The Broken Stick Problem

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June 2, 2023

1 Introduction

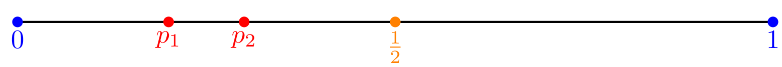
We analyze the problem in two variants.

1.1 First variant

Question 1.1. *Assume you take a stick of length 1 and you break it uniformly at random into three parts. What is the probability that the three pieces can be used to form a triangle?*

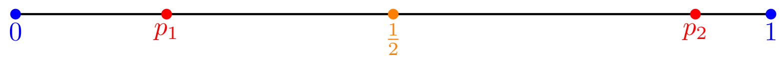
First of all, let us remember from high school knowledge that a triangle can be built if its longest size is not longer than the sum of the other two sides. This condition can be translated to the fact that none of the sticks must overcome the length of $\frac{1}{2}$.

In this formulation, two points are sampled are random along the stick. Let us define these two points as p_1 and p_2. We can observe that if p_1 and p_2 both fall on the same side of the midpoint, either on its left or its right, then no triangle is possible, because, in that case, the length of one of the pieces would be greater than $\frac{1}{2}$. This is displayed in the following figure:



The probabilities that p_1 and p_2 are on the same side is simply $\frac{1}{2}$.

Now, assume that p_1 and p_2 fall on different sides of the midpoint. If p_1 is further to the left in its half than p_2 is in its half, then no triangle is possible in that case, since then the part lying between p_1 and p_2 would have a length strictly greater than 0.5. This is illustrated in the next figure:



This has a $\frac{1}{2}$ chance of occurring by a simple symmetry argument, but it is conditional on p_1 and p_2 being on different sides of the midpoint, an outcome which itself has a $\frac{1}{2}$ chance of occurring. Therefore, this case occurs with probability $\frac{1}{2} * \frac{1}{2} = \frac{1}{4}$.

These two cases represent all cases in which no valid triangle can be formed; thus it follows that probability of a valid triangle being formed equals $1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

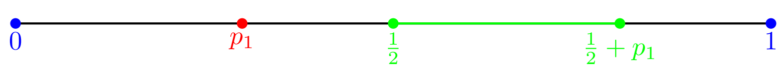
1.2 Second variant

Question 1.2. *Assume you take a stick of length 1 and you break it into three parts, following this procedure:*

1. *First, you break the stick into two pieces uniformly at random. The left part of the stick constitutes the first side of the triangle*
2. *Then you break the remaining part of the stick into two pieces again uniformly at random.*

What is the probability that the three pieces can be used to form a triangle?

To answer this question, as observed before, the length of the first piece can not be greater than $\frac{1}{2}$. If we denote by p_1 the first point drawn uniformly at random from $[0, 1]$ then we have to assume that this will be in the interval $\left[0, \frac{1}{2}\right]$. Then, in the subsequent break of the remaining part of the stick, we must impose that no part is, again, of length greater than $\frac{1}{2}$. This is illustrated in the following picture:



If we consider p_1 as set, the green interval denotes the possible locations for p_2 that allow a triangle to be formed. Notice that the green line has length p_1. Thus, we can compute the probability of forming a triangle, which we denote by T, by conditioning on all possible valid positions of the first point p_1:

$$\begin{aligned}\mathbb{P}(T) &= \int_0^{\frac{1}{2}} \mathrm{p}(T|p_1)\, \mathrm{p}(p_1)\, \mathrm{dp}_1 = \int_0^{\frac{1}{2}} \frac{p_1}{1-p_1}\, \mathrm{dp}_1 = \int_0^{\frac{1}{2}} \left(-1 + \frac{1}{1-p_1}\right) \mathrm{dp}_1 \\ &= \left[-p_1 - \log(1-p_1)\right]_0^{\frac{1}{2}} = -\frac{1}{2} + \log 2 \approx 0.19\end{aligned}$$