Lost in Manhattan

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1 Introduction

Angelo and Leonardo are out on a trip in New York City. However, after a few hours they are already struggling with Manhattan grid pattern and they get lost. Leonardo is just coming out of a trip from Rockfeller center and is now located at the intersection between 5th Avenue and 47th Street. Angelo instead happens to be a little be more downtown west at the intersection between 9th Avenue and 43th Street. Can you help them find their way?

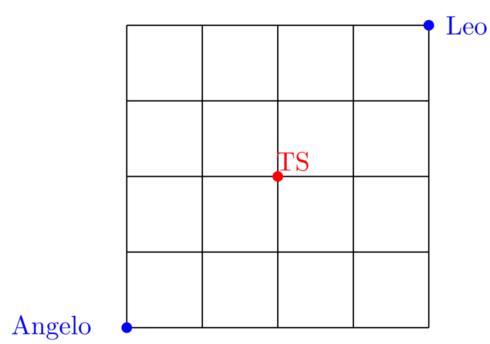
In particular, we ask you 3 questions.

1. How many possible shortest routes does Angelo have to reach Leonardo?

   Let's assume Angelo and Leonardo start moving towards each other at the same time. We can also assume that they walk at the same speed. Of course Angelo is only moving north and east, while Leo only goes south and west.

2. What is the probability that thy bump into each other at Times Square (7th Avenue, 45th Street)?

3. What is the probability that they happen to bump into each other?



2 Solution - Question 1

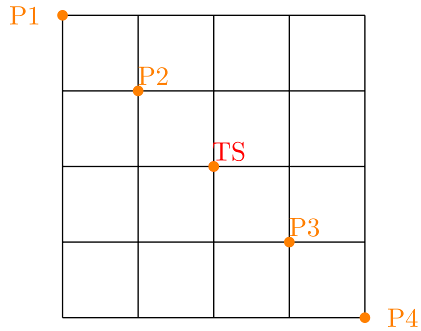
The first question is a simple problem of combinatorics. In order to reach Leo in the shortest possible way, Angelo has to make exactly 4 moves up and 4 moves east. So a possible route by Angelo might be: $\uparrow$, $\rightarrow$, $\rightarrow$, $\uparrow$, $\uparrow$, $\rightarrow$, $\rightarrow$,$\uparrow$. How do we count all the routes?
If we chose the positions of the moves towards north ($\uparrow$), the moves towards eats are set as well. So the choice of the 4 moves north completely determines the full path. The number of ways we can pick the moves north are the number of ways we can pick 4 positions out of a total of 8 positions, which is equal to: $\binom{8}{4} = 70$.

3 Solution - Question 2

So according to the decription, Angelo and Leonardo now both move towards each other. We can observe that their possibilities are symmetrical and thus we just need to study the moves of one of them. We have already established from Question 1 that Angelo has 70 possible paths towards Leonardo. How many of them pass through Times Square?
For a path to go thought Times Square, we need that among the first 4 moves by Angelo, two of them are eastwards($\rightarrow$) and two of them upwards($\uparrow$). So, with a similar argument to Question 1, we can state there are $\binom{4}{2} = 6$ ways for Angelo to reach Times Square in 4 moves. Angelo's path from Time Square towards Leo's position can continue in 6 other possible ways. So 36 of the possible 70 routes at Angelo's disposal traverse Times Square.
In order for Leo and Angelo to meet at Times Square, both of them must pick a route passing through it, so the requested probability is: $(\frac{36}{70})^2 \approx 0.2645$

4 Solution - Question 3

We have already established that Angelo can bump into Leo at Times Square with probability $(\frac{36}{70})^2$. Where else can they meet?
If they move at the same speed, the possible locations where they can bump into each other are marked in green in the next figure:



For each of the possible meeting point, we can compute the probability that the meeting happens with the same reasoning of question 2).

$\mathbb{P}(\text{meeting in P1}) = (\frac{1}{70})^2$

$\mathbb{P}(\text{meeting in P2}) = (\frac{16}{70})^2$

$\mathbb{P}(\text{meeting in TS}) = (\frac{36}{70})^2$

$\mathbb{P}(\text{meeting in P3}) = (\frac{16}{70})^2$

$\mathbb{P}(\text{meeting in P4}) = (\frac{1}{70})^2$

Summing all up, we get:

$\mathbb{P}(meeting) = \mathbb{P}(\text{meeting in P1}) + \mathbb{P}(\text{meeting in P2}) + \mathbb{P}(\text{meeting in TS}) + \mathbb{P}(\text{meeting in P3}) + \mathbb{P}(\text{meeting in P4}) \approx 0.3693$