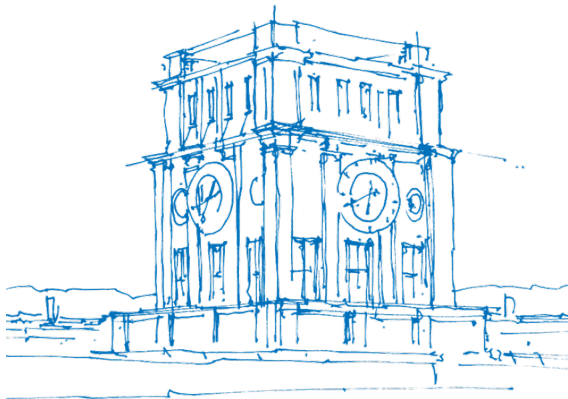


# Clifford Tableaus and the Stabilizer Algorithm

**Leonard Uscinowicz**

Technical University of Munich

December 20<sup>th</sup>, 2024



## 1 Preliminary Definitions

## 2 Stabilizer Formalism

## 3 Stabilizer Algorithm

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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# Pauli Matrices

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## Products of Pauli matrices:

$$\begin{aligned} I^2 &= X^2 = Y^2 = Z^2 = I \\ IX &= XI = X & IY &= YI = Y & IZ &= ZI = Z \\ XY &= iZ & YX &= -iZ \\ YZ &= iX & ZY &= -iX \\ ZX &= iY & XZ &= -iY \end{aligned}$$

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■ **Inverse:**  $\forall g \in G \implies \exists g^{-1} \in G$  such that  $g \cdot g^{-1} = g^{-1} \cdot g = e$

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# Pauli Group

## Definitions

$\mathcal{P}_n$  is defined as the group of  $n$ -qubit Pauli operators.

It consists of all tensor products of  $n$  Pauli matrices, with a phase factor  $\pm 1$  or  $\pm i$ .

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$$\mathcal{P}_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$$

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$$\mathcal{P}_n = \left\{ i^m \bigotimes_{j=1}^n \sigma_{k_j} \mid m, k_j \in \{0, 1, 2, 3\}, \sigma_0 = I, \sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z \right\}$$

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Size of a Pauli Group:  $|\mathcal{P}_n| = 4^{n+1}$

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# Pauli Group

## Operations

Given two Pauli operators  $P = i^{m_P} \bigotimes_{j=1}^n P_j$  and  $Q = i^{m_Q} \bigotimes_{j=1}^n Q_j$ , their product, as necessitated by Group Definition, is:

$$P \cdot Q = i^{m_P+m_Q} \bigotimes_{j=1}^n P_j Q_j$$

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$P$  commutes with  $Q$  if the number of indices  $j$  such that  $P_j$  anti-commutes with  $Q_j$  is even.

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# Group Generators

A set of  $l$  elements  $\{g_i\}_{1 \leq i \leq l}$  generates a group  $G$  if every element  $g \in G$  can be written as a product of the generators.

In this case, the group  $G$  can be written in terms of its generators:

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**Examples:**

$$\begin{aligned} \mathcal{P}_1 &= \langle X, Z, iI \rangle \\ \langle X \rangle &= \{I, X\} \end{aligned}$$

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- Element  $g \in \mathcal{P}_n$  **stabilizes**  $|\psi\rangle$  iff  $g|\psi\rangle = |\psi\rangle$ .  
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- $S \triangleq$  Subgroup of the Pauli Group  $\mathcal{P}_n$ :  $S \subseteq \mathcal{P}_n$ .

- $V_S \triangleq$  Set of  $n$ -qubit states stabilized by  $S$ :

$$V_S = \{|\psi\rangle \mid S \subseteq \mathcal{P}_n, \forall g \in S \text{ holds: } g|\psi\rangle = |\psi\rangle\}$$

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Not just any subgroup  $S$  of the Pauli group can be used as the stabilizer for a non-trivial vector space  $V_S$ .

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# Stabilizer Conditions

## Commutativity Proof

Let  $V_S$  be non-trivial and let  $g_1, g_2 \in S$ .

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$-I \in S, iI \in S, -iI \in S$  lead to contradictions.

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## Check Matrix Structure

Suppose  $S = \langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$ .

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Example:

$$l \left\{ \left[ \begin{array}{cccccccc|cccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right] \right.$$

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## Interpretation

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More explicitly, with  $h_{i,j}$  denoting the element of  $H_S$  at row  $i$  and column  $j$ :

- If  $g_i$  contains  $I$  on the  $j^{\text{th}}$  qubit  $\implies h_{i,j} = 0$  and  $h_{i,n+j} = 0$ .

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- If  $g_i$  contains  $Y$  on the  $j^{\text{th}}$  qubit  $\implies h_{i,j} = 1$  and  $h_{i,n+j} = 1$ .

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## Example Steane Code

For Readability tensor product operator signs are left out.  $\sigma_i \sigma_j$  corresponds to  $\sigma_i \otimes \sigma_j$ .

$$\left[ \begin{array}{cccccc|cccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right] \cong$$

Generator	Operator
$g_1$	$IIIXXXX$
$g_2$	$IXXIIXX$
$g_3$	$XIXIXIX$
$g_4$	$IIIZZZZ$
$g_5$	$IZZIIZZ$
$g_6$	$ZIZIZIZ$

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# Unitary Operations

## Main Revelation

Suppose  $U$  is a unitary operator,  $|\psi\rangle \in V_S$  and  $g \in S$ .

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$\implies$  State  $U |\psi\rangle$  is stabilized by  $U g U^\dagger$ .

$\implies$  If we can describe a state by its stabilizers, we can easily compute the stabilizers of the state that emerges from the previous state under a unitary operation.

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# Unitary Operations

## Advantages for Computation

For certain special unitary operations  $U$  this transformation of the generators takes on a particularly appealing form.

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### Example:

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→ Apply Hadamard gate  $H$  to  $|\psi\rangle$ .

⇒ Resulting (Unkown) state  $|\psi'\rangle$  stabilized by  $Z$ .

# Unitary Operations

## Transformation under Conjugation

Operation	Input	Output
$CX$	$X_1$	$X_1X_2$
	$X_2$	$X_2$
	$Z_1$	$Z_1$
	$Z_2$	$Z_1Z_2$
$H$	$X$	$Z$
	$Z$	$X$
$S$	$X$	$Y$
	$Z$	$Z$

Operation	Input	Output
$X$	$X$	$X$
	$Z$	$-Z$
$Y$	$X$	$-X$
	$Z$	$-Z$
$Z$	$X$	$-X$
	$Z$	$Z$

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# Measurement

## Main Principles

We want to measure observable  $g \in \mathcal{P}_n$  of state  $|\psi\rangle$ , stabilized by  $\langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$ .

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In both cases the measurement does not disturb the state of the system, and leaves the stabilizer invariant.

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$\implies g$  only commutes with  $g_1$ .

Because  $g$  has eigenvalues  $\pm 1$ , the measurement operators are:  $M_{\pm g} = \frac{I \pm g}{2}$

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# Measurement

## Non-deterministic case continuation

Measurement probabilities:

$$p(+1) = \text{tr} \left( \frac{I + g}{2} |\psi\rangle \langle\psi| \right) \quad \wedge \quad p(-1) = \text{tr} \left( \frac{I - g}{2} |\psi\rangle \langle\psi| \right)$$

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$$p(+1) = p(-1) \text{ and } p(+1) + p(-1) = 1 \implies p(+1) = p(-1) = \frac{1}{2}$$

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- 1 Preliminary Definitions
- 2 Stabilizer Formalism
- 3 Stabilizer Algorithm**

# Gottesman–Knill Theorem

Suppose a quantum computation is performed which involves only the following elements:

- State preparations in the computational basis
- Hadamard gates
- Phase gates
- Controlled-NOT gates
- Pauli gates
- Measurements of observables in the Pauli group

Together with the possibility of classical control conditioned on the outcome of such measurements. Such a computation may be efficiently simulated on a classical computer.

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# Stabilizer Circuit

**Mainly:**

Circuit consisting solely of  $CX$ ,  $H$ ,  $S$  and  $M$  gates.

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$$Z = SS$$

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$$SWAP(a, b) = CX(a, b)CX(b, a)CX(a, b)$$

# Clifford Tableau Structure

Basically an expanded Check Matrix.

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

# Clifford Tableau

## Structure

Basically an expanded Check Matrix.

$$\left( \begin{array}{ccc|ccc|c} x_{1,1} & \cdots & x_{1,n} & z_{1,1} & \cdots & z_{1,n} & r_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n,1} & \cdots & x_{n,n} & z_{n,1} & \cdots & z_{n,n} & r_n \\ \hline x_{(n+1),1} & \cdots & x_{(n+1),n} & z_{(n+1),1} & \cdots & z_{(n+1),n} & r_{n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{(2n),1} & \cdots & x_{(2n),n} & z_{(2n),1} & \cdots & z_{(2n),n} & r_{2n} \\ \hline x_{(2n+1),1} & \cdots & x_{(2n+1),n} & z_{(2n+1),1} & \cdots & z_{(2n+1),n} & r_{2n+1} \end{array} \right)$$

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## Interpretation

Developers of the following algorithm introduce additional  $n$  "Destabilizer" generators, which are Pauli operators that together with the stabilizer generators generate the full Pauli group.

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- Rows 1 to  $n$  represent Destabilizers.

$x_{1,1}$	$\cdots$	$x_{1,n}$	$z_{1,1}$	$\cdots$	$z_{1,n}$	$r_1$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$x_{n,1}$	$\cdots$	$x_{n,n}$	$z_{n,1}$	$\cdots$	$z_{n,n}$	$r_n$
$x_{(n+1),1}$	$\cdots$	$x_{(n+1),n}$	$z_{(n+1),1}$	$\cdots$	$z_{(n+1),n}$	$r_{n+1}$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$x_{(2n),1}$	$\cdots$	$x_{(2n),n}$	$z_{(2n),1}$	$\cdots$	$z_{(2n),n}$	$r_{2n}$
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# Clifford Tableau

## Interpretation

Developers of the following algorithm introduce additional  $n$  "Destabilizer" generators, which are Pauli operators that together with the stabilizer generators generate the full Pauli group.

- Rows 1 to  $n$  represent Destabilizers.
- Rows  $n + 1$  to  $2n$  represent Stabilizers.

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 $r_i = 0$  for  $+1$  and  $r_i = 1$  for  $-1$ .

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## Initial Tableau

Pauli- $Z$  gates stabilize  $|0\rangle$  states and Pauli- $X$  gates destabilize  $|0\rangle$  states.

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Tableau for  $|00\rangle$  is

$$\left( \begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

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In the following slides, let  $R_i$  denote the  $i$ -th row of the stabilizer tableau.

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# Hadamard Gate Algorithm

Suppose we apply a Hadamard gate to qubit  $a$  of current state  $|\psi\rangle$ .

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# Measurement Algorithm

## Deterministic Case

Measuring qubit  $a$

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Measuring qubit  $a \implies$  Stabilizers have  $I$  or  $Z$  on  $a$  and  $g$  is either  $-Z_a$  or  $Z_a$ .

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 $Z_a$  anti-commutes with all  $R_i$  for which  $x_{i,a} = 1$ .

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Compute  $g$  in scratch-space row  $2n + 1$  and extract  $r_{2n+1}$ :

$$r_{2n+1} = 0 \implies g = Z_a \implies \text{Measurement: } 0$$

$$r_{2n+1} = 1 \implies g = -Z_a \implies \text{Measurement: } 1$$

Measurement result is just  $r_{2n+1}$ .

# Measurement Algorithm

## Non-deterministic Case

Measuring qubit  $a$ :  $\implies g$  is either  $-Z_a$  or  $Z_a$ .

# Measurement Algorithm

## Non-deterministic Case

Measuring qubit  $a$ :  $\implies g$  is either  $-Z_a$  or  $Z_a$ .

1. Find  $p \in [n + 1, 2n]$  such that  $x_{p,a} = 1$ .

This  $R_p$  corresponds to our previous  $g_1$ , which anti-commutes with  $g$ .

# Measurement Algorithm

## Non-deterministic Case

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1. Find  $p \in [n+1, 2n]$  such that  $x_{p,a} = 1$ .

This  $R_p$  corresponds to our previous  $g_1$ , which anti-commutes with  $g$ .

2. Multiply  $g_1$  on to all  $g_j$  for which  $x_{j,a} = 1$ .

This transforms all  $g_j$  anti-commuting with  $g$  to  $g'_j$  commuting with  $g$ .

# Measurement Algorithm

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4. Assign  $R_p = g$ , either  $Z_a$  or  $-Z_a$  with equal probability.

Retrieve the measurement result from  $r_p$  like before.



## References

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