Politecnico di Milano

Discrete Dynamical Models – A.Y. 2022-2023

# Analysis of the Italian population with the Leslie model

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- Introduction

This project will study the dynamics of the Italian population between 2002 and 2019 and will try to make a projection for the future.

First, there will be a general qualitative analysis, with the aim of several graphs, then the gender difference will come out, again in a qualitative way. Therefore, a theory introduction will be presented in order to understand the quantitative behaviour of the model. The next step will be an analysis using the Leslie model, and then some considerations on the future will conclude the project.

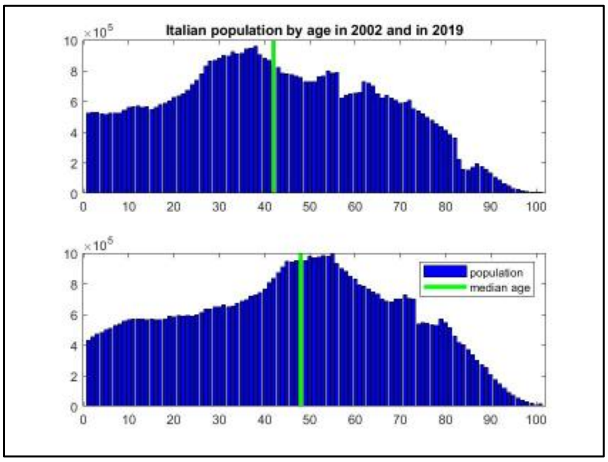
Regarding the theory part, the reference is to *"Discrete Dynamical Models, E. Salinelli, F. Tomarelli, Springer".*

The data used are open source and they came from the ISTAT (Italian national institute of statistics) web site. The other reference used is the ISTAT report on the Italian population from the year 2019. All the graphs and numbers are an elaboration of these data using excel and MATLAB.

This is the aspect of the excel table from which the analysis was made, as it can be seen every cell represents the total amount of people who in that year had the age at the left.

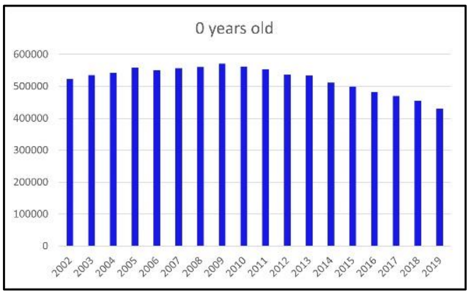
| | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Age | | | | | | | | | | | |
| 0 | 523007 | 535656 | 542323 | 558500 | 551127 | 556884 | 560700 | 570791 | 562322 | 553218 | 53734 |
| 1 | 529233 | 526577 | 538020 | 545596 | 559067 | 552870 | 560028 | 563021 | 569963 | 561694 | 55172 |
| 2 | 528131 | 533126 | 531043 | 540860 | 548487 | 560347 | 556146 | 562640 | 564317 | 569064 | 56094 |
| 3 | 518790 | 530847 | 537507 | 535545 | 543768 | 550808 | 563000 | 559193 | 564536 | 565219 | 56813 |
| 4 | 515957 | 522484 | 534064 | 542329 | 540131 | 545782 | 554754 | 565127 | 560899 | 565563 | 56562 |
| 5 | 521883 | 519874 | 526759 | 537687 | 547286 | 543899 | 549926 | 557963 | 566411 | 562064 | 56643 |
| 6 | 521774 | 525434 | 524440 | 531761 | 541502 | 551380 | 549632 | 553286 | 560048 | 566952 | 56316 |
| 7 | 524929 | 525094 | 529430 | 529267 | 536730 | 544564 | 557571 | 554779 | 555487 | 561735 | 56780 |
| 8 | 539637 | 528714 | 528612 | 533557 | 534646 | 540875 | 549836 | 562743 | 558705 | 557192 | 56316 |
| 9 | 562660 | 543267 | 532760 | 532755 | 538376 | 538956 | 546981 | 554495 | 567192 | 562052 | 55898 |
| 10 | 566374 | 565920 | 547054 | 537956 | 537418 | 542326 | 545130 | 552693 | 558196 | 571096 | 56479 |
| 11 | 571769 | 568684 | 569499 | 551622 | 542931 | 541177 | 547924 | 551007 | 557461 | 561199 | 57418 |
| 12 | 562281 | 573863 | 571180 | 573595 | 556540 | 547135 | 546663 | 553371 | 555780 | 561491 | 56395 |
| 13 | 565951 | 564359 | 576103 | 574651 | 578284 | 560460 | 553021 | 551938 | 557633 | 559950 | 56473 |
| 14 | 544761 | 568358 | 566610 | 578992 | 578169 | 582015 | 566019 | 558684 | 556355 | 561392 | 56325 |
| 15 | 559012 | 547025 | 570882 | 569709 | 582731 | 581012 | 587212 | 571616 | 563539 | 560238 | 56460 |
| 16 | 578356 | 559983 | 549909 | 574567 | 573866 | 585817 | 585422 | 592699 | 576646 | 567923 | 56339 |
| 17 | 586101 | 579525 | 561513 | 554212 | 579441 | 577712 | 590672 | 590575 | 597942 | 581230 | 57180 |

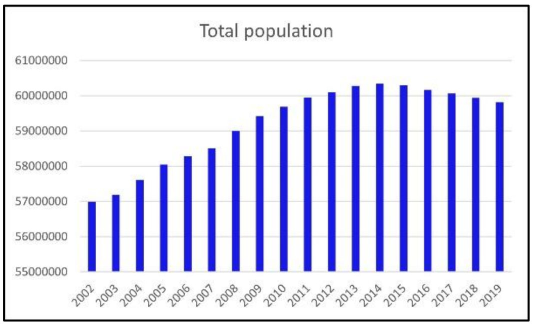
- Qualitative analysis




According to the ISTAT 2019 report, a significant growth in the older ages and a drop in new-borns characterize the Italian demographic chart. Two aspects that result in a constant aging of the population, as it can be seen from the graph which highlights the shape of the population groups in function of their age. On top of that, the median age of the Italian population increases from almost 42 to 48 years; this is an interesting parameter, in fact it represents that age for which there are the same amount of people younger and older than that age.

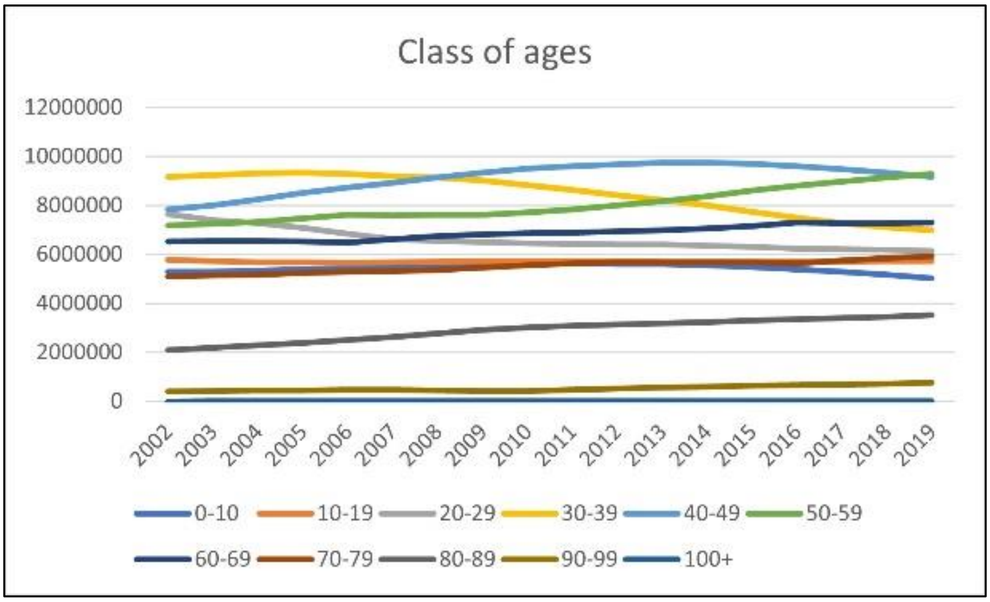
With respect to the births, despite a slight improvement in the earlier 2000th, they are plunging in our days: 0-years old people are rapidly decreasing by over 20% in the last 20 years. According to the report, the drop in the births is totally accountable to the drop in the grown female population which is capable to reproduce.




Despite this fact, the overall variation of the residents has a positive rate, by almost 5%. In fact, the older groups of people are soaring: all groups above 50 years are increasing their size. Life expectancy is sharply increasing, the percentage of over-80 years old population is considerable: above 7%, against the almost 4% in 2002. As a matter of fact, Italy has the record for over a hundred-years-old number of people all over the World.

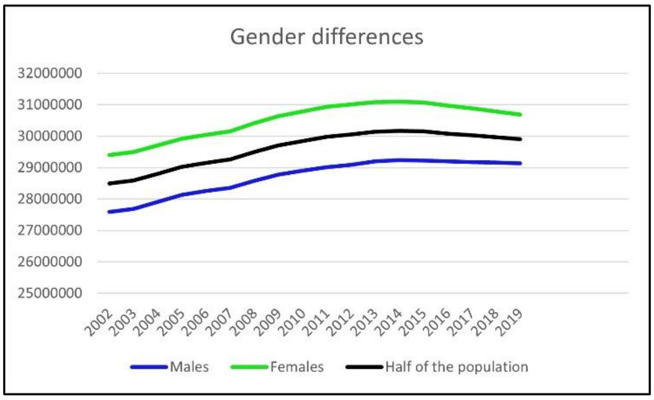
The "class of ages" chart highlights these results. 11 classes are created according to the ages: every class has a 10-year range, so that the first class represents people between 0 years old and 9 years old up to the tenth class with people aged from 90 to 99 years old, and they are plotted on the historical series in the 20 years of analysis.



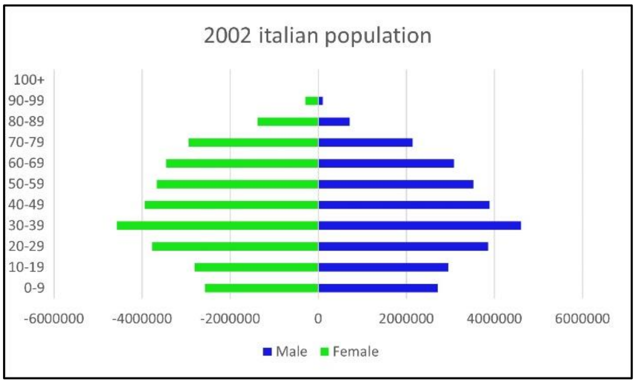

All groups below 40 years old are going down, the opposite happens for the other groups. Regarding the first group, despite in 2002 they were the same amount of the eighth group (70 to 79 years old), in 2019 they are far below the eighth group and moreover they seem to become comparable to the ninth group (80 to 89 years old). On the upper part of the graph, it can be seen how while in 2002 the most populated group is the fourth (30 to 39 years old), after 20 years the leadership has been taken from the sixth group (50 to 59 years old).

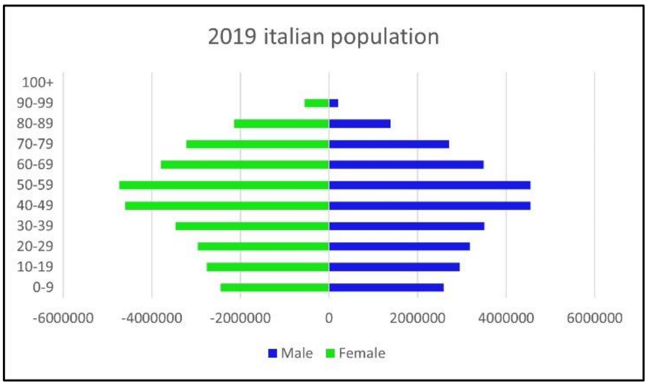
- Qualitative gender differences analysis

Regarding the gender differences, the main fact is that in Italy women are more than men. This fact is illustrated in the graph representing the male and female population, together with half of the total population in the historical series; it can be clearly seen that the pattern is equal for all the three lines, and that woman are steadily above the half-population line which is above men.




In order to have a more detailed analysis, two bar plots have been made using the most common kind of graph with respect to the study of populations: the magnitude of the age groups is plotted in a particular year differentiating between males and females. This kind of graph emphasises all the considerations already seen in only one picture. Women are more than men, especially in the older classes, and it is even more clear how the population is aging, creating a more accentuated figure in the "middle-age" classes.




Overall, it can be seen how going up on the age, the percentage of female became increasingly high. In fact, while the new-borns are in majority man, even from the third group (20 to 29 years old) the percentage of female get larger than the men's. The increment is significant, growing dramatically from the almost equality of the third group to above 70% of female in the tenth group (90 to 99 years-old). Repeating the analysis in 2019, the graph shows how the dynamics of the percentages does not change.

- Leslie's model

The Leslie's model is one of the most popular and simple models for describing the dynamics of a population. Formalized in 1940, it is based on the mortality and fertility coefficients for the age groups in which the population is divided. The peculiarity of this model, with respect to the Malthus' model, is such age groups; in fact, this technique allows us to differentiate the effect of the mortality and fertility in terms of oldness. For instance, mortality is not constant or homogeneous along the groups: obviously, older people tend to have a coefficient much higher than young people.

In the Malthus model, the population is not divided in groups, and so X is a scalar quantity, representing the total population. At every iteration, this quantity is multiplied by a coefficient μ creating the iterative model $X^{k+1} = \mu * X^k$.

This model instead is a vector linear homogeneous model. As is to say that, given a vector $\boldsymbol{X}^0 \in \boldsymbol{R}^n, \boldsymbol{X}^0 = (x_i^0)$ and a matrix $\mathbf{M} \in \boldsymbol{R}^{nxn}, \mathbf{M} = (m_{i,j})$, the linear homogeneous vector discrete dynamical system is given by the law $\boldsymbol{X}^{k+1} = \mathbf{M}\boldsymbol{X}^k$ . Namely,

$$\begin{cases} x_1^{k+1} = \sum_{j=1}^{n} m_{1,j} * x_j^k \\ x_i^{k+1} = \sum_{j=1}^{n} m_{i,j} * x_j^k \\ x_n^{k+1} = \sum_{j=1}^{n} m_{n,j} * x_j^k \end{cases} .$$

The solution (or orbit) of the system is given by the $\boldsymbol{X}^k$ results of the iterations.

In the Leslie model, the population is so divided into groups, according to the age, and a vector $\mathbf{X}$ is created with every element that corresponds to the total amount of people being in that group. A matrix $\mathbf{L}$ is then created, in order to define the linear vector discrete dynamical system $\boldsymbol{X}^{k+1}=\mathbf{L}\boldsymbol{X}^k, k \in \boldsymbol{N}$ ; which takes the name of Leslie's model. The index k stands for the discrete time, in fact starting from a particular moment ($k = 0$) the matrix multiplication can be iterated (and every iteration stands for one unit of time, usually one year) with k that stands for the "number of iterations" made, and so it says how many units of time the model has processed.

From now on it will be considered the particular form of the matrix of this particular project. In fact, in this project every age represents a group, so the first group will be the 0-years-old people and then up to the 99-years-old people with finally the last group which is the 100 years old group and as in the general case, the unit of time will be one year. The over 100 years old will not be considered and the assumption made is that after the year in the 100 years-old group all the people exit from the model.

Regarding the first row of the matrix, it has to consider the new-borns, with the $j^{th}$ element of the first row which corresponds to the fertility rate of the $j^{th}$ group of the population. The multiplication between the fertility rate and the total amount of people for that specific group is the total number of 0-years-old people which are born from that age class. Summing all the contributions, the result is the first element of the new vector, which represents the 0-years-old people living the next year. The fertility rates were collected by the *born for 1000 women* datum, and because of that, the datum had to be converted in *new-borns for each person*: the fertility rate given was so divided by 1000, and then multiplied by the ratio (total females in that group/total population in that group).

Regarding the rest of the population, the model takes into account only the passage between close groups: all the people in the $(i-1)^{th}$ group, in a year (an iteration of the model) will pass in the $i^{th}$ group and only in this group, other passages are not allowed, this is because every age is considered to be a group and so it is not possible to have an amount of people who stays for more than one iteration in the same group, or that go in the previous one; thus the elements of the first lower sub diagonal of the matrix are the only non-trivial elements. Because of this, in an inverse way the rest of the non-trivial coefficients of the matrix were computed: for the $i^{th}$ group $\boldsymbol{L}_{i,i-1} \cdot \boldsymbol{X}_{i-1}^{k} = \boldsymbol{X}_{i}^{k+1}$. Resulting in $\boldsymbol{L}_{i,i-1} = \boldsymbol{X}_{i}^{k+1}/\boldsymbol{X}_{i-1}^{k}$.

Due to migrations a problem was found in these computations: it happens that the number of people in the $(i)^{th}$ group for the year *(k+1)* were more than the number of people in the $(i-1)^{th}$ group for the year *k*, which is clearly impossible according to the model. So, to avoid this problem, an approximation was made: in such situations the coefficient was set to be 1, having no sense a study with a coefficient strictly more than 1.

The dynamics of the population is now totally described by the iteration $\boldsymbol{X}_{k+1}$=L$\boldsymbol{X}_{k}$.

- Theory prospective

A general theory introduction about the vector linear systems is needed in order to study the behaviour of the system, in particular it is possible to study the asymptotic shape of the population due to the special form of the matrix. In this analysis, the existence of a basis of eigenvectors will be given, because of the special form of the Leslie matrix: this discussion is delayed to the end of the paragraph.

We recall three important definitions from linear algebra. Let **M** be a square matrix of order *n*, which in this theory prospective will be the iteration matrix, $\begin{bmatrix} m_{1,1} & \cdots & m_{1,n} \\ \cdots & \cdots & \cdots \\ m_{n,1} & \cdots & m_{n,n} \end{bmatrix}$, and let **X** be a vector of length $n$, $\mathbf{X} = \begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix}$.

1. The <u>one norm</u> of the vector $\mathbf{X} \in \boldsymbol{R}^n$, is said to be $\sum_{i=1}^{n} x_i$, namely the sum of all the components.
2. The <u>eigenvalues</u> of **M** are said to be the roots (in the complex plane **C**) of its characteristic polynomial $\mathrm{P}(\lambda) = \det(\mathbf{M} - \lambda \mathbf{I}) : \{\lambda \in \mathbf{C} : \mathrm{P}(\lambda)=0\}$. According to the Fundamental Theorem of Algebra, a matrix **M** of order *n* has exactly *n* eigenvalues in the complex plane, if they are counted with the same multiplicity of the roots of P(λ).
3. A vector $\mathbf{V} \in \boldsymbol{C}^n \setminus \{\mathbf{0}\}$ is said to be an <u>eigenvector</u> of the square matrix **M** of order n if there exists a complex number λ such that $\mathbf{MV} = \lambda\mathbf{V}$. In this case λ is an eigenvalue of **M** and λ is said to be the eigenvalue associated to the eigenvector **V**.

It is natural to define equilibrium every vector $\mathbf{X}$ such that $\mathbf{X}=\mathbf{MX}$, namely the eigenvectors associated to the eigenvalue 1. Obviously, $\mathbf{0}$ is always an equilibrium: the study of the model is the understanding of the attractivity or stability of the equilibria, which determine the long-time course of the system.

The eigenvalues of the iteration matrix play a fundamental role in the dynamics of the system: their modulus impose a clear pattern to the evolution of the vector $\mathbf{X}$. In fact, if there exists a basis of eigenvectors $\boldsymbol{V}^j$ of $\mathbf{M}$, then for each $\mathbf{X}_0$ there exist (and are unique) n constants $c_1, \dots, c_n$ such that $\boldsymbol{X}_0 = \sum_{j=1}^{n} c_j \boldsymbol{V}^j$ . Then, the orbit of the system is uniquely determined by $\boldsymbol{X}_k = \sum_{j=1}^{n} c_j \, (\lambda_{\boldsymbol{j}})^{\boldsymbol{k}} \, \boldsymbol{V}^j$, because of the spectral decomposition of the square matrices with a basis of eigenvectors.

Because of that, the fundamental analysis became knowing if the modulus of the eigenvalues is more or less than 1. In fact, if $|\lambda_j| < 1 \ \forall j$ , then the orbit of the system will collapse to the unique equilibrium, which is $\mathbf{0}$. Moreover, if there exists an eigenvalue which have modulus strictly more than 1, then the iterative vector will diverge.There are three important theorems about the eigenvalues:

*a)* **Perron-Frobenius Theorem.** *If $\boldsymbol{M}$ is a strictly positive matrix (namely have all the elements strictly more than 0), then its eigenvalue of maximum modulus (which is called dominant eigenvalue) $\lambda_{\boldsymbol{M}}$ is unique, real, simple, and greater than 0. Furthermore, there exists a strictly positive eigenvector $\boldsymbol{V_M}$ (called dominant eigenvector) associated to $\lambda_{\boldsymbol{M}}$.*

*b)* **Theorem.** *If $\boldsymbol{M}$ is a weakly positive matrix (namely have all the elements more or equal than 0 with at least one element which is zero) but there exists an $h \in \boldsymbol{R}$ such that $\boldsymbol{M}^h$ is strictly positive, then the result of the Perron-Frobenius theorem holds.*

*c)* **Theorem.** *If $\boldsymbol{M}$ is a weakly positive matrix (namely have all the elements which are non-negative with at least one which is 0), then the existence of an eigenvalue such that $\lambda_{\boldsymbol{M}} \geq |\mu| \ \forall \mu$ other eigenvalue is guaranteed. Moreover, its uniqueness is guaranteed too, but the reality and the simplicity are not given in general.*

Thus, the analysis on the dominant eigenvalue became important, because if the dominant eigenvalue is strictly less than one, then all the eigenvalues will have the modulus strictly less than 1.

Another theorem is needed regarding the asymptotic behaviour.

d) **Theorem.** *If $\boldsymbol{X}$ is strictly bigger than $\boldsymbol{0}$ and $\boldsymbol{M}$ is strictly positive (or weakly positive and there exists h such that $\boldsymbol{M}^h$ is strictly positive), then the solution of the discrete dynamical system tends to align himself with the dominant eigenvector of the iteration matrix: $\lim_k \frac{\boldsymbol{X}_k}{\|\boldsymbol{X}_k\|} = \frac{\boldsymbol{V_M}}{\|\boldsymbol{V_M}\|}$. Finally, its coefficient in the expansion of $\boldsymbol{X}_0$, $c_{\boldsymbol{M}}$, is strictly bigger than 0.*

Now, taking into account the specific case of the Leslie matrix, it is clearly a weakly positive matrix, because all the coefficients are non-negative, but it is impossible to have a power of the matrix strictly positive. The existence of the dominant eigenvalue is assured, but the situation with the eigenvector (possibly plural) may be complicated.

$$\mathbb{L} = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & \blacksquare & 0 \\ \blacksquare & 0 & 0 & 0 & 0 & 0 \\ 0 & \blacksquare & 0 & 0 & 0 & 0 \\ 0 & 0 & \blacksquare & 0 & 0 & 0 \\ 0 & 0 & 0 & \blacksquare & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \blacksquare & 0 \end{array}\right].$$

Anyway, it is possible to say something about the eigenvector and about the asymptotic behaviour.

This at the left is the approximative form of the Leslie's matrix, in fact both the oldest and the youngest groups do not reproduce themselves, and so the only non-trivial fertility rates are associated to the intermediate classes.

Iterating the power of the matrix the result will be in a form like the one below.

$$\mathbb{L} = \left[\begin{array}{ccccc|c} \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & 0 \\ \blacksquare & 0 & 0 & 0 & 0 & 0 \\ 0 & \blacksquare & 0 & 0 & 0 & 0 \\ 0 & 0 & \blacksquare & 0 & 0 & 0 \\ 0 & 0 & 0 & \blacksquare & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \blacksquare & 0 \end{array}\right].$$

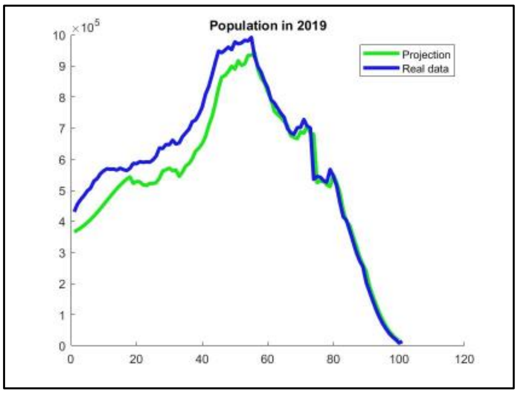
In this form, elevating to an increasing power the matrix, it results in a strictly positive matrix but the last column. Let's denote **B** the matrix obtained removing the last column and the last row. **B** is a matrix that verifies the assumptions of the corollary of the *Perron-Frobenius* theorem (if the mortality rates and the fertility rates are strictly positive, which is the case); det $\mathbf{L} = 0$ and det $\mathbf{B} \neq 0$. Therefore, the dominant eigenvalue $\lambda_{\boldsymbol{B}} > 0$ and the dominant eigenvector $\boldsymbol{V^B}$ of **B** are such that even for **L** (which adds to the eigenvalues of **B** the only zero eigenvalue because det $\mathbf{L} = 0$):

$$\lambda_{\boldsymbol{L}} = \lambda_{\boldsymbol{B}} \;\; ; \quad \boldsymbol{V^L} = \begin{bmatrix} \boldsymbol{V^B} \\ t \end{bmatrix} \;\; ; \;\; \mathrm{t} \in \mathrm{R}.$$

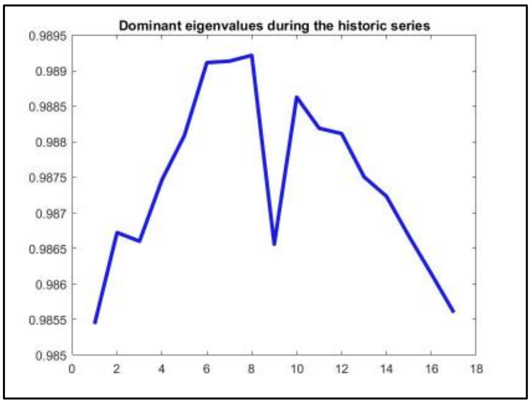
It remains to prove that $\mathrm{t} > 0$ (to obtain $\boldsymbol{V^L}$ strictly positive): explicitly from $\mathbf{L}\boldsymbol{V^L} = \lambda_{\boldsymbol{L}}\boldsymbol{V^L}$ it follows $\mathrm{t} = \lambda_{\boldsymbol{L}}{}^{-1}\, \sigma_n{}^{-1}\, \boldsymbol{V}^{\boldsymbol{B}}_{n-1} > 0$ (where $\sigma_n$ represent the survival rate of the $n^{th}$ group, namely the last one). By studying the dynamic of the system $\boldsymbol{X}_{k+1} = \mathbf{L}\boldsymbol{X}_k$ the corollary of the *Perron-Frobenius* theorem ensures that the age classes distribution that is observed in the long term is the one provided by the vector $\boldsymbol{V^L}$.

Because of this discussion, the initial matter of the existence of the eigenvector's basis is meaningless, being that every needed algebraic tool is guaranteed by the previous observations.

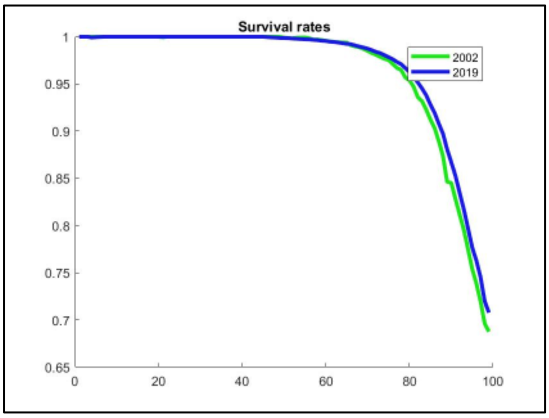
- Quantitative results



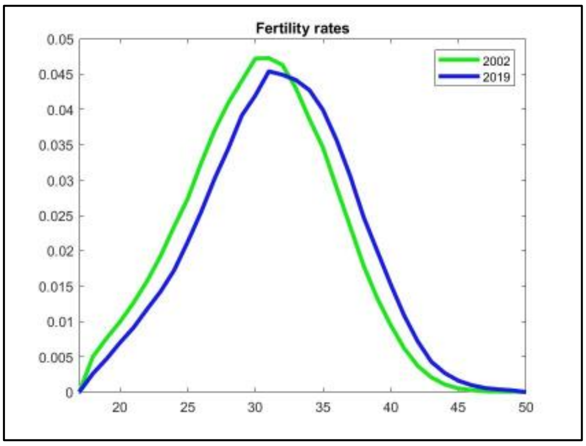

In order to analyse the results, the first argument to take into account is the goodness of the model and of the matrix. To study this behaviour, a simulation is performed using the matrix made with the last available data (the populations in 2019 and in 2018) and fitting this model with the starting population in 2002. Iterating for eighteen times the matrix multiplication, the resulting forecasted population should be the same as the one in 2019 (in the graph the two shapes of the populations' groups can be seen in function of the age of the groups). The overall error from the projection to the real data is 7,76%, an acceptable range for an almost 20-year forecast. Moreover, the migration phenomenon is relevant in this analysis and must be considered, in fact people who arrive from a foreign country are not included in the model and in the matrix (their presence is not in the survival or in the fertility rates), and moreover the approximation made in the construction of the matrix is completely due to this fact and produces almost the entire error. To strengthen the argument, it can be clearly seen how the error is not uniformly distributed along the graph, in fact the right part of the figure is forecasted better from the model than the left one, indeed, the older population is less inclined to migration than the younger one; and furthermore the real data is steadily upper than the projection, because the main migration channel is from the foreign countries to Italy.




The first quantitative observation that must be made is the one on the dominant eigenvalue, being its importance higher than any other aspect. In fact, if the dominant eigenvalue $\lambda_L < 1$ , then obviously $|\lambda_j| < 1 \ \forall j$ and so the dynamics will tend to the zero vector. The graph on the historic series oscillates considerably, but it is steadily less than 1, despite being close to it. This fact highlights how the overall population is decreasing during the years, and in particular this drop is caused by natural factors (as the ISTAT report says). In fact, even during the years of growing in the population, the dominant eigenvalue stays below 1, as saying that sooner or later the population will go down.

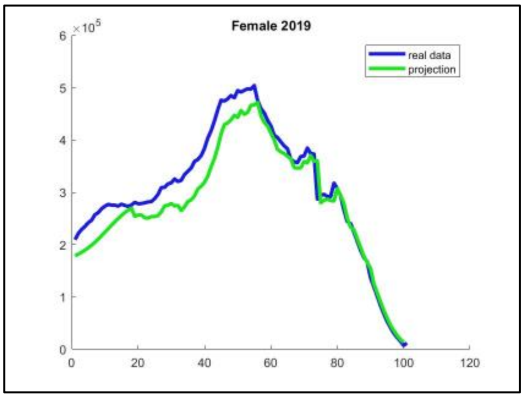


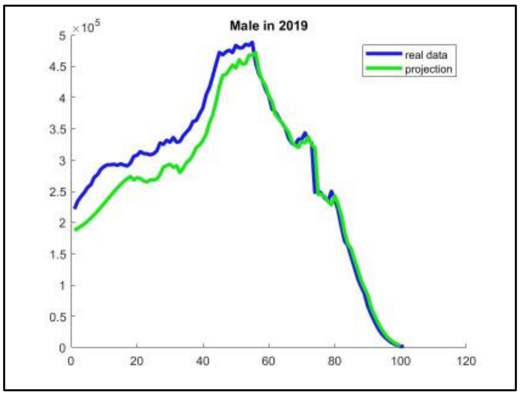

In order to understand the aging of the population, an analysis on the survival rates is made: from the 2002 and 2019 data the survival rates are extracted and plotted, in function of the age of the group for which the rate is extracted. Obviously, until around 50 years old the line is almost equal to 1, then there is a significant plunge. Despite being the two lines close, the pattern is clear, and the survival rates from the 2019 population are considerably bigger than the 2002's one.



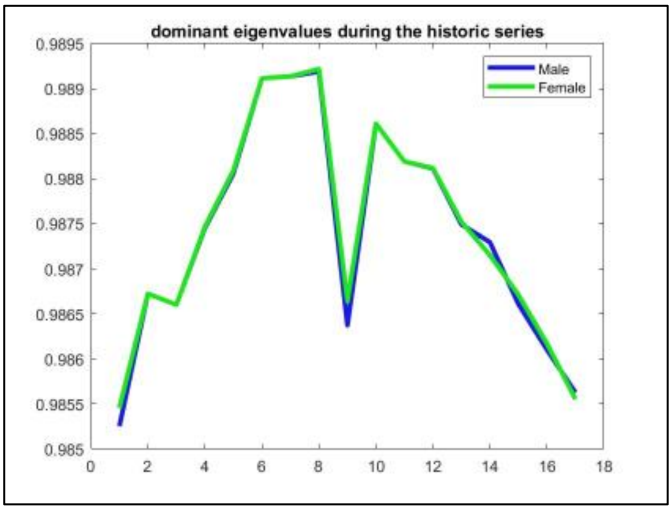

According to the ISTAT report, the drop in the new-borns is due to natural causes, because of the drop in fertile population. Because of this, the changing in the fertility rates should not be dramatic during the 20-years period. And this is the case, the rates are almost equal from the two years, the only significant difference is a translation of the graph (representing the fertility rates in function of the age of the parents) to the right side, but the pattern is the same. The report highlights how people tend to create their own family and to procreate more lately year after year, as the graph says.

Now the gender difference is taken into account: the same analysis made before is made both for males and females.



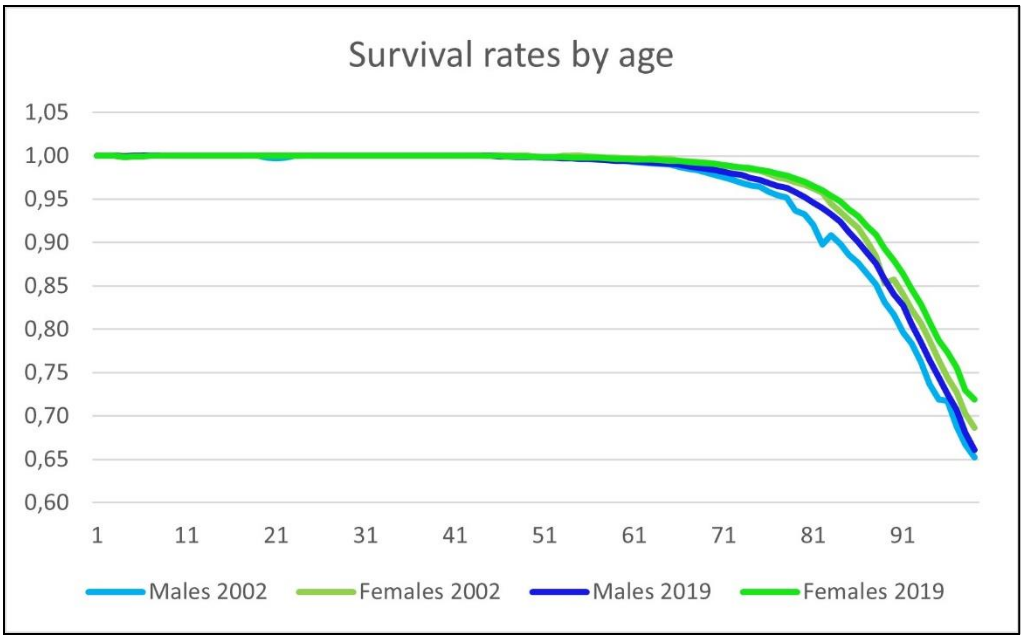




The first result is again about the goodness of the model and again the difference is due to migration because of all the arguments already discussed for the general case. Considering females, the error is 8,2%. Regarding the males, the difference is 7,27%.




Considering the dominant eigenvalue, the difference between males and females is negligible, and in both the populations the pattern is oscillating (as in the general case) and steadily below 1. The pattern is clearly the one of the general case, and now the absence of differences between the two genders is absolutely noticeable. This, with the next graph, are the most important result in terms of differences from males to female.

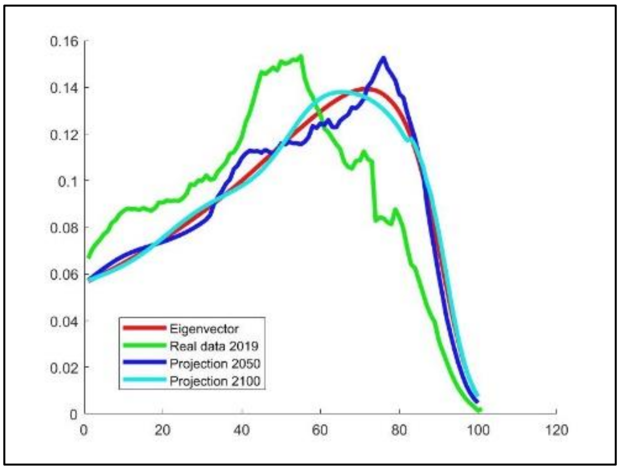
A general plot about the survival rates has been made: the rates from 2002 and from 2019 have been plotted together both for the females and the males, the rates are in function of the age of the corresponding group. The first thing to note is that for all the 4 lines the pattern is similar, and as already said the line is decreasing because of the more likely death of the older groups of people. In this chart it can be seen how overall the male population is more fragile than the female one, in fact the survival rates are lower both in 2002 and in 2019; this strengthens the fact that there are more woman than man, as already observed in the qualitative analysis. In both the two genders there are a constant increase of the rates during the 20-years period time, indeed, the 2019 males' rates are almost equal to the 2002 females', but the difference between the two lines is almost constant. In fact, from a numerical analysis, the difference between the 2002's line and the 2019's one for males is 1,2% while for females is 1,1%. Moreover, the difference from the male's line and the female's one remains almost constant, going from 1,9% in 2002 to 1,7% in 2019, so actually the two lines are getting closer but in a negligible way.



- Asymptotic analysis

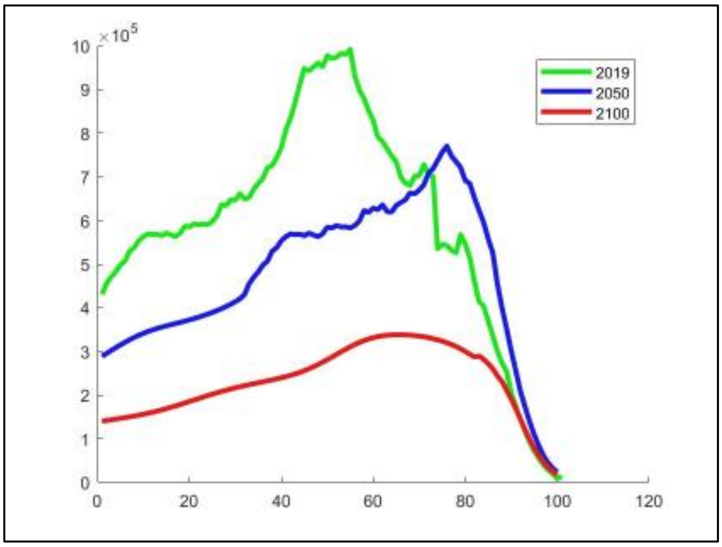
The first step of the asymptotic analysis is the *theorem d)*. It can be said that the long time claimed in the theorem is not arrived yet, in fact, despite the last massive change in the number of Italian citizens was due to the second World War, the last century was dense of changes in the social behaviour and in traditions, as the ISTAT report says, producing continuous variations in the coefficients of the Leslie matrix. Taking this into account, the asymptotic behaviour still can nor be seen and will present its characteristics in the future, clearly only if the trend remains like this.

The normalized eigenvector is plotted with the normalized population from 2019, the normalized population the model forecasts for 2050 and the one for 2100, all in function of the age of the groups. It is clear how the shape of the graph tends to become the one represented by the eigenvector, and how nowadays the asymptotic behaviour is yet to reach.



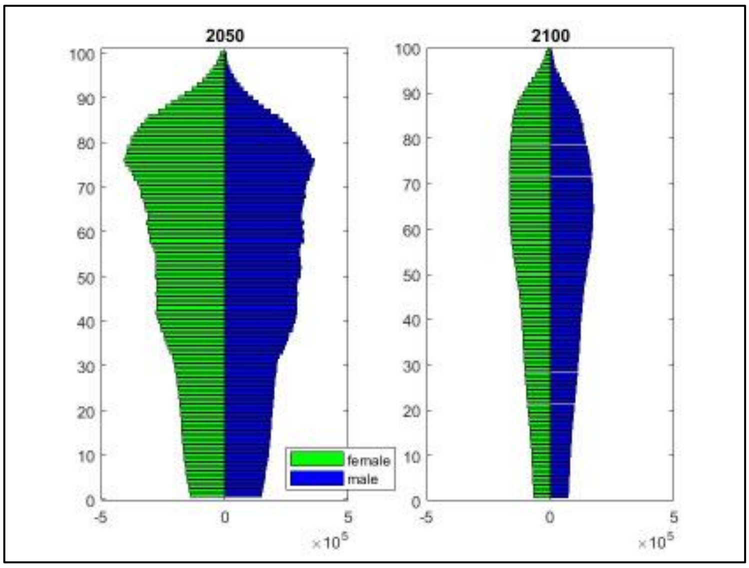

The shape will be close to the one provided by the eigenvector, but because of the eigenvalues, the population will collapse to zero, so let's take into account a quantitative analysis of the asymptotic behaviour. In these forecasts, the median age is relevant, it shifts from 48 years old in 2019, to 54 and 55 in 2050 and 2100, respectively. Actually, the interesting fact is that the median for the normalized eigenvector is 55 too, so it can be said that the median age tends to the median of the eigenvector and this fact strengthen how the asymptotic behaviour is not reached yet and how in the projection for 2100 it is achieved.

The one norm of the vector is here interesting, in fact, it says how many people will live in Italy, at least according to the model: the one norm of the population in 2050 is around 47 million people and in 2100 is around 23 million people. Even more interesting is the shifting of the peak of the image: while in 2019 the more represented age group is the 55-years-old one, in 2050 it become the 76 years-old one.

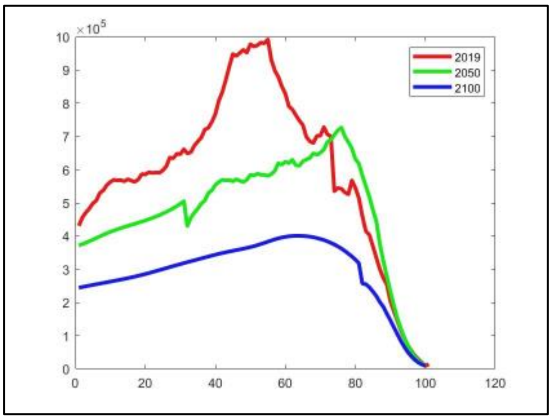



The most worrying aspect of this analysis is the possibility of collapse of social securities, in fact with the law in force the projection for retirements is unfeasible: the forecast in 2050 sees 21 million people capable of work and 19 million retired, a non-sustainable situation with a margin by only 12%, considering all people with a coherent age which work. Obviously, the situation is not improving in the projection for 2100, there will be around 10 million people working and 9 million retired people, with a margin of 11%.

The last graph summarizes all these considerations, it is made forecasting both the males and females' population again in 2050 and 2100. It is impressive how the population will plunge, and the aging, compared to the 2019 graph, is considerable, being the most populated groups going up. On the other hand, it seems to become negligible the difference between the number of males and females.



- Different scenario




This last analysis is made in order to stress again the magnitude of the problem inside the Italian population. In fact, the different scenario analysed is the one in which Italians would be able to increase their fertility rate, not a simple thing to do in the real life, but mathematically speaking the analysis is made increasing the fertility rates by 20%. The pattern is the same as before as it can be seen from the graph, and more importantly the dominant eigenvalue remains strictly smaller than one. So, the population will again collapse to zero.

The only aspect that change is the shape of the population and in particular the considerations on the retirements: in this scenario the margin for the workers is around 50%. It is out of the scope of this project to study if this number is enough. The median of the population in 2100 is now 51 years old.

- Conclusions

A more precise forecast and analysis must consider the migrations flows, in fact, the consideration of this phenomenon is not complete in this project, even because the lack of data available.

Nevertheless, many relevant results have been presented, and the trend that the population is facing is very clear. The fact that the eigenvalues stay below 1 even if the population is increasing in that period is the most relevant result: it shows the power and the reliability of the model, because in the subsequent years the trend actually change and the number of Italian residents starts to go down; the same happens for the new-borns, the model is able to predict the decrease of future new-borns because it is due to the fact that there will be less fertile women, and actually so it is. The last so relevant fact is the one about the different scenario, it highlights how the Italian population have in some sense crossed a non-returning point, because even with an incredible effort to increase some of the coefficients, it will be very difficult to reverse the trend.