Math 54 Differential Equations

Leo Villani

Summer 2024

1 Introduction and Definitions

Differential equations have many applications in physics, math and engineering. We begin the study of differential equation with soon to be simple looking constant coefficient, linear, homogeneous, second order differential equation.

$$ay''(t) + by'(t) + cy(t) = 0,\ a, b, c \in \mathbb{R},\ a \neq 0$$

The goal is to find a function $y(t) : \mathbb{R} \longrightarrow \mathbb{R}$ that satisfies the differential equation above.
First make note of the following definitions:

- **Constant Coefficient**: a, b, c are constant coefficients in $\mathbb{R}$ (not depending on t)
- **Homogeneous** The equation $= 0$, same definition as in linear systems.
- **Linear** This just means solutions to the above equation form a vector space, closed under addition and multiplication by scalars, and the 0 function always solves this because it is homogeneous.
- **Second Order** The highest derivative is the second derivative.

Note: For a second order differential equation like the one above we only have two real degrees of freedom. Similar to the dimension of a vector space, the solution space basis will only have 2 linearly independent solutions.

2 Solving Homogeneous 2nd Order DEs

To build up some intuition for guessing solutions later on lets think about what could possibly satisfy this differential equation. We need a function that when we add together itself, and its first and second derivatives everything cancels out to 0. In other words we want the derivatives to "look like" the function itself. Three main functions come to mind:

$$\alpha e^{\beta}x, sin\gamma x, cos\delta x$$

As we'll see soon combinations of these functions are exactly the general solutions to the original differential equation.
From now Ill be omitting the t as it is implied but make sure you know that we still are speaking of a function $y(t)$:

$$ay'' + by' + cy = 0, a, b, c \in \mathbb{R}, a \neq 0$$

The main idea in solving this is to set up the **auxiliary equation**. We do this by substituting $y^{(n)}$ with r^n (nth derivative of y becomes r^n). Hence we get

$$ar^2 + br + c = 0$$

Finding r can be done by quadratic formula, factoring, or even completing the square. Depending on $b^2 - 4ac$ (discriminant) for the auxiliary equation we split off into three cases:

- **Two Distinct Real Roots,** $b^2 - 4ac > 0$
  For the two distinct real root case we get auxiliary solution $r = r_1, r_2 \in \mathbb{R}$. The general solution is $c_1 e^{r_1 t} + c_2 e^{r_2 t}$, where $(e^{r_1 t}, e^{r_2 t})$ are the two linear independent solutions forming a basis for the solution space.

- **One Repeated Real Root,** $b^2 - 4ac = 0$
  For the repeated real root case we get auxiliary solution $r \in \mathbb{R}$. The general solution is $c_1 e^{rt} + c_2 t e^{rt}$, where (e^{rt}, te^{rt}) are the two linear independent solutions forming a basis for the solution space.

- **Complex Conjugate Pair Roots,** $b^2 - 4ac < 0$
  For the complex conjugate pair root case we get auxiliary solution $r = \alpha \pm \beta$ for $\alpha, \beta \in \mathbb{R}$. The general solution is $c_1 e^{\alpha t} cos(\beta t) + c_2 e^{\alpha t} sin(\beta t)$, where $(e^{\alpha t} cos(\beta t), e^{\alpha t} sin(\beta t))$ are the two linear independent solutions forming a basis for the solution space. The sin, cos functions may have appeared out of nowhere without knowledge of eulers formula. Go read about the proof if youre curious.

3 Homogeneous Initial Value Problems

We just observed that in general the solution for a second order differential equation ends up looking like:

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

with two linear independent solutions $(y_1(t), y_2(t))$. An **initial value problem**, or **IVP** consists of a differential equation along with two other pieces of information, $y(t_0) = a_0, y'(t_0) = a_1$, to fix what c_1, c_2 have to be.
Existence and Uniqueness: Homogeneous Case Theorem
For any real numbers $a \neq 0, b, c, t_0, Y_0, Y_1$ there exist a unique solution to the IVP:

$$ay'' + by' + cy = 0;\ y(t_0) = Y_0, y'(t_0) = Y_1$$

The solution is valid for all t in $(-\infty, +\infty)$
Note: In particular if a solution $y(t)$ and its derivative vanish simultaneously at a point t_0 then $y(t)$ must be the 0 function.
We need to find a condition on $y_1(t), y_2(t)$ so that it is always possible to find unique c_1, c_2 to achieve the unique solution. For example if $y_1(t)$ is the 0 function then the constants are not necessarily unique. What we look for ends up being linear independence.
Linear Independence of Two Functions
A pair of functions $y_1(t), y_2(t)$ is said to be **linearly independent on the interval I** if and only if neither of them is a constant multiple of the other on all of I. Linear dependence of functions is just if they are constant multiples.
So if $y_1(t), y_2(t)$ are two linearly independent solutions it is always true that we can find unique c_1, c_2 for the unique solution $y(t) = c_1 y_1(t) + c_2 y_2(t)$.
The **Wronskian** is the following determinant which is a function of t:

$$W(y_1, y_2) = det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} = y_1(t) y_2'(t) - y_2(t) y_1'(t)$$

Suppose that $y_1(t), y_2(t)$ solve the same differential equation. Then the Wronskian being the 0 function is the same as $y_1(t), y_2(t)$ being linearly dependent. Hence if there is point $\hat{t}$ where the Wronskian does not vanish then $y_1(t), y_2(t)$ are linearly independent solutions.

4 Non-Homogeneous Case

Our goal now is to solve equations of the form $ay'' + by' + cy = f$ for $f \neq 0$. Recall when solving linear systems $Ax = b$ we can solve for the homogeneous case $Ax = 0$ and find some solution space and then shift this solution space by any singular vector that solves $Ax = b$. This works because a matrix is a linear operator. Turns out we can do the same thing with the derivative linear operator.
Hence, we should solve the homogeneous case and then find a particular solution. Here are the steps:

1. Solve the complimentary equation. $ay'' + by' + cy = 0$ for general complimentary solution $y_c(t) = c_1 y_1(t) + c_2 y_2(t)$.
2. Find any solution to $ay'' + by' + cy = f$ say $y_p(t)$
3. Construct the general solution by adding both together $y(t) = y_c(t) + y_p(t)$

To most common way to find $y_p(t)$ is to use **the method of undetermined coefficients**.
The best way to explain this method is by thinking back on properties of derivatives and how we can make a good educated guess at what kind of functions could even generate f. A combination of the function $y(t)$ and its derivatives must add together to be f so there are only a few realistic guesses for what function $y(t)$ should be. Then we just plug in our guess and see what values make it work. Best shown by an example.

$$y'' + 3y' + 2y = 3t$$

First you should think about what functions derivatives could create a polynomial. Realistically the easiest would be to just use another polynomial. Recall we are looking for a single solution to attach to the complimentary so just think of the easiest case. Lets try $y(t) = At + B$ as our guess. As derivatives decrease the degree of a polynomial, any higher degree polynomial than $3t$ would always be left with a higher degree term on the left.
Plugging in the guess:

$$(At + B)'' + 3(At + B)' + 2(At + b) = 0 + 3A + 2At + 3B = 3t$$

Now matching terms on both sides we know $3A + 3B = 0$ as there is no constant term on the right, and $2A = 3$ to match the coefficients of t on both sides. Hence solving the system we get $A = \frac{3}{2}, B = \frac{-9}{4}$, and $y(t) = \frac{3}{2}t - \frac{9}{4}$ is our particular solution.
One problem you might run into is given in the following example. Lets solve the complimentary case before hand to see why.

$$y'' + y = sin(t)$$

An initial guess would probably be $Asin(t) + Bcos(t)$ as $sin(t)$ and $cos(t)$ are the only functions that could generate and cancel $sin(t)$ terms. But the complimentary case is:

$$y'' + y = 0$$

with $r^2 + 1 = 0$, $r = \pm i$ so $y_c(t) = Asin(t) + Bcos(t)$. This means our guess would end up generating 0 on the right because it is also a homogeneous solution. Hence we boost our guess by a factor of t to avoid collisions:

$$y_p(t) = Atsin(t) + Btcos(t)$$

and find A and B after plugging in. This "boosting" method is used when there is collisions with the complimentary case so always find the complimentary solution first and then "boost" your particular solution guess if you need to.
The general method is summarized in this graphic:

Method of Undetermined Coefficients

To find a particular solution to the differential equation

$$ay'' + by' + cy = Ct^m e^{rt},$$

where m is a nonnegative integer, use the form

(14) $$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt},$$

with

(i) $s = 0$ if r is not a root of the associated auxiliary equation;
(ii) $s = 1$ if r is a simple root of the associated auxiliary equation; and
(iii) $s = 2$ if r is a double root of the associated auxiliary equation.

To find a particular solution to the differential equation

$$ay'' + by' + cy = \begin{cases} Ct^m e^{\alpha t}\cos\beta t \\ \quad\text{or} \\ Ct^m e^{\alpha t}\sin\beta t \end{cases}$$

for $\beta \neq 0$, use the form

(15) $$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{\alpha t}\cos\beta t + t^s(B_m t^m + \cdots + B_1 t + B_0)e^{\alpha t}\sin\beta t,$$

with

(iv) $s = 0$ if $\alpha + i\beta$ is not a root of the associated auxiliary equation; and
(v) $s = 1$ if $\alpha + i\beta$ is a root of the associated auxiliary equation.

5 The Superposition Principle

The Superposition Principle
If y_1 is a solution to the differential equation

$$ay'' + by' + cy = f_1(t)$$

and y_2 is a solution to

$$ay'' + by' + cy = f_2(t)$$

then for any constant k_1, k_2 the function $k_1 y_1 + k_2 y_2$ is a solution to the differential equation

$$ay'' + by' + cy = k_1 f_1(t) + k_2 y_2(t)$$

This principle is the reason why we can "split up" work when solving differential equations. For example for the differential equation

$$y'' + 3y' + 2y = 3t + 10e^{3t}$$

I can find two particular solutions $y_1(t)$, $y_2(t)$ that solve

$$y'' + 3y' + 2y = 3t,\ y'' + 3y' + 2y = 10e^{3t}$$

respectively and then $y_1(t) + y_2(t)$ is my final particular solution.

6 Non-Homogeneous Initial Value Problems

We can also solve IVPs in the nonhomogeneous case. The idea is the same, just set up the general solution and then plug in conditions to solve for c_1, c_2 just now the general solution is of form $y_p(t) + y_c(t) = y_p(t) + c_1y_1(t) + c_2y_2(t)$.
Note: The coefficients determined inside of $y_p(t)$ should not be solved for at this step. That was already done in the method of undetermined coefficients.
Existence and Uniqueness: Nonhomogeneous Case
For any real numbers $a \neq 0, b, c, t_0, Y_0, Y_1$ suppose $y_p(t)$ is a particular solution in an interval I containing t_0 and that $y_1(t), y_2(t)$ are linearly independent solutions to the associated homogeneous equation in I. Then there exists a unique solution in I to the IVP

$$ay''(t) + by'(t) + cy(t) = f(t),\ y(t_0) = Y_0, y'(t_0) = Y_1$$

and it is given by $y_p(t) + c_1y_1(t) + c_2y_2(t)$ for appropriate c_1, c_2.

7 Variation of Parameters

We have seen that the method of undetermined coefficients is a simple procedure to find particular solutions when the equation is constant coefficient and the nonhomogeneous term is within our knowledge to guess. There is a more general method, **variation of paramters**, that helps when the nonhomogeneous term is something we dont know how to deal with yet.
Consider the typical

$$ay'' + by' + cy = f(t)$$

and let $(y_1(t), y_2(t))$ be two linearly independent solutions to the associated homogeneous equation. To find a particular solution to the nonhomogeneous equation we replace the constants in the general complimentary solution with functions that vary, hence the name.

$$y_c(t) = c_1y_1(t) + c_2y_2(t) \longrightarrow y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

Method of Variation of Parameters
To determine a particular solution to $ay'' + by' + cy = f(t)$:

1. Find two linearly independent solutions $(y_1(t), y_2(t))$ to the corresponding homogeneous equation and take $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$

2. Determine $v_1(t)$ and $v_2(t)$ by solving the system below for $v_1'(t)$ and $v_2'(t)$ and integrating

$$\begin{aligned} y_1v_1' + y_2v_2' &= 0 \\ y_1'v_1' + y_2'v_2' &= \tfrac{f(t)}{a} \end{aligned}$$

3. Substitute $v_1(t)$ and $v_2(t)$ into the expression for $y_p(t)$ to obtain a particular solution

8 Matrix Differential Equations

When the equations in the differential system are linear, matrix algebra provides a compact notation for expressing the system. In fact, the notation itself suggests new and elegant ways of characterizing the solution properties.

8.1 Writing a differential system as a Matrix

In general if a system of differential equations is expressed as

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + ... + a_{1n}(t)x_n \\ x_2' &= a_{21}(t)x_1 + ... + a_{2n}(t)x_n \\ &... \\ x_n' &= a_{n1}(t)x_1 + ... + a_{nn}(t)x_n \end{aligned}$$

it is said to be a linear homogeneous system in **normal form**. The matrix formulation of such a system is then

$$x' = Ax$$

where A is the coefficient matrix depending on t and x' is the derivative vector while x is the solution vector.

8.2 Homogeneous Linear Systems with Constant Coefficients

We now discuss a procedure for obtaining a general solution for the homogeneous system

$$x'(t) = Ax(t)$$

where A is a (real) constant nxn matrix. The general solution will be defined for all t as A's elements are not changing.
We might expect like before our solution should have form $x(t) = e^{rt}u$ where r is a constant and u is a constant vector. Substituting in we get

$$re^{rt}u = Ae^{rt}u = e^{rt}Au$$

where after cancelling the e^{rt} factor and rearranging we get

$$(A - rI)u = 0$$

Thus $e^{rt}u$ is a solution if and only if r and u satisfy $(A - rI)u = 0$. This equation should look familiar from the linear algebra section. Its just the eigenvalue/eigenvector definition.
Hence, let $A = [a_{ij}]$ be an nxn constant matrix. The **eigenvalues** of A are those (real or complex) numbers r for which $(A - rI)u = 0$ has at least one nontrivial (real or complex) solution u. The corresponding nontrivial solutions

u are called the **eigenvectors** of A associated with r.
We already know how to do eigenvector/eigenvalue calculations so we can move straight to a theorem.
n **Linearly Independent Eigenvectors Theorem**
Suppose the nxn constant matrix A has n linearly independent eigenvectors $u_1, ..., u_n$. Let r_i be the eigenvalue corresponding to u_i. Then

$$(e^{r_1 t}u_1, ..., e^{r_n t}u_n)$$

is a fundamental solution set on $(-\infty, \infty)$ for the homogeneous system $x' = Ax$. Consequently a general solution of $x' = Ax$ is

$$x(t) = c_1 e^{r_1 t}u_1 + ... + c_n e^{r_n t}u_n$$

where $c_1, ..., c_n$ are arbitrary constants.
Note: if all n-eigenvalues are distinct then the eigenvectors are guaranteed to be linearly independent allowing us to use the above theorem
Also note: for a real symmetric matrix A we are also guaranteed there are n linearly independent eigenvectors corresponding to real eigenvalues

8.3 Complex Eigenvalues

Recall when we have a real constant matrix A, complex eigenvalues come in complex conjugate pairs.
Complex Eigenvalue Theorem
If the real matrix A has complex conjugate eigenvalues $\alpha \pm i\beta$ with corresponding eigenvectors $a \pm ib$, then two linearly independent real vector solutions to $x'(t) = Ax(t)$ are

$$e^{\alpha t}cos(\beta t)a - e^{\alpha t}sin(\beta t)b,\ e^{\alpha t}sin(\beta t)a + e^{\alpha t}cos(\beta t)b$$

9 Fourier Series

Fourier series deals with decomposing a function/frequency into a trigonometric series. Before jumping in we need to review some relevant function properties: piecewise continuity, periodicity, and even and odd symmetry.
A **piecewise continuous function** on $[a, b]$ is a function that is continuous at every point in $[a, b]$ except possibly for a finite number of points at which the function has a **jump discontinuity**. Such functions are necessarily integrable over any finite interval on which they are piecewise continuous.
A function is **periodic of period** T if $f(x+T) = f(x)$ for all x in the domain of f. The smallest possible value of T is called the **fundamental period**. For example $sin(x), cos(x)$ are both periodic functions with fundamental period 2π.
A function f is **even** if it satisfies $f(-x) = f(x)$ for all x in the domain of f. Its graph is symmetric with respect to the y-axis.
A function f is **odd** if it satisfies $f(-x) = -f(x)$ for all x in the domain of f. Its graph is symmetric with respect to the origin.
Note: $sin(x), tan(x)$ are odd functions while $cos(x)$ is an even function

Properties of Symmetric Functions
If f is an even piecewise continuous function on $[-a.a]$ then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

If f is an odd piecewise continuous function on $[-a.a]$ then

$$\int_{-a}^{a} f(x)dx = 0$$

There are a couple of "orthogonality condition" integrals that build up to the main result so I will state them. You can verify the results with some knowledge of period and symmetry.

$$\int_{-L}^{L} sin(\tfrac{m\pi x}{L})cos(\tfrac{n\pi x}{L})dx = 0$$
$$\int_{-L}^{L} sin(\tfrac{m\pi x}{L})sin(\tfrac{n\pi x}{L})dx = 0 \text{ if } m \neq n$$
$$\int_{-L}^{L} sin(\tfrac{m\pi x}{L})sin(\tfrac{n\pi x}{L})dx = L \text{ if } m = n$$
$$\int_{-L}^{L} cos(\tfrac{m\pi x}{L})cos(\tfrac{n\pi x}{L})dx = 0 \text{ if } m \neq n$$
$$\int_{-L}^{L} cos(\tfrac{m\pi x}{L})cos(\tfrac{n\pi x}{L})dx = L \text{ if } m = n \neq 0$$
$$\int_{-L}^{L} cos(\tfrac{m\pi x}{L})cos(\tfrac{n\pi x}{L})dx = 2L \text{ if } m = n = 0$$

Note: if $f_1, ..., f_n$ are periodic with period T then so is any linear combination

$$c_1 f_1 + ... + c_n f_n$$

Just as we can associate a Taylor series with a function that has derivatives of all orders at a fixed point, we can identify a particular trigonometric series with a piecewise continuous function.
Fourier Series
Let f be a piecewise continuous function on the interval $[-L, L]$. The **Fourier Series** of f is the trigonometric series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L})\}$$

where the a_n's and b_n's are given by the formulas

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos(\frac{n\pi x}{L}),\ n = 0, 1, 2, ...$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin(\frac{n\pi x}{L}),\ n = 1, 2, 3, ...$$

The above formulas are called the **Euler-Fourier formulas**. The $\approx$ symbol was used to remind us that this series is associated to $f(x)$ but may not converge to $f(x)$.
Note: The process of trigonemtric decomposition can be generalized to any orthonormal system giving rise to the **generalized Fourier series**
Lets define orthogonality a little better:
A set of functions $\{f_n(x)\}_{n=1}^{\infty}$ is said to be an **orthogonal system** or just **orthogonal** with respect to the nonnegative weight function $w(x)$ on the interval $[a, b]$ if

$$\int_a^b f_m(x)f_n(x)w(x)dx = 0, \text{ whenever } m \neq n$$

If we define the **norm** of f as

$$\|f\| = [\int_a^b f^2(x)w(x)dx]^{\frac{1}{2}}$$

then we say that a set of functions $\{f_n(x)\}_{n=1}^{\infty}$ is an **orthonormal system with respect to** $w(x)$ if the orthogonality condition above holds and also $\|f_n\| = 1$ for each n. Equivelantly we can say the set is an orthonormal system if

$$\int_a^b f_m(x)f_n(x)w(x)dx = 0, \text{ whenever } m \neq n$$
$$\int_a^b f_m(x)f_n(x)w(x)dx = 1, \text{ whenever } m = n$$

Note: just like in the linear algebra section we can obtain an orthonormal system from an orthogonal system by dividing each element by its norm
The **generalized Fourier series** or **orthogonal expansion** seeks to answer whether we can expand a function $f(x)$ in terms of an orthogonal system $\{f_n(x)\}_{n=1}^{\infty}$. To determine the coefficients c_n in an expansion like

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + ...$$

we can derive it similar to how we did with the Euler formulas. If you're curious the formula is this (given certain formal assumptions)

$$c_n = \frac{\int_a^b f(x)f_n(x)w(x)dx}{\|f_n\|^2},\ n = 1, 2, 3, ...$$

Lets now talk about the convergence of Fourier Series:
Pointwise Convergence of Fourier Series
If f and f' are piecewise continuous on $[-L, L]$ then for any x in $(-L, L)$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L})\} = \frac{1}{2}[f(x^+) + f(x^-)]$$

where the a_n's and b_n's are given by the Euler-Fourier formulas. For $x = \pm L$, the series converges to $\frac{1}{2}[f(-L^+) + f(L^-)]$
In other words when f and f' are piecewise continuous on $[-L, L]$, the Fourier series converges to $f(x)$ whenever f is continuous on x and converges to the average of the left and right hand limits at points where f is discontinuous.
Under certain conditions we can have an even stronger kind of convergence:
Uniform Convergence of Fourier Series
Let f be a continuous function on $(-\infty, \infty)$ and periodic of period $2L$. If f' is piecewise continuous on $[-L, L]$ then the Fourier series for f converges uniformly to f on $[-L, L]$ and hence on any interval. That is for each $\epsilon > 0$ there exists an integer N_0 (that depends on ϵ) such that

$$|f(x) - [\tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty}\{a_n cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L})\}]| < \epsilon$$

for all $N \geq N_0$ and all $x \in (-\infty, \infty)$
Differentiation of Fourier Series
Let $f(x)$ be continuous on $(-\infty, \infty)$ and $2L$-periodic. Let $f'(x)$ and $f''(x)$ be piecewise continuous on $[-L, L]$. Then the Fourier series of $f'(x)$ can be obtained from the Fourier series for $f(x)$ by termwise differentiation. In particular if

$$f(x) = \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty}\{a_n cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L})\}$$

then

$$f'(x) \approx \textstyle\sum_{n=1}^{\infty} \frac{\pi n}{L}\{-a_n sin(\frac{n\pi x}{L}) + b_n cos(\frac{n\pi x}{L})\}$$

Termwise integration of a Fourier series is allowed under much weaker conditions.
Integration of Fourier Series
Let $f(x)$ be piecewise continuous on $[-L, L]$ with Fourier series

$$f(x) \approx \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty}\{a_n cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L})\}$$

Then for any x in $[-L, L]$

$$\textstyle\int_{-L}^{x} f(t)dt = \int_{-L}^{x} \frac{a_0}{2}dt + \sum_{n=1}^{\infty}\int_{-L}^{x}\{a_n cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L})\}dt$$

9.1 Fourier Cosine and Sine Series

This section addresses the problem of representing a function defined on some finite interval by a trigonometric series consisting of only sine functions or only cosine functions. Well, the Fourier series for an odd function defined on $[-L, L]$ consists entirely of sine terms, so we can achieve this by extending $f(x)$, $0 < x < L$ to the interval $(-L, L)$ in such a way that the extended function is odd. This extension is called the **odd $2L$-periodic extension** of $f(x)$ and we can similarly define the **even $2L$-periodic extension**.
Fourier Cosine and Sine Series Definitions
Let $f(x)$ be a piecewise continuous on the interval $[0, L]$. The **Fourier cosine series** of $f(x)$ on $[0, L]$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(\frac{n\pi x}{L})$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) cos(\frac{n\pi x}{L}) dx,\ n = 0, 1, ...$$

The **Fourier sine series** of $f(x)$ on $[0, L]$ is

$$\sum_{n=1}^{\infty} b_n sin(\frac{n\pi x}{L})$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) sin(\frac{n\pi x}{L}) dx,\ n = 1, 2, ...$$

Note: these are the Fourier series for the even and odd $2L$-periodic extensions of $f(x)$, also called the **half-range expansions** for $f(x)$.