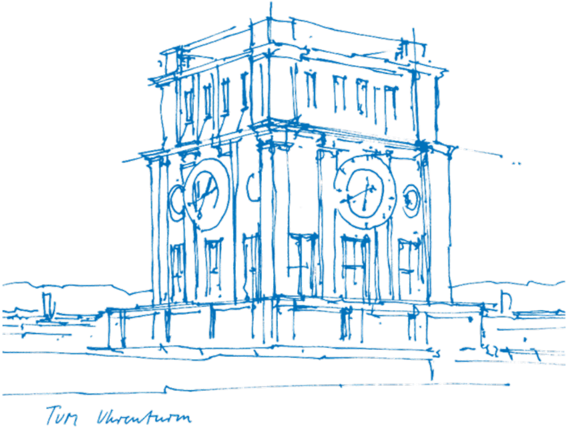
Clifford Tableaus and the Stabilizer Algorithm



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Outline

Pauli Matrixes

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Products of Pauli matrices:

$$I^2 = X^2 = Y^2 = Z^2 = I$$

$$IX = XI = X \qquad IY = YI = Y \qquad IZ = ZI = Z$$

$$XY = iZ \qquad YX = -iZ$$

$$YZ = iX \qquad ZY = -iX$$

$$ZX = iY \qquad XZ = -iY$$

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Group Theory

Group $(G, \cdot)$ is a non-empty set G
with a binary group multiplication operation ”$\cdot$”
with the properties:

- **Closure:** $\forall g_1, g_2 \in G \implies g_1 \cdot g_2 \in G$
- **Associativity:** $\forall g_1, g_2, g_3 \in G \implies g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$
- **Identity:** $\exists e \in G$ such that $\forall g \in G \implies e \cdot g = g \cdot e = g$
- **Inverse:** $\forall g \in G \implies \exists g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Pauli Group

Pauli Group $\mathcal{P}_n$ is defined as the group of n-qubit Pauli operators.
It consists of all tensor products of n Pauli matrices,
together with a phase factor of ± 1 or $\pm i$.

$$\mathcal{P}_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$$

$$\mathcal{P}_n = \left\{ i^m \bigotimes_{j=1}^{n} \sigma_{k_j} \middle| m, k_j \in \{0,1,2,3\}, \sigma_0 = I, \sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z \right\}$$

Size of a Pauli Group: $|\mathcal{P}_n| = 4^{n+1}$

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Pauli Group Operation

Given two Pauli operators $P = i^{m_P} \bigotimes_{j=1}^{n} P_j$ and $Q = i^{m_Q} \bigotimes_{j=1}^{n} Q_j$, their product, as necessitated by Group Definition, is:

$$P \cdot Q = i^{m_P + m_Q} \bigotimes_{j=1}^{n} P_j Q_j$$

P commutes with Q if the number of indices j such that P_j anti-commutes with Q_j is even.

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Group Generators

A set of l elements $\{g_i\}_{1\leq i\leq l}$ generates a group G if every element $g \in G$ can be written as a product of the generators.
In this case, the group G can be written in terms of its generators:

$$G = \langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$$

Examples:

$$\mathcal{P}_1 = \langle X, Z, iI \rangle$$
$$\langle X \rangle = \{I, X\}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Outline

Stabilizer Groups
Definitions

- Element $g \in \mathcal{P}_n$ **stabilizes** $|\psi\rangle$ iff $g|\psi\rangle = |\psi\rangle$.
  $|\psi\rangle$ is eigenstate of g with eigenvalue $+1$.

- $S \widehat{=}$Subgroup of the Pauli Group $\mathcal{P}_n$: $S \subseteq \mathcal{P}_n$.

- $V_S \widehat{=}$Set of n-qubit states stabilized by S:

$$V_S = \{|\psi\rangle \mid S \subseteq \mathcal{P}_n, \forall g \in S \text{ holds: } g|\psi\rangle = |\psi\rangle\}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Groups
Properties

Not just any subgroup S of the Pauli group can be used as the stabilizer for a non-trivial vector space V_S.

Example: $S = \{\pm I, \pm X\}$

$$(-I) \in S \text{ and } (-I)\left|\psi\right\rangle = -\left|\psi\right\rangle \implies \left|\psi\right\rangle = \vec{0} \implies V_S = \left\{\vec{0}\right\} \text{ (trivial)}$$

Conditions for S such that V_S not trivial:

- **Commutativity:** $\forall g_1, g_2 \in S$ holds: $g_1 g_2 = g_2 g_1$
- **Strict Identity:** $-I \notin S$, $iI \notin S$, $-iI \notin S$,

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Conditions

Commutativity Proof

Let V_S be non-trivial.

Let $g_1, g_2 \in S$.

$\implies g_1$ and g_2 are tensor products of Pauli matrices.

$\implies g_1$ and g_2 must either commute or anti-commute.

Suppose g_1 and g_2 anti-commute:

$$|\psi\rangle = g_1 g_2 |\psi\rangle = -g_2 g_1 |\psi\rangle = -|\psi\rangle \iff |\psi\rangle = \vec{0} \implies V_S \text{ is trivial.}$$

$\implies g_1$ and g_2 anti-commuting leads to a contradiction.

$\implies g_1$ and g_2 commute.

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Refernces

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