The First Ace

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1 Introduction

You shuffle an ordinary deck of 52 playing cards containing 4 aces. From the top, you turn up cards until the first ace appears. We ask you two questions:

1. On average, how many cards are required to see the first ace?
2. If you have to bet on this number once, what would you bet?

2 Solution 1 - Math Based

Let X be the random variable which counts how many cards we need to turn up before seeing an ace. We are asked to compute $\mathbb{E}[X]$. Notice that is a discrete random variable with support [1,2,..., 49]. X = 49 happens in the unlucky case where the 4 aces are all at the end of deck.
Let F_x be the cumulative distribution function of X. We have that:

$\mathbb{P}(X = 1) = \frac{4}{52}$

$F_x(1) = \mathbb{P}(X = 1) = \frac{4}{52}$

Then, $\forall i = 2, ..., 49$ it holds that;

$\mathbb{P}(X = i) = (1 - F_x(i - 1)) \cdot \frac{4}{52-(i-1)}$

$F_x(i) = F_x(i - 1) + \mathbb{P}(X = i)$

The probability of getting the first ace at round i can be computed as the probability of having avoided an ace until round i - 1 times the probability of getting it at exactly round i (where we have already turned up i - 1 cards).

Therefore, we can setup an iterative algorithm as follows:

1. Initialize:
   - $\mathbb{P}(X = 1) = \frac{4}{52}$
   - $F_x(1) = \frac{4}{52}$
   - $\mathbb{E}[X] = \mathbb{P}(X = 1)$
2. $\forall i = 2, ..., 49$
   - $\mathbb{P}(X = i) = (1 - F_x(i-1)) \cdot \frac{4}{52-(i-1)}$
   - $F_x(i) = F_x(i-1) + \mathbb{P}(X = i)$
   - $\mathbb{E}[X] = \mathbb{E}[X] + i \cdot \mathbb{P}(X = i)$
3. Return: $\mathbb{E}[X]$

The algorithm is linear in the number of cards (O(52)) and uses constant space (O(1)), as the locality of the recursive relation allows us to store only the previous values of the pdf and cdf.
By running the algorithm on a computer, we can obtain the exact expected value:

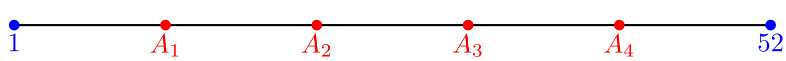
$\mathbb{E}[X] = 10.6$

3 Solution 1 - Intuition Based

Do we really need a computer to solve this problem?

We have 4 aces and 48 other cards in the deck.

In the average case, the cards will be distributed as follows:



A_j for j = 1,2,3,4 denotes the position of the j-th ace in the deck.
We basically need to compute the length of the first segment. The cards before the first ace on average are $\frac{48}{5} = 9.6$, since there are 5 equal intervals of the 48 other non-ace cards.
We also need to lift another card (the first ace itself) before seeing the first ace, so the answer is indeed 9.6 + 1 = 10.6.

4 Solution 2

The second question is asking to compute the mode of the distribution of X. Of course, we can compute all the probabilities with the iterative algorithm of *Section 2* to obtain the distribution of X. Taking the biggest of these probabilities yields the mode of the distribution.

But we can make a smarter consideration:
$\forall i = [2, ..., 49]$ we have:

$$\begin{aligned}
\mathbb{P}(X = i+1) &= (1 - F_x(i)) \cdot \frac{4}{52-i} = (1 - F_x(i-1) - \mathbb{P}(X = i)) \cdot \frac{4}{52-i} \\
&= (1 - F_x(i-1)) \cdot \frac{4}{52-i} - \mathbb{P}(X = i)) \cdot \frac{4}{52-i} \\
&= (1 - F_x(i-1)) \cdot \frac{4}{52-(i-1)} \cdot \frac{52-(i-1)}{52-i} - \mathbb{P}(X = i)) \cdot \frac{4}{52-i} \\
&= \mathbb{P}(X = i)(\frac{52-(i-1)}{52-i} - \frac{4}{52-i}) = \mathbb{P}(X = i)(\frac{49-i}{52-i})
\end{aligned}$$

We have found another recursive formula to compute the $\mathbb{P}(X = i)$. Moreover, it clearly holds that:

$\mathbb{P}(X = i+1) < \mathbb{P}(X = i), \forall i = [2, ..., 49]$

Thus,the pdf of X is strictly decreasing, which means the mode is at X = 1.

So if you had one shot to guess how many cards you have to wait to get an ace, you are better off by saying that the ace will be the first card to be turned.

In the following *Figure 1*, the ditributioon of X is plotted.

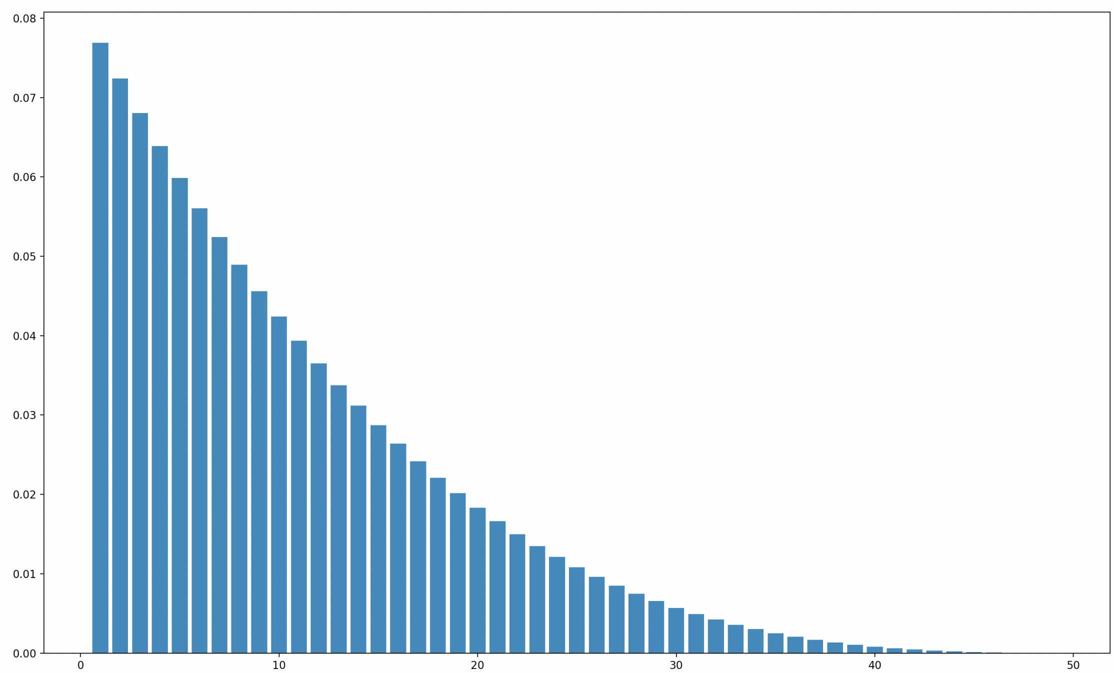


Figure 1: *Distribution of the random variable X*