Filling the Circle

Angelo Gnazzo, Leonardo Pagani

1 Introduction

You have to uniformly fill a circle of radius R with N 2D points. You are only provided with a uniform sampler U(0,1). Devise a strategy to achieve the task.

2 Uniform Distribution of Points

What does it mean that the N points are uniformly distributed in the circle? We can use the following definition:
Let ρ be the random variable denoting the distance of a sample from the center of the circle. To achieve a uniform distribution, we need that $\forall r \in (0, R)$:

$\mathrm{P}(\rho \leq r) \propto \frac{\pi r^2}{\pi R^2} \propto \frac{r^2}{R^2}$

The probability of a point being in an inner circle of radius r must be proportional to the ratio of the areas of the inner circle and the full circle.

3 Strategy 1 - Uniform radius sampling

As a first idea, we switch to a polar coordinate system centered in the center of the circle. We can try a simple strategy:

1. $\forall i = 1, ..., N$
    - Sample $u1$ from U(0,1)
    - Set $r = u1 \cdot R$
    - Sample $u2$ from U(0,1)
    - Set $\theta = u2 \cdot 2\pi$
    - Set $x_i = r \cos\theta$
    - Set $y_i = r \sin\theta$

This allows us to sample all points inside the circle. But are they uniformly distributed?

$\mathbb{P}(\rho \leq r) = \frac{r}{R}$

as the radius is simply picked by scaling the uniform distribution. So, **this is not a successful strategy**.

4 Strategy 2 - Cartesian Uniform Sampling with discard

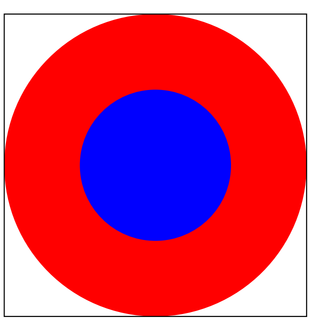
This second strategy is built in a cartesian coordinate system, centered in the center of the circle. It works as follows:

1. Initialize i = 0
2. While i < N:
   - Sample $u1$ from U(0,1)
   - Sample $u2$ from U(0,1)
   - Set $x_i = -R + u1 \cdot 2R$
   - Set $y_i = -R + u2 \cdot 2R$
   - If $x_i^2 + y_i^2 \leq R^2$: i += 1

Basically, we independently sample the x and y coordinates, while discarding all the points outside the circle. Is this a valid sampling strategy? Let X and Y be the random variables denoting the cartesian coordinates of a random point.

$\mathbb{P}(\rho \leq r) = \mathbb{P}(x^2 + y^2 \leq r^2 | x^2 + y^2 \leq R^2) = \frac{r^2}{R^2}$

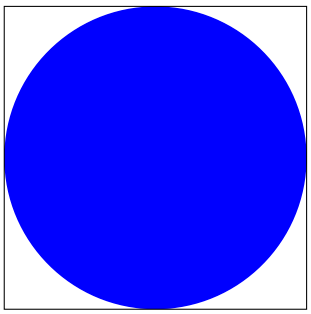
Notice that we can condition on $x^2 + y^2 \leq R^2$ because we discard a point if it lands outside the circle. Thus, we can compute $\mathbb{P}(\rho \leq r)$ by a simple geometrical argument and noticing that it equals the ratio of the areas.



This is a **valid sampling strategy, but no very efficient**. In fact, the discarding procedure leads to a lot of samples being wasted. What is the probability that a point will be discarded.

$\mathbb{P}(discard) = \mathbb{P}(x^2 + y^2 \geq R^2) = 1 - \mathbb{P}(x^2 + y^2 \leq R^2) = 1 - \frac{\pi R^2}{(2R)^2} = \frac{4-\pi}{4}$

Again, we have used the geometrical argument to compute $\mathbb{P}(x^2 + y^2 \leq R^2)$ as the ratio between the circle and the square of length 2R.



This means that on average to get N samples in the circle, we have to sample $\frac{4N}{\pi}$ points.

5 Strategy 3 - Inverse Tranform Method

We can devise an efficient sampling strategy by the following reasoning. From section 2, we want:

$\mathbb{P}(\rho \leq r) \propto \frac{\pi r^2}{\pi R^2} \propto \frac{r^2}{R^2}$

By definition, this last expression equals the cumulative distribution of ρ. Since $\mathbb{P}(\rho \leq R^2) = 1$, we have that:

$\mathbb{P}(\rho \leq r) = \frac{r^2}{R^2}$

Then, if U is our uniform random variable, we can set:

$\mathbb{P}(U \leq u) = u = \frac{r^2}{R^2}$

yielding:

$r = R\sqrt{u}$

Thus an **efficient and valid sampling strategy** is obtained with the following algorithm:

1. $\forall i = 1, ..., N$
   - Sample $u1$ from U(0,1)
   - Set $r = \sqrt{u1} \cdot R$
   - Sample $u2$ from U(0,1)
   - Set $\theta = u2 \cdot 2\pi$
   - Set $x_i = r \cos\theta$
   - Set $y_i = r \sin\theta$