L2 Random Variables

1 Random Variable Basics

Definition (Random Variable) A random variable is a function $X : \Omega \to \mathbb{R}$ with the property that $\{\omega \in \Omega : X(\omega) \in \Sigma\}$ for each $x \in \mathbb{R}$. Such a function is said to be **Σ-measurable**.

Definition (Distribution Functions) The **distribution function** of a random variable X is the function $F : \mathbb{R} \to [0, 1]$ given by $F(x) = \Pr(X \leq x)$.

Lemma (Properties of Distribution Functions)

1. $\lim_{x \to -\infty} F(x) = 0, \lim_{x \to \infty} F(x) = 1$
2. if $x < y$ then $F(x) \leq F(y)$
3. F is right-continuous, that is, $F(x + h) \to F(x)$ as $h \downarrow 0$.

Note: In order to prove the third property, the theorem of **continuity of probability measures** is needed.

Definition (Independence of Random Variable) Random variables X and Y are called **independent** if $\{X \leq x\}$ and $\{Y \leq y\}$ are independent events for all $x, y \in \mathbb{R}$.

1.1 Discrete Random Variable

Definition (Discrete Random Variable) The random variable X is called **discrete** if it takes values in some *countable* subset $\{x_1, x_2, \dots\}$, only, of $\mathbb{R}$. The **discrete random variable** X has **(probability) mass function (pmf)** $f : \mathbb{R} \to [0, 1]$ given by $f(x) = \Pr(X = x)$.

1.2 Continous Random Variable

Definition (Continous Random Variable) The random variable X is called **continuous** if its distribution function can be expressed as

$$F(x) = \int_{-\infty}^{x} f(u)du, x \in \mathbb{R}$$

for some integrable function $f : \mathbb{R} \to [0, \infty)$ called the (**probability**) **density function (pdf)** of X.

> Note that the word ‘continuous’ is a misnomer when used in this regard: in describing X as continuous, we are referring to a property of its distribution function rather than of the random variable (function) X itself.

1.3 Moment and Deviation

Definition (Expectation) The **mean value**, or **expectation**, or **expected value** of the random variable X with mass function f is defined to be

$$\mathbb{E}(X) = \sum_{x:f(x)>0} x f(x)$$

whenever this sum is absolutely convergent.

Note: $\mathbb{E}(X)$ can be denoted as $\mathbb{E}X$.

Theorem (Linearity of Expectation) if $a, b \in R$ then $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$

Definition (Variance) The **variance** of the random variable X with mass function f is defined to be

$$\mathrm{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2) - \mathbb{E}X^2$$

Theorem. X and Y are **independent** and both have finite variances, then

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

2 Elementary Models

2.1 Discrete Models

2.1.1 Bernoulli Trials

Definition (Bernoulli trials) A random variable X takes values 1 and 0 with probabilities p and $1 - p$, respectively. $\mathbb{E}(X) = p, \mathrm{Var}(X) = p(1 - p)$.

2.1.2 Binomial Distribution

Definition (Binomial distribution) We perform n independent Bernoulli trials $X_1, X_2, \ldots, X_n$ and count the total number of successes $Y = X_1 + X_2 + \cdots + X_n$. The mass function of Y is:

$$f(k) = \binom{n}{k} p^k (1 - p)^k, k = 0, 1, 2, \ldots, n.$$

$\mathbb{E}(Y) = np, \mathrm{Var}(Y) = np(1 - p)$.

2.1.3 Geometric Distribution

Definition (Geometric distribution) A *geometric* variable X is a random variable with the geometric mass function

$$f(k) = p(1-p)^{k-1}, k = 1, 2, \ldots$$

for some number p in $(0, 1)$. $\mathbb{E}(X) = p^{-1}, \mathrm{Var}(X) = (1-p)p^{-2}$.

2.1.4 Poisson Distribution

Definition (Poisson distribution) A *Poisson* variable is a random variable with the Poisson mass function

$$f(k) = \frac{\lambda^k}{k!}e^{-\lambda}$$

for some $\lambda > 0$. $\mathbb{E}(X) = \lambda, \mathrm{Var}(X) = \lambda$.

2.2 Continous Models

2.2.1 Uniform Distribution (Continous)

Definition (Uniform Distribution) The random variable X is uniform on a, b if it has density function f:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < x < b, \\ 0 & \text{otherwise,} \end{cases}$$

$\mathbb{E}(X) = (a+b)/2, \mathrm{Var}(X) = \dfrac{(b-a)^2}{12}$.

2.2.2 Exponential Distribution

Definition (Exponential distribution) The random variable X is *exponential* with parameter $\lambda(> 0)$ if it has density function f:

$$f(x) = \lambda e^{-\lambda x}, \text{for } x \geq 0$$

$\mathbb{E}(X) = 1/\lambda, \mathrm{Var}(X) = 1/\lambda^2$.

2.2.3 Normal Distribution

The normal (or Gaussian) distribution with two parameters μ and σ^2 has density function f:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}(x-\mu)^2}, -\infty < x < \infty$$

$\mathbb{E}(X) = \mu, \mathrm{Var}(X) = \sigma^2$.

If $\mu = 0$ and $\sigma^2 = 1$ then $f(x)$:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, -\infty < x < \infty$$

is the density of the *standard* normal distribution.

2.3 Summary

Distribution	Parameter	pmf/pdf	$\mathbb{E}(X)$	$\mathrm{Var}(X)$
Bernoulli	p	$f(k) = p^k(1-p)^{(1-k)}, k = 0, 1$	p	$p(1-p)$
Binomial	n, k, p	$f(k) = \binom{n}{k} p^k(1-p)^k, k = 0, 1, 2, \ldots, n.$	np	$np(1-p)$
Geometric	p	$f(k) = p(1-p)^{k-1}, k = 1, 2, \ldots$	p^{-1}	$(1-p)p^{-2}$
Poisson	λ	$f(k) = \frac{\lambda^k}{k!} e^{-\lambda}$	λ	λ
Uniform (continuous)	$a, b \in \mathbb{N}$	$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b, \\ 0 & \text{otherwise,} \end{cases}$	$(a+b)/2$	$\frac{(b-a)^2}{12}$
Exponential	λ	$f(x) = \lambda e^{-\lambda x}, x \geq 0$	$1/\lambda$	$1/\lambda^2$
Normal $N(\mu, \sigma)$	μ, σ	$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, -\infty < x < \infty$	μ	σ^2