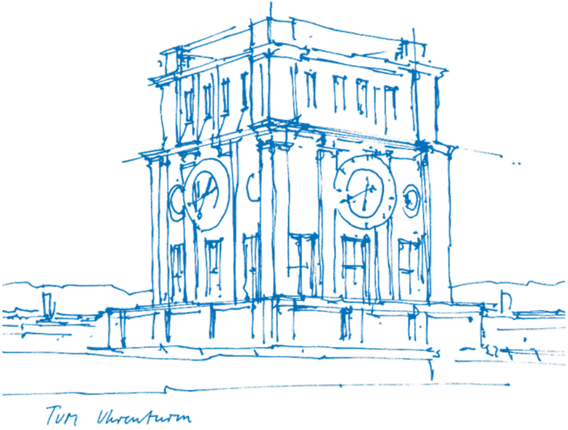
Clifford Tableaus and the Stabilizer Algorithm



Leonard Uscinowicz

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December 20th, 2024

Outline

Pauli Matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Products of Pauli matrices:

$$I^2 = X^2 = Y^2 = Z^2 = I$$

$$IX = XI = X \qquad IY = YI = Y \qquad IZ = ZI = Z$$

$$XY = iZ \qquad YX = -iZ$$

$$YZ = iX \qquad ZY = -iX$$

$$ZX = iY \qquad XZ = -iY$$

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Group Theory

Group $(G, \cdot)$ is a non-empty set G with a binary group multiplication operation $" \cdot "$.

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- **Identity:** $\exists e \in G$ such that $\forall g \in G \implies e \cdot g = g \cdot e = g$
- **Inverse:** $\forall g \in G \implies \exists g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$

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Pauli Group
Definitions

$\mathcal{P}_n$ is defined as the group of n-qubit Pauli operators.
It consists of all tensor products of n Pauli matrices, with a phase factor ± 1 or $\pm i$.

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$$\mathcal{P}_n = \left\{ i^m \bigotimes_{j=1}^{n} \sigma_{k_j} \middle| m, k_j \in \{0,1,2,3\}, \sigma_0 = I, \sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z \right\}$$

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Size of a Pauli Group: $|\mathcal{P}_n| = 4^{n+1}$

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Pauli Group
Operations

Given two Pauli operators $P = i^{m_P} \bigotimes_{j=1}^{n} P_j$ and $Q = i^{m_Q} \bigotimes_{j=1}^{n} Q_j$, their product, as necessitated by Group Definition, is:

$$P \cdot Q = i^{m_P + m_Q} \bigotimes_{j=1}^{n} P_j Q_j$$

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P commutes with Q if the number of indices j such that P_j anti-commutes with Q_j is even.

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Group Generators

A set of l elements $\{g_i\}_{1\leq i\leq l}$ generates a group G if every element $g \in G$ can be written as a product of the generators.
In this case, the group G can be written in terms of its generators:

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Examples:

$$\mathcal{P}_1 = \langle X, Z, iI \rangle$$
$$\langle X \rangle = \{I, X\}$$

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Stabilizer Groups
Definitions

- Element $g \in \mathcal{P}_n$ **stabilizes** $|\psi\rangle$ iff $g\,|\psi\rangle = |\psi\rangle$.
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- $V_S \widehat{=}$Set of n-qubit states stabilized by S:

$$V_S = \{|\psi\rangle \mid S \subseteq \mathcal{P}_n, \forall g \in S \text{ holds: } g|\psi\rangle = |\psi\rangle\}$$

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Stabilizer Groups
Properties

Not just any subgroup S of the Pauli group can be used as the stabilizer for a non-trivial vector space V_S.

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$$(-I) \in S \text{ and } (-I)\left|\psi\right\rangle = -\left|\psi\right\rangle \implies \left|\psi\right\rangle = \vec{0} \implies V_S = \left\{\vec{0}\right\} \text{ (trivial)}$$

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Conditions for S such that V_S not trivial:

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- **Strict Identity:** $-I \notin S$, $iI \notin S$, $-iI \notin S$

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Stabilizer Conditions

Commutativity Proof

Let V_S be non-trivial and let $g_1, g_2 \in S$.

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$-I \in S$, $iI \in S$, $-iI \in S$ lead to contradictions.

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Check Matrix
Structure

Suppose $S = \langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$.

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Example:
$$
l \left\{ \vphantom{\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}} \right.
\begin{bmatrix}
\underbrace{\begin{matrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}}_{n}
&\Bigg|&
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0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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More explicitly, with $h_{i,j}$ denoting the element of H_S at row i and column j:

- If g_i contains I on the j^{th} qubit $\Longrightarrow h_{i,j} = 0$ and $h_{i,n+j} = 0$.

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Check Matrix
Example Steane Code

For Readability tensor product operator signs are left out. $\sigma_i\sigma_j$ corresponds to $\sigma_i \otimes \sigma_j$.

$$\left[\begin{array}{ccccccc|ccccccc} 0&0&0&1&1&1&1&0&0&0&0&0&0&0\\ 0&1&1&0&0&1&1&0&0&0&0&0&0&0\\ 1&0&1&0&1&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&0&0&1&1\\ 0&0&0&0&0&0&0&1&0&1&0&1&0&1 \end{array}\right] \widehat{=}$$

Generator	Operator
g_1	$I\,I\,I\,X\,X\,X\,X$
g_2	$I\,X\,X\,I\,I\,X\,X$
g_3	$X\,I\,X\,I\,X\,I\,X$
g_4	$I\,I\,I\,Z\,Z\,Z\,Z$
g_5	$I\,Z\,Z\,I\,I\,Z\,Z$
g_6	$Z\,I\,Z\,I\,Z\,I\,Z$

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Unitary Operations
Main Revelation

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$\implies$ If we can describe a state by its stabilizers, we can easily compute the stabilizers of the state that emerges from the previous state under a unitary operation.

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Unitary Operations

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$$HXH^\dagger = Z \qquad HYH^\dagger = -Y \qquad HZH^\dagger = X$$

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Example:

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Example:

(Unkown) State $|\psi\rangle$ stabilized by X.

$\longrightarrow$ Apply Hadamard gate H to $|\psi\rangle$.

$\Longrightarrow$ Resulting (Unkown) state $|\psi'\rangle$ stabilized by Z.

Unitary Operations
Transformation under Conjugation

Operation	Input	Output
CX	X_1	X_1X_2
	X_2	X_2
	Z_1	Z_1
	Z_2	Z_1Z_2
H	X	Z
	Z	X
S	X	Y
	Z	Z

Operation	Input	Output
X	X	X
	Z	$-Z$
Y	X	$-X$
	Z	$-Z$
Z	X	$-X$
	Z	Z

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement
Main Principles

We want to measure observable $g \in \mathcal{P}_n$ of state $|\psi\rangle$, stabilized by $\langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$.

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Measurement

Deterministic case

g commutes with all g_i and assume g does not have a global phase.

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$$\forall i \text{ holds: } g_i g |\psi\rangle = g g_i |\psi\rangle = g |\psi\rangle \quad \Longrightarrow \quad g |\psi\rangle \in V_S$$

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In both cases the measurement does not disturb the state of the system, and leaves the stabilizer invariant.

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$\implies g$ only commutes with g_1.

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Because g has eigenvalues ± 1, the measurement operators are: $M_{\pm g} = \frac{I \pm g}{2}$

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Measurement

Non-deterministic case continuation

Measurement probabilities:

$$p(+1) = \text{tr}\left(\frac{I+g}{2}\left|\psi\right\rangle\left\langle\psi\right|\right) \qquad \wedge \qquad p(-1) = \text{tr}\left(\frac{I-g}{2}\left|\psi\right\rangle\left\langle\psi\right|\right)$$

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$$p(+1) = p(-1) \text{ and } p(+1) + p(-1) = 1 \implies p(+1) = p(-1) = \frac{1}{2}$$

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Outline

Gottesman–Knill Theorem

Suppose a quantum computation is performed which involves only the following elements:

- State preparations in the computational basis
- Hadamard gates
- Phase gates
- Controlled-NOT gates
- Pauli gates
- Measurements of observables in the Pauli group

Together with the possibility of classical control conditioned on the outcome of such measurements. Such a computation may be efficiently simulated on a classical computer.

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Stabilizer Circuit

Mainly:

Circuit consisting solely of CX, H, S and M gates.

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$$Z = SS$$
$$X = HZH$$
$$Y = SXSSS$$
$$SWAP(a, b) = CX(a, b)CX(b, a)CX(a, b)$$

Clifford Tableau

Structure

Basically an expanded Check Matrix.

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$$
\left(\begin{array}{ccc|ccc|c}
x_{1,1} & \cdots & x_{1,n} & z_{1,1} & \cdots & z_{1,n} & r_1 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x_{n,1} & \cdots & x_{n,n} & z_{n,1} & \cdots & z_{n,n} & r_n \\
\hline
x_{(n+1),1} & \cdots & x_{(n+1),n} & z_{(n+1),1} & \cdots & z_{(n+1),n} & r_{n+1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x_{(2n),1} & \cdots & x_{(2n),n} & z_{(2n),1} & \cdots & z_{(2n),n} & r_{2n} \\
\hline
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Developers of the following algorithm introduce additional n "Destabilizer" generators, which are Pauli operators that together with the stabilizer generators generate the full Pauli group.

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- Rows 1 to n represent Destabilizers.

$$\left(\begin{array}{ccc|ccc|c}
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\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
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- Row $2n+1$ is scratch-space.
- r_i of row i represents the global phase, $r_i = 0$ for $+1$ and $r_i = 1$ for -1.

$$\left(\begin{array}{ccc|ccc|c}
x_{1,1} & \cdots & x_{1,n} & z_{1,1} & \cdots & z_{1,n} & r_1 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x_{n,1} & \cdots & x_{n,n} & z_{n,1} & \cdots & z_{n,n} & r_n \\
\hline
x_{(n+1),1} & \cdots & x_{(n+1),n} & z_{(n+1),1} & \cdots & z_{(n+1),n} & r_{n+1} \\
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Clifford Tableau
Initial Tableau

Pauli-Z gates stabilize $|0\rangle$ states and Pauli-X gates destabilize $|0\rangle$ states.

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$$\text{Tableau for } |00\rangle \text{ is } \left(\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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In the following slides, let R_i denote the i-th row of the stabilizer tableau.

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Hadamard Gate Algorithm

Suppose we apply a Hadamard gate to qubit a of current state $|\psi\rangle$.

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Operation summary: $x'_{i,a} = z_{i,a} \quad \land \quad z'_{i,a} = x_{i,a} \quad \land \quad r'_i = r_i \oplus (x_{i,a} \cdot z_{i,a})$

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Measurement Algorithm

Deterministic Case

Measuring qubit a

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Measuring qubit $a \implies$ Stabilizers have I or Z on a and g is either $-Z_a$ or Z_a.

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Invariants: $\forall\, i \in [1, n]$ holds: R_i anti-commutes with R_{n+i} and commutes with all $R_{j \neq n+i}$

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g the product of R_{n+i} for such $i \longleftrightarrow$ it anti-commutes exactly with these R_i.

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Compute g in scratch-space row $2n+1$ and extract r_{2n+1}:

$$\begin{aligned} r_{2n+1} = 0 &\implies g = Z_a &&\implies \text{Measurement: } 0 \\ r_{2n+1} = 1 &\implies g = -Z_a &&\implies \text{Measurement: } 1 \end{aligned}$$

Measurement result is just r_{2n+1}.

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Measurement Algorithm
Non-deterministic Case

Measuring qubit a: $\implies g$ is either $-Z_a$ or Z_a.

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Measurement Algorithm

Non-deterministic Case

Measuring qubit a: $\implies g$ is either $-Z_a$ or Z_a.

1. Find $p \in [n+1, 2n]$ such that $x_{p,a} = 1$.
   This R_p corresponds to our previous g_1, which anti-commutes with g.

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3. Push R_p onto R_{p-n}.
   After measurement, the stabilizer anti-commuting with g becomes a destabilizer.

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3. Push R_p onto R_{p-n}.
   After measurement, the stabilizer anti-commuting with g becomes a destabilizer.
4. Assign $R_p = g$, either Z_a or $-Z_a$ with equal probability.
   Retrieve the measurement result from r_p like before.

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Advantages

Simulation of stabilizer circuit runs

in polynomial $O\left(n^2\right)$ time complexity.

Measurement and state update during simulation run

in polynomial $O\left(n^2\right)$ time complexity.

References

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* (2004).

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010.