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A New Approach to Gal's Theory of Search Games on Weakly Eulerian Networks

Steve Alpern

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Abstract A network is called weakly Eulerian if it consists of a finite number of disjoint Eulerian networks which are connected in a tree-like fashion. S. Gal and others developed a theory of (zero-sum) hide-and-seek games on such networks. The minimax search time for a network is called its search value. A network is called simply searchable if its search value is half the minimum time to tour it. A celebrated result of Gal is that a network is simply searchable if and only if it is weakly Eulerian. This expository article presents a new approach to the Gal theory, based on ideas borrowed from the author's recent extension of Gal's theory to networks which can be searched at speeds depending on the location and direction in the network. Most of the proofs are new.

Keywords Search theory · Tree · Zero-sum game · Search game

1 Introduction

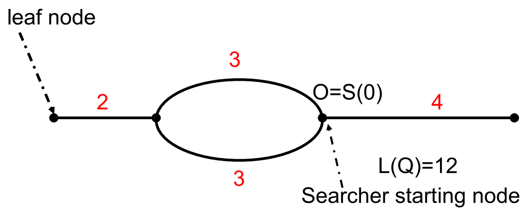
In the last chapter of his well known book, *Differential Games* [13], Rufus Isaacs proposed the following *search game with immobile hider*. The game takes place on a network Q consisting of finitely many nodes and edges, where each edge e is assigned a finite length $L(e)$. We can view L as a measure on Q (called length) and then define a metric d on Q as the length of the shortest path between two points. A unit speed path is then simply any path $S(t)$ satisfying $d(S(t), S(t')) \leq |t - t'|$. Given a searcher starting point O in Q, we define the zero-sum game $G = G(Q, O)$ as follows. The maximizing Hider picks a point H in Q and the minimizing Searcher picks a unit speed covering path $S(t)$ on Q with $S(0) = O$. The payoff is the *capture time*

$$T = \min\{t : S(t) = H\}.$$

The existence of a value $V = V(Q, O)$, an optimal mixed strategy for the Searcher and an ε-optimal mixed strategy for the Hider was established by Gal [10] and now follows from

S. Alpern (✉)
Mathematics Department, London School of Economics, London, UK
e-mail: S.Alpern@lse.ac.uk

Fig. 1 A rooted network (Q, O)



a subsequent general minimax theorem of Alpern and Gal [5]. We call $V = V(Q, O)$ the *search value* of Q (or of Q, O if O is relevant). We will denote mixed strategies of the Searcher and Hider by lower case s and h, and the corresponding *expected* payoff (mean capture time) by $T(s, h)$.

As an example consider the case where the network is the circle C of circumference μ, with some starting point O. If the Hider picks the antipodal point O' to O, it will take the Searcher at least time $\mu/2$ to reach him, so $V \geq \mu/2$. Suppose the Hider picks some point at distance x clockwise of O. If the Searcher goes at unit speed equiprobably clockwise or counterclockwise, then T is equiprobably x or $\mu - x$, so the mean time is $T = \mu/2$. Consequently for the circle C we have

$$V(C) = \mu/2. \tag{1}$$

In the network of Fig. 1, the total length of Q is $L(Q) = 12$, which we denote by μ. There is a minimum length (Chinese Postman, or CP) tour $\bar{S}(t)$ which goes from O along (say) the lower arc, then to the left leaf node, then back to O along the upper arc, and then to the right leaf node and back to O. The length of $\bar{S}$ is 18. In general, we will denote the total length $L(Q)$ of a network Q by $\mu = L(Q)$ and the length of the minimal tour by $\bar{\mu}$. So in this example we have $\mu = 12$ and $\bar{\mu} = 18$. We always have $V \leq \bar{\mu}/2$ because if the a CP tour $\bar{S}$ satisfies $T(\bar{S}, H) = t_1$ then for the reverse path $\bar{S}^{-1}(t) = \bar{S}(\bar{\mu} - t)$ we have $\bar{S}^{-1}(\bar{\mu} - t_1) = \bar{S}(\bar{\mu} - t_1) = H$, and hence $T(\bar{S}^{-1}, H) \leq \bar{\mu} - t_1$. Thus the average value of $T(\bar{S}, H)$ and $T(\bar{S}^{-1}, H)$ is no more than $\bar{\mu}/2$. So for the network above, we have $V \leq \bar{\mu}/2 = 9$.

Next suppose the Hider chooses the left leaf node with probability 5/9 and the right leaf node with probability 4/9. An optimal Searcher response is clearly to go to one leaf end and then the other, by shortest paths. Going to the left leaf node first gives $T = (5/9)5 + (4/9)14 = 9$, and the other way around gives $T = (5/9)13 + (4/9)4 = 9$. Thus for the network of Fig. 1 the value of the search game is $9 = \bar{\mu}/2$. We shall call networks Q satisfying $V(Q, O) = \bar{\mu}/2$ for some O *simply searchable*. (If this is satisfied for some O, it turns out it is satisfied for any O.) That is, a network is simply searchable if a randomized CP tour (equiprobable mixture of a CP tour and its reverse tour) is an optimal search strategy.

By Gal's theory of network search games (with an immobile Hider) we mean his topological (combinatorial) characterization of simply searchable networks as those which are *weakly Eulerian* [11]. Roughly speaking, a network is weakly Eulerian is it consists of a disjoint family of Eulerian networks which are connected in a tree-like fashion. More formally, it is weakly Eulerian if the removal of all (open) edges which disconnect it leaves an Eulerian network (in general, not connected). Since the network of Fig. 1 has $V = \bar{\mu}/2 = 9$, it is simply searchable and by Gal's characterization it must be weakly Eulerian. Indeed removing the two arcs which disconnect it (those of lengths 2 and 4) leaves the Eulerian network consisting of a circle of length 6 and the two leaf nodes (which have even degree 0). In fact the network of Fig. 1 has the stronger property of being *weakly cyclic*, which means that between any two points there are at most two disjoint paths. There are two precursors to Gal's

Theorem. First, there is Gal's 1979 result [9] that trees are simply searchable; next there was the important 1993 result of Reijnierse and Potters [17] that weakly cyclic networks are simply searchable (with Gal's Theorem as a conjecture). The reader should note that for networks which are not weakly Eulerian, even such an apparently simple one like two nodes connected by three equal length arcs (see the detailed analysis of Pavlovic [16]) the optimal Searcher and Hider strategies can be extremely complicated, and the search paths may not be combinatorial ones.

This note gives an alternative presentation of the theory of simply searchable networks. We make use of two simple lemmas, presented in Sect. 2. The first, the Search Density Lemma (Lemma 1) says that in quite general search situations *with a known Hider distribution h*, when faced with two alternative disjoint searches (which start and end at a common point), one should first perform the search with higher *search density*. (The search density of a region is simply the probability that the Hider will be found in this region divided by the time taken by the search.) This observation was used extensively in the theory of *alternating search at two locations* developed by Alpern and Howard [7], where one seeks to minimize the time taken to find an object hidden at one of two locations according a known distribution, and at each location the search must be done in a particular order (like digging for the object in two tunnels). The second result we use is the simple Edge-Adding Lemma (Lemma 2) (a special case of a result in [4]) which analyzes the effect on the value of adding a new edge to a network, including edges of length zero (identifying points). This lemma makes it possible to analyze search values entirely in a network (rather than game) context (without referring to particular strategies). Both lemmas can be viewed as formalizations of ideas already implicit in earlier work.

The paper is organized as follows. The approach given in this paper is based on recent work of Alpern [1] and Alpern and Lidbetter [2] concerning a more general search model in which the speed of the Searcher depends on the direction in which he is traveling. Specializing techniques developed in that context leads to the simplified treatment of the original search problem. Section 2 presents the Search-Density and Edge-Adding Lemmas 1 and 2, and gives some examples of their use. Section 3 gives a new proof that trees are simply searchable, relying on the Search-Density Lemma. Section 4 gives a proof via the Edge-Adding Lemma of Gal's Theorem: a network is simply searchable if and only if it is weakly Eulerian. Section 5 concludes.

The approach taken here is informed by the recent work of the author on search games on variable speed networks. These methods seem most suited for extending the Gal Theory to such generalized networks. In particular, we consider Searcher strategies based on branching probabilities at nodes, rather than Randomized Chinese Postman Tours.

2 Search Density and Edge-Adding Lemmas

This section presents two general results that we will use later in various contexts. We begin by fixing a network Q and a Hider distribution (mixed strategy) h on Q. If $S(t)$ is a Searcher path with cumulative capture distribution $F(x) = \Pr(T(S,h) \le x) = h(S[0,x])$, then we have $T(S,h) = \int_0^\infty t \, dF(t)$. Suppose that $a < b < c$ and $S(a) = S(b) = S(c)$ and that S searches (probabilistically) disjoint regions A and B of Q in time intervals $[a,b]$ and $[b,c]$, that is $h(S[a,c]) = h(S[a,b]) + h(S[b,c])$. The following lemma considers the question of when the Searcher will do better (reduce T) by searching in the opposite order. The answer is in terms of what we call the search density. The search density of A is the probability that the Hider is in A (that is, $h(A)$) divided by the time required to search A (in this case $b-a$).

Lemma 1 (Search density) *Fix a network Q and a Hider distribution h. Suppose S (with cumulative capture distribution F) searches disjoint regions A and B in time intervals $[a,b]$ and $[b,c]$, while S' searches in the other order (B during $[a,a+(c-b)]$ and A during $[a+(c-b),c]$).*

$$\text{If } \frac{F(c)-F(b)}{c-b} \geq \frac{F(b)-F(a)}{b-a}, \quad \text{then } T(S,h) \geq T(S',h).$$

In other words the search with higher search density should be carried out first. If two searches have the same search density they can be carried out consecutively in either order.

Proof The difference $T(S,h)-T(S',h)$ is given by

$$\left[\int_a^b t\,dF(t)+\int_b^c t\,dF(t)\right]-\left[\int_a^b \big(t+(c-b)\big)\,dF(t)+\int_b^c \big(t-(b-a)\big)\,dF(t)\right]$$
$$=\int_b^c (b-a)\,dF(t)-\int_a^b (c-b)\,dF(t)$$
$$=(b-a)\big(F(c)-F(b)\big)-(c-b)\big(F(b)-F(a)\big)$$
$$=(b-a)(c-b)\left(\frac{F(c)-F(b)}{c-b}-\frac{F(b)-F(a)}{b-a}\right)\geq 0.$$
□

Our second general lemma concerns the effect on the game value of adding new edges between two points of Q to create a new network Q' (with the same starting point O). This also covers the case where we identify the two points, since we can consider that the edge has length zero. Here d_Q is the metric on the original network Q. Part 3 is based on an observation of my MSc student V. Lalchand for a particular tree. Logically speaking, it doesn't belong in this Lemma, as we don't prove it here and don't use it to prove anything else (we will prove it in Corollary 10). However as a statement of fact, this is obviously the right place for it.

Lemma 2 (Edge-Adding) *Let the network Q' be obtained from a network Q by adding an edge a of length $l \geq 0$ between two points x and y of Q. Then*

1. $V(Q') \leq V(Q)+2l$, *so that* $V(Q') \leq V(Q)$ *if we simply identify the nodes* ($l=0$). *Note that the first part applies even if a is attached only at one of its ends.*
2. *If $l \geq d_Q(x,y)$, then $V(Q') \geq V(Q)$. In particular, any hiding strategy on Q does equally well (in the worst case) on Q'.*
3. *If Q is a tree then $V(Q')-V(Q)=(l-d_Q(x,y))/2$, so that in particular if $l \leq d_Q(x,y)$, then $V(Q') \leq V(Q)$.*

Proof (1) Let s be an optimal strategy in Q, so that $T(s,x) \leq V$. Let s' be the strategy based on s which modifies each pure S so that after the first time it reaches x it goes to y and back along a before continuing with the original path. Then points on a are found in expected time $\leq V+l$ and points in Q are found in expected time $\leq V+2l$, giving (i).

(2) If $l \geq d_Q(x,y)$ then the optimal strategy h in Q guarantees V in Q', as every strategy S' in Q' is no better against h than the strategy S obtained by replacing any use of a by a shortest path in Q between x and y. □

3 Search on Trees

We call Q a tree (more specifically a rooted tree, with root O) if it has no cycles. On a tree, the leaf nodes together dominate all other hiding positions, so we can assume the Hider is at one of these. Every arc a of a tree determines a unique subtree called the *branch* $\sigma(a)$, the set of all arcs (including a) which are connected to the root through a. We define the *Equal Branch Density* (EBD) hider distribution e as the distribution on the leaves which assigns to each set of branches $\sigma(a_1), \dots, \sigma(a_k)$ at a node a probability proportional to the time required to search it (twice its length). That is, the search density $e(\sigma(a_i))/(2L(\sigma(a_i)))$ is the same on all the branches of a node. If $b_1, b_2, \dots, b_m$ is the arc path leading from O to a leaf w, then a recursive calculation shows that

$$e(w) = \frac{L(b_m)}{\mu} \prod_{i=1}^{m-1} \frac{L(\sigma(b_i))}{L(\sigma(b_i)) - L(b_i)}.$$

Figure 2 illustrates the calculation of the EBD distribution on the leaf nodes of a particular tree. The two branches at O both have length 5, so they get equal measure. This continues on the right. But on the left one leaf node gets twice the measure of the other.

A *depth-first* (*DF*) path is a tour S of Q from the root O which always leaves a node by an untraversed forward arc (away from O) whenever possible; otherwise it takes the unique backward arc. The *Random Depth First* (*RDF*) strategy r is the mixed Searcher strategy determined by the behavioral rule of choosing equiprobably among the untraversed forward arcs at a node. Its sample paths are the DF searches. It turns out that the EBD Hider strategy e and the RDF Searcher strategy r form an equilibrium pair. To show this, the next two lemmas show that each is an optimal response to the other. The following result is taken from [2, 3]. A version for networks with more general travel time assumptions is given in [4].

Lemma 3 *Against the EBD Hider distribution e, the optimal responses for the Searcher are the DF paths. Hence any mixture of these, such as the RDF strategy r, is also an optimal response.*

Proof Suppose S is optimal against e and is not DF. Let X be a node furthest from O for which S goes on the backwards arc c from X to its predecessor Y before some forward arc b at X has been traversed. (An illustration is given below in Fig. 3.) Suppose S reaches X at time t_1, then goes

$$\sigma_a, cS'c, \sigma_b, \tag{2}$$

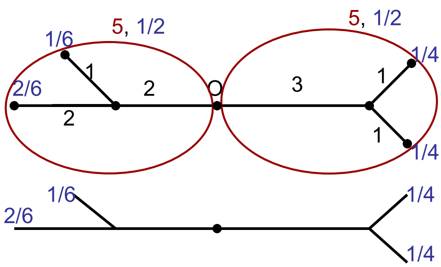
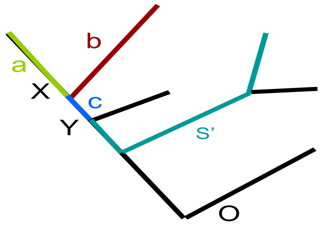


Fig. 2 Calculating the EBD distribution e on a tree

Fig. 3 From Y, S goes $c, \sigma(a), c, S', c, \sigma(b)$



for some path S'. The paths separated by commas are three searches (with disjoint leaves) which start and end at the node X, so they could be carried out in any order. Since S is optimal, the Search Density Lemma implies that their search densities must be nonincreasing. The definition of e ensures that the search densities of σ_a and σ_b are the same, so by the previous remark their density must be the same as that of the intervening search $cS'c$. Now consider the behavior of S from the earlier time that Y is reached just before S goes to X. Here there are three leaf disjoint paths, starting and ending at Y, carried out in the order

$$c\sigma_a c,\, S',\, c\sigma_b c. \tag{3}$$

But these are not in nonincreasing order of search density, as S' has a higher search density than $cS'c$ and the other two have lower search densities than σ_a and σ_b. Thus S is not optimal.

Next we want to show that all DF searches give the same value of T against e, so that they are all optimal responses. Consider the graph with the DF searches as nodes and two searches $S_{a,b}$ and $S_{b,a}$ adjacent if they are identical except that at one node X the first searches two branches σ_a and σ_b consecutively in that order, while the second uses the other order. Note that by the Edge-Adding Lemma and the definition of e, $S_{a,b}$ and $S_{b,a}$ give the same value of T against e. Since this graph is connected, all DF searches have the same value of T against e. □

Lemma 4 *The RDF Search strategy r on a tree Q reaches all leaves in expected time $\mu = \mu(Q)$. Thus all the leaf nodes are optimal Hider responses to r. Hence any mixture of these, such as the EBD strategy e, is also an optimal response.*

Proof (The reader may wish to refer to Fig. 4.) The proposition is obvious when Q has $m = 1$ edge, so assume it is true for networks with fewer than m edges and suppose Q has m edges. Suppose the first branch node is at distance $l \geq 0$ from O and that there are k branches Q_i, with $\mu_i = \mu(Q_i)$. Let v be any leaf node of Q, which we may assume belongs to Q_1. Let T denote the expected time to reach v, and T_i the expected value of T given that branch Q_i is chosen first. By the inductive hypothesis applied to Q_1, we have $T_1 = l + \mu_1$; applied to $Q - Q_i$ we have $T_i = (l + 2\mu_i) + \mu(Q - Q_i) = \mu + \mu_i,\ i > 1$. Hence

$$T = \frac{1}{k}\left((l + \mu_1) + \sum_{i=2}^{k}(\mu + \mu_i)\right) = \mu.$$

□

We note Lemmas 3 and 4 remain valid if the Searcher strategy r is replaced by an equiprobable mixture of any DF tour and its reverse tour. Such strategies based on two DF tours can also be called Random Chinese Postman tours, because for trees the DF tours have minimal length. This is the approach taken by Gal, and it has the advantage that similar ideas can be

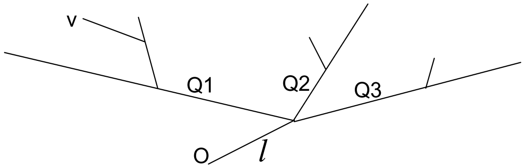


Fig. 4 Illustration for Lemma 4

applied to any network, with μ replaced by $\bar{\mu}/2$, where $\bar{\mu}$ is the minimal time for a tour. Our approach has the advantage that it can be modified to solve the game for trees where the travel times between points may depend on the direction (asymmetric travel times). For such networks, it is not sufficient for the Searcher to employ a single DF tour in a random direction, even if the directional probabilities are not taken to be equiprobable. However by varying the equiprobable assumption on forward untraversed arcs, we can indeed find optimal branching functions for the Searcher on such networks.

Theorem 5 *For any rooted tree Q, O, the mixed strategies (r, e) form an equilibrium pair and the value of the game $\Gamma(Q, O)$ is μ.*

Proof Lemmas 3 and 4 establish that (r, e) is an equilibrium pair. Hence each is optimal and the value of the game is $T(r, e)$, which is μ, by Lemma 4. □

4 Bounds on the Value

In this section we show that every network can be associated with two trees: one with a larger value and one with a smaller value, according to the Edge Adding Lemma.

Theorem 6 *To every network Q of length μ, we can find trees W_1 and W_2, with $L(W_1) = \mu$ and $L(W_2) = \mu/2$, such that*

$$V(W_2) \leq V(Q) \leq V(W_1).$$

Proof To obtain the upper bound, let W_1 be any tree from which Q can be obtained by identification of nodes. (We may make cuts to eliminate every cycle.) Since this process doesn't alter the total length, both Q and W_1 have the same length μ. By the first part of the Edge-Adding Lemma (Lemma 2), we have $V(Q) \leq V(W_1)$.

To obtain the lower bound, let Q^* be the network obtained from Q by identifying all its nodes (including O), so that Q^* consists of a single node O^* with loops of lengths l_i. By the first part of the Edge-Adding Lemma, we have $V(Q^*) \leq V(Q)$. Next let W_2 be the star tree with root O^* and rays of lengths $l_i/2$, so that Q^* can be obtained from W_2 by doubling all the rays, and $L(W_2) = L(Q^*)/2 = \mu/2$. Thus $V(Q^*) \geq V(W_2)$ by the second part of the Edge-Adding Lemma. □

Since Theorem 5 says that the search value of a tree is equal to its length, we have the following consequence.

Corollary 7 *For any network Q of total length μ, we have*

$$\mu/2 \leq V(Q) \leq \mu. \tag{4}$$

Our proof of the lower bound is distinct from Gal's original proof [9], where it is obtained by showing that the uniformly distributed Hider cannot be found in expected time less than $\mu/2$. Our proof is not simpler, but does not involve integration or any explicit notion of strategies—it is entirely by comparisons of networks.

Suppose E is an Eulerian network, with some Eulerian tour S which maps the circle $C = [0, \mu]$ mod μ onto E. We can obtain E from C by identifying points of C with the same image in E, and so by the Edge-Adding Lemma we have $V(E) \le V(C) = \mu/2$, by the observation in (1). Since we showed above that $\mu/2$ is a lower bound for the value, we have

$$V(E) = \mu/2, \quad \text{for any Eulerian network } E. \tag{5}$$

We can improve the upper bound for the value given in (4) by the following argument. For any network Q, let $\bar{E} = \bar{E}_Q$ be the Eulerian network of minimum length $\bar{\mu} = \bar{\mu}(Q)$ which can be obtained from Q by doubling some of its edges. By the second part of the Edge-Adding Lemma, doubling edges cannot decrease the value, so we have by (5) that

$$V(Q) \le V(\bar{E}) = \bar{\mu}/2 \quad \text{for any network } Q.$$

5 Weakly Eulerian Networks

In this section we give a proof of Gal's celebrated result on characterization of the simply searchable networks as the weakly Eulerian networks.

Definition 8 If $V(Q) = \bar{\mu}/2$, we say that Q is simply searchable.

Note that our previous work shows that both trees and Eulerian networks are simply searchable. We now establish the search value of weakly Eulerian networks.

Theorem 9 *If Q is weakly Eulerian then $V(Q) = \bar{\mu}/2$.*

Proof (The reader may wish to refer to Fig. 5.) Given Q, we define two new networks Q^* and Q^{**} with the same $\bar{\mu}$ and values denoted by V^* and V^{**}. Define Q^* by identifying all nodes on each of the Eulerian subgraphs E_i of Q and calling the new node v_i. Since Q^* is obtained by identifying nodes of Q we have $V^* \le V$ by part (i) of the Edge-Adding Lemma. The arcs of E_i become in Q^* loops λ_{ij} at v_i which we denote as two equal length arcs a_{ij} and a'_{ij} between v_i and the midpoint of the loop. Define Q^{**} by deleting the arcs a'_{ij} of Q^* to leave a tree, so $V^{**} = \bar{\mu}/2$. Reversing the construction, Q^* is obtained from Q^{**} by doubling the arcs a_{ij}, so it follows from part (ii) of the 'Arc-adding Lemma' that $V^* \ge V^{**}$. Recalling that we always have $V \le \bar{\mu}/2$, it follows from our two previous inequalities that

$$\bar{\mu}/2 = V^{**} \le V^* \le V \le \bar{\mu}/2, \quad \text{so } V = \bar{\mu}/2. \qquad \square$$

The networks Q^* and Q^{**} corresponding to a particular weakly Eulerian network Q are shown below in Fig. 5. We call this process *identify-cut-delete*.

We can now very simply establish part 3 of the Edge-Adding Lemma.

Corollary 10 *Suppose Q' is obtained from a tree Q by adding an arc e between two points x and y of Q. Then $V(Q') - V(Q) = (l - d_Q(x, y))/2$, so that in particular if $l \le d_Q(x, y)$, then $V(Q') \le V(Q)$.*

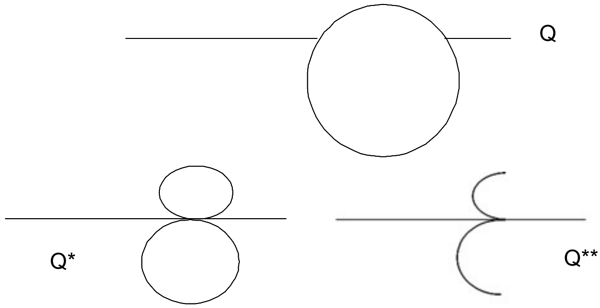


Fig. 5 Identify-cut-delete applied to Q

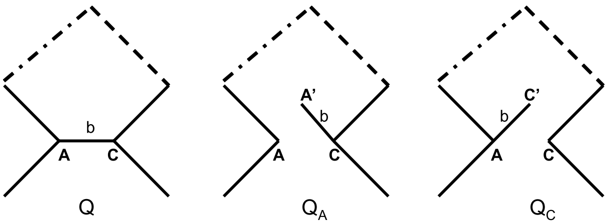


Fig. 6 Networks Q_A and Q_C for Q

Proof The network Q' comprises of a cycle of length $d+l$, where $d = d_Q(x, y)$, consisting of the path between x and y in Q and the edge e, together with disjoint trees of total length $\mu - d$ attached to the cycle at single points. Consequently Q' is weakly Eulerian and $\bar{\mu}' = (d+l) + 2(\mu - d)$ and we have

$$V(Q') = \frac{(d+l) + 2(\mu - d)}{2} = \mu + \frac{l-d}{2} = V(Q) + \frac{l-d}{2}. \qquad \square$$

To see that this result does not hold in general, let Q consist of two unit arcs between the root node O and another node A, so that $V(Q) = 1$. If we add another unit length arc e between O and A, we have the celebrated 3−arc network Q' whose value V' is at least 3/2 by Corollary 7. In fact, L. Pavlovic has shown that $V' = (4 + \ln 2)/3 = 1.5644$, a deep result involving complicated backtracking strategies.

Theorem 11 *If h is a mixed hiding strategy on Q with $T(S, h) \geq \bar{\mu}/2$ for all S, then Q is weakly Eulerian.*

Proof Fix any CP Tour of Q and let Q_1 and Q_2 denote the arcs traversed once and twice by it. We know that $Q - Q_2$ is the union of disjoint Eulerian tours and hence that $Q - a$ is connected if a is in Q_1. So by the (alternate) definition of WE, we must show that arcs in Q_2 disconnect Q. So suppose, on the contrary, that there is an arc $b = AC$ of Q_2 such that $Q - b$ is connected. Let Q_A be the network resulting from disconnecting b from A and adding a new leaf node A', with Q_C the same for C. All three networks Q, Q_A and Q_B have the same value of $\bar{\mu} = L(Q_1) + 2L(Q_2)$. (See Fig. 6.)

Since h guarantees that $T \geq \bar{\mu}/2$ in Q,it does the same in both Q_A and Q_B and hence is optimal in both. In $G(Q_A)$ points $b - A'$ are dominated and in $G(Q_C)$ those in $b - C'$ are dominated. So no point of b is undominated in both games, and hence h gives zero probability to hiding in b. But for such an h, a randomized CPT of $Q - b$ guarantees an expected capture time $T \leq (\bar{\mu} - 2\,L(b))/2 < \bar{\mu}/2$, contradicting the definition of h. $\square$

If Q has the value $\bar{\mu}/2$ and there is an optimal (rather than ε-optimal) strategy for the Hider, then Theorem 11 show that Q must be weakly Eulerian. But the proof is easily modified to the case where the Hider only has an ε-optimal strategy, leading to a new proof of the following.

Theorem 12 (Gal) *If $V = \bar{\mu}/2$ then Q is weakly Eulerian.*

The same proof as above works if h guarantees that $T(S,h) \geq \bar{\mu}/2 \; - \varepsilon$. Let $p = p_\varepsilon = h(b)$ be the probability the Hider is in b and let $l = L(b)$ denote the length of b. Without loss of generality we may assume that the center of gravity of h on b is at least as close to C as to A for a sequence of ε tending to 0. Let s be a RCP Tour of Q_A, so that $T(s,H) \leq \bar{\mu}/2$ for all H. For H in b we have $T(s,H) = T(s,A) - d(A,H)$ and the mean on $H \in b$ of $d(A,H)$ is at least $l/2$. Hence

$$T(s,h) \leq (1-p)\frac{\bar{\mu}}{2} + p\left(\frac{\bar{\mu}}{2} - \frac{l}{2}\right) = \frac{\bar{\mu}}{2} - \frac{pl}{2}, \quad \text{so } p_\varepsilon \leq \frac{2\varepsilon}{l} \to 0.$$

Let s' be a RCP tour of $Q - b$ followed by a path which reaches A' at some finite time M $(\leq 3\bar{\mu})$. Then

$$T(s',h) \leq (1-p_\varepsilon)\frac{\bar{\mu}-2l}{2} + p_\varepsilon M = \frac{\bar{\mu}}{2} - l + p_\varepsilon\left(M - \frac{\bar{\mu}-2l}{2}\right) \to \frac{\bar{\mu}}{2} - l$$

which leads to the same contradiction as in the proof of Theorem 11 for ε sufficiently small.

Related work on search games with immobile Hiders can be found in [8, 12, 14, 15].

6 Conclusion

The constructions and concepts that have been developed to extend Gal's theory of search games to networks with variable travel times [1, 2, 4] can also be used to give new insights into Gal's original context. Learning the primary theory in this manner will simplify the reading and understanding of the newer results. It should be emphasized however that there are still many outstanding questions for networks which are not weakly Eulerian even in the original setting.

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