The Sylow Theorems

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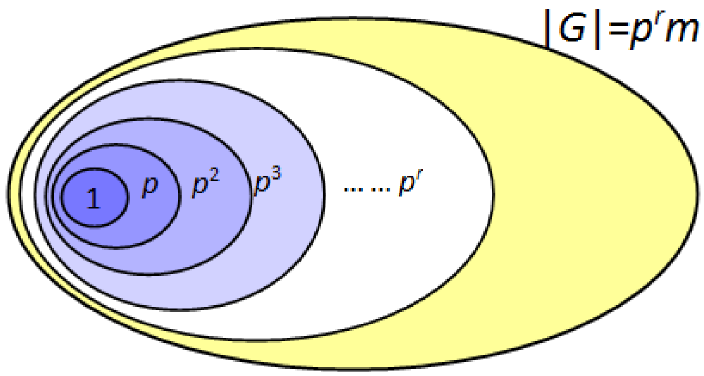
September 11, 2024



1 Introduction

Named after Peter Sylow the Sylow theorems are a collection of theorems that describes subgroups of finite groups. Suppose that p is a prime number, and G is a finite group with mp^r elements, where m is not divisible by p. A p-Sylow subgroup of G is a subgroup with p^r elements. The Sylow theorems describe some properties of such subgroups.

2 Definitions

- A cyclic group is a group generated by a single element
- If p is a prime number, a finite group G is a p-group if $|G| = p^a$ for some $a \epsilon \mathbb{N}$.
- For a prime number p we define the p-Sylow subgroups as the maximal p subgroup of G. The set of all p-Sylow subgroups for a finite group G is denoted $Syl_p(G)$.

3 Comparison to Lagrange's Theorem

Lagrange's Theorem says that for a finite group G, for an arbitrary subgroup of G say H then the order of H divides G. The Sylow Theorems on the other hand say that given a finite group G of order $|G|$ with some prime factorization there exist p-Sylow subgroups for every prime factor of order p^n where p^n is the maximal power of p that divides G. While there exist subgroups of order p^r with $0 <= r <= n$ the p-Sylow subgroups are the largest possible p-groups of G.

4 Theorems

1. p-Sylow subgroups always exist
2. all p-Sylow subgroups of G are conjugate
3. the number of p-Sylow subgroups is congruent to $1 \mod p$

4.1 Corollaries

- Any p-subgroup H of G is contained in a p-Sylow subgroup
- A p-Sylow subgroup of G is normal if and only if it is unique (Proof: being normal $\Leftrightarrow xPx^{-1} = P$ for all x $\Leftrightarrow$ each p-Sylow subgroups is P (by Thm 2))

5 Applications to Classification of Groups

- In general it is difficult to classify finite order groups which includes p-groups. But there is only one group of size p. (By Cauchy's Theorem the group contains an element of order p which thereby generates a subgroup say H of order p, where H $\cong C_p$ (cyclic group of order p)
- If a group G has no repeated prime factors that p-Sylow subgroups must be isomorphic to C_p, which in certain cases can completely determine G.

5.1 Important Theorem

Suppose $N_1, N_2 \trianglelefteq G$ such that $N_1 N_2 = G$ and $N_1 \cap N_2 = \{e\}$ Then $G \cong N_1 \times N_2$ (i). Using this theorem lets look at an example group G with order 15. By the 3rd Sylow Theorem $|Syl_3(G)| = 1$ mod 3. Also by the 2nd Sylow Theorem G acts transitively $Syl_3(G)$ (by conjugation) thus by the Orbit Stabilizer Theorem $|Syl_3(G)|||G|$ so $|Syl_3(G)|$ must be a divisor of 15, either $\{1, 3, 5, 15\}$ so $|Syl_3(G)| = 1$ by modular condition so there is a unique 3-sylow group and thus must be normal in G. Similarly there is a unique and thus normal 5-sylow subgroup. For prime order, groups are isomorphic to the cyclic group of the same order. Thus by (i) any group G after checking their intersection is the identity and their product is the total group we know that any group G of order 15 has: $G \cong P_3 \times P_5$ where P_i denotes the i-Sylow subgroup and as the P's are of prime order they are also isomorphic to the cyclic groups of 3 and 5. Thus for any G with $|G| = 15, G \cong C_3 \times C_5$.

5.2 Road map

Thus as we can see in the example in 5.1, given the order of G we can see subtleties in the group structure:

- $|Syl_p(G)| \equiv 1 \mod p$, but by Orbit Stabilizer Relation $|Syl_p(G)|||G|$
- Hopefully, $|Syl_p(G)| = 1$ which then makes the p-Sylow subgroup normal
- Apply the logic in 5.1

5.3 Abelian Case and Applications

Note that if we were in an abelian group all subgroups are normal as the left and right cosets are the same and thus all p-Sylow subgroups are normal so we don't have to worry about edge cases in the 5.2 Road map. These theorems for finite abelian groups help show that any finite abelian group is the product of its p-Sylow subgroups and p-Sylow subgroups are the product of cyclic p-groups. This leads to a factorization of sorts of groups into cyclic groups. Consequently, the fundamental theorem of finite Abelian groups states that a finite Abelian group is isomorphic to a direct product of cyclic p-groups, where the decomposition is unique up reordering because of 5.1 and the decomposition of p-Sylow groups being unique up to reordering. Warning: While this factorization into cyclic p-groups is unique it is possible that there are other ways to write it using non prime order cyclic groups.

6 Conclusion

In conclusion, the Sylow Theorems who only know about the prime factorization of the order G have uncovered some underlying structure about the group.

References: Math 113 Online Lectures Christopher Ryba