Game Theory and Control - Assignment 2

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1 Exercise 1 Demand-Side Management game

a) Let us recall the definition of a **convex game**. Consider an N player game with continuous action space K_i. A convex game is characterized by the following:

- the action spaces K_i are compact and convex.
- J_i are continuous in $x \in K$
- J_i are convex in x_i for fixed $x_{\text{-i}}$

Our constrained action spaces K_i are surely compact and convex as they are expressed by a linear equation, where each term is bounded. In fact we know that:

$x_{\mathrm{i}}{}^{\mathrm{h}} \in [0, \overline{x}_{\mathrm{i}}{}^{\mathrm{h}}]$.

Moreover we have:

$$\sum_h x_{\mathrm{i}}{}^{\mathrm{h}} = e_{\mathrm{i}} \tag{1}$$

This set of equation describes indeed a convex action space.

Let us now analyze the cost functions J_{i}:
We can express J_{i} as follows:

$$J_{\mathrm{i}} = p_{\mathrm{i}1} x_{\mathrm{i}}{}^{\mathrm{T}} x_{\mathrm{i}} + p_{\mathrm{i}1} \sum_{j \neq i} x_{\mathrm{j}}{}^{\mathrm{T}} x_{\mathrm{i}} + p_{\mathrm{i}2} 1^{24\ \mathrm{T}} x_{\mathrm{i}} \tag{2}$$

This function is clearly continuous in $x \in K$. Let us now analyze its convexity. We will first compute the gradient and then the Hessian of the cost function:
The gradient $\nabla J_{\mathrm{i}} \in R^{24}$.

$$\nabla J_{\mathrm{i}} = 2 p_{\mathrm{i}1} x_{\mathrm{i}} + p_{\mathrm{i}1} \sum_{j \neq i} x_{\mathrm{j}} + p_{\mathrm{i}2} 1^{24} \tag{3}$$

The Hessian $\nabla^2 J_{\mathrm{i}}$ is instead a 24 X 24 matrix, with the following diagonal structure:

$$\nabla^2 J_{\mathrm{i}} = \begin{bmatrix} 2p_{\mathrm{i}1} & ... & ... & 0 \\ 0 & 2p_{\mathrm{i}1} & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 2p_{\mathrm{i}1} \end{bmatrix}$$

We know that the function J_{i} is convex if and only if the Hessian $\nabla^2 J_{\mathrm{i}}$ is positive semi-definite. Since the Hessian is symmetric, we can check the condition on its eigenvalues, which are the elements of the diagonal. A symmetric matrix is positive semidefinite if and only if all its eigenvalues are positive. In order for the game to be a symmetric matrix, we need to have:

$p_{\mathrm{i}1} \geq 0$
$\forall i$

The parameters $\mathrm{p}_{\mathrm{i}2}$ instead are free and do not determine convexity.

We can immediately notice that there exists a trivial counterexample for which the Nash Equilibrium is not unique in such ranges. We have just shown that the game is convex for $p_{\mathrm{i}1} = 0$ and $p_{\mathrm{i}2} = 0 \ \forall i$. This corresponds to the case where energy is basically free and the cost functions are always equal to zero regardless of each player strategy. So for the parameters choices $p_{\mathrm{i}1} = 0$ and $p_{\mathrm{i}2} = 0 \ \forall i$ the game is convex but does not have a unique Nash Equilibrium.

b) A sufficient condition for the game to be a potential game is that the parameters $p_{\mathrm{i}1}$ are all equal to each other. In formulas:

$p_{\mathrm{i}1} = p_1$

$\forall i$

where p_1 is an arbitrary constant satisfying the convex constraint, $p_1 \geq 0$

Let's recall that in order for a function P to be an exact potential for the game, it has to hold:

$\mathrm{P}(\mathrm{x'}_{\mathrm{i}}, \mathrm{x}_{\text{-i}})$ - $\mathrm{P}(\mathrm{x''}_{\mathrm{i}}, \mathrm{x}_{\text{-i}}) = \mathrm{J}_{\mathrm{i}}(\mathrm{x'}_{\mathrm{i}}, \mathrm{x}_{\text{-i}})$ - $\mathrm{J}_{\mathrm{i}}(\mathrm{x''}_{\mathrm{i}}, \mathrm{x}_{\text{-i}})$,$\forall i$ (4)

Here $\mathrm{x'}_{\mathrm{i}}$ and $\mathrm{x''}_{\mathrm{i}}$ are two different strategies by player i.
Let us first calculate the right hand side of equation 4 for any player i.

$J_{\mathrm{i}}(x'_{\mathrm{i}}, x_{\text{-i}}) - J_{\mathrm{i}}(x''_{\mathrm{i}}, x_{\text{-i}}) = p_1(x'^{\mathrm{T}}_{\mathrm{i}} x'_{\mathrm{i}} - x''^{\mathrm{T}}_{\mathrm{i}} x''_{\mathrm{i}}) + p_1 \sum_{j \neq i} x_{\mathrm{j}}^{\mathrm{T}} (x'_{\mathrm{i}} - x''_{\mathrm{i}}) + p_{\mathrm{i}2} 1^{24\ \mathrm{T}} (x'_{\mathrm{i}} - x''_{\mathrm{i}})$

We now propose a potential function P which is:

$P(x) = \sum_j (p_1 x_\mathrm{j}{}^\mathrm{T} x_\mathrm{j} + \frac{p_1}{2} \sum_{k \neq j} (x_\mathrm{k}{}^\mathrm{T} x_\mathrm{j}) + p_{\mathrm{j}2} 1^{24\ \mathrm{T}} x_\mathrm{j})$

Let us now compute the left hand side of equation 4, $P(x'_\mathrm{i}, x_{\text{-i}}) - P(x''_\mathrm{i}, x_{\text{-i}})$

$P(x'_\mathrm{i}, x_{\text{-i}}) - P(x''_\mathrm{i}, x_{\text{-i}}) = p_1 x'_\mathrm{i}{}^\mathrm{T} x'_\mathrm{i} + p_1 \sum_{j \neq i} (x_j{}^\mathrm{T} x_\mathrm{j}) + \frac{p_1}{2} \sum_{k \neq i} (x_\mathrm{k}{}^\mathrm{T} x'_\mathrm{i}) + \frac{p_1}{2} \sum_{j \neq i} \sum_{k \neq i,j} (x_\mathrm{k}{}^\mathrm{T} x_\mathrm{j}) + \frac{p_1}{2} \sum_{j \neq i} (x'_\mathrm{i}{}^\mathrm{T} x_\mathrm{j}) + p_{\mathrm{i}2} 1^{24\ \mathrm{T}} x'_\mathrm{i} + \sum_{j \neq i} (p_{\mathrm{j}2} 1^{24\ \mathrm{T}} x'_\mathrm{i}) - p_1 x''_\mathrm{i}{}^\mathrm{T} x''_\mathrm{i} - p_1 \sum_{j \neq i} (x_\mathrm{j}{}^\mathrm{T} x_\mathrm{j}) - \frac{p_1}{2} \sum_{k \neq i} (x_\mathrm{k}{}^\mathrm{T} x''_\mathrm{i}) - \frac{p_1}{2} \sum_{j \neq i} \sum_{k \neq i,j} (x_\mathrm{k}{}^\mathrm{T} x_\mathrm{j}) - \frac{p_1}{2} \sum_{j \neq i} (x''_\mathrm{i}{}^\mathrm{T} x_\mathrm{j}) - p_{\mathrm{i}2} 1^{24\ \mathrm{T}} x''_\mathrm{i} - \sum_{j \neq i} (p_{\mathrm{j}2} 1^{24\ \mathrm{T}} x'_\mathrm{i}) =$

$p_1 (x'_\mathrm{i}{}^\mathrm{T} x'_\mathrm{i} - x''_\mathrm{i}{}^\mathrm{T} x''_\mathrm{i}) + \frac{p_1}{2} \sum_{k \neq i} (x_\mathrm{k}{}^\mathrm{T} (x'_\mathrm{i} - x''_\mathrm{i})) + \frac{p_1}{2} \sum_{j \neq i} ((x'_\mathrm{i}{}^\mathrm{T} - x''_\mathrm{i}{}^\mathrm{T}) x_\mathrm{j}) + p_{\mathrm{i}2} 1^{24\ \mathrm{T}} (x'_\mathrm{i} - x''_\mathrm{i}) =$

$p_1 (x'_\mathrm{i}{}^\mathrm{T} x'_\mathrm{i} - x''_\mathrm{i}{}^\mathrm{T} x''_\mathrm{i}) + p_1 \sum_{j \neq i} x_\mathrm{j}{}^\mathrm{T} (x'_\mathrm{i} - x''_\mathrm{i}) + p_{\mathrm{i}2} 1^{24\ \mathrm{T}} (x'_\mathrm{i} - x''_\mathrm{i}) =$

$J_\mathrm{i}(x'_\mathrm{i}, x_{\text{-i}}) - J_\mathrm{i}(x''_\mathrm{i}, x_{\text{-i}})$

We have therefore shown that our proposed P is a potential function whenever:

$p_{\mathrm{i}1} = p_1$

$\forall i$

c)To assess the existence and the uniqueness of the Nash Equilibrium in the proposed game, we can apply the following reasoning. At first, to assess the existence of the Equilibrium it is sufficient to show that the game is convex for the choice of the parameters $p_{\mathrm{i}1} = 0$ and $p_{\mathrm{i}2}$. Since all $p_{\mathrm{i}1}$ are actually positive, the game is indeed convex and therefore it exists a Nash Equilibrium. Let us now evaluate the uniqueness of the Nash Equilibrium. From the lectures, we know that to guarantee that the game has a unique NE, it is sufficient to prove the strict monotonicity of the game map:

$$F(x) = (\nabla_{\mathrm{x}1} J_1(x); \nabla_{\mathrm{x}2} J_2(x); ...; \nabla_{\mathrm{xn}} J_\mathrm{n}(x)) \tag{5}$$

The game map F belongs to R^{2400} and looks as follows:

$$\mathrm{F} = \begin{bmatrix} \nabla J_1 \\ \nabla J_2 \\ ... \\ \nabla J_{100} \end{bmatrix} = \begin{bmatrix} 2p_{11} x_1 + p_{11} \sum_{j \neq 1} x_\mathrm{j} + p_{12} 1^{24} \\ 2p_{21} x_2 + p_{21} \sum_{j \neq 2} x_\mathrm{j} + p_{22} 1^{24} \\ ... \\ 2p_{100/1} x_{100} + p_{100/1} \sum_{j \neq 100} x_\mathrm{j} + p_{100/2} 1^{24} \end{bmatrix}$$

We can now compute the Jacobian of the game map ∇F. It is a $2400X2400$ matrix with the following structure:

$$\nabla F = \begin{bmatrix} \nabla^2{}_{\mathrm{x1}} J_1 & \nabla_{\mathrm{x1}} \nabla_{\mathrm{x2}} J_1 & ... & \nabla_{\mathrm{x1}} \nabla_{\mathrm{x100}} J_1 \\ \nabla_{\mathrm{x2}} \nabla_{\mathrm{x1}} J_2 & \nabla^2{}_{\mathrm{x2}} J_2 & ... & \nabla_{\mathrm{x2}} \nabla_{\mathrm{x100}} J_2 \\ ... & ... & ... & ... \\ \nabla_{\mathrm{x100}} \nabla_{\mathrm{x1}} J_{100} & \nabla_{\mathrm{x100}} \nabla_{\mathrm{x2}} J_{100} & ... & \nabla^2{}_{\mathrm{x100}} J_{100} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\partial^2 J_1}{\partial({x_1}^1 \partial {x_1}^1)} & \frac{\partial^2 J_1}{\partial({x_1}^1 \partial {x_1}^2)} & ... & \frac{\partial^2 J_1}{\partial({x_1}^1 \partial {x_1}^{24})} & ... & ... & \frac{\partial^2 J_1}{\partial({x_1}^1 \partial {x_{100}}^{24})} \\ \frac{\partial^2 J_1}{\partial({x_1}^2 \partial {x_1}^1)} & \frac{\partial^2 J_1}{\partial({x_1}^2 \partial {x_1}^2)} & ... & \frac{\partial^2 J_1}{\partial({x_1}^2 \partial {x_1}^{24})} & ... & ... & \frac{\partial^2 J_1}{\partial({x_1}^2 \partial {x_{100}}^{24})} \\ ... & ... & ... & ... & ... & ... & ... \\ \frac{\partial^2 J_1}{\partial({x_1}^{24} \partial {x_1}^1)} & \frac{\partial^2 J_1}{\partial({x_1}^{24} \partial {x_1}^2)} & ... & \frac{\partial^2 J_1}{\partial({x_1}^{24} \partial {x_1}^{24})} & ... & ... & \frac{\partial^2 J_1}{\partial({x_1}^{24} \partial {x_{100}}^{24})} \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... \\ \frac{\partial^2 J_{100}}{\partial({x_{100}}^{24} \partial {x_1}^1)} & \frac{\partial^2 J_1}{\partial({x_{100}}^{24} \partial {x_1}^2)} & ... & \frac{\partial^2 J_1}{\partial({x_{100}}^{24} \partial {x_1}^{24})} & ... & ... & \frac{\partial^2 J_1}{\partial({x_{100}}^{24} \partial {x_{100}}^{24})} \end{bmatrix}$$

We can observe that if $i \neq j$, then:

$$\nabla_{\mathrm{xi}} \nabla_{\mathrm{xj}} J_{\mathrm{i}} = \begin{bmatrix} p_{\mathrm{i1}} & 0 & ... & 0 \\ 0 & p_{\mathrm{i1}} & ... & 0 \\ 0 & 0 & ... & p_{\mathrm{i1}} \end{bmatrix} \tag{6}$$

while:

$$\nabla^2{}_{\mathrm{xi}} J_{\mathrm{i}} = \begin{bmatrix} 2p_{\mathrm{i1}} & 0 & ... & 0 \\ 0 & 2p_{\mathrm{i1}} & ... & 0 \\ 0 & 0 & ... & 2p_{\mathrm{i1}} \end{bmatrix} \tag{7}$$

That means we can easily construct the matrix ∇F on Matlab. The matrix ∇F is not a symmetric matrix but we can compute its symmetric part $(\frac{\nabla F + \nabla F^{\mathrm{T}})}{2}$. We know that if this is positive definite, then the matrix ∇F is positive definite. Also, if the Jacobian of the game map F is positive definite, we know that F is strictly monotone and that the game has a unique Nash Equilibrium. Checking with MatLab, we find out that the eigenvalues of $(\frac{\nabla F + \nabla F^{\mathrm{T}})}{2}$ are indeed strictly positive, therefore the game has a unique NE.

d) We can solve this question by implementing the projected game map algorithm as explained in lecture 4. The algorithm consists of:

- Random initialization of the vector x(0)
- Gradient descent step $z(t+1) = x(t) - \gamma F(x(t))$

- Projection of x(t+1) onto the feasible set of the action spaces K, via the solution of a quadratic program.

  x(t+1) $= \min_x(||x - z(t+1)||)$ (8)

As we have shown in the lecture, such an algorithm is guaranteed to converge for a proper choice of γ. In particular it converges for $\gamma \in (0, \frac{2\mu}{L^2})$ Here μ and L^2 denote the strong-monotonicity coefficient and the Lipschitz constant of the game map.

We can express the game map F as an affine function. (We actually use this formulation in the algorithm):

$F(x) = A * x + B$

where A coincides with the previously computed Jacobian of the game map ∇F, whereas B $\in R^{2400}$ that looks as follows:

$$\begin{bmatrix} p_{\text{i1}} \\ p_{\text{i1}} \\ \vdots \\ p_{\text{i1}} \\ p_{\text{i2}} \\ \vdots \\ p_{\text{i2}} \\ \vdots \\ \vdots \\ p_{\text{i100}} \\ \vdots \\ p_{\text{i100}} \end{bmatrix}$$

This B vector contains each p_{i1} price parameter 24 times.

So we have just proved that the game map is an affine function. From Tutorial 5, we know how to compute the strong monotonicity coefficient μ for a linear map. We know that:

$\mu = \lambda_{\min}(\frac{A^{\mathrm{T}}+A}{2}) = 0.0087$

Moreover we can compute the Lipschitz constant via the following consideration:

$$||F(x) - F(y)|| = ||A(x) + B - A(y) - B)|| \leq ||A||||x - y||$$

Thus, it holds that $L = ||A|| = 1.2778$

So, in order for the projected game map algorithm to converge to the unique NE, we need to set $\gamma \in (0, \frac{2\mu}{L^2}) = (0, 0.0106)$. Setting γ within this range is only a sufficient condition for convergence.

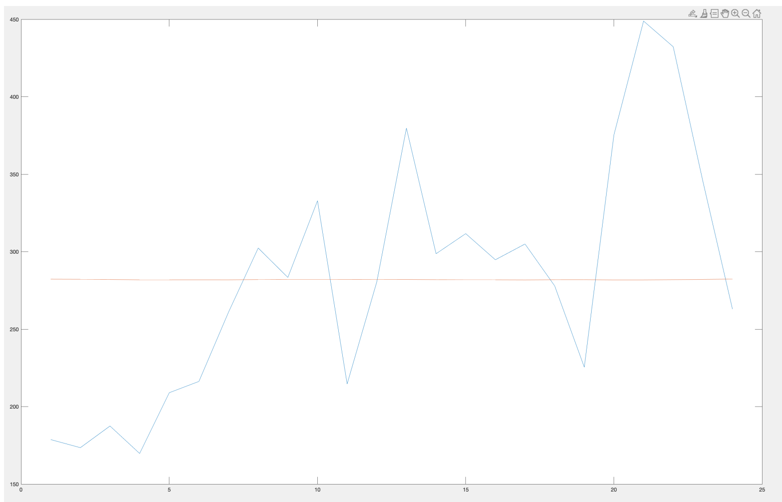

Figure 1: Aggregate nominal energy consumption (blue) and aggregate energy consumption at the Nash equilibrium (orange).

We can make some interesting observations on this plot. The Equilibrium corresponds to a situation where the aggregate energy consumption is constant during the 24 hours. There are no peaks of total consumption; instead the energy demand is uniformly distributed along the day. On the other hand, the nominal energy consumption clearly have some peaks and lows during the day.

2 Exercise 2 Security cameras

a) The game can indeed be formulated as a **security game**. The attacker is played by the thief, whereas the defender is the designer of the camera feeds. The attacker **targets** are the kinds of merchandise in the supermarket:

$$\Sigma = \begin{bmatrix} \sigma 1 \\ \sigma 2 \\ \sigma 3 \\ \sigma 4 \\ \sigma 5 \\ \sigma 6 \\ \sigma 7 \\ \sigma 8 \\ \sigma 9 \end{bmatrix} = \begin{bmatrix} Zurisack \\ Cigarettes \\ Alcohol \\ Newspapers \\ Stationery \\ PersonalHygiene \\ Dairy \\ Meat \\ Fruit \end{bmatrix}$$

The **coverage choices** are basically represented by the cameras displayed on the security monitor:

$$\Gamma = \begin{bmatrix} \gamma 1 \\ \gamma 2 \\ \gamma 3 \\ \gamma 4 \\ \gamma 5 \\ \gamma 6 \\ \gamma 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Remember that each coverage choice γ_i tells whether the targets are covered or not.

Let $\mathrm{y_i}$ be the probability of implementing coverage choice γ_i. We can compute each $\mathrm{y_i}$ as the fraction of time for which the monitor is fed with camera *i*. Let us now compute the coverage vector $c \in R^9$, where the element $\mathrm{c_j}$ represents the probability that the target j is covered:

$$\mathrm{c} = \begin{bmatrix} y_1 \\ y_1 + y_2 \\ y_2 + y_3 \\ y_3 \\ y_4 \\ y_4 + y_5 \\ y_5 + y_6 \\ y_6 \\ y_7 \end{bmatrix}$$

Since we assume that the theft happens in a single instant, we can observe that the only meaningful parameters for the defenders are the coverage choices.

The defender basically randomizes the coverage by choosing which camera to feed to the monitor at every instant. If we consider a timestamp of T = 60 seconds, the coverage choice $\mathrm{y_i}$ would be the fraction of time $\mathrm{t_i}$ in which camera i is fed to the monitor. Of course it has to hold:

$\sum_i t_{\mathrm{i}} = 60$

$t_{\mathrm{i}} \leq 60 \ \ \forall i$

Then, it holds:

$y_{\mathrm{i}} = \frac{t_{\mathrm{i}}}{T} \ \ \forall i$

So, the only thing that matters is the ratio $\frac{t_{\mathrm{i}}}{T}$ which defines the probability $\mathrm{y_i}$. The order in which the cameras are fed to the monitor is not relevant. b) Let A and B be the payoff matrices respectively for the defender(supermarket) and the attacker(thief). Both the players are, from now on, perceived as minimizers. Given the provided estimates we can formalize the game as follows:

$$\mathrm{A} = \begin{bmatrix} 0 & 0 & 10 & 2 & 3 & 5 & 4 & 10 & 8 \\ 20 & 0 & 0 & 2 & 3 & 5 & 4 & 10 & 8 \\ 20 & 15 & 0 & 0 & 3 & 5 & 4 & 10 & 8 \\ 20 & 15 & 10 & 2 & 0 & 0 & 4 & 10 & 8 \\ 20 & 15 & 10 & 2 & 3 & 0 & 0 & 10 & 8 \\ 20 & 15 & 10 & 2 & 3 & 5 & 0 & 0 & 8 \\ 20 & 15 & 10 & 2 & 3 & 5 & 4 & 10 & 0 \end{bmatrix}$$

$$\mathrm{B} = \begin{bmatrix} 0 & 0 & -5 & -1 & -2 & -3 & 0 & 0 & -1 \\ -20 & 0 & 0 & -1 & -2 & -3 & 0 & 0 & -1 \\ -20 & -15 & 0 & 0 & -2 & -3 & 0 & 0 & -1 \\ -20 & -15 & -5 & -1 & 0 & 0 & 0 & 0 & -1 \\ -20 & -15 & -5 & -1 & -2 & 0 & 0 & 0 & -1 \\ -20 & -15 & -5 & -1 & -2 & -3 & 0 & 0 & -1 \\ -20 & -15 & -5 & -1 & -2 & -3 & 0 & 0 & 0 \end{bmatrix}$$

If camera 1 is fed to the monitor all the time, that means the thief has not even a chance to steal Zurisack or Cigarettes, without getting caught. It follows that the best response for the attacker to the defender's pure strategy of watching camera 1 is to go for the Alcohol. Formally:

$R_a(\gamma_1) = argmin_j(B_{1j}) = \sigma_3 = Alcohol$

c) To answer this question, it is possible to make some observations in order to simplify the game. There is plenty of dominated strategy both for the attacker and the defender. First of all dairy and meat (σ_7, σ_8) provide no positive value for the thief and are therefore dominated by all other strategies. Then, we can observe that it makes no sense for the defender to look at camera 6, since the thief is stealing neither the meat nor the dairy (γ_6 is dominated). Strategy γ_5 for the defender is also dominated by γ_4: since the thief is not going to steal dairy, it is better to protect the stationery. From this, it follows that action σ_5 (Stationery) is also dominated by σ_6 (Personal Hygene): they are both guarded only by camera 4 (camera 5 is dominated) but Personal Hygene is worth more for the attacker. Finally, σ_2, is a dominated strategy from the start. It makes no sense for the thief to try to steal the cigarettes as they are always more protected (but less valuable) than the Zurisack.

To force the thief best response to be $\sigma_3 = Alcohol$, we have to guard the Zurisack as much as possible to make sure that for the thief it is not convenient to stole it.

Let us suppose our mixed strategy to be [p, 0, 1-p, 0, 0, 0, 0]. That means we only feed camera 1 and 3 to the monitor with the idea of protecting the Zurisack and the Alcohol. If the attacker best response is to steal the Zurisack, we have that:

$E(Zurisack) = -20*(1-p)$, corresponding to the case where the thief manages to get away with stealing the Zurisack.

Instead, if the attacker best response is to steal Alcohol:

$E(Alcohol) = -5*p$, corresponding to the case where the thief manages to get away with stealing Alcohol.

We can set $-20*(1-p) = -5*p$, which yields $p = 0.8$. That means, that, by setting $p = 0.8$, attacker best response is the tuple $[\sigma_1, \sigma_3]$ If we want the best response of the thief to be σ_3 (Alcohol), we can set $p = 0.8 + \epsilon$. Notice that the expected value of the strategy $\sigma_6 = PersonalHygene$ is:

$E(PersonalHygene) = -3$, as the thief is certainly not going to get caught.

Even then the best response for the thief is to still go for Alcohol, as $-3 \geq -4$. We have therefore computed the best strategy for the defender that guarantees that the attacker will steal alcohol:

$$Y_1 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 0.8+\epsilon \\ 0 \\ 0.2-\epsilon \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Actually there is another strategy that achieves the same result, which involves watching camera 2 instead of camera 3 (attacker is not going to steal newspaper anyway). Infact, camera 2 still protects the Alcohol. So another possible strategy to obtain the same result would be:

$$Y_2 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 0.8+\epsilon \\ 0.2-\epsilon \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

d) The question asks to find the Stackelberg equilibrium of a non-zero sum game. We can actually implement the **divide and conquer algorithm** on Matlab to solve the game. However, it is useful to provide some intuition, using the results that we have already achieved. First of all, let us analyze how the attacker should in general respond. With simple calculations we can easily check that, regardless of the defender's strategy, attacker best response is always going to be σ_1 or σ_3 (Zurisack or Alcohol). We have already proved that σ_2 (Cigarettes) is a dominated strategy as they are always protected more (or equally) than Zurisack, while being less valuable. Moreover, the next most appealing item for the thief is the Personal Hygene (σ_6). We are about to show that there is no way to force the thief best response to be σ_6. One can show that in order to prevent the attacker to steal Zurisack, camera 1 has to be on monitor at least 85% of the time. However, if for the remaining 15% of the time we are watching camera 2 or 3 (to protect the Alcohol), that is still not going to prevent the attacker from performing σ_3 (Alcohol). Let us formalize this reasoning:

To compute the minimum time fraction y_1 to prevent the thief from stealing Zurisack instead of Personal Hygene, we can set:

$-3 = -20 * (1 - y_1)$

which yields:

$y_1 = 0.85$

If we set for example $y_2 = 0.15$, we have that:

$E[alcohol] = (1 - y_2) * (-5) = 0.85 * -5 = -4.15 < -3$

That means that the best response for the attacker is actually to steal Alcohol

(σ_3). We have just shown that attacker's best response is always going to be either σ_1 or σ_3.

Moreover, Camera 2 and Camera 3 basically have the exact same purpose, which is to guard against the thief stealing the Alcohol. Feeding Camera 2 or Camera 3 to the monitor produces exactly the same results. The Stackelberg equilibrium strategy for the defender is therefore going to look like:

$$\begin{bmatrix} y_1 & y_2 & y_3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We have already shown that for $y_1 < 0.8$, best response for the attacker is σ_1 (Zurisack), while for $y_1 > 0.8$, best response for the attacker is σ_3 (Alcohol). We have to remember, however, that this is not a zero-sum game and the supermarket is not playing against the thief. We can show by implementing divide and conquer that all the actions of the form:

$$Y = \begin{bmatrix} 0.8 \\ y_2 \\ y_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \sigma(Y_1) = \sigma_1, \quad \text{where } y_2 + y_3 = 0.2$$

are Stackelberg Equilibria. The supermarket prefers the attacker to go for Zurisack instead of Alcohol. With the value of $y_1 = 0.8$ we have already shown that the best response for the attacker is the tuple $[\sigma_1, \sigma_3]$. The supermarket however benefits the most from the attacker choosing σ_1 as:

$$E_{\text{def}}[\sigma_1, Y_1] = (1 - p) * (20) = 4$$

while,

$$E_{\text{def}}[\sigma_3, Y_1] = p * (10) = 8.$$

So if the supermarket wants to nullify the risk of the attacker responding with σ_3 it is more sensible to play, for example:

$$Y = \begin{bmatrix} 0.8 - \epsilon \\ 0.2 + \epsilon \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \sigma(Y_1) = \sigma_1$$

This time, attacker best response is the singleton $[\sigma_1]$.

So to answer the other questions, the cost for the supermarket at the Stackelberg Equilibrium is equal to 4 and the target of the thieves are the Zurisacks. Moreover, a possible sequence that implements the equilibrium is to check Camera 1 for the first 48 seconds of the timestamp and Camera 2 for the remaining 12 seconds.

e)To compute a Nash Equilibrium for the game we can construct an auxiliary zero-sum security game by modifying the outcome for the defender as follows:

$D^{\mathrm{j}}{}_{\mathrm{c}} = -A^{\mathrm{j}}{}_{\mathrm{c}}$
$D^{\mathrm{j}}{}_{\mathrm{u}} = -A^{\mathrm{j}}{}_{\mathrm{u}}$

where the superscripts j represents whether the target is covered or uncovered. Therefore, the auxiliary zero-sum game is represented by the following A matrix:

$$\mathrm{A} = \begin{bmatrix} 0 & 0 & 5 & 1 & 2 & 3 & 0 & 0 & 1 \\ 20 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 1 \\ 20 & 15 & 0 & 0 & 2 & 3 & 0 & 0 & 1 \\ 20 & 15 & 5 & 1 & 0 & 0 & 0 & 0 & 1 \\ 20 & 15 & 5 & 1 & 2 & 0 & 0 & 0 & 1 \\ 20 & 15 & 5 & 1 & 2 & 3 & 0 & 0 & 1 \\ 20 & 15 & 5 & 1 & 2 & 3 & 0 & 0 & 0 \end{bmatrix}$$

We can compute one Nash Equilibrium for this zero-sum game by solving a linear program. We obtain the following NE. Let us denote with y' the mixed strategy for the defender and with z' the mixed strategy for the attacker.

$$y'_{\mathrm{NE}} = \begin{bmatrix} 0.8 \\ 0.2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad z'_{\mathrm{NE}} = \begin{bmatrix} 0.2 \\ 0 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can now use the relationship, as explained in the lecture, between the Nash Equilibria of the Auxiliary game and the NE of the original game.

We know that:

$z^*{}_{\mathrm{j}} \sim z'{}_{\mathrm{j}} \frac{A_{\mathrm{j}}{}^{\mathrm{c}} - A_{\mathrm{j}}{}^{\mathrm{u}}}{D_{\mathrm{j}}{}^{\mathrm{u}} - D_{\mathrm{j}}{}^{\mathrm{c}}}$

$y^*{}_{\mathrm{j}} = y'$

Finally, the corresponding Nash Equilibrium of the original game is:

$$y^*_{\mathrm{NE}} = \begin{bmatrix} 0.8 \\ 0.2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad y_{*\mathrm{NE}} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{2}{3} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let us now compare the costs of the Nash and the Stackelberg Equilibria for both the attacker and the defender.

Let us denote with $\mathrm{J_A}^{\mathrm{S}}$ and $\mathrm{J_D}^{\mathrm{S}}$ the payoffs at the Stackelberg equilibria respectively for the attacker and the defender. We have that

$$J_{\mathrm{A}}{}^{\mathrm{S}} = 0.2 * (-20) = -4$$

$$J_{\mathrm{D}}{}^{\mathrm{S}} = 0.2 * (20) = 4$$

Instead let's denote with $\mathrm{J_A}^{\mathrm{NE}}$ and $\mathrm{J_D}^{\mathrm{NE}}$ the payoffs at the Nash Equilibrium. Then:

$$J_{\mathrm{A}}{}^{\mathrm{NE}} = y^{*\mathrm{T}}{}_{\mathrm{NE}} * A * z^*{}_{\mathrm{NE}} = -4$$

$$J_{\mathrm{D}}{}^{\mathrm{NE}} = y^{*\mathrm{T}}{}_{\mathrm{NE}} * B * z^*{}_{\mathrm{NE}} = \frac{20}{3}$$

Since the players are minimizers, it is more convenient for the defender to announce their commitment, therefore playing a Stackelberg game.

Finally, the defender Stackelberg Equilibrium strategy is indeed a Nash Equilibrium strategy. We have shown that:

$$y \;= \begin{bmatrix} 0.8 \\ 0.2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

is a plausible defender strategy for both the Stackelberg and the Nash Equilibrium