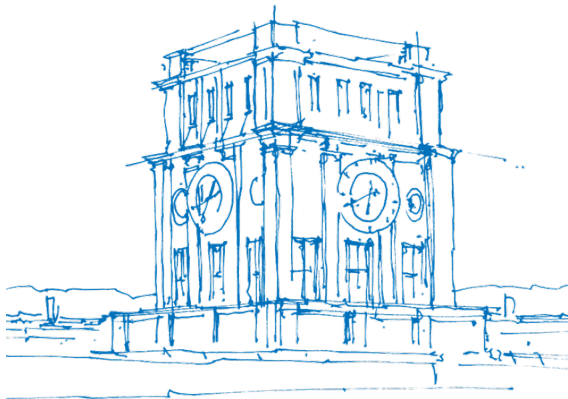


Clifford Tableaus and the Stabilizer Algorithm

Leonard Uscinowicz

Technical University of Munich

December 20th, 2024



1 Preliminary Definitions

2 Stabilizer Formalism

3 Stabilizer Algorithm

Pauli Matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Pauli Matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Products of Pauli matrices:

$$\begin{aligned} I^2 &= X^2 = Y^2 = Z^2 = I \\ IX &= XI = X & IY &= YI = Y & IZ &= ZI = Z \\ XY &= iZ & YX &= -iZ \\ YZ &= iX & ZY &= -iX \\ ZX &= iY & XZ &= -iY \end{aligned}$$

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Group (G, \cdot) is a non-empty set G with a binary group multiplication operation " \cdot ".

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Group Theory

Group (G, \cdot) is a non-empty set G with a binary group multiplication operation " \cdot ".

Properties:

■ **Closure:** $\forall g_1, g_2 \in G \implies g_1 \cdot g_2 \in G$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Group Theory

Group (G, \cdot) is a non-empty set G with a binary group multiplication operation " \cdot ".

Properties:

■ **Closure:** $\forall g_1, g_2 \in G \implies g_1 \cdot g_2 \in G$

■ **Associativity:** $\forall g_1, g_2, g_3 \in G \implies g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Group Theory

Group (G, \cdot) is a non-empty set G with a binary group multiplication operation " \cdot ".

Properties:

■ **Closure:** $\forall g_1, g_2 \in G \implies g_1 \cdot g_2 \in G$

■ **Associativity:** $\forall g_1, g_2, g_3 \in G \implies g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

■ **Identity:** $\exists e \in G$ such that $\forall g \in G \implies e \cdot g = g \cdot e = g$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Group Theory

Group (G, \cdot) is a non-empty set G with a binary group multiplication operation " \cdot ".

Properties:

■ **Closure:** $\forall g_1, g_2 \in G \implies g_1 \cdot g_2 \in G$

■ **Associativity:** $\forall g_1, g_2, g_3 \in G \implies g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

■ **Identity:** $\exists e \in G$ such that $\forall g \in G \implies e \cdot g = g \cdot e = g$

■ **Inverse:** $\forall g \in G \implies \exists g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Pauli Group

Definitions

\mathcal{P}_n is defined as the group of n -qubit Pauli operators.

It consists of all tensor products of n Pauli matrices, with a phase factor ± 1 or $\pm i$.

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Pauli Group

Definitions

\mathcal{P}_n is defined as the group of n -qubit Pauli operators.

It consists of all tensor products of n Pauli matrices, with a phase factor ± 1 or $\pm i$.

$$\mathcal{P}_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$$

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Pauli Group

Definitions

\mathcal{P}_n is defined as the group of n -qubit Pauli operators.

It consists of all tensor products of n Pauli matrices, with a phase factor ± 1 or $\pm i$.

$$\mathcal{P}_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$$

$$\mathcal{P}_n = \left\{ i^m \bigotimes_{j=1}^n \sigma_{k_j} \mid m, k_j \in \{0, 1, 2, 3\}, \sigma_0 = I, \sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z \right\}$$

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Pauli Group

Definitions

\mathcal{P}_n is defined as the group of n -qubit Pauli operators.

It consists of all tensor products of n Pauli matrices, with a phase factor ± 1 or $\pm i$.

$$\mathcal{P}_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$$

$$\mathcal{P}_n = \left\{ i^m \bigotimes_{j=1}^n \sigma_{k_j} \mid m, k_j \in \{0, 1, 2, 3\}, \sigma_0 = I, \sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z \right\}$$

Size of a Pauli Group: $|\mathcal{P}_n| = 4^{n+1}$

[1] [Scott Aaronson and Daniel Gottesman](#). “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

[2] [Michael A Nielsen and Isaac L Chuang](#). *Quantum computation and quantum information*. Cambridge university press, 2010

Pauli Group

Operations

Given two Pauli operators $P = i^{m_P} \bigotimes_{j=1}^n P_j$ and $Q = i^{m_Q} \bigotimes_{j=1}^n Q_j$, their product, as necessitated by Group Definition, is:

$$P \cdot Q = i^{m_P+m_Q} \bigotimes_{j=1}^n P_j Q_j$$

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Pauli Group Operations

Given two Pauli operators $P = i^{m_P} \bigotimes_{j=1}^n P_j$ and $Q = i^{m_Q} \bigotimes_{j=1}^n Q_j$, their product, as necessitated by Group Definition, is:

$$P \cdot Q = i^{m_P+m_Q} \bigotimes_{j=1}^n P_j Q_j$$

P commutes with Q if the number of indices j such that P_j anti-commutes with Q_j is even.

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Group Generators

A set of l elements $\{g_i\}_{1 \leq i \leq l}$ generates a group G if every element $g \in G$ can be written as a product of the generators.

In this case, the group G can be written in terms of its generators:

$$G = \langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Group Generators

A set of l elements $\{g_i\}_{1 \leq i \leq l}$ generates a group G if every element $g \in G$ can be written as a product of the generators.

In this case, the group G can be written in terms of its generators:

$$G = \langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$$

Examples:

$$\begin{aligned} \mathcal{P}_1 &= \langle X, Z, iI \rangle \\ \langle X \rangle &= \{I, X\} \end{aligned}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

- 1 Preliminary Definitions
- 2 Stabilizer Formalism**
- 3 Stabilizer Algorithm

Stabilizer Groups

Definitions

- Element $g \in \mathcal{P}_n$ **stabilizes** $|\psi\rangle$ iff $g|\psi\rangle = |\psi\rangle$.
 $|\psi\rangle$ is eigenstate of g with eigenvalue $+1$.

Stabilizer Groups

Definitions

- Element $g \in \mathcal{P}_n$ **stabilizes** $|\psi\rangle$ iff $g|\psi\rangle = |\psi\rangle$.
 $|\psi\rangle$ is eigenstate of g with eigenvalue $+1$.
- $S \triangleq$ Subgroup of the Pauli Group \mathcal{P}_n : $S \subseteq \mathcal{P}_n$.

Stabilizer Groups

Definitions

- Element $g \in \mathcal{P}_n$ **stabilizes** $|\psi\rangle$ iff $g|\psi\rangle = |\psi\rangle$.
 $|\psi\rangle$ is eigenstate of g with eigenvalue $+1$.

- $S \triangleq$ Subgroup of the Pauli Group \mathcal{P}_n : $S \subseteq \mathcal{P}_n$.

- $V_S \triangleq$ Set of n -qubit states stabilized by S :

$$V_S = \{|\psi\rangle \mid S \subseteq \mathcal{P}_n, \forall g \in S \text{ holds: } g|\psi\rangle = |\psi\rangle\}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Groups

Properties

Not just any subgroup S of the Pauli group can be used as the stabilizer for a non-trivial vector space V_S .

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Groups

Properties

Not just any subgroup S of the Pauli group can be used as the stabilizer for a non-trivial vector space V_S .

Example: $S = \{\pm I, \pm X\}$

Stabilizer Groups

Properties

Not just any subgroup S of the Pauli group can be used as the stabilizer for a non-trivial vector space V_S .

Example: $S = \{\pm I, \pm X\}$

$$(-I) \in S \text{ and } (-I) |\psi\rangle = -|\psi\rangle \implies |\psi\rangle = \vec{0} \implies V_S = \{\vec{0}\} \text{ (trivial)}$$

Stabilizer Groups

Properties

Not just any subgroup S of the Pauli group can be used as the stabilizer for a non-trivial vector space V_S .

Example: $S = \{\pm I, \pm X\}$

$$(-I) \in S \text{ and } (-I) |\psi\rangle = -|\psi\rangle \implies |\psi\rangle = \vec{0} \implies V_S = \{\vec{0}\} \text{ (trivial)}$$

Conditions for S such that V_S not trivial:

■ **Commutativity:** $\forall g_1, g_2 \in S$ holds: $g_1 g_2 = g_2 g_1$

Stabilizer Groups

Properties

Not just any subgroup S of the Pauli group can be used as the stabilizer for a non-trivial vector space V_S .

Example: $S = \{\pm I, \pm X\}$

$$(-I) \in S \text{ and } (-I) |\psi\rangle = -|\psi\rangle \implies |\psi\rangle = \vec{0} \implies V_S = \{\vec{0}\} \text{ (trivial)}$$

Conditions for S such that V_S not trivial:

- **Commutativity:** $\forall g_1, g_2 \in S$ holds: $g_1 g_2 = g_2 g_1$
- **Strict Identity:** $-I \notin S, iI \notin S, -iI \notin S$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Conditions

Commutativity Proof

Let V_S be non-trivial and let $g_1, g_2 \in S$.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Conditions

Commutativity Proof

Let V_S be non-trivial and let $g_1, g_2 \in S$.

$\implies g_1$ and g_2 are tensor products of Pauli matrices.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Conditions

Commutativity Proof

Let V_S be non-trivial and let $g_1, g_2 \in S$.

$\implies g_1$ and g_2 are tensor products of Pauli matrices.

$\implies g_1$ and g_2 must either commute or anti-commute.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Conditions

Commutativity Proof

Let V_S be non-trivial and let $g_1, g_2 \in S$.

$\implies g_1$ and g_2 are tensor products of Pauli matrices.

$\implies g_1$ and g_2 must either commute or anti-commute.

Suppose g_1 and g_2 anti-commute:

$$|\psi\rangle = g_1 g_2 |\psi\rangle = -g_2 g_1 |\psi\rangle = -|\psi\rangle \iff |\psi\rangle = \vec{0} \implies V_S \text{ is trivial.}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Conditions

Commutativity Proof

Let V_S be non-trivial and let $g_1, g_2 \in S$.

$\implies g_1$ and g_2 are tensor products of Pauli matrices.

$\implies g_1$ and g_2 must either commute or anti-commute.

Suppose g_1 and g_2 anti-commute:

$$|\psi\rangle = g_1 g_2 |\psi\rangle = -g_2 g_1 |\psi\rangle = -|\psi\rangle \iff |\psi\rangle = \vec{0} \implies V_S \text{ is trivial.}$$

$\implies g_1$ and g_2 anti-commuting leads to a contradiction.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Conditions

Commutativity Proof

Let V_S be non-trivial and let $g_1, g_2 \in S$.

$\implies g_1$ and g_2 are tensor products of Pauli matrices.

$\implies g_1$ and g_2 must either commute or anti-commute.

Suppose g_1 and g_2 anti-commute:

$$|\psi\rangle = g_1 g_2 |\psi\rangle = -g_2 g_1 |\psi\rangle = -|\psi\rangle \iff |\psi\rangle = \vec{0} \implies V_S \text{ is trivial.}$$

$\implies g_1$ and g_2 anti-commuting leads to a contradiction.

$\implies g_1$ and g_2 commute.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Conditions

Strict Identity Proof

Let V_S be non-trivial.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Conditions

Strict Identity Proof

Let V_S be non-trivial.

$$(-I) \in S \quad \implies \quad |\psi\rangle = (-I) |\psi\rangle = -|\psi\rangle \quad \iff \quad |\psi\rangle = \vec{0} \quad \implies \quad V_S \text{ is trivial.}$$

$$(iI) \in S \quad \implies \quad |\psi\rangle = (iI) |\psi\rangle = i|\psi\rangle \quad \iff \quad |\psi\rangle = \vec{0} \quad \implies \quad V_S \text{ is trivial.}$$

$$(-iI) \in S \quad \implies \quad |\psi\rangle = (-iI) |\psi\rangle = -i|\psi\rangle \quad \iff \quad |\psi\rangle = \vec{0} \quad \implies \quad V_S \text{ is trivial.}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Conditions

Strict Identity Proof

Let V_S be non-trivial.

$$(-I) \in S \quad \implies \quad |\psi\rangle = (-I) |\psi\rangle = -|\psi\rangle \quad \iff \quad |\psi\rangle = \vec{0} \quad \implies \quad V_S \text{ is trivial.}$$

$$(iI) \in S \quad \implies \quad |\psi\rangle = (iI) |\psi\rangle = i|\psi\rangle \quad \iff \quad |\psi\rangle = \vec{0} \quad \implies \quad V_S \text{ is trivial.}$$

$$(-iI) \in S \quad \implies \quad |\psi\rangle = (-iI) |\psi\rangle = -i|\psi\rangle \quad \iff \quad |\psi\rangle = \vec{0} \quad \implies \quad V_S \text{ is trivial.}$$

$-I \in S, iI \in S, -iI \in S$ lead to contradictions.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix Structure

Suppose $S = \langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix Structure

Suppose $S = \langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$.

Extremely useful way of presenting the generators: **Check Matrix** H_S

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix Structure

Suppose $S = \langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$.

Extremely useful way of presenting the generators: **Check Matrix** H_S

H_S is an $l \times 2n$ binary matrix whose rows correspond to the generators g_1 through g_l .

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix Structure

Suppose $S = \langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$.

Extremely useful way of presenting the generators: **Check Matrix** H_S

H_S is an $l \times 2n$ binary matrix whose rows correspond to the generators g_1 through g_l .

Example:

$$l \left\{ \left[\begin{array}{cccccccc|cccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right] \right.$$

$\underbrace{\hspace{10em}}_n \qquad \underbrace{\hspace{10em}}_n$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix

Interpretation

- Row i corresponds to generator $g_i \in S$.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix

Interpretation

- Row i corresponds to generator $g_i \in S$.
- Left $l \times n$ submatrix contains 1s to indicate which generators contain X s.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix

Interpretation

- Row i corresponds to generator $g_i \in S$.
- Left $l \times n$ submatrix contains 1s to indicate which generators contain X s.
- Right $l \times n$ submatrix contains 1s to indicate which generators contain Z s.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix

Interpretation

- Row i corresponds to generator $g_i \in S$.
- Left $l \times n$ submatrix contains 1s to indicate which generators contain X s.
- Right $l \times n$ submatrix contains 1s to indicate which generators contain Z s.
- Presence of 1 in both submatrices indicates Y in that generator.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix

Interpretation

- Row i corresponds to generator $g_i \in S$.
- Left $l \times n$ submatrix contains 1s to indicate which generators contain X s.
- Right $l \times n$ submatrix contains 1s to indicate which generators contain Z s.
- Presence of 1 in both submatrices indicates Y in that generator.

More explicitly, with $h_{i,j}$ denoting the element of H_S at row i and column j :

- If g_i contains I on the j^{th} qubit $\implies h_{i,j} = 0$ and $h_{i,n+j} = 0$.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix

Interpretation

- Row i corresponds to generator $g_i \in S$.
- Left $l \times n$ submatrix contains 1s to indicate which generators contain X s.
- Right $l \times n$ submatrix contains 1s to indicate which generators contain Z s.
- Presence of 1 in both submatrices indicates Y in that generator.

More explicitly, with $h_{i,j}$ denoting the element of H_S at row i and column j :

- If g_i contains I on the j^{th} qubit $\implies h_{i,j} = 0$ and $h_{i,n+j} = 0$.
- If g_i contains X on the j^{th} qubit $\implies h_{i,j} = 1$ and $h_{i,n+j} = 0$.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix

Interpretation

- Row i corresponds to generator $g_i \in S$.
- Left $l \times n$ submatrix contains 1s to indicate which generators contain X s.
- Right $l \times n$ submatrix contains 1s to indicate which generators contain Z s.
- Presence of 1 in both submatrices indicates Y in that generator.

More explicitly, with $h_{i,j}$ denoting the element of H_S at row i and column j :

- If g_i contains I on the j^{th} qubit $\implies h_{i,j} = 0$ and $h_{i,n+j} = 0$.
- If g_i contains X on the j^{th} qubit $\implies h_{i,j} = 1$ and $h_{i,n+j} = 0$.
- If g_i contains Z on the j^{th} qubit $\implies h_{i,j} = 0$ and $h_{i,n+j} = 1$.

[2] [Michael A Nielsen and Isaac L Chuang](#). *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix

Interpretation

- Row i corresponds to generator $g_i \in S$.
- Left $l \times n$ submatrix contains 1s to indicate which generators contain X s.
- Right $l \times n$ submatrix contains 1s to indicate which generators contain Z s.
- Presence of 1 in both submatrices indicates Y in that generator.

More explicitly, with $h_{i,j}$ denoting the element of H_S at row i and column j :

- If g_i contains I on the j^{th} qubit $\implies h_{i,j} = 0$ and $h_{i,n+j} = 0$.
- If g_i contains X on the j^{th} qubit $\implies h_{i,j} = 1$ and $h_{i,n+j} = 0$.
- If g_i contains Z on the j^{th} qubit $\implies h_{i,j} = 0$ and $h_{i,n+j} = 1$.
- If g_i contains Y on the j^{th} qubit $\implies h_{i,j} = 1$ and $h_{i,n+j} = 1$.

[2] [Michael A Nielsen and Isaac L Chuang](#). *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix

Example Steane Code

For Readability tensor product operator signs are left out. $\sigma_i \sigma_j$ corresponds to $\sigma_i \otimes \sigma_j$.

$$\left[\begin{array}{cccccc|cccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right] \cong \begin{array}{|c|c|} \hline \text{Generator} & \text{Operator} \\ \hline g_1 & III XXXX \\ g_2 & IXX I IXX \\ g_3 & XI XI XI X \\ g_4 & III ZZZZ \\ g_5 & IZZ I IZZ \\ g_6 & ZI ZI ZI Z \\ \hline \end{array}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Unitary Operations

Main Revelation

Suppose U is a unitary operator, $|\psi\rangle \in V_S$ and $g \in S$.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Unitary Operations

Main Revelation

Suppose U is a unitary operator, $|\psi\rangle \in V_S$ and $g \in S$.

$$U |\psi\rangle = U g |\psi\rangle = U g I |\psi\rangle = U g U^\dagger U |\psi\rangle = (U g U^\dagger) U |\psi\rangle$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Unitary Operations

Main Revelation

Suppose U is a unitary operator, $|\psi\rangle \in V_S$ and $g \in S$.

$$U |\psi\rangle = U g |\psi\rangle = U g I |\psi\rangle = U g U^\dagger U |\psi\rangle = (U g U^\dagger) U |\psi\rangle$$

\implies State $U |\psi\rangle$ is stabilized by $U g U^\dagger$.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Unitary Operations

Main Revelation

Suppose U is a unitary operator, $|\psi\rangle \in V_S$ and $g \in S$.

$$U |\psi\rangle = U g |\psi\rangle = U g I |\psi\rangle = U g U^\dagger U |\psi\rangle = (U g U^\dagger) U |\psi\rangle$$

\implies State $U |\psi\rangle$ is stabilized by $U g U^\dagger$.

\implies If we can describe a state by its stabilizers, we can easily compute the stabilizers of the state that emerges from the previous state under a unitary operation.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Unitary Operations

Advantages for Computation

For certain special unitary operations U this transformation of the generators takes on a particularly appealing form.

Unitary Operations

Advantages for Computation

For certain special unitary operations U this transformation of the generators takes on a particularly appealing form.

$$HXH^\dagger = Z \quad HYH^\dagger = -Y \quad HZH^\dagger = X$$

Unitary Operations

Advantages for Computation

For certain special unitary operations U this transformation of the generators takes on a particularly appealing form.

$$HXH^\dagger = Z \quad HYH^\dagger = -Y \quad HZH^\dagger = X$$

Example:

(Unkown) State $|\psi\rangle$ stabilized by X .

Unitary Operations

Advantages for Computation

For certain special unitary operations U this transformation of the generators takes on a particularly appealing form.

$$HXH^\dagger = Z \quad HYH^\dagger = -Y \quad HZH^\dagger = X$$

Example:

(Unknown) State $|\psi\rangle$ stabilized by X .

→ Apply Hadamard gate H to $|\psi\rangle$.

Unitary Operations

Advantages for Computation

For certain special unitary operations U this transformation of the generators takes on a particularly appealing form.

$$HXH^\dagger = Z \quad HYH^\dagger = -Y \quad HZH^\dagger = X$$

Example:

(Unkown) State $|\psi\rangle$ stabilized by X .

→ Apply Hadamard gate H to $|\psi\rangle$.

⇒ Resulting (Unkown) state $|\psi'\rangle$ stabilized by Z .

Unitary Operations

Transformation under Conjugation

Operation	Input	Output
CX	X_1	X_1X_2
	X_2	X_2
	Z_1	Z_1
	Z_2	Z_1Z_2
H	X	Z
	Z	X
S	X	Y
	Z	Z

Operation	Input	Output
X	X	X
	Z	$-Z$
Y	X	$-X$
	Z	$-Z$
Z	X	$-X$
	Z	Z

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Main Principles

We want to measure observable $g \in \mathcal{P}_n$ of state $|\psi\rangle$, stabilized by $\langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Main Principles

We want to measure observable $g \in \mathcal{P}_n$ of state $|\psi\rangle$, stabilized by $\langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$.

Two possibilities:

1. g commutes with all generators of the stabilizer.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Main Principles

We want to measure observable $g \in \mathcal{P}_n$ of state $|\psi\rangle$, stabilized by $\langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$.

Two possibilities:

1. g commutes with all generators of the stabilizer.
 \implies Measurement outcome is deterministic.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Main Principles

We want to measure observable $g \in \mathcal{P}_n$ of state $|\psi\rangle$, stabilized by $\langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$.

Two possibilities:

1. g commutes with all generators of the stabilizer.
 \implies Measurement outcome is deterministic.
2. g anti-commutes with at least 1 generator of the stabilizer.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Main Principles

We want to measure observable $g \in \mathcal{P}_n$ of state $|\psi\rangle$, stabilized by $\langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$.

Two possibilities:

1. g commutes with all generators of the stabilizer.
 \implies Measurement outcome is deterministic.
2. g anti-commutes with at least 1 generator of the stabilizer.
 \implies Measurement outcome is not deterministic.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Deterministic case

g commutes with all g_i and assume g does not have a global phase.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Deterministic case

g commutes with all g_i and assume g does not have a global phase.

$$\forall i \text{ holds: } g_i g |\psi\rangle = g g_i |\psi\rangle = g |\psi\rangle \implies g |\psi\rangle \in V_S$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Deterministic case

g commutes with all g_i and assume g does not have a global phase.

$$\forall i \text{ holds: } g_i g |\psi\rangle = g g_i |\psi\rangle = g |\psi\rangle \implies g |\psi\rangle \in V_S$$

$$g^2 |\psi\rangle = I |\psi\rangle = |\psi\rangle \implies g |\psi\rangle = \pm |\psi\rangle \implies g \in S \vee (-g) \in S$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Deterministic case

g commutes with all g_i and assume g does not have a global phase.

$$\forall i \text{ holds: } g_i g |\psi\rangle = g g_i |\psi\rangle = g |\psi\rangle \implies g |\psi\rangle \in V_S$$

$$g^2 |\psi\rangle = I |\psi\rangle = |\psi\rangle \implies g |\psi\rangle = \pm |\psi\rangle \implies g \in S \vee (-g) \in S$$

$$g \in S \implies g |\psi\rangle = |\psi\rangle \implies \text{Measurement yields } +1$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Deterministic case

g commutes with all g_i and assume g does not have a global phase.

$$\forall i \text{ holds: } g_i g |\psi\rangle = g g_i |\psi\rangle = g |\psi\rangle \implies g |\psi\rangle \in V_S$$

$$g^2 |\psi\rangle = I |\psi\rangle = |\psi\rangle \implies g |\psi\rangle = \pm |\psi\rangle \implies g \in S \vee (-g) \in S$$

$$g \in S \implies g |\psi\rangle = |\psi\rangle \implies \text{Measurement yields } +1$$

$$(-g) \in S \implies g |\psi\rangle = -|\psi\rangle \implies \text{Measurement yields } -1$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Deterministic case

g commutes with all g_i and assume g does not have a global phase.

$$\forall i \text{ holds: } g_i g |\psi\rangle = g g_i |\psi\rangle = g |\psi\rangle \implies g |\psi\rangle \in V_S$$

$$g^2 |\psi\rangle = I |\psi\rangle = |\psi\rangle \implies g |\psi\rangle = \pm |\psi\rangle \implies g \in S \vee (-g) \in S$$

$$g \in S \implies g |\psi\rangle = |\psi\rangle \implies \text{Measurement yields } +1$$

$$(-g) \in S \implies g |\psi\rangle = -|\psi\rangle \implies \text{Measurement yields } -1$$

In both cases the measurement does not disturb the state of the system, and leaves the stabilizer invariant.

[2] [Michael A Nielsen and Isaac L Chuang](#). *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case preliminaries

Without loss of generality, let g anti-commute with g_1 and g does not have a global phase.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case preliminaries

Without loss of generality, let g anti-commute with g_1 and g does not have a global phase.

$\forall g_j$ with $j \neq 1$ and $g_j g = -g g_j$: Replace g_j with $g'_j = g_1 g_j$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case preliminaries

Without loss of generality, let g anti-commute with g_1 and g does not have a global phase.

$$\begin{aligned} \forall g_j \text{ with } j \neq 1 \text{ and } g_j g = -g g_j : \text{ Replace } g_j \text{ with } g'_j = g_1 g_j \\ \implies g'_j g = g_1 g_j g = -g_1 g g_j = g g_1 g_j = g g'_j \end{aligned}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case preliminaries

Without loss of generality, let g anti-commute with g_1 and g does not have a global phase.

$$\begin{aligned} \forall g_j \text{ with } j \neq 1 \text{ and } g_j g = -g g_j : & \text{ Replace } g_j \text{ with } g'_j = g_1 g_j \\ \implies g'_j g = g_1 g_j g = -g_1 g g_j = g g_1 g_j = g g'_j \\ \implies g \text{ commutes with } g'_j \end{aligned}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case preliminaries

Without loss of generality, let g anti-commute with g_1 and g does not have a global phase.

$\forall g_j$ with $j \neq 1$ and $g_j g = -g g_j$: Replace g_j with $g'_j = g_1 g_j$

$$\implies g'_j g = g_1 g_j g = -g_1 g g_j = g g_1 g_j = g g'_j$$

$$\implies g \text{ commutes with } g'_j$$

$\implies g$ only commutes with g_1 .

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case preliminaries

Without loss of generality, let g anti-commute with g_1 and g does not have a global phase.

$$\begin{aligned} \forall g_j \text{ with } j \neq 1 \text{ and } g_j g = -g g_j : & \text{ Replace } g_j \text{ with } g'_j = g_1 g_j \\ \implies g'_j g = g_1 g_j g = -g_1 g g_j = g g_1 g_j = g g'_j \\ \implies g \text{ commutes with } g'_j \end{aligned}$$

$\implies g$ only commutes with g_1 .

Because g has eigenvalues ± 1 , the measurement operators are: $M_{\pm g} = \frac{I \pm g}{2}$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case continuation

Measurement probabilities:

$$p(+1) = \text{tr} \left(\frac{I + g}{2} |\psi\rangle \langle\psi| \right) \quad \wedge \quad p(-1) = \text{tr} \left(\frac{I - g}{2} |\psi\rangle \langle\psi| \right)$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case continuation

Measurement probabilities:

$$p(+1) = \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) \quad \wedge \quad p(-1) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| \right)$$

$$p(+1) = \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right)$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case continuation

Measurement probabilities:

$$p(+1) = \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) \quad \wedge \quad p(-1) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| \right)$$

$$p(+1) = \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) = \text{tr} \left(\frac{I+g}{2} g_1 |\psi\rangle \langle\psi| \right)$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case continuation

Measurement probabilities:

$$p(+1) = \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) \quad \wedge \quad p(-1) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| \right)$$

$$p(+1) = \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) = \text{tr} \left(\frac{I+g}{2} g_1 |\psi\rangle \langle\psi| \right) = \text{tr} \left(g_1 \frac{I-g}{2} |\psi\rangle \langle\psi| \right)$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case continuation

Measurement probabilities:

$$p(+1) = \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) \quad \wedge \quad p(-1) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| \right)$$

$$\begin{aligned} p(+1) &= \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) = \text{tr} \left(\frac{I+g}{2} g_1 |\psi\rangle \langle\psi| \right) = \text{tr} \left(g_1 \frac{I-g}{2} |\psi\rangle \langle\psi| \right) \\ &= \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| g_1 \right) \end{aligned}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case continuation

Measurement probabilities:

$$p(+1) = \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) \quad \wedge \quad p(-1) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| \right)$$

$$\begin{aligned} p(+1) &= \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) = \text{tr} \left(\frac{I+g}{2} g_1 |\psi\rangle \langle\psi| \right) = \text{tr} \left(g_1 \frac{I-g}{2} |\psi\rangle \langle\psi| \right) \\ &= \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| g_1 \right) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| g_1^\dagger \right) \end{aligned}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case continuation

Measurement probabilities:

$$p(+1) = \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) \quad \wedge \quad p(-1) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| \right)$$

$$\begin{aligned} p(+1) &= \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) = \text{tr} \left(\frac{I+g}{2} g_1 |\psi\rangle \langle\psi| \right) = \text{tr} \left(g_1 \frac{I-g}{2} |\psi\rangle \langle\psi| \right) \\ &= \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| g_1 \right) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| g_1^\dagger \right) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| \right) \end{aligned}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case continuation

Measurement probabilities:

$$p(+1) = \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) \quad \wedge \quad p(-1) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| \right)$$

$$\begin{aligned} p(+1) &= \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) = \text{tr} \left(\frac{I+g}{2} g_1 |\psi\rangle \langle\psi| \right) = \text{tr} \left(g_1 \frac{I-g}{2} |\psi\rangle \langle\psi| \right) \\ &= \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| g_1 \right) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| g_1^\dagger \right) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| \right) = p(-1) \end{aligned}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Non-deterministic case continuation

Measurement probabilities:

$$p(+1) = \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) \quad \wedge \quad p(-1) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| \right)$$

$$\begin{aligned} p(+1) &= \text{tr} \left(\frac{I+g}{2} |\psi\rangle \langle\psi| \right) = \text{tr} \left(\frac{I+g}{2} g_1 |\psi\rangle \langle\psi| \right) = \text{tr} \left(g_1 \frac{I-g}{2} |\psi\rangle \langle\psi| \right) \\ &= \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| g_1 \right) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| g_1^\dagger \right) = \text{tr} \left(\frac{I-g}{2} |\psi\rangle \langle\psi| \right) = p(-1) \end{aligned}$$

$$p(+1) = p(-1) \text{ and } p(+1) + p(-1) = 1 \implies p(+1) = p(-1) = \frac{1}{2}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

- 1 Preliminary Definitions
- 2 Stabilizer Formalism
- 3 Stabilizer Algorithm**

Gottesman–Knill Theorem

Suppose a quantum computation is performed which involves only the following elements:

- State preparations in the computational basis
- Hadamard gates
- Phase gates
- Controlled-NOT gates
- Pauli gates
- Measurements of observables in the Pauli group

Together with the possibility of classical control conditioned on the outcome of such measurements. Such a computation may be efficiently simulated on a classical computer.

[2] [Michael A Nielsen and Isaac L Chuang](#). *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Circuit

Mainly:

Circuit consisting solely of CX , H , S and M gates.

Stabilizer Circuit

Mainly:

Circuit consisting solely of CX , H , S and M gates.

Can actually also implement X , Y , Z and $SWAP$.

Stabilizer Circuit

Mainly:

Circuit consisting solely of CX , H , S and M gates.

Can actually also implement X , Y , Z and $SWAP$.

$$Z = SS$$

Stabilizer Circuit

Mainly:

Circuit consisting solely of CX , H , S and M gates.

Can actually also implement X , Y , Z and $SWAP$.

$$Z = SS$$

$$X = HZH$$

Stabilizer Circuit

Mainly:

Circuit consisting solely of CX , H , S and M gates.

Can actually also implement X , Y , Z and $SWAP$.

$$Z = SS$$

$$X = HZH$$

$$Y = SXSSS$$

Stabilizer Circuit

Mainly:

Circuit consisting solely of CX , H , S and M gates.

Can actually also implement X , Y , Z and $SWAP$.

$$Z = SS$$

$$X = HZH$$

$$Y = SXSSS$$

$$SWAP(a, b) = CX(a, b)CX(b, a)CX(a, b)$$

Clifford Tableau Structure

Basically an expanded Check Matrix.

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Clifford Tableau

Structure

Basically an expanded Check Matrix.

$$\left(\begin{array}{ccc|ccc|c} x_{1,1} & \cdots & x_{1,n} & z_{1,1} & \cdots & z_{1,n} & r_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n,1} & \cdots & x_{n,n} & z_{n,1} & \cdots & z_{n,n} & r_n \\ \hline x_{(n+1),1} & \cdots & x_{(n+1),n} & z_{(n+1),1} & \cdots & z_{(n+1),n} & r_{n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{(2n),1} & \cdots & x_{(2n),n} & z_{(2n),1} & \cdots & z_{(2n),n} & r_{2n} \\ \hline x_{(2n+1),1} & \cdots & x_{(2n+1),n} & z_{(2n+1),1} & \cdots & z_{(2n+1),n} & r_{2n+1} \end{array} \right)$$

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Clifford Tableau

Interpretation

Developers of the following algorithm introduce additional n "Destabilizer" generators, which are Pauli operators that together with the stabilizer generators generate the full Pauli group.

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Clifford Tableau

Interpretation

Developers of the following algorithm introduce additional n "Destabilizer" generators, which are Pauli operators that together with the stabilizer generators generate the full Pauli group.

- Rows 1 to n represent Destabilizers.

$x_{1,1}$	\cdots	$x_{1,n}$	$z_{1,1}$	\cdots	$z_{1,n}$	r_1
\vdots	\ddots	\vdots	\vdots	\ddots	\vdots	\vdots
$x_{n,1}$	\cdots	$x_{n,n}$	$z_{n,1}$	\cdots	$z_{n,n}$	r_n
$x_{(n+1),1}$	\cdots	$x_{(n+1),n}$	$z_{(n+1),1}$	\cdots	$z_{(n+1),n}$	r_{n+1}
\vdots	\ddots	\vdots	\vdots	\ddots	\vdots	\vdots
$x_{(2n),1}$	\cdots	$x_{(2n),n}$	$z_{(2n),1}$	\cdots	$z_{(2n),n}$	r_{2n}
$x_{(2n+1),1}$	\cdots	$x_{(2n+1),n}$	$z_{(2n+1),1}$	\cdots	$z_{(2n+1),n}$	r_{2n+1}

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Clifford Tableau

Interpretation

Developers of the following algorithm introduce additional n "Destabilizer" generators, which are Pauli operators that together with the stabilizer generators generate the full Pauli group.

- Rows 1 to n represent Destabilizers.
- Rows $n + 1$ to $2n$ represent Stabilizers.

$x_{1,1}$	\cdots	$x_{1,n}$	$z_{1,1}$	\cdots	$z_{1,n}$	r_1
\vdots	\ddots	\vdots	\vdots	\ddots	\vdots	\vdots
$x_{n,1}$	\cdots	$x_{n,n}$	$z_{n,1}$	\cdots	$z_{n,n}$	r_n
$x_{(n+1),1}$	\cdots	$x_{(n+1),n}$	$z_{(n+1),1}$	\cdots	$z_{(n+1),n}$	r_{n+1}
\vdots	\ddots	\vdots	\vdots	\ddots	\vdots	\vdots
$x_{(2n),1}$	\cdots	$x_{(2n),n}$	$z_{(2n),1}$	\cdots	$z_{(2n),n}$	r_{2n}
$x_{(2n+1),1}$	\cdots	$x_{(2n+1),n}$	$z_{(2n+1),1}$	\cdots	$z_{(2n+1),n}$	r_{2n+1}

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Clifford Tableau

Interpretation

Developers of the following algorithm introduce additional n "Destabilizer" generators, which are Pauli operators that together with the stabilizer generators generate the full Pauli group.

- Rows 1 to n represent Destabilizers.
- Rows $n + 1$ to $2n$ represent Stabilizers.
- Row $2n + 1$ is scratch-space.

$$\left(\begin{array}{ccc|ccc|c} x_{1,1} & \cdots & x_{1,n} & z_{1,1} & \cdots & z_{1,n} & r_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n,1} & \cdots & x_{n,n} & z_{n,1} & \cdots & z_{n,n} & r_n \\ \hline x_{(n+1),1} & \cdots & x_{(n+1),n} & z_{(n+1),1} & \cdots & z_{(n+1),n} & r_{n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{(2n),1} & \cdots & x_{(2n),n} & z_{(2n),1} & \cdots & z_{(2n),n} & r_{2n} \\ \hline x_{(2n+1),1} & \cdots & x_{(2n+1),n} & z_{(2n+1),1} & \cdots & z_{(2n+1),n} & r_{2n+1} \end{array} \right)$$

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Clifford Tableau

Interpretation

Developers of the following algorithm introduce additional n "Destabilizer" generators, which are Pauli operators that together with the stabilizer generators generate the full Pauli group.

- Rows 1 to n represent Destabilizers.
- Rows $n + 1$ to $2n$ represent Stabilizers.
- Row $2n + 1$ is scratch-space.
- r_i of row i represents the global phase,
 $r_i = 0$ for $+1$ and $r_i = 1$ for -1 .

$$\left(\begin{array}{ccc|ccc|c} x_{1,1} & \cdots & x_{1,n} & z_{1,1} & \cdots & z_{1,n} & r_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n,1} & \cdots & x_{n,n} & z_{n,1} & \cdots & z_{n,n} & r_n \\ \hline x_{(n+1),1} & \cdots & x_{(n+1),n} & z_{(n+1),1} & \cdots & z_{(n+1),n} & r_{n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{(2n),1} & \cdots & x_{(2n),n} & z_{(2n),1} & \cdots & z_{(2n),n} & r_{2n} \\ \hline x_{(2n+1),1} & \cdots & x_{(2n+1),n} & z_{(2n+1),1} & \cdots & z_{(2n+1),n} & r_{2n+1} \end{array} \right)$$

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Clifford Tableau

Initial Tableau

Pauli- Z gates stabilize $|0\rangle$ states and Pauli- X gates destabilize $|0\rangle$ states.

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Clifford Tableau

Initial Tableau

Pauli- Z gates stabilize $|0\rangle$ states and Pauli- X gates destabilize $|0\rangle$ states.

\implies Tableau for $|0\rangle^{\otimes n}$ has an Identity submatrix for its first $(2n) \times (2n)$ submatrix.

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Clifford Tableau

Initial Tableau

Pauli- Z gates stabilize $|0\rangle$ states and Pauli- X gates destabilize $|0\rangle$ states.

\implies Tableau for $|0\rangle^{\otimes n}$ has an Identity submatrix for its first $(2n) \times (2n)$ submatrix.

Tableau for $|00\rangle$ is

$$\left(\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Clifford Tableau

Initial Tableau

Pauli- Z gates stabilize $|0\rangle$ states and Pauli- X gates destabilize $|0\rangle$ states.

\implies Tableau for $|0\rangle^{\otimes n}$ has an Identity submatrix for its first $(2n) \times (2n)$ submatrix.

$$\text{Tableau for } |00\rangle \text{ is } \left(\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

In the following slides, let R_i denote the i -th row of the stabilizer tableau.

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Hadamard Gate Algorithm

Suppose we apply a Hadamard gate to qubit a of current state $|\psi\rangle$.

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Hadamard Gate Algorithm

Suppose we apply a Hadamard gate to qubit a of current state $|\psi\rangle$.

\implies For all stabilizers and destabilizers R_i the entries $x_{i,a}$ and $z_{i,a}$ determine the entries $x'_{i,a}$ and $z'_{i,a}$ of the new stabilizers and destabilizers R'_i after the Hadamard gate application in the following way:

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Hadamard Gate Algorithm

Suppose we apply a Hadamard gate to qubit a of current state $|\psi\rangle$.

\implies For all stabilizers and destabilizers R_i the entries $x_{i,a}$ and $z_{i,a}$ determine the entries $x'_{i,a}$ and $z'_{i,a}$ of the new stabilizers and destabilizers R'_i after the Hadamard gate application in the following way:

$$\blacksquare \quad x_{i,a}z_{i,a} = 00 \implies P_{i,a} = I \implies P'_{i,a} = H I H^\dagger = I \implies x'_{i,a}z'_{i,a} = 00 \wedge r'_i = r_i$$

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Hadamard Gate Algorithm

Suppose we apply a Hadamard gate to qubit a of current state $|\psi\rangle$.

\implies For all stabilizers and destabilizers R_i the entries $x_{i,a}$ and $z_{i,a}$ determine the entries $x'_{i,a}$ and $z'_{i,a}$ of the new stabilizers and destabilizers R'_i after the Hadamard gate application in the following way:

$$\blacksquare \quad x_{i,a}z_{i,a} = 00 \implies P_{i,a} = I \implies P'_{i,a} = H I H^\dagger = I \implies x'_{i,a}z'_{i,a} = 00 \wedge r'_i = r_i$$

$$\blacksquare \quad x_{i,a}z_{i,a} = 01 \implies P_{i,a} = Z \implies P'_{i,a} = H Z H^\dagger = X \implies x'_{i,a}z'_{i,a} = 10 \wedge r'_i = r_i$$

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Hadamard Gate Algorithm

Suppose we apply a Hadamard gate to qubit a of current state $|\psi\rangle$.

\implies For all stabilizers and destabilizers R_i the entries $x_{i,a}$ and $z_{i,a}$ determine the entries $x'_{i,a}$ and $z'_{i,a}$ of the new stabilizers and destabilizers R'_i after the Hadamard gate application in the following way:

- $x_{i,a}z_{i,a} = 00 \implies P_{i,a} = I \implies P'_{i,a} = H I H^\dagger = I \implies x'_{i,a}z'_{i,a} = 00 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 01 \implies P_{i,a} = Z \implies P'_{i,a} = H Z H^\dagger = X \implies x'_{i,a}z'_{i,a} = 10 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 10 \implies P_{i,a} = X \implies P'_{i,a} = H X H^\dagger = Z \implies x'_{i,a}z'_{i,a} = 01 \wedge r'_i = r_i$

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Hadamard Gate Algorithm

Suppose we apply a Hadamard gate to qubit a of current state $|\psi\rangle$.

\implies For all stabilizers and destabilizers R_i the entries $x_{i,a}$ and $z_{i,a}$ determine the entries $x'_{i,a}$ and $z'_{i,a}$ of the new stabilizers and destabilizers R'_i after the Hadamard gate application in the following way:

- $x_{i,a}z_{i,a} = 00 \implies P_{i,a} = I \implies P'_{i,a} = H I H^\dagger = I \implies x'_{i,a}z'_{i,a} = 00 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 01 \implies P_{i,a} = Z \implies P'_{i,a} = H Z H^\dagger = X \implies x'_{i,a}z'_{i,a} = 10 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 10 \implies P_{i,a} = X \implies P'_{i,a} = H X H^\dagger = Z \implies x'_{i,a}z'_{i,a} = 01 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 11 \implies P_{i,a} = Y \implies P'_{i,a} = H Y H^\dagger = -Y \implies x'_{i,a}z'_{i,a} = 11 \wedge r'_i = \neg r_i$

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Hadamard Gate Algorithm

Suppose we apply a Hadamard gate to qubit a of current state $|\psi\rangle$.

\implies For all stabilizers and destabilizers R_i the entries $x_{i,a}$ and $z_{i,a}$ determine the entries $x'_{i,a}$ and $z'_{i,a}$ of the new stabilizers and destabilizers R'_i after the Hadamard gate application in the following way:

- $x_{i,a}z_{i,a} = 00 \implies P_{i,a} = I \implies P'_{i,a} = H I H^\dagger = I \implies x'_{i,a}z'_{i,a} = 00 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 01 \implies P_{i,a} = Z \implies P'_{i,a} = H Z H^\dagger = X \implies x'_{i,a}z'_{i,a} = 10 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 10 \implies P_{i,a} = X \implies P'_{i,a} = H X H^\dagger = Z \implies x'_{i,a}z'_{i,a} = 01 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 11 \implies P_{i,a} = Y \implies P'_{i,a} = H Y H^\dagger = -Y \implies x'_{i,a}z'_{i,a} = 11 \wedge r'_i = \neg r_i$

Operation summary: $x'_{i,a} = z_{i,a} \quad \wedge \quad z'_{i,a} = x_{i,a} \quad \wedge \quad r'_i = r_i \oplus (x_{i,a} \cdot z_{i,a})$

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Phase Gate Algorithm

Suppose we apply a Phase gate to qubit a of current state $|\psi\rangle$.

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Phase Gate Algorithm

Suppose we apply a Phase gate to qubit a of current state $|\psi\rangle$.

\implies For all stabilizers and destabilizers R_i the entries $x_{i,a}$ and $z_{i,a}$ determine the entries $x'_{i,a}$ and $z'_{i,a}$ of the new stabilizers and destabilizers R'_i after the Phase gate application in the following way:

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Phase Gate Algorithm

Suppose we apply a Phase gate to qubit a of current state $|\psi\rangle$.

\implies For all stabilizers and destabilizers R_i the entries $x_{i,a}$ and $z_{i,a}$ determine the entries $x'_{i,a}$ and $z'_{i,a}$ of the new stabilizers and destabilizers R'_i after the Phase gate application in the following way:

$$\blacksquare \quad x_{i,a}z_{i,a} = 00 \implies P_{i,a} = I \implies P'_{i,a} = SIS^\dagger = I \implies x'_{i,a}z'_{i,a} = 00 \wedge r'_i = r_i$$

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Phase Gate Algorithm

Suppose we apply a Phase gate to qubit a of current state $|\psi\rangle$.

\implies For all stabilizers and destabilizers R_i the entries $x_{i,a}$ and $z_{i,a}$ determine the entries $x'_{i,a}$ and $z'_{i,a}$ of the new stabilizers and destabilizers R'_i after the Phase gate application in the following way:

$$\blacksquare \quad x_{i,a}z_{i,a} = 00 \implies P_{i,a} = I \implies P'_{i,a} = SIS^\dagger = I \implies x'_{i,a}z'_{i,a} = 00 \wedge r'_i = r_i$$

$$\blacksquare \quad x_{i,a}z_{i,a} = 01 \implies P_{i,a} = Z \implies P'_{i,a} = SZS^\dagger = Z \implies x'_{i,a}z'_{i,a} = 01 \wedge r'_i = r_i$$

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Phase Gate Algorithm

Suppose we apply a Phase gate to qubit a of current state $|\psi\rangle$.

\implies For all stabilizers and destabilizers R_i the entries $x_{i,a}$ and $z_{i,a}$ determine the entries $x'_{i,a}$ and $z'_{i,a}$ of the new stabilizers and destabilizers R'_i after the Phase gate application in the following way:

- $x_{i,a}z_{i,a} = 00 \implies P_{i,a} = I \implies P'_{i,a} = SIS^\dagger = I \implies x'_{i,a}z'_{i,a} = 00 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 01 \implies P_{i,a} = Z \implies P'_{i,a} = SZS^\dagger = Z \implies x'_{i,a}z'_{i,a} = 01 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 10 \implies P_{i,a} = X \implies P'_{i,a} = SXS^\dagger = Y \implies x'_{i,a}z'_{i,a} = 11 \wedge r'_i = r_i$

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Phase Gate Algorithm

Suppose we apply a Phase gate to qubit a of current state $|\psi\rangle$.

\implies For all stabilizers and destabilizers R_i the entries $x_{i,a}$ and $z_{i,a}$ determine the entries $x'_{i,a}$ and $z'_{i,a}$ of the new stabilizers and destabilizers R'_i after the Phase gate application in the following way:

- $x_{i,a}z_{i,a} = 00 \implies P_{i,a} = I \implies P'_{i,a} = SIS^\dagger = I \implies x'_{i,a}z'_{i,a} = 00 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 01 \implies P_{i,a} = Z \implies P'_{i,a} = SZS^\dagger = Z \implies x'_{i,a}z'_{i,a} = 01 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 10 \implies P_{i,a} = X \implies P'_{i,a} = SXS^\dagger = Y \implies x'_{i,a}z'_{i,a} = 11 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 11 \implies P_{i,a} = Y \implies P'_{i,a} = SY S^\dagger = -X \implies x'_{i,a}z'_{i,a} = 10 \wedge r'_i = \neg r_i$

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Phase Gate Algorithm

Suppose we apply a Phase gate to qubit a of current state $|\psi\rangle$.

\implies For all stabilizers and destabilizers R_i the entries $x_{i,a}$ and $z_{i,a}$ determine the entries $x'_{i,a}$ and $z'_{i,a}$ of the new stabilizers and destabilizers R'_i after the Phase gate application in the following way:

- $x_{i,a}z_{i,a} = 00 \implies P_{i,a} = I \implies P'_{i,a} = SIS^\dagger = I \implies x'_{i,a}z'_{i,a} = 00 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 01 \implies P_{i,a} = Z \implies P'_{i,a} = SZS^\dagger = Z \implies x'_{i,a}z'_{i,a} = 01 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 10 \implies P_{i,a} = X \implies P'_{i,a} = SXS^\dagger = Y \implies x'_{i,a}z'_{i,a} = 11 \wedge r'_i = r_i$
- $x_{i,a}z_{i,a} = 11 \implies P_{i,a} = Y \implies P'_{i,a} = SY S^\dagger = -X \implies x'_{i,a}z'_{i,a} = 10 \wedge r'_i = \neg r_i$

Operation summary: $x'_{i,a} = x_{i,a} \quad \wedge \quad z'_{i,a} = x_{i,a} \oplus z_{i,a} \quad \wedge \quad r'_i = r_i \oplus (x_{i,a} \cdot z_{i,a})$

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

References

- [1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328.
- [2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010.