Algolab 2022 — Week 5

Greedy Algorithms and Split & List

Overview

Today's tutorial: 'Advanced' Techniques

- Greedy Algorithms
  - Example 1: Knapsack
  - Proof Technique: Exchange Argument
  - Example 2: Interval Scheduling
  - Proof Technique: Staying Ahead
- Split & List

Greedy Algorithms

'Greed is good.

Greed is right.

Greed works.'

— Wall Street 1987

by Gordon Gekko

- Sometimes **locally optimal** choices result in a **globally optimal** solution.
- This is when we can apply **greedy algorithms**.
- **However** often choices that seem best in a particular moment turn out to not be optimal in the long run (e.g. in chess, life, etc.).

A greedy approach typically has the following steps:

1. **Modelling**: realise that your task requires you to construct a set that is in some sense **globally optimal**.

2. **Greedy choice**: given already chosen elements $c_1, \ldots, c_{k-1}$, decide how to choose c_k, based on some **local optimality criterion**.

3. **Prove** that elements obtained in this way result in a **globally optimal** set.

4. **Implement** the greedy choice to be as efficient as possible.

Example: The Knapsack Problem

Knapsack

Given an integer W and a set of n items, the i-th item has weight w_i and value v_i

maximise $\sum v_i x_i$

subject to $\sum w_i x_i \leqslant W$ and $x_i \in \{0, 1\}$

Fractional Knapsack

Given an integer W and a set of n items, the i-th item has weight w_i and value v_i

maximise $\sum v_i x_i$

subject to $\sum w_i x_i \leqslant W$ and $x_i \in [0, 1]$

Example: The Fractional Knapsack Problem

1. **Modelling** done for us in the problem description.

Example: The Fractional Knapsack Problem

2. Greedy choice

Idea:

Sort items **decreasingly** according to $\frac{v_i}{w_i}$ ratio and choose as much of item i as possible then move on to the next until knapsack is **full**.

Example: The Fractional Knapsack Problem

3. Prove that this yields an optimal solution.

General method: Exchange Argument

- Let A be the choices made by the greedy algorithm.
- Let O be an optimal solution.
- **Goal:** Assuming A and O are 'not equal', modify O to create O' such that
  1. O' is at least as good as O, and
  2. O' is 'more like' A.

Tip: One good way to do the last bit is to assume O is an optimal solution which **'follows A the longest'**, that is has the **longest common prefix with A**.

Look at the first point at which O differs from A and exchange some (further) element to get O' which agrees with A at that point as well.

Example: The Fractional Knapsack Problem

3. Prove that this yields an optimal solution.

Proof Sketch

- Suppose $\frac{v_1}{w_1} \geqslant \frac{v_2}{w_2} \geqslant \ldots \geqslant \frac{v_n}{w_n}$.
- Let $A = (x_1, \ldots, x_n)$ be the choices made by the greedy algorithm, where $x_i \in [0,1]$ stands for the fraction of item i we took.
- Let $O = (x'_1, \ldots, x'_n)$ be an optimal solution (which shares the longest prefix with A).
- If $A = O$ we are done.
- Let $i \in [n-1]$ be the **smallest index** such that:

$$x_j = x'_j \qquad \text{for all } j < i \qquad \text{and} \qquad x_i \neq x'_i$$

$$A = (x_1, \ldots, x_{i-1}, x_i, \ldots, x_n) \qquad O = (x_1, \ldots, x_{i-1}, x'_i, \ldots, x'_n)$$

Example: The Fractional Knapsack Problem

Proof Sketch (cont.)

- Because of the greedy choice: $x_i > x'_i$
- Because of optimality: exists $j > i$ with $x'_j > x_j$ (this implies $i < n$)
- How to make O 'closer' to A? O' obtained from O by using x_i of item i and $x'_j - (x_i - x'_i)\frac{w_i}{w_j}$ of item j (assuming $x'_j - (x_i - x'_i)\frac{w_i}{w_j} \geqslant 0$; otherwise 'distribute' the extra weight among some $j_1, \ldots, j_t > i$)
- difference of weight in O' and O:

$$(x_i w_i + x'_j w_j - (x_i - x'_i) w_i) - (x'_i w_i + x'_j w_j) = 0$$

$$w(O') - w(O) = 0$$

- difference of value in O' and O: $(x_i v_i + x'_j v_j - (x_i - x'_i)\frac{w_i}{w_j} v_j) - (x'_i v_i + x'_j v_j) = (x_i - x'_i)(\frac{v_i}{w_i} - \frac{v_j}{w_j}) w_i \geqslant 0$

$$v(O') - v(O) = (x_i - x'_i)\left(\frac{v_i}{w_i} - \frac{v_j}{w_j}\right) w_i \geqslant 0$$

- O' is **valid** and at least as good as O! **Contradiction!** $\square$

$$O = (\ldots, x'_i, \ldots, x'_j, \ldots, x'_n) \qquad O' = (\ldots, x_i, \ldots, x'_j - (x_i - x'_i)\frac{w_i}{w_j}, \ldots, x'_n)$$

Example: The Fractional Knapsack Problem

4. Implement the algorithm efficiently.

1. **Sort** the items according to value to weight ratio.
2. Iterate over the items in this order.
3. While there is space, add the items to the knapsack.

Running time: $O(n \log n)$

Example: The Knapsack Problem

Knapsack

Given an integer W and a set of n items, the i-th item has weight w_i and value v_i

maximise $\sum v_i x_i$

subject to $\sum w_i x_i \leqslant W$ and $x_i \in \{0, 1\}$

Fractional Knapsack

Given an integer W and a set of n items, the i-th item has weight w_i and value v_i

maximise $\sum v_i x_i$

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Greedy algorithm

Example: The Knapsack Problem

Knapsack

Given an integer W and a set of n items, the i-th item has weight w_i and value v_i

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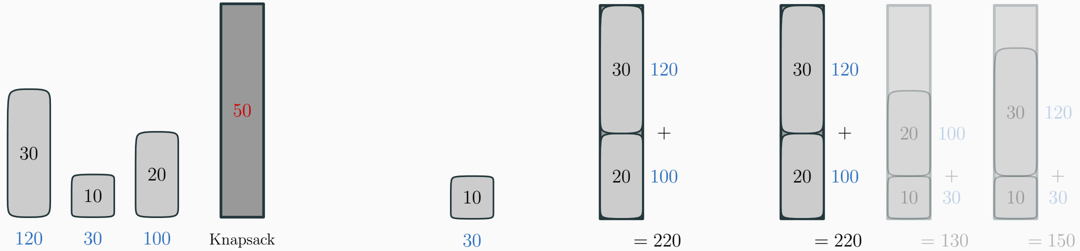


$$\frac{100}{20} > \frac{120}{30} > \frac{30}{10}$$

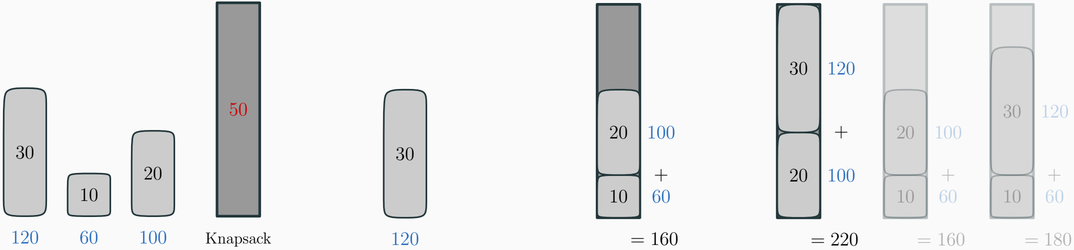
Example: The Knapsack Problem

Knapsack

Given an integer W and a set of n items, the i-th item has weight w_i and value v_i

maximise $\sum v_i x_i$

subject to $\sum w_i x_i \leqslant W$ and $x_i \in \{0,1\}$



$$\frac{60}{10} > \frac{100}{20} > \frac{120}{30}$$

Example: The Knapsack Problem

Knapsack

Given an integer W and a set of n items, the i-th item has weight w_i and value v_i

maximise $\sum v_i x_i$

subject to $\sum w_i x_i \leqslant W$ and $x_i \in \{0, 1\}$

NP-complete :(

Fractional Knapsack

Given an integer W and a set of n items, the i-th item has weight w_i and value v_i

maximise $\sum v_i x_i$

subject to $\sum w_i x_i \leqslant W$ and $x_i \in [0, 1]$

Greedy algorithm

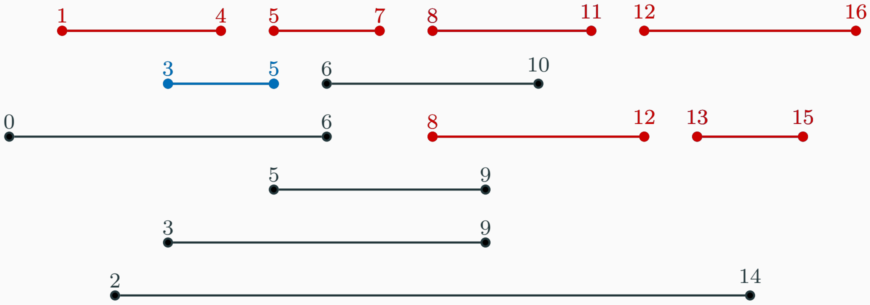
WARNING!

- Greedy **usually works** for many instances of most problems (e.g. many variations of the knapsack problem)
- Easy to convince yourself that the greedy approach **‘works’**
- The greedy approach **rarely yields optimal solutions** (i.e. many problems **do not have** a greedy solution)!
- The greedy approach **always needs proof!**

Example: Interval Scheduling

- Your CPU needs to execute N jobs described by time intervals $[s_i, f_i]$.
- Job i starts at time s_i and ends at time f_i.
- Two jobs are **compatible** if their intervals are disjoint.
- **Goal:** find the maximum number of mutually compatible jobs.

Example: Interval Scheduling



$$A = \{[3,5],[8,11],[13,15]\}$$

Optimal:

$B = \{[1,4],[5,7],[8,11],[12,16]\}$ also $C = \{[1,4],[5,7],[8,12],[13,15]\}$

Example: Interval Scheduling

1. Modelling done for us in the problem description: find the maximum set of compatible jobs.

Example: Interval Scheduling

2. Greedy choice: decide how to choose the job i_k given already chosen jobs $i_1, \ldots, i_{k-1}$.

Natural candidates:

- **Earliest start time** — among compatible jobs, take the one with smallest s_k.
- **Earliest finish time** — among compatible jobs, take the one with smallest f_k.
- **Shortest length** — among compatible jobs, take the one with smallest $f_k - s_k$.
- **Fewest conflicts** — among compatible jobs, take the one which conflicts with the least number of other compatible jobs.

Example: Interval Scheduling

Earliest start time

Earliest finish time

Shortest length

Fewest conflicts

Which one do you think will work?

Example: Interval Scheduling

Earliest start time

WRONG

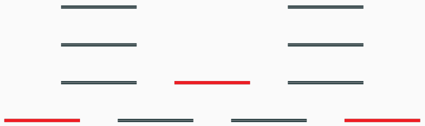
Example: Interval Scheduling

Shortest length

WRONG

Example: Interval Scheduling

Fewest conflicts



WRONG

Example: Interval Scheduling

Earliest finish time

Maybe???

Example: Interval Scheduling

3. Prove that earliest finish time gives an optimal solution.

General method: Staying Ahead

- Let $A = \{A_1, \ldots, A_n\}$ be the jobs chosen according to earliest finish time.
- Let $O = \{O_1, \ldots, O_m\}$ be an optimal solution (sorted by finish time).
- If $n = m$ we are done. Assume $n < m$.
- **Goal:** Show that for all $k \leqslant n$ we have $f_{A_k} \leqslant f_{O_k}$ (that is, A 'stays ahead').

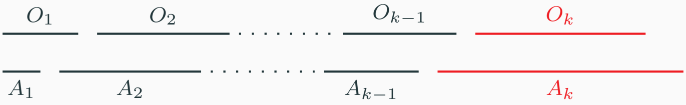
Example: Interval Scheduling

3. Prove that earliest finish time gives an optimal solution.

Proof Sketch

- **Goal:** Show that for all $k \leqslant n$ we have $f_{A_k} \leqslant f_{O_k}$ (that is, A 'stays ahead').
- Proof by induction on k.
- **Base case,** $k = 1$: **Clearly holds**!
- Let $k > 1$ and assume it holds for $k - 1$ (i.e. $f_{A_{k-1}} \leqslant f_{O_{k-1}}$).
- Could it happen that $f_{A_k} > f_{O_k}$? **NO!**

  WHY?! $f_{A_{k-1}} \leqslant f_{O_{k-1}}$ and O_k is **compatible** with O_{k-1}, thus with A_{k-1} as well. The greedy algorithm would select O_k instead of A_k.



Example: Interval Scheduling

3. Prove that earliest finish time gives an optimal solution.

Proof Sketch (cont.)

- **Goal:** Show that for all $k \leqslant n$ we have $f_{A_k} \leqslant f_{O_k}$ (that is, A 'stays ahead').
- For all $k \leqslant n = |O|$, we have $f_{A_k} \leqslant f_{O_k}$.
- Since $|O| > |A|$, there is O_{n+1} in O with:

$$s_{O_{n+1}} > f_{O_n} \qquad \text{and thus} \qquad s_{O_{n+1}} > f_{A_n}.$$

- Therefore, O_{n+1} is **compatible** with $A_1, \ldots, A_n$, but **does not** belong to A.

Contradiction! $\square$

Example: Interval Scheduling

4. Implement the algorithm efficiently.

1. **Sort** the jobs according to increasing finish time.
2. Iterate over the jobs in this order.
3. For each job with interval $[s_i, f_i]$, add the job if s_i is greater than the finish time of the last job that was added.

Running time: $O(N \log N)$

Example: Checking Change

ATM has bills with values $1, 10$, and 25 and is supposed to give you 42. What is the minimum number of bills used?

Greedy choice

$$1 \times 25 + 1 \times 10 + 7 \times 1 = 42$$

Bills used: **9**.

Optimal

$$4 \times 10 + 2 \times 1 = 42$$

Bills used: **6**.

Conclusion

- Some (but not all!) problems can be solved with a greedy approach.
- Deciding how to make the greedy choice can be non-obvious.
- We can check whether the greedy solution works using an **exchange argument** or a **staying ahead** argument.
- We can disprove a greedy solution via a counterexample.
- Convincing yourself greedy 'works' is easy (**even when it does not!**)

Split & List[1]

[1] *This is an advanced technique intended for students aiming for top grades. If it appears in the exam, it will only be necessary for the last 20 or 40 points of an exercise.*

Brute Force

Brute force: some problems are **hard** and we only know how to solve them by **trying everything**.

However, one can often do it a **little bit smarter**:

1. Heuristics (important in practice, not in AlgoLab)
2. **Improve worst case complexity :)**

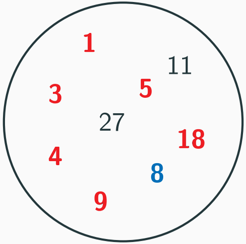
We will see a technique called **Split & List** (also known as **meet-in-the-middle**).

Example: Subset Sum

Problem: Given a set $S = \{s_1, \ldots, s_n\} \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, is there a subset $S' \subseteq S$ such that $\sum_{s \in S'} s = k$?

special case of **knapsack** *with $v_i = w_i$ and 'hit' exactly W*

- $S = \{1, 3, 4, 5, 8, 9, 11, 18, 27\}$
- $k = 8$? **YES!** $S' = \{1, 3, 4\}$ or $S' = \{8\}$
- $k = 1000$? **NO!**
- $k = 37$? **YES!** $S' = \{1, 4, 5, 9, 18\}$



NP-Complete :(

n is small: brute force

Check **all subsets**!

Recursive/Iterative algorithm

k is small: DP

EXERCISE!

Subset Sum — Recursive

Problem: Given a set $S = \{s_1, \ldots, s_n\} \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, is there a subset $S' \subseteq S$ such that $\sum_{s \in S'} s = k$?

We want a recursive definition of $f(i, j) :=$ 'is there $S' \subseteq \{s_1, \ldots, s_i\}$ s.t. $\sum_{s \in S'} s = j$'. Final answer is then $f(n, k)$.

- Base cases:

  $f(i, 0) =$ `true`, for all i, and

  $f(0, j) =$ `false`, for all $j > 0$.
- $f(i, j) = f(i - 1, j - s_i) \vee f(i - 1, j)$

Recursive algorithm:

```
bool f(int i, int j) {
    if (j == 0) return true;
    if ((i == 0 && j > 0) || j < 0) return false;
    return f(i - 1, j - elements[i]) || f(i - 1, j);
}
```

Running time: $O(2^n)$, ok for $n \approx 25$.

Subset Sum — Iterative

How can we iterate over all subsets of an n element set?

Trick: encode the set in an integer.

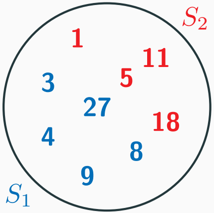
```
bool subsetsum(int k) {
    for (int s = 0; s < 1<<n; ++s) { // Iterate through all subsets
        int sum = 0;
        for (int i = 0; i < n; ++i) {
            if (s & 1<<i) sum += elements[i]; // If i-th element in subset
        }
        if (sum == k) return true;
    }
    return false;
}
```

Running time: $O(n \cdot 2^n)$, ok for $n \approx 25$.

Subset Sum — Faster? Split & List

Split S into $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \varnothing$ of size $\approx \frac{n}{2}$.

List all subset sums of S_1 and S_2 into L_1 and L_2



Lemma: The following statements are equivalent:

- There is a $S' \subseteq S$ with $\sum_{s \in S'} s = k$
- There are $S_1' \subseteq S_1$ and $S_2' \subseteq S_2$ such that $\sum_{s \in S_1'} s + \sum_{s \in S_2'} s = k$

Idea: use second statement to check the first.

Algorithm sketch:

- Sort L_2
- For each k_1 in L_1 check if there is k_2 in L_2 (**binary search**!) such that $k_1 + k_2 = k$.

Running time: $O(n \cdot 2^{n/2})$, ok for $n \approx 50$.