Game Theory & Control
Assignment 1

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Please submit your solutions on Moodle.

You can either type them (in LaTeX, for example), or scan your handwritten solutions.

If you are scanning your handwritten solutions, please create a PDF that is readable and small in size. You can use any smartphone app that allows to scan documents to PDF (this feature is present in both the Dropbox app and the Google Drive app, plus many others). Do not send plain photos of the pages.

The deadline for this assignment is 1 April 2022, at 23:59.

Please write your solutions in a legible form, include all the steps of your reasoning, indicate what question your are answering, and don't include any solution attempt that you don't want us to grade.

Verify your answers by trying to get to the same results in different ways, checking if they are compatible with the theoretical results that you know and with your intuition. You have plenty of time. Typos and errors by distraction are not acceptable.

This assignment contributes towards 10% of the final score for the course, and must be done independently by each student. No group work is allowed.

Each "letter" question is worth 1 point.
Make sure you answer everything to get the full point for each question. No fraction of points are awarded.
Take your time to double-check that you have not missed any part of each question.

Exercise 1 Many Nash Equilibria

Consider the non-zero-sum two-player (minimizer) simultaneous game described by the two matrices

$$A = \begin{bmatrix} 0 & -10 & 0 & 0 \\ -10 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -10 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

a) Compute all *pure* Nash Equilibria and all *completely mixed* Nash Equilibria of the game.

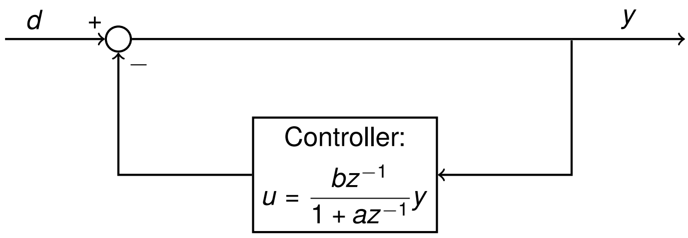
b) Compute all *mixed* Nash Equilibria of the game.

   Hint. *Assume that the players only mix their first two actions. What would be a mixed Nash Equilibrium? What would be the expected outcome? Prove that this mixed strategy is also a Nash Equilibrium for the game with all the four actions.*

c) Draw the Hasse diagram of all the Nash Equilibria and identify the admissible Nash Equilibria.

Exercise 2 Robust tuning of an active damper

You are the engineer in charge of tuning the controller of an active damping system in a car. The model of the closed-loop system is as follows:



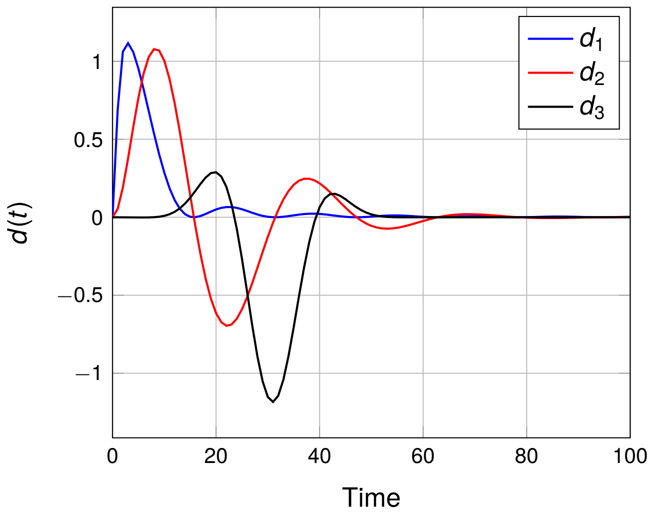
More specifically:

- d is the disturbance coming from the road;
- y is the displacement of the wheel;
- u is the output of your controller, given by

$$u = \frac{bz^{-1}}{1+az^{-1}}\,y,$$

  where z^{-1} represent the unit time delay operator (or equivalently $u(k) = -au(k-1) + by(k-1)$);
- a and b are two scalar parameters that you are allowed to tune.

By recording hours of data from a test car, you came to the conclusion that there are only three kinds of disturbances that can hit the wheel, corresponding to three different kinds of bumps. They are represented below, and are available as a `csv` file on Moodle. The data is as follows: the first column is time, the second column is the first disturbance, and so on.



Your goal is to pick the values of the parameters a and b in the discrete sets

$$a \in \{0,\ 0.4,\ 0.8\}, \qquad b \in \{0,\ 0.3,\ 0.6\},$$

so that the total control effort $\sum_{t=0}^{T} u(t)^2$ is minimized **while maintaining the output** $y(t)$ **below 1 for all times** (i.e., $y(t) \le 1$ for all $t = 0, 1, \ldots, T$). Consider an horizon of $T = 100$.

a)
- How can you formalize this control design problem as a game?
- What are the players? What are the actions available to each player?
- Do players act simultaneously or sequentially?

b) Solve the control design problem by studying the resulting game. Explain your solution in detail, and plot the output y and the input u with the parameters resulting from the control design.

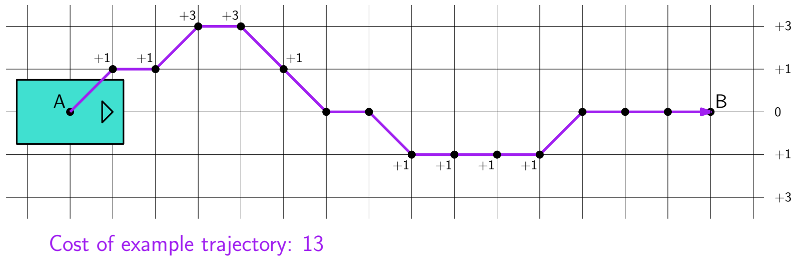
Hint. *If you use Matlab, the commands* `filt`, `feedback`, *and* `lsim` *might be helpful.*

c) Suppose now that the vision system of the car is scanning the road ahead. Assume it can accurately identify which disturbance is hitting the wheel, and that after having identified the disturbance you can instantly tune the controller.

- What are the optimal controller parameters for each disturbance?
- How would you tune your controller if the vision system of the car sometimes fails to identify the disturbance (i.e., you cannot always trust the prediction of the vision system)?

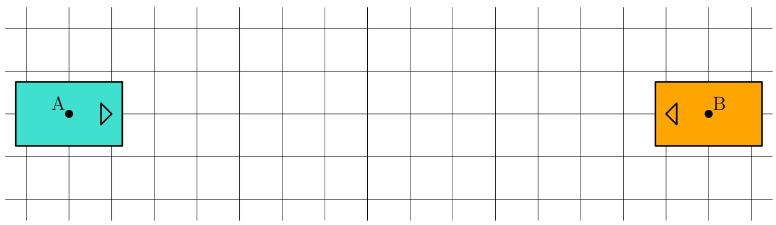
Exercise 3 Trajectory planning game

Consider the simplified trajectory planning problem represented in this figure.

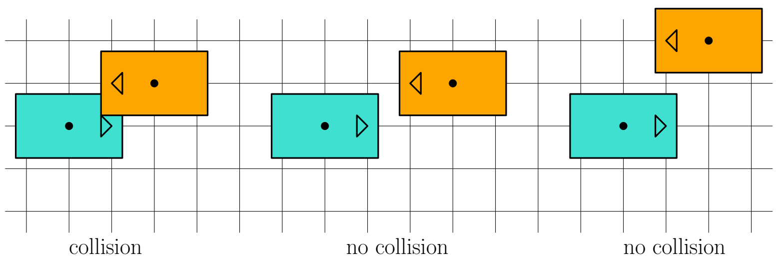


The blue car needs to plan a trajectory from A to B. The car can move one square (also diagonally) at each time step. The resulting cost depends on how much time the car spends outside of the middle of the road: +1 for every time step in which the car is 1 cell off the middle line, and +3 for every time step in which the car is 2 cells off the middle line. See the example trajectory in the figure.

Consider then the 2-player version of this problem, represented in this figure.



The blue car needs to plan a trajectory from A to B, while the orange car needs to plan a trajectory from B to A. Their cost is computed as before, but it now includes also a collision cost of 100. The collision term is present if at any time the footprints of the two cars overlap (that is, if at the same instant in time their longitudinal distance is ≤ 2 and their lateral distance is ≤ 1). See the following examples.



We interpret this problem as a two-player game where the two players have to take their decision simultaneously, and their action is the trajectory that they choose. Once they have decided a trajectory, they commit to that and they are not allowed to change it.

Perform the following tasks.

a) Let the blue car choose its action without accounting for the presence of the orange car, and let call this action γ^*_{blue}. What is γ^*_{blue}? What is the optimal reaction set $\mathcal{R}_{\text{orange}}(\gamma^*_{\text{blue}})$ of the orange car?

b) Pick $\hat{\gamma}_{\text{orange}} \in \mathcal{R}_{\text{orange}}(\gamma^*_{\text{blue}})$. Is $(\gamma^*_{\text{blue}}, \hat{\gamma}_{\text{orange}})$ a Nash equilibrium? What is the resulting cost for the two players? Via a symmetric argument, can you find another Nash Equilibrium for this game?

c) Consider the welfare cost defined as the sum of the costs for the two cars. What is a pair of trajectories $(\bar{\gamma}_{\text{blue}}, \bar{\gamma}_{\text{orange}})$ that minimizes this welfare cost? Is $(\bar{\gamma}_{\text{blue}}, \bar{\gamma}_{\text{orange}})$ a Nash Equilibrium?

d) Prove that

- the game is a potential game
- the Price of Anarchy is strictly greater than 1
- the Price of Stability is 1

where the Price of Anarchy is defined in the lecture slides, while the Price of Stability is defined as

$$PoS := \frac{\min_{\gamma \in \Gamma_{\mathrm{NE}}} W(\gamma)}{\min_{\gamma \in \Gamma} W(\gamma)}$$