Choosing the Largest Dowry

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1 Introduction

Question 1.1. *The king, to test Leonardo, a candidate for the position of wise man, offers him a chance to marry the young lady in the court with the largest dowry. The amounts of dowries are written on slips of paper and mixed. A slip is drawn at random and Leonardo must decide whether that is the largest dowry or not. If he decides it is, he gets the lady and her dowry if he is correct; otherwise, he gets nothing. If Leonardo decides against the amount written on the first slip, he must choose or refuse the next slip, and so on until he chooses one, or else the slips are exhausted. In all, 100 attractive young ladies participate, each with a different dowry. How should Leonardo make his decision?*

2 The solution

By choosing a random one of the young ladies that presents himself, Leonardo has obviously a chance equal to $\frac{1}{100}$ to choose the one with the largest dowry. However, he wants to find the optimal strategy, such that he would be able to improve his chances. To understand how he could do this, let's look at a simplified problem. Instead of having to choose between 100 ladies and their dowries, let's look at 2 different dowries to choose from, which we call by their ranks (i.e. 1 and 2). In the following table, you can look at the possible order of drawings of the two girls.

1	2
2	1

In this case, there are two possible strategies:

- choosing a random (probability is $\frac{1}{2}$)
- deciding to not choose the first lady drawn and, later, choose the first lady whose dowry exceeds the first drawn (probability is $\frac{1}{2}$)

As observed, with 2 ladies the two strategies make no difference, but already with 3 (and 6 different orderings) it is possible that the second strategy improves the probability of winning:

$$
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 3 & 2 \\
2 & 1 & 3
\end{array}
\quad
\begin{array}{ccc}
2 & 3 & 1 \\
3 & 1 & 2 \\
3 & 2 & 1
\end{array}
$$

With the second strategy, Leonardo would win 3 times over 6 possible different outcomes, while with the first strategy only 2 times over 6. So, we already found a strategy better than random guess, at least for the case where there are 3 ladies to choose from.

Continuing on the same line of reasoning, we could define a different strategy:

- deciding to not choose the first two ladies drawn and, later, choose the first lady whose dowry exceeds the first two drawn

This can even be generalized more, by:

- deciding to not choose the first $\ell - 1$ ladies drawn and, later, choose the first lady whose dowry exceeds the first $\ell - 1$ drawn

We want to actually show that this is the best strategy for Leonardo (and find which is the optimal ℓ after which we stop)!

Claim 2.1. *We say that a candidate i is the optimal one if the probability of the lady i having the largest dowry is larger than the probability of finding the largest dowry at a later stage:*

$$
\begin{gathered}
\mathbb{P}\left(\textit{find largest dowry at draw } i\right) > \\
\mathbb{P}\left(\textit{find largest dowry with the optimal strategy later}\right)
\end{gathered}
$$

We claim:

- *the first probability increases as i increases*

- *the second probability decreases as i increases*

Proof. Observe that, being at the early stages of the game makes Leonardo lose no possible strategies available later on, as he could always skip the first ladies until he gets into a position he wants to be. Therefore, it follows that the second probability must decrease (or stay constant) as i increases. For $i = 0$, this is the optimal probability (which we are looking for) at the beginning of the game, for $i = n - 1 = 99$ the probability is $\frac{1}{n} = \frac{1}{100}$, as we passed all the draws so we have only one chance of finding the largest dowry.
For the first probability, instead, observe the following:
$\mathbb{P}\left(\text{find largest dowry at draw } i\right)$ is the probability of finding the maximum among the first i draws (which is equal to $\frac{i}{n}$), and this function increases with i from $\frac{1}{n}$ to 1. $\square$

Now, we want to find the optimal probability and the optimal strategy: we want to find the point i^* at which the first probability is larger than the second so that Leonardo can choose i^* to select the best candidate to have the largest dowry.

2.1 Finding the optimal strategy

Recall that we chose the optimal to be of the form: let the first $\ell - 1$ go by and then choose the first lady whose dowry is larger than the first $\ell - 1$ seen. With this strategy, we have to choose the optimal ℓ so we have very large flexibility and we include the optimal strategies for $n = 2, 3$, etc., as we saw in the examples at the beginning of the section.
The probability of a win is the probability of there being only one candidate from draw l to draw n. The probability that the maximum slip is at draw ℓ is $1/n$. The probability that the maximum of the first $\ell - 1$ draws appears in the first $k - 1$ is $\frac{\ell-1}{k-1}$ (This condition allows us to identify ℓ as a candidate).
Then, the probability of a win at draw $\ell, k \leq \ell \ \leq n$ is given by: $\frac{k-1}{n(\ell-1)}$.
Let us define as $\pi_n(\ell)$, the probability of winning when using the optimal strategy and rejecting the first $\ell - 1$ dowries.

$$\begin{aligned}\pi_n(\ell) &= \frac{1}{n}\sum_{k=\ell}^{n}\frac{\ell-1}{k-1} = \frac{\ell-1}{n}\sum_{k=\ell-1}^{n-1}\frac{1}{\ell} \\ &= \frac{\ell-1}{n}\left(\frac{1}{\ell-1}+\frac{1}{\ell}+\cdots+\frac{1}{n-1}\right)\end{aligned}$$

for $\quad 1 < \ell \leq n$.

If we choose the first draw as our candidate ($\ell = 1$, observe that then the probability of finding the true maximum with this choice is: $\frac{1}{n}$. For $k = 2, 3$, etc. we recover the results we found in the example above. To find the optimal value ℓ^*, we have to solve (for the smallest ℓ), our initial inequalities:

$\frac{l}{n} > \pi_n(\ell + 1)$, which means:

$$\tfrac{\ell}{n} > \tfrac{\ell}{n}\left(\tfrac{1}{\ell}+\tfrac{1}{\ell+1}+\cdots+\tfrac{1}{n-1}\right)$$

Equivalently, we can find that ℓ for which:

$$\tfrac{1}{\ell}+\tfrac{1}{\ell+1}+\cdots+\tfrac{1}{n-1} < 1 < \tfrac{1}{\ell-1}+\tfrac{1}{\ell}+\cdots+\tfrac{1}{n-1}$$

By solving this (numerically), for n = 100, we find the optimal $\ell^* = 38$.

2.1.1 Conclusion

Leonardo should skip the first 37 ladies that are presented to him and then choose the next candidate. He wins the game with probability $\pi_{100}(38) \approx 0.371$

2.1.2 Large Number Approximations

For large numbers n, we can approximate $\sum_{k=1}^{n}\frac{1}{k} \approx \log(n) + \gamma$, where γ is the Euler–Mascheroni constant. Using this approximation, we get:

$\pi_n(\ell) \approx \frac{\ell-1}{n} \log(\frac{n-1}{\ell-1}) \approx \frac{\ell}{n} \log(\frac{n}{\ell})$

We also know that:

$\log(\frac{n}{\ell^*}) \approx 1$

and so $\ell^* \approx \frac{n}{e}$.

Substituting these results, we find out that:

$lim_{n\to\infty}(\pi_n(\ell^*)) = \frac{1}{e} = 0.368$

where $\ell^* = \frac{n}{e}$.

To sum up for large values of n, the optimal strategy passes approximately the fraction $\frac{1}{e}$ of the slips and chooses the first candidate thereafter. The probability of winning is approximately $\frac{1}{e}$.