Game Theory and Control - Assignment 1

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1 Exercise 1 - Many Nash Equilibria

Consider the non-zero-sum two-player (minimizer) simultaneous game described by the two matrices:

$$A = \begin{bmatrix} 0 & -10 & 0 & 0 \\ -10 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \; B = \begin{bmatrix} -10 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Matrix A represents the cost assigned to Player 1, while matrix B represents the cost assigned to Player 2.

a) To find the **pure Nash Equilibria** of the game we can simply proceed by inspection of the two matrices. It is clear that the two pair of strategies:

$(\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix})$
$(\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix})$

corresponding to the couples of pure actions:
(3,3)
(4,4)
are Nash Equilibria, since no player can improve their outcome by unilaterally chenging their decision. In fact, action 3 is the best Player 1 can do given that Player 2 has also played action 3 and at the same time action 3 is the best Player 2 can do given that Player 1 has also played action 3. The same reasoning holds for action 4.
To find the **completely mixed Nash Equilibria**, we can use the **principle of indifference**. Notice that the principle of indifference is a necessary and sufficient condition for completely mixed Nash Equilibria. Let us indeed define:

$$\mathrm{y}^* = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} z^* = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix},$$

the completely mixed strategies corresponding to a Nash Equilibrium, respectively for Player 1 and Player 2. Then, the principle of indifference states that y^* and z^* need to satisfy:

$Az^* = p^*1$
$y^{*T}B = q^*1^T$
$1^T y^* = 1$
$1^T z^* = 1$

where p^* and q^* are the values of the equilibrium respectively for player 1 and player 2.

Solving:

$y^{*T}B = q^*1^T$

we get:

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} -10 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} q^* & q^* & q^* & q^* \end{bmatrix} \tag{1}$$

which combined with the equation:

$y_1 + y_2 + y_3 + y_4 = 1$

yields the solution:

$$\mathrm{y}^* = \begin{bmatrix} \frac{1}{17} \\ \\ \frac{1}{17} \\ \\ \frac{10}{17} \\ \\ \frac{5}{17} \end{bmatrix} \mathrm{q}^* = -\frac{10}{17}$$

Similarly, solving:

$Az^* = p^*1$
$z_1 + z_2 + z_3 + z_4 = 1$

we get the solution:

$$\mathrm{z}^* = \begin{bmatrix} \frac{1}{17} \\ \frac{1}{17} \\ \frac{5}{17} \\ \frac{10}{17} \end{bmatrix} \mathrm{p}^* = -\frac{10}{17}$$

b) Computing **mixed Nash Equilibria** can be a challenging task even for a simple bi-matrix game like this. It was properly suggested in the assignment to try to reduce the game to fewer strategies in order to compute non-pure non-completely mixed NE. Let's see an example on how to do that:

Let's assume players only mix between their first two actions. The game is reduced to a new smaller game represented by the following matrices:

$$\overline{A} = \begin{bmatrix} 0 & -10 \\ -10 & 0 \end{bmatrix} \overline{B} = \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix}$$

This is now an example of a 2 action game for both players. For such a game **each mixed Nash Equilibrium will necessarily be completely mixed**. That means that an equilibrium when one player plays a pure strategy and the other mixes their actions can never exist for two action games. In formulas:

$$\begin{aligned}
&(\begin{bmatrix}1 & 0\end{bmatrix} \begin{bmatrix}z & 1-z)\end{bmatrix}) \\
&(\begin{bmatrix}0 & 1\end{bmatrix} \begin{bmatrix}z & 1-z\end{bmatrix}) \\
&(\begin{bmatrix}y & 1-y\end{bmatrix} \begin{bmatrix}1 & 0\end{bmatrix}) \\
&(\begin{bmatrix}y & 1-y\end{bmatrix} \begin{bmatrix}0 & 1\end{bmatrix})
\end{aligned}$$

can never be Nash Equilibria for every $y, z > 0$. Let's prove the result for one case:

Assume that:

$$(\begin{bmatrix}1 & 0\end{bmatrix} \begin{bmatrix}z & 1-z)\end{bmatrix})$$

is a Nash Equilibrium.

Then the expected payoff for Player 2 can be computed as:

$$y^T Bz = \begin{bmatrix}1 & 0\end{bmatrix} \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} z \\ 1-z \end{bmatrix} = -10z \tag{2}$$

That means Player 2 can minimize the payoff by setting z = 1, playing always action 1. This result is also consistent with our intuition: if Player 1 plays

always action 1, best response for Player 2 will be to always play action 2, without mixing between the two actions. This prove can be easily extended to the other cases. We have therefore shown that in this reduced game a mixed Nash Equilibrium is always completely mixed.
We can therefore compute the completely mixed equilibrium of this reduced game by using the principle of indifference. Again, solving the system:

$\overline{A}z^* = p^*1$
$y^{*T}\overline{B} = q^*1^T$
$1^T y^* = 1$
$1^T z^* = 1$

we come up with the solution:

$\text{y}^* = \begin{bmatrix} \frac{1}{2} \\ \\ \frac{1}{2} \end{bmatrix} \text{p}^* = -5$
$\text{z}^* = \begin{bmatrix} \frac{1}{2} \\ \\ \frac{1}{2} \end{bmatrix} \text{q}^* = -5$

The corresponding mixed NE for the complete game is:

$(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix})$

with the two players never picking the last two actions. We now have to prove that this is a Nash Equilibrium for the complete game. We will use the definition of Nash Equilibrium and prove that:

$y^{*T}Az^* \leq y^T Az^*$
$y^{*T}Bz^* \leq y^{*T}Bz$

for every other strategies y and z. Let us prove the first inequality:

$$-5 = y^{*T}Az^* \leq y^T Az = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 0 & -10 & 0 & 0 \\ -10 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} = \tag{3}$$

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} -5 \\ -5 \\ 0 \\ 0 \end{bmatrix} = -5(y_1 + y_2) \tag{4}$$

Notice that the inequality holds because $-5 \leq -5(y_1+y_2)$ as $0 \leq (y_1+y_2) \leq 1$. Remember that y_1 and y_2 are probabilities. Showing that:

$y^{*T}Bz^* \leq y^{*T}Bz$

is analogous. We were thus able to show that:

$$\left(\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\end{bmatrix} \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\end{bmatrix}\right)$$

is a mixed equilibrium for the original game.
Reducing the game to a game with fewer actions can be a valuable strategies for computing mixed Nash Equilibria. The idea here is to try all the possible combinations of two action games [(1,2)(1,3)(1,4),(2,3),(2,4),(3,4)] to find more Nash Equilibria. However, it is not always the case that the equilibrium for the reduced game is also an equilibrium for the complete game. For instance, let's consider the reduced game where players are allowed to pick only between action 1 and 3. Then the matrices representing this game are:

$$\overline{A} = \begin{bmatrix}0 & 0\\0 & -2\end{bmatrix} \overline{B} = \begin{bmatrix}-10 & 0\\0 & -1\end{bmatrix}$$

If we apply once again the principle of indifference on this game, we come up with the equilibrium:

$$\left(\begin{bmatrix}\frac{1}{11} & \frac{10}{11}\end{bmatrix} \begin{bmatrix}1 & 0\end{bmatrix}\right)$$

However, we can immediately see that the extended strategies:

$$\left(\begin{bmatrix}\frac{1}{11} & 0 & \frac{10}{11} & 0\end{bmatrix} \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\right)$$

with expected payoff (0, -10/11) do not form a Nash Equilibrium for the complete game. In fact, we can immediately come up with an improving strategy for Player 1. If Player 2 plays action 1 with probability 1, best response of Player 1 will of course be to play action 2 all the time, leading to a minimized cost of -10. Notice that action 2 is not available in the reduced game.
We have therefore shown how to extend the results of a two action sub-game on the complete game: by applying similar reasoning to all the combinations of two actions, we can find out that the only mixed Nash Equilibria associated to 2 action sub-games are:

$\left(\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\end{bmatrix} \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\end{bmatrix}\right)$ with value (-5,-5)

$\left(\begin{bmatrix}0 & 0 & \frac{2}{3} & \frac{1}{3}\end{bmatrix} \begin{bmatrix}0 & 0 & \frac{1}{3} & \frac{2}{3}\end{bmatrix}\right)$ with value (-2/3,-2/3)

To complete the analysis we have to analyze the reduced games with three actions. For example,let's observe the game when the two players can only choose between actions 1,2,3. The payoff matrices are:

$$\overline{A} = \begin{bmatrix} 0 & -10 & 0 \\ -10 & 0 & 0 \\ -10 & 0 & 0 \end{bmatrix} \overline{B} = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

At first, we can make some considerations on the structure of the Nash Equilibria for this reduced game. Again, the equilibria are either pure or completely mixed. Let's suppose Player 1 plays a completely mixed strategy $[y_1, y_2, y_3]$ with $y_1, y_2, y_3 > 0$. If Player 2 responds by only playing two strategies, e.g $[z_1, z_2, 0]$ with $z_1, z_2 > 0$, then the expected payoff for Player 1 is:

$$y^T Az = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & -10 & 0 \\ -10 & 0 & 0 \\ -10 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = -10(y_1 z_2 + y_2 z_1) \tag{5}$$

which is minimized by maximizing y_1 and y_2, thus setting $y_3 = 0$. However, by setting $y_3 = 0$ we come back to the previously analyzed two action game. The same can be said if Player 2 responds to by playing a pure strategy (e.g.$[z_1, 0, 0]$): at that point Player 1 should also play a pure strategy to minimize the payoff. This shows that the new equilibria to be found in reduced 3 action games are completely mixed. Notice that this result is mainly due to the **sparsity** of the matrices A and B.

We can find the completely mixed Nash Equilibrium of the game via the principle of Indifference and extend it to the complete game with similar reasoning as before: using the definition and finding a lower bound for the expected value. Computation is not reported here to avoid lengthy repetitions.

$(\begin{bmatrix} \frac{1}{12} & \frac{1}{12} & \frac{5}{6} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{7} & \frac{1}{7} & \frac{5}{7} & 0 \end{bmatrix})$

is the associated mixed NE for the complete game. We should apply the same procedure for all possible triplets of actions ([1,2,3],[1,2,4],[1,3,4],[2,3,4]). Again, not all reduced equilibria can be extended to the complete game (those can be quickly dismissed via the best response argument in the complete game). Overall, the mixed NE associated to 3 action sub-games are:

$(\begin{bmatrix} \frac{1}{12} & \frac{1}{12} & \frac{5}{6} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{7} & \frac{1}{7} & \frac{5}{7} & 0 \end{bmatrix})$ with value (-10/7, -5/6)

$(\begin{bmatrix} \frac{1}{7} & \frac{1}{7} & 0 & \frac{5}{7} \end{bmatrix} \begin{bmatrix} \frac{1}{12} & \frac{1}{12} & 0 & \frac{5}{6} \end{bmatrix})$ with value (-5/6, -10/7)

c) Overall, the game features the following **7 Nash Equilibria**:

1. $\left(\begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix} \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}\right)$ with value (-2,-1)

2. $\left(\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix} \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}\right)$with value (-1,-2)

3. $\left(\begin{bmatrix}\frac{1}{17} & \frac{1}{17} & \frac{10}{17} & \frac{5}{17}\end{bmatrix} \begin{bmatrix}\frac{1}{17} & \frac{1}{17} & \frac{5}{17} & \frac{10}{17}\end{bmatrix}\right)$ with value (-10/17, -10/17)

4. $\left(\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\end{bmatrix} \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\end{bmatrix}\right)$ with value (-5,-5)

5. $\left(\begin{bmatrix}0 & 0 & \frac{2}{3} & \frac{1}{3}\end{bmatrix} \begin{bmatrix}0 & 0 & \frac{1}{3} & \frac{2}{3}\end{bmatrix}\right)$ with value (-2/3,-2/3)

6. $\left(\begin{bmatrix}\frac{1}{12} & \frac{1}{12} & \frac{5}{6} & 0\end{bmatrix} \begin{bmatrix}\frac{1}{7} & \frac{1}{7} & \frac{5}{7} & 0\end{bmatrix}\right)$ with value (-10/7, -5/6)

7. $\left(\begin{bmatrix}\frac{1}{7} & \frac{1}{7} & 0 & \frac{5}{7}\end{bmatrix} \begin{bmatrix}\frac{1}{12} & \frac{1}{12} & 0 & \frac{5}{6}\end{bmatrix}\right)$ with value (-5/6, -10/7)

We can now rank the values of these equililibria in the following **Hasse diagram**:

$$
\begin{tikzcd}[column sep=small]
 & \left(-\frac{10}{17}, -\frac{10}{17}\right) \arrow[d, dash] & \\
 & \left(-\frac{2}{3}, -\frac{2}{3}\right) \arrow[dl, dash] \arrow[dr, dash] & \\
\left(-\frac{5}{6}, -\frac{10}{7}\right) \arrow[d, dash] & & \left(-\frac{10}{7}, -\frac{5}{6}\right) \arrow[d, dash] \\
(-1, -2) \arrow[dr, dash] & & (-2, -1) \arrow[dl, dash] \\
 & (-5, -5) &
\end{tikzcd}
$$

It is clear that the only **admissible NE** is:

$\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}\right)$ with value (-5,-5)

2 Exercise 2 - Robust tuning of an active damper

a) This control design problem can be formalized as a **zero - sum game**. It is common in control engineering to formalize such problems as zero sum games. The idea is that we want to guarantee the minimization of the cost even in the worst case, meaning that it is reasonable to assume that the road is playing against us. The behavior of the road can be assumed to be completely adversarial to the car, as we want to guarantee **robust control**. We can formalize the game as follows:

Player 1: Project designer of the car
Player 2: Road with its bumps

The actions available to Player 1 are the 9 possible combinations of the parameters of the controller a and b, while the actions of Player 2 are the possible disturbances, corresponding to three kinds of bumps hitting the wheel. Let us formalize the action spaces Γ and Σ respectively for Player 1 and Player 2 as:

$\Gamma = ([0, 0], [0, 0.3], [0, 0.6], [0.4, 0], [0.4, 0.3], [0.4, 0.6], [0.8, 0], [0.8, 0.3], [0.8, 0.6])$

$\Sigma = ([d_1, d_2, d_3])$

We can assume that the two players act **simultaneously** as the bumps on the road do not depend on our project and at the same time the project designer cannot know which disturbance will hit the wheel. Players pick their action without knowing what the other player has chosen.

b)MatLab was used to solve the control problem. In particular, after having properly designed the closed-loop transfer function, I have simulated the response y of the system for every input disturbance d and for every possible choice of the parameters a and b. The payoff matrix A was computed as follows: entrance a_{ij} is equal to infinity if the actions of the two players (i,j) result in the output y overcoming 1 in any timestep of the response. Otherwise, the entrance a_{ij} was set to the control effort, computed as the squared sum of the control output signal u, as suggested in the assignment. The following payoff matrix was obtained:

$$A = \begin{bmatrix} Inf & Inf & 0 \\ 0.3625 & 0.6570 & 0.5398 \\ 0.9786 & 1.7393 & 1.4293 \\ Inf & Inf & 0 \\ 0.2191 & 0.3853 & 0.3166 \\ 1.9596 & 1.1139 & 0.9154 \\ Inf & Inf & 0 \\ Inf & Inf & 0.4581 \\ Inf & Inf & Inf \end{bmatrix}$$

Given this payoff matrix, we can compute the security levels and the corresponding security strategies for both players. Let's define the security strategies for Player 1 and Player 2 with $\overline{V}$ and $\underline{V}$. Then:

$\overline{V} = min_i max_j a_{ij} = 0.3853$
$\underline{V} = max_j min_i a_{ij} = 0.3853$

A well known result states that a zero-sum games defined by A has a saddle-point Nash Equilibrium if and only if $\overline{V} = \underline{V}$, and that the corresponding security strategies define the saddle-point equilibrium. Let's compute the security strategies $\bar{i}$ and $\underline{j}$:

$\bar{i} = argmin_i(max_j a_{ij}) = [0.4, 0.3]$
$\underline{j} = argmax_j(min_i a_{ij}) = [d_2]$

Therefore, the Nash Equilibrium for this zero-sum game is $([0.4, 0.3], [d_2])$. That means that a robust choice of the parameters a and b would be to set $a = 0.4$ and $b = 0.3$.

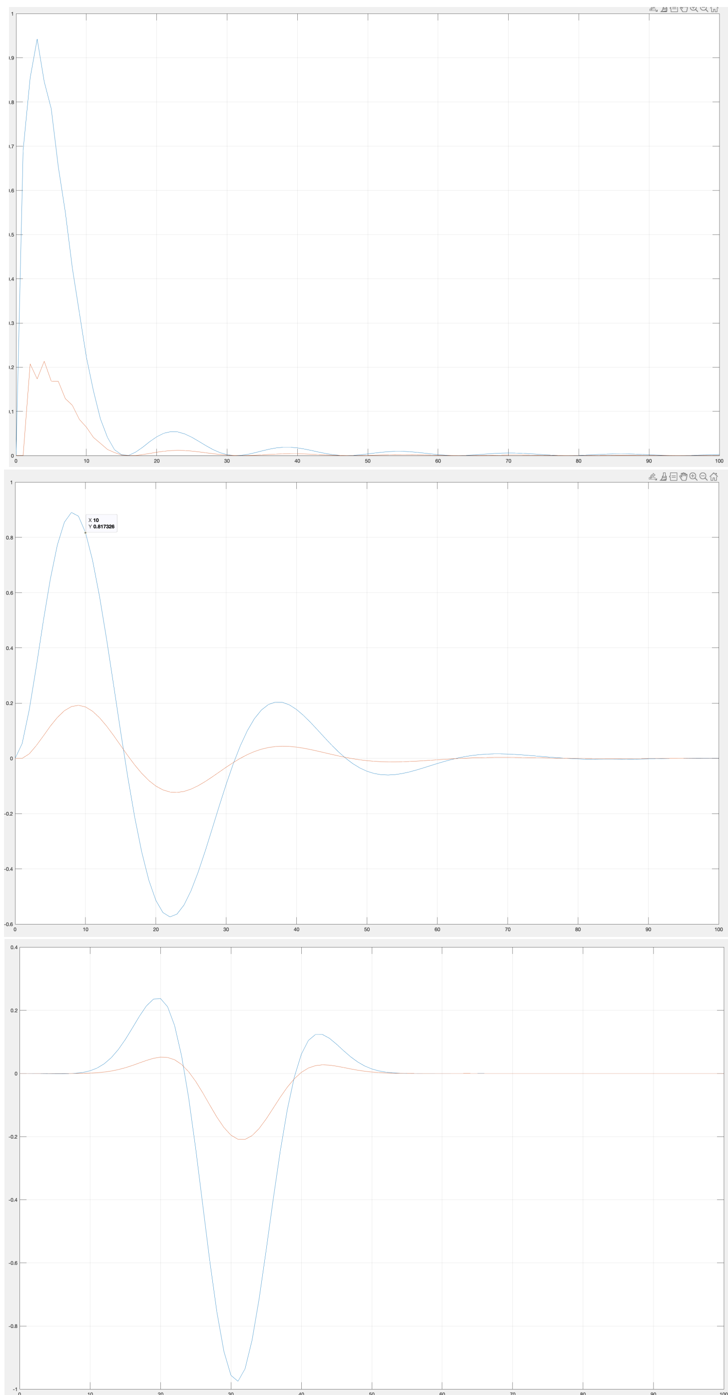


Figure 1: In blue the response of the output y and in red the behaviour of the control variable u for the three different disturbances d_1, d_2, d_3 for the robust choice of the parameters $a = 0.4$, $b = 0.3$.

c) Let's now suppose the car is equipped with a vision system allowing us to identify the disturbance in advance and to tune the parameters accordingly. The game is no longer simultaneous but can be modelled as a **sequential** game. Player 2 (the road) plays first and we can tune the controller accordingly. We can answer the question by computing Player 1 **best response R** to each of Player 2 actions. We then have:

$R_1(d_1) = argmin(a_{i1}) = [0.4, 0.3]$
$R_1(d_2) = argmin(a_{i2}) = [0.4, 0.3]$
$R_1(d_3) = argmin(a_{i3}) = ([0, 0], [0.4, 0], [0.8, 0])$

We notice that Player 1 best responses to d_1 and d_2 actually coincide with the security strategy previously computed, tuning the parameters $a = 0.4$ and $b = 0.3$. However, the best action to take when disturbance d_3 hits the wheels is to set parameter $b = 0$. We observe that this corresponds to eliminate the feedback in the control loop, leading to a 0 cost. This is reasonable since the disturbance d_3 never crosses the critical threshold of 1 and we can actually rely on the open loop system without providing a stabilizing feedback control.

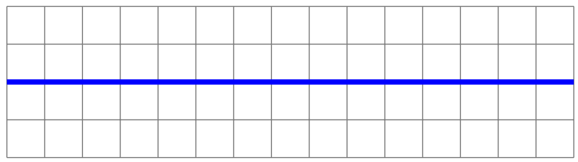
If the vision system sometimes fails to identify the right disturbance, the soundest idea is to tune the controller with $a = 0.4$ and $b = 0.3$ because this is the security strategy. If the vision system predicts either a disturbance d_1 or d_2 the best response is indeed to set $a = 0.4$ and $b = 0.3$. If the vision system predicts a disturbance *d3* coming, let's define as p the probability that this disturbance is not d_3, so it's either d_1 or d_2. Let's now calculate the expected payoff for Player 1 for two different strategies:

1. Player 1 sets $b = 0$; then the expected payoff J is:
   $J = Inf * p + 0 * (1 - p) = Inf$ for every strictly positive value of p
2. Player 1 sets $[a = 0.4,\ b = 0.3]$; then the expected payoff J is computed as:
   $J = (0.2191 + 0.3853)/2 * p + 0.3166 * (1 - p)$,which is a finite number for every p in (0,1), where we are assuming that the disturbances d_1 and d_2 share the same probability of hitting the wheel.

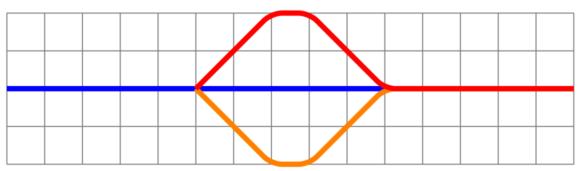
Moreover, setting $a = 0.4$ and $b = 0.3$ is the second best response to d_3. Since we have shown that an open loop (no feedback) control makes the system diverge for d_1 or d_2, setting $a = 0.4$ and $b = 0.3$ has to be the best action for Player 1 in case of uncertainties in the vision system.

3 Exercise 3 - Trajectory planning game

a) Let the blue car choose its action without accounting for the presence of the orange car, and let's call this action ${\gamma^*}_{\text{blue}}$. Clearly the trajectory ${\gamma^*}_{\text{blue}}$ will be a straight line joining A and B, yielding 0 cost.



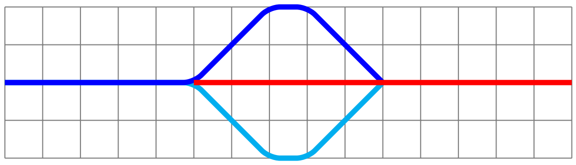
To compute the optimal set $R_{\text{orange}}({\gamma^*}_{\text{blue}})$ we can perform basic reasoning to avoid any collision between the two cars. The optimal set $R_{\text{orange}}({\gamma^*}_{\text{blue}})$ consists of two trajectories which are depicted in red and orange in the following figure: collision is avoided in the center of the grid by swerving upward (or downward) and cost is equal to 8.



b)Let's define as $\hat{\gamma}_{\text{orange}}$ one of the two trajectories belonging to $R_{\text{orange}}({\gamma^*}_{\text{blue}})$. We can use the **best-response** argument to prove that the set $({\gamma^*}_{\text{blue}},\hat{\gamma}_{\text{orange}})$ is a Nash Equilibrium. It is fairly easy (and intuitive) to show that:

${\gamma^*}_{\text{blue}} \in R_{blue}(\hat{\gamma}_{\text{orange}})$
$\hat{\gamma}_{\text{orange}} \in R_{orange}({\gamma^*}_{\text{blue}})$

It is clear that none of the two players have any regrets. Given that the blue car has gone straight, the orange car could not have done better than swerving in the middle of the grid to avoid the collision. On the other hand, given that the orange car has swerved, the blue car best strategy is indeed to nullify the cost by going straight into the middle of the road for the entire time. It follows that $({\gamma^*}_{\text{blue}},\hat{\gamma}_{\text{orange}})$ is indeed a Nash Equilibrium with resulting cost (0,8), respectively for the blue car and the orange car. No player can improve their outcome by unilaterally changing their decision.
By making a simple symmetric argument, it is clear that there are at least two more symmetric Nash Equilibria. This time, the orange car goes straight, while the blue car swerves (either upward or downward). Nash Equilibria trajectories are shown in the following figure:

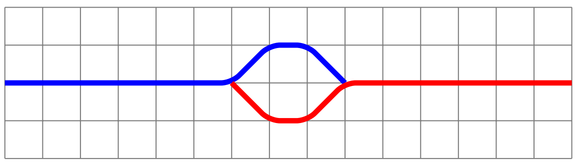


The resulting costs associated to this new Nash Equilibrium are (8,0), respectively for the blue car and the orange car.

c) Let's now define the welfare W cost as the sum of the costs for the two cars. We have:

$W = J_{blue} + J_{orange}$

where J_{blue} and J_{orange} are the costs that the blue and the orange car want to minimize. The previously found Nash Equilibria have a welfare cost W = 8, with the entire cost attributed to a single player.
We notice that if both players make the effort to swerve a bit (by one square), collision can be avoided in the middle of the board and total welfare cost is minimized. In the following figure we show one of the trajectories minimizing the welfare (the other one simply consists in blue car swerving downward and orange car swerving upward).



Let's denote with $\overline{\gamma}_{\text{blue}}$ and $\overline{\gamma}_{\text{orange}}$ these two trajectories. Again we can show via the **best response** argument that ($\overline{\gamma}_{\text{blue}}$,$\overline{\gamma}_{\text{orange}}$) is indeed a Nash Equilibrium with values (2,2) and W = 4. It holds:

$\overline{\gamma}_{\text{blue}} \in R_{blue}(\overline{\gamma}_{\text{orange}})$
$\overline{\gamma}_{\text{orange}} \in R_{orange}(\overline{\gamma}_{\text{blue}})$

d) Now we want to prove that the game is a **potential game**. According to the definition, a game is a potential game if it admits a potential function.

Let $\gamma_{\text{blue}}, \gamma_{\text{orange}}$ be two pure strategies respectively for the blue car and for the orange car. Let us first define the function:

$$\text{Collision}(\gamma_{\text{blue}}, \gamma_{\text{orange}}) = \begin{cases} 0, & \text{if } \gamma_{\text{blue}}, \gamma_{\text{orange}} \text{ do not generate a collision} \\ 100, & \text{if } \gamma_{\text{blue}}, \gamma_{\text{orange}} \text{ generate a collision} \end{cases}$$

Furthermore, let us define the functions $S_{blue}(\gamma_{\text{blue}})$ and $S_{orange}(\gamma_{\text{orange}})$ as the costs assigned to the two players only taking into account the distance from the center of the road in their trajectories, without considering any collision cost. For example if both cars opt for a straight trajectory from A to B, then $S_{blue}(\gamma_{\text{blue}}) = 0$ and $S_{orange}(\gamma_{\text{orange}}) = 0$, regardless of the collision which will eventually happen in the middle of the road. Notice that $S_{blue}(\gamma_{\text{blue}})$ is only a function of the trajectory γ_{blue} and $S_{orange}(\gamma_{\text{orange}})$ is only a function of the trajectory γ_{orange}, as this cost only depends on each player own trajectory.

We can now observe that:

$J_{blue}(\gamma_{\text{blue}}, \gamma_{\text{orange}}) = S_{blue}(\gamma_{\text{blue}}) + Collision(\gamma_{\text{blue}}, \gamma_{\text{orange}})$

$J_{orange}(\gamma_{\text{blue}}, \gamma_{\text{orange}}) = S_{orange}(\gamma_{\text{orange}}) + Collision(\gamma_{\text{blue}}, \gamma_{\text{orange}})$

Let us now define the function:

$$P(\gamma_{\text{blue}}, \gamma_{\text{orange}}) = W(\gamma_{\text{blue}}, \gamma_{\text{orange}}) - Collision(\gamma_{\text{blue}}, \gamma_{\text{orange}})$$

Let us prove that this is an **exact potential function**:
Let ${\gamma^1}_{\text{blue}}$ and ${\gamma^2}_{\text{blue}}$ be two trajectories for the blue car and γ_{orange} a trajectory for the orange car. Then, it holds:

$J_{blue}({\gamma^1}_{\text{blue}}, \gamma_{\text{orange}}) - J_{blue}({\gamma^2}_{\text{blue}}, \gamma_{\text{orange}}) =$

$W({\gamma^1}_{\text{blue}}, \gamma_{\text{orange}}) - J_{orange}({\gamma^1}_{\text{blue}}, \gamma_{\text{orange}}) - W({\gamma^2}_{\text{blue}}, \gamma_{\text{orange}}) + J_{orange}({\gamma^2}_{\text{blue}}, \gamma_{\text{orange}}) =$

$W({\gamma^1}_{\text{blue}}, \gamma_{\text{orange}}) - W({\gamma^2}_{\text{blue}}, \gamma_{\text{orange}}) - [J_{orange}({\gamma^1}_{\text{blue}}, \gamma_{\text{orange}}) - J_{orange}({\gamma^2}_{\text{blue}}, \gamma_{\text{orange}})] =$

$W({\gamma^1}_{\text{blue}}, \gamma_{\text{orange}}) - W({\gamma^2}_{\text{blue}}, \gamma_{\text{orange}}) - [S_{orange}(\gamma_{\text{orange}}) + Collision({\gamma^1}_{\text{blue}}, \gamma_{\text{orange}}) -$
$S_{orange}(\gamma_{\text{orange}}) - Collision({\gamma^2}_{\text{blue}}, \gamma_{\text{orange}})] =$

$W({\gamma^1}_{\text{blue}}, \gamma_{\text{orange}}) - Collision({\gamma^1}_{\text{blue}}, \gamma_{\text{orange}}) - [W({\gamma^2}_{\text{blue}}, \gamma_{\text{orange}}) - Collision({\gamma^2}_{\text{blue}}, \gamma_{\text{orange}})] =$

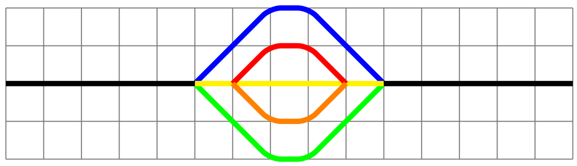
$P({\gamma^1}_{\text{blue}}, \gamma_{\text{orange}}) - P({\gamma^2}_{\text{blue}}, \gamma_{\text{orange}})$

We can similarly show that:

$J_{orange}(\gamma_{\text{blue}}, {\gamma^1}_{\text{orange}}) - J_{orange}(\gamma_{\text{blue}}, {\gamma^2}_{\text{orange}}) = P(\gamma_{\text{blue}}, {\gamma^1}_{\text{orange}}) - P(\gamma_{\text{blue}}, {\gamma^2}_{\text{orange}})$

for two different trajectories ${\gamma^1}_{\text{orange}}$ and ${\gamma^2}_{\text{orange}}$, which means P is an exact potential function.

If we further analyze the game, we notice that each player has only five dominant strategies. All the other strategies happen to be **dominated**. In the following figure we represent the five dominant strategies at disposal of each player. Blue car and orange car have the same dominant strategies as the game is completely symmetric. Therefore, the game is reduced considerably and can be described by the following matrix M:



$$\text{M} = \begin{array}{cccccc} // & Blue & Red & Yellow & Orange & Green \\ Blue & (108,108) & (108,102) & (8,0) & (8,2) & (8,8) \\ Red & (102,108) & (102,102) & (102,100) & (2,2) & (2,8) \\ Yellow & (0,8) & (100,102) & (100,100) & (100,102) & (0,8) \\ Orange & (2,8) & (2,2) & (102,100) & (102,102) & (102,108) \\ Green & (8,8) & (8,2) & (8,0) & (108,102) & (108,108) \end{array}$$

where the entrance $\mathrm{m_{ij}}$ represents the costs $J_{blue}(\gamma_{\text{blue}}, \gamma_{\text{orange}})$ and $J_{orange}(\gamma_{\text{blue}}, \gamma_{\text{orange}})$ assigned to the blue car and to the orange car when playing actions γ_{blue} and γ_{orange}. By analyzing the matrix M we can easily find all the previously computed Nash Equilibria.
Moreover, our proposed potential:
$P(\gamma_{\text{blue}}, \gamma_{\text{orange}}) = W(\gamma_{\text{blue}}, \gamma_{\text{orange}}) - Collision(\gamma_{\text{blue}}, \gamma_{\text{orange}})$
looks like:

$$\text{P} = \begin{array}{cccccc} // & Blue & Red & Yellow & Orange & Green \\ Blue & 116 & 110 & 8 & 10 & 16 \\ Red & 110 & 104 & 102 & 4 & 10 \\ Yellow & 8 & 102 & 100 & 102 & 8 \\ Orange & 10 & 4 & 102 & 104 & 110 \\ Green & 16 & 10 & 8 & 110 & 116 \end{array}$$

If we try to compute the improvement along several paths of length 4, we always end up with 0, thus confirming that P is a potential.
The PoA is the ratio between the maximum welfare cost achievable in the set of Nash Equilibria and the minimum welfare of the game (among all possible strategies). We have already shown that there exist at least two Nash Equilibria with W = 8, therefore since the minimum welfare achievable is $W_{min} = 4$ (as shown in point c) we have that the price of Anarchy is strictly grater than one (at least 2 in our case). The price of Stability, as defined in the assignment is exactly equal to 1 as we have already shown that there exists a Nash Equilibrium identified by ($\overline{\gamma}_{\text{blue}}$,$\overline{\gamma}_{\text{orange}}$) that achieves the minimum welfare. Of course, by definition the PoS can never be strictly smaller than 1.