Exercice 1

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For
- $a>0$: $F_X(t) = \left(1-e^{-\lambda t^{1/a}}\right) \mathbb{1}\{t \geq 1\}$ $\quad p_X(y) = \frac{\lambda}{a} t^{\frac{1}{a}-1} e^{-\lambda t^{1/a}}$, $t \geq 0$
- $a=0$: $F_X(t) = \mathbb{1}\{t \geq 1\}$ $\quad p_X(y) = \delta(y-1)$
- $a<0$: $F_X(t) = e^{-\lambda t^{1/a}} \mathbb{1}\{t \geq 1\}$ $\quad p_X(y) = -\frac{\lambda}{a} t^{\frac{1}{a}-1} e^{-\lambda t^{1/a}}$ $t \geq 0$

a) $E[X] = \int_0^\infty p_X(y) \cdot y \, \delta y$

For $a>0$, we have $\int_0^\infty \frac{\lambda}{a} t^{\frac{1}{a}-1} e^{-\lambda t^{1/a}} t \, \delta t = \frac{\lambda}{a} \int_0^\infty t^{1/a} e^{-\lambda t^{1/a}} \delta t$

Now, we define $\tau = t^{1/a} \Rightarrow \tau^a = t$ ($\forall \tau, t$ since $t \geq 0$)

$a\tau^{a-1}\delta\tau = \delta t$ $\quad t \to 0, \tau \to 0$; $t \to \infty, \tau \to \infty$

This holds: $\frac{\lambda}{a} \int_0^\infty \tau e^{-\lambda\tau} a\tau^{a-1} \delta\tau$

$$= \lambda \int_0^\infty \tau^a e^{-\lambda\tau} \delta\tau = \lambda \left[\underbrace{\left(-\tau^a \frac{e^{-\lambda\tau}}{\lambda}\right)_0^\infty}_{\to 0} - \int_0^\infty a\tau^{a-1} \frac{e^{-\lambda\tau}}{-\lambda} \delta\tau \right]$$

$$= a \int_0^\infty \tau^{a-1} e^{-\lambda\tau} \delta\tau$$

This means that for $a \in \mathbb{R}$, $a = k+\varepsilon$, where $k \in \mathbb{N}$ & $\varepsilon \in [0,1[$

we have $$\lambda \int_0^\infty \tau^a e^{-\lambda\tau} \delta\tau = \lambda \frac{1}{\lambda^k} \prod_{i=0}^{k-1}(a-i) \int_0^\infty \tau^\varepsilon e^{-\lambda\tau} \delta\tau$$

(if $k=0$, we have $\prod_{i=0}^{k-1}(a-i) = 1$)

$$= \frac{\prod_{i=0}^{k-1}(a-i)}{\lambda^{k-1}} \int_0^\infty \tau^\varepsilon e^{-\lambda\tau} \delta\tau \qquad \text{Let } M_k = \frac{\prod_{i=0}^{k-1}(a-i)}{\lambda^{k-1}}$$

we have $$M_k \int_0^\infty \tau^\varepsilon e^{-\lambda\tau} \delta\tau = M_k \left[\int_0^1 \overbrace{\tau^\varepsilon e^{-\lambda\tau}}^{\leq 1} \delta\tau + \int_1^\infty \overbrace{\tau^\varepsilon}^{\leq \tau} e^{-\lambda\tau} \delta\tau \right]$$

$$\leq M_k \left[\int_0^1 1 \, \delta\tau + \overbrace{\int_1^\infty \tau e^{-\lambda\tau} \delta\tau}^{\leq E[\mathcal{E}(\lambda)]} \right] \leq M_k \left[1 + E[X]\right] = M_k \left[1 + \frac{1}{\lambda}\right]$$

which is finite $\forall a > 0$

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For $\underline{\underline{a=0}}$, we have

$$E[X] = \int_0^\infty y\,\delta(y-1)\,dy = 1 < \infty$$

For $\underline{\underline{a<0}}$, we have

$$E[X] = \int_0^\infty y\left(-\frac{\lambda}{a}\right) y^{\frac{1}{a}-1} e^{-\lambda y^{1/a}}\,dy = -\frac{\lambda}{a}\int_0^\infty y^{1/a} e^{-\lambda y^{1/a}}\,dy$$

Let $x = y^{1/a}$ $\quad x^a = y \quad$ $y \to 0,\ x \to \infty$; $y \to \infty,\ x \to 0$ $\quad dy = a x^{a-1}\,dx$

This holds: $$-\frac{\lambda}{a}\int_\infty^0 x e^{-\lambda x} a x^{a-1}\,dx = \lambda\int_0^\infty x^a e^{-\lambda x}\,dx$$

$$= \lambda\left[\left(\frac{x^{a+1}}{a+1}e^{-\lambda x}\right)_0^\infty - \int_0^\infty \frac{x^{a+1}}{a+1}(-\lambda)e^{-\lambda x}\,dx\right]$$

↳ infinite at 0 so for $a \leq -1$, we don't have a defined mean. (even for $a=-1$, since then, $a+1=0$)

Now, for $a \in]-1,0]$, we have:

$$\lambda\int_0^\infty x^a e^{-\lambda x}\,dx = \lambda\int_0^1 x^a \underbrace{e^{-\lambda x}}_{\leq 1}\,dx + \lambda\int_1^\infty x^a e^{-\lambda x}\,dx$$

$\leq \int_1^\infty e^{-\lambda x}\,dx$ which converges

$$\leq \lambda\int_0^1 x^a\,dx + \lambda\int_1^\infty e^{-\lambda x}\,dx$$

$$= \lambda\left[\int_0^1 x^a\,dx + \frac{e^{-\lambda x}}{-\lambda}\Big|_1^\infty\right] = \lambda\int_0^1 x^a\,dx + e^{-\lambda}$$

and $\int_0^1 \frac{1}{x^b}\,dx$, $b \in]0,1[$ converges. So this converges

$$\int_0^1 \frac{1}{x^b}\,dx = x^{1-b}\cdot\frac{1}{1-b}\Big|_0^1 = \frac{x^{1-b}}{1-b}\Big|_0^1 = \frac{1}{1-b} - 0 = \frac{1}{1-b} = \frac{1}{1+a} < \infty$$

So for $a \in]-1, \infty[$, $E[X]$ exists

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b) $Var(Y) = E[Y^2] - E[Y]^2$

For $\underline{a > 0}$

$$E[Y^2] = \int_0^\infty y^2 \frac{\lambda}{a} y^{\frac{1}{a}-1} e^{-\lambda y^{1/a}} dy = \frac{\lambda}{a}\int_0^\infty y^{\frac{1}{a}+1} e^{-\lambda y^{1/a}} dy$$

$$J = y^{1/a} \quad J^a = y \quad a J^{a-1} dJ = dy$$

$$\frac{\lambda}{a}\int_0^\infty (J^a)^{\frac{1}{a}+1} e^{-\lambda J} a J^{a-1} dJ = \lambda \int_0^\infty J^{2a} e^{-\lambda J} dJ$$ -> we can do like for previous question but $2k$ times instead and still get a finite result. So $Var(Y) < \infty$, $a > 0$

For $\underline{a = 0}$

$$E[Y^2] = \int_0^\infty y^2 \delta(y-1) dy = 1 \quad \rightarrow \quad Var(Y) = 1-1 = 0 \text{ for } a = 0$$

(logic since Y is deterministic

For $\underline{a < 0}$

Since $E[Y] \rightarrow \infty$ for $a \leq -1$, we only have to consider $a \in]-1, 0[$.

We have: $E[Y^2] = \int_0^\infty \left(-\frac{\lambda}{a}\right) y^2 y^{\frac{1}{a}-1} e^{-\lambda y^{\frac{1}{a}}} dy$

$$= -\frac{\lambda}{a}\int_0^\infty y^{\frac{1}{a}+1} e^{-\lambda y^{1/a}} dy \qquad J = y^{1/a} \quad J^a = y \quad a J^{a-1} dJ = dy$$

$$y \rightarrow 0 \quad J \rightarrow \infty \quad y \rightarrow \infty \quad J \rightarrow 0$$

$$= \frac{\lambda}{\alpha}\int_0^\infty (J^a)^{\frac{1}{a}+1} e^{-\lambda J} \alpha J^{a-1} dJ = \lambda \int_0^\infty J^{2a} e^{-\lambda J} dJ$$

we need $2a > -1 \Rightarrow a > -\frac{1}{2}$

So for $a > -\frac{1}{2}$, $Var(Y) < \infty$

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c) We saw that for $a > -\frac{1}{2}$ $Var(Y)$ is well defined and finite

Now, there are 2 cases

1) $a \in]-1, -\frac{1}{2}]$
$E[Y]$ finite
$E[Y^2] \to \infty$ so infinite

2) $a \in]-\infty, -1]$
$\hookrightarrow E[Y] \to \infty$
$E[Y^2] \to \infty$ so undefined

d) We know for $a = 0$ $Var(Y) = 0$ and for $a \in \mathbb{Z} \leq -1$, it is either infinite or undefined.

For $a \in \mathbb{Z} > 0$, we have

$$E[Y] = \lambda \int y^a e^{-\lambda y} \, dy = \frac{\prod_{i=0}^{k-1} (a-i)}{\lambda^{k-1}} \int_0^\infty y^\varepsilon e^{-\lambda y} \, dy \text{ but } \varepsilon = 0 \text{ here}$$

this means $$E[Y] = \frac{a!}{\lambda^{a-1}} \int_0^\infty e^{-\lambda y} \, dy = \frac{a!}{\lambda^{a-1}} \left. \frac{e^{-\lambda y}}{-\lambda} \right|_0^\infty$$

$$= \frac{a!}{\lambda^{a-1}} \cdot \frac{1}{\lambda} = \frac{a!}{\lambda^a}$$

And $$E[Y^2] = \lambda \int_0^\infty y^{2a} e^{-\lambda y} \, dy = \frac{(2a)!}{\lambda^{2a}}$$

So $$Var(Y) = \frac{(2a)!}{\lambda^{2a}} - \left(\frac{a!}{\lambda^a} \right)^2 = \frac{(2a)! - (a!)^2}{\lambda^{2a}}$$

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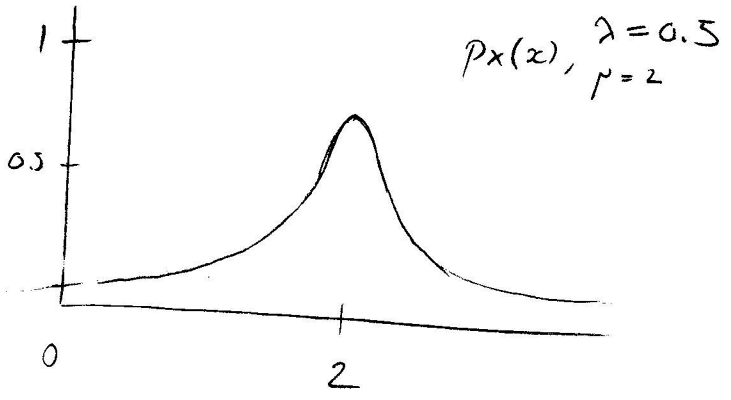
Exercice 2

a) $$\int_{-\infty}^{\infty} \frac{c}{\lambda^2+(x-\mu)^2}\, dx = \frac{c}{\lambda^2}\int_{-\infty}^{\infty} \frac{1}{1+\left(\frac{x-\mu}{\lambda}\right)^2}\, dx$$

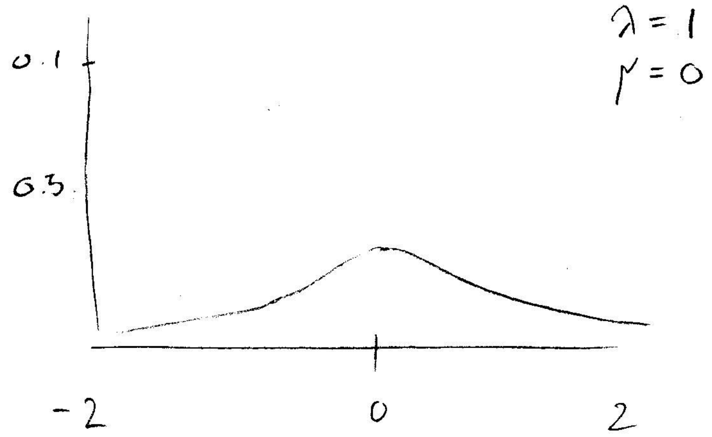
$$= \frac{c}{\lambda^2}\left[\arctan\left(\frac{x-\mu}{\lambda}\right)\lambda\right]_{-\infty}^{\infty} = \frac{c}{\lambda}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right] = \frac{c\pi}{\lambda}$$

$$c\,\frac{\pi}{\lambda} = 1 \Rightarrow c = \frac{\lambda}{\pi}$$

$$\rightarrow p_X(x) = \frac{\lambda}{\pi}\,\frac{1}{\lambda^2+(x-\mu)^2} \quad \forall x \in \mathbb{R}$$



c) μ is the center of the probability as it is symetric in μ and λ is the "spread" of the distribution, λ gets large mean more spread $p_X(x)$ and conversely



b) $$E[X] = \int_{-\infty}^{\infty} \frac{\lambda}{\pi}\,\frac{x}{\lambda^2+(x-\mu)^2}\, dx$$

$$= \frac{1}{\pi\lambda}\int_{-\infty}^{\infty}\frac{x}{1+\left(\frac{x-\mu}{\lambda}\right)^2}\, dx = \frac{1}{\pi\lambda}\ln\left(1+\left(\frac{x-\mu}{\lambda}\right)^2\right)\Bigg|_{-\infty}^{\infty}\cdot\frac{\lambda^2}{2} = \frac{\lambda}{2\pi}\ln\left(1+\left(\frac{x-\mu}{\lambda}\right)^2\right)\Bigg|_{-\infty}^{\infty}$$

$$= \frac{\lambda}{2\pi}\left[\infty - \infty\right]$$ so it is undefined

and same goes for $Var(X)$ in consequence

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d) $P(Y\le t) = P(\frac{1}{X}\le t) = P(X\ge 1/t) = 1-P(X\le 1/t)$

$$= 1-\int_{-\infty}^{1/t} p_X(x)\,\delta x = 1-\int_{-\infty}^{1/t}\frac{\lambda}{\pi}\frac{1}{\lambda^2+x^2}\,\delta x = 1-\int_{-\infty}^{1/t}\frac{1}{\pi\lambda}\frac{1}{1+(\frac{x}{\lambda})^2}\,\delta x$$

$$= 1-\frac{1}{\pi\lambda}\left[\arctan\left(\frac{x}{\lambda}\right)\lambda\right]_{-\infty}^{1/t} = 1-\frac{1}{\pi}\left[\arctan\left(\frac{1}{t\lambda}\right)+\frac{\pi}{2}\right]$$

$$P(Y=t) = \frac{\delta(1-P(X\le 1/t))}{\delta t} = -\frac{\delta(\frac{1}{\pi}\arctan(\frac{1}{t\lambda}))}{\delta t} = \frac{-1}{\pi}\frac{1}{1+\left(\frac{1}{t\lambda}\right)^2}\left(-\frac{1}{\lambda}\cdot\frac{1}{t^2}\right)$$

$$= \frac{1}{\pi\lambda}\frac{(\lambda t)^2}{1+(\lambda t)^2}\cdot\frac{1}{t^2} = \frac{\lambda}{\pi}\frac{1}{1+(\lambda t)^2} = \frac{\lambda}{\pi}\frac{1}{\lambda^2(\frac{1}{\lambda^2}+t^2)}$$

$$= \frac{1}{\pi\lambda}\frac{1}{\frac{1}{\lambda^2}+t^2}$$ So by setting $r=\frac{1}{\lambda}$, we have

$$\frac{r}{\pi}\frac{1}{r^2+t^2}$$ -> so we inverted the spread

Exercice 3

a) We have $P(X\le t) = (1-e^{-\lambda t})\mathbb{1}\{t\ge 0\}$

So $E[X] = \int_0^\infty 1-(1-e^{-\lambda t})\,\delta t = \int_0^\infty e^{-\lambda t}\,\delta t = \left.\frac{e^{-\lambda t}}{-\lambda}\right|_0^\infty = \frac{1}{\lambda}$

b) $X(\omega) = \sum_{i=0}^{X(\omega)-1} 1 = \sum_{i=0}^{\infty}\mathbb{1}\{i\le X(\omega)-1\}$

So $E[X] = E\left[\sum_{i=0}^{\infty}\mathbb{1}\{i\le X(\omega)-1\}\right] = \sum_{i=0}^{\infty} E[\mathbb{1}\{i\le X(\omega)-1\}]$

$$= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} p_X(k)\mathbb{1}\{i\le k-1\} = \sum_{i=0}^{\infty}\sum_{k=i+1}^{\infty} p_X(k) = \sum_{i=0}^{\infty} P(X\ge i+1)$$

c) Bernoulli -> $E[X] = \sum_{i=0}^{\infty} P(X\ge i+1) = P(X\ge 1) = p$

Geometric -> $E[X] = \sum_{i=0}^{\infty} P(X\ge i+1) = \sum_{i=0}^{\infty}\sum_{k=i+1}^{\infty}(1-p)p^k = (1-p)\sum_{i=0}^{\infty}\left[\sum_{k=0}^{\infty}p^k - \sum_{k=0}^{i}p^k\right]$

$$= (1-p)\sum_{i=0}^{\infty}\frac{1}{1-p} - \frac{1-p^{i+1}}{1-p} = \sum_{i=0}^{\infty}p^{i+1} = p\sum_{i=0}^{\infty}p^i = p\cdot\frac{1}{1-p} = \frac{p}{1-p}$$

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Exercice 4

a) First note $P(Z \notin [0,1]) = 0$ since $X, Y \geq 0$

$$P(Z \leq z) = P\left(\frac{X}{X+Y} \leq z\right) = P(X \leq z(X+Y))$$

$$= \int_0^\infty P(X \leq z(X+Y) \mid Y=y) \cdot P(Y=y)\, dy$$

$$= \int_0^\infty P(X \leq zX + zy) \cdot P(Y=y)\, dy = \int_0^\infty P(X(1-z) \leq zy)\, P(Y=y)\, dy$$

$$= \int_0^\infty P\left(X \leq \frac{zy}{1-z}\right) P(Y=y)\, dy \qquad \text{Let } \frac{z}{1-z} = r.$$

We have $\displaystyle\int_0^\infty (1 - e^{-\lambda r y})\, \mu e^{-\mu y}\, dy = \mu \int_0^\infty e^{-\mu y} - e^{-y(\lambda r + \mu)}\, dy$

$$= \mu \left[\frac{e^{-\mu y}}{-\mu} + \frac{e^{-y(\lambda r+\mu)}}{(\lambda r + \mu)} \right]_0^\infty = \mu\left[(0+0) - \left(-\frac{1}{\mu} + \frac{1}{\mu + \lambda r}\right)\right]$$

$$= 1 - \frac{\mu}{\mu + \lambda r} = \frac{\lambda r}{\mu + \lambda r} = \frac{\lambda \frac{z}{1-z}}{\mu + \lambda \frac{z}{1-z}} = \frac{z\lambda}{\mu(1-z) + z\lambda}$$

$$= \frac{z}{\frac{\mu}{\lambda} + z\left(1 - \frac{\mu}{\lambda}\right)}$$

b) $\lambda = \mu \Rightarrow \frac{\mu}{\lambda} = 1 \Rightarrow P(Z \leq z) = z$ so it is a uniform distribution on $[0,1]$

and it doesn't depend on λ

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c) Let T_i = time of waiting for bus i

We want $P(T_1 \leq s \mid T_1 + T_2 = \omega)$ (where $\omega = 3$)

This holds:

$$\int_0^\infty P(T_1 \leq s \mid T_1 + T_2 = \omega, T_2 = R) \cdot P(T_2 = R)\, \delta R$$

$$= \int_0^\infty P(T_1 \leq s \mid T_1 = \omega - R) \cdot P(T_2 = R)\, \delta R$$

$$= \int_0^\infty \mathbb{1}\{\omega - R \leq s\} \cdot P(T_2 = R)\, \delta R = \int_0^\infty \mathbb{1}\{R \geq \omega - s\}\, \lambda e^{-\lambda R}\, \delta R$$

$$= \int_{\omega - s}^\infty \lambda e^{-\lambda R}\, \delta R = \lambda \left[\frac{e^{-\lambda R}}{-\lambda} \right]_{\omega - s}^\infty = \frac{\lambda}{\lambda}\left[0 + e^{-\lambda(\omega - s)} \right]$$

$$= e^{-\lambda(\omega - s)} \Big|_{\substack{\lambda = 1 \\ \omega = 3}} = e^{-1(3-s)} = e^{s-3} \quad \text{for } s \in [0, 3]$$

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c) $p_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} e^{-|t|} \delta t$

$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-itx} e^{t} \delta t + \int_{0}^{\infty} e^{-itx} e^{-t}\right]$

$= \frac{1}{2\pi}\left[\int_{0}^{\infty} e^{-itx} e^{-t} \delta t - \int_{0}^{-\infty} e^{-itx} e^{t} \delta t\right] = \frac{1}{2\pi}\left[\int_{0}^{\infty} e^{-itx} e^{-t} \delta t + \int_{0}^{\infty} e^{itx} e^{-t} \delta t\right]$

$= \frac{1}{2\pi}\int_{0}^{\infty} e^{-t}\left(e^{itx} + e^{-itx}\right) \delta t = \frac{2}{2\pi}\int_{0}^{\infty} e^{-t} \frac{e^{itx} + e^{-itx}}{2} \delta t$

$= \frac{1}{\pi}\int_{0}^{\infty} e^{-t} \cos(tx) \delta t$

$I =$

$\int_{0}^{\infty} e^{-t}\cos(tx)\delta t = \underbrace{\left[-e^{-t}\cos(tx)\right]_{0}^{\infty}}_{=1} - \int_{0}^{\infty} -e^{-t}(-\sin(tx))x\,\delta t = 1 - x\int_{0}^{\infty} e^{-t}\sin(tx)\delta t$

$= 1 - x\left(\underbrace{\left[-e^{-t}\sin(tx)\right]_{0}^{\infty}}_{=0} - \int_{0}^{\infty}(-e^{-t})\cos(tx)x\,\delta t\right)$

$= 1 - x^2\int_{0}^{\infty} e^{-t}\cos(tx)\delta t \implies I = 1 - x^2 I$

$\implies I(1+x^2) = 1 \quad I = \frac{1}{1+x^2}$

So $p_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ so X is a Cauchy r.v. with 0 mean and 1 variance