1. Wave equation in 1D

The wave equation in one dimension is $\frac{1}{c^2}\frac{\partial^2\psi(x,t)}{\partial t^2} = \frac{\partial^2\psi(x,t)}{\partial x^2}$.

We consider a rubber band of length $L = 1\,\mathrm{m}$. The speed of mechanical waves is assumed to be $c = 20\,\mathrm{m/s}$. The rubber band is fixed on both ends. At $t = 0$ the initial condition is $\psi(x, 0) = 5\sin(4\pi x/L)$. (The band is at rest at this time.)

- We want to discretize the problem. What condition for Δt and Δx has to be fulfilled for the algorithm to be stable? Why?
- Give the relation for $\psi(x, t)$ in terms of discretized quantities.
- To calculate the values $\psi(x_n, t_{n+1})$ the algorithm needs values for $\psi(x_n, t_n)$ and $\psi(x_n, t_{n-1})$. How can you calculate the first step?
- Implement the algorithm and create a 3D plot of $\psi(x, t)$.

2. Wave equation in 2D

We want to generalise our algorithm to two spatial dimensions. The wave equation is modified by adding a term $\partial^2\psi(x, y, t)/\partial y^2$ to the RHS. We consider a square membrane of dimension 20×20 which is fixed at $x = -10, 10$ and $y = -10, 10$. The initial excitation is described by

$$\psi(x, y, t = 0) = \exp\left(-\frac{x^2 + y^2}{50}\right) \sin\left(2\pi\frac{x}{10}\right) \sin\left(2\pi\frac{y}{10}\right) .$$

The speed of mechanical waves is $c = 20\,\mathrm{m/s}$. Now we want to solve the time evolution of this setup on a 200×200 grid for the spatial components.

- Generalize the formula for the 1D case to two spatial dimensions and choose appropriate values for $\Delta x, \Delta y$ and Δt.
- Find a formula for the first time step.
- Solve the wave equation for 500 time steps, plot the final result and make some intermediate plots. (One every 50 time steps).
- * Since a real drum is round we should take this into account.
  Either transform the PDE: Develop the wave equation in polar coordinates, discretize the drum membrane on a (r, ϕ)-grid and determine the temporal evolution of an initial excitation that you may choose yourself (something like $\psi(r, \phi, t = 0) \sim -\exp(-r^2/2R)$). Set $\psi(r = R, \phi, t) = 0$ and impose periodic boundary conditions in ϕ direction. Ensure continuity of ψ at the inner edge of the membrane, i.e. at $r = 0$.
  Or use a rectangular grid: Use your code for the square membrane and implement the new boundary conditions on this grid. (Numerically less accurate.)
- ** Can you hear the shape of a drum? See
  `http://en.wikipedia.org/wiki/Hearing_the_shape_of_a_drum`
  Plot spectra of both types of membranes for comparable initial excitations.