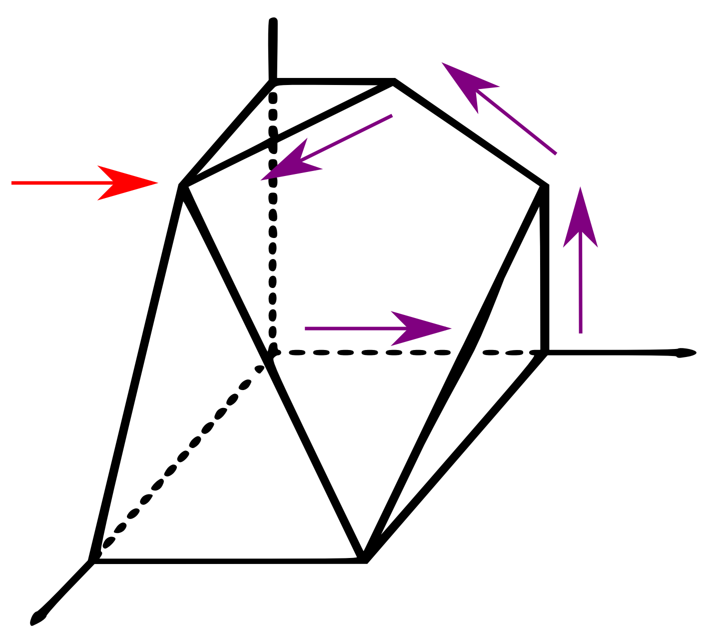
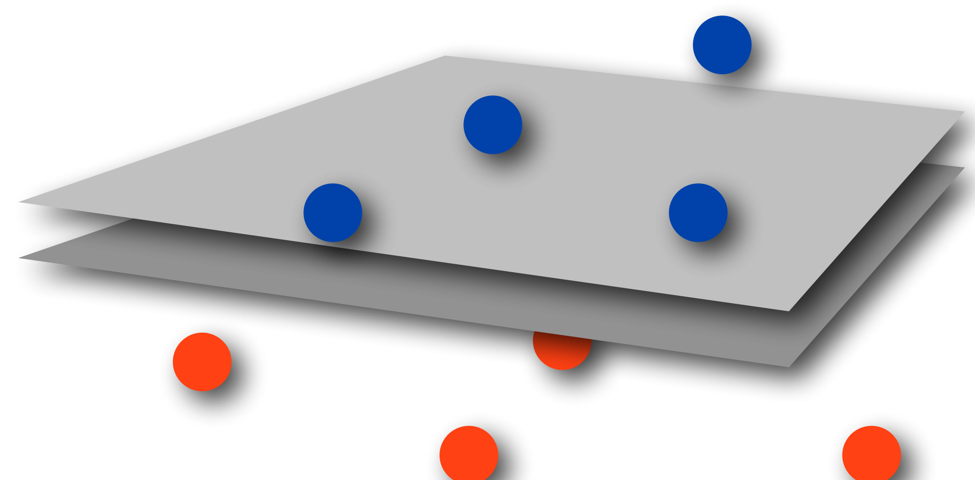
Linear Programming with CGAL





AlgoLab HS22 Tutorial — Nicolas Zucchet

Based on work by Luis Barba, Bernd Gärtner and Sebastian Stich.

Today

- Quick introduction to Linear Programming (formulation and representation)
- Linear Programming in CGAL
- Examples

Linear Programming (LP)

- Central topic in optimization
- Powerful modeling tool in many applications
- Attracted a lot of attention in optimization during the last six decades for two main reasons:
  - Applicability: Many real-world problems can be modeled using LP;
  - Solvability: Theoretically and practically efficient techniques for solving large-scale problems.

Linear Programming (LP)

An optimization problem subject to constraints.

- **Variables:** parameters whose values we can choose;
- **Objective function:** the criterion to minimize (e.g. cost) or maximize (e.g. profit);
- **Constraints:** limitations for choosing values for the

Obs. Considering minimization is sufficient because minimizing -f is equivalent to maximizing f.

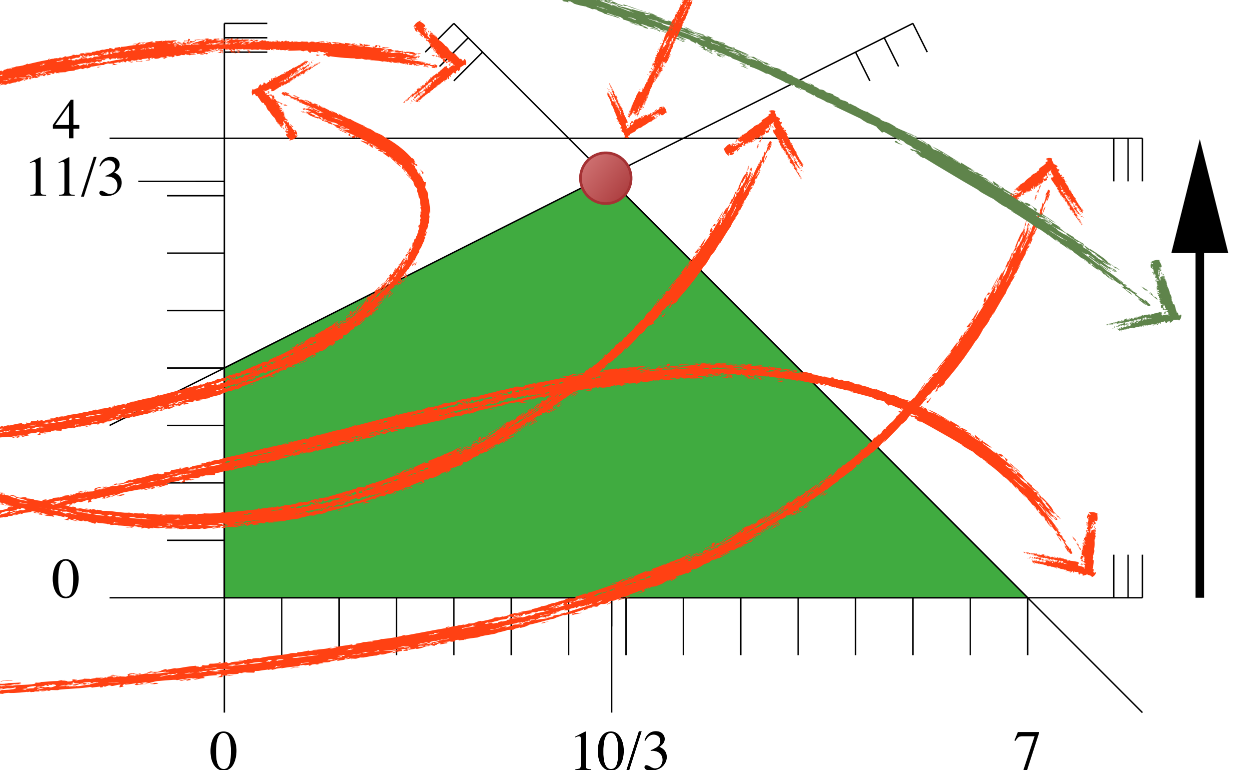
LP—Mathematical Formulation

- **Problem:** Minimize a linear function in *n* variables subject to *m* linear (in)equality constraints.

- **Example** (*n*=2, *m*=5):

$$
\begin{aligned}
&\text{minimize} \\
&-32y + 64 \quad \text{subject to} \\
&\qquad x + y \le 7 \\
&\qquad -x + 2y \le 4 \\
&\qquad x \ge 0 \\
&\qquad y \ge 0 \\
&\qquad y \le 4
\end{aligned}
$$

Optimal solution (value -160/3)



LP—Mathematical Formulation

- **Problem:** Minimize a linear function in *n* variables subject to *m* linear (in)equality constraints.

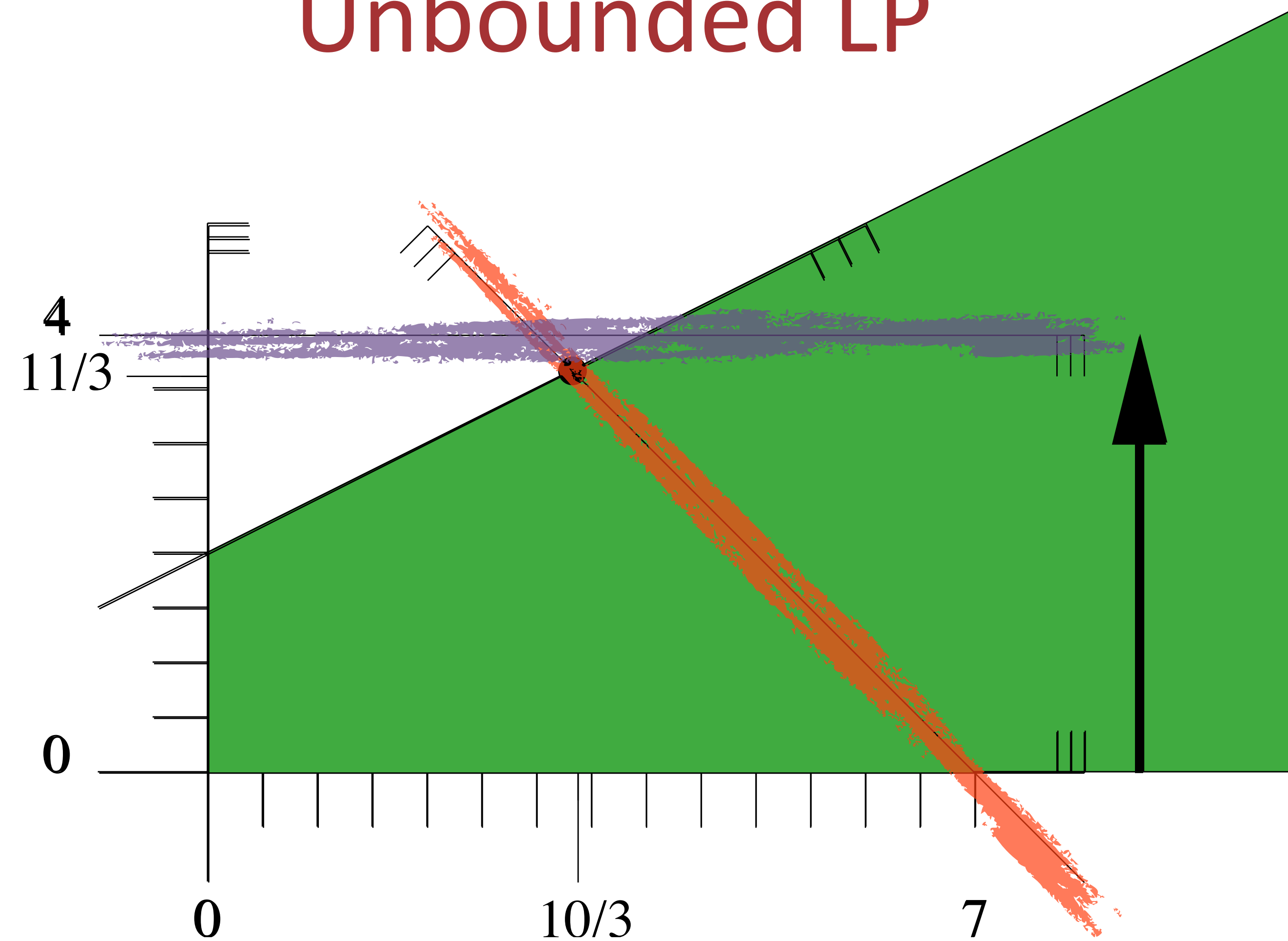
- **Example** (*n*=2, *m*=5):

minimize

$$
-32y + 64 \;\; \text{subject to}
$$

$$
\begin{aligned}
\cancel{x + y} &\;\cancel{\leq}\; \cancel{7} \\
-x + 2y &\leq 4 \\
x &\geq 0 \\
y &\geq 0 \\
\cancel{y} &\;\cancel{\leq}\; \cancel{4}
\end{aligned}
$$

Unbounded LP



LP—Mathematical Formulation

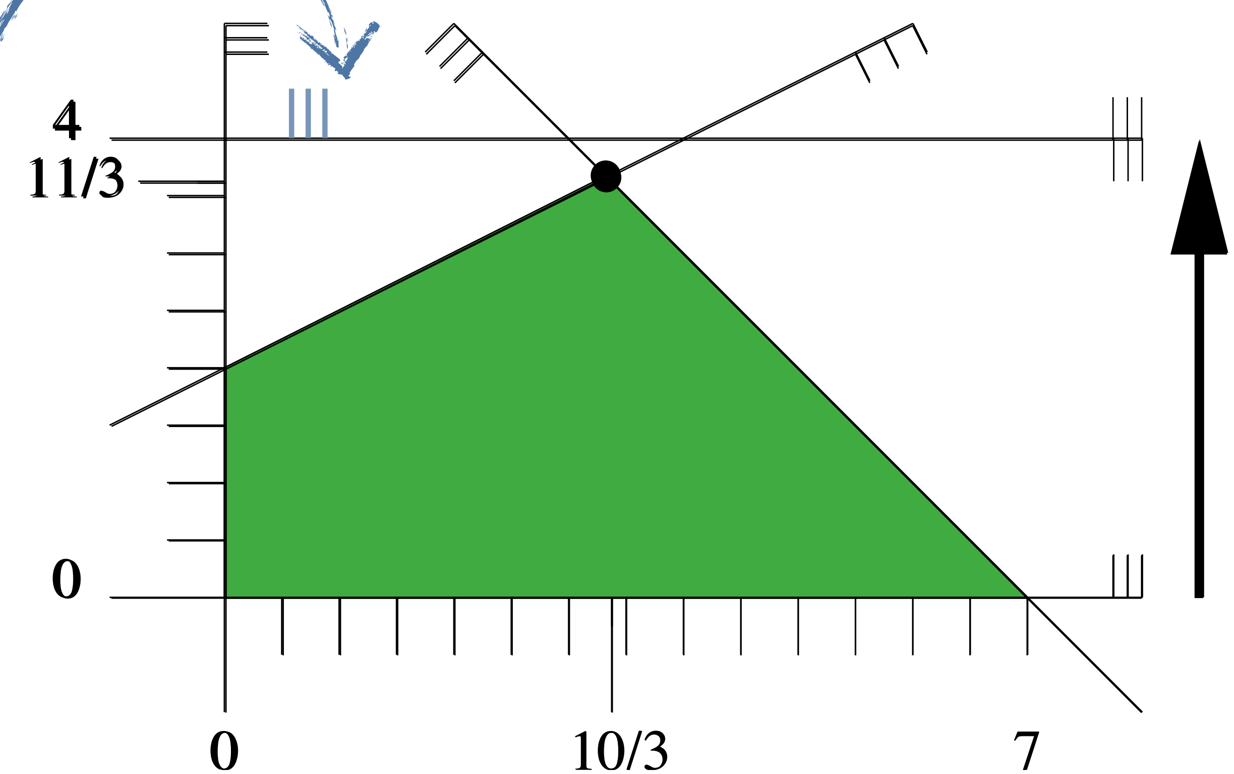
- **Problem:** Minimize a linear function in *n* variables subject to *m* linear (in)equality constraints.

- **Example** (*n*=2, *m*=5):

minimize

$$
\begin{array}{rcl}
-32y + 64 & \text{subject to} & \\
x + y & \leq & 7 \\
-x + 2y & \leq & 4 \\
x & \geq & 0 \\
y & \geq & 0 \\
y \geq 4 \quad \cancel{y} & \cancel{\leq} & \cancel{4}
\end{array}
$$

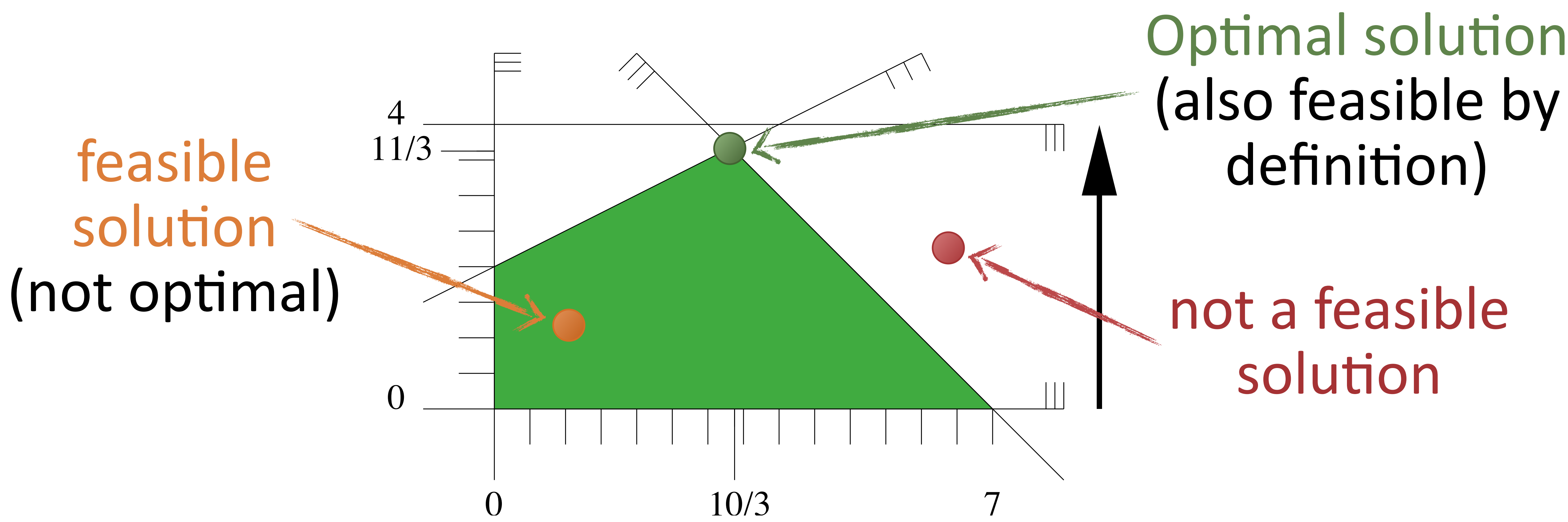
Infeasible LP



Feasible and Optimal Solutions

A feasible solution is an assignment of values to the variables of an LP that satisfies all constraints.

A feasible solution is optimal if it minimizes the objective function (among all feasible solutions).



Types of Linear Programs

A linear program is of one of the following three types:

- **Optimal:** there exists an optimal solution (possibly many) that attains the unique minimum of the objective function;
- **Unbounded:** there exist feasible solutions that attain arbitrarily small values of the objective function;
- **Infeasible:** There is no feasible solution.

LP—Matrix Formulation

- **Problem:** Minimize a linear function in *n* variables subject to *m* linear (in)equality constraints!

- **Example** (*n*=2, *m*=5):

convert "≥" to "≤" by inverting coefficients

minimize

$$-32y + 64 \quad \text{subject to}$$

$$
\begin{array}{rrcl}
 & x + y & \leq & 7 \\
 & -x + 2y & \leq & 4 \\
-x \leq 0 & \cancel{x} & \cancel{\geq} & \cancel{0} \\
-y \leq 0 & \cancel{y} & \cancel{\geq} & \cancel{0} \\
 & y & \leq & 4
\end{array}
$$

$$
\begin{array}{ccccc}
\text{A} & & & & \text{b} \\
\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} & \leq & & \begin{bmatrix} 7 \\ 4 \\ 0 \\ 0 \\ 4 \end{bmatrix}
\end{array}
$$

LP—Matrix Formulation

- **Problem:** Minimize a linear function in *n* variables subject to *m* linear (in)equality constraints!

- **Example** (*n*=2, *m*=5)**:**

$$\begin{array}{rcl} \text{minimize} & & \\ \boxed{-32y + 64} \ \text{subject to} & & \\ x + y & \leq & 7 \\ -x + 2y & \leq & 4 \\ -x & \leq & 0 \\ -y & \leq & 0 \\ y & \leq & 4 \end{array}$$

$$\underbrace{\begin{pmatrix} 0 \\ -32 \end{pmatrix}}_{c^T} \begin{pmatrix} x & y \end{pmatrix} + \underbrace{64}_{c_0}$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 2 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} \leq \underbrace{\begin{pmatrix} 7 \\ 4 \\ 0 \\ 0 \\ 4 \end{pmatrix}}_{b}$$

LP—General Form in CGAL

≤, =, or ≥ (individually for each constraint)

$$
\begin{array}{ll}
\text{minimize} & c^T x + c_0 \\
\text{subject to} & Ax \gtreqless b \\
 & \ell \leq x \leq u
\end{array}
$$

variables

$$(x, c, \ell, u \in \mathbb{R}^n, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad c_0 \in \mathbb{R})$$

objective function

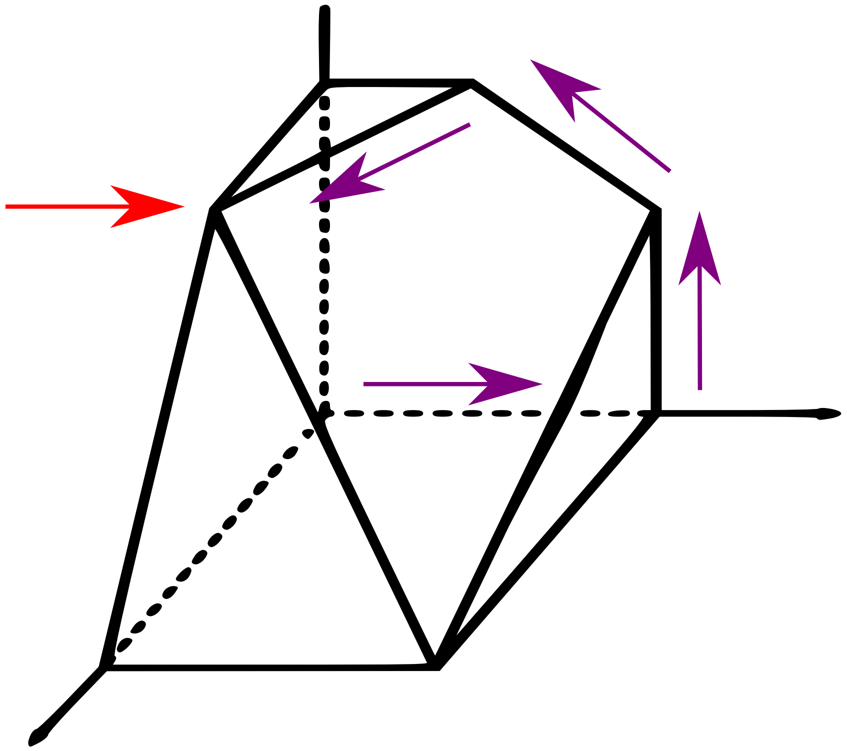
shift

lower and upper bounds

right-hand side

constraint matrix

LP—Geometric Interpretation

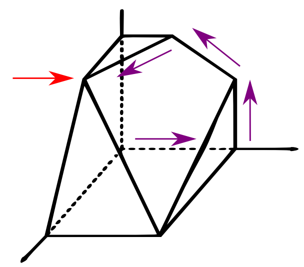


$$\begin{array}{ll} \text{minimize} & c^T x + c_0 \\ \text{subject to} & Ax \begin{smallmatrix}\geq \\ = \\ \leq\end{smallmatrix} b \\ & \ell \leq x \leq u \end{array}$$

$$P = \{x | Ax \leq b\}$$

- The constraints define an *n*-dimensional convex polyhedron *P* with ≤ *m* *faces*.
- Optimum objective value is attained at a vertex of *P*.
- But #vertices(*P*) can be exponential in *m* and *n*.
  => Trying all vertices not feasible.

LP—Complexity of CGAL Solver



$$\begin{array}{ll} \text{minimize} & c^T x + c_0 \\ \text{subject to} & Ax \gtreqqless b \\ & \ell \leq x \leq u \end{array}$$

$$P = \{x | Ax \leq b\}$$

- The constraints define an n-dimensional convex polyhedron P with ≤ m *faces*.
- CGAL solver uses a simplex-type algorithm: Walk from vertex to vertex along edges of P while improving the objective value.
- Worst-case complexity is exponential in n and m.

For min $\{n, m\}$ small, the complexity is O(max $\{n, m\}$).

LP in CGAL—Preamble

- Choose input type (in this order of preference):
  - `int`, `long`
  - `CGAL::Gmpz` (arbitrary precision integer)
  - `CGAL::Gmpq` (arbitrary precision rational)

  use the GNU Multi-Precision Library (GMP)
- Choose exact type for solver:
  - `CGAL::Gmpq` (if this is the input type)
  - `CGAL::Gmpz` (in all other cases)

```
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpz.h>

// choose input type (input coefficients must fit)
typedef int IT;
// choose exact type for solver (CGAL::Gmpz or CGAL::Gmpq)
typedef CGAL::Gmpz ET;
```

LP in CGAL—Preamble (2)

- Define program and solution type:

```
// program and solution types
typedef CGAL::Quadratic_program<IT> Program;
typedef CGAL::Quadratic_program_solution<ET> Solution;
```

- Yep, it is called Quadratic… (more general concept) but also used for linear programs.
- Internally, the solver uses `CGAL::Quotient<ET>`.

LP in CGAL—Setup

- Enter the program data:

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \end{array}$$

$$Ax \leq b \left\{ \begin{array}{rcl} x + y & \leq & 7 \\ -x + 2y & \leq & 4 \\ x & \geq & 0 \\ y & \geq & 0 \\ y & \leq & 4 \end{array} \right.$$

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

// set the coefficients of A and b
const int X = 0;
const int Y = 1;
lp.set_a(X, 0,  1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);  //  x + y  <= 7
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);  // -x + 2y <= 4
// set upper bound
lp.set_u(Y, true, 4);                                   //      y <= 4
// objective function
lp.set_c(Y, -32);                                       // -32y
lp.set_c0(64);                                          // +64
```

LP in CGAL—Setup

- Enter the program data:

$$
\begin{array}{ll}
\text{minimize} & -32y + 64 = c^T x + c_0 \\
\text{subject to} & \\
Ax \leq b & \left\{ \begin{array}{rcl} x + y & \leq & 7 \\ -x + 2y & \leq & 4 \\ x & \geq & 0 \\ y & \geq & 0 \\ y & \leq & 4 \end{array} \right.
\end{array}
$$

Default inequality is ≤.

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

// set the coefficients of A and b
const int X = 0;
const int Y = 1;
lp.set_a(X, 0,  1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);  //  x + y  <= 7
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);  // -x + 2y <= 4
// set upper bound
lp.set_u(Y, true, 4);                                   //       y <= 4
// objective function
lp.set_c(Y, -32);                                       // -32y
lp.set_c0(64);                                          // +64
```

LP in CGAL—Setup

- Enter the program data:

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \\ Ax \leq b & \left\{ \begin{array}{rcl} x + y & \leq & 7 \\ -x + 2y & \leq & 4 \\ x & \geq & 0 \\ y & \geq & 0 \\ y & \leq & 4 \end{array} \right. \end{array}$$

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

// set the coefficients of A and b
const int X = 0;
const int Y = 1;
lp.set_a(X, 0,  1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);  //  x + y  <= 7
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);  // -x + 2y <= 4
// set upper bound
lp.set_u(Y, true, 4);                                   //       y <= 4
// objective function
lp.set_c(Y, -32);                                       // -32y
lp.set_c0(64);                                          // +64
```

Lower bounds: All variables must be ≥ zero.

LP in CGAL—Setup

- Enter the program data:

$$
\begin{array}{ll}
\text{minimize} & -32y + 64 = c^T x + c_0 \\
\text{subject to} & \\
Ax \leq b & \left\{ \begin{array}{rcl} x + y & \leq & 7 \\ -x + 2y & \leq & 4 \\ x & \geq & 0 \\ y & \geq & 0 \\ y & \leq & 4 \end{array} \right.
\end{array}
$$

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

// set the coefficients of A and b
const int X = 0;
const int Y = 1;
lp.set_a(X, 0,  1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);  //  x + y  <= 7
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);  // -x + 2y <= 4
// set upper bound
lp.set_u(Y, true, 4);                                    //       y <= 4
// objective function
lp.set_c(Y, -32);                                        // -32y
lp.set_c0(64);                                           // +64
```

No upper bounds.

LP in CGAL—Setup

- Enter the program data:

$$
\begin{array}{ll}
\text{minimize} & -32y + 64 = c^T x + c_0 \\
\text{subject to} & \\
Ax \le b & \left\{ \begin{array}{rcl} x + y & \le & 7 \\ -x + 2y & \le & 4 \\ x & \ge & 0 \\ y & \ge & 0 \\ y & \le & 4 \end{array} \right.
\end{array}
$$

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

// set the coefficients of A and b
const int X = 0;
const int Y = 1;
lp.set_a(X, 0,  1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);  //  x + y  <= 7
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);  // -x + 2y <= 4
// set upper bound
lp.set_u(Y, true, 4);                                   //       y <= 4
// objective function
lp.set_c(Y, -32);                                       // -32y
lp.set_c0(64);                                          // +64
```

Convenient alias (index) for variables.

LP in CGAL—Setup

- Enter the program data:

$$
\begin{array}{ll}
\text{minimize} & -32y + 64 = c^T x + c_0 \\
\text{subject to} & \\
Ax \le b & \left\{ \begin{array}{rcl} x + y & \le & 7 \\ -x + 2y & \le & 4 \\ x & \ge & 0 \\ y & \ge & 0 \\ y & \le & 4 \end{array} \right.
\end{array}
$$

Constraint/inequality #0.

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

// set the coefficients of A and b
const int X = 0;
const int Y = 1;
lp.set_a(X, 0,  1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);  //  x + y  <= 7
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);  // -x + 2y <= 4
// set upper bound
lp.set_u(Y, true, 4);                                    //       y <= 4
// objective function
lp.set_c(Y, -32);                                        // -32y
lp.set_c0(64);                                           // +64
```

LP in CGAL—Setup

- Enter the program data:

$$
\begin{array}{ll}
\text{minimize} & -32y + 64 = c^T x + c_0 \\
\text{subject to} & \\
Ax \le b & \left\{ \begin{array}{rcl} x + y & \le & 7 \\ -x + 2y & \le & 4 \\ x & \ge & 0 \\ y & \ge & 0 \\ y & \le & 4 \end{array} \right.
\end{array}
$$

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

// set the coefficients of A and b
const int X = 0;
const int Y = 1;
lp.set_a(X, 0,  1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);  //  x + y  <= 7
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);  // -x + 2y <= 4
// set upper bound
lp.set_u(Y, true, 4);                                   //       y <= 4
// objective function
lp.set_c(Y, -32);                                       // -32y
lp.set_c0(64);                                          // +64
```

Constraint/inequality #1.

LP in CGAL—Setup

- Enter the program data:

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \\ Ax \leq b \left\{ \begin{array}{rcl} x + y & \leq & 7 \\ -x + 2y & \leq & 4 \\ x & \geq & 0 \\ y & \geq & 0 \\ y & \leq & 4 \end{array} \right. & \end{array}$$

You can use set_a() and set_b() for the upper bound, too. But we recommend set_u() for convenience and efficiency.

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

// set the coefficients of A and b
const int X = 0;
const int Y = 1;
lp.set_a(X, 0,  1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);  //  x + y  <= 7
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);  // -x + 2y <= 4
// set upper bound
lp.set_u(Y, true, 4);                                   //      y  <= 4
// objective function
lp.set_c(Y, -32);                                       // -32y
lp.set_c0(64);                                          // +64
```

Upper bound for y.

LP in CGAL—Setup

- Enter the program data:

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \\ Ax \leq b & \left\{ \begin{array}{rcl} x + y & \leq & 7 \\ -x + 2y & \leq & 4 \\ x & \geq & 0 \\ y & \geq & 0 \\ y & \leq & 4 \end{array} \right. \end{array}$$

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

// set the coefficients of A and b
const int X = 0;
const int Y = 1;
lp.set_a(X, 0,  1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);  //  x + y  <= 7
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);  // -x + 2y <= 4
// set upper bound
lp.set_u(Y, true, 4);                                   //       y <= 4
// objective function
lp.set_c(Y, -32);                                       // -32y
lp.set_c0(64);                                          // +64
```

LP in CGAL—Setup

- Enter the program data:

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \end{array}$$

$$Ax \le b \left\{ \begin{array}{rcl} x + y & \le & 7 \\ -x + 2y & \le & 4 \\ x & \ge & 0 \\ y & \ge & 0 \\ y & \le & 4 \end{array} \right.$$

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

// set the coefficients of A and b
const int X = 0;
const int Y = 1;
lp.set_a(X, 0,  1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);  //  x + y  <= 7
lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);  // -x + 2y <= 4
// set upper bound
lp.set_u(Y, true, 4);                                   //       y <= 4
// objective function
lp.set_c(Y, -32);                                       // -32y
lp.set_c0(64);                                          // +64
```

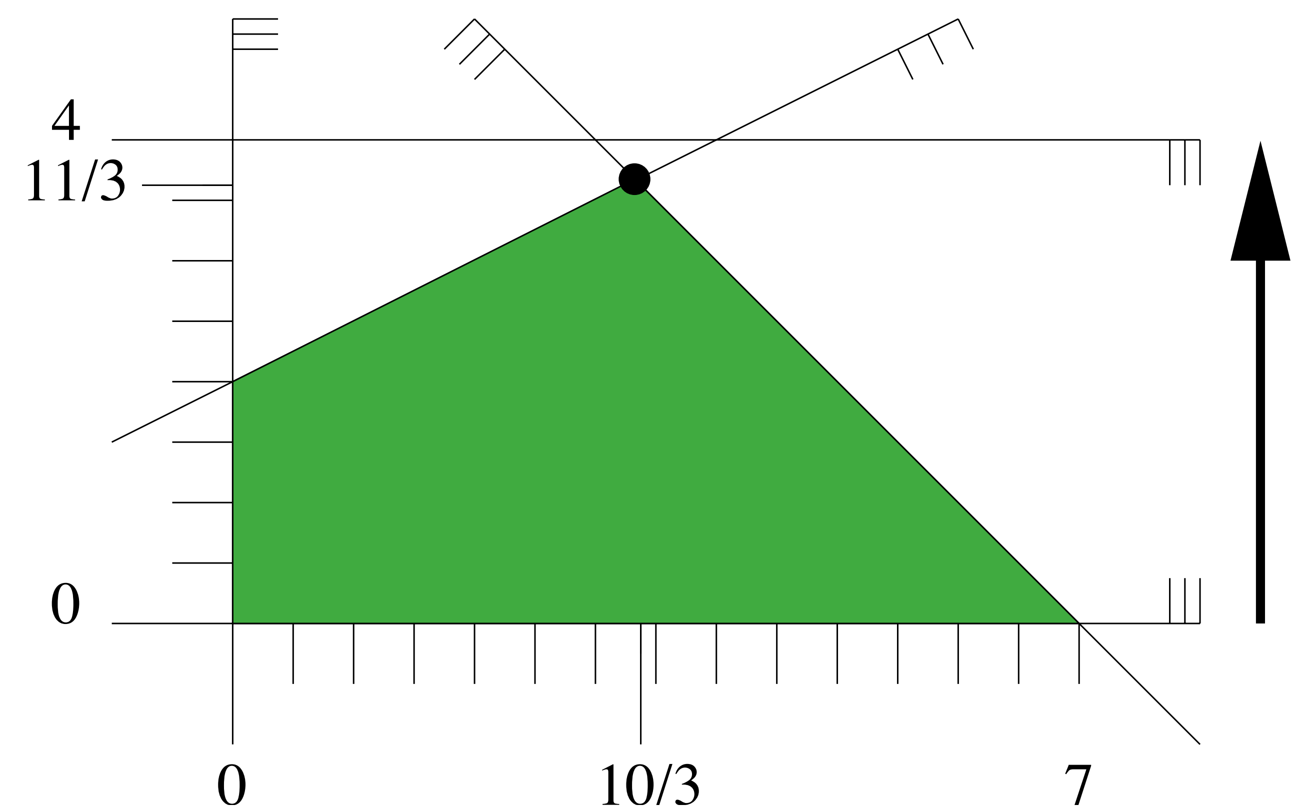
LP in CGAL—Solve

- Call the LP solver and output the solution:

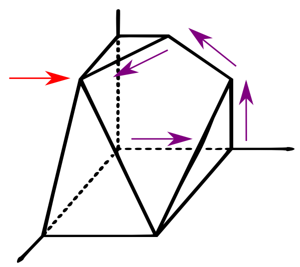
```
// solve the program, using ET as the exact type
Solution s = CGAL::solve_linear_program(lp, ET());

// output solution
std::cout << s;
```

- Output:

```
status:          OPTIMAL
objective value: -160/3
variable values:
  0: 10/3
  1: 11/3
```



When to apply LP?



$$\begin{array}{ll} \text{minimize} & c^T x + c_0 \\ \text{subject to} & Ax \gtreqless b \\ & \ell \le x \le u \end{array}$$

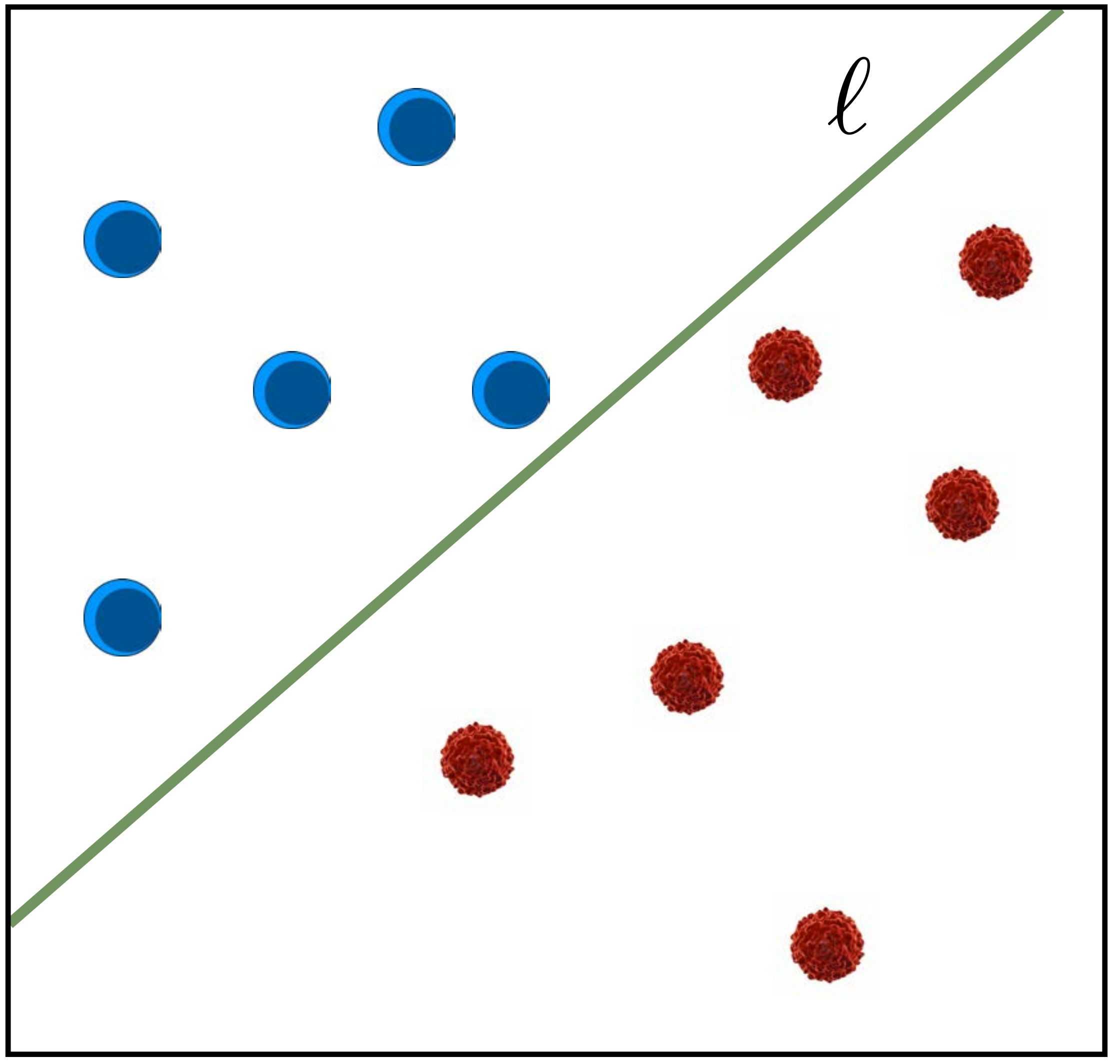
$$P = \{x | Ax \le b\}$$

- If a problem can be modeled using *n* variables and *m* constraints s.t.
  - constraints are **linear** (in)equalities in the variables
  - at least **one of** *n* or *m* is **small** (s.a.).
- Agenda/Checklist:

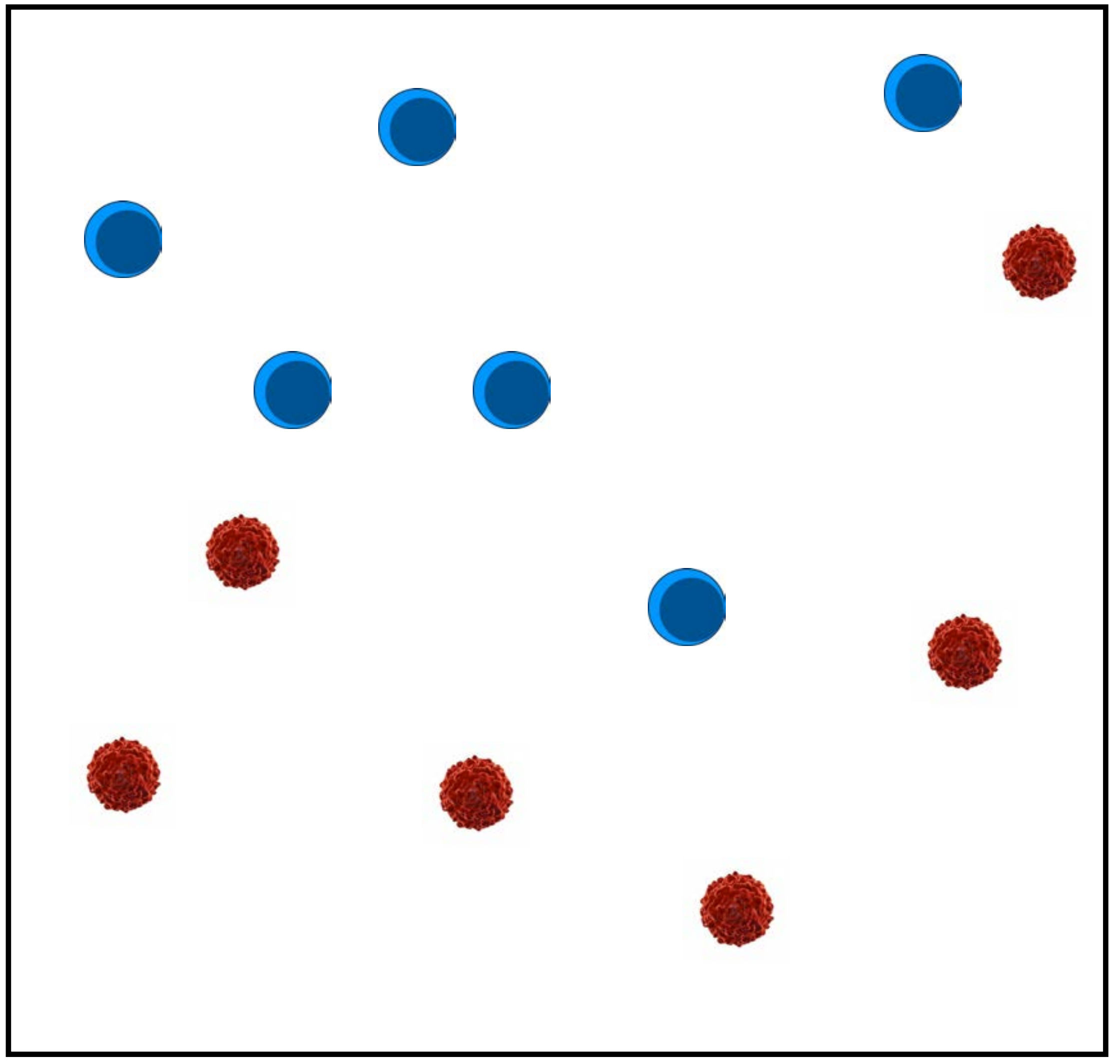
  Output Input
  - What are variables, what are constraints?
  - What is *n*, what is *m*? Is one of *n* or *m* small?
  - Are all constraints linear in the variables?

Ex. I: Linear Separation

- **Given:** A set R of red points and a set B of blue points.
- **Want:** A line ℓ that separates R and B.



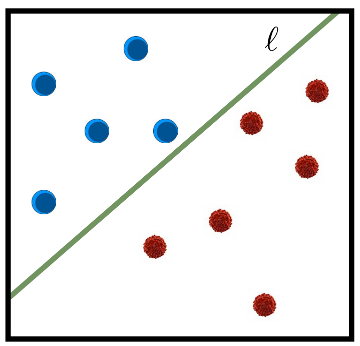
Linearly separable



No solution 😞

Ex. I: Linear Separation

- **Given:** A set R of red points and a set B of blue points.
- **Want:** A line ℓ that separates R and B.



- Agenda/Checklist:
- What are variables, what are constraints?
- What is *n*, what is *m*? Is one of *n* or *m* small?
- Are all constraints linear in the variables?

n=3 => small

m=|R|+|B|

Yes!

- **Variables/Output:** We are looking for a line ℓ.

  ℓ: $\{(x,y) : ax+by+c=0\}$, for a,b,c unknown.

- **Constraints/Input:** R and B on different sides of ℓ.

  $a p_x + b p_y \leq c$, for $(p_x,p_y) \in R$ and

  $a q_x + b q_y \geq c$, for $(q_x,q_y) \in B$.

Ex. I: Linear Separation

```
// example: decide whether two point sets R and B can be
// separated by a line

#include <iostream>
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpz.h>

// choose input type (input coefficients must fit)
typedef int IT;
// choose exact type for solver (CGAL::Gmpz or CGAL::Gmpq)
typedef CGAL::Gmpz ET;

// program and solution types
typedef CGAL::Quadratic_program<IT> Program;
typedef CGAL::Quadratic_program_solution<ET> Solution;

int main()
{
    // create an LP with Ax <= b and no lower/upper bounds
    Program lp (CGAL::SMALLER, false, 0, false, 0);
    const int a = 0;
    const int b = 1;
    const int c = 2;

    // number of red and blue points
    int m; std::cin >> m;
    int n; std::cin >> n;
```

```
    // read the red points
    for (int i = 0; i < m; ++i) {
        int x; std::cin >> x;
        int y; std::cin >> y;
        // set up constraint a x + b y + c <= 0
        lp.set_a(a, i, x);
        lp.set_a(b, i, y);
        lp.set_a(c, i, 1);
    }
    // read the blue points
    for (int i = 0; i < n; ++i) {
        int x; std::cin >> x;
        int y; std::cin >> y;
        // set up constraint a x + b y + c >= 0
        lp.set_a(a, m+i, -x);
        lp.set_a(b, m+i, -y);
        lp.set_a(c, m+i, -1);
    }
    // no objective function, just feasibility

    // solve the program, using ET as the exact type
    Solution s = CGAL::solve_linear_program(lp, ET());
    assert(s.solves_linear_program(lp));

    // output solution
    std::cout << s;
}
```

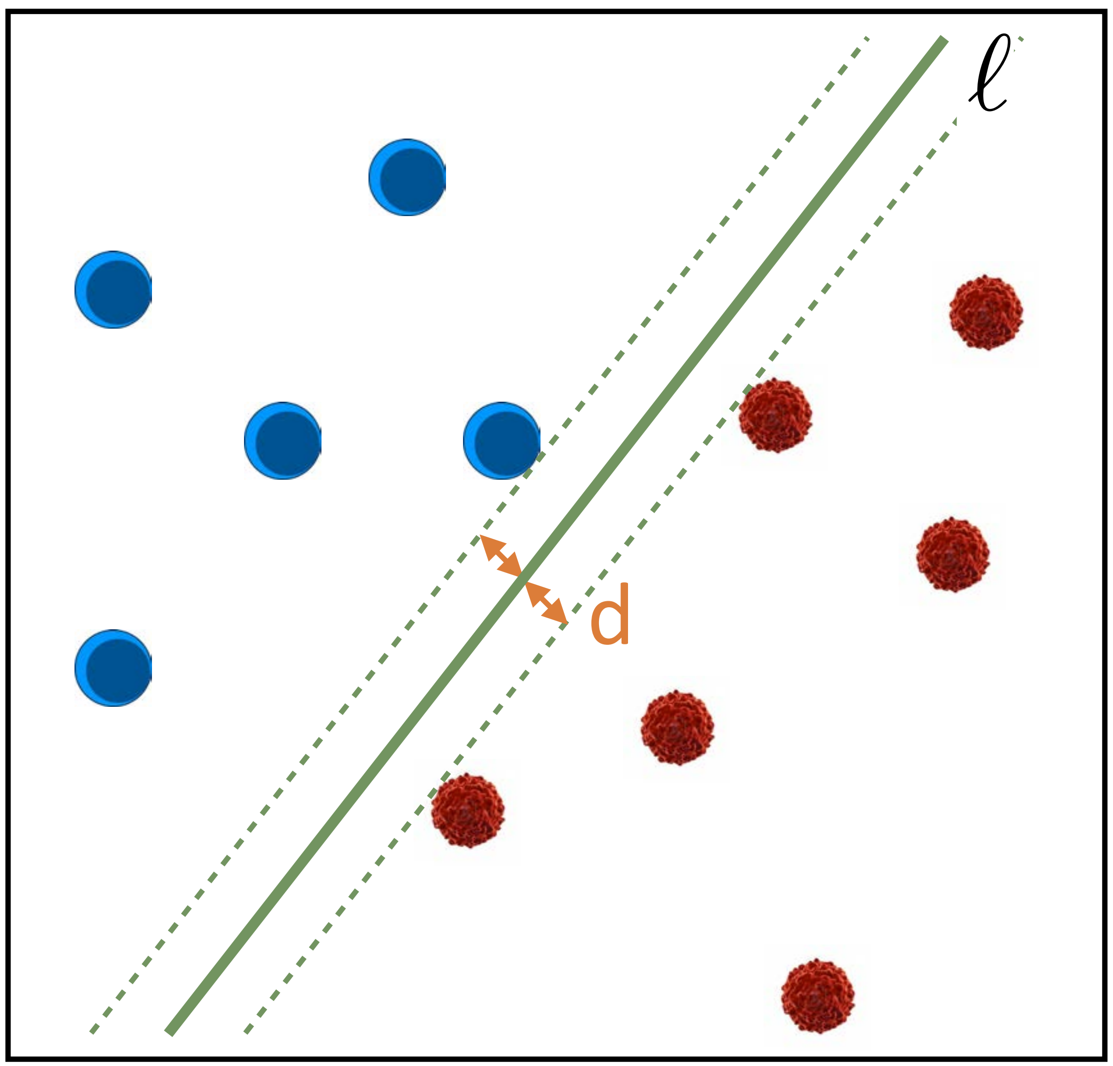
Output for R={(0,0)} and B={(1,1)}:

```
status:          OPTIMAL
objective value: 0/1
variable values:
  0: 0/1
  1: 0/1
  2: 0/1
```

Oops, not a line...

Ex. I: Linear Separation

- **Given:** A set R of red points and a set B of blue points.
- **Want:** A line ℓ that strictly separates R and B.



- We introduce a margin d around the line
- The constraints become

  $a\,p_x + b\,p_y \le c - d$, for $(p_x, p_y) \in R$ and
  $a\,q_x + b\,q_y \ge c + d$, for $(q_x, q_y) \in B$.

  status: UNBOUNDED

- We normalize by d

  $a\,p_x + b\,p_y \le c - 1$, for $(p_x, p_y) \in R$ and
  $a\,q_x + b\,q_y \ge c + 1$, for $(q_x, q_y) \in B$.

Ex. I: Linear Separation

```cpp
// example: decide whether two point sets R and B can be
// strictly separated by a line

#include <iostream>
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpz.h>

// choose input type (input coefficients must fit)
typedef int IT;
// choose exact type for solver (CGAL::Gmpz or CGAL::Gmpq)
typedef CGAL::Gmpz ET;

// program and solution types
typedef CGAL::Quadratic_program<IT> Program;
typedef CGAL::Quadratic_program_solution<ET> Solution;

int main()
{
    // create an LP with Ax <= b and no lower/upper bounds
    Program lp (CGAL::SMALLER, false, 0, false, 0);
    const int a = 0;
    const int b = 1;
    const int c = 2;

    // number of red and blue points
    int m; std::cin >> m;
    int n; std::cin >> n;
```

```cpp
    // read the red points
    for (int i = 0; i < m; ++i) {
        int x; std::cin >> x;
        int y; std::cin >> y;
        // set up constraint a x + b y + c <= -1
        lp.set_a(a, i, x);
        lp.set_a(b, i, y);
        lp.set_a(c, i, 1);
        lp.set_b(i, -1);
    }
    // read the blue points
    for (int i = 0; i < n; ++i) {
        int x; std::cin >> x;
        int y; std::cin >> y;
        // set up constraint a x + b y + c >= 1
        lp.set_a(a, m+i, -x);
        lp.set_a(b, m+i, -y);
        lp.set_a(c, m+i, -1);
        lp.set_b(m+i, -1);
    }

    // solve the program, using ET as the exact type
    Solution s = CGAL::solve_linear_program(lp, ET());
    assert(s.solves_linear_program(lp));

    // output solution
    std::cout << s;
}
```

Output for

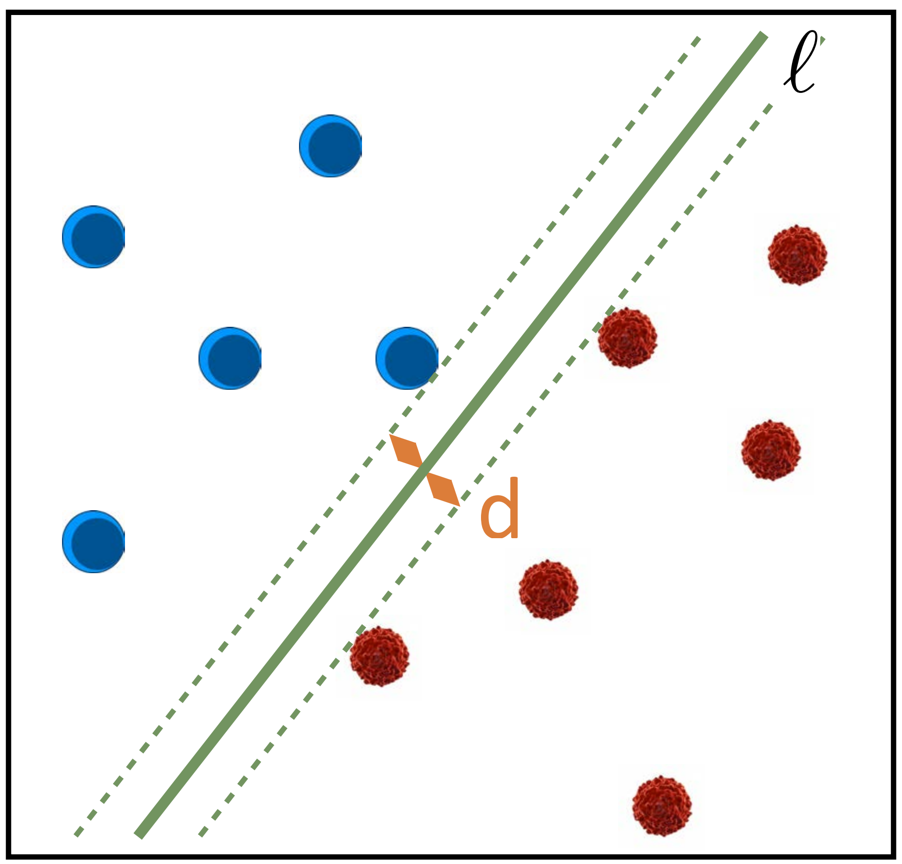
R={(0,0)} and

B={(1,1)}:

```
status:          OPTIMAL
objective value: 0/1
variable values:
    0: 2/1
    1: 0/1
    2: -1/1
```

Ex. I: Linear Separation

- **Given:** A set R of red points and a set B of blue points.
- **Want:** A line ℓ that <u>optimally</u> separates R and B.

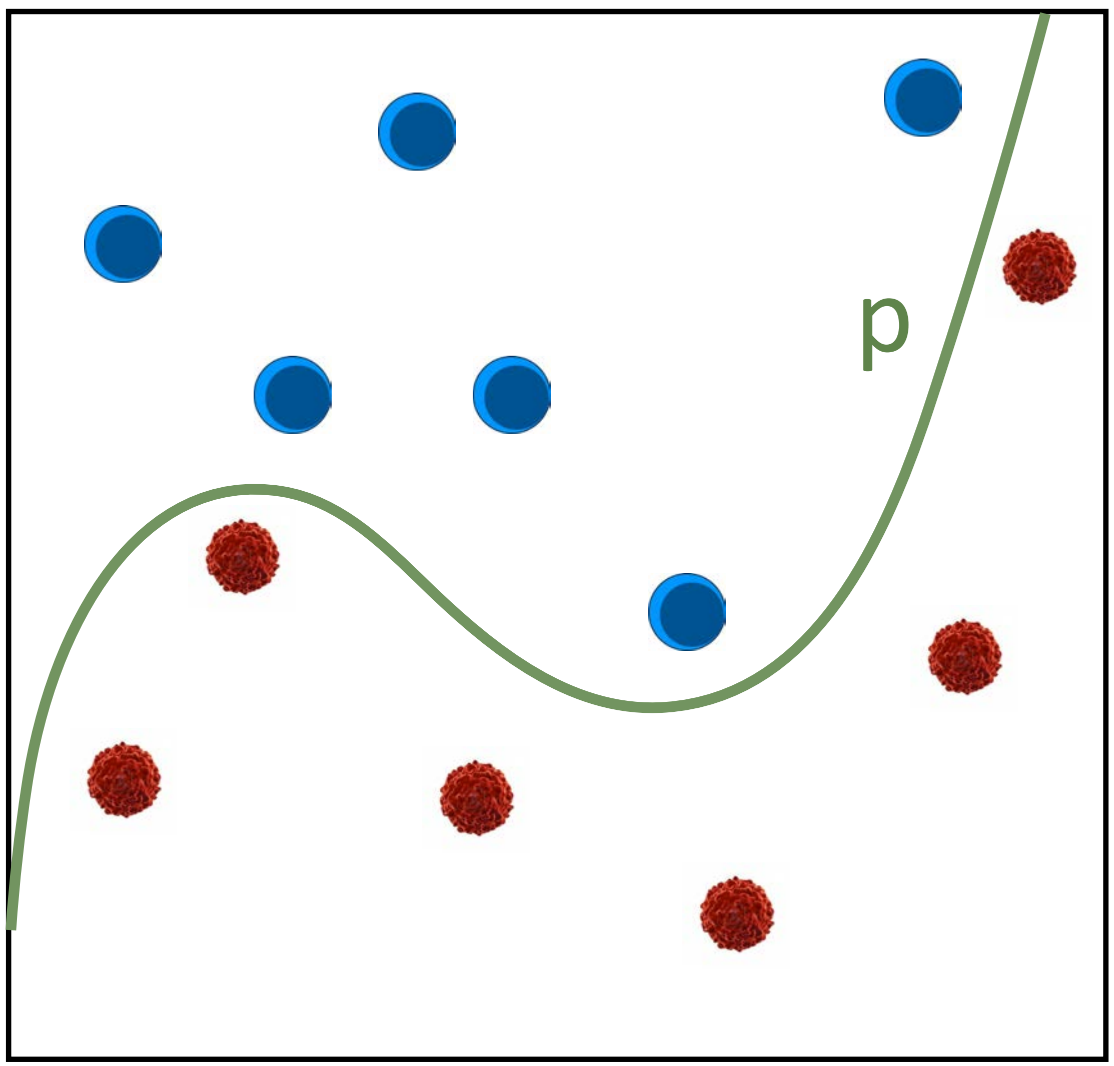


$a\,p_x + b\,p_y \leq c\text{-}d$, for $(p_x, p_y) \in R$ and
$a\,q_x + b\,q_y \geq c + d$, for $(q_x, q_y) \in B$.

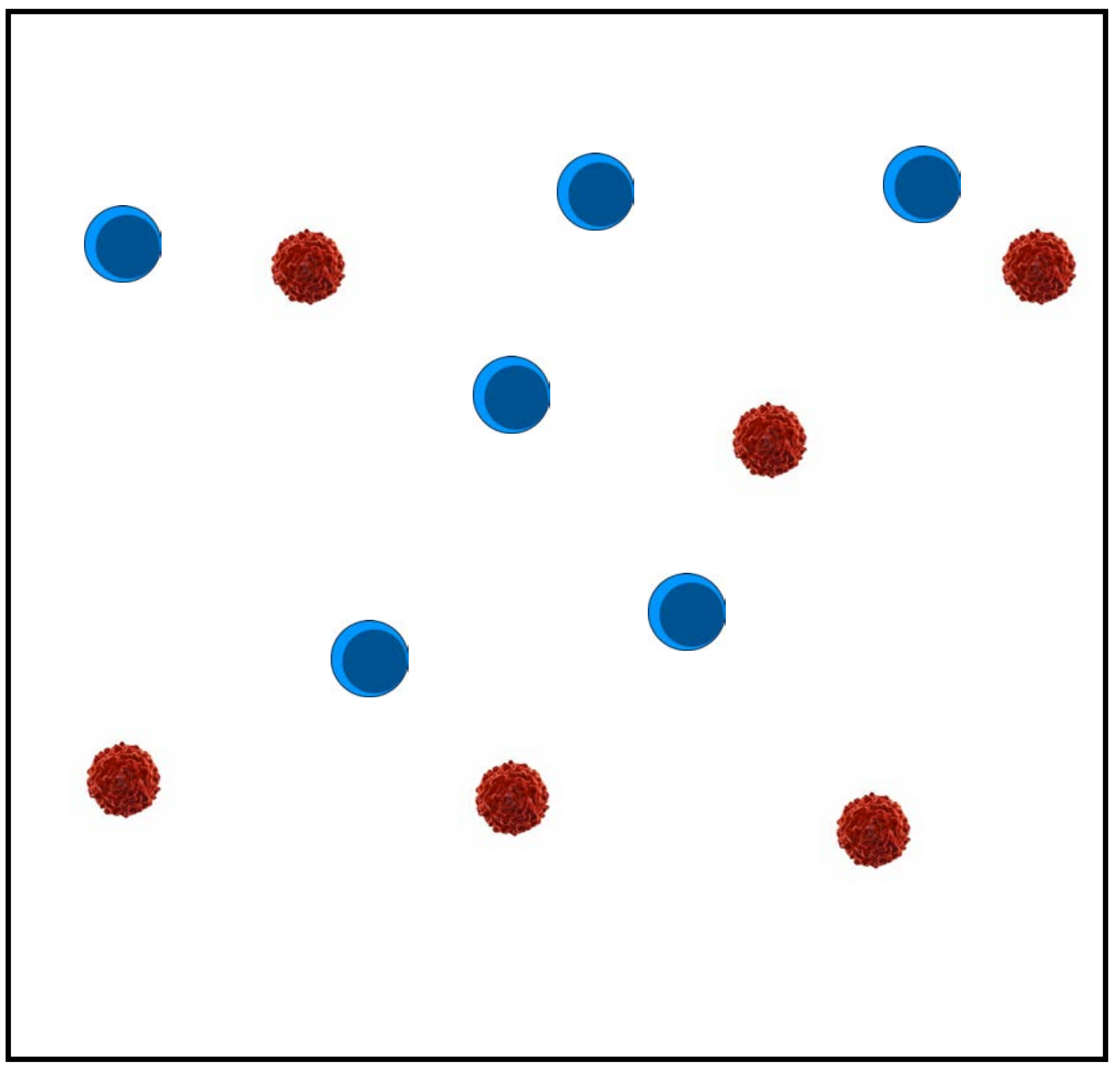
- If we normalize by a, b or c instead of d, we can maximize the margin.
- But we disregard
  - a: horizontal lines
  - b: vertical lines
  - c: lines through the origin

Ex. 2: Cubic Separation

- **Given:** A set R of red points and a set B of blue points.
- **Want:** A cubic polynomial p that separates R and B.



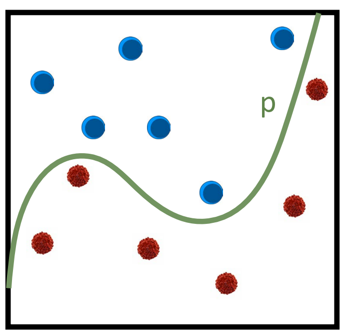
Separable



No solution 😞

Ex. 2: Cubic Separation

- **Given:** A set R of red points and a set B of blue points.
- **Want:** A cubic polynomial p that separates R and B.



- Agenda/Checklist:
- What are variables, what are constraints?
- What is *n*, what is *m*? Is one of *n* or *m* small?
- Are all constraints linear in the variables?

n=10 => small
m=|R|+|B|

Yes!

- **Variables/Output:** We want a cubic polynomial p :

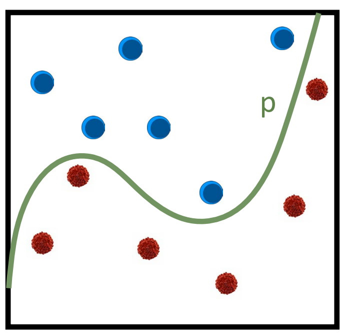
$\{(x,y) : a x^3+b x^2y+c xy^2+d y^3+e x^2+f xy+g y^2+h x+i y+j\}$,
for a,…,j unknown.

- **Constraints/Input:** R and B on different sides of p.

$a p_x^3+b p_x^2p_y+c p_xp_y^2+d p_y^3+e p_x^2+f p_xp_y+g p_y^2+h p_x+i p_y \le j,\ \forall(p_x,p_y)\in R,$
$a q_x^3+b q_x^2q_y+c q_xq_y^2+d q_y^3+e q_x^2+f q_xq_y+g q_y^2+h q_x+i q_y \ge j,\ \forall(q_x,q_y)\in B.$

Ex. 2: Cubic Separation

- **Given:** A set R of red points and a set B of blue points.
- **Want:** A cubic polynomial p that separates R and B.



- Agenda/Checklist:
- What are variables, what are constraints?
- What is n, what is m? Is one of n or m small?
- Are all constraints linear in the variables?

n=10 => small
m=|R|+|B|

Why? There are powers here…

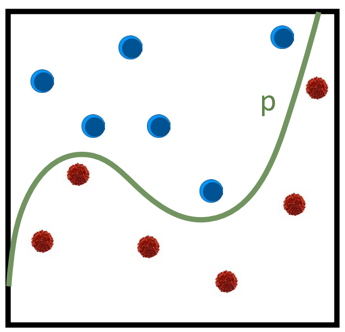
- **Variables:** a,b,c,d,e,f,g,h,i,j.
- **Constraints:** R and B on different sides of p.

$$a\,p_x^3+b\,p_x^2p_y+c\,p_xp_y^2+d\,p_y^3+e\,p_x^2+f\,p_xp_y+g\,p_y^2+h\,p_x+i\,p_y\le j,\ \forall(p_x,p_y)\in R,$$
$$a\,q_x^3+b\,q_x^2q_y+c\,q_xq_y^2+d\,q_y^3+e\,q_x^2+f\,q_xq_y+g\,q_y^2+h\,q_x+i\,q_y\ge j,\ \forall(q_x,q_y)\in B.$$

- Constraints are linear in a,b,c,d,e,f,g,h,i,j.

Ex. 2: Cubic Separation

- **Given:** A set R of red points and a set B of blue points.
- **Want:** A cubic polynomial p that separates R and B.



- Agenda/Checklist:
- What are variables, what are constraints?
- What is n, what is m? Is one of n or m small?
- Are all constraints linear in the variables?

Why? There are powers here…

- **Ex.** For $(2,3) \in R$ the constraint

$$a\,p_x^3 + b\,p_x^2 p_y + c\,p_x p_y^2 + d\,p_y^3 + e\,p_x^2 + f\,p_x p_y + g\,p_y^2 + h\,p_x + i\,p_y \le j$$

reads

$$8a + 12b + 18c + 27d + 4e + 6f + 9g + 2h + 3i \le j.$$

Careful with Indices

- The matrix A is adjusted dynamically to the entries and indices you provide.

```
// create an LP with Ax <= b, lower bound 0 and no upper bounds
Program lp (CGAL::SMALLER, true, 0, false, 0);

const int X = 10000;
lp.set_a(X, 0,  1);
// now A has >= 10000 columns…
```

- Even if all these coefficients are zero, this affects efficiency (and space usage).

Nonnegative Solver

- There is a special solver for the case where all variables are nonnegative.

```
// solve the program, using ET as the exact type
Solution s = CGAL::solve_nonnegative_linear_program(lp, ET());
```

- It may (or may not) be more efficient than the general solver on specific instances.
- It is not needed to pass the test sets.
- Attention! The nonnegative solver ignores all other lower and upper bounds (specified with the constructor or with set_l or set_u).

Processing Solutions

- There are various functions to access the solution.

```
// solve the program, using ET as the exact type
Solution s = CGAL::solve_linear_program(lp, ET());
if (s.is_optimal()) { ... }
if (s.objective_value() < 0) { ... }
if (s.is_unbounded()) { ... }
if (s.is_infeasible()) { ... }

typedef Solution::Variable_value_iterator SVI;
for (SVI opt = s.variable_values_begin(); opt < s.variable_values_end(); ++opt) {
  // keep in mind that the values are of type Quotient<ET>
  CGAL::Quotient<ET> res = *opt;
  std::cout << res.numerator() << "/" << res.denominator() << “\n”;
}
```

- In a maximization problem you probably want to invert the objective value…

How to avoid cycling

- The solver is driven by a pricing strategy.
- By default this strategy is deterministic and may cycle => run into an infinite loop.
- To avoid cycling, use Bland's Rule (see code below).

```
CGAL::Quadratic_program_options options;
options.set_pricing_strategy(CGAL::QP_BLAND);
Solution s = CGAL::solve_linear_program(lp, ET(), options);
```

- Bland's Rule is slower => only use where needed.
- Usually you will notice on the public test sets.

Known Bug :=(

- Do not copy or assign objects of type **CGAL::Quadratic_program_solution<ET>**!
- **Workaround 1:** If you want to pass or return such an instance to/from a function, pass a pointer to the instance instead!
- **Workaround 2:** Just don't do it!