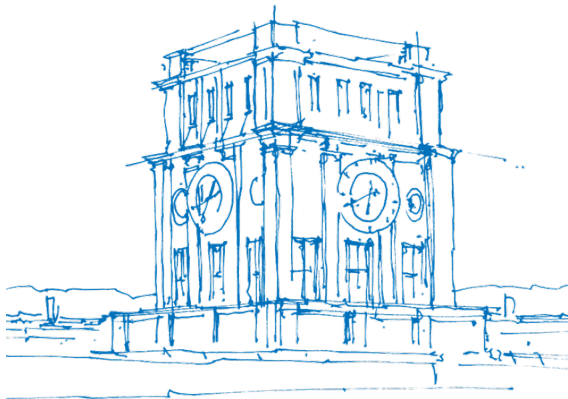


Clifford Tableaus and the Stabilizer Algorithm

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December 20th, 2024



1 Preliminary Definitions

2 Stabilizer Formalism

3 Stabilizer Algorithm

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Products of Pauli matrices:

$$\begin{aligned} I^2 &= X^2 = Y^2 = Z^2 = I \\ IX &= XI = X & IY &= YI = Y & IZ &= ZI = Z \\ XY &= iZ & YX &= -iZ \\ YZ &= iX & ZY &= -iX \\ ZX &= iY & XZ &= -iY \end{aligned}$$

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Pauli Group

Definitions

\mathcal{P}_n is defined as the group of n -qubit Pauli operators.

It consists of all tensor products of n Pauli matrices, with a phase factor ± 1 or $\pm i$.

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$$\mathcal{P}_n = \left\{ i^m \bigotimes_{j=1}^n \sigma_{k_j} \mid m, k_j \in \{0, 1, 2, 3\}, \sigma_0 = I, \sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z \right\}$$

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Size of a Pauli Group: $|\mathcal{P}_n| = 4^{n+1}$

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Pauli Group Operations

Given two Pauli operators $P = i^{m_P} \bigotimes_{j=1}^n P_j$ and $Q = i^{m_Q} \bigotimes_{j=1}^n Q_j$, their product, as necessitated by Group Definition, is:

$$P \cdot Q = i^{m_P+m_Q} \bigotimes_{j=1}^n P_j Q_j$$

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P commutes with Q if the number of indices j such that P_j anti-commutes with Q_j is even.

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Group Generators

A set of l elements $\{g_i\}_{1 \leq i \leq l}$ generates a group G if every element $g \in G$ can be written as a product of the generators.

In this case, the group G can be written in terms of its generators:

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Examples:

$$\begin{aligned} \mathcal{P}_1 &= \langle X, Z, iI \rangle \\ \langle X \rangle &= \{I, X\} \end{aligned}$$

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- $S \triangleq$ Subgroup of the Pauli Group \mathcal{P}_n : $S \subseteq \mathcal{P}_n$.

- $V_S \triangleq$ Set of n -qubit states stabilized by S :

$$V_S = \{|\psi\rangle \mid S \subseteq \mathcal{P}_n, \forall g \in S \text{ holds: } g|\psi\rangle = |\psi\rangle\}$$

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Stabilizer Groups

Properties

Not just any subgroup S of the Pauli group can be used as the stabilizer for a non-trivial vector space V_S .

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■ **Strict Identity:** $-I \notin S, iI \notin S, -iI \notin S$

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Stabilizer Conditions

Commutativity Proof

Let V_S be non-trivial and let $g_1, g_2 \in S$.

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Check Matrix Structure

Suppose $S = \langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$.

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Example:

$$l \left\{ \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right] \right.$$

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Interpretation

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More explicitly, with $h_{i,j}$ denoting the element of H_S at row i and column j :

- If g_i contains I on the j^{th} qubit $\implies h_{i,j} = 0$ and $h_{i,n+j} = 0$.

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- If g_i contains Y on the j^{th} qubit $\implies h_{i,j} = 1$ and $h_{i,n+j} = 1$.

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Check Matrix

Example Steane Code

For Readability tensor product operator signs are left out. $\sigma_i \sigma_j$ corresponds to $\sigma_i \otimes \sigma_j$.

$$\left[\begin{array}{cccccc|cccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right] \cong \begin{array}{|c|c|} \hline \text{Generator} & \text{Operator} \\ \hline g_1 & III XXXX \\ g_2 & IXX I IXX \\ g_3 & XI XI XI X \\ g_4 & III ZZZZ \\ g_5 & IZZ I IZZ \\ g_6 & ZI ZI ZI Z \\ \hline \end{array}$$

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Unitary Operations

Main Revelation

Suppose U is a unitary operator, $|\psi\rangle \in V_S$ and $g \in S$.

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\implies State $U |\psi\rangle$ is stabilized by $U g U^\dagger$.

\implies If we can describe a state by its stabilizers, we can easily compute the stabilizers of the state that emerges from the previous state under a unitary operation.

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Unitary Operations

Advantages for Computation

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(Unkown) State $|\psi\rangle$ stabilized by X .

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→ Apply Hadamard gate H to $|\psi\rangle$.

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Advantages for Computation

For certain special unitary operations U this transformation of the generators takes on a particularly appealing form.

$$HXH^\dagger = Z \quad HYH^\dagger = -Y \quad HZH^\dagger = X$$

Example:

(Unkown) State $|\psi\rangle$ stabilized by X .

→ Apply Hadamard gate H to $|\psi\rangle$.

⇒ Resulting (Unkown) state $|\psi'\rangle$ stabilized by Z .

Unitary Operations

Transformation under Conjugation

Operation	Input	Output
CX	X_1	X_1X_2
	X_2	X_2
	Z_1	Z_1
	Z_2	Z_1Z_2
H	X	Z
	Z	X
S	X	Y
	Z	Z

Operation	Input	Output
X	X	X
	Z	$-Z$
Y	X	$-X$
	Z	$-Z$
Z	X	$-X$
	Z	Z

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Measurement

Main Principles

We want to measure observable $g \in \mathcal{P}_n$ of state $|\psi\rangle$, stabilized by $\langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$.

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Measurement

Deterministic case

g commutes with all g_i and assume g does not have a global phase.

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$$\forall i \text{ holds: } g_i g |\psi\rangle = g g_i |\psi\rangle = g |\psi\rangle \implies g |\psi\rangle \in V_S$$

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In both cases the measurement does not disturb the state of the system, and leaves the stabilizer invariant.

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$\implies g$ only commutes with g_1 .

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Because g has eigenvalues ± 1 , the measurement operators are: $M_{\pm g} = \frac{I \pm g}{2}$

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Measurement

Non-deterministic case continuation

Measurement probabilities:

$$p(+1) = \text{tr} \left(\frac{I + g}{2} |\psi\rangle \langle\psi| \right) \quad \wedge \quad p(-1) = \text{tr} \left(\frac{I - g}{2} |\psi\rangle \langle\psi| \right)$$

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$$p(+1) = p(-1) \text{ and } p(+1) + p(-1) = 1 \implies p(+1) = p(-1) = \frac{1}{2}$$

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- 1 Preliminary Definitions
- 2 Stabilizer Formalism
- 3 Stabilizer Algorithm**

Gottesman–Knill Theorem

Suppose a quantum computation is performed which involves only the following elements:

- State preparations in the computational basis
- Hadamard gates
- Phase gates
- Controlled-NOT gates
- Pauli gates
- Measurements of observables in the Pauli group

Together with the possibility of classical control conditioned on the outcome of such measurements. Such a computation may be efficiently simulated on a classical computer.

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Stabilizer Circuit

Mainly:

Circuit consisting solely of CX , H , S and M gates.

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Can actually also implement X , Y , Z and $SWAP$.

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$$Z = SS$$

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$$SWAP(a, b) = CX(a, b)CX(b, a)CX(a, b)$$

Clifford Tableau Structure

Basically an expanded Check Matrix.

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Clifford Tableau

Structure

Basically an expanded Check Matrix.

$$\left(\begin{array}{ccc|ccc|c} x_{1,1} & \cdots & x_{1,n} & z_{1,1} & \cdots & z_{1,n} & r_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n,1} & \cdots & x_{n,n} & z_{n,1} & \cdots & z_{n,n} & r_n \\ \hline x_{(n+1),1} & \cdots & x_{(n+1),n} & z_{(n+1),1} & \cdots & z_{(n+1),n} & r_{n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{(2n),1} & \cdots & x_{(2n),n} & z_{(2n),1} & \cdots & z_{(2n),n} & r_{2n} \\ \hline x_{(2n+1),1} & \cdots & x_{(2n+1),n} & z_{(2n+1),1} & \cdots & z_{(2n+1),n} & r_{2n+1} \end{array} \right)$$

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Interpretation

Developers of the following algorithm introduce additional n "Destabilizer" generators, which are Pauli operators that together with the stabilizer generators generate the full Pauli group.

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- Rows 1 to n represent Destabilizers.

$x_{1,1}$	\cdots	$x_{1,n}$	$z_{1,1}$	\cdots	$z_{1,n}$	r_1
\vdots	\ddots	\vdots	\vdots	\ddots	\vdots	\vdots
$x_{n,1}$	\cdots	$x_{n,n}$	$z_{n,1}$	\cdots	$z_{n,n}$	r_n
$x_{(n+1),1}$	\cdots	$x_{(n+1),n}$	$z_{(n+1),1}$	\cdots	$z_{(n+1),n}$	r_{n+1}
\vdots	\ddots	\vdots	\vdots	\ddots	\vdots	\vdots
$x_{(2n),1}$	\cdots	$x_{(2n),n}$	$z_{(2n),1}$	\cdots	$z_{(2n),n}$	r_{2n}
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- Row $2n + 1$ is scratch-space.
- r_i of row i represents the global phase,
 $r_i = 0$ for $+1$ and $r_i = 1$ for -1 .

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Pauli- Z gates stabilize $|0\rangle$ states and Pauli- X gates destabilize $|0\rangle$ states.

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Tableau for $|00\rangle$ is

$$\left(\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

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Clifford Tableau

Initial Tableau

Pauli- Z gates stabilize $|0\rangle$ states and Pauli- X gates destabilize $|0\rangle$ states.

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In the following slides, let R_i denote the i -th row of the stabilizer tableau.

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Operation summary: $x'_{i,a} = z_{i,a} \quad \wedge \quad z'_{i,a} = x_{i,a} \quad \wedge \quad r'_i = r_i \oplus (x_{i,a} \cdot z_{i,a})$

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Measurement Algorithm

Deterministic Case

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Compute g in scratch-space row $2n + 1$ and extract r_{2n+1} :

$$r_{2n+1} = 0 \implies g = Z_a \implies \text{Measurement: } 0$$

$$r_{2n+1} = 1 \implies g = -Z_a \implies \text{Measurement: } 1$$

Measurement result is just r_{2n+1} .

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1. Find $p \in [n + 1, 2n]$ such that $x_{p,a} = 1$.

This R_p corresponds to our previous g_1 , which anti-commutes with g .

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After measurement, the stabilizer anti-commuting with g becomes a destabilizer.

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After measurement, the stabilizer anti-commuting with g becomes a destabilizer.

4. Assign $R_p = g$, either Z_a or $-Z_a$ with equal probability.

Retrieve the measurement result from r_p like before.

Simulation of stabilizer circuit runs
in polynomial $O(n^2)$ time complexity.

Measurement and state update during simulation run
in polynomial $O(n^2)$ time complexity.

References

- [1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328.
- [2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010.