Real Analysis — Problem Set 5

1. Let K be a compact convex set in $\mathbb{R}^n$. For any two points $x, y \in K$, let $\gamma_{x,y}$ be the line segment that connects these two points. Suppose that f is a continuous function on K. Show that there exists a constant M_f depending on the function f so that

$$\boxed{\left| \int_{\gamma_{x,y}} f \, ds \right| \le M_f |\vec{x} - \vec{y}|}$$

$|\vec{x}-\vec{y}|$ = distance between the two points $\vec{x}, \vec{y} \in \mathbb{R}^n$

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2. Consider the surface in $\mathbb{R}^3$ defined by $z = (1+x^2+y^2)\, e^{-x^2-y^2}$ and calculate the volume enclosed by this surface and the xy-plane. Here we take x, y, z as the variables in $\mathbb{R}^3$.

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3. Let $\alpha(x)$ be a monotone increasing function on the real line. Explain why the function α is Riemann integrable on the unit interval $[0,1]$. Suppose further that $\alpha(x)$ is continuous at the point $x=1$. Show that the integral $\boxed{\int_0^1 \alpha(x)\,dx}$ can be represented in terms of a Riemann-Stieltjes integral.

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4. Suppose f is a bounded real-valued function on $[0,1]$ and that f^2 is Riemann integrable on $[0,1]$. Does it follow that f is Riemann integrable? Explain your answer. How does this question relate to composition of continuous functions? See Rudin, page 86 or Rosenlicht, page 71.

5. Trigonometric integrals on the half-line $[0,\infty)$ have interesting properties related to cancellation and absolute convergence. Show the following relations and explain the interesting character of these forms:

(a) $$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \frac{1-\cos x}{x^2}\,dx$$

(b) $$\int_0^\infty \frac{\cos x}{1+x}\,dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx = \int_0^\infty \frac{1+\sin x}{(1+x)^2}\,dx - 1$$

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6. Let p, q be positive real numbers $1<p<\infty$, $1<q<\infty$ that satisfy $\boxed{\frac{1}{p}+\frac{1}{q}=1}$ equivalently $p=\frac{q}{q-1}$

For $u \geq 0$ & $v \geq 0$ show that $\boxed{uv \leq \frac{u^p}{p}+\frac{v^q}{q}}$

Explain why this is a one-variable calculus problem.
