How long can you make it?

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1 Introduction

You are invited to play a game. You will be repeatedly extracting values from a uniform distribution $\mathbb{U}(0,1)$. You keep drawing as long as the sequence you are getting is monotonically increasing. You win an amount of money which equals the length of the sequence (excluding the last value which breaks the monotonicity). How much should you pay to play the game?

2 Solution

$\forall i = 1, 2, ...N$ let us define a random variable X_i. $X_i = 1$ if the sequence is still increasing after draw i, otherwise $X_i = 0$.
Consider some arbitrary i. In order to have a monotonically increasing sequence of values, the i values extracted must be following an increasing order. Out of the $i!$ possible permutations of values only 1 is valid. Therefore:

$\mathbb{P}(X_i = 1) = \mathbb{E}[X_i] = \frac{1}{i!}$

Now let Y be the random variable measuring the length of the monotonically increasing sequence. We have that:

$\mathbb{E}[Y] = \sum_{i=1}^{+\infty}(\mathbb{E}[X_i]) = \sum_{i=1}^{+\infty}(\frac{1}{i!}) = e - 1$

This is the answer to the question: paying more than this value to play the game is not convenient.

3 Follow Up

How do you compute the probability of having a sequence of exactly length y?

We can reason as follows. Let F denote the the cumulative distribution function of the discrete random variable Y.

It holds that $\forall y \in \mathbf{N}$

$F(y) = \mathbb{P}(Y <= y) = \mathbb{P}(X_{i+1} = 0) = 1 - \mathbb{P}(X_{i+1} = 1) = 1 - \frac{1}{(y+1)!}$

Then, for discrete random variables:

$\mathbb{P}(Y = y) = F(y) - F(y-1) = 1 - \frac{1}{(y+1)!} - 1 + \frac{1}{y!} = \frac{1}{y!} - \frac{1}{(y+1)!} = \frac{y}{(y+1)!}$

As a last step, let us verify that the computed probabilities define a valid probability distribution. First we have that: $\forall y \in \mathbf{N}$

$\frac{y}{(y+1)!} >= 0$

Then let us check that the probabilities sum up to 1.

$\sum_{i=1}^{+\infty}(\frac{i}{(i+1)!}) = \sum_{i=1}^{+\infty}(\frac{i+1}{(i+1)!}) - \sum_{i=1}^{+\infty}(\frac{1}{(i+1)!}) = \sum_{i=1}^{+\infty}(\frac{1}{(i)!}) - \sum_{j=2}^{+\infty}(\frac{1}{j!}) =$

$(e-1) - (e-1-1) = 1$