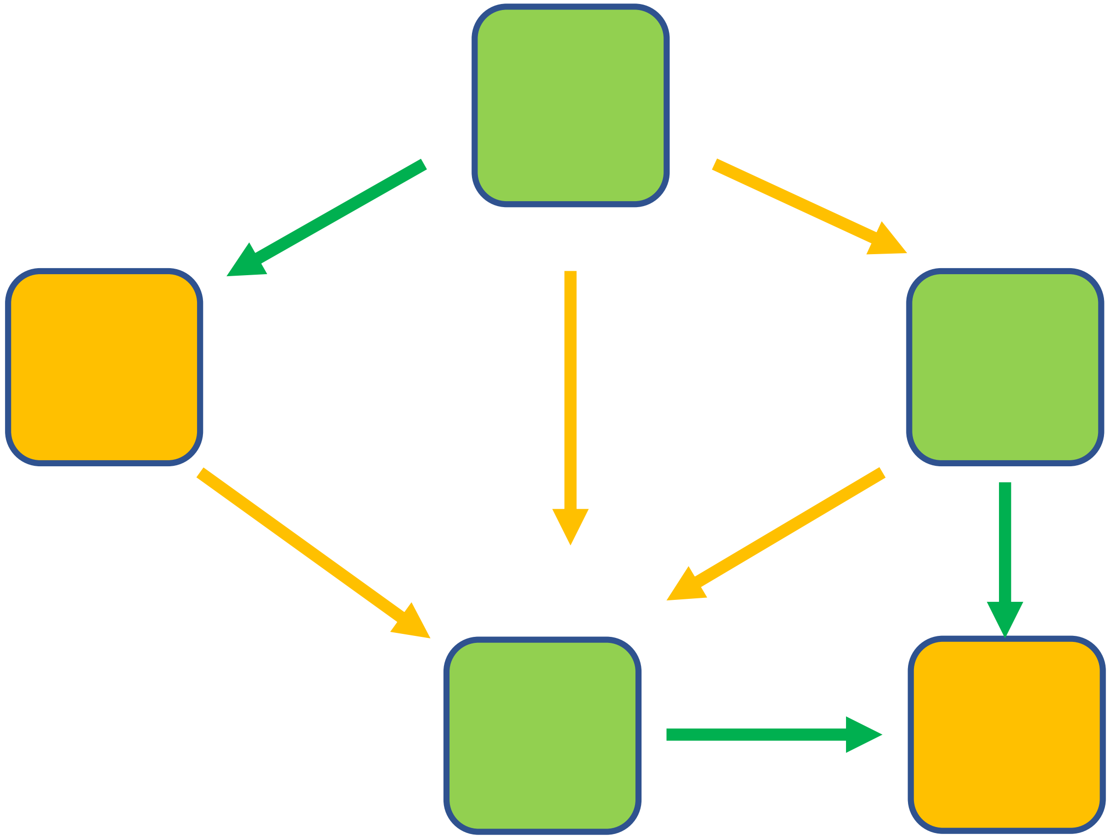


Directed Graph Game – Winning Arrows (moves) green



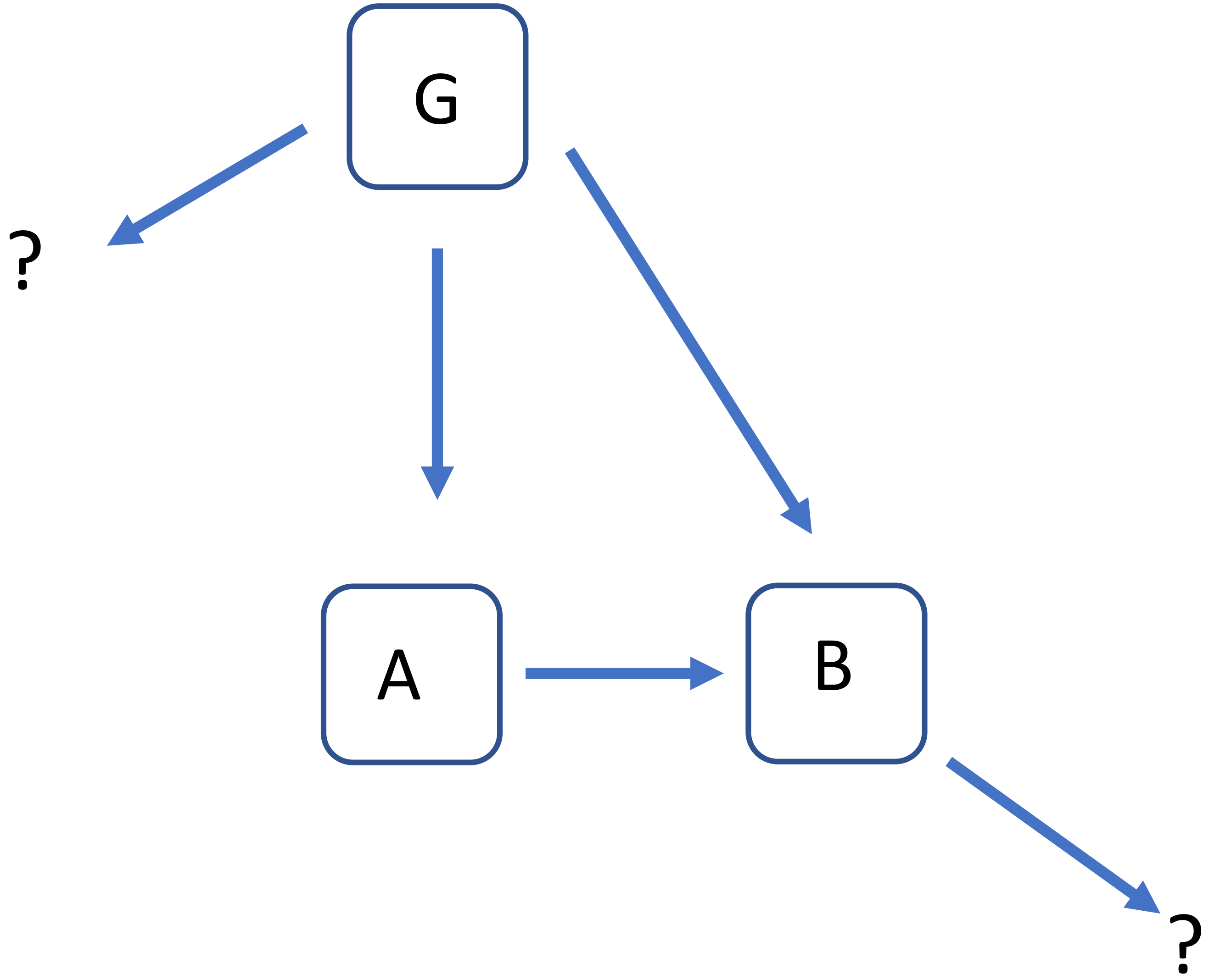
So top is a winning position, winning move is left down.

Directed Graph Game – General Colouring Algorithm



Some box has no out arrows (why?), colour such boxed orange. For boxes with successors all labelled, colour them green if at least one successor is orange, colour them orange if all successors green. Eventually top is labeled. If green, it is a winning position (winning game). If orang it is a losing game. Note: Induction is hidden here.

Brain Teaser – Is G winning or losing? Think about it for next week.

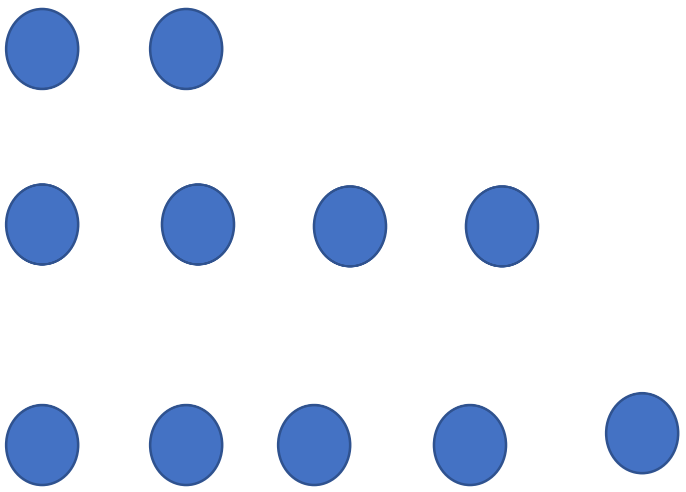


Lecture 1C: Using Binary Numbers to Solve Nim

We show how every Nim position can be represented by a grid of binary numbers. We then add the 0s and 1s

In each column in a special way and show how this ‘nim sum’ of numbers determines whether the game is winning or losing.

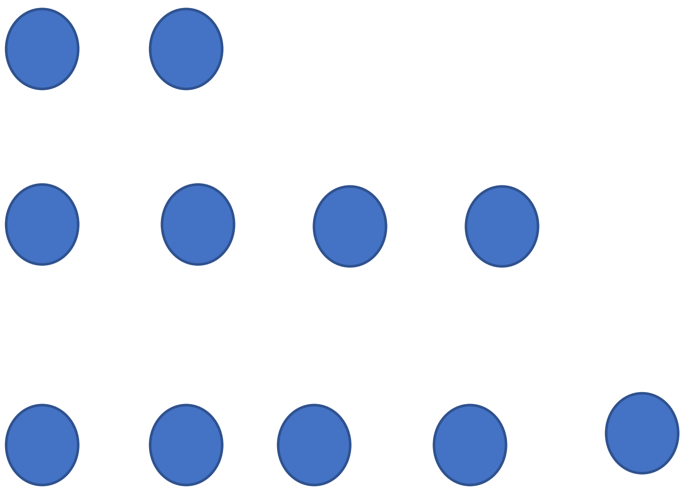
Representing a Nim position as grid of binary numbers.



	8	4	2	1
2				
4				
5				
sum				

Column sum 1 of odd number of 1s; if not, 0.

Representing a Nim position as grid of binary numbers.



$2 \oplus 4 \oplus 5 = 3$ (for later)

	8	4	2	1
2			1	0
4		1	0	0
5		1	0	1
3		0	1	1

Balanced Positions in Nim

A Nim position is called balanced if all columns in the binary grid have an even number of 1s. This means the column sums are all 0s. The nim sum $\oplus$ is 0. If a position is not balanced it is called unbalanced. (So [4, 5,7, 8] is unbalanced, while

[4, 4] is balanced.) In previous slide, [2,4,5] is unbalanced. Note that the losing position [n,n] is balanced (why?).

	8	4	2	1
4		1	0	0
5		1		1
7		1	1	1
8	1			
nim sum	1	1	1	0

[4,5,7,8] is unbalanced

A few observations.

1. The one immediate losing position (with no moves) is balanced
2. *n* piles of a single chip is winning if n is odd (last column sum 1, so unbalanced).
3. Games with duplication, like [n,n] are losing by copycat and are balanced.

So we Conjecture: Balance positions are losing, unbalanced winning

Two Lemmas for you to Prove For Next Week

Lemma 1: Given an unbalanced position, a player can always convert it to a balanced position.

Example: [4, 5,7, 8] -> [4, 5, 7, 6] (taking two chips from the pile of 8).

Nim Calc	8	4	2	1
4	0	1	0	0
5	0	1	0	1
7	0	1	1	1
8	1	0	0	0
14	1	1	1	0

Nim Calc	8	4	2	1
4	0	1	0	0
5	0	1	0	1
7	0	1	1	1
6	0	1	1	0
0	0	0	0	0

Lemma 2: Given a balanced position, any move converts it to an unbalanced position.

Example: [1 , 2 , 3] (balanced) ->
one of [0 , 2 , 3]* , [1 , 1 , 3] , [1,0,3] , [1,2,2] , [1,2,1] , [1 2,0] :

Nim Calc	8	4	2	1
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
0	0	0	0	0
0	0	0	0	0

Nim Calc	8	4	2	1
1	0	0	0	0
2	0	0	1	0
3	0	0	1	1
0	0	0	0	0
0	0	0	0	1

THM: Nim Position Winning if and only if Unbalanced.

L1: Player in unbalanced position can balance it.
L2: Any move from balanced position is unbalanced.

Proof: Winning Strategy from unbalanced: balance it. This is possible on first move by L1. Player II gets balanced position, so by L2 must return an unbalanced position to Player I. Continuing like this, I always gets an unbalanced position and II always gets a balanced position. Eventually, the immediately losing (no move) position is reached. Since this is balanced, it is for II to move. So I wins.