Mathematical Game Theory, Autumn 2020

Steve Alpern

Notes for first half of the course.

For the first part of the course, on Combinatorial Games like Nim, the following notes of Prof Bernhard von Stengel from a course we jointly taught at the LSE will be useful. Prof. von Stengel has kindly given me permission to use these notes as well as the lecture slides from the relevant portion of the LSE course. (The second part of the course, on Search Games, will be based on my book with S. Gal and my own slides.)

Game Theory Basics

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February 15, 2011

The material in this book has been adapted and developed from material originally produced for the BSc Mathematics and Economics by distance learning offered by the University of London External Programme (www.londonexternal.ac.uk).

I am grateful to George Zouros for his valuable comments, and to Rahul Savani for his continuing interest in this project, his thorough reading of earlier drafts, and the fun we have in joint research and writing on game theory.

Preface

Game theory is the formal study of conflict and cooperation. It is concerned with situations where "players" interact, so that it matters to each player what the other players do. Game theory provides mathematical tools to model, structure and analyse such interactive scenarios. The players may be, for example, competing firms, political voters, mating animals, or buyers and sellers on the internet. The language and concepts of game theory are widely used in economics, political science, biology, and computer science, to name just a few disciplines.

Game theory helps to understand effects of interaction that seem puzzling at first. For example, the famous "prisoners' dilemma" explains why fishers can exhaust their resources by over-fishing: they hurt themselves collectively, but each fisher on his own cannot really change this and still profits by fishing as much as possible. Other insights come from the way of looking at interactive situations. Game theory treats players equally and recommends to each player how to play well, given what the other players do. This mindset is useful in strategic questions of management, because "you put yourself in your opponent's shoes".

Game theory is fascinating as a topic because of its diverse applications. The ideas of game theory started with mathematicians, most notably the outstanding mathematician John von Neumann (1903–1957). In the 1950s, a group of young researchers in mathematics at Princeton developed game theory further, among them John Nash, Harold Kuhn and Lloyd Shapley, and these pioneers can still, over 50 years later, be met at conferences on game theory. Most research in game theory is now done by economists and other social scientists.

My own interests are in the mathematics of games, so I see my own research in the tradition of early game theory. My research specialty is the connection of game theory to computer science. In particular, I develop methods to find equilibria of games, which make use of insights from geometry. That interest is partly reflected in the choice of topics in this book. This book has also a strong emphasis on *methods*, so you will learn a lot of "tricks" that allow you to understand games quickly. With these methods at hand, you will be in a position to analyse games that you can create for applications to management or economics.

Structure of this book

Chapter 1 on combinatorial games is on playing and winning games with perfect information defined by rules, in particular a simple game called "nim", which has a central role in that theory. This chapter introduces abstract mathematics with a fun topic. You would probably not learn its content otherwise, because it does not have a typical application in economics. However, every game theorist should know the basics of combinatorial games. In fact, this can be seen as the motto for the contents of this book: *what every game theorist should know*.

Chapter 1 on combinatorial games is independent of the other material. This topic is deliberately put at the beginning because it is more mathematical and formal than the other topics, so that it can be used to test whether you can cope with the abstract parts of game theory, and with the mathematical style of this subject.

Chapters 2, 3, and 4 successively build on each other. These chapters cover the main concepts and methods of non-cooperative game theory.

With the exception of imperfect information, the fundamentals of non-cooperative game theory are laid out in chapter 2. This part of game theory provides ways to model in detail the agents in an interactive situation, their possible actions, and their incentives. The model is called a *game* and the agents are called *players*. There are two types of games, called *game trees* and games in *strategic form*. The game tree (also called the *extensive form* of a game) describes in depth the actions that are available to the players, how these evolve over time, and what the players know or do not know about the game. (Games with imperfect information are treated in chapter 4.) The players' incentives are modelled by payoffs that these players want to maximise, which is the sole guiding principle in non-cooperative game theory. In contrast, cooperative game theory studies, for example, how players should split their proceeds when they decide to cooperate, but leaves it open how they enforce an agreement. (A simple example of this cooperative approach is explained in chapter 5 on bargaining.)

Chapter 3 shows that in certain games it may be useful to leave your actions uncertain. A nice example is the football penalty kick, which serves as our introduction to zero-sum games (see figure 3.11). The striker should not always kick into the same corner, nor should the goalkeeper always jump into the same corner, even if they are better at scoring or saving a penalty there. It is better to be unpredictable! Game theory tells the players how to choose optimal *probabilities* for each of their available strategies, which are then used to *mix* these strategies randomly. With the help of mixed strategies, every game has an *equilibrium*. This is the central result of John Nash, who discovered the equilibrium concept in 1950 for general games. For zero-sum games, this was already found earlier by John von Neumann. It is easier to prove for zero-sum games that they have an equilibrium than for general games. In a logical progression of topics, in particular when starting from win/lose games like nim, we could have treated zero-sum games before general games. However, we choose to treat zero-sum games later, as special cases of general games, because the latter are much more important in economics. In a course on game theory, one could omit zero-sum games and their special properties, which is why we treat them in the last section of chapter 3. However, one could not omit the concept of Nash equilibrium, which is therefore given prominence early on.

Chapter 4 explains how to model the *information* that players have in a game. This is done by means of so-called information sets in game trees, introduced in 1953 by Harold Kuhn. The central result of this chapter is called Kuhn's theorem. Essentially, this result states that players can choose a "behaviour strategy", which is a way of playing the game that is not too complicated, provided they do not forget what they knew and did earlier. This result is typically considered as technical, and given short shrift in many game theory texts. We go into great detail in explaining this result. The first reason is that it is a beautiful result of discrete mathematics, because elementary concepts like the game tree and the information sets are combined naturally to give a new result. Secondly, the result is used in other, more elaborate "dynamic" games that develop over time. For more advanced studies of game theory, it is therefore useful to have a solid understanding of game trees with imperfect information.

The final chapter 5 on bargaining is a particularly interesting application of non-cooperative theory. It is partly independent of chapters 3 and 4, so to a large extent it can be understood after chapter 2. A first model provides conditions – called axioms – that an acceptable "solution" to bargaining situations should fulfil, and shows that these axioms lead to a unique solution of this kind. A second model of "alternating offers" is more detailed and uses, in particular, the analysis of game trees with perfect information introduced in chapter 2. The "bargaining solution" is thereby given an incentive-based justification with a more detailed non-cooperative model.

Game theory, and this text, use only a few *prerequisites* from elementary linear algebra, probability theory, and some analysis. You should know that vectors and points have a *geometric interpretation* (for example, as either points or vectors in three-dimensional space if the dimension is three). It should be clear how to multiply matrices, and how to multiply a matrix with a vector. The required notions from probability theory are that of *expected value* of a function (that is, function values weighted with their probabilities), and that *independent* events have a probability that is the product of the probabilities of the individual events. The concepts from analysis are those of a *continuous* function, which, for example, assumes its maximum on a compact (closed and bounded) domain. None of these concepts is difficult, and you are reminded of their basic ideas in the text whenever they are needed.

We introduce and illustrate each concept with examples, many pictures, and a minimum of notation. Not every concept is defined with a formal definition (although many concepts are), but instead explained by means of an example. It should in each case be clear how the concept applies in similar settings, and in general. On the other hand, this requires some maturity in dealing with mathematical concepts, and in being able to generalise from examples.

Distinction to other textbooks

Game theory has seen an explosion in interest in recent years. Correspondingly, many new textbooks on game theory have recently appeared. The present text is complementary (or "orthogonal") to most existing textbooks.

First of all, it is mathematical in spirit, meaning it can be used as a textbook for teaching game theory as a mathematics course. On the other hand, the mathematics in this book should be accessible enough to students of economics, management, and other social sciences.

The book is relatively swift in treating games in strategic form and game trees in a single chapter (chapter 3), which are often considered separately. I find these concepts simple enough to allow for that approach.

Mixed strategies and the best response condition are very useful for finding equilibria in games, so they are given detailed treatment. Similarly, game trees with imperfect information, and the concepts of perfect recall and behaviour strategies, and Kuhn's theorem, are also given an in-depth treatment, because this not found in many textbooks at this level.

The book does not try to be comprehensive in presenting the most relevant economic applications of game theory. Corresponding "stories" are only told for illustration of the concepts.

One book served as a starting point for the selection of the material, namely

- K. Binmore, *Fun and Games*, D. C. Heath, 1991. Its recent, significantly revised version is *Playing for Real*, Oxford University Press, 2007.

This book describes nim as a game that one can learn to play perfectly. In turn, this led me to the question of why the binary system has such an important role, and towards the study of combinatorial games. The topic of bargaining is also motivated by Binmore's book. At the same time, the present text is very different from Binmore's book, because it tries to treat as few side topics as possible.

Two short sections can be considered as slightly non-standard side topics, namely the sections 2.9, Symmetries involving strategies, and 4.9 Perfect recall and order of moves, both marked with a star as "can be skipped at first reading". I included them as topics that I have not found elsewhere. Section 4.9 demonstrates how to reason carefully about moves in extensive games, and should help understand the perfect recall condition.

Methods, not philosophy

The general emphasis of the book is to teach *methods, not philosophy*. Game theorists tend to question and to justify the approaches they take, for example the concept of Nash equilibrium, and the assumed common knowledge of all players about the rules of the game. These questions are of course very important. In fact, in practice these are probably the very issues that make a game-theoretic analysis questionable. However, this problem is not remedied by a lengthy discussion of why one should play Nash equilibrium.

I think a student who learns about game theory should first become fluent in knowing the modelling tools (such as game trees and the strategic form) and in analysing games (finding their Nash equilibria). That toolbox will then be useful when comparing different game-theoretic models and looking at their implications. In many disciplines that are further away from mathematics, these possible implications are typically more interesting than the mathematical model itself (the model is necessarily imperfect, so there is no point in being dogmatic about the analysis).

Contents

Chapter 1

Nim and combinatorial games

1.1 Aims of the chapter

This chapter introduces the basics of combinatorial games, and explains the central role of the game nim. A detailed summary of the chapter is given in section 1.4.

Furthermore, this chapter demonstrates the use of abstract mathematics in game theory. This chapter is written more formally than the other chapters, in parts in the traditional mathematical style of definitions, theorems and proofs. One reason for doing this, and why we start with combinatorial games, is that this topic and style serves as a warning shot to those who think that game theory, and this text in particular, is "easy". If we started with the well-known "prisoner's dilemma" (which makes its due appearance in Chapter 2), the less formally inclined student might be lulled into a false sense of familiarity and "understanding". We therefore start deliberately with an unfamiliar topic.

This is a mathematics text, with great emphasis on rigour and clarity, and on using mathematical notions precisely. As mathematical prerequisites, game theory requires only the very basics of linear algebra, calculus and probability theory. However, game theory provides its own conceptual tools that are used to model and analyse interactive situations. This text emphasises the mathematical structure of these concepts, which belong to "discrete mathematics". Learning a number of new mathematical concepts is exemplified by combinatorial game theory, and it will continue in the study of classical game theory in the later chapters.

1.2 Learning objectives

After studying this chapter, you should be able to:

- play nim optimally;
- explain the concepts of game-sums, equivalent games, nim values and the mex rule;
- apply these concepts to play other impartial games like those described in the exercises.

1.3 Further reading

Very few textbooks on game theory deal with combinatorial games. An exception is chapter 1 of the following book:

- Mendelson, Elliot *Introducing Game Theory and Its Applications.* (Chapman & Hall / CRC, 2004) [ISBN 1584883006].

The winning strategy for the game nim based on the binary system was first described in the following article, which is available electronically from the JSTOR archive:

- Bouton, Charles "Nim, a game with a complete mathematical theory." *The Annals of Mathematics*, 2nd Ser., Vol. 3, No. 1/4 (1902), pp. 35–39.

The definitive text on combinatorial game theory is the set of volumes "Winning Ways" by Berlekamp, Conway and Guy. The material of this chapter appears in the first volume:

- Berlekamp, Elwyn R., John H. Conway and Richard K. Guy *Winning Ways for Your Mathematical Plays, Volume 1*, second edition. (A. K. Peters, 2001) [ISBN 1568811306].

Some small pieces of that text have been copied here nearly verbatim, for example in Sections 1.6, 1.8, and 1.11 below.

The four volumes of "Winning Ways" are beautiful books. However, they are not suitable reading for a beginner, because the mathematics is hard, and the reader is confronted with a wealth of material. The introduction to combinatorial game theory given here represents a very small fraction of that body of work, but may invite you to study it further.

A very informative and entertaining mathematical tour of parlour games is

- Bewersdorff, Jörg *Logic, Luck and White Lies.* (A. K. Peters, 2005) [ISBN 1568812108].

Combinatorial games are treated in part II of that book.

1.4 What is combinatorial game theory?

This chapter is on the topic of *combinatorial games*. These are games with two players, perfect information, and no chance moves, specified by certain rules. Familiar games of this sort are chess, go, checkers, tic-tac-toe, dots-and-boxes, and nim. Such games can be played perfectly in the sense that either one player can force a win or both can force a draw. In reality, games like chess and go are too complex to find an optimal strategy, and they derive their attraction from the fact that (so far) it is not known how to play them perfectly. We will, however, learn how to play nim perfectly.

There is a "classical" game theory with applications in economics which is very different from combinatorial game theory. The games in classical game theory are typically formal models of conflict *and* cooperation which cannot only be lost or won, and in which there is often no perfect information about past and future moves. To the economist, combinatorial games are not very interesting. Chapters 2–5 of the book are concerned with classical game theory.

Why, then, study combinatorial games at all in a text that is mostly about classical game theory, and which aims to provide an insight into the theory of games as used in economics? The reason is that combinatorial games have a rich and interesting mathematical theory. We will explain the basics of that theory, in particular the central role of the game nim for impartial games. It is non-trivial mathematics, it is fun, and you, the student, will have learned something that you would most likely not have learned otherwise.

The first "trick" from combinatorial game theory is how to win in the game nim, using the binary system. Historically, that winning strategy was discovered first (published by Charles Bouton in 1902). Only later did the central importance of nim, in what is known as the Sprague–Grundy theory of impartial games, become apparent. It also revealed why the binary system is important (and not, say, the ternary system, where numbers are written in base three), and learning that is more satisfying than just learning how to use it.

In this chapter, we first define the game nim and more general classes of games with perfect information. These are games where every player knows exactly the state of the game. We then define and study the concepts listed in the learning outcomes above, which are the concepts of game-sums, equivalent games, nim values and the mex rule. It is best to learn these concepts by following the chapter in detail. We give a brief summary here, which will make more sense, and should be re-consulted, after a first study of the chapter (so do not despair if you do not understand this summary).

Mathematically, any game is defined by other "games" that a player can reach in his first move. These games are called the *options* of the game. This seemingly circular definition of a "game" is sound because the options are *simpler* games, which need fewer moves in total until they end. The definition is therefore not circular, but *recursive*, and the mathematical tool to argue about such games is that of mathematical *induction*, which will be used extensively (it will also recur in chapter 2 as "backward induction" for game trees). Here, it is very helpful to be familiar with mathematical induction for proving statements about natural numbers.

We focus here on *impartial* games, where the available moves are the same no matter whether player I or player II is the player to make a move. Games are "combined" by the simple rule that a player can make a move in exactly one of the games, which defines a *sum* of these games. In a "losing game", the first player to move loses (assuming, as always, that both players play as well as they can). An impartial game added to itself is always losing, because any move can be copied in the other game, so that the second player always has a move left. This is known as the "copycat" principle (lemma 1.6). An important observation is that a losing game can be "added" (via the game-sum operation) to any game without changing the winning or losing properties of the original game.

In section 1.10, the central theorem 1.10 explains the winning strategy in nim. The importance of nim for impartial games is then developed in section 1.11 via the beautiful mex rule. After the comparatively hard work of the earlier sections, we almost instantly obtain that any impartial game is equivalent to a nim heap (corollary 1.13).

At the end of the chapter, the sizes of these equivalent nim heaps (called nim values) are computed for some examples of impartial games. Many other examples are studied in the exercises.

Our exposition is distinct from the classic text “Winning Ways” in the following respects: First, we only consider impartial games, even though many aspects carry over to more general combinatorial games. Secondly, we use a precise definition of *equivalent* games (see section 1.9), because a game where you are bound to lose against a smart opponent is not the *same* as a game where you have already lost. Two such games are merely equivalent, and the notion of equivalent games is helpful in understanding the theory. So this text is much more restricted, but to some extent more precise than “Winning Ways”, which should help make this topic accessible and enjoyable.

1.5 Nim – rules

The game nim is played with heaps (or piles) of chips (or counters, beans, pebbles, matches). Players alternate in making a move, by removing some chips from one of the heaps (at least one chip, possibly the entire heap). The first player who cannot move any more loses the game.

The players will be called, rather unimaginatively, player I and player II, with player I to start the game.

For example, consider three heaps of size $1,1,2$. What is a good move? Removing one of the chips from the heap with two chips will create the position $1,1,1$, then player II must move to $1,1$, then player I to 1, and then player II takes the last chip and wins. So this is not a good opening move. The winning move is to remove all chips from the heap of size 2, to reach position $1,1$, and then player I will win. Hence we call $1,1,2$ a *winning position*, and $1,1$ a *losing position*.

When moving in a winning position, the player to move can win by playing well, by moving to a losing position of the other player. In a losing position, the player to move will lose no matter what move she chooses, if her opponent plays well. This means that *all* moves from a losing position lead to a winning position of the opponent. In contrast, one needs only *one* good move from a winning position that goes to a losing position of the next player.

Another winning position consists of three nim heaps of sizes $1,1,1$. Here all moves result in the same position and player I always wins. In general, a player in a winning position must play well by picking the right move. We assume that players play well, forcing a win if they can.

Suppose nim is played with only two heaps. If the two heaps have equal size, for example in position $4,4$, then the first player to move loses (so this is a losing position), because player II can always *copy* player I’s move by equalising the two heaps. If the two heaps have different sizes, then player I can equalise them by removing an appropriate number of chips from the larger heap, putting player II in a losing position. The rule for 2-heap nim is therefore:

Lemma 1.1 *The nim position m,n is winning if and only if $m \neq n$, otherwise losing, for all $m,n \geq 0$.*

This lemma applies also when $m = 0$ or $n = 0$, and thus includes the cases that one or both heap sizes are zero (meaning only one heap or no heap at all).

With three or more heaps, nim becomes more difficult. For example, it is not immediately clear if, say, positions $1,4,5$ or $2,3,6$ are winning or losing positions.

⇒ At this point, you should try exercise 1.1(a) on page 20.

1.6 Combinatorial games, in particular impartial games

The games we study in this chapter have, like nim, the following properties:

1. There are just two players.
2. There are several, usually finitely many, *positions*, and sometimes a particular *starting position*.
3. There are clearly defined *rules* that specify the *moves* that either player can make from a given position to the possible new positions, which are called the *options* of that position.
4. The two players move alternately, in the game as a whole.
5. In the *normal play* convention a player unable to move loses.
6. The rules are such that play will always come to an end because some player will be unable to move. This is called the *ending condition*. So there can be no games which are drawn by repetition of moves.
7. Both players know what is going on, so there is *perfect information*.
8. There are no *chance moves* such as rolling dice or shuffling cards.
9. The game is *impartial*, that is, the possible moves of a player only depend on the position but not on the player.

As a negation of condition 5, there is also the *misère play* convention where a player unable to move *wins*. In the surrealist (and unsettling) movie “Last year at Marienbad” by Alain Resnais from 1962, misère nim is played, several times, with rows of matches of sizes $1,3,5,7$. If you have a chance, try to watch that movie and spot when the other player (not the guy who brought the matches) makes a mistake! Note that this is misère nim, not nim, but you will be able to find out how to play it once you know how to play nim. (For games other than nim, normal play and misère versions are typically not so similar.)

In contrast to condition 9, games where the available moves depend on the player (as in chess where one player can only move white pieces and the other only black pieces) are called *partisan* games. Much of combinatorial game theory is about partisan games, which we do not consider to keep matters simple.

Chess, and the somewhat simpler tic-tac-toe, also fail condition 6 because they may end in a tie or draw. The card game poker does not have perfect information (as required in 7) and would lose all its interest if it had. The analysis of poker, although it is also a win-or-lose game, leads to the "classical" theory of zero-sum games (with imperfect information) that we will consider later. The board game backgammon is a game with perfect information but with chance moves (violating condition 8) because dice are rolled.

We will be relatively informal in style, but our notions are precise. In condition 3 above, for example, the term *option* refers to a position that is reachable in one move from the current position; do not use "option" when you mean "move". Similarly, we will later use the term *strategy* to define a plan of moves, one for every position that can occur in the game. Do not use "strategy" when you mean "move". However, we will take some liberty in identifying a game with its starting position when the rules of the game are clear.

⇒ Try now exercises 1.2 and 1.3 starting on page 20.

1.7 Simpler games and notation for nim heaps

A game, like nim, is defined by its rules, and a particular starting position. Let G be such a particular instance of nim, say with the starting position $1,1,2$. Knowing the rules, we can identify G with its starting position. Then the options of G are $1,2$, and $1,1,1$, and $1,1$. Here, position $1,2$ is obtained by removing either the first or the second heap with one chip only, which gives the same result. Positions $1,1,1$ and $1,1$ are obtained by making a move in the heap of size two. It is useful to list the options systematically, considering one heap to move in at a time, so as not to overlook any option.

Each of the options of G is the starting position of another instance of nim, defining one of the new games H, J, K, say. We can also say that G is defined by the moves to these games H, J, K, and we call these *games* also the *options* of G (by identifying them with their starting positions; recall that the term "option" has been defined in point 3 of section 1.6).

That is, we can define a game as follows: Either the game has no move, and the player to move loses, or a game is given by one or several possible moves to new games, in which the other player makes the initial move. In our example, G is defined by the possible moves to H, J, or K. With this definition, the entire game is completely specified by listing the initial moves and what games they lead to, because all subsequent use of the rules is encoded in those games.

This is a *recursive* definition because a "game" is defined in terms of "game" itself. We have to add the *ending* condition that states that every sequence of moves in a game must eventually end, to make sure that a game cannot go on indefinitely.

This recursive condition is similar to defining the set of natural numbers as follows: (a) 0 is a natural number; (b) if n is a natural number, then so is $n+1$; and (c) all natural numbers are obtained in this way, starting from 0. Condition (c) can be formalised by the principle of induction that says: if a property $P(n)$ is true for $n = 0$, and if the property $P(n)$ implies $P(n+1)$, then it is true for all natural numbers.

We use the following *notation for nim heaps*. If G is a single nim heap with n chips, $n \geq 0$, then we denote this game by $*n$. This game is completely specified by its options, and they are:

$$\text{options of } *n: \quad *0,\ *1,\ *2, \ldots,\ *(n-1). \tag{1.1}$$

Note that $*0$ is the empty heap with no chips, which allows no moves. It is invisible when playing nim, but it is useful to have a notation for it because it defines the most basic losing position. (In combinatorial game theory, the game with no moves, which is the empty nim heap $*0$, is often simply denoted as 0.)

We could use (1.1) as the definition of $*n$; for example, the game $*4$ is defined by its options $*0, *1, *2, *3$. It is very important to include $*0$ in that list of options, because it means that $*4$ has a winning move. Condition (1.1) is a recursive definition of the game $*n$, because its options are also defined by reference to such games $*k$, for numbers k smaller than n. This game fulfils the ending condition because the heap gets successively smaller in any sequence of moves.

If G is a game and H is a game reachable by one or more successive moves from the starting position of G, then the game H is called *simpler* than G. We will often prove a property of games inductively, using the assumption that the property applies to all simpler games. An example is the – already stated and rather obvious – property that one of the two players can force a win. (Note that this applies to games where winning or losing are the only two outcomes for a player, as implied by the "normal play" convention in 5 above.)

Lemma 1.2 *In any game G, either the starting player I can force a win, or player II can force a win.*

Proof. When the game has no moves, player I loses and player II wins. Now assume that G does have options, which are simpler games. By inductive assumption, in each of these games one of the two players can force a win. If, in all of them, the starting player (which is player II in G) can force a win, then she will win in G by playing accordingly. Otherwise, at least one of the starting moves in G leads to a game G' where the second-moving player in G' (which is player I in G) can force a win, and by making that move, player I will force a win in G. □

If in G, player I can force a win, its starting position is a winning position, and we call G a *winning game*. If player II can force a win, G starts with a losing position, and we call G a *losing game*.

1.8 Sums of games

We continue our discussion of nim. Suppose the starting position has heap sizes $1, 5, 5$. Then the obvious good move is to option $5, 5$, which is losing.

What about nim with four heaps of sizes $2,2,6,6$? This is losing, because $2,2$ and $6,6$ independently are losing positions, and any move in a heap of size 2 can be copied in the other heap of size 2, and similarly for the heaps of size 6. There is a second way of looking at this example, where it is not just two losing games put together: consider the game with heap sizes $2,6$. This is a winning game. However, two such winning games, put together to give the game $2,6,2,6$, result in a losing game, because any move in one of the games $2,6$, for example to $2,4$, can be copied in the other game, also to $2,4$, giving the new position $2,4,2,4$. So the second player, who plays "copycat", always has a move left (the copying move) and hence cannot lose.

Definition 1.3 The *sum* of two games G and H, written $G+H$, is defined as follows: The player may move in either G or H as allowed in that game, leaving the position in the other game unchanged.

Note that $G+H$ is a notation that applies here to *games* and not to numbers, even if the games are in some way defined using numbers (for example as nim heaps). The result is a new game.

More formally, assume that G and H are defined in terms of their options (via moves from the starting position) $G_1, G_2, \ldots, G_k$ and $H_1, H_2, \ldots, H_m$, respectively. Then the options of $G+H$ are given as

$$\text{options of } G+H: \quad G_1+H, \ldots, G_k+H, G+H_1, \ldots, G+H_m\,. \tag{1.2}$$

The first list of options G_1+H, $G_2+H, \ldots, G_k+H$ in (1.2) simply means that the player makes his move in G, the second list $G+H_1$, $G+H_2, \ldots, G+H_m$ that he makes his move in H.

We can define the game nim as a sum of nim heaps, where any single nim heap is recursively defined in terms of its options by (1.1). So the game nim with heaps of size $1,4,6$ is written as $*1+*4+*6$.

The "addition" of games with the abstract $+$ operation leads to an interesting connection of combinatorial games with abstract algebra. If you are somewhat familiar with the concept of an abstract *group*, you will enjoy this connection; if not, you do not need to worry, because this connection it is not essential for our development of the theory.

A group is a set with a binary operation $+$ that fulfils three properties:

1. The operation $+$ is *associative*, that is, $G+(J+K) = (G+J)+K$ holds for all G,J,K.
2. The operation $+$ has a *neutral element* 0, so that $G+0 = G$ and $0+G = G$ for all G.
3. Every element G has an *inverse* $-G$ so that $G+(-G) = 0$.

Furthermore,

4. The group is called *commutative* (or "abelian") if $G+H = H+G$ holds for all G,H.

Familiar groups in mathematics are, for example, the set of integers with addition, or the set of positive real numbers with multiplication (where the multiplication operation is written as $\cdot$, the neutral element is 1, and the inverse of G is written as G^{-1}).

The games that we consider form a group as well. In the way the sum of two games G and H is defined, $G+H$ and $H+G$ define the same game, so $+$ is commutative. Moreover, when one of these games is itself a sum of games, for example $H=J+K$, then $G+H$ is $G+(J+K)$ which means the player can make a move in exactly one of the games G, J, or K. This means obviously the same as the sum of games $(G+J)+K$, that is, $+$ is associative. The sum $G+(J+K)$, which is the same as $(G+J)+K$, can therefore be written unambiguously as $G+J+K$.

An obvious neutral element is the empty nim heap $*0$, because it is "invisible" (it allows no moves), and adding it to any game G does not change the game.

However, there is no direct way to get an inverse operation because for any game G which has some options, if one adds any other game H to it (the intention being that H is the inverse $-G$), then $G+H$ will have some options (namely at least the options of moving in G and leaving H unchanged), so that $G+H$ is not equal to the empty nim heap.

The way out of this is to identify games that are "equivalent" in a certain sense. We will see shortly that if $G+H$ is a losing game (where the first player to move cannot force a win), then that losing game is "equivalent" to $*0$, so that H fulfils the role of an inverse of G.

1.9 Equivalent games

There is a neutral element that can be added to any game G without changing it. By definition, because it allows no moves, it is the empty nim heap $*0$:

$$G+*0=G. \tag{1.3}$$

However, other games can also serve as neutral elements for the addition of games. We will see that any losing game can serve that purpose, provided we consider certain games as equivalent according to the following definition.

Definition 1.4 Two games G,H are called *equivalent*, written $G\equiv H$, if and only if for any other game J, the sum $G+J$ is losing if and only if $H+J$ is losing.

In definition 1.4, we can also say that $G\equiv H$ if for any other game J, the sum $G+J$ is winning if and only if $H+J$ is winning. In other words, G is equivalent to H if, whenever G appears in a sum $G+J$ of games, then G can be replaced by H without changing whether $G+J$ is winning or losing.

One can verify easily that $\equiv$ is indeed an equivalence relation, meaning it is reflexive ($G\equiv G$), symmetric ($G\equiv H$ implies $H\equiv G$), and transitive ($G\equiv H$ and $H\equiv K$ imply $G\equiv K$; all these conditions hold for all games G,H,K).

Using $J=*0$ in definition 1.4 and (1.3), $G\equiv H$ implies that G is losing if and only if H is losing. The converse is not quite true: just because two games are winning does not mean they are equivalent, as we will see shortly. However, any two *losing* games are equivalent, because they are all equivalent to $*0$:

Lemma 1.5 *If G is a losing game (the second player to move can force a win), then $G \equiv *0$.*

Proof. Let G be a losing game. We want to show $G \equiv *0$ By definition 1.4, this is true if and only if for any other game J, the game $G+J$ is losing if and only if $*0+J$ is losing. According to (1.3), this holds if and only if J is losing.

So let J be any other game; we want to show that $G+J$ is losing if and only if J is losing. Intuitively, adding the losing game G to J does not change which player in J can force a win, because any intermediate move in G by his opponent is simply countered by the winning player, until the moves in G are exhausted.

Formally, we first prove by induction the simpler claim that for all games J, if J is losing, then $G+J$ is losing. (So we first ignore the "only if" part.) Our inductive assumptions for this simpler claim are: for all losing games G'' that are simpler than G, if J is losing, then $G''+J$ is losing; and for all games J'' that are simpler than J, if J'' is losing, then $G+J''$ is losing.

So suppose that J is losing. We want to show that $G+J$ is losing. Any initial move in J leads to an option J' which is winning, which means that there is a corresponding option J'' of J' (by player II's reply) where J'' is losing. Hence, when player I makes the corresponding initial move from $G+J$ to $G+J'$, player II can counter by moving to $G+J''$. By inductive assumption, this is losing because J'' is losing. Alternatively, player I may move from $G+J$ to $G'+J$. Because G is a losing game, there is a move by player II from G' to G'' where G'' is again a losing game, and hence $G''+J$ is also losing, by inductive assumption, because J is losing. This completes the induction and proves the claim.

What is missing is to show that if $G+J$ is losing, so is J. If J was winning, then there would be a winning move to some option J' of J where J' is losing, but then, by our claim (the "if" part that we just proved), $G+J'$ is losing, which would be a winning option in $G+J$ for player I. But this is a contradiction. This completes the proof. $\square$

The preceding lemma says that any losing game Z, say, can be added to a game G without changing whether G is winning or losing (in lemma 1.5, Z is called G). That is, extending (1.3),

$$Z \text{ losing} \quad \Longrightarrow \quad G+Z \equiv G. \tag{1.4}$$

As an example, consider $Z = *1 + *2 + *3$, which is nim with three heaps of sizes $1,2,3$. To see that Z is losing, we examine the options of Z and show that all of them are winning games. Removing an entire heap leaves two unequal heaps, which is a winning position by lemma 1.1. Any other move produces three heaps, two of which have equal size. Because two equal heaps define a losing nim game Z, they can be ignored by (1.4), meaning that all these options are like single nim heaps and therefore winning positions, too.

So $Z = *1 + *2 + *3$ is losing. The game $G = *4 + *5$ is clearly winning. By (1.4), the game $G+Z$ is equivalent to G and is also winning. However, verifying directly that $*1 + *2 + *3 + *4 + *5$ is winning would not be easy to see without using (1.4).

It is an easy exercise to show that in sums of games, games can be replaced by equivalent games, resulting in an equivalent sum. That is, for all games G,H,J,

$$G \equiv H \quad \Longrightarrow \quad G+J \equiv H+J. \tag{1.5}$$

Note that (1.5) is not merely a re-statement of definition 1.4, because equivalence of the games $G+J$ and $H+J$ means more than just that the games are either both winning or both losing (see the comments before lemma 1.9 below).

Lemma 1.6 (The copycat principle) $G+G \equiv *0$ *for any impartial game* G.

Proof. Given G, we assume by induction that the claim holds for all simpler games G'. Any option of $G+G$ is of the form $G'+G$ for an option G' of G. This is winning by moving to the game $G'+G'$ which is losing, by inductive assumption. So $G+G$ is indeed a losing game, and therefore equivalent to $*0$ by lemma 1.5. □

We now come back to the issue of inverse elements in abstract groups, mentioned at the end of section 1.8. If we identify equivalent games, then the addition $+$ of games defines indeed a group operation. The neutral element is $*0$, or any equivalent game (that is, a losing game).

The inverse of a game G, written as the negative $-G$, fulfils

$$G+(-G) \equiv *0. \tag{1.6}$$

Lemma 1.6 shows that for an impartial game, $-G$ is simply G itself.

Side remark: For games that are not impartial, that is, partisan games, $-G$ exists also. It is G but with the roles of the two players exchanged, so that whatever move was available to player I is now available to player II and vice versa. As an example, consider the game checkers (with the rule that whoever can no longer make a move loses), and let G be a certain configuration of pieces on the checkerboard. Then $-G$ is the same configuration with the white and black pieces interchanged. Then in the game $G+(-G)$, player II (who can move the black pieces, say), can also play "copycat". Namely, if player I makes a move in either G or $-G$ with a white piece, then player II copies that move with a black piece on the other board ($-G$ or G, respectively). Consequently, player II always has a move available and will win the game, so that $G+(-G)$ is indeed a losing game for the starting player I, that is, $G+(-G) \equiv *0$. However, we only consider impartial games, where $-G = G$.

The following condition is very useful to prove that two games are equivalent.

Lemma 1.7 *Two impartial games* G,H *are equivalent if and only if* $G+H \equiv *0$.

Proof. If $G \equiv H$, then by (1.5) and lemma 1.6, $G+H \equiv H+H \equiv *0$. Conversely, $G+H \equiv *0$ implies $G \equiv G+H+H \equiv *0+H \equiv H$. □

Sometimes, we want to prove equivalence inductively, where the following observation is useful.

Lemma 1.8 *Two games G and H are equivalent if all their options are equivalent, that is, for every option of G there is an equivalent option of H and vice versa.*

Proof. Assume that for every option of G there is an equivalent option of H and vice versa. We want to show $G+H \equiv *0$. If player I moves from $G+H$ to $G'+H$ where G' is an option in G, then there is an equivalent option H' of H, that is, $G'+H' \equiv *0$ by lemma 1.7. Moving there defines a winning move in $G'+H$ for player II. Similarly, player II has a winning move if player I moves to $G+H'$ where H' is an option of H, namely to $G'+H'$ where G' is an option of G that is equivalent to H'. So $G+H$ is a losing game as claimed, and $G \equiv H$ by lemma 1.5 and lemma 1.7. □

Note that lemma 1.8 states only a sufficient condition for the equivalence of G and H. Games can be equivalent without that property. For example, $G+G \equiv *0$, but $*0$ has no options whereas $G+G$ has many.

We conclude this section with an important point. Equivalence of two games is a finer distinction than whether the games are both losing or both winning, because that property has to be preserved in sums of games as well. Unlike losing games, winning games are in general *not* equivalent.

Lemma 1.9 *Two nim heaps are equivalent only if they have equal size: $*n \equiv *m \Longrightarrow n = m$.*

Proof. By lemma 1.7, $*n \equiv *m$ if and only if $*n + *m$ is a losing position. By lemma 1.1, this implies $n = m$. □

That is, different nim heaps are not equivalent. In a sense, this is due to the different amount of "freedom" in making a move, depending on the size of the heap. However, all the relevant freedom in making a move in an impartial game can be captured by a nim heap. We will later show that *any impartial game* is equivalent to *some* nim heap.

1.10 Sums of nim heaps

Before we show how impartial games can be represented as nim heaps, we consider the game of nim itself. We show in this section how any nim game, which is a sum of nim heaps, is equivalent to a single nim heap. As an example, we know that $*1+*2+*3 \equiv *0$, so by lemma 1.7, $*1+*2$ is equivalent to $*3$. In general, however, the sizes of the nim heaps cannot simply be added to obtain the equivalent nim heap (which by lemma 1.9 has a unique size). For example, as shown after (1.4), $*1+*2+*3 \equiv *0$, that is, $*1+*2+*3$ is a losing game and not equivalent to the nim heap $*6$. Adding the game $*2$ to both sides of the equivalence $*1+*2+*3 \equiv *0$ gives $*1+*3 \equiv *2$, and in a similar way $*2+*3 \equiv *1$, so any two heaps from sizes $1,2,3$ has the third size as its equivalent single heap. This rule is very useful in simplifying nim positions with small heap sizes.

If $*k \equiv *n + *m$, we also call k the *nim sum* of n and m, written $k = n \oplus m$. The following theorem states that for distinct powers of two, their nim sum is the ordinary sum. For example, $1 = 2^0$ and $2 = 2^1$, so $1 \oplus 2 = 1 + 2 = 3$.

Theorem 1.10 *Let $n \geq 1$, and $n = 2^a + 2^b + 2^c + \cdots$, where $a > b > c > \cdots \geq 0$. Then*

$$* n \equiv *(2^a) + *(2^b) + *(2^c) + \cdots. \tag{1.7}$$

We first discuss the implications of this theorem, and then prove it. The right-hand side of (1.7) is a sum of games, whereas n itself is represented as a sum of powers of two. Any n is uniquely given as such a sum. This amounts to the binary representation of n, which, if $n < 2^{a+1}$, gives n as the sum of all powers of two $2^a, 2^{a-1}, 2^{a-2}, \ldots, 2^0$, each power multiplied with one or zero. These ones and zeros are then the digits in the binary representation of n. For example,

$$13 = 8 + 4 + 1 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0,$$

so that 13 in decimal is written as 1101 in binary. Theorem 1.10 uses only the powers of two $2^a, 2^b, 2^c, \ldots$ that correspond to the digits "one" in the binary representation of n.

Equation (1.7) shows that $*n$ is equivalent to the game sum of many nim heaps, all of which have a size that is a power of two. Any other nim heap $*m$ is also a sum of such games, so that $*n + *m$ is a game sum of several heaps, where equal heaps cancel out in pairs. The remaining heap sizes are all distinct powers of two, which can be added to give the size of the single nim heap $*k$ that is equivalent to $*n + *m$. As an example, let $n = 13 = 8 + 4 + 1$ and $m = 11 = 8 + 2 + 1$. Then $*n + *m \equiv *8 + *4 + *1 + *8 + *2 + *1 \equiv *4 + *2 \equiv *6$, which we can also write as $13 \oplus 11 = 6$. In particular, $*13 + *11 + *6$ is a losing game, which would be very laborious to show without the theorem.

One consequence of theorem 1.10 is that the nim sum of two numbers never exceeds their ordinary sum. Moreover, if both numbers are less than some power of two, then so is their nim sum.

Lemma 1.11 *Let $0 \leq p, q < 2^a$. Then $*p + *q \equiv *r$ where $0 \leq r < 2^a$, that is, $r = p \oplus q < 2^a$.*

Proof. Both p and q are sums of distinct powers of two, all smaller than 2^a. By theorem 1.10, r is also a sum of such powers of two, where those that appear in both p and q cancel out, so that $r < 2^a$. $\square$

The following proof may be best understood by considering it along with an example, say $n = 7$.

Proof of theorem 1.10. We proceed by induction. Consider some n, and assume that the theorem holds for all smaller n. Let $n = 2^a + q$ where $q = 2^b + 2^c + \cdots$. If $q = 0$, the claim holds trivially (n is just a single power of two), so let $q > 0$. We have $q < 2^a$. By inductive assumption, $*q \equiv *(2^b) + *(2^c) + \cdots$, so all we have to prove is that $*n \equiv *(2^a) + *q$ in order to show (1.7). We show this using lemma 1.8, that is, by showing that the options of the games $*n$ and $*(2^a) + *q$ are all equivalent. The options of $*n$ are $*0, *1, *2, \ldots, *(n-1)$.

The options of $*(2^a) + *q$ are of two kinds, depending on whether the player moves in the nim heap $*(2^a)$ or $*q$. The first kind of options are given by

$$\begin{aligned}
*0 + *q &\equiv *r_0 \\
*1 + *q &\equiv *r_1 \\
&\vdots \\
*(2^a - 1) + *q &\equiv *r_{2^a - 1}
\end{aligned} \tag{1.8}$$

where the equivalence of $*i + *q$ with some nim heap $*r_i$, for $0 \le i < 2^a$, holds by inductive assumption. Moreover, by lemma 1.11 (which is a consequence of theorem 1.10 which can be used by inductive assumption), both i and q are less than 2^a so that also $r_i < 2^a$. On the right-hand side in (1.8), there are 2^a many nim heaps $*r_i$ for $0 \le i < 2^a$. We claim they are all different, so that these options form exactly the set $\{*0, *1, *2, \ldots, *(2^a - 1)\}$. Namely, by adding the game $*q$ to the heap $*r_i$, (1.5) implies $*r_i + *q \equiv *i + *q + *q \equiv *i$, so that $*r_i \equiv *r_j$ implies $*r_i + *q \equiv *r_j + *q$, that is, $*i \equiv *j$ and hence $i = j$ by lemma 1.9, for $0 \le i, j < 2^a$.

The second kind of options of $*(2^a) + *q$ are of the form

$$\begin{aligned}
*(2^a) + *0 &\equiv *(2^a + 0) \\
*(2^a) + *1 &\equiv *(2^a + 1) \\
&\vdots \\
*(2^a) + *(q - 1) &\equiv *(2^a + q - 1),
\end{aligned}$$

where the heap sizes on the right-hand sides are given again by inductive assumption. These heaps form the set $\{*(2^a), *(2^a + 1), \ldots, *(n - 1)\}$. Together with the first kind of options, they are exactly the options of $*n$. This shows that the options of $*n$ and of the game sum $*(2^a) + *q$ are indeed equivalent, which completes the proof. $\square$

The nim sum of any set of numbers can be obtained by writing each number as a sum of distinct powers of two and then cancelling repetitions in pairs. For example,

$$6 \oplus 4 = (4+2) \oplus 4 = 2, \qquad \text{or} \qquad 11 \oplus 16 \oplus 18 = (8+2+1) \oplus 16 \oplus (16+2) = 8+1 = 9.$$

This is usually described as "write the numbers in binary and add without carrying", which comes to the same thing. In the following tables, the top row shows the powers of two needed in the binary representation for the numbers beneath; the bottom row gives the resulting nim sum.

	4	2	1
6 =	1	1	0
4 =	1	0	0
2 =	0	1	0

	16	8	4	2	1
11 =		1	0	1	1
16 =	1	0	0	0	0
18 =	1	0	0	1	0
9 =	0	1	0	0	1

However, using only the powers of two that are used and cancelling repetitions is easier to do in your head, and is less prone to error.

How does theorem 1.10 translate into playing nim? When the nim sum of the heap sizes is zero, then the player is in a losing position. (Such nim positions are sometimes called *balanced* positions.) All moves will lead to a winning position, and in practice the best advice may only be not to move to a winning position that is too obvious, like one where two heap sizes are equal, in the hope that the opponent makes a mistake.

If the nim sum of the heap sizes is not zero, it is some sum of powers of two, say $s = 2^a + 2^b + 2^c + \cdots$, like for example $11 \oplus 16 \oplus 18 = 9 = 8 + 1$ above. The winning move is then obtained as follows:

1. Identify a heap of size n which uses 2^a, the largest power of 2 in the nim sum; at least one such heap must exist. In the example, that heap has size 11 (so it is not always the largest heap).

2. Compute $n \oplus s$. In that nim sum, the power 2^a appears in both n and s, and it cancels out, so the result is some number m that is smaller than n. In the example, $m = 11 \oplus 9 = (8+2+1) \oplus (8+1) = 2$.

3. Reduce the heap of size n to size m, in the example from size 11 to size 2. The resulting heap $*m$ is equivalent to $*n + *s$, so when it replaces $*n$ in the original sum, $*s$ is added and cancels with $*s$, and the result is equivalent to $*0$, a losing position.

On paper, the binary representation may be easier to use. In step 2 above, computing $n \oplus s$ amounts to "flipping the bits" in the binary representation of the heap n whenever the corresponding bit in the binary representation of the sum s is one. In this way, a player in a winning position moves always from an "unbalanced" position (with nonzero nim sum) to a balanced position (nim sum zero), which is losing because any move will create again an unbalanced position. This is the way nim is usually explained. The method was discovered by Bouton who published it in 1902.

$\Rightarrow$ You are now in a position to answer all of exercise 1.1 on page 20.

So far, it is not fully clear why powers of two appear in the computation of nim sums. One reason is provided by the proof of theorem 1.10. The options of moving from $*(2^a) + *q$ in (1.8) neatly produce exactly the numbers $0, 1, \ldots, 2^a - 1$, which would not work when replacing 2^a with something else.

As a second reason, the copycat principle $G + G \equiv *0$ shows that the impartial games form a group where every element is its own inverse. There is essentially only one mathematical structure that has these particular properties, namely the addition of binary vectors. In each component, such vectors are separately added modulo two, where $1 \oplus 1 = 0$. Here, the binary vectors translate into binary numbers for the sizes of the nim heaps. The "addition without carry" of binary vectors defines exactly the winning strategy in nim, as stated in theorem 1.10. However, the proof of this theorem is stated directly and without any recourse to abstract algebra and groups.

A third reason uses the construction of equivalent nim heaps for any impartial game, in particular a sum of two nim heaps, which we explain next; see also figure 1.2 below.

1.11 Poker nim and the mex rule

Poker nim is played with heaps of poker chips. Just as in ordinary nim, either player may reduce the size of any heap by removing some of the chips. But alternatively, a player may also *increase* the size of some heap by adding to it some of the chips he acquired in earlier moves. These two kinds of moves are the only ones allowed.

Let's suppose that there are three heaps, of sizes $3,4,5$, and that the game has been going on for some time, so that both players have accumulated substantial reserves of chips. It's player I's turn, who moves to $1,4,5$ because that is a good move in ordinary nim. But now player II adds 50 chips to the heap of size 4, creating position $1,54,5$, which seems complicated.

What should player I do? After a moment's thought, he just removes the 50 chips player II has just added to the heap, reverting to the previous position. Player II may keep adding chips, but will eventually run out of them, no matter how many she acquires in between, and then player I can proceed as in ordinary nim.

So a player who can win a position in ordinary nim can still win in poker nim. He replies to the opponent's reducing moves just as he would in ordinary nim, and reverses the effect of any increasing move by using a reducing move to restore the heap to the same size again. Strictly speaking, the ending condition (see condition 6 in section 1.6) is violated in poker nim because in theory the game could go on forever. However, a player in a winning position wants to end the game with his victory, and never has to put back any chips; then the losing player will eventually run out of chips that she can add to a heap, so that the game terminates.

Consider now an impartial game where the options of player I are games that are equivalent to the nim heaps $*0$, $*1$, $*2$, $*5$, $*9$. This can be regarded as a rather peculiar nim heap of size 3 which can be reduced to any of the sizes $0,1,2$, but which can also be increased to size 5 or 9. The poker nim argument shows that this extra freedom is in fact of no use whatsoever.

The *mex rule* says that if the options of a game G are equivalent to nim heaps with sizes from a set S (like $S = \{0,1,2,5,9\}$ above), then G is equivalent to a nim heap of size m, where m is the *smallest non-negative integer not contained in S*. This number m is written $\text{mex}(S)$, where mex stands for "minimum excludant". That is,

$$m = \text{mex}(S) = \min\{k \geq 0 \mid k \notin S\}. \tag{1.9}$$

For example, $\text{mex}(\{0,1,2,3,5,6\}) = 4$, $\text{mex}(\{1,2,3,4,5\}) = 0$, and $\text{mex}(\emptyset) = 0$.

$\Rightarrow$ Which game has the empty set $\emptyset$ as its set of options?

Theorem 1.12 (The mex rule) *Let the impartial game G have the set of options that are equivalent to $\{*s \mid s \in S\}$ for some set S of non-negative integers (assuming S is not the set of all non-negative integers, for example if S is finite). Then $G \equiv *(\text{mex}(S))$.*

Proof. Let $m = \text{mex}(S)$. We show $G + *m \equiv *0$, which proves the theorem by lemma 1.7. If player I moves from $G + *m$ to $G + *k$ for some $k < m$, then $k \in S$ and there is an option

K of G so that $K \equiv *k$ by assumption, so player II can counter by moving from $G + *k$ to the losing position $K + *k$. Otherwise, player I may move to $K + *m$, where K is some option K which is equivalent to $*k$ for some $k \in S$. If $k < m$, then player II counters by moving to $K + *k$. If $k > m$, then player II counters by moving to $M + *m$ where M is the option of K that is equivalent to $*m$ (because $K \equiv *k$). The case $k = m$ is excluded by the definition of $m = \text{mex}(S)$. This shows $G + *m$ is a losing game. □

A special case of the preceding theorem is that $\text{mex}(S) = 0$, which means that all options of G are equivalent to positive nim heaps, so they are all winning positions, or that G has no options at all. Then G is a losing game, and indeed $G \equiv *0$.

Corollary 1.13 *Any impartial game G is equivalent to some nim heap $*n$.*

Proof. We can assume by induction that this holds for all games that are simpler than G, in particular the options of G. They are equivalent to nim heaps whose sizes form the set S (which we assume is not the set of all non-negative integers). Theorem 1.12 then shows $G \equiv *m$ for $m = \text{mex}(S)$. □

⇒ Do exercise 1.7 on page 23, which provides an excellent way to understand the mex rule.

1.12 Finding nim values

By corollary 1.13, any impartial game can be played like nim, provided the equivalent nim heaps of the positions of the game are known. This forms the basis of the *Sprague–Grundy* theory of impartial games, named after the independent discoveries of this principle by R. P. Sprague in 1936 and P. M. Grundy in 1939. Any sum of such games is then evaluated by taking the nim sum of the sizes of the corresponding nim heaps.

The nim values of the positions can be evaluated by the mex rule (theorem 1.12). This is illustrated in the "rook-move" game in figure 1.1. Place a rook on a chess board of given arbitrary size. In one move, the rook is moved either horizontally to the left or vertically upwards, for any number of squares (at least one) as long as it stays on the board. The first player who can no longer move loses, when the rook is on the top left square of the board. We number the rows and columns of the board by $0, 1, 2, \ldots$ starting from the top left.

Figure 1.2 gives the nim values for the positions of the rook on the chess board. The top left square is equivalent to $*0$ because the rook can no longer move. The square below that allows only to reach the square with $*0$ on it, so it is equivalent to $*1$ because $\text{mex}\{0\} = 1$. The square below gets $*2$ because its options are equivalent to $*0$ and $*1$. From any square in the leftmost column in figure 1.2, the rook can only move upwards, so any such square in row i corresponds obviously to a nim heap $*i$. Similarly, the topmost row has entry $*j$ in column j.

In general, a position on the board is evaluated knowing all nim values for the squares to the left and the top of it, which are the options of that position. As an example, consider

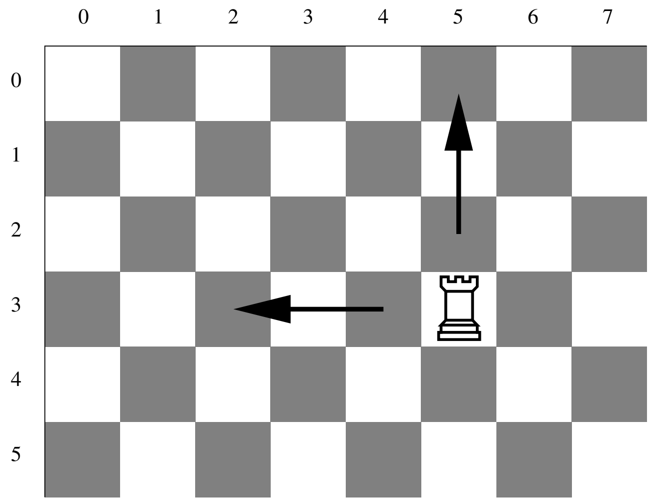


Figure 1.1 Rook move game, where the player may move the rook on the chess board in the direction of the arrows.

the square in row 3 and column 2. To the left of that square, the entries $*3$ and $*2$ are found, and to the top the entries $*2, *3, *0$. So the square itself is equivalent to $*1$ because $\text{mex}(\{0,2,3\}) = 1$. The square in row 3 and column 5, where the rook is placed in figure 1.1, gets entry $*6$.

	0	1	2	3	4	5	6	7	8	9	10
0	$*0$	$*1$	$*2$	$*3$	$*4$	$*5$	$*6$	$*7$	$*8$	$*9$	$*10$
1	$*1$	$*0$	$*3$	$*2$	$*5$	$*4$	$*7$	$*6$	$*9$	$*8$	$*11$
2	$*2$	$*3$	$*0$	$*1$	$*6$	$*7$	$*4$	$*5$	$*10$	$*11$	$*8$
3	$*3$	$*2$	$*1$	$*0$	$*7$	$*6$	$*5$	$*4$	$*11$	$*10$	$*9$
4	$*4$	$*5$	$*6$	$*7$	$*0$	$*1$	$*2$	$*3$	$*12$	$*13$	$*14$
5	$*5$	$*4$	$*7$	$*6$	$*1$	$*0$	$*3$	$*2$	$*13$	$*12$	$*15$
6	$*6$	$*7$	$*4$	$*5$	$*2$	$*3$	$*0$	$*1$	$*14$	$*15$	$*12$
7	$*7$	$*6$	$*5$	$*4$	$*3$	$*2$	$*1$	$*0$	$*15$	$*14$	$*13$
8	$*8$	$*9$	$*10$	$*11$	$*12$	$*13$	$*14$	$*15$	$*0$	$*1$	$*2$
9	$*9$	$*8$	$*11$	$*10$	$*13$	$*12$	$*15$	$*14$	$*1$	$*0$	$*3$
10	$*10$	$*11$	$*8$	$*9$	$*14$	$*15$	$*12$	$*13$	$*2$	$*3$	$*0$

Figure 1.2 Equivalent nim heaps $*n$ for positions of the rook move game.

The astute reader will have noticed that the rook move game is just nim with two heaps. A rook positioned in row i and column j can either diminish i by moving left, or j

by moving up. So this position is the sum of nim heaps $*i + *j$. It is a losing position if and only if $i = j$, where the rook is on the diagonal leading to the top left square. Therefore, figure 1.2 represents the computation of nim heaps equivalent to $*i + *j$, or, by omitting the stars, the nim sums $i \oplus j$, for all $i, j \geq 0$.

The nim addition table figure 1.2 is computed by the mex rule, and does not require theorem 1.10. Given this nim addition table, one can conjecture (1.7). You may find it useful to go back to the proof of theorem 1.10 using figure 1.2 and check the options for a position of the form $*(2^a) + *q$ for $q < 2^a$, as a square in row 2^a and column q.

Another impartial game is shown in figure 1.3 where the rook is replaced by a queen, which may also move diagonally. The squares on the main diagonal are therefore no longer losing positions. This game can also be played with two heaps of chips where in one move, the player may either take chips from one heap as in nim, or reduce *both* heaps by the *same* number of chips (so this is no longer a sum of two games!). In order to illustrate that we are not just interested in the winning and losing squares, we add to this game a nim heap of size 4.

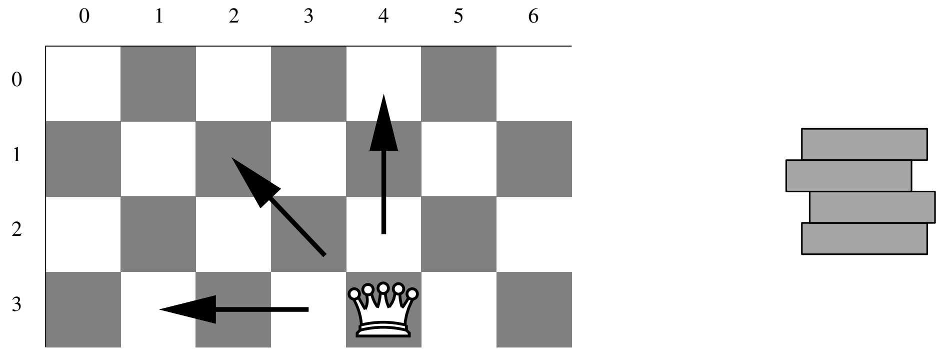


Figure 1.3 Sum of a queen move game and a nim heap. The player may *either* move the queen in the direction of the arrows *or* take some of the 4 chips from the heap.

Figure 1.4 shows the equivalent nim heaps for the positions of the queen move game, determined by the mex rule. The square in row 3 and column 4 occupied by the queen in figure 1.3 has entry $*2$. So a winning move is to remove 2 chips from the nim heap to turn it into the heap $*2$, creating the losing position $*2 + *2$.

	0	1	2	3	4	5	6
0	$*0$	$*1$	$*2$	$*3$	$*4$		
1	$*1$	$*2$	$*0$	$*4$	$*5$		
2	$*2$	$*0$	$*1$	$*5$	$*3$		
3	$*3$	$*4$	$*5$	$*6$	$*2$		

Figure 1.4 Equivalent nim heaps for positions of the queen move game.

This concludes our introduction to combinatorial games. Further examples will be given in the exercises.

⇒ Do the remaining exercises 1.4–1.11, starting on page 21, which show how to use the mex rule and what you learned about nim and combinatorial games.

1.13 Exercises for chapter 1

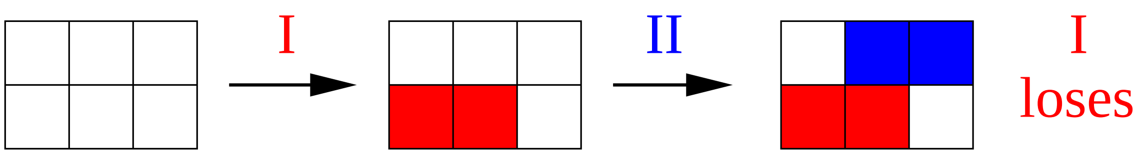
In this chapter, which is more abstract than the others, the exercises are particularly important. Exercise 1.1 is a standard question on nim, where part (a) can be answered even without the theory. Exercise 1.2 is an example of an impartial game, which can also be answered without much theory. Exercise 1.3 is difficult – beware not to rush into any quick and false application of nim values here; part (c) of this exercise is particularly challenging. Exercise 1.4 tests your understanding of the queen move game. For exercise 1.5, remember the concept of a sum of games, which applies here naturally. In exercise 1.6, try to see how nim heaps are hidden in the game. Exercise 1.7 is very instructive for understanding the mex rule. In exercise 1.8, it is essential that you understand nim values. It takes some work to investigate all the options in the game. Exercise 1.9 is another familiar game where you have to find out nim values. In exercise 1.10, a new game is defined that you can analyse with the mex rule. Exercise 1.11 is an impartial game that is rather different from the previous games. In the challenging part (c) of that exercise, you should first formulate a conjecture and then prove it precisely.

Exercise 1.1 Consider the game nim with heaps of chips. The players alternately remove some chips from one of the heaps. The player to remove the last chip wins.

(a) For all positions with three heaps, where one of the heaps has only *one* chip, describe exactly the *losing* positions. Justify your answer, for example with a proof by induction, or by theorems on nim.
[*Hint: Start with the easy cases to find the pattern.*]

(b) Determine all initial winning moves for nim with three heaps of size 6, 10 and 15, using the theorem on nim where the heap sizes are represented as sums of powers of two.

Exercise 1.2 The game *dominos* is played on a board of $m \times n$ squares, where players alternately place a domino on the board which covers two adjacent squares that are free (not yet occupied by a domino), vertically or horizontally. The first player who cannot place a domino any more loses. Example play for a 2×3 board:



(a) Who will win in 3×3 dominos?
[*Hint: Use the symmetry of the game to investigate possible moves, and remember that it suffices to find one winning strategy.*]

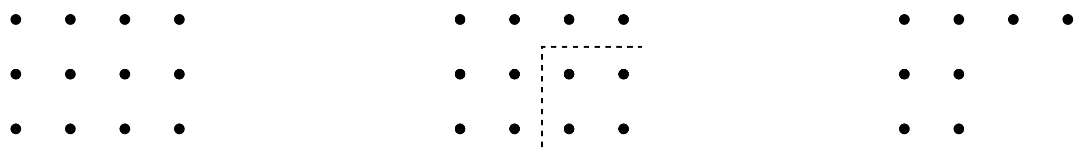
(b) Who will win in $m \times n$ dominos when both m and n are even?

(c) Who will win in $m \times n$ dominos when m is odd and n is even?

Justify your answers.

Note (not a question): Because of the known answers from (b) and (c), this game is more interesting for "real play" on an $m \times n$ board where both m and n are odd. Play it with your friends on a 5×5 board, for example. The situation often decomposes into independent parts, like contiguous fields of 2, 3, 4, 5, 6 squares, that have a known winner, which may help you analyse the situation.

Exercise 1.3 Consider the following game chomp: A rectangular array of $m \times n$ dots is given, in m rows and n columns, like 3×4 in the next picture on the left. A dot in row i and column j is named (i, j). A move consists in picking a dot (i, j) and removing it and *all other dots to the right and below it*, which means removing all dots (i', j') with $i' \geq i$ and $j' \geq j$, as shown for $(i, j) = (2,3)$ in the middle picture, resulting in the picture on the right:



Player I is the first player to move, players alternate, and the last player who removes a dot *loses*.

An alternative way is to think of these dots as (real) cookies: a move is to eat a cookie and all those to the right and below it, but the top left cookie is poisoned. See also `http://www.stolaf.edu/people/molnar/games/chomp/`

(a) Assuming optimal play, determine the winning player and a winning move for chomp of size 2×2, size 2×3, size $2 \times n$, and size $m \times m$, where $m \geq 3$. Justify your answers.

(b) In the way described here, chomp is a misère game where the last player to make a move loses. Suppose we want to play the same game so that the normal play convention applies, where the last player to move wins. (This would be a boring game with the board as given, by simply taking the top left dot $(1,1)$.) Explain how this can be done by removing one dot from the initial array of dots.

(c) Show that when chomp is played for a game of any size $m \times n$, player I can always win.
[*Hint: You only have to show that a winning move* exists, *but you do not have to describe that winning move.*]

Exercise 1.4

(a) Complete the entries of equivalent nim heaps for the queen move game in columns 5 and 6, rows 0 to 3, in the table in figure 1.4.

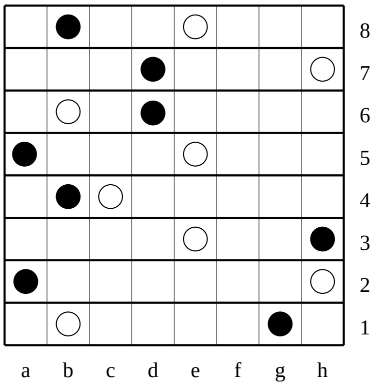
(b) Describe *all* winning moves in the game-sum of the queen move game and the nim heap in figure 1.3.

Exercise 1.5 Consider the game dominos from exercise 1.2, played on a $1 \times n$ board for $n \geq 2$. Let D_n be the nim value of that game, so that the starting position of the $1 \times n$ board is equivalent to a nim heap of size D_n. For example, $D_2 = 1$ because the 1×2 board is equivalent to $*1$.

(a) How is D_n computed from smaller values D_k for $k < n$?
[*Hint: Use sums of games and the mex rule. The notation for nim sums is* $\oplus$, *where* $a \oplus b = c$ *if and only if* $*a + *b \equiv *c$.]

(b) Give the values of D_n up to $n = 10$ (or more, if you are ambitious – higher values come at $n = 16$, and at some point they even repeat, but before you detect that you will probably have run out of patience). For which values of n, where $1 \leq n \leq 10$, is dominos on a $1 \times n$ board a losing game?

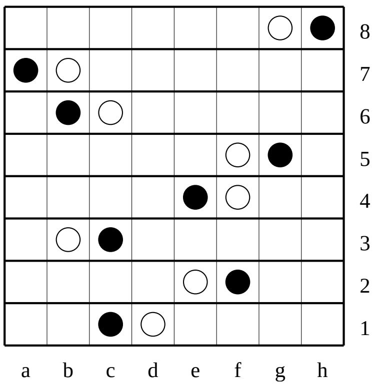
Exercise 1.6 Consider the following game on a rectangular board where a white and a black counter are placed in each row, like in this example:



(*)

Player I is white and starts, and player II is black. Players take turns. In a move, a player moves a counter of his colour to any other square within its row, but may not jump over the other counter. For example, in (*) above, in row 8 white may move from e8 to any of the squares c8, d8, f8, g8, or h8. The player who can no longer move loses.

(a) Who will win in the following position?

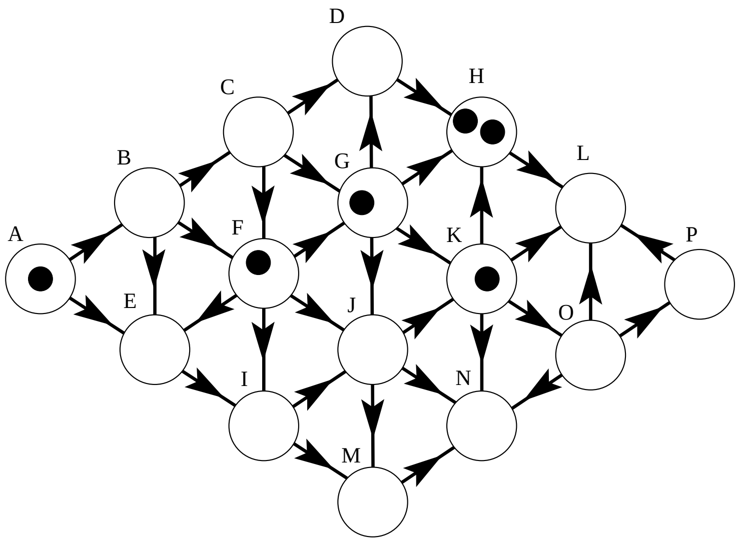


(b) Show that white can win in position (*) above. Give at least two winning moves from that position.

Justify your answers.
[*Hint: Compare this with another game that is not impartial and that also violates the*

ending condition, but that nevertheless is close to nim and ends in finite time when played well.]

Exercise 1.7 Consider the following network (in technical terms, a directed graph or "digraph"). Each circle, here marked with one of the letters A–P, represents a *node* of the network. Some of these nodes (here A, F, G, H, and K) have counters on them, which are allowed to share a node, like the two counters on H. In a move, one of the counters is moved to a neighbouring node in the direction of the arrow as indicated, for example from F to I (but not from F to C, nor directly from F to D, say). Players alternate, and the last player no longer able to move loses.

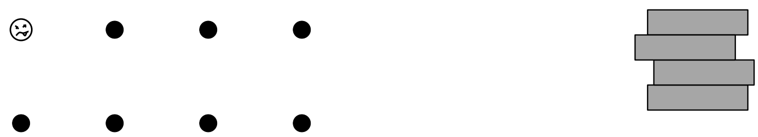


(a) Explain why this game fulfils the ending condition.

(b) Who is winning in the above position? If it is player I (the first player to move), describe *all* possible winning moves. Justify your answer.

(c) How does the answer to (b) change when the arrow from J to K is reversed so that it points from K to J instead?

Exercise 1.8 Consider the game chomp from Exercise 1.3 of size 2×4, added to a nim heap of size 4.



What are the winning moves of the starting player I, if any?
[*Hint: Represent chomp as a game in the normal play convention (see exercise 1.3(b), by changing the dot pattern), so that the losing player is not the player who takes the "poisoned cookie", but the player who can no longer move. This will simplify finding the various nim values.*]

Exercise 1.9 In $m \times n$ *dominos* (see exercise 1.2), a rectangular board of $m \times n$ squares is given. The two players alternately place a domino either horizontally or vertically on two unoccupied adjacent squares, which then become occupied. The last player to be able to place a domino wins.

(a) Find the nim-value (size of the equivalent nim heap) of 2×3 dominos.

(b) Find all winning moves, if any, for the *game-sum* of a 2×3 domino game and a 1×4 domino game.

Exercise 1.10 Consider the following variant of nim called split-nim, which is played with heaps of chips as in nim. Like in nim, a player can remove some chips from one of the heaps, or else *split* a heap into two (not necessarily equal) new heaps. For example, a heap of size 4 can be reduced to size 3, 2, 1, or 0 as in ordinary nim, or be split into two heaps of sizes 1 and 3, respectively, or into two heaps of sizes 2 and 2. As usual, the last player to be able to move wins.

(a) Find the nim-values (size of equivalent nim heaps) of the single split-nim heaps of size 1, 2, 3, and 4, respectively.

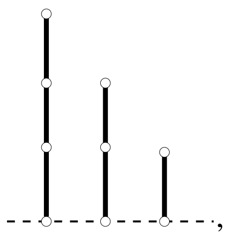
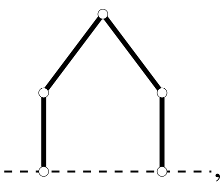
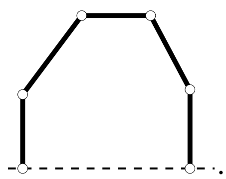
(b) Find all winning moves, if any, when split-nim is played with three heaps of sizes 1, 2, 3.

(c) Find all winning moves, if any, when split-nim is played with three heaps of sizes 1, 2, 4.

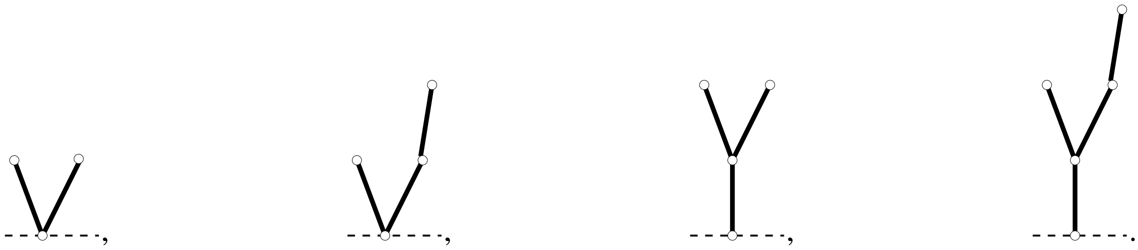
(d) Determine if the following statement is true or false: "Two heaps in split-nim are equivalent if and only if they have equal size." Justify your answer.

Exercise 1.11 The game *Hackenbush* is played on a figure consisting of dots connected with lines (called *edges*) that are connected to the ground (the dotted line in the pictures below). A move is to remove ("chop off") an edge, and with it all the edges that are then no longer connected to the ground. For example, in the leftmost figure below, one can remove any edge in one of the three stalks. Removing the second edge from the stalk consisting of three edges takes the topmost edge with it, leaving only a single edge. As usual, players alternate and the last player able to move wins.

(a) Compute the nim values for the following three Hackenbush figures (using what you know about nim, and the mex rule):

(b) Compute the nim values for the following four Hackenbush figures:



(c) Based on the observation in (b), give a rule how to compute the nim value of a “tree” which is constructed by putting several “branches” with known nim values on top of a “stem” of size n, for example $n = 2$ in the picture below, where three branches of height 3, 2, and 4 are put on top of the “stem”. Prove that rule (you may find it advantageous to glue the branches together at the bottom, as in (b)). Use this to find the nim value of the rightmost picture.