The notion of connection in the twentieth century and the departure from the tangent bundle. *

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1 Introduction

A connection is a mathematical object that makes it possible to define the notion of parallel transport and the differentiation of geometric objects in a curved space. Emerging from the desire to define a differentiation operator invariant under coordinate changes, the notion of connection plays a central role in computing geodesic curves, torsion, and curvature on so-called Riemannian manifolds. Once generalized, connections provide a powerful tool for defining parallelism on spaces of various dimensions and geometries. Connections thus appear at the point of contact between non-Euclidean geometries and infinitesimal calculus, at the birth of differential geometry. The scope of this notion in mathematics (differential geometry, fiber bundles, cohomology, etc.) and in physics (connections of Yang et Mills (1954), of Berry (1984), general relativity (Einstein 1915), etc.) is absolutely enormous, making it an indispensable concept in contemporary geometry.

We will present here how this concept emerged and became structured during the development of modern differential geometry. In particular, we will seek to understand how the notion of connection, initially inseparably linked to the metric — i.e. to the tangent bundle — was able to emancipate itself from it during the twentieth century, allowing the emergence of the general notion of fiber bundles through figures such as Weyl, Cartan, and Ehresmann. We will defend the thesis that this transformation occurred under the influence of a double necessity : a first necessity arising from physical theories, in response to the need to extend general relativity and to generalize gauge theories to the context of the then-nascent quantum mechanics, and a second geometrical necessity, stemming from the need to reconcile two apparently incompatible conceptions of geometric space proposed by Klein and Riemann. As we shall see, the weight of this dual necessity provides a particularly striking case study of the interdependence between developments in physics and mathematics. During this generalization, crucial properties such as curvature and torsion ceased to be properties of spaces themselves and became properties of the connections, thereby further amplifying the importance of this concept.

In Section 2, we will begin with a brief contextual overview presenting the status of geometry within which the notion of connection could emerge at the end of the nineteenth century. Then, in Section 3, we will present the genesis of the notion of connection within the framework of Riemannian geometry, from the introduction of Christoffel symbols in 1869 to Levi-Civita's notion of parallel transport in 1917. In Section 4, which constitutes the main contribution of this essay, we will describe how this notion was generalized during the twentieth century. Finally, we will conclude in Section 5. Additional definitions that may assist the reader are provided in Appendix A, followed by a formal presentation of the different contemporary approaches to defining connections in Appendix B. Finally, several elements of H. Weyl's Weltgeometrie are presented in Appendix C. I already cite here the valuable articles of Bourguignon (1992) and Freeman (2011), which served as secondary sources for the structure of the present essay.

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2 The status of geometry up to the end of the nineteenth century : the path of geodesics

The notion of connection naturally appears within the framework of differential geometry, representing the union of differential calculus and so-called non-Euclidean geometry. As we will briefly argue here, this encounter took place beginning in the eighteenth century, partly as a result of the scientific community's interest in the study of extremal curves (minimizing and maximizing), and in particular those curves that extremize distance on surfaces, called geodesics [1] . At the end of this intellectual journey, connections would emerge as the ultimate differential operator enabling the derivation of the geodesic equation on surfaces and, more generally, in all types of spaces.

At the end of the seventeenth century, the powerful differential calculus appeared as a new tool allowing global results about functions and curves to be derived from the study of their local properties (i.e. on infinitesimal elements). The introduction of such a tool is due to Leibniz (1684) and Newton (1669), notably in response to needs arising from mechanics. In their developments, the two authors proposed approaches that differed greatly both technically and conceptually, which would later be shown to be equivalent [2].

With the aim of demonstrating the full power of differential calculus, Jean Bernoulli (1696), a friend and mentor of Leibniz, challenged the community of geometers to determine the mechanical trajectory corresponding to the shortest time for a point mass, subject only to its own weight, to travel between two points (the brachistochrone problem). Such a challenge indeed provided an ideal testing ground for the application of differential calculus methods, attracting experts in the field such as Leibniz and Newton, as well as Johann Bernoulli, Jean's brother, the Marquis de l'Hôpital, and Ehrenfried Walther von Tschirnhaus. In a context of rivalry between the Bernoulli brothers, challenges concerning curves with special properties followed one another, notably the study of isoperimetric curves, proposed as a challenge by Jacques Bernoulli (1697a) to his brother and the origin of numerous public disputes. That same year, Bernoulli (1697b) challenged the community to determine the geodesic curves on arbitrary surfaces. The differential equation providing the solution would only be published more than thirty years later, independently by Euler (1732), at the request of his mentor Johann Bernoulli, and by Clairaut (1733). To derive his solution, Euler was the first to characterize a general surface by a differential expression of the form $\mathrm{d}S(x, y, z) = 0$ in a coordinate system (x, y, z), anticipating later developments in which surfaces would be characterized by quadratic forms (metrics) [3]. It then gradually became clear that geodesics form special curves allowing the properties of the surfaces themselves to be studied. Gauss (1828) used these curves to study and define the notion of curvature and emphasized the intrinsic character of this quantity [4]. Gauss's investigations of surfaces represented a genuine conceptual turning point, foreshadowing most of the modern tools of differential geometry, such as the differential linear element of length $\mathrm{d}s$.

Following Gauss, the extensive study of Euclid's fifth postulate during the nineteenth century gradually led mathematicians to conceive alternative geometries, known as "non-Euclidean." Gauss was among the first to recognize the significance of these geometries, which he called "anti-Euclidean," but he refrained from publishing on the subject for fear of controversy. The Russian and Hungarian mathematicians Nikolai Lobachevsky and János Bolyai studied spaces with particular properties in 1829–1830 and 1832 respectively, developing what would later become known as hyperbolic geometry.

In parallel, the methods of differential calculus were progressively refined, notably through the δ-calculus proposed by Lagrange (1806) and extensively and systematically studied by Euler [5]. These new tools were further elevated by the mechanical studies proposed by William Rowan Hamilton (1805–1865), Joseph Liouville (1809–1882), and Gaston Darboux (1842–1917), among others. They

1. I had the opportunity to discuss the first stages of the modern study of geodesic curves in the short essay available here.
2. For further details on the history of differential calculus, see Boyer (1959) and Freguglia et Giaquinta (2016).
3. For a complete history of geodesic curves, see in particular Eneström (1899).
4. On this topic, see the analysis of Nabonnand (1995).
5. On this subject, see Woodhouse (1810) and Carathéodory (1937).

gave rise to modern analytical mechanics, in which geodesics and minimizing curves still play a fundamental role, since they characterize the trajectories of physical systems [6] (now referred to as Lagrangian and Hamiltonian mechanics).

The convergence of all these developments led Riemann (1854) to introduce the notion of an "n-fach ausgedehnte Mannigfaltigkeit," today called an n-dimensional manifold, during his habilitation lecture, at the request of Gauss, who had previously supervised his doctoral work (the manuscript would only be published after Riemann's death in 1868). This work, highly conceptual and almost devoid of formulas, provided the foundations for generalizing Gauss's work on surfaces to arbitrary dimensions. Riemann partially formalized these ideas in a later unfinished essay in Latin dealing with thermodynamic considerations, written in response to a competition organized by the Paris Academy of Sciences (Riemann 1861). In this new framework, distances on the manifold are given by a quadratic form $\mathrm{d}s^2$ generalizing Gauss's linear element. $\mathrm{d}s^2$ would later be expressed through the metric tensor g, from which curvature can be defined [7]. As we shall see, it is within this context that the notion of connection emerged to account for differentiation and subsequently parallelism on Riemannian manifolds.

3 The genesis of connections, from Riemann to Levi-Civita

3.1 Christoffel 1869

After their introduction, Riemann's works remained little recognized, except by his doctoral advisor Gauss. Gradually, however, the formal study of the new geometrical framework he proposed began to take shape, and many mathematicians produced substantial developments, such as Lamé, Beltrami, and Helmholtz. Major mathematical advances were also brought about by developments in physics, such as the formulation of electromagnetism proposed by James Clerk Maxwell [8]. In this context, Erwin Bruno Christoffel (1869) and Rudolf Lipschitz (1869) studied the transformation properties of quadratic forms under coordinate changes. Christoffel considers the relation (Eq. 1 p.47)

$$\sum \omega_{ik}\partial x_i \partial x_k = \sum \omega'_{ik}\partial x'_i \partial x_k, \tag{1}$$

expressing the invariance of the differential form ω under a change of coordinate system $f : x \to x'$. One may typically think of the particularly interesting case where $\omega = g$ is the metric. Essentially, Christoffel seeks here to construct covariant expressions (which do not change form under coordinate transformations). By studying which constraints must be satisfied by the derivatives of the transformation $f(x_i)$ in order to fulfill the condition given by equation (1), Christoffel introduces, for convenience, the object $\left[\begin{smallmatrix} ij \\ k \end{smallmatrix}\right]$, later called the "Christoffel symbols." It is defined in terms of derivatives of ω as $\left[\begin{smallmatrix} ij \\ k \end{smallmatrix}\right] = \omega^{kl}(\partial_i\omega_{jl} + \partial_j\omega_{il} - \partial_l\omega_{ji})/2$ (Eq. 4 p.48). In the case where $\omega = g$, those familiar with Riemannian geometry will recognize in $\left[\begin{smallmatrix} ij \\ k \end{smallmatrix}\right]$ the expression of the coefficients of the Levi-Civita connection ${\Gamma^k}_{ij}$ [9] (for a definition see Appendix B).

However, $\left[\begin{smallmatrix} ij \\ k \end{smallmatrix}\right]$ is not yet interpreted as being related to the notion of parallel transport, and Christoffel introduces this coefficient solely in the search for an algebraic reduction of differential invariants [10].

6. In Lagrangian mechanics, system trajectories extremize the action, while in Hamiltonian mechanics they correspond to forms of geodesics on phase space viewed as a symplectic manifold.

7. The curvature tensor already appears in Riemann's writings as a second-order term in the expression of $\mathrm{d}s^2$. However, it is important to understand that the notion of a tensor did not yet exist, and one instead spoke of "symbols." For more details on the emergence of the curvature tensor, see Darrigol (2015).

8. For a complete history of mathematics following Riemann, see Part II of Struik (1933) and the very comprehensive exposition of Spivak (1999), which also includes an analysis of the papers of Gauss and Riemann

9. As argued in this discussion, the notation ${\Gamma^i}_{jk}$ appears in Einstein (1914) (p.1058) Einstein (1915) (p. 783). The choice of the letter gamma may be associated with the word "gravitation." During the twentieth century, Christoffel's bracket notations $\left[\begin{smallmatrix} ij \\ k \end{smallmatrix}\right]$ and $\left\{\begin{smallmatrix} ij \\ k \end{smallmatrix}\right\}$ were progressively replaced in the literature by ${\Gamma^i}_{jk}$.

10. There is in fact absolutely no discussion of differential calculus on manifolds in Christoffel's works, which rely

3.2 Ricci-Curbastro 1901

One must wait for the developments of Gregorio Ricci-Curbastro (Ricci 1887 ; Ricci 1888), followed by the central manuscript co-written with his student Tullio Levi-Civita : "Méthodes de calcul différentiel absolu et leurs applications" (Ricci et Levi-Civita 1901). The essay is striking for its clear and modern style and is considered the first to have formally introduced the notion of the tensor [12] and the distinction between covariant and contravariant quantities.

In their manuscript, Ricci and Levi-Civita build upon the invariance properties studied by Christoffel [13] to define a differentiation operator on Riemannian manifolds that is invariant under coordinate changes. For the particular case of a vector $X_r = \partial X / \partial x_r$, where X is a function and x_r is a coordinate, the authors define the covariant derivative of X_r with respect to x_s according to the "forme fondamentale" (the metric) as (Eq. 19 to 19"' p.138)

$$X_{rs} = \frac{\partial X_r}{\partial x_s} - \sum_{q=1}^{n} \left\{ {rs \atop q} \right\} X_q. \tag{2}$$

Thus, it is shown that the object Xrs, today commonly denoted $\nabla_s X_r$, can be understood as a derivative of the vector X_r in the direction of a vector pointing along x_s, whose value is invariant under a change of reference frame and preserves the form of the metric. A major conceptual leap takes place here : the notions of differential calculus are extended to geometric objects such as vectors and tensors in a curved space. The entire purpose of the "calcul differentiel absolu" is therefore to define derivatives that are independent of the choice of coordinates, motivated by the desire to reason about geometric objects whose nature is independent of an arbitrary choice of basis in which they are expressed.

In Ricci et Levi-Civita (1901), the crucial link is also established with geodesic curves and the trajectories of Lagrangian analytical mechanics, which are understood as curves whose covariant derivative $\nabla_u u$ of the tangent vector u vanishes at every point (see the discussion in Chapter V, p.178). This therefore marks the birth of the notion of covariant differentiation, and consequently that of connection.

However, as Levi-Civita himself would later declare, "for many more years it was used almost exclusively by its inventor and a few of his students," until in general relativity "Ricci's calculus revealed itself to be not only useful but truly indispensable" [14]. Indeed, the formalism proposed by Ricci and Levi-Civita would prove to be an indispensable tool for the development of the theory of general relativity by Albert Einstein (1915), thereby representing a tremendous success for absolute differential calculus. This physical theory describes the force of gravity as the manifestation of a curved spacetime (modeled by a Riemannian manifold [15] in four dimensions) on which objects move along geodesic motions. However, according to Bourguignon (1992), this constituted a "missed opportunity" for connections, since they were neither geometrically interpreted nor even studied

exclusively on algebraic methods (in contrast to Lipschitz (1869), who uses an approach based on the calculus of variations). The study of invariance properties of quadratic forms under changes of reference frame was then a central topic of the period (particularly in Italy) and had already been initiated by other mathematicians such as Casorati (1862) [11].

12. One may argue that such tools had already been developed, at least partially, with the exterior calculus of Grassmann (1844).

13. The authors are very explicit on this point, and it seems important to quote the following passage : "M. Christoffel a remarqué le premier que si un système d'ordre m, $X_{r_1,r_2\ldots r_m}$ est covariant, le système d'ordre $m+1$

$$X_{r_1,r_2\ldots r_m,r_{m+1}} = \frac{\partial X_{r_1,r_2\ldots r_m}}{\partial x_{r_{m+1}}} - \sum_{l=1}^{m}\sum_{q=1}^{n} \left\{ {r_l r_{m+1} \atop q} \right\} X_{r_1,r_2\ldots r_{l-1} q r_{l+1}\ldots r_m}$$

est aussi covariant. Nous appelons dérivation covariante [...] l'opération par laquelle, cette forme aidant, on passe d'un système donné $X_{r_1,r_2\ldots r_m}$ au système $X_{r_1,r_2\ldots r_m,r_{m+1}}$" (Ricci et Levi-Civita (1901), p.138).

14. Quotations taken from Freeman (2011).

15. More precisely, a "pseudo-Riemannian" manifold, since the scalar product defined by the metric at each point is not positive-definite.

individually : there existed only one way to differentiate objects on spacetime, through the use of Christoffel symbols (Eq. 2), and this was above all an algebraic rather than a geometric operation.

3.3 Levi-Civita 1917

The next major step was achieved by Levi-Civita (1917) [16], who proposed a geometric interpretation of the formal objects of Ricci's calculus associated with Riemannian geometry [17]. He arrived at such results while originally seeking to simplify the complexity of the derivations associated with absolute differential calculus. Levi-Civita begins his essay by attempting to generalize "ordinary parallelism"—defined in Euclidean space—onto a Riemannian manifold V_n. To do so, he expresses the condition of ordinary parallelism in Euclidean space between two directions α and α' located at two infinitesimally close points P and P' as the preservation of angles $\widehat{(f)(\alpha)} = \widehat{(f)(\alpha')}\ \forall f$. The challenge is then to require that this condition be satisfied for vectors belonging to the tangent space of V_n at P and P' ($\alpha, f \in TV_n$ [18]).

Denoting by ξ^i the coordinates of a vector defining the direction α at P, by $\xi^i + \mathrm{d}\xi^i$ those of a vector associated with the direction α' at P', and by $\mathrm{d}x_j$ the infinitesimal distance PP', Levi-Civita shows that the condition of parallelism becomes (Eq. A p. 174)

$$\mathrm{d}\xi^{(i)} + \sum_{jl}^{n} \left\{ {jl \atop i} \right\} \mathrm{d}x_j \xi^{(l)} = 0. \tag{3}$$

Considering then a parametric curve $x_j(s)$ connecting P and P', equation (3) is equivalent to the vanishing of the covariant derivative $\nabla_v \xi = 0$, where $v = (\mathrm{d}x^j/\mathrm{d}s), \partial_j$ is the tangent vector to $x(s)$ (in modern notation). This allowed Levi-Civita to define the "parallel transport" of the vector ξ in the direction v, corresponding to the rolling without slipping of the vector along the manifold, and thus to understand that it is the connection that defines parallelism on a Riemannian manifold through the covariant derivative.

By considering that it is always possible to immerse the manifold V_n into a Euclidean space S_N such that $N > n$ [19], Levi-Civita understood that the covariant derivative corresponds to the orthogonal projection of the variation of the vector in S_N, that is $\nabla_u u = P_{\gamma(\tau)} \frac{\mathrm{d}u}{\mathrm{d}t}$ where $P : S_N \to TV_n$ is the orthogonal projector from the ambient Euclidean space onto the tangent space [20], thus the vector is differentiated in the ambient space and then "pressed" onto the manifold as rigidly as possible, corresponding to the experience of moving an object as parallel as possible along a curved surface. The geodesic equation $\nabla_u u = 0$ is therefore understood as stating that "the velocity u of a geodesic is transported as parallel as possible along the curve."

This final development therefore represents a major step, providing a geometric visualization of ∇ and gradually leading to thinking of connections as objects in their own right. Building on these results, Levi-Civita also gave a geometric interpretation of the curvature tensor. By considering a geodesic parallelogram $PQQ'P'$, he showed that the curvature tensor allows one to compute the difference in length $P'Q'^2 - PQ^2$.

16. See the English translation in Godina (2022).
17. Similar developments were also undertaken independently by G. Hessenberg and J.A. Schouten (Reich 1992 ; Cogliati 2016).
18. The angle $\widehat{(u)(v)}$ between two vectors u and v is defined by the metric g in each tangent space as $\cos\left[\widehat{(u)(v)}\right] = g(u,v)/g(u,u)/g(v,v)$.
19. This observation is in fact Levi-Civita's starting point, and he uses the fact that all vectors of TV_n are also vectors of S_N to derive equation (3).
20. For a detailed analysis of the covariant derivative defined in this way, see the excellent lecture notes of Faure (2021a).

4 Connections in the Twentieth Century and the Departure from the Tangent Bundle

The notion of connection introduced so far is inseparably defined and linked to the metric and thus to the geometric properties of the space in which it is defined. It makes it possible to compute geodesics and curvature and — as Levi-Civita showed — defines a notion of parallel transport of geometric objects consistent with the intuition of transporting an object (e.g. a vector) as "rigidly" as possible through space without inducing additional rotation. As crucial as this concept is, it is nevertheless today understood as a particular case of a connection, called the Levi-Civita connection, defined on the tangent bundle (where vectors and tensors live). A major conceptual leap then occurred during the twentieth century, making it possible to define parallelism and parallel transport for more abstract objects. As we aim to argue here, this transition took place under the influence of two independent necessities : a need motivated by the physical description of interactions beyond Einsteinian gravity and the desire to unify the geometry proposed by Riemann with that proposed by Klein.

4.1 The Necessity of Physics : From Weyl to Yang–Mills

The first two decades of the twentieth century witnessed the success of general relativity and the beginnings of quantum physics. It is within this new framework that connections more general than the Levi-Civita connection appeared, first in a failed attempt to extend general relativity to include electromagnetism, before being imposed by the formalism of relativistic quantum mechanics and particle physics. The German mathematician and physicist Hermann Weyl (1918c) was the first to formally define affine connections independently of the metric, thereby freeing the concept from the unique Levi-Civita connection studied until then and allowing the consideration of other possible connections, turning them into mathematical objects in their own right. In this crucial essay, he was also the first to call the object associated with the Christoffel symbols a "Zusammenhang," which may be translated as "connection." Formally, Weyl realized that the notion of parallelism is not contained in the manifold (M, g) alone, but requires the addition of an extra structure Γ defining parallelism [21], i.e. it is possible to define Γ independently of g, of which the Levi-Civita connection is only a particular case [22] [23].

Immediately following these first mathematical reflections, Weyl provided an example of an alternative affine connection in Weyl (1918a) and later in the book Weyl (1918b) [24], while attempting to unify electromagnetism and gravitation within a single unified field theory : Weltgeometrie. Weyl's original motivation stemmed from the fact that the use of the Levi-Civita affine connection Γ to describe spacetime appeared unnatural both mathematically — because it allows a change in the direction of vectors under parallel transport but not in their length, and is therefore not the most general case — and physically, because it allows lengths to be defined absolutely throughout spacetime, whereas the physical principles guiding the emergence of special and general relativity favored a fully local theory [25].

As detailed in Appendix C, Weyl then constructed his theory by invoking a general form of invariance of spacetime properties, called "Massstab Invarianz" and later "Eich Invarianz" [26], which would become the modern precursor of gauge invariance and would guide many subsequent developments in physics (O'Raifeartaigh et Straumann 1998 ; Jackson et Okun 2001). In addition to

21. On this subject see also the supplements associated with Bell et Korté (2016).

22. During these same mathematical developments, Weyl also introduced C, today known as the "Weyl tensor," an indispensable tool for classifying different types of spacetime in general relativity.

23. Shortly afterward, other mathematicians such as Schouten (1922) undertook the generalization of the notion of parallelism proposed by Levi-Civita (for a complete discussion see e.g. Struik (1933) and Cogliati (2016)).

24. The book discusses Weltgeometrie only starting from its third edition in 1921.

25. Namely, that spacetime points are maximally independent from one another and that information propagates locally from point to point at speeds below that of light.

26. This name was suggested following exchanges with Einstein (Scholz 2004).

changes of reference frames in spacetime, the theory is required to remain invariant under the joint transformation [27] $g \to e^{\Lambda} g$ and $A = A - \mathrm{d}\Lambda$, where $\Lambda(x)$ is an arbitrary function on spacetime. The transformation of g (today called a conformal transformation) expresses the impossibility of defining lengths absolutely throughout spacetime while preserving angles, and the transformation of A corresponds to the classical gauge transformation of the electromagnetic potential. Before being identified with the electromagnetic potential, A was originally introduced into the theory in order to uniquely define the parallel transport of objects through a new affine connection (the so-called "Weyl connection") involving both the Levi-Civita connection Γ and the electromagnetic potential A. Whereas Γ induces rotations of vectors under parallel transport, the contribution of A can modify their length. Thus, A alone constitutes a new type of connection introduced by Weyl, governing the parallel transport of lengths. From a physical standpoint, the electromagnetic field, like the gravitational field, thereby becomes a property of spacetime itself (and influences measurements of time and space). The proposal of Weltgeometrie was therefore particularly revolutionary, as it predates both the formalization of quantum mechanics and the modern understanding of electromagnetism as a gauge field.

Einstein described Weltgeometrie as "a coup of genius of the first rate..." but had to admit to Weyl that "although your idea is so beautiful, I have to declare frankly that, in my opinion, it is impossible that the theory corresponds to nature" [28]. Indeed, Einstein observed that, since the length of objects and the time measured by clocks depend on their trajectory in spacetime within Weltgeometrie, the frequency of atomic clocks should depend on the position and past history of atoms, in blatant contradiction with experiment. Weyl nevertheless continued to defend his theory vigorously for more than ten years, motivated by its mathematical beauty. The advent of quantum mechanics, which allowed the selection of a universal unit of length through $\hbar$, eventually convinced Weyl that length could indeed be chosen globally, leading him to abandon Weltgeometrie in the early 1920s. He then focused on purely mathematical studies until 1926.

However, Weyl's ideas were widely reused in physics, notably in later developments by Kaluza and Klein, in which electromagnetism emerges from a fifth dimension of spacetime (Kaluza 1921) [29], but also in the search for conformal (or Weyl-invariant) theories in quantum field theory, cosmology, and string theory [30].

From the perspective of connections, Weyl's first contribution is therefore substantial : on the one hand, he isolated the connection as an object in its own right, central to both physics and mathematics, and, motivated by a physical theory, he provided an example of a connection alternative to the Levi-Civita connection.

In the mid-1920s, the advent of the modern formalization of quantum mechanics once again profoundly transformed the role played by connections in physics and greatly generalized them, although physicists were not yet aware of this. Schrödinger (1922) noticed that the theory proposed by Weyl could describe not the metric, but the properties of the electron wavefunction, provided that the proposed gauge transformation were given by a complex phase (rather than a real one as proposed by Weyl). Similar considerations were later carried out by Fock (1926) and London (1927) [31]. As Schrödinger (1926) developed his reformulation of quantum mechanics in terms of wavefunctions through his eponymous equation, it became increasingly clear to physicists that electromagnetism enters quantum theory through the momentum operator $p_\mu = \partial_\mu - ieA_\mu$.

A major turning point occurred when Dirac (1928) demonstrated that, in order to comply with special relativity [32], the wavefunction describing the electron must have four components (it is a

27. Here written in modern notation.

28. Quotations taken from Straumann (1996) (p.1).

29. For a formal introduction see e.g. Coquereaux et Jadczyk (1988).

30. For a discussion of the modern resurgence of Weyl's theory, see e.g. Sanomiya et al. (2020), Farnsworth, Luty et Prilepina (2017), Barceló, Carballo-Rubio et Garay (2018), Scholz (2018) et Scholz (2020).

31. On this subject, see the complete discussion given by Yang (1987).

32. The situation is somewhat more complex, since Klein (1926) and Gordon (1926) had already proposed a relativistic Schrödinger equation for a scalar wavefunction. Dirac sought instead a relativistic equation that would be linear.

spinor[33]). Weyl then proposed the following year a generalization of the Levi-Civita connection to spinors[34] [35] (Weyl 1929b ; Weyl 1929a). Connections were thus extended to act on objects not living on the tangent bundle (vectors and tensors) but on complex spaces (associated with the tangent bundle) on which alternative representations of the rotation group act.

This development reached its culmination when, at the end of the paper, Weyl crucially demonstrated that the electromagnetic interaction can be derived by requiring the theory to respect a gauge symmetry. Indeed, once a reference frame is fixed on spacetime, Weyl observed that there remains a freedom in the definition of the Dirac field ψ leaving the theory invariant under the transformation $\psi \to e^{i\Lambda}\psi$. Weyl argued that, just as one may make an arbitrary choice of reference frame at each point of spacetime, one should be able to do the same with the choice of the spinor phase, and Λ must therefore be a function of x. To accommodate this choice and preserve the invariance of the term $\partial_\mu \psi$ in the Dirac equation[36], it becomes necessary to introduce a new field A into the theory, entering the momentum $\partial_\mu - ieA_\mu$ and transforming as $A_\mu \to A_\mu - \partial_\mu \Lambda$ under a phase change of ψ (again called a gauge transformation here). A can then be identified as the electromagnetic field. One thus recovers a structure similar to that of Weltgeometrie, where ψ plays the role of g and the exponential of the conformal transformation contains an additional imaginary factor. Using the theorem of Noether (1918), Weyl also associated gauge invariance with conservation of electric charge, as he had already attempted to do in 1918[37].

In light of these new results, Weyl remained cautious after the disappointment of his first great unification attempt, and the visibility of his work remained limited compared to the magnitude of his achievement[38]. Moreover, although Weyl unquestionably recognized that A is an affine connection in his 1918 theory, he does not appear to identify A again as a connection in 1929, whose F would be the curvature (see discussion in Section 4.3). He thus unknowingly took a giant step beyond the tangent bundle in the theory of connections.

The procedure employed by Weyl to generate an interaction from a gauge symmetry was slowly assimilated by physicists and later reused to model all known fundamental interactions. Yang et Mills (1954) proposed a similar modeling of the strong interaction with a gauge transformation acting on the proton/neutron doublet through the non-Abelian group SU(2), modeling isospin symmetry[39]. This same model was later independently adapted by Glashow (1961), Weinberg (1967) et Salam (1969) to propose the existence of the electroweak interaction unifying electromagnetic and weak nuclear forces within a single formalism, and also to model the strong interaction through the color symmetry of a quark triplet under the action of the group SU(3) (Gell-Mann 1964). However, the weak interaction appears to violate gauge symmetry due to the non-zero mass of the vector bosons W. It is therefore remarkable that, in order to preserve the gauge principle introduced and defended by Weyl, the existence of a new scalar field — the Higgs boson — was independently predicted in 1964 by several physicists (Englert et Brout 1964 ; Higgs 1964 ; Guralnik, Hagen et Kibble 1964). It was finally detected at the Large Hadron Collider (LHC) in 2012 (CMS Collaboration 2012 ; Atlas Collaboration 2012).

The status of gauge theories today is that physical matter fields ψ live in fiber bundles asso-

33. An object already discovered in a geometrical context by Cartan (1913), who will be one of the main figures of the next section.

34. Fock (1929) proposed a similar development. For a comparison of the approaches of Weyl and Fock, see Scholz (2004).

35. Since Dirac spinors transform under the $(1/2, 1/2)$ representation of the Lorentz group O(1,3), Weyl had to introduce tetrads or moving frames extensively studied by Cartan in order to define the connection (see next section).

36. For definitions, see Appendix A.

37. For a very enlightening discussion on the history of the links between gauge transformations and charge conservation, see Brading (2002).

38. As noted by Yang (1987), generalizing Einstein's objection to Weltgeometrie to Weyl's new gauge theory predicts the possibility of measuring (e.g. through interference) a phase difference between electrons having followed different trajectories. This is in fact the prediction of the well-known effect presented by Aharonov et Bohm (1959) and later observed by Tonomura et al. (1982).

39. It does not appear that Yang and Mills were aware of Weyl's work and instead cite a 1941 paper by Pauli for gauge theories.

ciated with structure groups G=U(1) for electromagnetism, G=SU(2) for the weak interaction, and G=SU(3) for the strong interaction, while the connections A are the mediator boson fields (respectively photons, the $W^{\pm}$ and Z bosons, and gluons) living in the adjoint bundle. Gauge transformations then simply express invariance under coordinate changes in the fiber. However, it seems clear that, until the 1970s, few physicists explicitly identified gauge fields with connections in the same sense as the Levi-Civita connection on spacetime (see again the discussion in Sec. 4.3).

To summarize, we quote Dyson (1983) : "So the story of gauge fields is full of ironies. A fashionable idea, invented for a purpose which turns out to be ephemeral, survives a long period of obscurity and emerges finally as a corner-stone of physics," thus illustrating the irony that Weyl first triumphantly introduced gauge fields as connections alternative to the Levi-Civita connection in 1918, unifying all known physics at the time, before abandoning his original idea and reintroducing them more cautiously in 1929 while, almost certainly unknowingly, making a discovery that would revolutionize physics a few decades later. One may add to this irony that Weyl reintroduced gauge fields without identifying them as connections, after which mathematicians and physicists pursued, in parallel, the extraordinarily promising paths of connections without immediately realizing that they were speaking about the same object.

4.2 The Geometrical Necessity : From Cartan to Ehresmann

Let us therefore return to the starting point of Levi-Civita's 1917 paper in order to study the developments undertaken independently in connection theory under the weight of a geometrical necessity. Beyond the major impact of Riemann's proposals and the developments discussed above, the status of geometry at the beginning of the twentieth century was shaped by a second major development : the Erlangen Program initiated by the German mathematician and physicist Felix Klein (1893). It is based on the groups and algebras of Lie (1888) [40], which are groups depending on a continuous parameter [41]. They can thus encode continuous transformations (e.g. translations and rotations), and it is possible to define notions of differential calculus on them through the notion of infinitesimal transformations [42].

The Erlangen Program pursued by Klein, rooted in projective geometry, proposes to characterize the spaces of geometry by one or more transformation groups acting on them and leaving them invariant. Thus, Euclidean space is characterized by the orthogonal group O(3), which leaves lengths invariant, and Minkowski space of special relativity by the Poincaré group O(1,3), which leaves the spacetime interval invariant. However, Riemann's curved spaces, for which the invariances (given by the metric) are defined only locally, do not seem to fit into the geometrical framework proposed by Klein.

It is to the French mathematician and physicist Élie Cartan that we owe the unification of these two approaches through the introduction of new spaces, which he calls "generalized spaces" or "non-holonomic spaces," which would become precursors of fiber bundles. In doing so, Cartan was also motivated by the desire to generalize Riemannian geometry and Levi-Civita's parallelism to other types of spaces through purified geometrical developments, in opposition to the heavy analytic computations of Ricci, which he described as an "orgy of indices" (Cartan 1928).

The first major revolution introduced by Cartan consists in studying the behavior of moving frames [43], considering in particular how such a frame behaves when it is parallel transported along curves or loops. The interest of this approach lies in the fact that specifying an orthonormal frame at each point of a space also defines the notion of orthonormality at each point, which is equivalent to specifying a metric. In addition, the frame contains information about orientation, and one can

40. The name was allegedly given by Weyl in 1930. Note here that Weyl also carried out major work on Lie groups and their application to quantum physics (Weyl 1928).

41. For a history of Lie groups, see Hawkins (2000).

42. Each Lie group is in fact assimilable to a differentiable manifold.

43. Also called trihedra by Cartan. One also finds the terms tetrads, vierbein, or n-beins depending on the dimension of the space in which they are defined.

apply rotations to it at each point, i.e. it transforms under the action of the Lie symmetry groups of the space, in the spirit of Klein's Erlangen Program. Reasoning in this way about moving frames would gradually lead to viewing the frame bundle rather than the tangent space as the object of interest for extending geometry, and thus explains the central place occupied by the notion of principal bundles in the modern definition of connections [44]. The method of moving frames has its origins in the mechanical applications of Darboux (1894), and Cartan applied it as early as 1910 to the study of Lie groups and Klein spaces [45] (Cartan 1910). He then partially formalized generalized spaces in Cartan (1922), where he discusses the mathematical foundations of general relativity. He also offers here a new definition and a geometric interpretation of the Levi-Civita connection and of curvature through their actions on moving frames. All these considerations were refined in numerous publications during 1922 and 1923, primarily centered on the mathematics of general relativity and its extension, resulting from ideas that had matured since 1910 [46]. In Cartan (1923), he notably emphasizes the importance of frames as representatives of groups for defining connections and understands that a connection makes it possible to identify two (affine) tangent spaces that are infinitely close by giving the Lie-group transformations that allow one to pass from one to the other : "An affine-connection manifold is a manifold which, in the immediate neighborhood of each point, has all the characteristics of an affine space, and for which one has a rule for referencing the domains surrounding two infinitely close points : this means that if, at each point, one chooses a system of Cartesian coordinates having that point as origin, one knows the transformation formulas (of the same nature as in affine space) that allow one to pass from one reference system to any other reference system with an infinitely close origin". Connections thus "glue" tangent spaces to one another. Additionally, Cartan provides an extensive study of connections with non-zero torsion [47], which are proposals for affine connections different from the unique Levi-Civita connection but which do not change the geodesic equation [48].

Cartan thus gradually realizes the following absolutely crucial points : 1) the spaces on which it is possible to define parallel transport can differ from tangent spaces, e.g. be Klein spaces defined at each point of a manifold ; 2) groups can act on these Klein spaces without acting on the manifold M, and thus play a central and independent role.

He develops his theory of generalized spaces in Cartan (1924a), and in Cartan (1924b) he considers a manifold to which, at each point, a projective space associated with a frame is attached. This is indeed an example of a generalized space in which a Klein space is given at each point of a Riemannian space. He then defines a connection on this space as the mathematical object that makes it possible to relate, point by point, the projective spaces to one another, i.e. to identify points belonging to two infinitesimally close projective spaces. We are thus faced with an incredibly modern definition of a connection, defined by its action on frames (i.e. the principal bundle) and not on the tangent bundle. The step outside the tangent bundle for connections is thus conscious and fully accomplished in mathematics.

Cartan perfectly summarizes his indispensable contribution to the generalization of connections since Levi-Civita as : [49] : "The fundamental idea is connected to the notion of parallelism that Mr. T. Levi-Civita introduced so fruitfully. The many authors who have generalized the theory of metric spaces all started from Mr. Levi-Civita's fundamental idea, but, it seems, without being able to detach it from the idea of a vector. This causes no inconvenience when dealing with affine-connection manifolds [...] But it seemed to forbid any hope of founding an autonomous theory of conformal or projective connection manifolds. In fact, what is essential in Mr. Levi-Civita's idea is that it provides

44. See Appendix B.

45. These were then referred to as "continuous groups" and "homogeneous spaces," respectively.

46. For a detailed history of generalized spaces in Cartan, see Nabonnand (2009) et Nabonnand (2016) and the associated references, as well as Marle (2014) et Cogliati et Mastrolia (2018) for a connection-centered approach.

47. see again Appendix B

48. These developments would have major impacts in modern and contemporary physics, where torsion is frequently invoked in extensions of general relativity such as so-called "Einstein–Cartan" theories and their generalizations (e.g. $f(T)$ theories) which can in particular explain dark energy or inflation ; see e.g. Trautman (2006).

49. Attributed to Cartan in 1924 and taken from Marle (2014) (I have not been able to find the text associated with the primary source).

a means to connect two infinitely close small pieces of a manifold, and it is this idea of *connection* that is fruitful."

In Cartan (1926), he defines and studies extensively holonomy groups on generalized spaces, namely the angle acquired by a vector at the same point when it has been parallel transported with a connection along a loop. This development enables the modern and geometric definition of curvature and had an enormous impact on the future study of geometry in order to characterize spaces (loop theory, Kähler spaces, etc.). Following these major developments, Cartan summarizes and completes his theory of generalized spaces and moving frames in the book Cartan et Leray (1937).

Weyl would also be a major actor following the developments undertaken by Cartan, in particular by seeking to popularize and unify the different proposals made to define connections and parallelism. He addressed criticisms to Cartan's approach as early as 1925 and published in 1929 a more detailed critique associated with a presentation at Princeton (Weyl 1929c). Weyl then considered that Cartan's theory requires an additional relation between the Klein spaces and the tangent spaces to M at each point, in order to link these spaces to the original manifold M, in the same way that tangent spaces can be determined solely from the differentiable structure of M. Cartan was absolutely dissatisfied with Weyl's critique, and he wrote to him in a letter in 1930 : "I do not believe the criticisms you address to my theory of projective-connection spaces are well founded ... The exposition you give of my theory does not quite correspond to my point of view."[50]. Indeed, Cartan wished to give his work the most general possible direction, anticipating the developments of fiber bundles, and wanted Klein spaces to be as independent as possible from the manifold to which they are associated. Weyl's criticism can be partially explained by the fact that Klein spaces were called by Cartan "tangent projective spaces," a source of confusion because it implies a link with the tangent space. Moreover, following Weyl's critique and connecting Klein spaces with the manifold M would lead to rich and important developments known today as "Cartan geometry." The two authors eventually gradually came to agreement through exchanges of letters, but Weyl (1938) reiterated similar criticisms of Cartan's approach before reconsidering in Weyl (1949), where he now praises Cartan's approach and the independence between frames defined in Klein spaces and the choice of coordinates on M, and realizes that he must have unconsciously relied on Cartan's approach when introducing spinors on spacetime in Weyl (1929b)[51].

Cartan's generalized spaces were then absorbed into the more general definition of fiber bundles by Seifert (1933)[52], which then naturally enabled Jean-Louis Koszul (1950) and finally Cartan's student Charles Ehresmann (1951) to define connections on these spaces. Whereas Koszul's definition generalizes covariant differentiation, that of Ehresmann, as a choice of a subspace of the tangent space of a principal bundle[53], makes it an abstract but purely geometric object, completely independent of any link with M and its metric. In this new framework, already largely initiated by Cartan, curvature and torsion become properties of the connection rather than of the space, thus making it an absolutely central object. Note once again that it is interesting that these developments took place completely independently of the developments in physics after general relativity described in the previous section, even though Weyl was a major actor in both developments.

4.3 The Point of Contact

Cartan and Weyl wear the dual hat of physicist and mathematician and advance both disciplines in tandem by developing the mathematics of general relativity (and of quantum physics in Weyl's case). Curiously, however, it seems that the mathematical and physical developments after the 1920s, which brought connections "outside the tangent bundle," were carried out independently by mathematicians and physicists. While Weyl in 1918 does indeed speak of a connection (of length) for A, he

50. Quotation taken from Scholz (2022).
51. For a detailed history of the debate between Weyl and Cartan, see Bell et Korté (2016) and Scholz (2022).
52. For a complete history of fiber bundles see McCleary (2011).
53. See Appendix B.

no longer uses this term in 1929 [54]. Yet one often credits Weyl with the identification of connections and gauge fields and with understanding the quantum operator $\partial\mu - ieA_\mu$ as a covariant derivative. Cartan had already developed, at least in part, his theory of connections on generalized spaces in 1924, and this was already well known to Weyl in 1929. One may then venture to explain Weyl's inability to see A as a connection by the disagreements between Cartan and Weyl discussed in the previous section and by his inability to conceive of a connection that is not intimately linked to the manifold on which it is defined and to its tangent bundle.

Identifying who first managed to understand that gauge fields were indeed connections is thus a crucial problem that is nevertheless not easy to resolve, and the subject is surprisingly little discussed in the literature [55]. One can clearly state that the identification was established in the 1970s and can certainly credit Wu et Yang (1975) with popularizing this identification among physicists (see Table I, which bridges gauge theories and fiber bundles). There remains, then, a gigantic gap between 1930 and 1975.

This discovery is sometimes attributed to Trautman (1970). There is indeed no doubt here that Trautman had realized this, since he clearly states that "classical electrodynamics may be interpreted as a theory of an infinitesimal connection in a principal fibre bundle with the structure group U(1)" (p.30). In these lecture notes, Trautman seems to treat this fact as obvious, with a certain contempt for physicists who failed to see it : "Few words have been abused by physicists more than relativity, symmetry, covariance, invariance and gauge or coordinate transformations. These notions used extensively since the advent of the theory of relativity, are hardly ever precisely defined in physical texts. [...] fibre bundles provide a convenient framework for discussing the concepts of relativity, invariance, and gauge transformations." (p.29).

One may then legitimately ask whether Trautman was the first to make this observation. Fragments of the gauge-field/connection identification can be found earlier in the paper by Lubkin (1963), which is rather technical and certainly not very accessible to physicists. I have not been able to find other intermediate articles dealing with the topic, except that of Utiyama (1955), in which he derives equations very similar to those of Yang–Mills while seeking to establish general rules for introducing new interaction-mediating fields from gauge-invariance properties, exactly as Weyl did in 1929 (and probably independently of him, since he does not cite him). He then draws strong parallels between the Levi-Civita affine connection acting on spinors [56] and gauge fields [57], but he does not clearly identify the two concepts, and it seems to me that this paper can simply be seen as a technical generalization of Weyl's paper [58], rather than a conceptual one. A more detailed analysis of how gauge fields came to be identified as connections would seem extremely desirable, but would unfortunately go too far beyond the scope of the present essay. From about 1975 onward, therefore, the bridge between gauge fields and connections is clearly established, and the theory of connections is very widely developed in its contemporary form.

5 Conclusion

We have seen how, starting from simple coefficients appearing in relations associated with the transformations of quadratic forms in Riemannian geometry, the notion of connection could gradually establish itself as a pillar of contemporary geometry and physics. Driven by a dual necessity, physical and geometrical, connections emerged as the indispensable object for defining differentiation and

54. In Weyl (1929a), the word "curvature" appears only three times, two of which refer to Riemann curvature and once to scalar curvature (the trace of the Ricci tensor $R = g^{\mu\nu}R^{\lambda}{}_{\mu\lambda\nu}$). The word "connection" appears five times, only in the ordinary sense, i.e. outside the mathematical context, e.g. "the connection between Einstein's theory of teleparallelism and the spin theory of the electron".

55. I thank here the anonymous people who gave me some leads for reflection in this discussion.

56. Expressing the invariance of the differentiation of spinors under the action of the spin-1/2 representations of SO(3,1).

57. Expressing the invariance of the differentiation of spinors under the action of a certain general Lie group G.

58. Generalization to non-Abelian Lie groups.

parallelism of geometric objects in an invariant manner in all kinds of spaces. By virtue of its many facets and richness, it required numerous developments in physics and mathematics before it could converge. Mathematical and physical developments first proceeded in tandem from 1861 to the 1920s within the framework of Riemannian geometry and general relativity, then separated and proceeded independently within the framework of fiber-bundle theory on the one hand and quantum mechanics, then particle physics, on the other, before coming together again during the 1970s. Throughout the twentieth century, the notion of connection thus allowed mathematicians to unify the geometries of Klein and Riemann and allowed physicists to unify all known fundamental interactions within a single formalism, providing—as its name indicates—an indispensable "binder" for contemporary science.

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A Useful Definitions

Mathematics :

- **A fiber bundle** is a triple (E, M, π) where E and M are two differentiable manifolds called, respectively, the total space and the base space. $\pi : E \to M$ is called the projection. The space $F = \pi^{-1}(x)$ for $x \in M$ is called the fiber at (over) the point x.
- Let $\phi : M \to N$ a smooth application between two manifolds M and N. Let $f \in C^\infty(N)$ be a smooth function $f : N \to \mathbb{R}$. The pullback of f by ϕ, $\phi^* f$ is the map $\phi^* f : M \to \mathbb{R}$ defined as $\phi^* f = f \circ \phi$. The pushforward allow to transform a vector $v \in TM$ in a vector $\phi_* v \in TN$ defined as $\phi_* v(f) = v(\phi^* f)\, \forall f \in C^\infty(M)$. Finally, the pullback of ϕ can be used to transform a 1-form (covector) ω on TN^* to a 1-form $\phi^*\omega$ on TM^* as $(\phi^*\omega)v = \omega(\phi_* v)\, \forall v \in TN$.
- The **exterior derivative** (or differential) of a function is defined as the 1-form $\mathrm{d}f(v) = v(f), \forall v \in TM, f \in C^\infty(M)$. It can be generalized to act on forms of any degree, by imposing the condition $\mathrm{d}^2 = 0$.

Physics :

- In quantum physics, the **wavefunction** $\psi(x) : \mathbb{R}^3 \to \mathbb{C}$ of a system describes the probability of observing the system at position x during a measurement as $P(x) = \psi^*\psi$ where ψ^* is complex conjugation.
- **The Schrödinger equation** describes the time evolution of wavefunctions associated with a system as $\hat{H}\psi = i\hbar\partial_t\psi$, where $\mathcal{H}$ is the Hamiltonian operator (the energy) of the system and t is time (this equation can in fact be seen, in terms of Lie groups, as defining energy as the

generator of time translations).

— **The Dirac equation** generalizes the Schrödinger equation to the case including special relativity. It is written $(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$ where γ^μ are the "Dirac matrices," representing the basis of the Clifford algebra of spacetime, and ψ is here a generalization of the wavefunction to 4 components (a Dirac spinor). To include electromagnetism, the term $\partial_\mu\psi$ is replaced by the covariant derivative $\partial_\mu - ieA_\mu$, where A_μ plays the role of a connection.

B 50 Shades of Connections

The notion of a connection can be defined and conceived in a multitude of different ways, reflecting a wide diversity of possible applications and making it a subtle concept to grasp. We present here a few commonly used definitions to help with reading the text. I assume here a certain familiarity on the part of the reader with Riemannian geometry and the theory of fiber bundles. Otherwise, I strongly recommend reading Baez et Muniain (1994), Coquereaux (2002) et Faure (2021b) (and the video lectures of Schuller (2016)) for pedagogical introductions and Bleeker (1981), Kobayashi et Nomizu (1996) et Nakahara (2003) for a more advanced treatment. Condensed lecture notes can be found in Marsh (2014) and Marsh (2016) for Riemannian geometry and fiber bundles respectively. We will also rely on Disney-Hogg (2019) to define connections in what follows. The presentation below claims neither to be exhaustive nor perfectly rigorous, and aims only to introduce the different notions discussed in the main text in order to better grasp their conceptual content.

In what follows, we will use Einstein's convention on repeated indices : $x_a y^a = \sum_a x_a y^a$. We will also do our best to respect the following conventions : Greek indices μ, ν, λ, ρ are reserved for objects belonging to the tangent bundle, and Latin indices i, j, k are reserved for internal indices describing the fiber of a fiber bundle. The index α is reserved for the decomposition of Lie-algebra elements on the basis of generators.

B.1 Affine connections on the tangent bundle and the Levi-Civita connection

Let M be a differentiable manifold, TM its tangent bundle[59] and $\Gamma(TM)$ the sections of the tangent bundle, i.e. vector fields. An **affine connection** $\nabla : X, Y \to \nabla_X Y$ is a bilinear map $\Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ such that for every function $f \in C^\infty(M)$ and for all $X, Y \in \Gamma(M)$:

— $\nabla_{fX} Y = f\nabla_X Y$ (linearity in the first slot)

— $\nabla_X(fY) = X(f)Y + f\nabla_X Y$ (Leibniz rule allowing one to identify ∇ with a differentiation).

$\nabla_X Y$ thus makes it possible to define on M how to differentiate (and move) a vector Y in a direction given by X (this definition also generalizes to tensors).

The **curvature** of ∇ is a 2-form (an antisymmetric rank-(2,0) tensor) with values in $\mathrm{End}(TM)$: $TM \to TM$ acting on a vector Z as

$$R_{X,Y} Z = \left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) Z. \tag{4}$$

At a point of M, R allows one to compute the holonomy associated with an infinitesimal loop (the rotation induced on a vector parallel transported around the loop) or the deviation between two nearby geodesics.

The **torsion** is a rank-2 tensor acting on a pair of vectors (X, Y) as

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{5}$$

It quantifies the tendency of a frame to rotate in the plane perpendicular to a curve when it is parallel transported along it.

Now let $g : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ be a (pseudo)-Riemannian metric on M. The **Levi-Civita connection** is the unique affine connection compatible with the metric and with vanishing torsion, i.e. satisfying :

59. A point $p \in TM$ of the tangent bundle is given locally by a pair (x, v) where x is a point of M and v is a vector associated with x (by locally we will always mean "in a choice of local trivialization").

— $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$: compatibility with the metric : parallel transport according to ∇ does not change the length of vectors (more generally, $\nabla g = 0$).

— $\nabla_X Y - \nabla_Y X = [X,Y]$ with the Lie bracket $X, Y = X(Y(f)) - Y(X(f))$.

As Levi-Civita showed in 1917, this connection defines parallel transport in an "intuitive" way and corresponds to transporting the vector on M in the most "parallel" possible manner (without inducing rotation) and without changing its length.

Let now $e_\mu = \partial_\mu$ be a basis of the tangent bundle on a chart $U \subseteq M$, and define the components of the connection, Γ, as :

$$\nabla_{e_\mu} e_\nu = \Gamma^\lambda_{\ \mu\nu} e_\lambda \tag{6}$$

Then, by the Leibniz rule and decomposing vectors in a basis of the tangent bundle $X = X^\mu e_\mu$ and $Y = Y^\nu e_\nu$:

$$\nabla_X Y = (X^\nu \partial_\nu Y^\lambda + \Gamma^\lambda_{\ \mu\nu} Y^\mu X^\nu) e_\lambda \tag{7}$$

For the Levi-Civita connection, one can show that Γ is computed from first derivatives of g as :

$$\Gamma^\lambda_{\ \mu\nu} = \frac{1}{2} g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}) \tag{8}$$

where the symmetry condition $\Gamma^\lambda_{\ \mu\nu} = \Gamma^\lambda_{\ \nu\mu}$ expresses the vanishing of torsion. f

B.2 Connections on the principal bundle : Ehresmann and connection forms

Let P be a principal bundle $P \xrightarrow{\pi} M$ with structure group G. One may think of P as the set of possible frames of a vector space at each point of M (related to one another by transformations of the Lie group G). In the case of the tangent bundle, P is the set of bases e_μ in which to decompose tangent vectors, related to one another at each point by transformations of $G = \mathrm{GL}(n)$. Locally, a point $p = (e, x) \in P$ corresponds to a choice of frame e over the point x.

Let $\mathfrak{g}$ be the Lie algebra of G (one can think of G itself as a manifold and $\mathfrak{g}$ as its tangent space at the identity T_eG). Let T_pP be the tangent space of P at the point p. The vectors $X_p \in T_pP$ represent infinitesimal changes of frames. The projection map $\pi : P \to M$, $\pi(p) = x$, returns the point $x \in M$ to which the frame e is "associated." Likewise, π can be used to obtain a vector $v \in TM$ over x from a vector X_p in T_pP as $\pi_(X_p) = v \in TM$ (using the pushforward defined above). Define the vertical subspace V_pP at p as

$$V_pP = \ker((\pi_) p) \qquad = X^\xi \in T_pP, /, (\pi *) p(X_p) = 0 \tag{9}$$

V_pP corresponds to the set of frame transformations corresponding to "pure rotations" (motions in the fiber) that are not associated with any translation of the frame on M. To each $X^\xi \in V_pP$, one can associate an element ξ of $\mathfrak{g} = T_eG$ via its action on functions :

$$X^\xi_p f = \frac{\mathrm{d}}{\mathrm{d}t}(f(pe^{t\xi}))\bigg| t = 0 = \mathbb{i}(\xi)(f) \tag{10}$$

where we introduced the map $\mathbb{i} : \mathfrak{g} \to T_pP$, $\xi \to X^\xi_p$.

An **Ehresmann connection** on P at the point p is a choice of horizontal space $H_p \subset T_pP$ such that $T_pP = V_pP \oplus H_pP$ and satisfying $\forall g \in G, p \in P$ and $X_p \in H_pP : g_* X_p \in H_{pg}P$, i.e. acting with g on a horizontal vector X_p at the point p yields a horizontal vector at the point pg. Once such a choice has been made, any vector $X_p \in T_pP$ can thus be decomposed as $X_p = \mathrm{Ver}(X_p) + \mathrm{Hor}(X_p)$with $\mathrm{Ver}(X_p) \in V_pP$ and $\mathrm{Hor}(X_p) \in H_pP$ (Attention here ! Even though V_pP is given by π and does not depend on the connection, the values of $\mathrm{Ver}(X_p)$ and $\mathrm{Hor}(X_p)$ both depend on the choice of the connection (see illustration in Schuller (2016), lecture number 21)).

In this way, the choice of a connection indeed defines the notion of "horizontality" associated with translations of bases on M (and the associated parallel transport) by locally identifying points on infinitesimally close fibers (verticality being defined naturally by rotations of bases around a same point of M).

A **Connection form** is a $\mathfrak{g}$-valued 1-form, $\omega : TP \to \mathfrak{g}$ satisfying

— $\forall p \in P, \omega_p : T_pP \to \mathfrak{g}$ is a vector-space isomorphism.

— $\forall g \in G, :, g^*\omega = \mathrm{Ad}(g^{-1})\omega$.

— $\forall \xi \in \mathfrak{g}, \omega(X^\xi) = \xi$.

with the adjoint map acting on the group on itself as $\mathrm{Ad}(g) : G \to G, h \to ghg^{-1}$ (thus forming a representation of G).

From an Ehresmann connection, one can associate the connection form as the $\mathfrak{g}$-valued 1-form satisfying

— $\omega(X^\xi) = \xi$ if $X^\xi \in V_pP$

— $\omega(X) = 0$ if $X \in H_pP$

or again, using the map $\mathbb{i} : \mathfrak{g} \to T_pP$, $\mathbb{i}(\xi) \to X_p^\xi$ defined above :

$$\omega_p(X_p) = \mathbb{i}^{-1}(\mathrm{Ver}(X_p)) \tag{11}$$

H_pP can be recovered from ω as $H_pP = \ker(\omega_p)$.

B.3 Connections on the associated vector bundle : Koszul connections

The definition given above of an affine connection is in fact modernized and presented as a special case of a more general connection : a Koszul connection.

Let E be a vector bundle (associated bundle $P \times_\rho V \to M$), associating to each point of M a vector space V (fiber) such that at each point of M, $u \in E$ is defined locally by the pair $u = (e, \psi) \in P \times V$ obeying the equivalence relation $(e, \psi) \sim (eg, \rho(g^{-1})\psi)$, where ρ is a representation of G on V.

Let now $s \in \Gamma(E)$ be a section of E. Define the Koszul connection $\nabla_X(s)$ as the map $\Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ satisfying

— $\nabla_{fX}s = f\nabla_X s$

— $\nabla_X(fs) = X(f)s + f\nabla_X s$.

Thus, the affine connection as we defined it above is a particular case of a Koszul connection for which the associated vector bundle is the tangent bundle TM.

Let us now see how an Ehresmann connection given by a connection form ω on P induces a Koszul connection on E. Let $e : U \to P$ be a local section of P (such that $\pi \circ e = \mathrm{Id}$), e.g. a choice of frames over a region $U \subset M$. One can obtain a Koszul connection from a Cartan 1-form ω as $\mathcal{A} = e^\omega$, that is

$$\mathcal{A}(v) = \omega(e_(v)) \tag{12}$$

$\mathcal{A}$ is then a 1-form on M with values in $\mathfrak{g}$, $\mathcal{A} = \mathcal{A}^\alpha_\mu \mathrm{d}x^\mu \otimes \xi_\alpha$ (where ξ_α is a basis of $\mathfrak{g}$).

One can let $\mathcal{A}$ act on E using the representation ρ

$$\rho(\mathcal{A}) = (\mathcal{A}^\alpha{}_\mu)^i{}_j \mathrm{d}x^\mu \otimes \rho(\xi_\alpha)^j{}_i \tag{13}$$

finally, to arrive at a Koszul connection on E, one imposes [60]

$$\nabla e_\mu e_i = e_j \mathcal{A}^j{}_{\mu i} \tag{14}$$

with $\mathcal{A}^j{}_{\mu i} = (\mathcal{A}^\alpha{}_\mu)^i{}_j \rho(\xi_\alpha)^j{}_i$. Using the Leibniz rule, one can then write locally the action of ∇_v on a section $s = s^i e_i$ as

$$\nabla_v s = e_i(\mathcal{A}^i{}_{j\mu} s^j + \partial_\mu s^i) v^\mu \tag{15}$$

60. It is possible to rigorously derive the equivalence between the Ehresmann connection on P and the Koszul connection on E thus proposed—and the associated geometric interpretation—by introducing the notion of horizontal lift (Nakahara (2003), Sec. 10.4) and/or of exterior covariant derivative on equivariant functions (Schuller 2016; Disney-Hogg 2019). There are multiple ways to proceed. One may, for instance, define $\nabla_v \psi|p = (\mathrm{d}\psi)p(\mathrm{hor}(X^v))$, where $X^v \in T_pP$ such that $\pi * X^v = v$ and $\psi \in C^\infty_G(P, V)$ is the representation of $s \in E$ given by an equivariant function on P.

The curvature of the connection on E is defined as

$$F = \mathrm{d}\mathcal{A} + [\mathcal{A} \wedge \mathcal{A}]\mathfrak{g} \tag{16}$$

where the bracket is defined on the Lie algebra : $[\omega \wedge \mu]\mathfrak{g}(v, w) := [\omega(v), \mu(w)] = \omega(v)\mu(w) - \mu(w)\omega(v)$.

In the case where $P = LM$ and $E = TM$, in a choice of coordinates x on $U \subset M$, a natural choice of section $e : U \to LM$ is defined as $e = \partial_i$. $G = \mathfrak{g} = \mathrm{GL}(\dim(M), \mathbb{R})$. The connection $\mathcal{A} = e^*\omega$ is then the particular case of the affine connection defined above : $\mathcal{A}^i{}_{jk} = \Gamma^i{}_{jk}$.

Only in this case can one define torsion

$$T = \mathrm{d}\theta + \Gamma \wedge \theta \tag{17}$$

where $\theta : TM \to TM$ is the "solder form" $\theta = e_\mu \otimes \epsilon^\mu$ where ϵ^μ is the cotangent basis associated with e_μ, i.e. $\epsilon^\mu(e_\nu) = \delta^\mu_{,\nu}$ (see 4.4 of Coquereaux (2002)). Even though we will not detail it here, it is possible to show that equations (16) and (17) are respectively equivalent to equations (4) and (5) when acting on vectors, in the case of an affine connection.

C Elements of Weltgeometrie

We present here a few elements of the Weltgeometrie proposed by Weyl in order to better understand the stakes of such a theory for the development of the notion of connection. For a more complete presentation, we refer to Chap. 6 of Ryckman (2005) as well as Bell et Korté (2016), O'Raifeartaigh et Straumann (1998) et Scholz (2022) and the very clear presentation of Straub (2005).

The following developments were proposed by Weyl gradually from 1918 to 1924. He was then seeking to generalize the Riemannian geometry at the heart of general relativity in order to present a theory of spacetime that would be purely local. In Riemannian geometry, the length ℓ of a vector v is defined globally on a manifold by the data of a metric g as $\ell = g(v, v)$. Weyl sought to abolish this absolute notion, which he saw as a legacy of Euclidean geometry, by proposing that lengths can be compared only at one and the same point. To do so, he sought to establish a geometry invariant under so-called conformal transformations (preserving angles), corresponding to metric transformations of the form

$$\bar{g} = \Omega^2 g \tag{18}$$

where $\Omega(x)$ is an arbitrary function on the manifold M. In such a geometry, it becomes impossible to compare lengths at distant points, and only ratios of lengths are clearly defined. Length therefore becomes a "relative" notion (like velocity) rather than absolute as in Riemannian geometry. In this new framework, however, the connection Γ preserving the metric is no longer unique, and any connection related by a transformation of the form

$$\bar{\Gamma}^\lambda_{,\mu\nu} = \Gamma^\lambda_{,\mu\nu} + \frac{1}{2}(\delta^\lambda_\mu \theta_\lambda + \delta^\lambda \nu \theta_\mu - g_{\mu\nu} g^{\lambda\rho} \theta_\rho) \tag{19}$$

— where $\theta = \theta_\mu \mathrm{d}x^\mu$ is an arbitrary 1-form — is also valid.

Weyl then sought a way to determine a unique connection Γ, defining parallel transport on M. To do so, Weyl had to introduce a new 1-form $A_\mu \mathrm{d}x^\mu$ that he called the metric connection (today often called the length connection (Bell et Korté 2016)), which determines how the length $\ell = g(v, v) = g_{\mu\nu} v^\mu v^\nu$ of a vector v varies under parallel transport :

$$\mathrm{d}\ell = -\ell A_\mu \mathrm{d}x^\mu \tag{20}$$

It then becomes possible to define a unique connection, called the Weyl connection, as

$${}_{\mathrm{W}}\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}) + \frac{1}{2}(\delta^\lambda_\mu A_\nu + \delta^\lambda_\nu A_\mu - g_{\mu\nu} g^{\lambda\rho} A_\rho) \tag{21}$$

$_W\Gamma$ is thus the sum of the Levi-Civita connection Γ, quantifying the change in orientation of vectors under parallel transport, and another contribution depending on A. This unique connection is invariant under the joint transformation

$$\begin{cases} \bar{g}_{\mu\nu} & = e^{\Lambda(x)} g_{\mu\nu}(x) \\ \bar{A}_{\nu} & = A_{\nu} - \partial_{\nu}\Lambda \end{cases} \tag{22}$$

where $\Lambda(x)$ is an arbitrary function. The metric transformation is nothing but a rewriting of the conformal transformation originally required by Weyl by writing $\Lambda(x) = \ln(\Omega^2)$. The transformation of A can be identified with a gauge transformation characteristic of the electromagnetic vector potential and associated with the conservation of electric charge. Weyl then interprets A as the electromagnetic potential, to which one can associate the curvature

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\mu}A_{\nu} = (\mathrm{d}A)_{\mu\nu}, \tag{23}$$

which is nothing but the electromagnetic tensor, quantifying the evolution of the electromagnetic field and its interaction with matter (the relations $\mathrm{d}F = \star\mathrm{d}\star F = 0$ recovering Maxwell's equations in vacuum, with $\star$ the Hodge dual).

The theory thus proposed indeed unifies gravity and electromagnetism within a single and unique geometrical framework. The contribution of A changes the length of vectors under parallel transport. The electromagnetic field $F_{\mu\nu}$ is then interpreted as a "length curvature" (Streckenkrümmung). Parallel transport changes direction through the Levi-Civita contribution and length (through A).