King's Arthur Round Table

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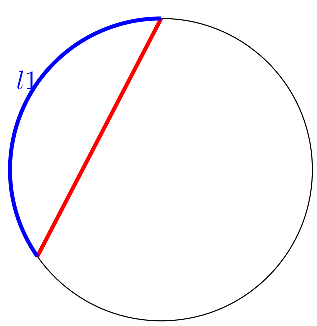
1 Introduction

Angelo and Leonardo have been invited to King's Arthur Round Table in Camelot Castle. Arthur asks them to use his sword (the legendary Sword in the Stone indeed) to draw two chords across the table. What is the probability that these two chords intercept each other?

2 Solution 1 - Calculus

We can forget for a moment the mythological setting and realize that we have been asked to compute the probability that two random chords of a circle intercept.
First, a chord is completely defined by its two end points on the circle.
For now, let us suppose that the first chord has already been drawn. Without loss of generality, we can always mark the first end of the first chord on the top of the circle.



Let us define L1 the random variable which takes into account the the length of the arc corresponding to the first chord. L1 corresponds to the blue arc in the figure. Notice that since we have assumed the first end of the first chord to always be at the top, the length L1 is only determined by the position of the second end.

Furthermore, let I be the event denoting the interception. Finally let C denote the entire length of the circle.

Given that the first chord is already drawn, to have an interception we need that the ends of the second chord land on opposite side of the chord. Since the two ends are independent from each other and their position follow a uniform distribution across the entire length of the circle, the following holds:

$\mathbb{P}(I|L1 = l1) = 2 \cdot \frac{l1}{C} \cdot \frac{C-l1}{C}$

Then, we can compute $\mathbb{P}(I)$ by integrating over all possible length $l1$. Knowing that p($l1$) also follows a uniform distribution:

$$\mathbb{P}(I) = \int_0^C \mathbb{P}(I|L1 = l1)\,\mathrm{p}(l_1)\,\mathrm{dl}_1 = \int_0^C 2 \cdot \frac{l1}{C} \cdot \frac{C - l1}{C} \cdot \frac{1}{C}\,\mathrm{dl}_1 = \frac{2}{C^3} \int_0^C l1 \cdot C - l1^2\,\mathrm{dl}_1$$
$$= \frac{2}{C^3} \cdot \left[C \cdot \frac{l1^2}{2} - \frac{l1^3}{3} \right]_0^C = \frac{1}{3}$$

Notice that, as common sense suggests, this probability does not depend on the radius of the circle.

3 Solution 2 - Combinatorics

We can also find a solution to this problem without using calculus. As mentioned before, a chord is entirely defined by its two end points. Two chords are therefore defined by 4 points. Let s_1, e_1, s_2, e_2 define the start and end points respectively for chord 1 and 2. These 4 points can be ordered in $4! = 24$ possible ways along the circle. How many of these permutations create an interception? Let's count them:

1. s_1, s_2, e_1, e_2
2. s_1, e_2, e_1, s_2
3. e_1, s_2, s_1, e_2
4. e_1, e_2, s_1, s_2
5. s_2, s_1, e_2, e_1
6. s_2, e_1, e_2, s_1
7. e_2, s_1, s_2, e_1
8. e_2, e_1, s_2, s_1

Therefore, the probability of interception is $\mathbb{P}(I) = \frac{8}{24} = \frac{1}{3}$