Penalty Shoot-out

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1 The problem

Consider a classic penalty shootout in football (or soccer if you are from the States ;)).

Question 1.1. *Let us define the probability of scoring a penalty as p. What is the expected number of penalty turns in a shoot-out?*

2 Some clarifications

Penalty shoot-out is used in football to break a tie. Both teams first kick 5 penalties each. If the score is still tied, the shoot-out continues each penalty at a time until one of the teams gets an advantage.

3 The solution

Let us define as X the random variable that counts the number of turns in a penalty shoot-out. Notice that the support domain of X is the set of natural integers greater than 5 ([5,6,7,...]). We are asked to compute $\mathbb{E}[X]$. We can split the computation of $\mathbb{E}[X]$ as follows:

$$\mathbb{E}[X] = (\mathbb{E}[X]|X = 5) * \mathbb{P}(X = 5) + (\mathbb{E}[X]|X > 5) * \mathbb{P}(X > 5)$$

Basically, we can split the problem in two: first consider the case where 5 five penalties are enough to determine a winner, then consider the case where more penalties are needed.

Let us consider the first case. We know that $(\mathbb{E}[X]|X = 5) = 5$, by definition. How do we compute $\mathbb{P}(X = 5)$?
For both teams, the number of penalties scored within the first five follows a Binomial distribution.
We will call this random variable $Y \sim \mathrm{Bi}(5, p)$. The game is still tied after 5 penalties if both teams score the same number of penalties.

We can write:

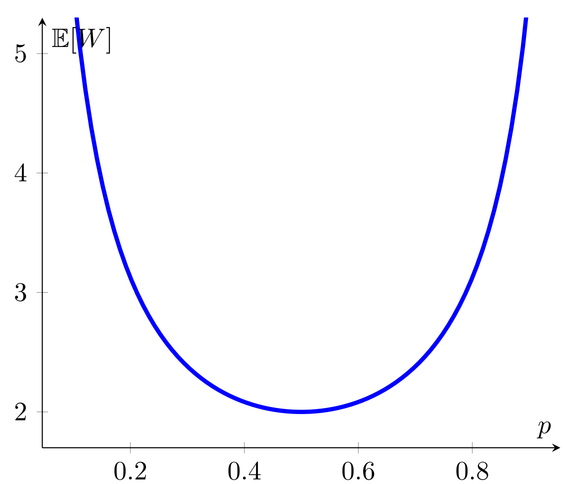
$$\mathbb{P}(X > 5) = \sum_{k=0}^{5} (\mathbb{P}(Y = k))^2 = \sum_{k=0}^{5} \left(\binom{5}{k} p^k (1-p)^{(1-k)} \right)^2$$

This value can be computed analytically for every $p \in [0, 1]$. We will refer to this value as $Q := \mathbb{P}(X > 5)$.

We now need to compute $\mathbb{E}\left[X|X > 5\right]$. We notice that if the game is tied after 5 turns, it will be either tied again at the next turn or it will end. Given that the game is tied, the probability of one team winning it at the next turn is $2 \cdot p \cdot (1 - p)$, corresponding to the case when one team scores and the other misses. Let us call W the number of turns needed after the game is still tied after 5 penalties.

We have that $W \sim \mathcal{G}(2 \cdot p \cdot (1 - p))$. W is a geometric random variable as at each turn the probabilities are independent from the previous turns. Thus, we know that $\mathbb{E}[W] = \frac{1}{2 \cdot p \cdot (1-p)}$.

If we plot $\mathbb{E}[W]$ with p we notice it is symmetrical around $p = \frac{1}{2}$. That is also the minimum of the function. Notice that for $p = 0$ and $p = 1$, te game never ends as every player either always score or always miss.



Then:

$$\mathbb{E}\left[X|X > 5\right] = 5 + \mathbb{E}[W] = 5 + \frac{1}{2 \cdot p \cdot (1 - p)}$$

So overall, we have $\forall\ p \in (0, 1)$:

$$\mathbb{E}[X] = (1 - Q) \cdot 5 + Q \cdot (5 + \frac{1}{2 \cdot p \cdot (1 - p)})$$

In the GitHub repo it is possible to solve the problem and compute $\mathbb{E}[X]$ $\forall p \in (0, 1)$.