Clifford Tableaus and the Stabilizer Algorithm

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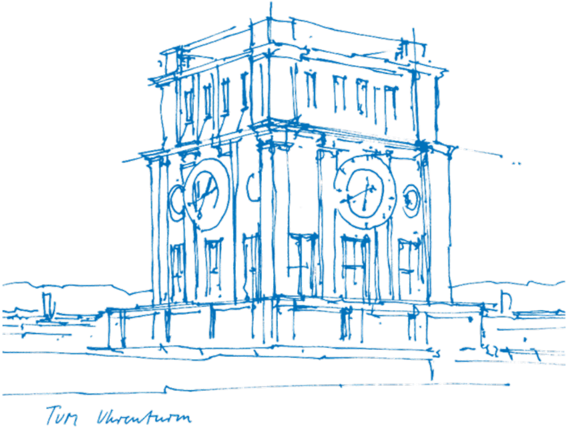

Outline

1. Preliminary Definitions
2. Stabilizer Formalism

Pauli Matrixes

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Products of Pauli matrices:

$$I^2 = X^2 = Y^2 = Z^2 = I$$

$$IX = XI = X \qquad IY = YI = Y \qquad IZ = ZI = Z$$

$$XY = iZ \qquad YX = -iZ$$

$$YZ = iX \qquad ZY = -iX$$

$$ZX = iY \qquad XZ = -iY$$

[1] Scott Aaronson and Daniel Gottesman. “Improved simulation of stabilizer circuits”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Group Theory

Group $(G, \cdot)$ is a non-empty set G
with a binary group multiplication operation $" \cdot "$
with the properties:

- **Closure:** $\forall g_1, g_2 \in G \implies g_1 \cdot g_2 \in G$
- **Associativity:** $\forall g_1, g_2, g_3 \in G \implies g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$
- **Identity:** $\exists e \in G$ such that $\forall g \in G \implies e \cdot g = g \cdot e = g$
- **Inverse:** $\forall g \in G \implies \exists g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Pauli Group
Definitions

$\mathcal{P}_n$ is defined as the group of n-qubit Pauli operators.
It consists of all tensor products of n Pauli matrices,
together with a phase factor of ± 1 or $\pm i$.

$$\mathcal{P}_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$$

$$\mathcal{P}_n = \left\{ i^m \bigotimes_{j=1}^{n} \sigma_{k_j} \,\middle|\, m, k_j \in \{0,1,2,3\}, \sigma_0 = I, \sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z \right\}$$

Size of a Pauli Group: $|\mathcal{P}_n| = 4^{n+1}$

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Pauli Group
Operations

Given two Pauli operators $P = i^{m_P} \bigotimes_{j=1}^{n} P_j$ and $Q = i^{m_Q} \bigotimes_{j=1}^{n} Q_j$, their product, as necessitated by Group Definition, is:

$$P \cdot Q = i^{m_P + m_Q} \bigotimes_{j=1}^{n} P_j Q_j$$

P commutes with Q if the number of indices j such that P_j anti-commutes with Q_j is even.

[1] Scott Aaronson and Daniel Gottesman. "Improved simulation of stabilizer circuits". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 70.5 (2004), p. 052328

Group Generators

A set of l elements $\{g_i\}_{1\leq i\leq l}$ generates a group G if every element $g \in G$ can be written as a product of the generators.
In this case, the group G can be written in terms of its generators:

$$G = \langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$$

Examples:

$$\mathcal{P}_1 = \langle X, Z, iI \rangle$$
$$\langle X \rangle = \{I, X\}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Outline

1 Preliminary Definitions

2 Stabilizer Formalism

Stabilizer Groups
Definitions

- Element $g \in \mathcal{P}_n$ **stabilizes** $|\psi\rangle$ iff $g|\psi\rangle = |\psi\rangle$.
  $|\psi\rangle$ is eigenstate of g with eigenvalue $+1$.

- $S \widehat{=}$Subgroup of the Pauli Group $\mathcal{P}_n$: $S \subseteq \mathcal{P}_n$.

- $V_S \widehat{=}$Set of n-qubit states stabilized by S:

$$V_S = \{|\psi\rangle \mid S \subseteq \mathcal{P}_n, \forall g \in S \text{ holds: } g|\psi\rangle = |\psi\rangle\}$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Groups
Properties

Not just any subgroup S of the Pauli group can be used as the stabilizer for a non-trivial vector space V_S.

Example: $S = \{\pm I, \pm X\}$

$$(-I) \in S \text{ and } (-I)\,|\psi\rangle = -\,|\psi\rangle \implies |\psi\rangle = \vec{0} \implies V_S = \left\{\vec{0}\right\} \text{ (trivial)}$$

Conditions for S such that V_S not trivial:

- **Commutativity:** $\forall g_1, g_2 \in S$ holds: $g_1 g_2 = g_2 g_1$
- **Strict Identity:** $-I \notin S$, $iI \notin S$, $-iI \notin S$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Conditions

Commutativity Proof

Let V_S be non-trivial.

Let $g_1, g_2 \in S$.

$\implies g_1$ and g_2 are tensor products of Pauli matrices.

$\implies g_1$ and g_2 must either commute or anti-commute.

Suppose g_1 and g_2 anti-commute:

$$|\psi\rangle = g_1 g_2 |\psi\rangle = -g_2 g_1 |\psi\rangle = -|\psi\rangle \iff |\psi\rangle = \vec{0} \implies V_S \text{ is trivial.}$$

$\implies g_1$ and g_2 anti-commuting leads to a contradiction.

$\implies g_1$ and g_2 commute.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Stabilizer Conditions

Strict Identity Proof

Let V_S be non-trivial.

$$
\begin{array}{lllll}
(-I) \in S & \implies & |\psi\rangle = (-I)\,|\psi\rangle = -\,|\psi\rangle & \iff & |\psi\rangle = \vec{0} & \implies & V_S \text{ is trivial.} \\
(iI) \in S & \implies & |\psi\rangle = (iI)\,|\psi\rangle = i\,|\psi\rangle & \iff & |\psi\rangle = \vec{0} & \implies & V_S \text{ is trivial.} \\
(-iI) \in S & \implies & |\psi\rangle = (-iI)\,|\psi\rangle = -i\,|\psi\rangle & \iff & |\psi\rangle = \vec{0} & \implies & V_S \text{ is trivial.}
\end{array}
$$

$-I \in S,\ iI \in S,\ -iI \in S$ lead to contradictions.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix
Structure

Suppose $S = \langle g_i \mid i \in \mathbb{N}, 1 \leq i \leq l \rangle$.

Check Matrix $H_S \widehat{=}$ Extremely useful way of presenting the generators

H_S is an $l \times 2n$ binary matrix whose rows correspond to the generators g_1 through g_l.

Example: $$l \left\{ \begin{bmatrix} \underbrace{\begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}}_{n} & \Bigg| & \underbrace{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{matrix}}_{n} \end{bmatrix} \right.$$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix
Interpretation

- Row i corresponds to generator $g_i \in S$.
- Left $l \times n$ submatrix contains 1s to indicate which generators contain Xs.
- Right $l \times n$ submatrix contains 1s to indicate which generators contain Zs.
- Presence of 1 in both submatrices indicates Y in that generator.

More explicitly, with $h_{i,j}$ denoting the element of H_S at row i and column j:

- If g_i contains I on the j^{th} qubit $\implies h_{i,j} = 0$ and $h_{i,n+j} = 0$.
- If g_i contains X on the j^{th} qubit $\implies h_{i,j} = 1$ and $h_{i,n+j} = 0$.
- If g_i contains Z on the j^{th} qubit $\implies h_{i,j} = 0$ and $h_{i,n+j} = 1$.
- If g_i contains Y on the j^{th} qubit $\implies h_{i,j} = 1$ and $h_{i,n+j} = 1$.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

Check Matrix
Example Steane Code

For Readability tensor product operator signs are left out. $\sigma_i\sigma_j$ corresponds to $\sigma_i \otimes \sigma_j$.

$$\left[\begin{array}{ccccccc|ccccccc}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}\right] \widehat{=}$$

Generator	Operator
g_1	$I\,I\,I\,X\,X\,X\,X$
g_2	$I\,X\,X\,I\,I\,X\,X$
g_3	$X\,I\,X\,I\,X\,I\,X$
g_4	$I\,I\,I\,Z\,Z\,Z\,Z$
g_5	$I\,Z\,Z\,I\,I\,Z\,Z$
g_6	$Z\,I\,Z\,I\,Z\,I\,Z$

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010

References

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