Conditional Probability Theory

Michael Betancourt

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Conditional probability theory provides a rigorous way to decompose probability distributions over a space X into a collection of probability distributions over subsets of X. This decomposition introduces two powerful new operations into probability theory. Firstly it allows us to reduce complicated probabilistic calculations over all of X into a sequence of potentially-simpler calculations over the smaller subsets. At the same time it also encodes the information that we lose when pushing probability distributions forward along non-bijective transformations. This, in turn, facilitates the practical construction of probability distributions by allowing us to build them up from more manageable, lower-dimensional components.

That said conditional probability theory can be subtle, and to avoid any confusion our introduction we will need to proceed carefully. We will first learn how to decompose spaces into subsets before discussing how probability distributions can be decomposed across those subsets. Finally we will dedicate a good bit of time working out how to decompose the probability density functions that are so critical to practical applications.

1 Decomposing Spaces With Partitions

In Chapter 6, Section 1.2.1 we introduced the notion of a *partition* (Figure 1b): a collection of subsets

$$\mathcal{P} = \{\mathsf{c}_1, \ldots, \mathsf{c}_i, \ldots\},$$

that are non-empty,

$$\mathsf{c}_i \neq \emptyset,$$

are mutually disjoint,

$$\mathsf{c}_i \cap \mathsf{c}_{i' \neq i} = \emptyset,$$

and cover the entire space,

$$\cup_i \mathsf{c}_i = X.$$

A collection of subsets that cover X but intersect with each other do not form a valid partition (Figure 1c), nor does a collection of disjoint subsets that don't cover all of X (Figure 1d).

The individual subsets that form a partition are known as the **cells** of the partition. A partition can contain a finite number of cells, a countably infinite number of cells, or even an uncountably infinite number of cells. I will refer to partitions with a finite, countably infinite, and uncountable infinite number of cells as finite, countable, and uncountable partitions, respectively.

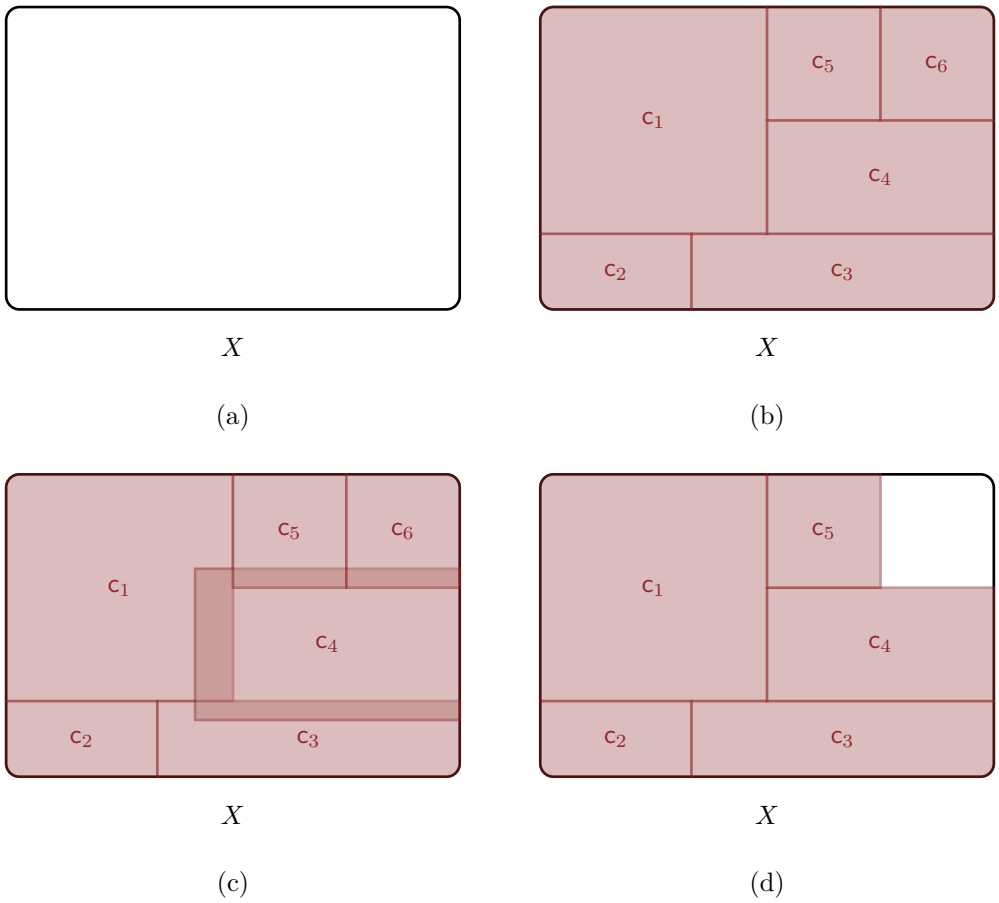


Figure 1: A partition is a decomposition of (a) an ambient space X into (b) a collection of disjoint subsets. (c) Overlapping subsets that cover X do not form a proper partition, nor do (d) disjoint subsets that do not fully cover X. We can categorize partitions by how many subsets they contain as well as the kinds of subsets they contain. For example a measurable partition consists entirely of disjoint subsets from the ambient σ-algebra, $\mathcal{X}$.

A finite partition can always be defined as an explicit list of cells, but this isn't practical for countable or uncountable partitions which would require infinitely long lists. In all of these cases, however, we can *implicitly* define a partition from the level sets of an appropriate function.

Consider, for example, a finite partition $\mathcal{P}$ defined as an explicit list of I subsets,

$$\mathcal{P} = \{\mathsf{c}_1, \dots, \mathsf{c}_i, \dots \mathsf{c}_I\}.$$

In order to distinguish between the individual cells I have assigned them each a unique numerical label or **index** from the integers $\{1, \dots, I\}$. Beyond a notational convenience, we can also use this indexing to define the partition itself.

The indexing implicitly defines a bijective **index function** that maps each cell to its corresponding integer index,

$$\begin{aligned} \chi_{\mathcal{P}} : \mathcal{P} &\to \{1, \dots, I\} \\ \mathsf{c}_i &\mapsto i \qquad\qquad . \end{aligned}$$

At the same time we can also define an **inclusion function** that maps each point in the ambient space $x \in X$ into the partition cell that contains it,

$$\begin{aligned} \iota_{\mathcal{P}} : X &\to \mathcal{P} \\ x &\mapsto \{\mathsf{c}_i \in \mathcal{P} \mid x \in \mathsf{c}_i\}. \end{aligned}$$

Composing these two functions together defines a third function that maps points in the ambient space to partition cell indices (Figure 2),

$$\begin{aligned} \phi_{\mathcal{P}} = \chi_{\mathcal{P}} \circ \iota_{\mathcal{P}} : X &\to \{1, \dots, I\} \\ x &\mapsto \{i \in \{1, \dots, I\} \mid x \in \mathsf{c}_i \in \mathcal{P}\}. \end{aligned}$$

Because the partition cells are, by definition, disjoint and cover all of X each point $x \in X$ falls into one, and only one, partition cell. In other words each point is associated with one and only one partition cell index and $\phi_{\mathcal{P}}$ will always be a surjective function.

The level set of $\phi_{\mathcal{P}}$ for a given index i is then the subset of all input points that fall into the ith partition cell,

$$\phi_{\mathcal{P}}^{-1}(i) = \{x \in X \mid \varpi_{\mathcal{P}}(x) = i\} = \mathsf{c}_i.$$

Consequently we can completely reconstruct the cells of the partition $\mathcal{P}$ from these level sets (Figure 3),

$$\mathcal{P} = \{\mathsf{c}_1 = \varpi_{\mathcal{P}}^{-1}(1), \dots, \mathsf{c}_i = \varpi_{\mathcal{P}}^{-1}(i), \dots, \mathsf{c}_I = \varpi_{\mathcal{P}}^{-1}(I)\}!$$

Because the cells in a partition are unordered the exact indexing we use is arbitrary. Different permutations of the labels define different index functions $\chi_{\mathcal{P}}$ and hence different composite

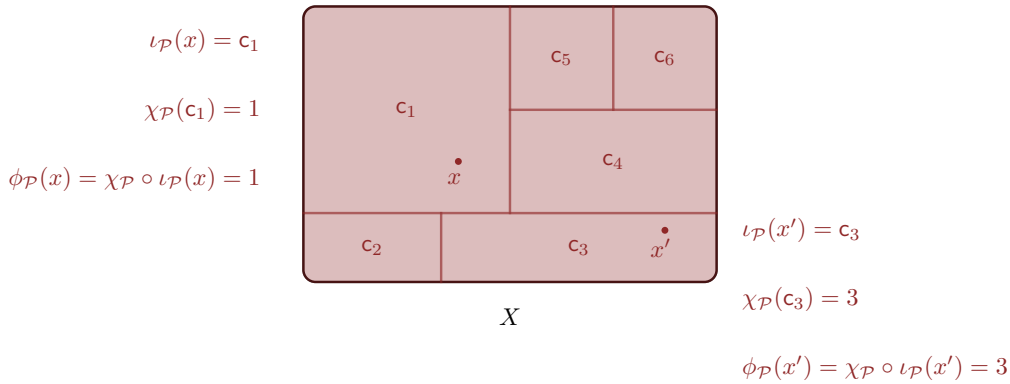


Figure 2: Any finite partition of a space X, here $\mathcal{P} = \{\mathsf{c}_1, \dots, \mathsf{c}_6\}$, implicitly defines three functions. The function $\iota_{\mathcal{P}}$ maps each point X to the partition cell that contains it while the function $\chi_{\mathcal{P}}$ maps each partition cell to its integer index. The composition $\phi_{\mathcal{P}} = \chi_{\mathcal{P}} \circ \iota_{\mathcal{P}}$ maps each point directly to the corresponding index.

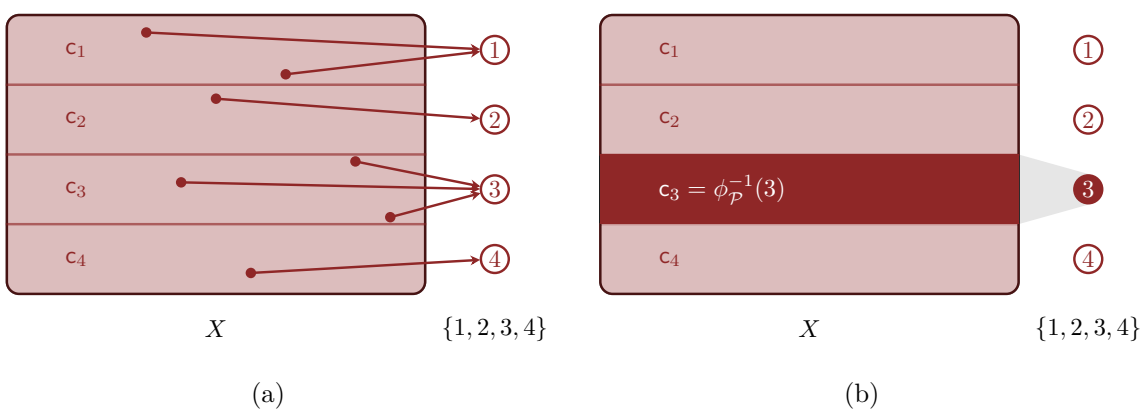


Figure 3: The composite function (a) $\phi_{\mathcal{P}}$ maps points $x \in X$ to the indices of the partition cells that contain them. (b) The level sets $\phi_{\mathcal{P}}^{-1}(n)$ map each index to all of the points contained in the corresponding partition cell. Collectively these level sets completely reconstruct the initial partition, $\{\phi_{\mathcal{P}}^{-1}(1), \phi_{\mathcal{P}}^{-1}(2), \phi_{\mathcal{P}}^{-1}(3), \phi_{\mathcal{P}}^{-1}(4)\} = \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3, \mathsf{c}_4\} = \mathcal{P}$.

functions $\phi_{\mathcal{P}}$. The level sets of these functions, however, are always the same, allowing us to work with whichever indexing might be most convenient in any given application.

Let's take a breath and summarize what we've done so far. A finite partition $\mathcal{P}$ can be *explicitly* defined as a list of disjoint subsets or *implicitly* defined by an appropriate surjective function. The advantage of this implicit definition is that it immediately generalizes to any type of partitions.

Every function $f : X \to Y$ decomposes the input space X into level sets $f^{-1}(y)$ that are not only disjoint but also cover all of X, space,

$$X = \bigcup_{y \in Y} f^{-1}(y).$$

If f is surjective then every one if its level sets will also be non-empty, $f^{-1}(y) \neq \emptyset$ for all $y \in Y$. Consequently the level sets of *every* surjective function implicitly defines a partition where each cell is indexed by a unique output value.

If the output space Y contains a finite number of points then the level sets of f define a finite partition (Figure 4). On the other hand if Y contains a countably infinite number of points then the level sets define a countable partition even though we cannot exhaustively list every cell in practice. Similarly if Y contains an uncountably infinite number of points then the level sets define an uncountable partition (Figure 5).

To demonstrate uncountable partitions let's consider a few examples over the space $X = \mathbb{R}^2$. The surjective function

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x_1, x_2) &\mapsto x_1 \end{aligned}$$

implicitly defines a partition that decomposes X into an uncountable number of real lines, each of which can be visualized by a vertical line (Figure 6a). Similarly the surjective function

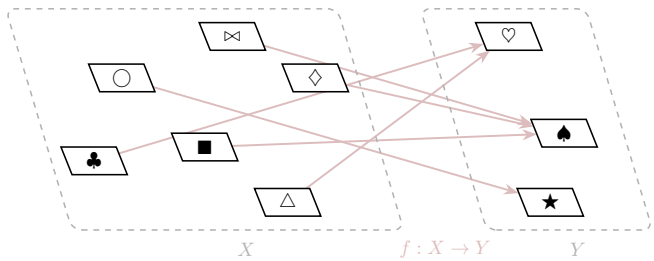
$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R}^+ \\ (x_1, x_2) &\mapsto r = \sqrt{x_1^2 + x_2^2} \end{aligned}$$

implicitly defines a partition that decomposes X into an uncountable number of concentric arcs with a fixed radii (Figure 6b).

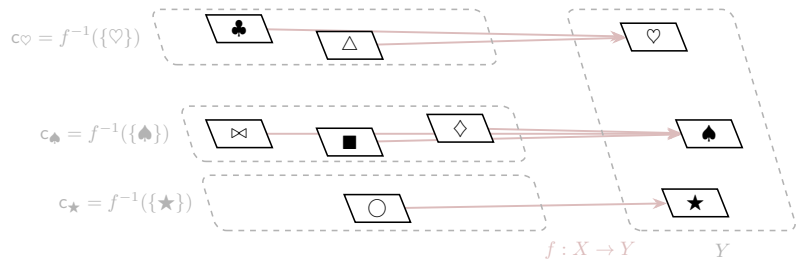
Partitions comprised of measurable subsets are particularly important in probability theory. When the cells of a partition $\mathcal{P}$ are all $\mathcal{X}$-measurable the partition itself becomes a subset of the defining σ-algebra, $\mathcal{P} \subset \mathcal{X}$. Unsurprisingly these partitions are referred to as **$\mathcal{X}$-measurable partitions**, or simply **measurable partitions** when the relevant σ-algebra is unambiguous.

Even if a surjective function is measurable it may not define a measurable partition. Only if the output space is equipped with a σ-algebra $\mathcal{Y}$ that includes all of the atomic subsets,

$$\{y\} \in \mathcal{Y}$$



(a)



(b)

Figure 4: Every (a) surjective function $f : X \to Y$ with a finite output space defines (b) a finite partition of the input space.

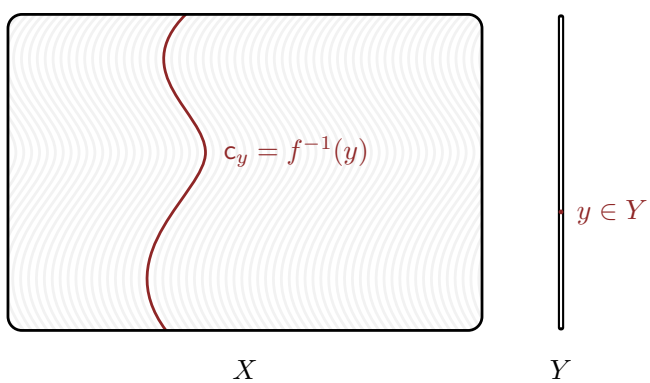


Figure 5: Every surjective function $f : X \to Y$ with an uncountably infinite output space defines an uncountable partition of the input space.

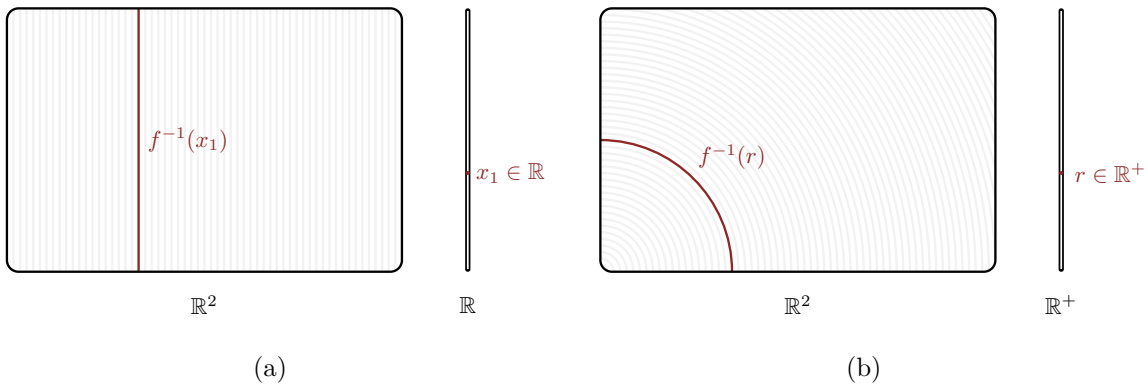


Figure 6: The partitions defined by surjective functions often admit convenient geometric interpretations. (a) For example the function $f : (x_1, x_2) \mapsto x_1$ decomposes the ambient space $\mathbb{R}^2$ into copies of $\mathbb{R}$, one for each output point $x_1 \in \mathbb{R}$. (b) Likewise the function $f : (x_1, x_2) \mapsto \sqrt{x_1^2 + x_2^2}$ decomposes $\mathbb{R}^2$ into concentric arcs.

for all $y \in Y$, will the level sets of a measurable function $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ always be $\mathcal{X}$-measurable subsets of the input space,

$$f^{-1}(y) = f^*(\{y\}) \in \mathcal{X}.$$

If $\mathcal{Y}$ does contain all of the atomic subsets, however, then every surjective and $(\mathcal{X}, \mathcal{Y})$-measurable will define measurable partition cells, and hence a measurable partition.

A σ-algebras that contains of all of the atomic subsets in the ambient space is known as a **Hausdorff σ-algebra**. Similarly a space paired with a Hausdorff σ-algebra is known as a **Hausdorff measurable space**.

Fortunately all but the most pathological σ-algebras are Hausdorff, and we can pretty safely assume that every σ-algebra that we will encounter in practice will satisfy this property. Because all of the functions that we will work with will be measurable we can also safely assume that the partitions implicitly defined by any surjective function we encounter in practice will be measurable.

2 Conditioning on Countable And Explicit Partitions

Whether defined explicitly or implicitly, a partition decomposes a space X into a collection of non-empty, non-overlapping subsets. This spatial decomposition then provides the basis for decomposing probability distributions over X into a collection of probability distributions confined to those subsets. Before tackling the full generality of this procedure we'll first build up intuition in the simples case of countable partitions explicitly defined as lists of subsets.

2.1 The Law of Total Probability

As we saw in Chapter Four Kolmogorov's axioms define a probability distribution as consistent allocation of probability over measurable subsets. Consequently in order to decompose a probability distribution we need to be able to decompose measurables subset and the probabilities allocated to them.

Any measurable set $\mathsf{x} \in \mathcal{X}$ can be immediately decomposed into its intersections with the cells of a given partition $\mathcal{P}$ (Figure 7b),

$$\mathsf{x} = \bigcup_{\mathsf{c} \in \mathcal{P}} (\mathsf{x} \cap \mathsf{c}) \,.$$

Because the partition cells are mutually disjoint these intersections will also be mutually disjoint: if $\mathsf{c}_1 \in \mathcal{P}$ and $\mathsf{c}_2 \in \mathcal{P}$ are two distinct partition cells then

$$\begin{aligned} (\mathsf{x} \cap \mathsf{c}_1) \cap (\mathsf{x} \cap \mathsf{c}_2) &= (\mathsf{x} \cap \mathsf{c}_1) \cap (\mathsf{c}_2 \cap \mathsf{x}) \\ &= \mathsf{x} \cap (\mathsf{c}_1 \cap \mathsf{c}_2) \cap \mathsf{x} \\ &= \mathsf{x} \cap \emptyset \cap \mathsf{x} \\ &= \emptyset. \end{aligned}$$

If the partition $\mathcal{P}$ is countable then any measurable subset $\mathsf{x} \in \mathcal{X}$ will decompose into a countable number of components. Moreover because σ-algebras are closed under countable unions then each of these components will also be measurable whenever the partition is measurable. Consequently we can apply the countable additivity of probability distributions to the decomposition of any measurable subset induced by a measurable, countable partition (Figure 7c).

In other words the probability allocated to $\mathsf{x} \in \mathcal{X}$ decomposes into a sum of the probabilities allocated to the disjoint partition intersections,

$$\begin{aligned} \pi(\mathsf{x}) &= \pi\left(\bigcup_{\mathsf{c} \in \mathcal{P}} (\mathsf{x} \cap \mathsf{c}) \right) \\ &= \sum_{\mathsf{c} \in \mathcal{P}} \pi(\mathsf{x} \cap \mathsf{c}). \end{aligned}$$

This decomposition of probability allocations is referred to as **the law of total probability**.

2.2 Conditional Probabilities

Now that we can decompose the probabilities allocated to individual measurable subsets we can consider how to decompose entire probability distributions. To make our first steps towards this decomposition more manageable let's begin, however, with a simplifying restriction on the partition.

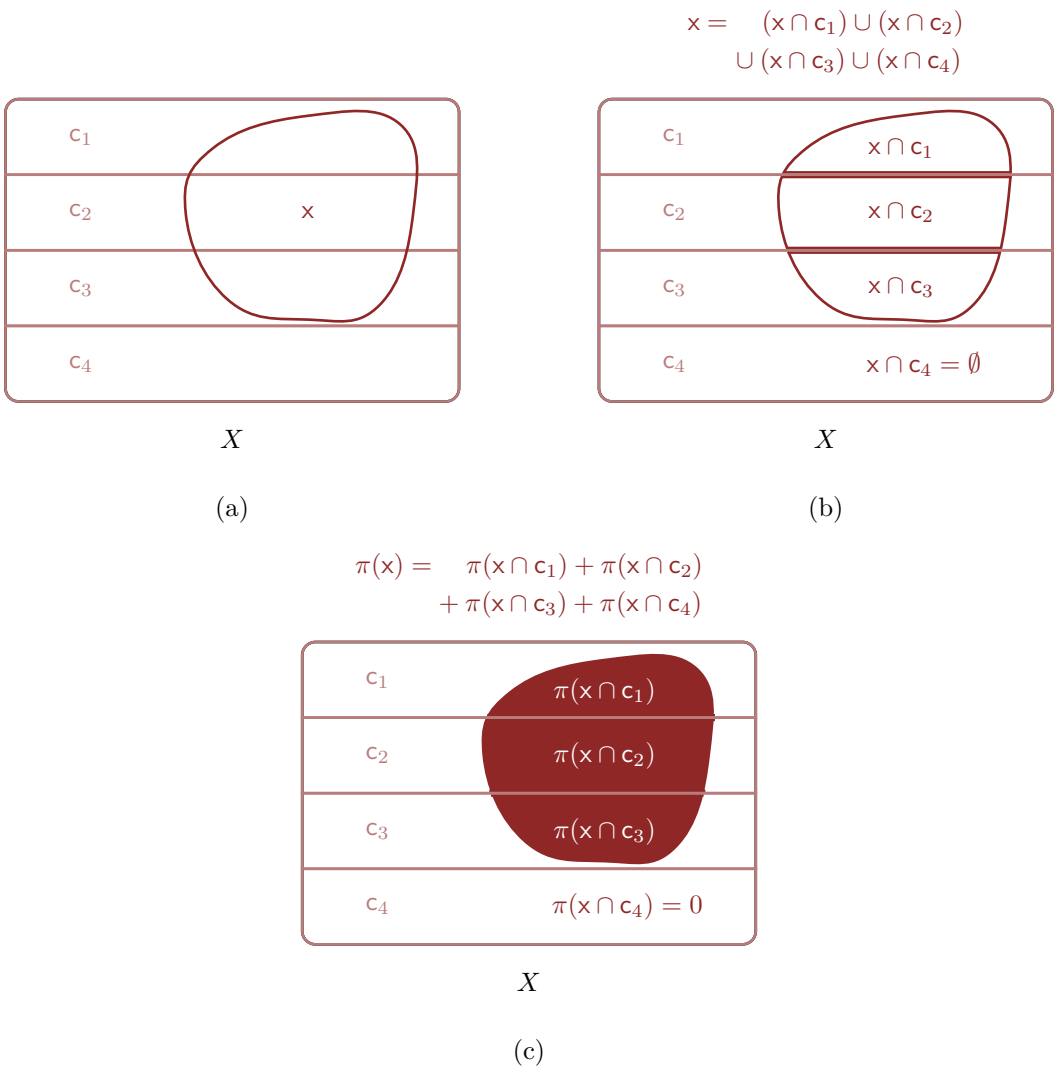


Figure 7: Countable, measurable partitions allow us to decompose probability allocations. (a) A measurable partition decomposes X into four measurable subsets. (b) Every measurable subset $\mathsf{x} \subset \mathcal{X}$ decomposes into disjoint intersections with the partition cells. (c) The probability allocated to x then decomposes into a sum of probabilities allocated to each of these intersections.

Partitions whose cells are all allocated non-zero probabilities allow us to multiply and divide by the cell probabilities without fear of dividing by zero. I will refer to a partition where every cell is not only measurable but also allocated a non-zero probability by the probability distribution π,

$$\pi(\mathsf{c}) > 0$$

for all $\mathsf{c} \in \mathcal{P}$, as a **$\pi$-non-null** partition.

Taking advantage of this flexibility we can rearrange each term in the law of total probability to give

$$\begin{aligned}
\pi(\mathsf{x}) &= \sum_{\mathsf{c} \in \mathcal{P}} \pi(\mathsf{x} \cap \mathsf{c}) \\
&= \sum_{\mathsf{c} \in \mathcal{P}} \pi(\mathsf{x} \cap \mathsf{c}) \cdot \frac{\pi(\mathsf{c})}{\pi(\mathsf{c})} \\
&= \sum_{\mathsf{c} \in \mathcal{P}} \frac{\pi(\mathsf{x} \cap \mathsf{c})}{\pi(\mathsf{c})} \cdot \pi(\mathsf{c}) \\
&\equiv \sum_{\mathsf{c} \in \mathcal{P}} \pi^{\mathcal{P}}(\mathsf{x} \mid \mathsf{c}) \cdot \pi(\mathsf{c}).
\end{aligned}$$

Here each **conditional probability**

$$\pi^{\mathcal{P}}(\mathsf{x} \mid \mathsf{c}) = \frac{\pi(\mathsf{x} \cap \mathsf{c})}{\pi(\mathsf{c})}$$

quantifies the *proportion* of the probability allocated to the intersection of x and the conditioning partition cell, $\pi(\mathsf{x} \cap \mathsf{c})$, relative to the total probability allocated to the conditioning partition cell, $\pi(\mathsf{c})$ (Figure Figure 8).

By definition a measurable subset $\mathsf{x} \in \mathcal{X}$ that doesn't overlap with the conditioning partition cell c is allocated zero conditional probability,

$$\begin{aligned}
\pi^{\mathcal{P}}(\mathsf{x} \mid \mathsf{c}) &= \frac{\pi(\mathsf{x} \cap \mathsf{c})}{\pi(\mathsf{c})} \\
&= \frac{\pi(\emptyset)}{\pi(\mathsf{c})} \\
&= \frac{0}{\pi(\mathsf{c})} \\
&= 0.
\end{aligned}$$

At the same time any measurable subset that completely overlaps with the conditioning par-

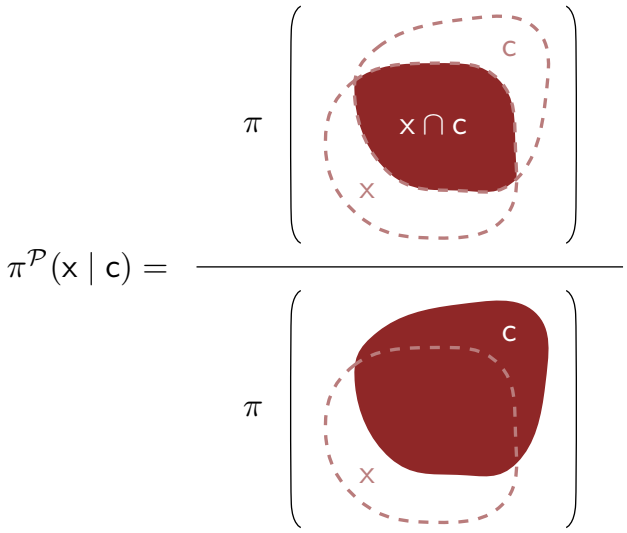


Figure 8: The conditional probability of a measurable subset x given a measurable partition cell c quantifies the proportion of probability allocated to the intersection $\mathsf{x} \cap \mathsf{c}$ relative to the total probability allocated to the partition cell c.

tition cell, $\mathsf{x} \cap \mathsf{c} = \mathsf{c}$, is allocated full conditional probability,

$$\begin{aligned}
\pi^{\mathcal{P}}(\mathsf{x} \mid \mathsf{c}) &= \frac{\pi(\mathsf{x} \cap \mathsf{c})}{\pi(\mathsf{c})} \\
&= \frac{\pi(\mathsf{c})}{\pi(\mathsf{c})} \\
&= 1.
\end{aligned}$$

Conditional probabilities look suspiciously like probability allocations that have been restricted to the domain of the conditioning partition cell. With a little more work we can show that this suspicion is in fact correct.

2.3 Conditional Probability Distributions Over The Ambient Space

Given a measurable subset $\mathsf{x} \in \mathcal{X}$ and a measurable, π-non-null partition cell $\mathsf{c} \in \mathcal{P}$ we can construct a single conditional probability $\pi^{\mathcal{P}}(\mathsf{x} \mid \mathsf{c})$. The collection of *all* conditional probabilities relative to a particular partition cell c defines a function from measurable subsets to conditional probabilities,

$$\begin{aligned}
\pi^{\mathcal{P}}_{\mathsf{c}} : \mathcal{X} &\to [0, 1] \\
\mathsf{x} &\mapsto \frac{\pi(\mathsf{x} \cap \mathsf{c})}{\pi(\mathsf{c})}.
\end{aligned}$$

The immediate question is whether or not this function defines a probability distribution. To answer that question we'll have to consider the Kolmogorov axioms.

We begin with the first Kolmogorov axiom which requires a function that maps measurable subsets into probabilities. This matches the inputs and output spaces of $\pi^{\mathcal{P}}_{\mathsf{c}}$.

In order to satisfy the second Kolmogorov axiom the probability allocated to the entire ambient set must be one. Indeed

$$\begin{aligned}
\pi^{\mathcal{P}}_{\mathsf{c}}(X) &= \frac{\pi(X \cap \mathsf{c})}{\pi(\mathsf{c})} \\
&= \frac{\pi(\mathsf{c})}{\pi(\mathsf{c})} \\
&= 1.
\end{aligned}$$

Finally we need $\pi^{\mathcal{P}}_{\mathsf{c}}$ to satisfy countable additivity. For any countable collection of measurable but disjoint subset sets

$$\{\mathsf{x}_1, \ldots, \mathsf{x}_j, \ldots\}$$

we have

$$\begin{aligned}
\pi_{\mathsf{c}}^{\mathcal{P}}(\cup_j \mathsf{x}_j) &= \frac{\pi(\,(\cup_j \mathsf{x}_j)\,\cap \mathsf{c})}{\pi(\mathsf{c})} \\
&= \frac{\pi(\cup_j (\mathsf{x}_j \cap \mathsf{c}))}{\pi(\mathsf{c})} \\
&= \frac{\sum_j \pi(\mathsf{x}_j \cap \mathsf{c})}{\pi(\mathsf{c})} \\
&= \sum_j \frac{\pi(\mathsf{x}_j \cap \mathsf{c})}{\pi(\mathsf{c})} \\
&= \sum_j \pi_{\mathsf{c}}^{\mathcal{P}}(\mathsf{x}_j),
\end{aligned}$$

as needed.

With all three Kolmogorov axioms verified we can now formally state that for any partition cell c the function defined by

$$\pi_{\mathsf{c}}^{\mathcal{P}}(\mathsf{x}) = \frac{\pi(\mathsf{x} \cap \mathsf{c})}{\pi(\mathsf{c})}$$

defines a probability distribution over the ambient space X. Formally we say that $\pi_{\mathsf{c}}^{\mathcal{P}}$ is a **conditional probability distribution**.

2.4 Conditional Probability Distributions Over Partition Cells

An important feature of conditional probability distributions is that they much more singular than we might would expect from a probability distribution over X. Recall that any measurable subset that doesn't intersect with the conditioning partition cell is always allocated zero probability,

$$\begin{aligned}
\pi_{\mathsf{c}}^{\mathcal{P}}(\mathsf{x}) &= \pi^{\mathcal{P}}(\mathsf{x} \mid \mathsf{c}) \\
&= \frac{\pi(\mathsf{x} \cap \mathsf{c})}{\pi(\mathsf{c})} \\
&= \frac{\pi(\emptyset)}{\pi(\mathsf{c})} \\
&= 0.
\end{aligned}$$

Instead *all* of the conditional probability concentrates within the conditioning partition cell itself,

$$\begin{aligned}
\pi^{\mathcal{P}}_{\mathsf{c}}(\mathsf{c}) &= \pi^{\mathcal{P}}(\mathsf{c} \mid \mathsf{c}) \\
&= \frac{\pi(\mathsf{c} \cap \mathsf{c})}{\pi(\mathsf{c})} \\
&= \frac{\pi(\mathsf{c})}{\pi(\mathsf{c})} \\
&= 1!
\end{aligned}$$

Intuitively this suggests that we can interpret a conditional probability distribution as a *restriction* of the initial probability distribution to a particular partition cell. To formalize this intuition, however, we have to define what it means to restrict not only the elements of $(X, \mathcal{X})$ to a partition cell but also the measurable subsets.

Taking the intersection of any subset $\mathsf{x} \subset X$ with a partition cell $\mathsf{c} \subset X$ gives a subset whose elements are entirely contained within the partition cell,

$$\mathsf{x} \cap \mathsf{c} \subset \mathsf{c}.$$

Moreover intersecting an entire collection of subsets

$$\{\mathsf{x}_1, \ldots, \mathsf{x}_j, \ldots\} \subset 2^X$$

with c gives a collection of subsets that are all contained witin the partition cell,

$$\{\mathsf{x}_1 \cap \mathsf{c}, \ldots, \mathsf{x}_j \cap \mathsf{c}, \ldots\} \subset 2^{\mathsf{c}}.$$

Importantly if the partition cell itself is an element of that initial collection of subsets then this restriction respects the subset operations. Formally if $\mathcal{X} \subset 2^X$ is a collection of subsets of X that contains c and is closed under complements, countable unions, and countable operations then the collection of intersections

$$\mathcal{X}_{\mathsf{c}} = \{\mathsf{x} \cap \mathsf{c} \text{ for all } \mathsf{x} \in \mathcal{X}\} \subset 2^{\mathsf{c}}$$

will also be a collection of subsets of c that is closed under complements, countable unions, and countable operations. In other words if $\mathcal{X}$ is a σ-algebra over X that contains c then $\mathcal{X}_{\mathsf{c}}$ will be a σ-algebra over c. This restricted σ-algebra is known as a **subspace σ-algebra**.

By construction every measurable subset in a restricted σ-algebra $\mathcal{X}_{\mathsf{c}}$ is also a measurable subset in the ambient σ-algebra $\mathcal{X}$. Consequently the probabilities defined by a conditional probability distribution over X also define probabilities over c. This allows us to define a new function

$$\begin{aligned}
\pi^{\mathcal{P}}_{\mathsf{c}} : \mathcal{X}_{\mathsf{c}} &\to [0, 1] \\
\mathsf{s} \quad &\mapsto \frac{\pi(\mathsf{s} \cap \mathsf{c})}{\pi(\mathsf{c})}
\end{aligned}$$

with

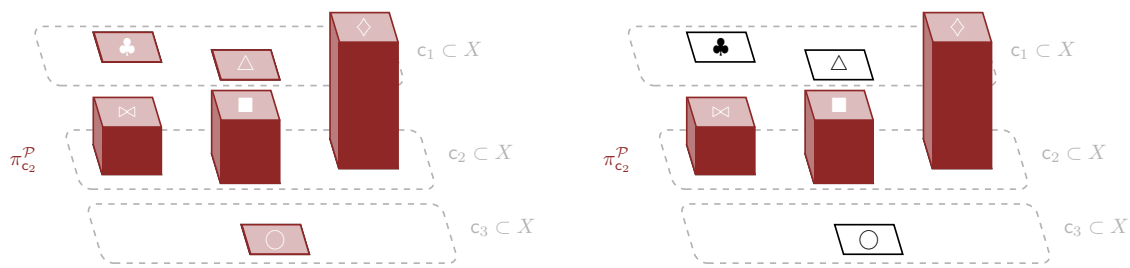
$$\pi^{\mathcal{P}}_{\mathsf{c}}(\mathsf{c}) = \pi^{\mathcal{P}}(\mathsf{c} \mid \mathsf{c}) = 1$$

and

$$\pi^{\mathcal{P}}_{\mathsf{c}}(\cup_j \mathsf{s}_j) = \sum_j \pi^{\mathcal{P}}_{\mathsf{c}}(\mathsf{s}_j),$$

which is exactly a probability distribution over the partition cell c!

All of this is to show that we have two equally valid interpretations of a conditional probability distribution. Firstly we can interpret a conditional probability distribution as a probability distribution over the full ambient space X which concentrates within a conditioning partition cell $\mathsf{c} \in \mathcal{P}$ (Figure 9a). Alternatively we can interpret a conditional probability distribution probability distribution over the conditioning partition cell itself (Figure 9b).



(a) (b)

Figure 9: Conditional probability distributions can be interpreted in two equally valid ways. (a) We can interpret a conditional probability distribution as a probability distribution over the entire space which concentrates all of its probability into to a single partition cell. Here $\pi^{\mathcal{P}}_{\mathsf{c}_2}$ allocates non-zero probability to only the elements of the partition cell $\mathsf{c}_2 = \{\blacksquare, \diamondsuit, \bowtie\}$. (b) Alternatively we can interpret a conditional probability distribution as a probability distribution defined over a single partition cell. From this perspective $\pi^{\mathcal{P}}_{\mathsf{c}_2}$ doesn't even consider elements outside of c_2.

The former interpretation is more common in technical mathematics. As we will see later on in this chapter and the next, however, the latter interpretation is more in line with how conditional probability distributions work are interpreted and used in more practical applications of probability theory.

2.5 Conditional Probability Kernels

We can push our organization one step further and collect all of the conditional probability distributions for all of the partition cells into a single mathematical object (Figure 10)

$$\begin{aligned} \pi^{\mathcal{P}}(\cdot \mid \cdot) : \mathcal{X} \times \mathcal{P} &\to [0, 1] \\ \mathsf{x}, \mathsf{c} \quad &\mapsto \frac{\pi(\mathsf{x} \cap \mathsf{c})}{\pi(\mathsf{c})}. \end{aligned}$$

I will refer to this binary function as a **conditional probability kernel**.

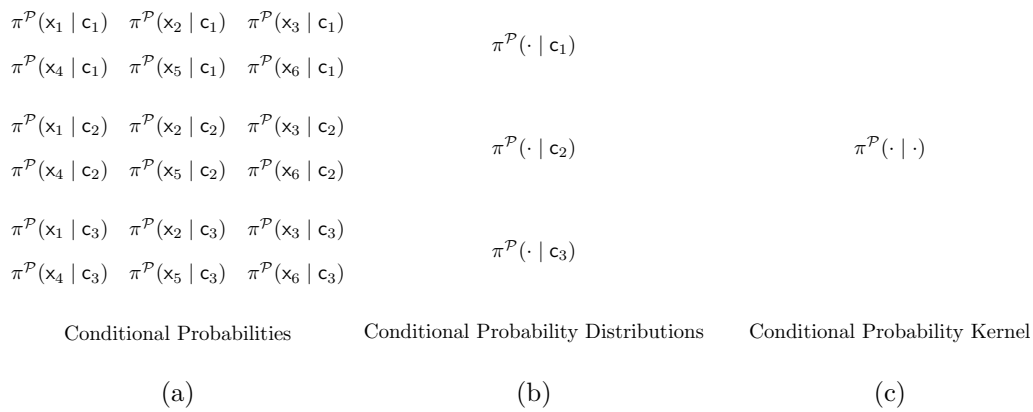


Figure 10: An important part of conditional probability theory is organization. (a) The conditional probabilities defined by π-non-null partition cells can be organized into (b) conditional probability distributions that return conditional probabilities when evaluated. Likewise the collection of conditonal probability distributions defined by all partition cells can be organized into a conditional probability kernel that returns conditonal probability distributions when partially evaluated. In order to generalize conditional probability theory to π-null partitions we will actually work backwards, showing first that a conditional probability kernel exists and then deriving conditonal probability distributions from the kernel and conditional probabilities from those distributions.

Partially evaluating a conditional probability kernel on a measurable subset in its first argument results in a measurable, unary function from each partition cell to the corresponding conditional probability,

$$\begin{aligned} p_{\mathsf{x}} = \pi^{\mathcal{P}}(\mathsf{x} \mid \cdot) : \mathcal{P} &\to [0, 1] \\ \mathsf{c} \quad &\mapsto \frac{\pi(\mathsf{x} \cap \mathsf{c})}{\pi(\mathsf{c})}. \end{aligned}$$

In words this partial evaluation quantifies how much the unconditional probability allocated to x contributes to the unconditional probability allocated to each partition cell. I will refer to this object as a **conditional probability function**.

On the other hand partially evaluating a conditional probability kernel on a partition cell in its second argument gives the corresponding conditional probability distribution,

$$\begin{aligned} \pi^{\mathcal{P}}_{\mathsf{c}} = \pi^{\mathcal{P}}(\cdot \mid \mathsf{c}) : \mathcal{X} &\to [0, 1] \\ \mathsf{x} &\mapsto \frac{\pi(\mathsf{x} \cap \mathsf{c})}{\pi(\mathsf{c})}. \end{aligned}$$

As is so often the case we have to be careful with the terminology here. I have used "conditional probability distribution" to refer to a particular probability distribution associated with a particular partition cell, and "conditional probability kernel" to refer to the collection of all probability distributions defined by all of the cells in a partition.

This convention, however, is by no means universal. Many references use "conditional probability distribution" to refer to the collection of probability distributions $\pi^{\mathcal{P}}$ instead of a particular probability distribution $\pi^{\mathcal{P}}_{\mathsf{c}}$, and some even use it to refer to both at the same time! Needless to say this latter overloaded and ambiguous terminology makes it very easy to confuse the two objects.

Again I will use "conditional probability distribution" and "conditional probability kernel" to avoid as much ambiguity in this book as possible, but when reading other texts you may want to be careful to identify to which object an author is referring as any given time. Moreover in our own writing there is no harm in being redundant in order clarify whether we are referring to $\pi^{\mathcal{P}}(\cdot \mid \cdot)$ or $\pi^{\mathcal{P}}(\cdot \mid \mathsf{c})$ in any given application.

2.6 The Law of Total Expectation

One of the benefits of conditional probability kernels is that they allow us to rewrite the law of total probability,

$$\pi(\mathsf{x}) = \sum_{\mathsf{c} \in \mathcal{P}} \pi^{\mathcal{P}}(\mathsf{x} \mid \mathsf{c}) \, \pi(\mathsf{c})$$

entirely in terms of expectation values.

If we write conditional probabilities as the outputs of a conditional probability function function,

$$\pi^{\mathcal{P}}(\mathsf{x} \mid \mathsf{c}) = p_{\mathsf{x}}(\mathsf{c}),$$

then the law of total probability becomes a discrete expectation value,

$$\begin{aligned} \pi(\mathsf{x}) &= \sum_{\mathsf{c} \in \mathcal{P}} \pi^{\mathcal{P}}(\mathsf{x} \mid \mathsf{c}) \, \pi(\mathsf{c}) \\ &= \sum_{\mathsf{c} \in \mathcal{P}} p_{\mathsf{x}}(\mathsf{c}) \, \pi(\mathsf{c}) \\ &= \mathbb{E}_{\pi_{\mathcal{P}}}[p_{\mathsf{x}}]. \end{aligned}$$

Here the probability distribution $\pi_{\mathcal{P}}$ is defined by the probability allocated to each partition cell,

$$\pi_{\mathcal{P}}(\mathsf{c}) = \pi(\mathsf{c}).$$

At the same time both the initial allocation and conditional probability function can be written in terms of expectation values of indicator functions,

$$\pi(\mathsf{x}) = \mathbb{E}_{\pi}[I_{\mathsf{x}}]$$

and

$$p_{\mathsf{x}}(\mathsf{c}) = \mathbb{E}_{\pi_{\mathsf{c}}^{\mathcal{P}}}[I_{\mathsf{x}}],$$

respectively. Consequently we can write the law of total probability as

$$\pi(\mathsf{x}) = \mathbb{E}_{\pi_{\mathcal{P}}}[p_{\mathsf{x}}]$$
$$\mathbb{E}_{\pi}[I_{\mathsf{x}}] = \mathbb{E}_{\pi_{\mathcal{P}}}[p_{\mathsf{x}}]$$

with

$$p_{\mathsf{x}}(\mathsf{c}) = \mathbb{E}_{\pi_{\mathsf{c}}^{\mathcal{P}}}[I_{\mathsf{x}}].$$

Conveniently this relationship between the various notions of expectation defined by an initial probability distribution π and a measurable partition $\mathcal{P}$ generalizes to arbitrary expectation values. The expectation value of *any* function

$$g : (X, \mathcal{X}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$$

can be written as

$$\mathbb{E}_{\pi}[g] = \mathbb{E}_{\pi_{\mathcal{P}}}[e_g],$$

where

$$e_g(\mathsf{c}) = \mathbb{E}_{\pi_{\mathsf{c}}^{\mathcal{P}}}[g].$$

Here the inner expectations e_g are known as **conditional expectation values** and the overall result is known as the **law of total expectation** or the **law of iterated expectation**.

With the laws of total probability and total expectation in hand we can decompose not only probability allocations but also expectation values along an explicit partition. If expectation values with respect to the initial probability distribution are difficult to compute but the conditional expectation values are more straightforward to work out then this iterative approach becomes a particularly-productive computational technique.

3 Conditioning On Implicit Partitions

The construction, and notation, of conditional probability distributions becomes particularly elegant when we implicitly define countable partitions through the level sets of a surjective function. This also paves the way for generalizing conditional probability theory to uncountable partitions implicitly defined by functions with an uncountable number of output points.

3.1 Conditioning On Countable Implicit Partitions

In Section 1 we learned that surjective functions $f : X \to Y$ implicitly define a partition of the input space X where the partition cells are defined by the non-empty level sets $f^{-1}(y) \subset X$. If f is also $(\mathcal{X}, \mathcal{Y})$-measurable and $\mathcal{Y}$ is a Hausdorff σ-algebra then these non-empty level sets will also be $\mathcal{X}$ measurable, allowing us to consistently allocate probability to them.

When Y contains a countable number of elements the partition defined by these level sets will be countable. Moreover if the probability allocated to each level set is non-zero,

$$\pi(f^{-1}(y)) > 0$$

for all $y \in Y$, then the partition will be π-non-null. In this case we can directly apply the conditional probability theory that we introduced in Section 2.

When working with partitions implicitly defined by a surjective function $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ I will denote the conditional probability kernel as

$$\begin{aligned} \pi^f : \mathcal{X} \times Y &\to [0, 1] \\ \mathsf{x}, y \quad &\mapsto \pi^f(\mathsf{x} \mid y) = \frac{\pi(\mathsf{x} \cap f^{-1}(y))}{\pi(f^{-1}(y))}. \end{aligned}$$

For each $\mathsf{x} \in \mathcal{X}$ the partial evaluation

$$p_{\mathsf{x}} = \pi^f(\mathsf{x} \mid \cdot) : Y \to [0, 1]$$

defines a $\mathcal{Y}$-measurable conditional probability function, and for each $y \in Y$ the partial evaluation

$$\begin{aligned} \pi^f_y = \pi^f(\cdot \mid y) : \mathcal{X} &\to [0, 1] \\ \mathsf{x} &\mapsto \pi^f(\mathsf{x} \mid y) \end{aligned}$$

defines a conditional probability distribution that concentrates entirely on the corresponding level set,

$$\begin{aligned} \pi^f_y(f^{-1}(y)) &= \pi^f(f^{-1}(y) \mid y) \\ &= \frac{\pi(f^{-1}(y) \cap f^{-1}(y))}{\pi(f^{-1}(y))} \\ &= \frac{\pi(f^{-1}(y))}{\pi(f^{-1}(y))} \\ &= 1. \end{aligned}$$

Equivalently we can interpret each π^f_y as a probability distribution over not the entire ambient space X but rather just the corresponding level set,

$$\pi^f_y : \mathcal{X}_y \to [0, 1],$$

where $\mathcal{X}_y$ denotes the subspace σ-algebra restricted to the level set $f^{-1}(y)$. Again we have the flexibility to interpret the conditional probability distributions induced by f as a collection of probability distributions over X that concentrate on particular level sets or as a collection of probability distributions over the particular level sets themselves.

At the same time because we have assumed that f is $(\mathcal{X}, \mathcal{Y})$-measurable we can push π forward along f to define a marginal probability distribution $f_* \pi$ over the output space. By definition the pushforward probability allocated to the output atomic subset $\{y\} \in \mathcal{Y}$ is equal to the input probability allocated to the corresponding level set,

$$f_* \pi(\{y\}) = \pi(f^{-1}(y)).$$

This means that a surjective function $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ induces a π-non null partition if and only if the pushforward probability allocated to every output atomic subset is non-zero,

$$f_* \pi(\{y\}) > 0$$

for all $y \in Y$.

For the countable and measurable partition implicitly defined by a sufficiently nice surjective function the law of total probability becomes

$$\begin{aligned} \pi(\mathsf{x}) &= \sum_{y \in Y} \pi^f(\mathsf{x} \mid y) \, \pi(f^{-1}(y)) \\ &= \sum_{y \in Y} \pi^f(\mathsf{x} \mid y) \, f_* \pi(\{y\}). \end{aligned}$$

This, however, is just a pushforward expectation value,

$$\begin{aligned} \pi(\mathsf{x}) &= \sum_{y \in Y} \pi^f(\mathsf{x} \mid y) \, f_* \pi(\{y\}) \\ &= \mathbb{E}_{f_* \pi}[p_{\mathsf{x}}], \end{aligned}$$

where

$$\begin{aligned} p_{\mathsf{x}} : Y &\to [0, 1] \\ y &\mapsto \pi^f(\mathsf{x} \mid y) = \frac{\pi(\mathsf{x} \cap f^{-1}(y))}{\pi(f^{-1}(y))} \end{aligned}$$

is just a conditional probability function.

Similarly the law of total expectation becomes

$$
\begin{aligned}
\mathbb{E}_{\pi}[g] &= \sum_{x \in X} \pi(\{x\})\, g(x) \\
&= \sum_{x \in X} \left[\sum_{y \in Y} \pi^{f}(\{x\} \mid y)\, f_{*}\pi(\{y\}) \right] g(x) \\
&= \sum_{y \in Y} \left[\sum_{x \in X} \pi^{f}(\{x\} \mid y)\, g(x) \right] f_{*}\pi(\{y\}) \\
&= \sum_{y \in Y} e_{g}\, f_{*}\pi(\{y\}) \\
&= \mathbb{E}_{f_{*}\pi}[e_{g}],
\end{aligned}
$$

where e_{g} is the conditional expectation function

$$
\begin{aligned}
e_{g} : Y &\to [0, 1] \\
y &\mapsto \mathbb{E}_{\pi^{f}_{y}}[g].
\end{aligned}
$$

Notice that a sufficiently-nice surjective function gives us two ways to manipulate a probability distribution over the input space: we can not only push it forward to a probability distribution on the output space but also decompose it into a collection of probability distributions across the level sets. Moreover the laws of total probability and total expectation show us that these operations complement each other in the sense that we can always recover any information about the initial probability distribution by combining the information from the pushforward and conditional probability distributions.

In other words conditional probability distributions encodes all of the information that we might lose when pushing a probability distribution forward along a surjective function. The pushforward probability distribution quantifies how much input probability is allocated to each level set while each conditional probability distribution quantifies how those allocations are distributed all across the corresponding level set (Figure 11).

Conditioning on implicit conditions suggests a variety of terminologies. We might, for example, say that we're conditioning the initial probability distribution π on a surjective function $f : X \to Y$, the output points $y \in Y$, the level sets defined by that point $f^{-1}(y) \in Y$, or even the subspace σ-algebras within that level set. All of this language, however, is just short-hand for conditioning on the *partition* implied by all of these intermediate objects.

3.2 Conditioning On General Implicit Partitions

Up to this point we have been able to define conditional probability distributions for measurable, π-non-null, and countable partitions that are defined either explicitly as a list of subsets

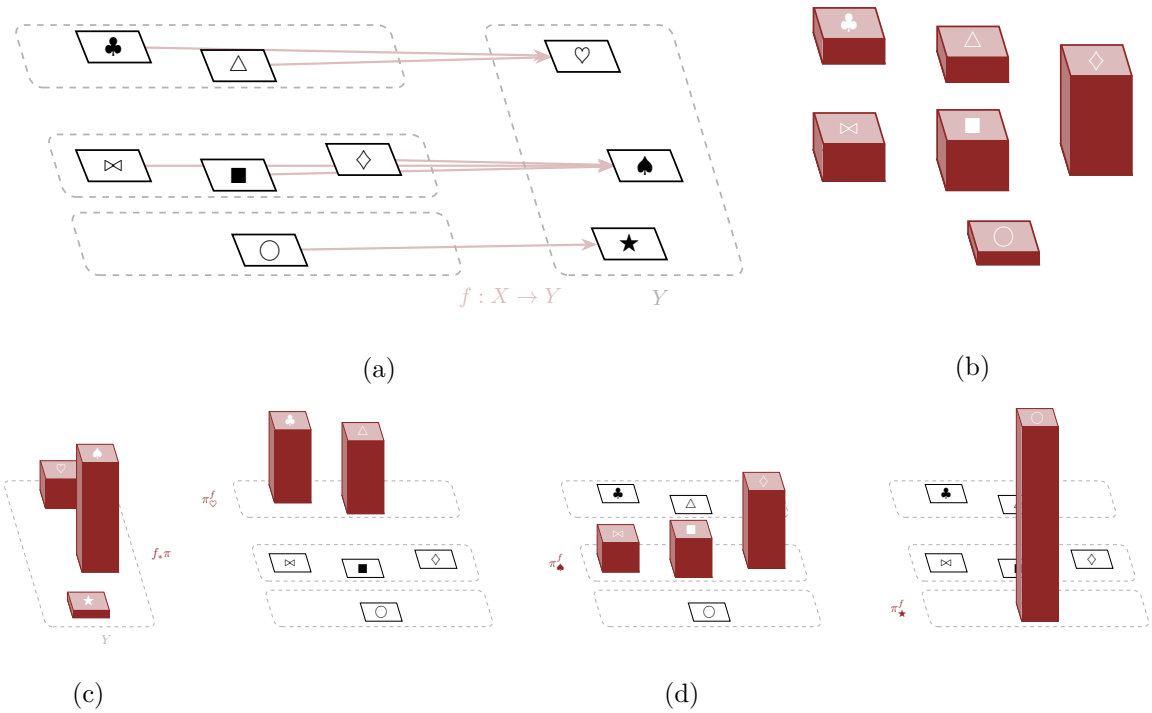


Figure 11: The partition implicitly defined by a surjective function can also be used to condition. (a) The level sets (b) decompose an initial probability distribution into (c) a pushforward probability distribution and (d) a conditional probability kernel. All of the information in the initial probability distribution is preserved one of these components. Moreover all probabilistic operations defined by the initial probability distribution can be computed using the pushforward and conditional probability distributions with the law of total expectation.

or implicitly as the level sets of a surjective function. Unfortunately this construction doesn't immediately generalize to the continuous spaces that dominate practical applications.

Consider for example a surjective function $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ where both the input space X and output space Y are continuous spaces with an uncountably infinite number of elements. This function implicitly defines a partition of the input space X into an uncountably infinite number of level sets $f^{-1}(y)$.

As in the countable case we can decompose any measurable subset $\mathsf{x} \in \mathcal{X}$ into its intersections with these level sets (Figure 12),

$$\mathsf{x} = \bigcup_{y \in Y} \left(\mathsf{x} \cap f^{-1}(y) \right).$$

Because there are an uncountably infinite number of intersections, however, we cannot write $\pi(\mathsf{x})$ as a sum over the intersection probabilities,

$$\pi(\mathsf{x}) = \pi \left(\bigcup_{y \in Y} \left(\mathsf{x} \cap f^{-1}(y) \right) \right) \neq \sum_{y \in Y} \pi(\mathsf{x} \cap f^{-1}(y)).$$

Remember that probability distributions are defined to have *countable* additivity but not uncountable additivity! Consequently we cannot define a law of total probability as a sum over individual output elements.

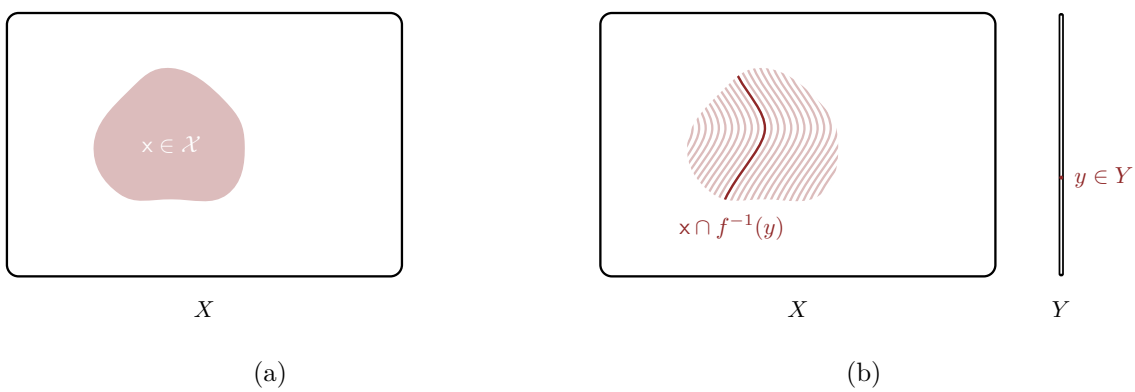


Figure 12: Given an uncountable partition (a) any measurable subset set (b) can be decomposed into an uncountable number of level set intersections. Because probability distributions are required to be only countably additive we cannot distribution probability allocations through this decomposition.

At the same time many probability distributions that we will encounter in practical applications of probability theory will allocate zero probability to either some or all of the level sets of the conditioning function,

$$\pi(f^{-1}(y)) = f_* \pi(\{y\}) = 0.$$

Consequently any attempt to directly define general conditional probabilities by the ratio

$$\pi^f(\mathsf{x} \mid y) = \pi^f_y(\mathsf{x}) = \frac{\pi(\mathsf{x} \cap f^{-1}(y))}{\pi(f^{-1}(y))}$$

would result with indefinite 0/0 outcomes.

Is there any hope for generalizing conditional probability to uncountable partitions? Fortunately the answer is yes. While we cannot *sum* over the individual level set probabilities we can define *expectations* over them. In particular the key is to generalize the expectation form of the law of total probability,

$$\pi(\mathsf{x}) = \mathbb{E}_{f_*\pi}[p_{\mathsf{x}}],$$

for some appropriate function

$$p_{\mathsf{x}} : Y \to [0, 1]$$

that we will have to define. Equivalently we can generalize the law of total expectation,

$$\mathbb{E}_\pi[g] = \mathbb{E}_{f_*\pi}[e_g],$$

for some appropriate function

$$e_g : Y \to [0, 1].$$

To formalize this generalization consider a surjective function $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ and a probability distribution $\pi : \mathcal{X} \to [0, 1]$. An intricate mathematical analysis (Chang and Pollard (1997); Leão Jr, Fragoso, and Ruffino (2004)) shows that if π is sufficiently well-behaved then, in addition to the pushforward distribution

$$f_*\pi : \mathcal{Y} \to [0, 1],$$

there always exists a conditional probability kernel

$$\begin{aligned} \pi^f(\cdot \mid \cdot) : \mathcal{X} \times Y &\to [0, 1] \subset \mathbb{R} \\ \mathsf{x}, y \quad &\mapsto \pi^f(\mathsf{x} \mid y) \ , \end{aligned}$$

that gives a $(\mathcal{Y}, \mathcal{B}_\mathbb{R})$-measurable conditional probability function for any partial evaluation on the first argument,

$$\pi^f(\mathsf{x} \mid \cdot) : Y \to [0, 1],$$

and a conditional probability distribution for $f_*\pi$-almost every partial evaluation on the second argument,

$$\begin{aligned} \pi^f_y : \mathcal{X} &\to [0, 1] \\ \mathsf{x} \ &\mapsto \pi^f(x \mid y). \end{aligned}$$

In particular the conditional probability distributions concentrate on the corresponding level set,

$$\pi^f_y(f^{-1}(y)) = \pi^f(f^{-1}(y) \mid y) = 1,$$

just as in the countable case.

Critically these partial evaluations satisfy a generalized law of total probability (Figure 13),

$$\pi(\mathsf{x}) = \mathbb{E}_{f_* \pi}[p_{\mathsf{x}}]$$

where $p_{\mathsf{x}}(y)$ is the conditional probability function. Moreover they also satisfy a generalized law of total expectation,

$$\mathbb{E}_{\pi}[g] = \mathbb{E}_{f_* \pi}[e_g]$$

where

$$\begin{aligned} e_g : Y &\to [0, 1] \\ y &\mapsto \mathbb{E}_{\pi_y^f}[g] \end{aligned}$$

is a conditional expectation value.

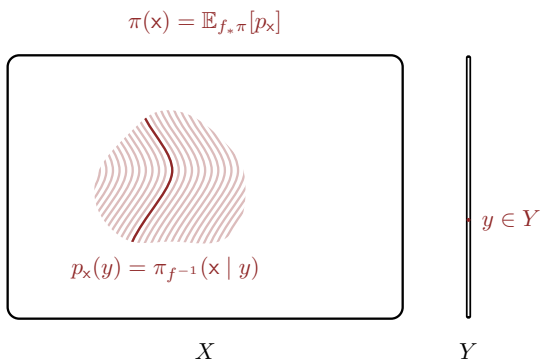


Figure 13: Although we can't always sum over the probabilities allocated to the intersection of a measurable subset $\mathsf{x} \in \mathcal{X}$ with the level sets of a sufficiently well-behaved function, we can take a pushforward expectation over the relative level set probabilities, $\pi(\mathsf{x}) = \mathbb{E}_{f_* \pi}[p_{\mathsf{x}}]$.

In the case where the output space, and hence the number of level sets, is countable these expectations reduce to discrete summations, and the general laws of total probability and expectation reduce to our initial laws of total probability and expectation. For example

$$\begin{aligned} \pi(\mathsf{x}) &= \mathbb{E}_{f_* \pi}[p_{\mathsf{x}}] \\ &= \sum_{y \in Y} f_* \pi(y) \, p_{\mathsf{x}}(y) \\ &= \sum_{y \in Y} \pi(f^{-1}(\{y\})) \, p^{f^{-1}}(\mathsf{x} \mid y) \\ &= \sum_{y \in Y} \pi(f^{-1}(\{y\})) \, \frac{\pi(\mathsf{x} \cap f^{-1}(\{y\}))}{\pi(f^{-1}(\{y\}))}. \end{aligned}$$

Any conditional probability kernel satisfying these properties is referred to as a **disintegration** of the probability distribution π with respect to f or, far less impressively, a **regular conditional probability distribution**, or **regular conditional probability kernel**. That said I find names like "regular conditional probability kernel" to be a bit of a mouthful. To streamline the terminology slightly I will use "conditional probability kernel" to refer to disintegrations generally, and "discrete conditional probability kernel" to refer to the special case of disintegrations with respect to functions that implicitly define countable partitions.

For countable output spaces a surjective function f and probability distribution π uniquely define a disintegration, and hence a discrete conditional probability kernel. More generally there will be infinitely many disintegrations compatible with a given function and probability distribution pair. The differences between these compatible disintegrations, however, are always confined to π-null subsets and, consequently, they all define equivalent probabilities and expectation values.

Disintegrations completely generalize the discrete conditional probability kernels that we derived for countable partitions. We can interpret disintegrations as a collection of probability distributions that concentrate on particular level sets or, equivalently, a collection of probability distributions defined directly on particular level sets. Moreover disintegrations can be also be interpreted as complementing the pushforward probability distribution, with the latter determining how much probability is allocated to each level set and the former determining how that total allocation unfurls across each level set.

There is one final technical detail that I have purposefully left ambiguous. Earlier I noted that disintegrations exist not for any probability distribution but rather only sufficiently "well-behaved" probability distributions. For those interested in exploring these details disintegrations can be derived only for a special class of measures known as **Radon measures**. Understanding what Radon measures are, and why they are needed to define disintegrations, goes far beyond the scope of this book. Fortunately every probability distribution we will encounter will be a Radon probability distribution, indeed non-Radon probability distributions are extremely weird mathematically, and we can consequently take this condition for granted.

4 Conditional Probability Density Functions

Conditional probability distributions are relatively straightforward to define abstractly. In practice, however, we will typically be working with not these probability distributions directly but rather their probability density function representations in the context of a convenient reference measure.

Unfortunately rigorously constructing conditional probability density functions is a bit more complicated. To do so properly we will need *all* of the measure theory tools that we have

developed to this point, and a few more that I will introduce below. Buckle up and make sure that you are aware of for your nearest emergency exit.

4.1 The Utility Of Integral Notation

Before diving into probability density functions let's take a second to ponder notation.

Recall that partially evaluating a regular conditional probability kernel on any $y \in Y$ yields a conditional probability distribution that concentrates on the level set $f^{-1}(y)$,

$$\begin{aligned} \pi_y^f : \mathcal{X} &\to [0, 1] \\ \mathsf{x} &\mapsto \pi^f(\mathsf{x} \mid y). \end{aligned}$$

When paired with an integrand $g : X \to \mathbb{R}$ the collection of all conditional probability distributions then defines a conditional expectation function,

$$\begin{aligned} e_g : Y &\to \mathbb{R} \\ y &\mapsto \mathbb{E}_{\pi_y^f}[g]. \end{aligned}$$

The law of total expectation ensures that the pushforward expectation of this conditional expectation function is always equal to the expectation value of the probability distribution that was disintegrated,

$$\mathbb{E}_\pi[g] = \mathbb{E}_{f_*\pi}\left[e_g\right].$$

Ideally we'd be able to write the law of total expectation more compactly, packing all of the contributions into a single line. Unfortunately any attempt at more compact notation is frustrated by the standard notational convention of ignoring arguments when denoting expectation values.

For example a notation that replaces y with a placeholder "$\cdot$",

$$\mathbb{E}_\pi[g] = \mathbb{E}_{f_*\pi}\left[e_g\right] = \mathbb{E}_{f_*\pi}\left[\mathbb{E}_{\pi^f_\cdot}[g]\right],$$

is not only hard to read but can be ambiguous in applications where there are multiple spaces on which f and g might act. At the same time a notation like

$$\mathbb{E}_\pi[g] = \mathbb{E}_{f_*\pi}\left[e_g\right] = \mathbb{E}_{f_*\pi}\left[\mathbb{E}_{\pi_y^f}[g(x)]\right]$$

fails to communicate on what spaces the probability distributions $f_*\pi$ and π_y^f are defined.

One way around these notational frustrations is to use the integral notation for expectation values that we first discussed in Chapter 5, Section 2.4 where appropriate variables specify the spaces on which all of the probability distributions and functions are defined.

For example if we interpret each conditional probability distribution π_y^f as being defined over the entirely of the ambient space X then the conditional expectation function can be written as

$$\begin{aligned} e_g(y) &= \mathbb{E}_{\pi_y^f}[g] \\ &= \int \pi^f(\mathrm{d}x \mid y)\, g(x), \end{aligned}$$

with the law of total expectation nesting measure-informed integrals over the entire ambient space within measure-informed integrals over the output space,

$$\begin{aligned} \mathbb{E}_\pi[g] &= \mathbb{E}_{f_*\pi}\left[e_g\right] \\ \int \pi(\mathrm{d}x)\, g(x) &= \int f_*\pi(\mathrm{d}y)\, e_g(y) \\ \int \pi(\mathrm{d}x)\, g(x) &= \int f_*\pi(\mathrm{d}y) \int \pi_y^f(\mathrm{d}x)\, g(x) \\ \int \pi(\mathrm{d}x)\, g(x) &= \int f_*\pi(\mathrm{d}y) \int \pi^f(\mathrm{d}x \mid y)\, g(x). \end{aligned}$$

To use the integral notation when we interpret each conditional probability distribution π_y^f as being defined over only the corresponding level set $f^{-1}(y) \subset X$ we need to be able to specify variables that take values in that level set. To that end let's introduce a **conditional variable** x_y that takes values in the level set corresponding to the output point $y \in Y$,

$$x_y \in f^{-1}(y) \subset X.$$

Conditional variables allow us to decompose any input variable $x \in X$ into an output variable and a corresponding conditional variable,

$$x = (y, x_y).$$

Note that the right hand-side isn't an ordered pair because the possible values of the second variable will in general depend on the choice of the first variable. For the mathematically-curious this construction is known as a **semi-direct product**.

Using conditional variables we can then write the conditional expectation function as

$$\begin{aligned} e_g(y) &= \mathbb{E}_{\pi_y^f}[g] \\ &= \int \pi_y^f(\mathrm{d}x_y)\, g(y, x_y) \\ &= \int \pi^f(\mathrm{d}x_y \mid y)\, g(y, x_y). \end{aligned}$$

The law of total expectation then nests measure-informed integrals over the level sets of f within an measure-informed integral over the output space,

$$\begin{aligned}
\mathbb{E}_{\pi}[g] &= \mathbb{E}_{f_* \pi}\left[e_g\right] \\
\int \pi(\mathrm{d}x)\, g(x) &= \int f_* \pi(\mathrm{d}y)\, e_g(y) \\
\int \pi(\mathrm{d}x)\, g(x) &= \int f_* \pi(\mathrm{d}y) \int \pi^f(\mathrm{d}x_y \mid y)\, g(y, x_y).
\end{aligned}$$

Conditional variables are by no means universal and there many other notational conventions for specifying measure-informed integrals over individual level sets that one might encounter. Some references, for example, decorate the integral sign with the relevant spaces,

$$\int_X \pi(\mathrm{d}x)\, g(x) = \int_Y f_* \pi(\mathrm{d}y) \int_{f^{-1}(y)} \pi^f(\mathrm{d}x \mid y)\, g(x),$$

while others use δ function-like shorthands such as

$$\int \pi(\mathrm{d}x)\, g(x) = \int f_* \pi(\mathrm{d}y) \int \pi^f(\mathrm{d}x \mid y)\, \delta(y - f(x))\, g(x)$$

or

$$\int \pi(\mathrm{d}x)\, g(x) = \int f_* \pi(\mathrm{d}y) \int \pi^f(\mathrm{d}x \mid y)\, \delta(f^{-1}(y))\, g(x)$$

to communicate the domain of integration. In this book, however, I will tend to favor the conditional variable notation as I find that it offers the best compromise between compactness and informativeness.

Finally the integral relationships implied by the law of total expectation are often simplified to relationships between the integrands. For example

$$\int \pi(\mathrm{d}x)\, g(x) = \int f_* \pi(\mathrm{d}y) \int \pi^f(\mathrm{d}x \mid y)\, g(x)$$

might be represented by

$$\pi(\mathrm{d}x) = f_* \pi(\mathrm{d}y) \pi^f(\mathrm{d}x \mid y),$$

and

$$\int \pi(\mathrm{d}x)\, g(x) = \int f_* \pi(\mathrm{d}y) \int \pi^f(\mathrm{d}x_y \mid y)\, g(x_y)$$

represented by

$$\pi(\mathrm{d}x) = f_* \pi(\mathrm{d}y) \pi^f(\mathrm{d}x_y \mid y).$$

We always have to be careful, however, to recognize that these simpler integrand equations are just notational shorthands for the full integral relationships and not misinterpret them otherwise. For example in general we do not have

$$\pi(\mathsf{x}) = f_* \pi(\mathsf{y}) \pi^f(\mathsf{x} \mid y)$$

for any combination of input subset $\mathsf{x} \in \mathcal{X}$, output subset $\mathsf{y} \in \mathcal{Y}$, and output point $y \in Y$.

4.2 Conditional Probability Density Functions For Non-Null Partitions

With our notational tools set let's make our first step into conditional probability density functions by considering the simplest case of a countable, non-null partition.

As usual we begin with an initial probability space $(X, \mathcal{X}, \pi)$. Next we'll introduce a surjective function $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ that maps the initial space into a countable output space such that each level set is allocated finite probability,

$$\pi(f^{-1}(y)) > 0.$$

Note that we don't need the output space to be countable for *some* level sets to be allocated finite probability, but we do need it to be countable for *all* level sets to be allocated finite probability.

With these assumptions the law of total expectation becomes

$$\begin{aligned}
\int \pi(\mathrm{d}x)\, g(x) &= \int f_* \pi(\mathrm{d}y) \int \pi^f(\mathrm{d}x \mid y)\, g(x) \\
\int \pi(\mathrm{d}x)\, g(x) &= \sum_{y \in Y} f_* \pi(\{y\}) \int \pi^f(\mathrm{d}x \mid y)\, g(x) \\
\int \pi(\mathrm{d}x)\, g(x) &= \sum_{y \in Y} \pi(f^{-1}(y)) \int \pi^f(\mathrm{d}x \mid y)\, g(x) \\
\int \pi(\mathrm{d}x)\, g(x) &= \sum_{y \in Y} \int \pi^f(\mathrm{d}x \mid y)\, \pi(f^{-1}(y))\, g(x).
\end{aligned}$$

To introduce probability density functions we need a sufficiently well-behaved reference measure. Let's assume a σ-finite reference measure μ that dominates our target probability distribution π and allows us to write the left-hand side as

$$\int \pi(\mathrm{d}x)\, g(x) = \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, g(x).$$

In order to write the conditional expectation values on the right-hand side as μ-informed integrals we need each π^f_y to also be absolutely continuous with respect to μ. Because each π^f_y completely concentrates on the corresponding level set $f^{-1}(y)$ absolutely continuity requires that μ allocates finite measure to each level set,

$$\mu(f^{-1}(y)) > 0.$$

Fortunately this is automatically guaranteed by our existing assumptions. If π is absolutely continuous with respect to μ then $\pi(\mathsf{x}) > 0$ only if $\mu(\mathsf{x}) > 0$. Consequently if $\pi(f^{-1}(y)) > 0$ then we have to have $\mu(f^{-1}(y)) > 0$ as well.

With absolutely continuity ensured we can write the right-hand side as

$$\sum_{y \in Y} \int \pi^f(\mathrm{d}x \mid y)\, \pi(f^{-1}(y))\, g(x) = \sum_{y \in Y} \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu}(x \mid y)\, \pi(f^{-1}(y))\, g(x).$$

Putting both sides together we have

$$\begin{aligned}
\int \pi(\mathrm{d}x)\, g(x) &= \sum_{y \in Y} \int \pi^f(\mathrm{d}x \mid y)\, \pi(f^{-1}(y))\, g(x) \\
\int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, g(x) &= \sum_{y \in Y} \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu}(x \mid y)\, \pi(f^{-1}(y))\, g(x),
\end{aligned}$$

where $\frac{\mathrm{d}\pi^f}{\mathrm{d}\mu}(x \mid y)$ is a collection of probability density functions over X indexed by output points in Y.

Unfortunately we still can't compare the integrands because of the sum over output elements on the right-hand side. To enable a proper comparison we will need to split the μ-informed integral on the left-hand side into a sum of μ-informed integrals for each output element $y \in Y$. One particularly nice way to do this is to take advantage of the fact that, because the level sets of f for a partition of X, the corresponding indicator functions always sum to one,

$$1 = \sum_{y \in Y} I_{f^{-1}(y)}(x)$$

for any input point $x \in X$.

Inserting a sum over output elements directly into the left-hand side of our current equation gives

$$\begin{aligned}
\int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, g(x) &= \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, 1\, g(x) \\
&= \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x) \left[\sum_{y \in Y} I_{f^{-1}(y)}(x) \right] g(x).
\end{aligned}$$

Because measure-informed integrals are countably linear we can then pull the summation outside of the measure-informed integral to give

$$\begin{aligned}
&\int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, g(x) \\
&\qquad = \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x) \left[\sum_{y \in Y} I_{f^{-1}(y)}(x) \right] g(x) \\
&\qquad = \sum_{y \in Y} \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, I_{f^{-1}(y)}(x)\, g(x).
\end{aligned}$$

After all of this work we now have

$$\int \pi(\mathrm{d}x)\, g(x) = \int f_* \pi(\mathrm{d}y) \int \pi^f(\mathrm{d}x \mid y)\, g(x)$$

$$\int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, g(x) = \sum_{y \in Y} \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu}(x \mid y)\, \pi(f^{-1}(y))\, g(x)$$

$$\sum_{y \in Y} \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, I_{f^{-1}(y)}(x)\, g(x) = \sum_{y \in Y} \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu}(x \mid y)\, \pi(f^{-1}(y))\, g(x).$$

In order for these summed integrals to be equal for any expectand $g : (X, \mathcal{X}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ we must have

$$\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, I_{f^{-1}(y)}(x) \overset{\mu}{=} \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu}(x \mid y)\, \pi(f^{-1}(y)),$$

or

$$\frac{\mathrm{d}\pi^f}{\mathrm{d}\mu}(x \mid y) \overset{\mu}{=} \frac{\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, I_{f^{-1}(y)}(x)}{\pi(f^{-1}(y))}.$$

Intuitively for any $y \in Y$ the corresponding conditional probability density function is given by truncating the initial probability density function $\mathrm{d}\pi/\mathrm{d}\mu$ to the level set $f^{-1}(y)$, zeroing the output for any inputs outside of $f^{-1}(y)$ and then correcting the normalization. Geometrically conditioning on a function with a countable output space is equivalent to *slicing* $\mathrm{d}\pi/\mathrm{d}\mu$ along the level sets boundaries and re-weighting the slices to ensure a proper normalization (Figure 14).

To double check our construction we can verity that this result is consistent with each conditional probability density function π^f_y completely concentrating on the corresponding level set,

$$\begin{aligned}
\pi^f_y(f^{-1}(y)) &= \pi^f(f^{-1}(y) \mid y) \\
&= \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu}(x \mid y)\, I_{f^{-1}(y)}(x) \\
&= \int \mu(\mathrm{d}x)\, \frac{\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, I_{f^{-1}(y)}(x)}{\pi(f^{-1}(y))}\, I_{f^{-1}(y)}(x) \\
&= \frac{1}{\pi(f^{-1}(y))} \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, \left(I_{f^{-1}(y)}(x)\right)^2 \\
&= \frac{1}{\pi(f^{-1}(y))} \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, I_{f^{-1}(y)}(x) \\
&= \frac{1}{\pi(f^{-1}(y))} \pi(f^{-1}(y)) \\
&= 1.
\end{aligned}$$

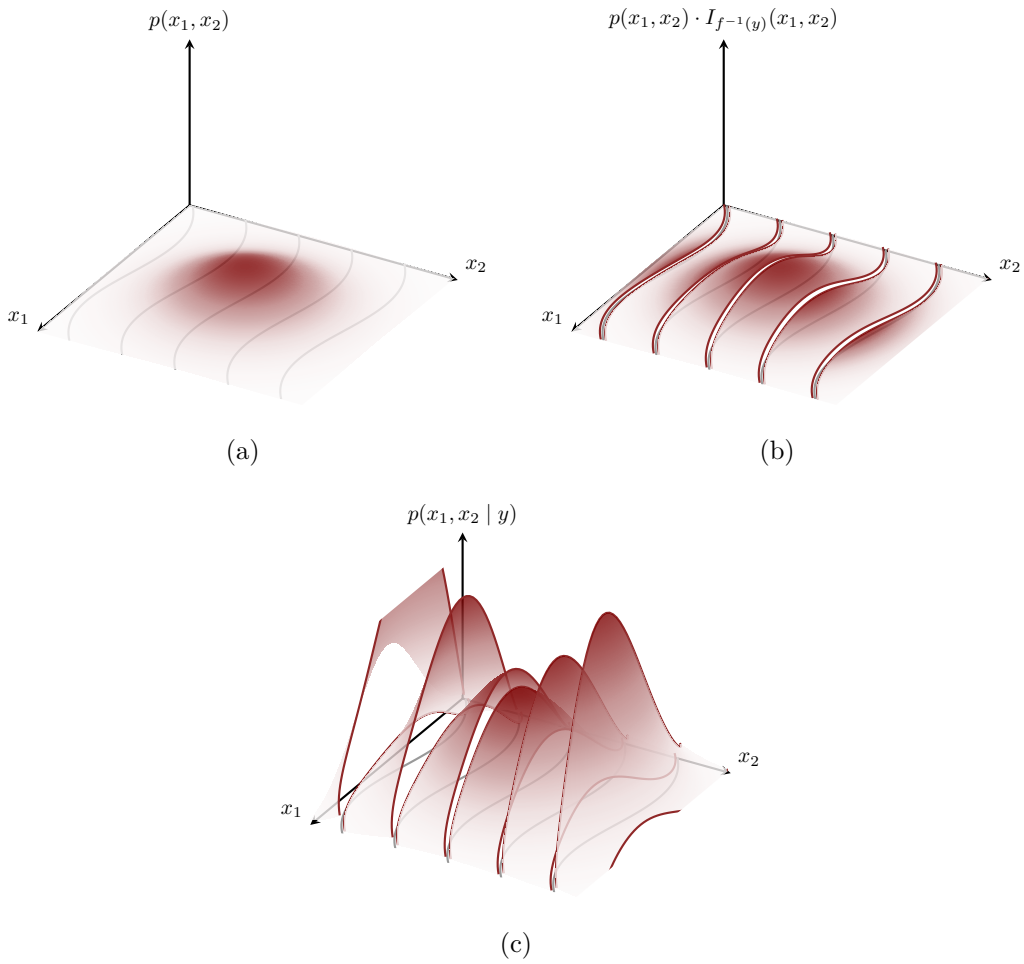


Figure 14: Conditional probability density functions are straightforward to construct for countable partitions. (a) A probability density function representing the initial probability distribution is first (b) sliced into density functions restricted to each level set. (c) Once properly normalized these density functions become conditional probability density functions that represent each conditional probability distribution.

Equivalently for any measurable subset $\mathsf{x} \in \mathcal{X}$ that is disjoint with a particular level set $\mathsf{x} \cap f^{-1}(y) = \emptyset$ we have

$$
\begin{aligned}
\pi_y^f(\mathsf{x}) &= \pi^f(\mathsf{x} \mid y) \\
&= \int \mu(\mathrm{d}x) \, \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu}(x \mid y) \, I_{\mathsf{x}}(x) \\
&= \int \mu(\mathrm{d}x) \, \frac{\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x) \, I_{f^{-1}(y)}(x)}{\pi(f^{-1}(y))} \, I_{\mathsf{x}}(x) \\
&= \frac{1}{\pi(f^{-1}(y))} \int \mu(\mathrm{d}x) \, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x) \, I_{f^{-1}(y)}(x) \, I_{\mathsf{x}}(x) \\
&= \frac{1}{\pi(f^{-1}(y))} \int \mu(\mathrm{d}x) \, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x) \, I_{\mathsf{x} \cap f^{-1}(y)}(x) \\
&= \frac{1}{\pi(f^{-1}(y))} \int \mu(\mathrm{d}x) \, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x) \, I_{\emptyset}(x) \\
&= \frac{1}{\pi(f^{-1}(y))} \cdot 0 \\
&= 0.
\end{aligned}
$$

4.3 The Problem With Null Partitions

Unfortunately this construction doesn't carry over to functions with more general output spaces that might contain an uncountably-infinite number of points. In this case at least some, if not all, of the level sets will be allocated vanishing probabilities,

$$\pi(f^{-1}(y)) = 0.$$

At the same time σ-finite reference measures will allocate vanishing measure to at least some, if not all, of the level sets,

$$\mu(f^{-1}(y)) = 0.$$

These behaviors immediately obstruct many steps in our construction of a discrete conditional probability density function. For example when $\pi(f^{-1}(y)) = 0$ the final definition of a discrete conditional probability density function

$$\frac{\mathrm{d}\pi^f}{\mathrm{d}\mu}(x \mid y) \overset{\mu}{=} \frac{\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x) \, I_{f^{-1}(y)}(x)}{\pi(f^{-1}(y))}$$

requires an ill-defined division by zero.

Problems, however, actually arise much earlier in the calculation. On the right-hand side of the law of total expectation we cannot convert the output expectation value over $f_* \pi$ into a

sum over individual output elements if Y is uncountable. Similarly when Y is uncountable we cannot apply the *countable* linearity of measure-informed integration to the constant function

$$1 = \sum_{y \in Y} I_{f^{-1}(y)}(x).$$

The most fundamental of our problems is that when $\mu(f^{-1}(y)) = 0$ any probability distribution that is absolutely continuous with respect to μ must also allocate zero probability to $f^{-1}(y)$. The conditional probability distributions π_y^f, however, allocate not just a non-zero probability to the level set $f^{-1}(y)$ but in fact all of their probability! In other words the conditional probability distributions are typically not absolutely continuous with respect to μ, preventing us from converting conditional expectation values into μ-informed integrals weighted by a conditional probability density function in the first place. Absolute continuity is easy to disregard as unnecessarily abstract, but it has important practical consequences like this!

Yet another way to see why we need a more general construction of conditional probability theory is to assume that a probability density function of a particular π_y^f with respect to μ does exist and show that a mathematical inconsistency arises. For example in order to ensure that

$$\pi_y^f(f^{-1}(y)) = \pi^f(f^{-1}(y) \mid y) = 1$$

we would need a conditional probability density function to satisfy

$$\begin{aligned} 1 &= \pi^f(f^{-1}(y) \mid y) \\ &= \int \pi^f(\mathrm{d}x \mid y)\, I_{f^{-1}(y)}(x) \\ &= \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu}(x \mid y)\, I_{f^{-1}(y)}(x). \end{aligned}$$

If, however, $\mu(f^{-1}(y)) = 0$ then the indicator function will be non-zero for only a μ-null subset of inputs. Consequently in terms of μ-informed integrals it should be equivalent to the zero function,

$$I_{f^{-1}(y)}(x) \overset{\mu}{=} 0.$$

This would require that

$$\begin{aligned} \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu}(x \mid y)\, I_{f^{-1}(y)}(x) &= \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu}(x \mid y) \cdot 0 \\ &= 0. \end{aligned}$$

Unfortunately

$$1 = \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu}(x \mid y)\, I_{f^{-1}(y)}(x) = 0$$

is a pretty immediate mathematical contradiction.

Notice the similarity of this inconsistent behavior with the awkward behavior we encountered when exploring the Dirac delta function in Chapter 6, Section 5.1. Intuitively when $f^{-1}(y)$ is a μ-null subset the corresponding conditional probability distribution π_y^f becomes singular, and probability density functions become ill-defined without opening our hearts and minds to generalized functions like the Dirac delta function.

One way to avoid this singular behavior is to embrace the interpretation of each conditional probability distribution π_y^f being defined over not all of the ambient space X but rather just the corresponding level set $f^{-1}(y)$. From this perspective we can define conditional probability density functions with respect to σ-finite reference measures defined on the level sets themselves,

$$\int \pi^f(\mathrm{d}x \mid y)\, g(x) = \int \nu_y(\mathrm{d}x)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\nu_y}(x \mid y)\, g(x).$$

Incorporating these probability functions into the law of total expectation, however, requires an explicit relationship between these level set reference measures ν_y to the ambient reference measure μ. This, in turn, requires extending the disintegration of probability measures to the disintegration of more general measures.

4.4 Disintegrating Measures

In Section 3.2 we introduced disintegrations of probability distributions. This definition pretty immediately generalizes to finite measures, which for mathematical purposes are equivalent to probability distributions, but it becomes problematic when working with non-finite measures. Even σ-finite measures require some extra care to decompose across null subsets.

The core mathematical issue is that a consistent disintegration of a measure μ with respect to a function $f : X \to Y$ requires not only that the initial measure μ is σ-finite but also that its pushforward $f_*\mu$ is σ-finite. Unfortunately this latter condition fails for many convenient reference measures.

Consider, for example, a rigid two-dimensional real space $\mathbb{R}^2$ equipped with the two-dimensional Lebesgue measure λ^2 and a projection function

$$\begin{aligned} \varpi_1 : \mathbb{R}^2 &\to \mathbb{R} \\ (x_1, x_2) &\mapsto x_1. \end{aligned}$$

The Lebesgue measure λ^2 is σ-finite, allocating finite measure to every measurable subset that can be encapsulated in a finite rectangle. Formally if

$$\mathsf{x} \subset [0, 1] \times [0, 1]$$

then

$$\begin{aligned}
\lambda^2(\mathsf{x}) &< \lambda^2([0,1] \times [0,1]) \\
&< l([0,1]) \cdot l([0,1]) \\
&< 1 \cdot 1 \\
&< 1.
\end{aligned}$$

Pushing λ^2 forward along ϖ_1, however, results in a measure that allocates *infinite* measure to finite intervals (Figure 15),

$$\begin{aligned}
(\varpi_1)_* \lambda^2([0,1]) &= \lambda^2(\varpi_1^*[0,1]) \\
&= \lambda^2([0,1] \times (-\infty, \infty)) \\
&= l([0,1]) \cdot l((-\infty, \infty)) \\
&= 1 \cdot \infty \\
&= \infty.
\end{aligned}$$

Consequently $(\varpi_1)_* \lambda^2$ cannot be σ-finite.

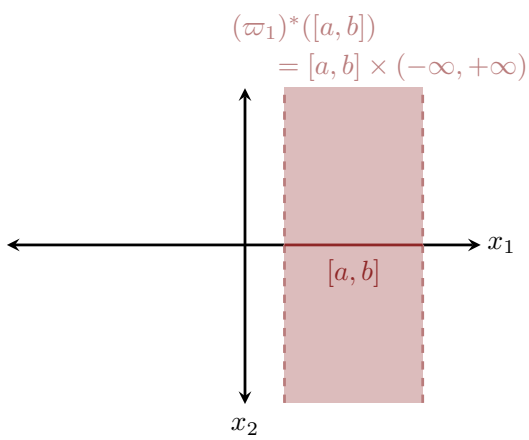


Figure 15: On $\mathbb{R}^2$ the projection function $\phi_1 : (x_1, x_2) \mapsto x_1$ pulls back finite intervals $[a,b]$ into infinite rectangles $[a,b] \times (-\infty, +\infty)$. Consequently the two dimensional Lebesgue measure λ^2 projects infinite measure onto finite intervals and the pushforward measure $(\phi_1)_* \lambda^2$ cannot be σ-finite. In particular λ^2 does *not* pushforward to a Lebesgue measure!

Fortunately disintigrations can be generalized to work with not only the pushforward of the target measure but also *any* convenient σ-finite measure on the output space. Mathematically if we have

1. an input measurable space $(X, \mathcal{X})$,

2. a σ-finite Radon measure $\mu : \mathcal{X} \to [0, \infty]$,
3. an output Hausdorff measurable space $(Y, \mathcal{Y})$.
4. a surjective measurable function $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$,
5. and finally a σ-finite measure $\nu : \mathcal{Y} \to [0, \infty]$

then there exists at least one **conditional measure kernel**

$$\begin{aligned} \mu^{f,\nu} : \mathcal{X} \times Y &\to [0, \infty] \\ \mathsf{x}, y \quad &\mapsto \mu^f(\mathsf{x} \mid y) \end{aligned}$$

that defines a $(\mathcal{Y}, \mathcal{B}_{\mathbb{R}})$-measurable function when partially evaluated on the first argument,

$$\begin{aligned} \mu^{f,\nu}_{\mathsf{x}} : Y &\to [0, \infty] \\ y &\mapsto \mu^{f,\nu}(\mathsf{x} \mid y) \end{aligned}$$

for all $\mathsf{x} \in \mathcal{X}$, and a σ-finite measure when partially evaluated on the second argument,

$$\begin{aligned} \mu^{f}_{y,\nu} : \mathcal{X} &\to [0, \infty] \\ \mathsf{x} &\mapsto \mu^{f,\nu}(\mathsf{x} \mid y) \end{aligned}$$

for ν-almost all $y \in Y$. A more technical discussion can be found in Chang and Pollard (1997).

The conditional measures derived from a conditional measure kernel behave very similarly to conditional probability distributions. For example they each concentrate on a particular level set,

$$\mu^{f,\nu}_y(f^{-1}(x)) \overset{\nu}{=} 1$$

with

$$\mu^{f,\nu}_y(\mathsf{x}) \overset{\nu}{=} 0$$

for any $\mathsf{x} \in \mathcal{X}$ with $\mathsf{x} \cap f^{-1}(y) = \emptyset$. They also satsify a law of total integration,

$$\int \mu(\mathrm{d}x)\, g(x) = \int \nu(\mathrm{d}y) \int \mu^{f,\nu}(\mathrm{d}x_y \mid y)\, g(x),$$

for any well-behaved integrand $g : X \to \mathbb{R}$.

In circumstances where $f_* \mu$ happens to be σ-finite we can always take $\nu = f_* \mu$ so that the law of total integration mirrors the law of total expectation. This is always possible if μ is a finite measure, and hence always possible when disintegrating probability distributions, but it is not always viable when μ is only σ-finite. In particular we have to be vigilent when attempting to disintegrate the Lebesgue and counting measures that we often turn to for convenient reference measures as they often pushforward to measures that are not σ-finite.

4.5 Conditional Probability Density Functions For General Implicit Partitions

Armed with a technique for disintegrating σ-finite measures we are now finally equipped with enough tools to construct conditional probability density functions for any conditional probability distribution and sufficiently well-behaved reference measure.

Recall that to construct a conditional probability distribution we need

1. an input measurable space $(X, \mathcal{X})$,
2. a Radon probability distribution $\pi : \mathcal{X} \to [0, 1]$,
3. an output Hausdorff measurable space $(Y, \mathcal{Y})$.
4. and a surjective measurable function $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$.

In order to construct conditional probability density functions we will also need convenient σ-finite Radon reference measures for

5. the input space, $\mu : \mathcal{X} \to [0, \infty]$,
6. the output space, $\nu : \mathcal{Y} \to [0, \infty]$,
7. and each level set, $\eta_y : \mathcal{F}_y \to [0, \infty]$.

As we have previously discussed we can safely take measurable functions, Hausdorff σ-algebras, and Radon measures for granted in practice. We will, however, have to be careful about the surjectivity of f and the σ-finiteness of the reference measures.

If $\pi \ll \mu$ then we can construct the probability density function

$$\frac{\mathrm{d}\pi}{\mathrm{d}\mu} : X \to \mathbb{R}^+,$$

and if $f_* \pi \ll \nu$ then we can construct the pushforward probability density function

$$\frac{\mathrm{d} f_* \pi}{\mathrm{d}\nu} : Y \to \mathbb{R}^+.$$

Upon disintegrating μ we can also construct the conditional probability density functions relative to the conditional measures,

$$\frac{\mathrm{d}\pi^f}{\mathrm{d}\mu^{f,\nu}} : X \times Y \to \mathbb{R}^+.$$

Finally if $\mu_y^{f,\nu} \ll \eta_y$ then we can convert these conditional probability density functions into conditional probability density functions relative to our level set reference measures,

$$\frac{\mathrm{d}\pi^f}{\mathrm{d}\eta_y}(x \mid y) \overset{\eta_y}{=} \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu^{f,\nu}}(x \mid y) \cdot \frac{\mathrm{d}\mu^{f,\nu}}{\mathrm{d}\eta_y}(x \mid y).$$

In many applications the conditional measures $\mu_y^{f,\nu}$ will be convenient reference measures and we will not need to consider an auxiliarycollection of reference measures η_y. That said here I will consider the more general scenario for completeness.

All that we're missing is a mathematical relationship that ties this menagerie of probability density functions together. That information is hidden within the law of total expectation,

$$\begin{aligned}\int \pi(\mathrm{d}x)\, g(x) &= \int f_* \pi(\mathrm{d}y) \int \pi^f(\mathrm{d}x_y \mid y)\, g(y, x_y) \\ L &= R.\end{aligned}$$

All we need to do is convert both sides of this equation into the same kind of measure-informed integral.

Let's start with the left-hand side,

$$\begin{aligned}L &= \int \pi(\mathrm{d}x)\, g(x) \\ &= \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, g(x).\end{aligned}$$

If we disintegrate μ with respect to f and ν this becomes

$$\begin{aligned}L &= \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, g(x) \\ &= \int \nu(\mathrm{d}y) \int \mu^{f,\nu}(\mathrm{d}x_y \mid y)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y)\, g(y, x_y).\end{aligned}$$

Incorporating the level set reference measures then gives

$$\begin{aligned}L &= \int \nu(\mathrm{d}y) \int \mu^{f,\nu}(\mathrm{d}x_y \mid y)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y)\, g(y, x_y) \\ &= \int \nu(\mathrm{d}y) \int \eta_y(\mathrm{d}x_y)\, \frac{\mathrm{d}\mu^{f,\nu}}{\mathrm{d}\eta_y}(x_y \mid y)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y)\, g(y, x_y) \\ &= \int \nu(\mathrm{d}y) \int \eta_y(\mathrm{d}x_y) \left[\frac{\mathrm{d}\mu^{f,\nu}}{\mathrm{d}\eta_y}(x_y \mid y)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y) \right] g(y, x_y).\end{aligned}$$

Over on the right-hand side we have

$$\begin{aligned}R &= \int f_* \pi(\mathrm{d}y) \int \pi^f(\mathrm{d}x_y \mid y)\, g(y, x_y) \\ &= \int \nu(\mathrm{d}y)\, \frac{\mathrm{d} f_* \pi}{\mathrm{d}\nu}(y) \int \pi^f(\mathrm{d}x_y \mid y)\, g(y, x_y) \\ &= \int \nu(\mathrm{d}y)\, \frac{\mathrm{d} f_* \pi}{\mathrm{d}\nu}(y) \int \eta_y(\mathrm{d}x_y)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\eta_y}(x_y \mid y)\, g(y, x_y).\end{aligned}$$

Because the domain of the inner measure-informed integral is single level set the pushforward probability density function $\mathrm{d}f_*\pi/\mathrm{d}\nu$ is a constant that can be pulled inside,

$$
\begin{aligned}
R &= \int f_*\pi(\mathrm{d}y) \int \pi^f(\mathrm{d}x_y \mid y)\, g(y, x_y) \\
&= \int \nu(\mathrm{d}y)\, \frac{\mathrm{d}f_*\pi}{\mathrm{d}\nu}(y) \int \eta_y(\mathrm{d}x_y)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\eta_y}(x_y \mid y)\, g(y, x_y) \\
&= \int \nu(\mathrm{d}y) \int \eta_y(\mathrm{d}x)\, \frac{\mathrm{d}f_*\pi}{\mathrm{d}\nu}(y)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\eta_y}(x_y \mid y)\, g(y, x_y) \\
&= \int \nu(\mathrm{d}y) \int \eta_y(\mathrm{d}x) \left[\frac{\mathrm{d}f_*\pi}{\mathrm{d}\nu}(y)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\eta_y}(x_y \mid y) \right] g(y, x_y).
\end{aligned}
$$

We can now put these two pieces back together,

$$
\begin{aligned}
L &= R \\
\int \pi(\mathrm{d}x)\, g(x) & \\
&= \int f_*\pi(\mathrm{d}y) \int \pi^f(\mathrm{d}x_y \mid y)\, g(y, x_y) \\
\int \nu(\mathrm{d}y) \int \eta_y(\mathrm{d}x_y) & \left[\frac{\mathrm{d}\mu^{f,\nu}}{\mathrm{d}\eta_y}(x_y \mid y)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y) \right] g(y, x_y) \\
&= \int \nu(\mathrm{d}y) \int \eta_y(\mathrm{d}x_y) \left[\frac{\mathrm{d}f_*\pi}{\mathrm{d}\nu}(y)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\eta_y}(x_y \mid y) \right] g(y, x_y).
\end{aligned}
$$

Because both sides of the equation are the same kind of measure-informed integral we have equality if and only if the integrands on both sides are equal up to null subsets. In particular we have equality for all integrands $g : X \to \mathbb{R}$ if and only if

$$
\frac{\mathrm{d}\mu^{f,\nu}}{\mathrm{d}\eta_y}(x_y \mid y)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y) \overset{\nu,\eta_y}{=} \frac{\mathrm{d}f_*\pi}{\mathrm{d}\nu}(y)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\eta_y}(x_y \mid y).
$$

Isolating the ambient probability density function gives

$$
\begin{aligned}
\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y) &\overset{\eta_y}{=} \frac{\mathrm{d}f_*\pi}{\mathrm{d}\nu}(y)\, \frac{\frac{\mathrm{d}\pi^f}{\mathrm{d}\eta_y}(x_y \mid y)}{\frac{\mathrm{d}\mu^{f,\nu}}{\mathrm{d}\eta_y}(x_y \mid y)} \\
\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y) &\overset{\mu}{=} \frac{\mathrm{d}f_*\pi}{\mathrm{d}\nu}(y)\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu^{f,\nu}}(x_y \mid y),
\end{aligned}
$$

or, in terms of unconditional variables,

$$
\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x) \overset{\mu}{=} \frac{\mathrm{d}f_*\pi}{\mathrm{d}\nu}(f(x))\, \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu^{f,\nu}}(x \mid f(x)).
$$

This relationship is known as the **product rule** for probability density functions. The product rule allows to construct the ambient probability density function, the conditional probability density function, or the pushforward probability density function given the other two. For example if we know the ambient probability density function and the pushforward probability density function then the conditional probability density function is given by

$$\frac{\mathrm{d}\pi^f}{\mathrm{d}\mu^{f,\nu}}(x_y \mid y) \overset{\mu}{=} \frac{\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y)}{\frac{\mathrm{d}f_*\pi}{\mathrm{d}\nu}(y)}.$$

In words we can conditional a probability density function $\mathrm{d}\pi/\mathrm{d}\mu$ to a particular output point $y \in Y$ in two steps. First we restrict the inputs of $\mathrm{d}\pi/\mathrm{d}\mu$ to the points in the level set $f^{-1}(y)$ (Figure 16b}). Then we divide by the outputs of $\mathrm{d}\pi/\mathrm{d}\mu$ by the pushforward probability density function evaluated at y, $\frac{\mathrm{d}f_*\pi}{\mathrm{d}\nu}(y)$ (Figure 16c).

Notice, however, that this last step doesn't change the *shape* of the conditional probability density function, just its height. In applications where we don't need to worry about the normalization we ignore this last step, and any difficulty in evaluating the pushforward probability density function, entirely.

To demonstrate this process consider a function $f : X \to \mathbb{N}$ that maps input points to output integers and induces a countable partition. Because the output space is discrete the counting measure is a natural output reference measure, $\nu = \chi$. Moreover if $f_*\mu(\{y\}) > 0$ for all $y \in \mathbb{N}$ then each $\mu_y^{f,\chi}$ becomes μ truncated to a particular level set,

$$\mu_y^{f,\chi} = \eta_y = I_{f^{-1}(y)} \cdot \mu.$$

In this case the product rule gives

$$\begin{aligned}
\frac{\mathrm{d}\pi^f}{\mathrm{d}\mu^{f,\chi}}(x_y \mid y) &\overset{\mu}{=} \frac{\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y)}{\frac{\mathrm{d}f_*\pi}{\mathrm{d}\chi}(y)} \\
&\overset{\mu}{=} \frac{\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y)}{f_*\pi(\{y\})} \\
&\overset{\mu}{=} \frac{\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y)}{\pi(f^{-1}(y))}.
\end{aligned}$$

For a given $y \in Y$ we can extend these conditional density functions to all inputs $x \in X$ by returning zero outside of the corresponding level set,

$$\frac{\mathrm{d}\pi^f}{\mathrm{d}\mu^{f,\chi}}(x \mid y) \overset{\mu}{=} \begin{cases} \frac{\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)}{\pi(f^{-1}(y))}, & x \in f^{-1}(y) \\ 0, & x \notin f^{-1}(y) \end{cases},$$

or more compactly,

$$\frac{\mathrm{d}\pi^f}{\mathrm{d}\mu^{f,\chi}}(x \mid y) \overset{\mu}{=} \frac{\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, I_{f^{-1}(y)}(x)}{\pi(f^{-1}(y))}.$$

This is encouragingly consistent with the result that we derived in Section 4.2.

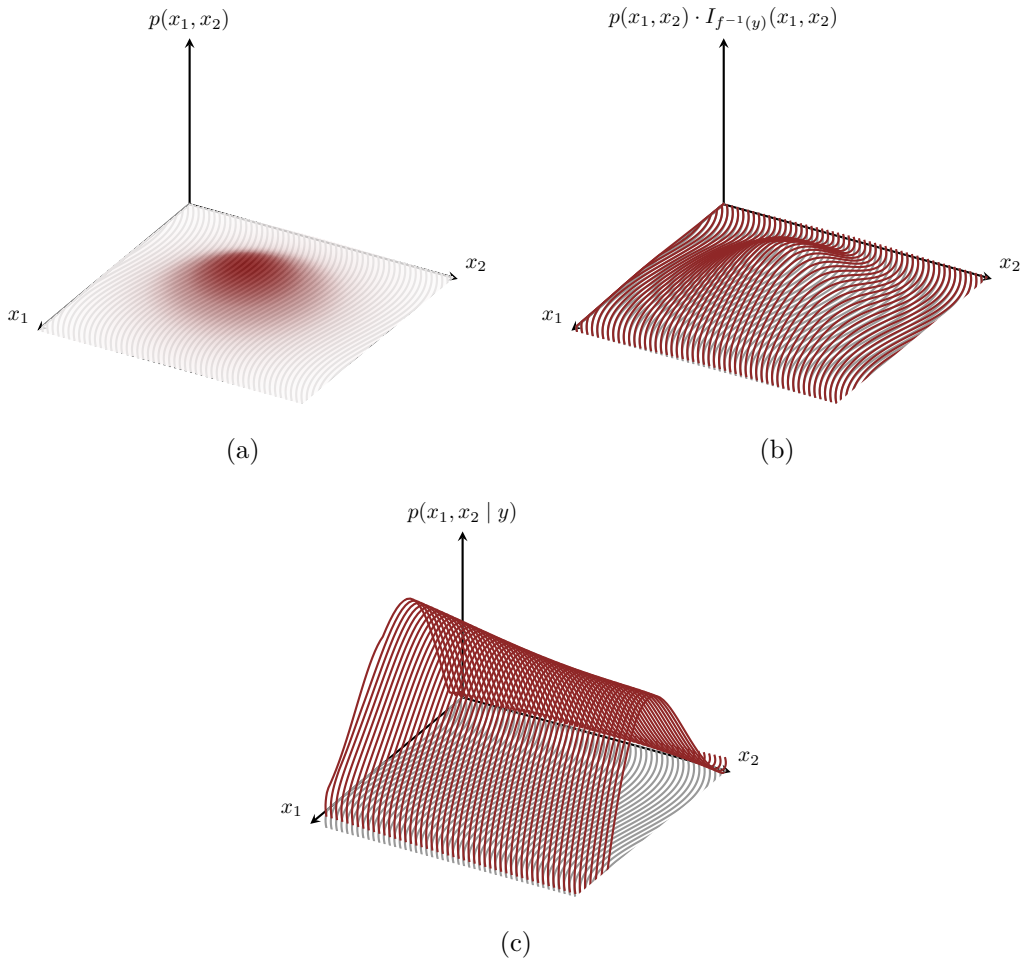


Figure 16: The product rule allows us to generalize the construction of conditional probability density functions that we first encountered in Section 4.2. (a) An initial probability density function is first (b) sliced into a collection of density functions by restricted inputs with non-zero outputs to particular level set. (c) Dividing by the corresponding pushforward probability density then normalizes these density functions into proper conditional probability density functions.

4.6 Explicit Formula For Pushforward Probability Density Functions

The ability to disintegrate reference measures also gives us a way to derive an explicit formula for pushforward probability density functions.

Here we need to start with the definition of pullback expectation values: for sufficiently-measurable functions $f : X \to Y$ and $h : Y \to \mathbb{R}$ we have

$$\begin{aligned} \mathbb{E}_{\pi}[h \circ f] &= \mathbb{E}_{f_* \pi}[h] \\ \mathbb{I}_{\mu}\left[\frac{\mathrm{d}\pi}{\mathrm{d}\mu} h \circ f\right] &= \mathbb{I}_{\nu}\left[\frac{\mathrm{d} f_* \pi}{\mathrm{d}\nu} h\right] \end{aligned}$$

or, equivalently,

$$\begin{aligned} \int \pi(\mathrm{d}x)\, h(f(x)) &= \int f_* \pi(\mathrm{d}y)\, h(y) \\ \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, h(f(x)) &= \int \nu(\mathrm{d}y)\, \frac{\mathrm{d} f_* \pi}{\mathrm{d}\nu}(y)\, h(y). \end{aligned}$$

Disintegrating μ with respect to f and ν allows us to write the left-hand side as

$$\begin{aligned} \int \pi(\mathrm{d}x)\, h(f(x)) &= \int \mu(\mathrm{d}x)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\, h(f(x)) \\ &= \int \nu(\mathrm{d}y) \int \mu^{f,\nu}(\mathrm{d}x_y \mid y)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y)\, h(y). \end{aligned}$$

Because the function $h \circ f : X \to \mathbb{R}$ yields the same output for any $x \in f^{-1}(y)$ it is a constant with respect to the inner integral that can be factored out of the inner measure-informed integral,

$$\begin{aligned} \int \pi(\mathrm{d}x)\, h(f(x)) &= \int \nu(\mathrm{d}y) \int \mu^{f,\nu}(\mathrm{d}x_y \mid y)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y)\, h(y) \\ &= \int \nu(\mathrm{d}y) \left[\int \mu^{f,\nu}(\mathrm{d}x_y \mid y)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y)\right] h(y). \end{aligned}$$

Consequently

$$\begin{aligned} \int \pi(\mathrm{d}x)\, h(f(x)) &= \int f_* \pi(\mathrm{d}y)\, h(y) \\ \int \nu(\mathrm{d}y) \left[\int \mu^{f,\nu}(\mathrm{d}x_y \mid y)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y)\right] h(y) &= \int \nu(\mathrm{d}y) \left[\frac{\mathrm{d} f_* \pi}{\mathrm{d}\nu}(y)\right] h(y). \end{aligned}$$

Because both sides of this equation are ν-informed integrals we have equality if and only if the integrands are equal up to ν-null subsets. In particular we have equality for any integrand $h : Y \to \mathbb{R}$ if and only if

$$\frac{\mathrm{d} f_* \pi}{\mathrm{d}\nu}(y) \overset{\nu}{=} \int \mu^{f,\nu}(\mathrm{d}x_y \mid y)\, \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y).$$

In theory this gives us an explicit formula for deriving pushforward probability density functions. Implementing this result in practice, however, will not be straightforward unless we happen to have an explicit method for integrating the initial probability density function over each level set of f.

For example consider the two-dimensional space $X = \mathbb{R}^+ \times \mathbb{R}^+$ equipped with a Lebesgue probability density function $p(x_1, x_2)$ and the radial function

$$\begin{aligned} f : X &\to \mathbb{R}^+ \\ (x_1, x_2) &\mapsto r = \sqrt{x_1^2 + x_2^2}. \end{aligned}$$

The level sets of f define angular arcs of constant radius. Conveniently these arcs are parameterized by a single variable if we transform to polar coordinates $\mathbb{R}^+ \times \left[0, \frac{\pi}{2}\right]$, with the map

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2} \\ \theta &= \arctan\left(\frac{x_2}{x_1}\right), \end{aligned}$$

or equivalently

$$\begin{aligned} x_1 &= r \cos\theta \\ x_2 &= r \sin\theta. \end{aligned}$$

In particular integrating over the variable θ implicitly for a particular r integrate over one of the angular level sets.

To take advantage of this in practice we need to first transform the initial probability density function $p(x_1, x_2)$ into the probability density function $p(r, \theta)$ using the Jacobian correction that we encountered in Chapter 7, Section 4.3.1. After this transformation we integrate over θ to derive a pushforward probability density function over the radial coordinate,

$$p(r) = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \, p(r, \theta).$$

Finally we can use the product rule to construct the conditional probability density function over the angular level sets

$$p(\theta_r \mid r) = \frac{p(r, \theta_r)}{p(r)}.$$

Implementing all of these calculations, however, is much easier said than done. For those with a taste for tricky integrals I work through an explicit example that requires some complicated mathematical functions in the Appendix.

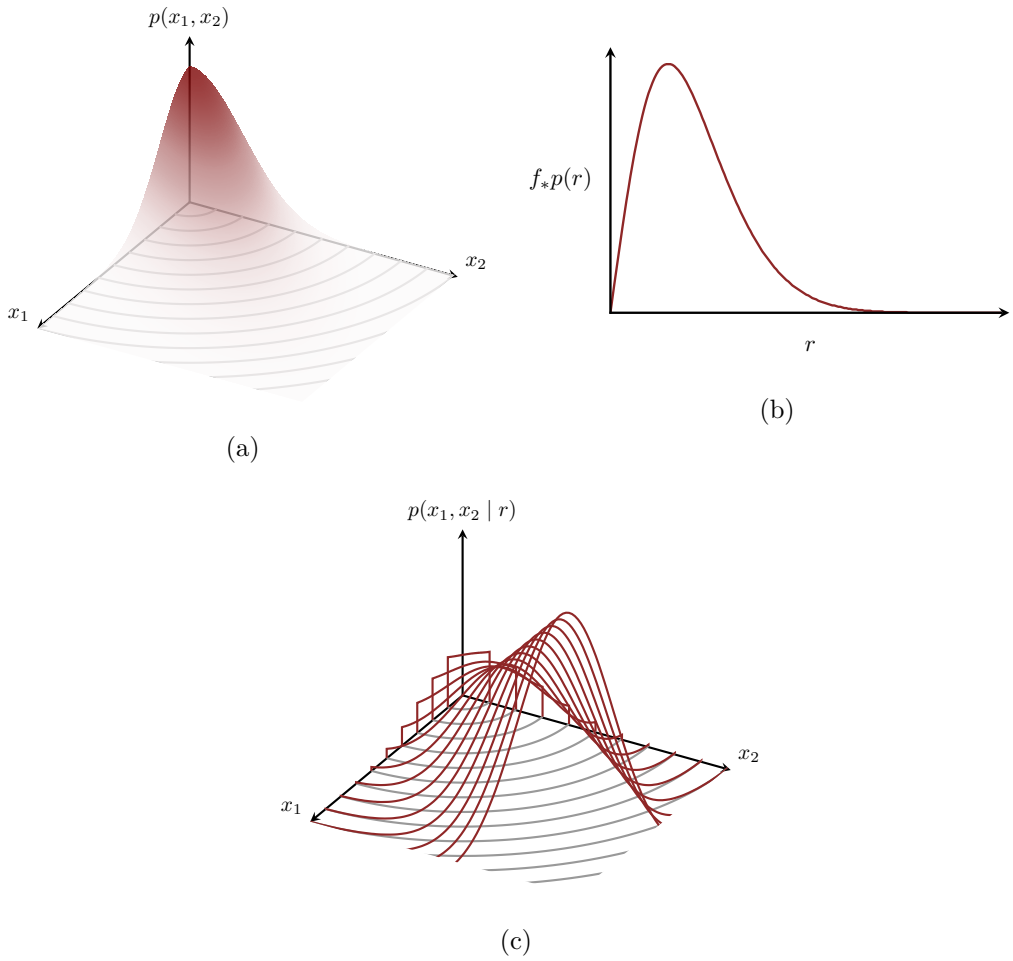


Figure 17: When we have the computational tools to integrate over level sets we can evaluate pushforward probability density functions, and hence conditional probability density functions. Here we integrate (a) an initial density function (b) over circular arcs to derive a pushforward probabilitiy density function over radii. (c) Restricting the initial probabilty density function to angular level sets and then dividing by pushforward probability densities then gives conditional probability density functions over each angular level set.

5 Conditional Building Blocks

To this point we have discussed conditional probability theory as a tool for *breaking* probability distributions down into simpler pieces. Conditional probability theory, however, can also be used to *build* probability distributions up from simpler pieces. Throughout I will continue to take the somewhat-obscure technical requirements of Radon measures and Hausdorff σ-algebras for granted.

Given a probability distribution π defined over the space X and a measurable function $f : X \to Y$ we can construct both a pushforward probability distribution $f_* \pi$ and a conditional probability kernel π^f. Through the laws of total probability and total expectation these two byproducts allow us to reconstruct the output of any probabilistic operation on π.

This construction also works the other way around. Given a measurable function $f : X \to Y$ any probability distribution over the output space ρ and conditional probability kernel

$$\begin{aligned} \tau : \mathcal{X} \times Y &\to [0, 1] \\ \mathsf{x}, y &\mapsto \tau(\mathsf{x} \mid y), \end{aligned}$$

with

$$\tau(f^{-1}(y) \mid y) = 1$$

uniquely define a probability distribution π over X through the law of total probability,

$$\pi(\mathsf{x}) = \mathbb{E}_\rho[t_\mathsf{x}]$$

where

$$\begin{aligned} t_\mathsf{x} : Y &\to [0, 1] \\ y &\mapsto \tau(\mathsf{x} \mid y). \end{aligned}$$

In this case we say that τ **lifts** ρ into a probability distribution over X.

Lifting allows us to construct probability distributions over X sequentially, first specifying a probability distribution over a Y and then filling in the missing information with conditional probability distributions across the level sets of a function $f : X \to Y$. If Y is a much simpler space than X, for example a lower-dimensional space with fewer degrees of freedom to consider, and the level sets of f are straightforward to interpret, then this sequential procedure can be much easier to implement in practice than trying to define a probability distribution directly on X.

Given a sequence of $N + 1$ spaces,

$$\{X_0, \dots, X_n, \dots, X_N\},$$

and functions relating them,

$$
\begin{aligned}
f_1 &: X_0 \to X_1 \\
&\dots \\
f_n &: X_{n-1} \to X_n \\
&\dots \\
f_N &: X_{N-1} \to X_N,
\end{aligned}
$$

we can even iterate this procedure, building up a probability distribution over X_0 from an initial probability distribution over X_N and a sequence of conditional probability kernels

$$\{\tau_N, \dots, \tau_n, \dots, \tau_1\}.$$

This allows us to incrementally build up sophisticated probability distributions over X_0 from much simpler pieces. Iteratively constructing probability distributions is particularly useful on product spaces, which will be the topic of **Chapter 9**.

We can also define a lifted probability distribution through its expectation values with the law of total expectation,

$$\int \pi(\mathrm{d}x)\, g(x) = \int \rho(\mathrm{d}y) \int \tau(\mathrm{d}x_y \mid y)\, g(x).$$

The advantage of this latter approach is that it allows us to implicitly define π through a sequence of probability density functions.

Given an output reference measure ν any sufficiently well-behaved function

$$r : Y \to \mathbb{R}^+$$

with $\mathbb{I}_\nu[r] = 1$ defines an output probability distribution $\rho = r\,\nu$ through the expectation values

$$\int \rho(\mathrm{d}y) h(y) = \int \nu(\mathrm{d}y)\, r(y)\, h(y).$$

Similarly given an input reference measure μ and its disintegration $\mu^{f,\nu}$ any sufficiently well-behaved binary function

$$t : X \times Y \to \mathbb{R}^+$$

with

$$\mathbb{I}_{\mu_y^{f,\nu}}[t] \overset{\nu}{=} 1$$

defines a conditional probability kernel $\tau = t\,\mu^{f,\nu}$ through the conditional expectation values

$$\int \tau(\mathrm{d}x_y \mid y)\, g(y, x_y) = \int \mu^{f,\nu}(\mathrm{d}x_y \mid y)\, \tau(x_y \mid y)\, g(y, x_y).$$

Finally the product of these two functions

$$p(x) = p(y, x_y) = t(x_y \mid y)\, r(y)$$

will always satisfy

$$\mathbb{I}_{\mu}[p] = 1$$

and hence define an input probability distribution $\pi = p\,\mu$ through the expectation values

$$\begin{aligned}\int \pi(\mathrm{d}x)\, g(x) &= \int \rho(\mathrm{d}y) \int \tau(\mathrm{d}x_y \mid y)\, g(x) \\ &= \int \nu(\mathrm{d}x)\, r(y) \int \mu^{f,\nu}(\mathrm{d}x_y \mid y)\, t(x_y \mid y)\, g(y, x_y).\end{aligned}$$

Iterating this construction over a sequence of spaces

$$\{X_0, \dots, X_n, \dots, X_N\},$$

requires a sequence of functions,

$$f_n : X_{n-1} \to X_n,$$

an probability density function over the terminal space,

$$p_N : X_N \to \mathbb{R}^+,$$

and a sequence of conditional probability density functions,

$$t_n : X_{n-1} \times X_n \to \mathbb{R}^+.$$

Applying the product rule once gives a probability density function over X_{N-1},

$$p_{N-1}(x_N, (x_{N-1})_{x_N}) = t_N((x_{N-1})_{x_N} \mid x_N)\, p_N(x_N).$$

Repeating the product rule $N-1$ more times then gives a probability density function over X_0.

The conditional variable notation becomes a bit cumbersome here, so I'll write this product as

$$\begin{aligned}p_0(x_0) &= t_1(x_0 \mid x_1) \cdots t_n(x_{n-1} \mid x_n) \cdots t_N(x_{N-1} \mid x_N)\, p_N(x_N) \\ &= \left[\prod_{n=1}^{N} t_n(x_{n-1} \mid x_n)\right] p_N(x_N)\end{aligned}$$

along with the recursive constraints

$$x_n = f_n(x_{n-1})$$

that completely fix the variables $\{x_1, \dots, x_N\}$ given a point x_0.

In **Chapter 9** we'll introduce a more elegant notation that works well in the special case of product spaces.

6 Independence

In general a probability distribution will induce different behavior on different level sets of the conditioning function. The exceptional cases, where the conditional behavior is the same for almost all level sets, arises often enough in practical applications to be worthy of its own terminology.

To start let's investigate what happens when conditioning not on a single subset. In particular consider two measurable subsets $\mathsf{x}_1 \in \mathcal{X}$ and $\mathsf{x}_2 \in \mathcal{X}$ that have non-zero overlap with each other,

$$\mathsf{x}_1 \cap \mathsf{x}_2 = \emptyset,$$

and are both allocated non-zero probability,

$$\pi(\mathsf{x}_1) > 0, \pi(\mathsf{x}_2) > 0.$$

The conditional probability of the first subset given the second is, by definition,

$$\pi(\mathsf{x}_1 \mid \mathsf{x}_2) = \frac{\pi(\mathsf{x}_1 \cap \mathsf{x}_2)}{\pi(\mathsf{x}_2)}$$

In order for the conditioning to have no affect on how probability is allocated to x_1 we need

$$\begin{aligned} \pi(\mathsf{x}_1) &= \pi(\mathsf{x}_1 \mid \mathsf{x}_2) \\ \pi(\mathsf{x}_1) &= \frac{\pi(\mathsf{x}_1 \cap \mathsf{x}_2)}{\pi(\mathsf{x}_2)} \end{aligned}$$

or

$$\pi(\mathsf{x}_1 \cap \mathsf{x}_2) = \pi(\mathsf{x}_1) \cdot \pi(\mathsf{x}_2)$$

When this condition holds we say that the two measurable subsets are **independent** of each other with respect to the probability distribution π.

The independence of subsets, however, doesn't tell us anything about how entire conditional probability distributions behave. For example we might be tempted to consider the case where *every* measurable subset $\mathsf{x} \in \mathcal{X}$ is independent of x_2,

$$\pi(\mathsf{x} \mid \mathsf{x}_2) = \pi(\mathsf{x}).$$

In this case the entire conditional probability distribution would reduce to the initial probability distribution. Unfortunately a condition this strong is hard to satisfy; in fact it holds only when $\mathsf{x}_2 = X$ and we're not really constraining the initial probability distribution in the first place.

A much more useful notion of independence is when almost all of the conditional probability distributions in a conditional probability kernel are equivalent, so the conditional behavior is independent of whichever partition cell, level set, or output point we consider. To rigorously

define this notion of independence, however, we need the level sets to be particularly well-behaved.

In general the level sets of a function don't need to share the same topology. Most functions, however, exhibit level sets with uniform or almost-uniform topologies. For example the level sets of the projection function

$$\begin{aligned} \varpi : \mathbb{R}^2 &\to \mathbb{R} \\ (x_1, x_2) &\mapsto x_1 \end{aligned}$$

are all real lines. Similarly the level sets of the radial function

$$\begin{aligned} r : \mathbb{R}^2 &\to \mathbb{R}^+ \\ (x_1, x_2) &\mapsto \sqrt{x_1^2 + x_2^2} \end{aligned}$$

are all circles except for the level set for $r(x_1, x_2) = 0$ which degenerates to a single point.

When the almost all of the level sets of a function share the same topology then we can treat them as equivalent representations of some common space L, which we write as

$$f^{-1}(y) \equiv L.$$

In this case we can, at least in theory, construct conditional probability kernels such that almost all of the conditional probability distribution are equivalent to some common probability distribution over L,

$$\pi_y^f(\mathsf{x}_y) = \rho(\mathsf{x}_y).$$

If the conditional probability kernel that we get by conditioning a probabilty distribution π on a function $f : X \to Y$ behaves in this way then we say that π is **independent** of f. Again this does not mean that the conditional probability distributions π_y^f behave exactly like π but rather that $f_*\pi$-almost all of them behave exactly like each other. In other words the behavior of π_y^f is independent of which level set $f^{-1}(y)$, and hence which output point $y \in Y$, we consider.

An immediate consequence of this definition is that if π is independent of f then any conditional probability density functions that we construct will not depend on y and the product rule becomes

$$\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(y, x_y) \overset{\mu}{=} \frac{\mathrm{d}\pi^f}{\mathrm{d}\mu^{f,\nu}}(x_y) \frac{\mathrm{d}f_*\pi}{\mathrm{d}\nu}(y).$$

Here all of the output dependence is isolated to the pushforward probability density function and all of the level set dependence is isolated to a single conditional probability density function.

This result suggests a straightforward procedure for constructing probability distributions that are independent of a given function $f : X \to Y$ whose level sets $f^{-1}(y)$ are almost all

equivalent to some common space L. Any function $r : Y \to \mathbb{R}^+$ with $\mathbb{I}_\nu[r] = 1$ implicitly defines a probability distribution over Y and any function $l : L \to \mathbb{R}^+$ with $\mathbb{I}_{\mu^{f,\nu}}[l] = 1$ defines a probability distribution over the common level set space. The product of these two functions $l(x_y) \cdot r(y)$ then defines a probability distribution over X that is always independent of f.

7 Conclusion

The intuition of conditional probability theory is relatively straightforward: decomposing probability distributions into simpler pieces. Implementing that intuition with consistent mathematics, however, is much more complicated.

In this chapter we've worked through the key foundations of conditional probability theory that will allow us to apply it to the discrete and continuous spaces that we'll regularly encounter in practical applications. That said the notation and terminology of this general theory can be frustratingly dense.

Fortunately much of this frustration will be resolved in the next chapter where we will apply conditional probability theory to product spaces and their natural projection functions. In this particular context much of the notation and terminology simplifies and conditional probability theory becomes a much more productive tool.

Appendix: "Explicit" Calculations

In this appendix I've sequestered some nasty integrals that arise when we try to integrate over angular level sets to construct the pushforward probability density function and the subsequent conditional probability density functions shown in Figure 17. This section is completely optional and can be ignored without any consequence for future chapters.

Consider the two-dimensional real space $X = \mathbb{R}^+ \times \mathbb{R}^+$, the Lebesgue probability density function

$$
\begin{aligned}
p(x_1, x_2) &= \frac{\exp\left(-\frac{1}{2\,s^2}\frac{1}{1-\rho^2}(x^2 - 2\,\rho\,x\,y + y^2)\right)}{\int_0^\infty \int_0^\infty \mathrm{d}x_1\,\mathrm{d}x_2\,\exp\left(-\frac{1}{2\,s^2}\frac{1}{1-\rho^2}(x^2 - 2\,\rho\,x\,y + y^2)\right)}. \\
&= C\,\exp\left(-\frac{1}{2\,s^2}\frac{1}{1-\rho^2}(x^2 - 2\,\rho\,x\,y + y^2)\right),
\end{aligned}
$$

and the radial function

$$
\begin{aligned}
f : X &\to \mathbb{R}^+ \\
(x_1, x_2) &\mapsto r = \sqrt{x_1^2 + x_2^2}.
\end{aligned}
$$

The level sets of f are given by angular arcs of constant radius. Calculations over these level sets become much easier when we reparameterize X into polar coordinates where the radial output becomes one of the component parameters, and the position along the corresponding arc becomes the other. This requires the transformation

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2} \\ \theta &= \arctan\left(\frac{x_2}{x_1}\right), \end{aligned}$$

or equivalently

$$\begin{aligned} x_1 &= r\,\cos\theta \\ x_2 &= r\,\sin\theta. \end{aligned}$$

Applying the transformation rule for Lebesgue probability density functions gives

$$\begin{aligned} p(r,\theta) &= p(x_1(r,\theta), x_2(r,\theta))\,\frac{1}{|\det \mathbf{J}(r,\theta)|} \\ &= C\,\exp\left(-\frac{1}{2\,s^2}\frac{1}{1-\rho^2}((r\,\cos\theta)^2 - 2\,\rho\,r\,\cos\theta\,r\,\sin\theta + (r\,\sin\theta)^2)\right)\,r \\ &= C\,r\,\exp\left(-\frac{1}{2\,s^2}\frac{1}{1-\rho^2}(r^2\,\cos^2\theta - 2\,\rho\,r^2\,\cos\theta\,\sin\theta + r^2\,\sin^2\theta)\right) \\ &= C\,r\,\exp\left(-\frac{1}{2\,s^2}\frac{r^2}{1-\rho^2}(\sin^2\theta + \cos^2\theta - 2\,\rho\,\sin\theta\,\cos\theta)\right) \\ &= C\,r\,\exp\left(-\frac{1}{2\,s^2}\frac{r^2}{1-\rho^2}(1 - 2\,\rho\,\sin\theta\,\cos\theta)\right) \\ &= C\,r\,\exp\left(-\frac{1}{2\,s^2}\frac{r^2}{1-\rho^2}(1 - \rho\,\sin 2\theta)\right). \end{aligned}$$

In theory we can construct the pushforward probability density function over the radial outputs of f by integrating out the angular parameter,

$$\begin{aligned} p(r) &= \int_0^{\frac{\pi}{2}} \mathrm{d}\theta\, p(r,\theta) \\ &= \int_0^{\frac{\pi}{2}} \mathrm{d}\theta\, C\,r\,\exp\left(-\frac{1}{2\,s^2}\frac{r^2}{1-\rho^2}(1 - \rho\,\sin 2\theta)\right) \\ &= C\,r\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\,\exp\left(-\frac{1}{2\,s^2}\frac{r^2}{1-\rho^2}\right)\,\exp\left(+\frac{1}{2\,s^2}\frac{r^2}{1-\rho^2}\,\rho\,\sin 2\theta\right) \\ &= C\,r\,\exp\left(-\frac{1}{2\,s^2}\frac{r^2}{1-\rho^2}\right)\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\,\exp\left(+\frac{r^2}{2\,s^2}\frac{\rho}{1-\rho^2}\,\sin 2\theta\right) \\ &= C\,r\,\exp\left(-\frac{1}{2\,s^2}\frac{r^2}{1-\rho^2}\right)\,\iota(r,\rho,\theta). \end{aligned}$$

Conveniently this integral can be reduced to special functions, albeit not necessarily common ones,

$$
\begin{aligned}
\iota(r, \rho, \theta) &= \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \, \exp\left(+ \frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \, \sin 2\theta \right) \\
&= \frac{1}{2} \int_0^{\pi} \mathrm{d}\phi \, \exp\left(+ \frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \, \sin \phi \right) \\
&= \frac{\pi}{2} \left(I_0 \left(\frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \right) + L_0 \left(\frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \right) \right)
\end{aligned}
$$

where $I_0(x)$ is the **zeroth-order modified Bessel function of the first kind** and $L_0(x)$ is the **zeroth-order modified Struve function**.

Using this result the radial probability density function becomes

$$
\begin{aligned}
p(r) &= C\, r \, \exp\left(- \frac{1}{2\,s^2} \frac{r^2}{1 - \rho^2} \right) \, \iota(r, \rho, \theta) \\
&= \frac{\pi}{2} \, C\, r \, \exp\left(- \frac{1}{2\,s^2} \frac{r^2}{1 - \rho^2} \right) \left(I_0 \left(\frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \right) + L_0 \left(\frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \right) \right).
\end{aligned}
$$

We can now use this pushforward probability density function to construct the conditional probability density function over the radial level sets,

$$
p(x_1, x_2 \mid r) = \frac{p(x_1, x_2)}{p(r)}.
$$

That said the conditional probability density functions simplify quite a bit if we work in polar coordinates where the angular coordinate completely parameterizes the level sets,

$$
\begin{aligned}
p(\theta \mid r) &= p(r, \theta \mid r) \\
&= \frac{p(r, \theta)}{p(r)} \\
&= \frac{C\, r \, \exp\left(- \frac{1}{2\,s^2} \frac{r^2}{1 - \rho^2} (1 - \rho \, \sin 2\theta) \right)}{\frac{\pi}{2} \, C\, r \, \exp\left(- \frac{1}{2\,s^2} \frac{r^2}{1 - \rho^2} \right) \left(I_0 \left(\frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \right) + L_0 \left(\frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \right) \right)} \\
&= \frac{\exp\left(- \frac{1}{2\,s^2} \frac{r^2}{1 - \rho^2} (1 - \rho \, \sin 2\theta) \right)}{\frac{\pi}{2} \, \exp\left(- \frac{1}{2\,s^2} \frac{r^2}{1 - \rho^2} \right) \left(I_0 \left(\frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \right) + L_0 \left(\frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \right) \right)} \\
&= \frac{2}{\pi} \, \frac{\exp\left(- \frac{1}{2\,s^2} \frac{r^2}{1 - \rho^2} (1 - \rho \, \sin 2\theta) + \frac{1}{2\,s^2} \frac{r^2}{1 - \rho^2} \right)}{I_0 \left(\frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \right) + L_0 \left(\frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \right)} \\
&= \frac{2}{\pi} \, \frac{\exp\left(- \frac{1}{2\,s^2} \frac{r^2}{1 - \rho^2} (- \rho \, \sin 2\theta) \right)}{I_0 \left(\frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \right) + L_0 \left(\frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \right)} \\
&= \frac{2}{\pi} \, \frac{\exp\left(+ \frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \, \sin 2\theta \right)}{I_0 \left(\frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \right) + L_0 \left(\frac{r^2}{2\,s^2} \frac{\rho}{1 - \rho^2} \right)}.
\end{aligned}
$$

Note that this construction ensures that each individual conditional probability density function is properly normalized,

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \, p(\theta \mid r) &= \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \, \frac{2}{\pi} \frac{\exp\left(+\alpha \, \sin 2\theta\right)}{I_0\left(\alpha\right) + L_0\left(\alpha\right)} \\
&= \frac{2}{\pi} \frac{1}{I_0\left(\alpha\right) + L_0\left(\alpha\right)} \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \, \exp\left(+\alpha \, \sin 2\theta\right) \\
&= \frac{1}{\pi} \frac{1}{I_0\left(\alpha\right) + L_0\left(\alpha\right)} \int_0^{\pi} \mathrm{d}\phi \, \exp\left(+\alpha \, \sin \phi\right) \\
&= \frac{1}{\pi} \frac{1}{I_0\left(\alpha\right) + L_0\left(\alpha\right)} \pi \left(I_0\left(\alpha\right) + L_0\left(\alpha\right)\right) \\
&= \frac{\pi}{\pi} \frac{I_0\left(\alpha\right) + L_0\left(\alpha\right)}{I_0\left(\alpha\right) + L_0\left(\alpha\right)} \\
&= 1,
\end{aligned}
$$

where

$$
\alpha = \frac{r^2}{2\, s^2} \frac{\rho}{1 - \rho^2}.
$$

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