Probability Density Functions

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As we saw in the last chapter, expectation values allow us to interrogate the behaviors of the probability distributions that we'll encounter in practical applications. At least, that is, in theory. In practice the expectation values of most probability distributions cannot be computed directly. Indeed integrals with respect to only a few exceptional measures can be computed directly at all.

By scaling those exceptional measures, however, we can sometimes use measure-informed integrals to indirectly compute expectation values from relevant probability distributions. If we can *engineer* scalings that reproduce expectation values of interest then we can use this indirect approach to implement probability theory in practice.

In this chapter we will formalize this procedure, identifying exactly when we can scale a given measure to reproduce the expectation values of a target probability distribution and how we can use scalings to specify new probability distributions in the context of a given measure. We will then investigate the detailed application of the method on discrete and real probability spaces.

1 Density Functions

Frustratingly we cannot translate integrals from one arbitrary measure to another. To avoid subtle mathematical inconsistencies we have to restrict our consideration to measures that are *compatible* with each other. In this section we will first motivate what kind of compatibility we might need on finite spaces before formalizing the compatibility requirements, and the systematic translation of integrals, on general spaces.

1.1 Finite Spaces

Let's start by considering a finite measurable space $(X, 2^X)$ and two measures, μ and ν. Given a real-valued function $f : X \to \mathbb{R}$ we can construct an integral with respect to both μ,

$$\mathbb{I}_{\mu}[f] = \sum_{x \in X} \mu(\{x\}) \cdot f(x),$$

and ν,

$$\mathbb{I}_{\nu}[f] = \sum_{x \in X} \nu(\{x\}) \cdot f(x).$$

To simplify the initial construction let's first assume that all of the atomic allocations are non-zero and finite,

$$\begin{aligned} 0 < \mu(\{x\}) < \infty \\ 0 < \nu(\{x\}) < \infty \end{aligned}$$

for all $x \in X$; this allow us to multiply and divide by the atomic allocations without running into ill-defined results. In particular we will always have

$$\frac{\nu(\{x\})}{\nu(\{x\})} = 1$$

so that we can write the μ integral as

$$\begin{aligned} \mathbb{I}_{\mu}[f] &= \sum_{x \in X} \mu(\{x\}) \cdot f(x) \\ &= \sum_{x \in X} \mu(\{x\}) \cdot 1 \cdot f(x) \\ &= \sum_{x \in X} \mu(\{x\}) \cdot \frac{\nu(\{x\})}{\nu(\{x\})} \cdot f(x) \\ &= \sum_{x \in X} \nu(\{x\}) \cdot \frac{\mu(\{x\})}{\nu(\{x\})} \cdot f(x). \end{aligned}$$

Now if we define a function that maps each element to the ratio of the atomic allocations,

$$\begin{aligned} r : X &\to \mathbb{R}^{+} \\ x &\mapsto \frac{\mu(\{x\})}{\nu(\{x\})}, \end{aligned}$$

then we can write the μ integral as

$$\begin{aligned} \mathbb{I}_{\mu}[f] &= \sum_{x \in X} \nu(\{x\}) \cdot \frac{\mu(\{x\})}{\nu(\{x\})} \cdot f(x) \\ &= \sum_{x \in X} \nu(\{x\}) \cdot r(x) \cdot f(x) \\ &= \mathbb{I}_{\nu}[r \cdot f]. \end{aligned}$$

In words the integral of any real-valued function f with respect to μ is equal to the integral of the *modified* function $r \cdot f$ with respect to ν.

More generally this result doesn't actually require that the ratio of atomic allocations

$$\frac{\nu(\{x\})}{\nu(\{x\})}$$

is well-defined for *all* $x \in X$. We need it to be well-defined for only the terms where $\mu(\{x\}) > 0$.

To see why let z_μ denote the subset of elements that are allocated zero measure,

$$\mathsf{z}_\mu = \{x \in X \mid \mu(\{x\}) = 0\},$$

with n_μ denoting the complementary subset of elements that are allocated non-zero measure,

$$\mathsf{n}_\mu = \{x \in X \mid \mu(\{x\}) > 0\}.$$

Using these subsets we can write any integral with respect to μ as

$$\begin{aligned}
\mathbb{I}_\mu[f] &= \sum_{x \in X} \mu(\{x\}) \cdot f(x) \\
&= \sum_{x \in \mathsf{z}_\mu} \mu(\{x\}) \cdot f(x) + \sum_{x \in \mathsf{n}_\mu} \mu(\{x\}) \cdot f(x) \\
&= \sum_{x \in \mathsf{z}_\mu} 0 \cdot f(x) + \sum_{x \in \mathsf{n}_\mu} \mu(\{x\}) \cdot f(x) \\
&= \sum_{x \in \mathsf{n}_\mu} \mu(\{x\}) \cdot f(x).
\end{aligned}$$

At the same time we can write the integral of the modified integrand with respect to ν as

$$\begin{aligned}
\mathbb{I}_\nu[r \cdot f] &= \sum_{x \in X} \nu(\{x\}) \cdot r(x) \cdot f(x) \\
&= \sum_{x \in \mathsf{z}_\nu} \nu(\{x\}) \cdot r(x) \cdot f(x) + \sum_{x \in \mathsf{n}_\nu} \nu(\{x\}) \cdot r(x) \cdot f(x) \\
&= \sum_{x \in \mathsf{z}_\nu} 0 \cdot r(x) \cdot f(x) + \sum_{x \in \mathsf{n}_\nu} \nu(\{x\}) \cdot r(x) \cdot f(x) \\
&= \sum_{x \in \mathsf{n}_\nu} \nu(\{x\}) \cdot r(x) \cdot f(x) \\
&= \sum_{x \in \mathsf{n}_\nu} \nu(\{x\}) \cdot \frac{\mu\{x\}}{\nu\{x\}} \cdot f(x) \\
&= \sum_{x \in \mathsf{n}_\nu} \mu(\{x\}) \cdot f(x).
\end{aligned}$$

In general n_μ and n_ν might contain different elements, in which case the two sums will not be equal,

$$\mathbb{I}_\mu[f] \neq \mathbb{I}_\nu[r \cdot f]$$

We have equality only when $\mu(\{x\}) = 0$ every time that $\nu(\{x\}) = 0$. In other words the two integrals will be equal only when

$$\mathsf{z}_\nu \subseteq \mathsf{z}_\mu.$$

Ultimately we can translate μ integrals into ν integrals only when all atomic allocations are finite and the null allocations of the two measures are consistent with each other (Figure 1).

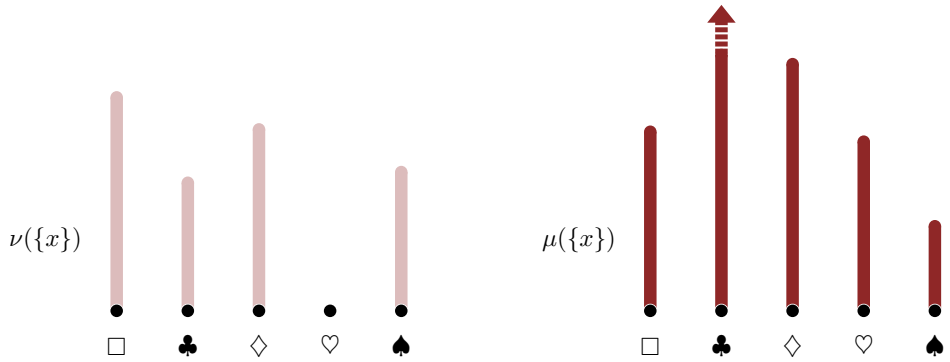


Figure 1: The atomic allocations from one discrete measure cannot be consistently scaled into the atomic allocations of any other discrete measure. For example there is no real number that will scale the finite allocation $\nu(\{\clubsuit\})$ into the infinite allocation $\mu(\{\clubsuit\})$. Similarly there is no real number that will consistently scale the zero allocation $\nu(\{\heartsuit\})$ into the non-zero allocation $\mu(\{\heartsuit\})$. Consistent scaling requires that all of the atomic allocations are finite and that the atomic allocations of the initial measure ν are zero only when the atomic allocations of the second measure μ are also zero.

Similarly we can translate ν integrals into μ integrals only when the atomic allocations are finite and

$$\mathsf{z}_\mu \subseteq \mathsf{z}_\nu.$$

Importantly these two conditions are not symmetric: we can translate back *and* forth between μ and ν integrals only when the null allocations are the same,

$$\mathsf{z}_\mu = \mathsf{z}_\nu.$$

1.2 General Spaces

Translating integrals between more general measure spaces requires a generalization of the atomic allocation conditions that we derived in the previous section to arbitrary allocations. Unsurprisingly this generalization will take a bit more care.

In this section we'll assume a measurable space $(X, \mathcal{X})$ and two measures $\mu : \mathcal{X} \to [0, \infty]$ and $\nu : \mathcal{X} \to [0, \infty]$.

1.2.1 σ-Finite Measures

Avoiding infinite atomic allocations is necessary on general measure spaces, but it is not sufficient for translating integrals from one measure to another. To avoid any mathematical inconsistencies we also need to limit exactly which subsets can be allocated infinite measure. Intuitively we need to avoid infinite "local" allocations even if the total measure is infinite.

To formalize this intuition we will first need to divide the ambient space into disjoint subsets that probe that local structure. A **partition** of X is any collection of subsets,

$$\mathcal{P} = \{\mathsf{x}_1, \ldots, \mathsf{x}_i, \ldots\},$$

that are mutually disjoint,

$$\mathsf{x}_i \cap \mathsf{x}_{i' \neq i} = \emptyset,$$

but cover the entire space,

$$\cup_i \mathsf{x}_i = X.$$

When working with measures we need to restrict attention to **measurable partitions**, which contain only measurable subsets, and **countable partitions**, which contain only a countable number of subsets.

For example the full set X by itself always defines a partition, as does the collection of atomic subsets. The collection of atomic subsets, however, will not always define a countable partition. On ordered metric spaces, such as real lines, measurable and countable partitions built up from intervals of constant length are particularly natural.

In general we can partition a measurable space in many different ways; some partitions will divide the space up into smaller subsets than others. If every subset in one measurable partition $\mathcal{P}$ is encapsulated by a subset in another measurable partition $\mathcal{P}'$, so that we can construct $\mathcal{P}$ by breaking up the subsets in $\mathcal{P}'$, then we say that $\mathcal{P}$ is a *refinement* of $\mathcal{P}'$. The more refined a measurable partition is the more sensitively it will be able to probe the "local" structure of a measure (Figure 2).

Every partition is a refinement of the partition defined by the full set X alone, making it the crudest possible partition. At the same time because the partition defined by the atomic subsets is a refinement of every other partition it is finest possible partition. Both of these extreme partitions are typically measurable, but the atomic partition will be countable on only countable spaces.

When the total measure is infinite then the allocations to the subsets in a crude measurable partition might also be infinite. If we break up those subsets into finer and finer pieces, however, then the infinite allocations might spread out into finite allocations. When we can construct a fine enough measurable partition such that *all* of the subset allocations are finite we will always able to avoid infinity entirely by working with small enough subsets. Moreover if that fine enough measurable partition is also countable then we will always be able to aggregate those smaller-subset allocations into any general allocation.

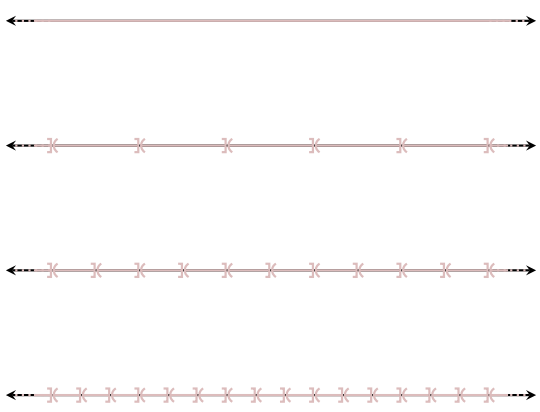


Figure 2: Every real line can be partitioned into half-open intervals of constant length. The narrower the intervals the more refined the partition will be, and the more sensitive the individual subsets will to the local structure of a measure. Decomposing a real line all the way into atomic elements results in a valid partition, but not a countable partition.

More formally a measure μ is **σ-finite** if we can construct *at least one* measurable and countable partition where each subset is allocated only a finite measure,

$$\mu(\mathsf{x}_i) < \infty$$

for all $\mathsf{x}_i \in \mathcal{P}$.

Every finite measure, including every probability distribution, is σ-finite because we can always take the trivial partition $\mathcal{P} = \{X\}$ with

$$\mu(X) < \infty.$$

On the other hand only exceptional infinite measures are also σ-finite. For example we can show that every counting measure χ is σ-finite by employing a countable atomic partition,

$$\mathcal{P} = \{\{x\} \mid x \in X\},$$

with

$$\chi(\mathsf{x}) = 1 < \infty$$

for all $\mathsf{x} \in \mathcal{P}$. Similarly we can show that every Lebesgue measure λ is σ-finite by partitioning a real line into intervals of constant length L,

$$\mathcal{P} = \{(L \cdot n, L \cdot (n + 1)] \mid n \in \dots, -1, 0, 1, \dots\},$$

with

$$\lambda(\mathsf{x}) = L < \infty$$

for all $\mathsf{x} \in \mathcal{P}$. Even though the local interval allocations are finite individually they aggregate into the infinite total Lebesgue measure.

One immediate obstruction to σ-finiteness is the allocation of infinite measure to a single atomic subset. Because we can't break an atomic subset down any further there's no way to diffuse that infinite allocation into something finite. Fortunately this extreme behavior rarely if ever show up in practical applications.

Every application that we will consider in this book will only ever use σ-finite measures.

1.2.2 Absolute Continuity

On finite spaces the compatibility between two measures was determined by the overlap of the null atomic subsets. On more general spaces we have to consider the overlap of *all* null subsets.

When every ν-null subset is a μ-null subset,

$$\mathcal{X}_{\nu=0} \subseteq \mathcal{X}_{\mu=0}$$

we can say that μ is **absolutely continuous** with respect to ν and write

$$\mu \ll \nu.$$

Equivalently we can say that ν **dominates** μ.

Because absolute continuity is defined by the null subsets, and not the precise allocations across non-null subsets, understanding of the null subsets alone allows us to characterize how compatible two measures are with each other. In particular when a collection of measures all share the *same* null subsets then each measure will be absolutely continuous with respect to every other measure, regardless of their idiosyncratic behaviors.

1.2.3 Radon-Nikodym Derivatives

With σ-finiteness and absolute continuity established we are now ready to define the most general circumstances under which we can translate integrals from one measure to another.

If μ and ν are two σ-finite measures on $(X, \mathcal{X})$ and μ is absolutely continuous with respect to ν,

$$\mu \ll \nu,$$

then at least one $\mathcal{X}$-measurable, positive real-valued function

$$\frac{\mathrm{d}\mu}{\mathrm{d}\nu} : X \to \mathbb{R}^{+}$$

exists such that

$$\mathbb{I}_{\mu}[f] = \mathbb{I}_{\nu}\left[\frac{\mathrm{d}\mu}{\mathrm{d}\nu} \cdot f\right]$$

for *every* $\mathcal{X}$-measurable, μ-integrable, real-valued function $f : X \to \mathbb{R}$.

Any such function $\mathrm{d}\mu / \mathrm{d}\nu$ that translates μ integrals into ν integrals is known as a **Radon-Nikodym derivative** of μ with respect to ν. In this case μ is denoted the **target** measure and ν is denoted the **reference** measure.

Because Radon-Nikodym derivatives are defined by measure-informed integrals they are not, in general, unique. Modifying the outputs of a Radon-Nikodym derivative on a ν-null subset of inputs results in the same ν integrals, and hence another valid Radon-Nikodym derivative. In practice we can usually restrict our consideration to Radon-Nikodym derivatives that also happen to be continuous or even smooth. These *structured* Radon-Nikodym derivatives are typically unique for each μ and ν, and we don't lose any generality by ignoring the others provided that we never try to interpret these functions outside of the shadow of an integral!

For example the unit function

$$\begin{aligned} u : X &\to \mathbb{R}^{+} \\ x &\mapsto 1 \end{aligned}$$

is one possible Radon-Nikodym derivative for any measure with respect to itself,

$$\frac{\mathrm{d}\nu}{\mathrm{d}\nu} \overset{\nu}{=} u.$$

At the same time any function that deviates from the unit function only on ν-null subsets is also a valid Radon-Nikodym derivative for ν with respect to itself. If $X = \mathbb{R}$, however, then the unit function is the only continuous Radon-Nikodym derivative.

Using the local scaling notation that we introduced in Chapter 5 we can also express the relationship between two compatible measures and the Radon-Nikodym derivative that links them together as

$$\mu = \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \cdot \nu.$$

Again this is just shorthand for the integral relationships

$$\mathbb{I}_{\mu}[f] = \mathbb{I}_{\nu}\left[\frac{\mathrm{d}\mu}{\mathrm{d}\nu} \cdot f\right].$$

1.2.4 Interpreting Radon-Nikodym Derivatives

One way to build intuition for Radon-Nikodym derivatives is to consider the integral of indicator functions. For any measurable subset $\mathsf{x} \in \mathcal{X}$ we have

$$\begin{aligned} \mu(\mathsf{x}) &= \mathbb{I}_{\mu}[I_{\mathsf{x}}] \\ &= \mathbb{I}_{\nu}\left[\frac{\mathrm{d}\mu}{\mathrm{d}\nu} \cdot I_{\mathsf{x}}\right]. \end{aligned}$$

If

$$\mathrm{d}\mu / \mathrm{d}\nu(x) > 1$$

for *almost all* points $x \in \mathsf{x}$ – remember that we always have to excuse deviant behavior over the null subsets of ν – then we will also have

$$\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x) \cdot I_{\mathsf{x}}(x) > I_{\mathsf{x}}(x)$$

for almost all $x \in X$. This implies that

$$\begin{aligned} \mu(\mathsf{x}) &= \mathbb{I}_{\nu}\left[\frac{\mathrm{d}\mu}{\mathrm{d}\nu} \cdot I_{\mathsf{x}}\right] \\ &> \mathbb{I}_{\nu}\left[I_{\mathsf{x}}\right] \\ &> \nu(\mathsf{x}), \end{aligned}$$

or

$$\mu(\mathsf{x}) > \nu(\mathsf{x}).$$

In other words if

$$\mathrm{d}\mu / \mathrm{d}\nu(x) > 1$$

across some subset of the ambient space then μ will allocate *more* measure there than ν. The larger the Radon-Nikodym derivative is the more excessive the μ allocation will be. Similarly if

$$\mathrm{d}\mu / \mathrm{d}\nu(x) < 1$$

across some subset then μ will allocate *less* measure there than ν. At the extreme any measurable collection of points with

$$\mathrm{d}\mu / \mathrm{d}\nu(x) \overset{\nu}{=} 0$$

always defines a μ-null subset. In between if

$$\mathrm{d}\mu / \mathrm{d}\nu(x) \overset{\nu}{=} 1$$

across a subset then μ and ν will allocate the *same* measure to that subset.

Consequently we can interpret a Radon-Nikodym derivative as quantifying how the allocations of μ are *locally* enhanced or suppressed relative to the allocations of ν. By integrating indicator functions we aggregate these local warpings into non-local changes in the subset allocations.

A common physical analogy is to interpret the reference measure ν as defining a sense of *volume* across the ambient space. The more measure ν allocates to a given subset the more background volume is spanned by that subset. If we analogize the measure allocated by μ to a physical *mass* then Radon-Nikodym derivatives become **density functions** that quantify how strongly mass concentrates within any given volume.

Personally I've always found the term "Radon-Nikodym derivative" to be aesthetically pleasing – it sounds like a term straight out of science fiction – but this density function analogy is typically much more accessible. Because of that I will use the term "density function" instead of "Radon-Nikodym derivative" as much as possible.

1.2.5 Operator Perspective

Speaking of terminology, why do we call a Radon-Nikodym derivative a "derivative" in the first place? It turns out that mathematically Radon-Nikodym derivatives do satisfy the properties of formal derivatives, but the theory needed to demonstrate that is pretty elaborate. In this section we will instead investigate some of the more superficial similarities between Radon-Nikodym derivative and the familiar derivatives from calculus.

Recall that in one-dimensional calculus differentiation is an operation that converts a smooth function $f : \mathbb{R} \to \mathbb{R}$ into a *new* function

$$\frac{\mathrm{d}f}{\mathrm{d}x} : \mathbb{R} \to \mathbb{R}.$$

This operation is linear,

$$\frac{\mathrm{d}}{\mathrm{d}x} (\alpha\, f + \beta\, g) = \alpha \frac{\mathrm{d}f}{\mathrm{d}x} + \beta \frac{\mathrm{d}g}{\mathrm{d}x},$$

and satisfies a chain rule,

$$\frac{\mathrm{d}(g \circ f)}{\mathrm{d}x}(x) = \frac{\mathrm{d}g}{\mathrm{d}y}(f(x)) \frac{\mathrm{d}f}{\mathrm{d}x}(x)$$

for any two functions $f : x \mapsto y$ and $g : y \mapsto z$.

The output of the derivative function at the point $x \in \mathbb{R}$,

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x),$$

quantifies how quickly f changes at that input (Figure 3). If

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) > 0$$

then f is increasing at x, if

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) < 0$$

then f is decreasing at x, and if

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = 0$$

then f is constant at x.

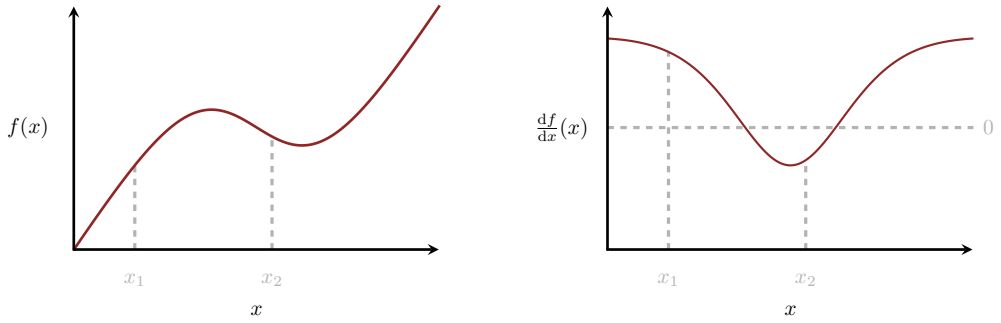


Figure 3: The derivative $\mathrm{d}f/\mathrm{d}x$ of a function f is itself a function that quantifies how f changes at each point $x \in \mathbb{R}$. Because f is increasing at x_1 the derivative is positive there, $\mathrm{d}f/\mathrm{d}x(x_1) > 0$. Similarly because f is decreasing at x_2 the derivative is negative there. We can also interpret the derivative as quantifying how much f changes *relative* to a constant function. Because $\mathrm{d}f/\mathrm{d}x(x_1) > 0$ the function f is increasing faster than the constant function at x_1. At the same time because $\mathrm{d}f/\mathrm{d}x(x_2) < 0$ the function f is increasing more slowly than the constant function at x_2.

Equivalently we can interpret a derivative function as quantifying how much the initial function changes *relative to a constant function.* If

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) > 0$$

then f is increasing *faster* than a constant function at x, if

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) < 0$$

then f is increasing *slower* than a constant function at x, and if

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = 0$$

then f is increasing *at the same rate* as a constant function at x. In other words we can interpret the derivative as an operator that gives an output function that quantifies how much the input function changes relative to some reference behavior.

Similarly we can interpret Radon-Nikodym derivatives as the output of an operator that gives output functions that quantify how much an input measure changes relative to some reference measure. Even more this operator is linear and satisfies a chain rule.

To formalize this similarity a bit more let's use $M(X, \mathcal{X})$ to denote the collection of all measures that we can define over the measurable space $(X, \mathcal{X})$ and $C^+(X, \mathcal{X}, \mu)$ to denote the collection of all $\mathcal{X}$-measurable functions from X to $\mathbb{R}^+$. We can then define **Radon-Nikodym differentiation** as a mapping

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\nu} : M(X, \mathcal{X}) &\to C^+(X, \mathcal{X}) \\ \mu &\mapsto \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \end{aligned}$$

that takes an input measure μ to an output function

$$\frac{\mathrm{d}\mu}{\mathrm{d}\nu} : X \to \mathbb{R}^+$$

that captures how μ varies relative to ν.

Admittedly we're being a bit mathematically sloppy here because Radon-Nikodym derivatives are defined only up to ν-null subsets. Without imposing additional constraints Radon-Nikodym differentiation doesn't specify a unique output function but rather a collection of output functions that are equivalent up to ν-null subsets. This sloppy notation, however, does allow us to show many of the useful properties of the operation.

For example Radon-Nikodym differentiation is linear. If we define a linear combination of measures by the allocations

$$(\alpha \cdot \mu + \beta \cdot \nu)(\mathsf{x}) = \alpha \cdot \mu(\mathsf{x}) + \beta \cdot \nu(\mathsf{x})$$

then

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\alpha \cdot \mu + \beta \cdot \nu) \overset{\nu}{=} \alpha \cdot \frac{\mathrm{d}\mu}{\mathrm{d}\lambda} + \beta \cdot \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}$$

whenever μ and ν are both absolutely continuous with respect to λ,

$$\mathcal{X}_{\lambda=0} \subseteq \mathcal{X}_{\mu=0} \cap \mathcal{X}_{\nu=0}.$$

Note that to be technically correct, the best kind of correct, equalities like these hold only up to ν-null subsets.

Similarly Radon-Nikodym differentiation satisfies a chain rule. If μ is absolutely continuous with respect to λ and λ is absolutely continuous with respect to ν,

$$\mathcal{X}_{\mu=0} \subseteq \mathcal{X}_{\lambda=0} \subseteq \mathcal{X}_{\nu=0},$$

then

$$\frac{\mathrm{d}\mu}{\mathrm{d}\nu} \overset{\nu}{=} \frac{\mathrm{d}\mu}{\mathrm{d}\lambda} \cdot \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}.$$

Again we have to be careful to acknowledge that this "equality" holds only up to ν-null subsets.

The Radon-Nikodym chain rule can be particularly useful in practice when we're interested in comparing two measures μ and ν to each other but a third measure λ is a more convenient reference measure. For example consider the simplified case where μ, ν, and λ all share the same null subsets,

$$\mathcal{X}_{\mu=0} = \mathcal{X}_{\nu=0} = \mathcal{X}_{\lambda=0} \equiv \mathcal{X}_0,$$

and hence are all absolutely continuous with respect to each other. In this case all of the Radon-Nikodym derivatives between these measures are almost everywhere non-zero and we can write

$$\begin{aligned} 1 &\overset{\nu}{=} \frac{\mathrm{d}\nu}{\mathrm{d}\nu}(x) \\ 1 &\overset{\nu}{=} \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}(x) \cdot \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(x) \end{aligned}$$

or

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(x) \overset{\nu}{=} \frac{1}{\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}(x)}$$

up to the common null subsets.

Consequently we can write the Radon-Nikodym derivative between μ and ν as a ratio of the Radon-Nikodym derivatives with respect to λ,

$$\begin{aligned} \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x) &\overset{\nu}{=} \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x) \cdot \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(x) \\ &\overset{\nu}{=} \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x) \cdot \frac{1}{\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}(x)} \\ &\overset{\nu}{=} \frac{\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x)}{\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}(x)}. \end{aligned}$$

In this way we can directly compare the behaviors of μ and ν to each other using only the more convenient Radon-Nikodym derivatives.

2 Specifying Probability Distributions With Density Functions

The initial use case for density functions, née Radon-Nikodym derivatives, requires the selection of a target measure and a convenient reference measure ν. In this setting we can translate implicit integrals with respect to the target measure to explicit integrals with respect to the reference measure,

$$\mathbb{I}_{\mu}[f] = \mathbb{I}_{\nu}\left[\frac{\mathrm{d}\pi}{\mathrm{d}\nu} \cdot f\right].$$

This construction, however, can also be reversed. Given a convenient reference measure, any measurable any positive-real valued function implicitly defines a target measure without having to directly specify any subset allocations.

Consider, for example, a measurable space $(X, \mathcal{X})$ and a convenient, σ-finite measure ν. Every non-negative, $\mathcal{X}$-measurable function $m : X \to R^+$ implicitly defines a measure μ with the measure-informed integrals

$$\mathbb{I}_{\mu}[f] = \mathbb{I}_{\nu}[m \cdot f].$$

The **normalization** of a density function is defined by its integral with respect to the reference measure, $\mathbb{I}_{\nu}[m]$. If

$$\mathbb{I}_{\nu}[m] = 1$$

then the total induced measure becomes

$$\begin{aligned} \mu(X) &= \mathbb{I}_{\mu}[I_X] \\ &= \mathbb{I}_{\nu}[m \cdot I_X] \\ &= \mathbb{I}_{\nu}[m] \\ &= 1; \end{aligned}$$

in this case the density function defines not just any measure but rather a probability distribution.

In other words every non-negative, $\mathcal{X}$-measurable function $p : X \to R^+$ that is properly normalized, $\mathbb{I}_{\nu}[p] = 1$, implicitly defines a probability distribution π with the expectation values

$$\mathbb{E}_{\pi}[f] = \mathbb{I}_{\nu}[p \cdot f].$$

Because we will be working with mostly probability distributions in practice, we will be working with mostly these probability density functions.

Note, while the probabilities given by evaluating a probability distribution can take only real values between zero and one, the probability densities given by evaluating a probability density function can take *any* non-negative real value Probability densities and probabilities are distinct objects!

In practice working with point-wise functions is much easier than working with subset-wise functions. For example when the ambient space is low-dimensional we can visualize point-wise functional behavior directly, allowing density functions to not define probability distributions but also communicate their behaviors. We'll explore this latter feature in the context of Lebesgue reference measures in Section 4.3.

This is not to say that probability density functions are not without their limitations. In particular probability density functions are defined only relative to the given reference measure. If the reference measure is at all ambiguous then a density function will not fully determine a probability distribution! At the same time if the reference measure every changes then probability density functions will also have to change if we want them to represent the same probability distributions.

We will use probability density functions to define relevant probability distributions almost exclusively in this book. This then raises a subtle question of notation. If we start with a probability distribution π and reference measure ν then the notation

$$\frac{\mathrm{d}\pi}{\mathrm{d}\nu}$$

for the corresponding probability density function is both clear and explicit. On the other hand if we start with the reference measure ν and a properly normalized function p then the notation

$$\pi = p \cdot \nu$$

for the corresponding probability distribution is also clear and explicit. These notations, however, are asymmetric and require the introduction of a third symbol,

$$p = \frac{\mathrm{d}\pi}{\mathrm{d}\nu}.$$

One way to make these two notations not only more symmetric but also more compact is to take the reference measure for granted and overload the *same symbol* to denote both the probability distribution and the probability density function, with only their arguments to distinguish between them. In other words $\pi(\mathsf{x})$ will denote a probability distribution mapping measurable subsets to probability allocations while $\pi(x)$ will denote a probability density function mapping individual points to local scalings of the reference measure.

To be honest this overloaded notation is not the best. Overloaded expressions like

$$\mathbb{E}_{\pi}[f] = \mathbb{I}_{\nu}[\pi \cdot f]$$
$$\int \pi(\mathrm{d}x)\, f(x) = \int \nu(\mathrm{d}x)\, \pi(x)\, f(x)$$

are awkward at best, and without care it can be easy to confuse probability distributions with probability density functions. At the same time it is not difficult to lose track of the underlying reference measure, especially when multiple reference measures are relevant at the same time. Really the only benefit to this notational shorthand is that it is typographically compact and doesn't take up much room on the page.

Unfortunately this convention has also become standard in many fields and consequently nearly impossible to avoid. To make this book as accessible as possible I will adopt this more overloaded notation but do my best to clarify the obscured details whenever they might be relevant. At times I might also fall back to the more explicit notations when there is any particular risk of confusion.

3 Probability Density Functions on Discrete Measure Spaces

Now that we have defined probability density functions in full generality we can study how they're used in the kinds of spaces that often arise in practical applications. We'll begin by looking at probability density functions on discrete measure spaces.

3.1 Probability Mass Functions

On discrete measurable spaces $(X, 2^X)$ we can always construct a uniform counting measure that allocates a unit measure to each atomic subset,

$$\chi(\{x\}) = 1.$$

Because a counting measure doesn't have any null subsets it dominates *every* measure that we could define on $(X, 2^X)$, making it a natural reference measure for any discrete probability distribution. Moreover the lack of null subsets also means that density functions will always be unique.

Following the steps we worked through in Section 1.1 we can derive an explicit result for the probability density function defined by any probability distribution π with respect to a counting measure. Translating an expectation value to an integral informed by the counting measure gives

$$\begin{aligned}
\mathbb{E}_{\pi}[f] &= \sum_{x \in X} \pi(\{x\}) \cdot f(x) \\
&= \sum_{x \in X} 1 \cdot \pi(\{x\}) \cdot f(x) \\
&= \sum_{x \in X} \chi(\{x\}) \cdot \pi(\{x\}) \cdot f(x) \\
&= \sum_{x \in X} \chi(\{x\}) \cdot (\pi(\{x\}) \cdot f(x)) \\
&= \mathbb{I}_{\chi}[\pi \cdot f],
\end{aligned}$$

where π in the last term denotes a function maps each element of X to its atomic allocation,

$$\begin{aligned}
\pi : X &\to [0, \infty] \\
x &\mapsto \pi(\{x\}).
\end{aligned}$$

In other words the density of any probability distribution with respect to the counting measure is just the corresponding probability mass function,

$$\frac{\mathrm{d}\pi}{\mathrm{d}\chi}(x) = \pi(x)!$$

Because of this association probability mass functions are also sometimes referred to as discrete probability density functions.

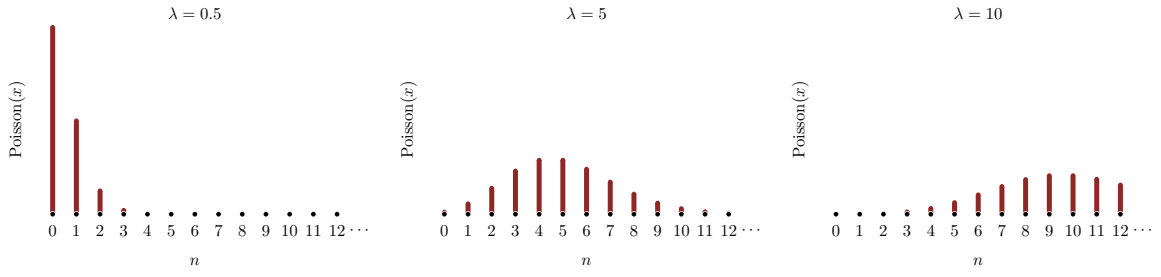
Identifying probability mass functions with Radon-Nikodym derivatives not only formalizes all of the more heuristic results that we've developed on countable spaces but also places them within the context of the more general probability theory. For example when we use the atomic allocations specified by a probability mass function to compute expectation values we're implicitly integrating with respect to a counting measure. Similarly when we visualize a probability distribution by plotting the atomic allocations we're communicating how much the probability distribution warps the uniform allocations defined by the counting measure.

3.2 The Poisson Family of Probability Mass Functions

To demonstrate the use of probability density functions on countable spaces let's consider the positive integers, $(\mathbb{N}, 2^{\mathbb{N}})$. Assuming the counting measure we can specify an entire *family* of probability distributions with the parametrized probability density function (Figure 4)

$$\text{Poisson}(n; \lambda) = \frac{\lambda^n e^{-\lambda}}{n!},$$

where $\lambda \in \mathbb{R}^+$.



(a) (b) (c)

Figure 4: For each value of the parameter λ the Poisson family $\text{Poisson}(n; \lambda)$ defines a discrete probability density function, and hence a probability distribution in the context of a reference counting measure. As λ increases from (a) to (c) the probability distributions concentrate are larger integers.

This family of probability mass functions, as well as the family of probability distributions they implicitly define, is known as the **Poisson** family. If we fix λ then $\text{Poisson}(n; \lambda)$ denotes a Poisson probability mass function, and implicitly Poisson distribution.

Expectation values with respect to a Poisson distribution are given by explicit summations,

$$\begin{aligned}\mathbb{E}_{\text{Poisson}}[f; \lambda] &= \mathbb{I}_{\chi}[\text{Poisson}(; \lambda) \cdot f] \\ &= \sum_{n=0}^{\infty} \text{Poisson}(n; \lambda) \, f(n).\end{aligned}$$

The sums for many common expectands can actually be worked out in closed form. I've isolated those calculations in the Appendix and will simply state some of the more important results here.

For example the normalization of each Poisson probability mass function is unity,

$$\begin{aligned}\text{Poisson}(X; \lambda) &= \mathbb{E}_{\text{Poisson}}[I_X; \lambda] \\ &= \mathbb{I}_{\chi}[\text{Poisson}(; \lambda) \cdot I_X] \\ &= \sum_{n=0}^{\infty} \text{Poisson}(n; \lambda) \cdot 1 \\ &= 1,\end{aligned}$$

as required.

Similarly the mean for each Poisson probability distribution is given by

$$\begin{aligned}\mathbb{M}(\lambda) &= \mathbb{E}_{\text{Poisson}}[\iota; \lambda] \\ &= \mathbb{I}_{\chi}[\text{Poisson}(; \lambda) \cdot \iota] \\ &= \sum_{n=0}^{\infty} \text{Poisson}(n; \lambda) \cdot \iota(n) \\ &= \lambda.\end{aligned}$$

Consequently the parameter λ moderates the centrality of probability distributions defined by each Poisson mass functions (Figure 5).

A slightly longer calculation shows that the variance of each Poisson distribution is also equal to λ. As we increase the parameter λ the individual Poisson distributions not only shift to larger values but also become more diffuse.

The Poisson cumulative distribution functions

$$\Pi(n) = \text{Poisson}([0, n]; \lambda) = \sum_{n'=0}^{n} \text{Poisson}(n'; \lambda)$$

can also be evaluated in closed form,

$$\Pi(n) = \frac{\Gamma(n + 1, \lambda)}{\Gamma(n + 1)},$$

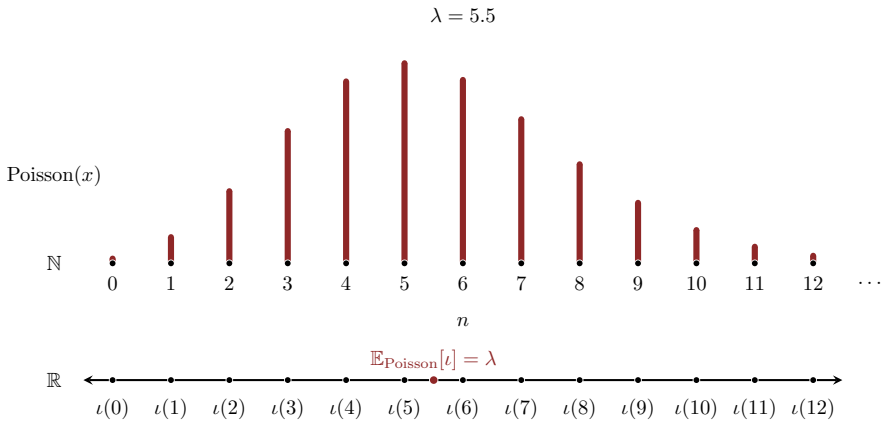


Figure 5: The positive real-valued parameter λ determines the centrality of each Poisson probability mass function, and hence each Poisson probability distribution. Recall that even though the ambient space is discrete the mean will in general take on real values.

where $\Gamma(x, y)$ is the **incomplete Gamma function** and $\Gamma(x)$ is the **gamma function**. Conveniently these two functions, if not the Poisson cumulative distribution function itself, are implemented in most programming languages, making them straightforward to use in practice.

These cumulative distribution functions give us two ways to evaluate interval probabilities (Figure 6). On one hand we can brute force an interval probability with direct summation,

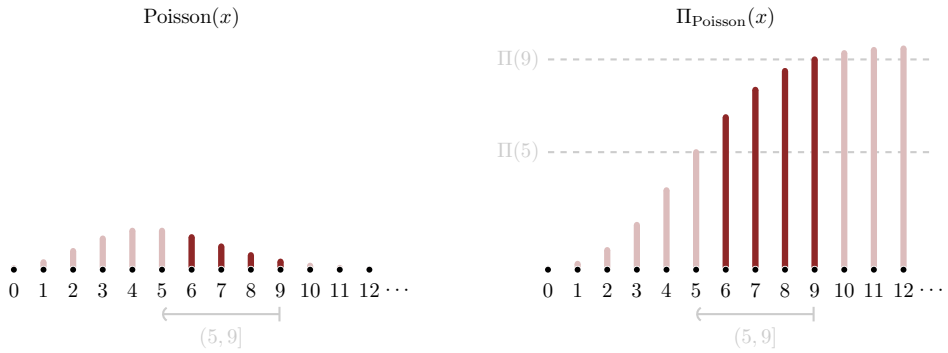
$$\text{Poisson}(\,(n_1, n_2]\,; \lambda) = \sum_{n'=n_1+1}^{n_2} \text{Poisson}(n'; \lambda).$$

On the other hand we can evaluate the cumulative distribution function at each boundary and subtract,

$$\text{Poisson}(\,(n_1, n_2]\,; \lambda) f = \Pi_{\text{Poisson}}(n_2) - \Pi_{\text{Poisson}}(n_1).$$

4 Probability Density Functions on Real Spaces

Having explored discrete measure spaces let's consider our prototypical uncountable spaces, the spaces of real numbers. On these spaces we don't have heuristics on which we can fall back, and the full machinery of Radon-Nikodym derivatives is needed to ensure consistent results. On these spaces the Lebesgue measure serves as a natural reference measure and most expectation values reduce not to summations but rather classic integrals.



(a) (b)

Figure 6: The interval probabilities allocated by Poisson probability distributions can be computed by (a) exhaustively summing over all of the points in an interval of (b) substracting the output of the cumulative distribution function at the interval boundaries.

4.1 Lebesgue Probability Density Functions

The metric structure of a real space makes Lebesgue measures particularly useful. In particular because Lebesgue measures are σ-finite they serve as immediate candidates for reference measures.

The only remaining obstruction to the construction of probability density functions is absolute continuity. For practical considerations the most important Lebesgue null subsets are the subsets consisting of only a countable number of points; any probability distribution that is absolutely continuous with respect to the Lebesgue measure must also allocate zero probability to atomic subsets and their countable unions. In theory there are other, more abstract, null subsets that need to be accommodated but the atomic subsets are the most practically relevant.

Given a particular D-dimensional real space, that is a particular rigid real space or particular parameterization of a flexible real space, and a compatible a compatible probability distribution π we can define a Lebesgue probability density function

$$\frac{\mathrm{d}\pi}{\mathrm{d}\lambda^D} : \mathbb{R}^D \to \mathbb{R}^+.$$

Alternatively any measurable, positive real-valued function

$$\pi : \mathbb{R}^D \to \mathbb{R}^+$$

that is appropriately normalized,

$$\mathbb{I}_{\lambda^D}[\pi] = \int \mathrm{d}^D x \, \pi(x_1, \ldots, x_d, \ldots, x_D) = 1,$$

will implicitly specify a probability distribution that is absolutely continuous with respect to the Lebesgue measure. By construction these engineered probability distributions will allocate zero probability to every atomic subset. In practice almost every probability distribution over real spaces that we will consider will be built up by scaling the Lebesgue measure in this way.

Given a one-dimensional Lebesgue probability density function we can compute the probability allocated to any interval subset with an appropriately bounded integral,

$$\begin{aligned}
\pi(\,[x_1, x_2]\,) &= \mathbb{E}_\pi \left[I_{[x_1, x_2]} \right] \\
&= \mathbb{I}_\lambda \left[\frac{\mathrm{d}\pi}{\mathrm{d}\lambda} \cdot I_{[x_1, x_2]} \right] \\
&= \int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{\mathrm{d}\pi}{\mathrm{d}\lambda}(x) \cdot I_{[x_1, x_2]}(x) \\
&= \int_{x_1}^{x_2} \mathrm{d}x \, \frac{\mathrm{d}\pi}{\mathrm{d}\lambda}(x).
\end{aligned}$$

In other words interval probabilities are equal to the *area* under the curve defined by the probability density function (Figure 7). For higher-dimensional real spaces subset probabilities become volumes under the surfaces defined by the higher-dimensional probability density functions.

When working with Lebesgue probability density functions in practice we have to be *very* careful to account for *which* Lebesgue measure we're using at any given time. Different real spaces, or different parameterizations of a single flexible real space, will in general feature different metrics which then give rise to different Lebesgue measures.

Because of this a fixed Lebesgue probability density function will define *different* probability distributions on different real spaces. Equivalently in order to represent a fixed target probability distribution π we need to use *different* Lebesgue probability density functions on every individual real space. Either way to avoid any ambiguity we need to clearly communicate which Lebesgue measure we're assuming.

In the next chapter we'll learn how to transform measures from one space to another. This will allow us to relate the Lebesgue measures, and the corresponding Lebesgue probability density functions, from one real space to another, or equivalently from one parameterization of a flexible real space to another.

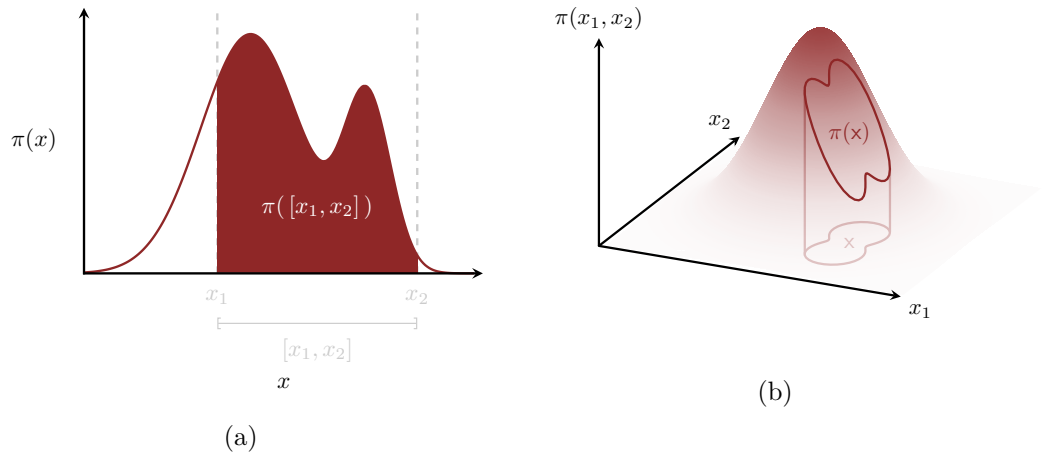


Figure 7: Subset probabilities are derived from Lebesgue probability density functions through classic integration. (a) One-dimensional interval probabilities are given by the area under the curve defined by a one-dimensional Lebesgue probability density function. (b) In higher-dimensions we have to compute the volume under the surface defined by a Lebesgue probability density function.

4.2 Abusing The Equals Sign

As with any Radon-Nikodym derivative, Lebesgue probability density functions are not unique. Any two probability density functions that differ only on countable subsets of input points will specify exactly the same probability distribution. Because of this we have to be careful whenever we're comparing probability density functions to each other.

For example if two probability distributions over $\mathbb{R}$ are equal to each other,

$$\pi = \rho,$$

then the expectation of every sufficiently well-behaved expectand f will be the same,

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{\mathrm{d}\pi}{\mathrm{d}\lambda}(x) \, f(x) = \int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{\mathrm{d}\rho}{\mathrm{d}\lambda}(x) \, f(x)$$

This does not, however, imply that the two probability density functions are equal to each other,

$$\frac{\mathrm{d}\pi}{\mathrm{d}\lambda}(x) = \frac{\mathrm{d}\rho}{\mathrm{d}\lambda}(x)$$

at every input point $x \in \mathbb{R}$. Rather they can deviate from each other on any Lebesgue null subset (Figure 8),

$$\frac{\mathrm{d}\pi}{\mathrm{d}\lambda}(x) \overset{\lambda}{=} \frac{\mathrm{d}\rho}{\mathrm{d}\lambda}.$$

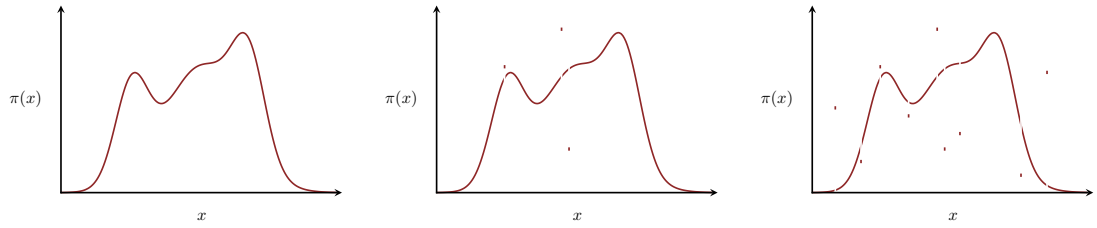


Figure 8: As with any density function Lebesgue probability density functions are defined only up to the null subsets of the reference measure; for the Lebesgue measure this includes atomic subsets and their countable unions. All three of these Lebesgue probability density functions are equivalent in the sense that they define exactly the same expectation values for any expectand. One way to break this ambiguity is to introduce an additional constraint; for example the left probability density function is the only continuous probability density function.

Most references take this subtlety for granted, abusing the equals sign by writing

$$\frac{\mathrm{d}\pi}{\mathrm{d}\lambda} = \frac{\mathrm{d}\rho}{\mathrm{d}\lambda}$$

to mean that

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{\mathrm{d}\pi}{\mathrm{d}\lambda}(x) \, f(x) = \int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{\mathrm{d}\rho}{\mathrm{d}\lambda}(x) \, f(x)$$

for any sufficiently well-behaved expectand f. Unfortunately this sloppy notation is ubiquitous and impossible to avoid in practice.

One way to make this convention a bit more rigorous is to impose additional constraints on the probability density functions when possible. For example if we can can restrict consideration to *continuous* probability density functions then

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{\mathrm{d}\pi}{\mathrm{d}\lambda}(x) \, f(x) = \int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{\mathrm{d}\rho}{\mathrm{d}\lambda}(x) \, f(x)$$

usually implies point-wise equality,

$$\frac{\mathrm{d}\pi}{\mathrm{d}\lambda}(x) = \frac{\mathrm{d}\rho}{\mathrm{d}\lambda}(x)$$

for all $x \in \mathbb{R}$.

Regardless we have to be careful with unstated assumptions. Directly equating probability density functions implies either that the equality holds only up to reference null subsets or that we're restricting consideration to certain structured probability density functions and not just any valid probability density functions.

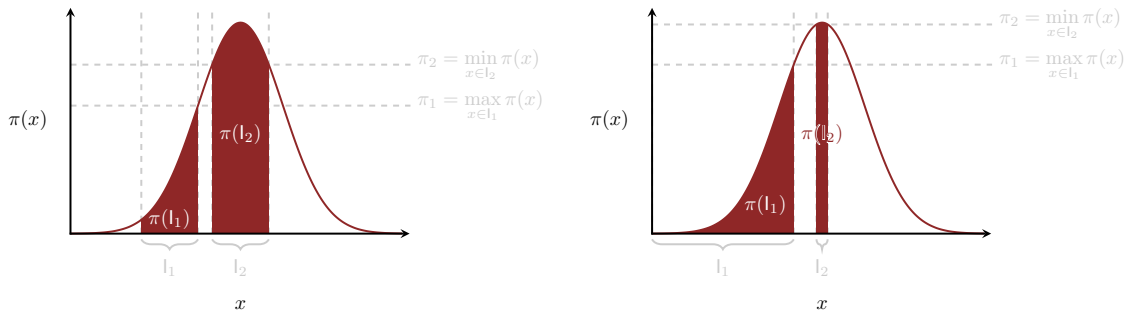
The safest approach is to never forget that, unlike regular functions, *Lebesgue probability density functions do not exist on their own.* Rather Lebesgue probability density functions *always* live under the shadow of integral signs. Anytime we see a bare probability density function we should recognize the implied integral context.

4.3 Lebesgue Probability Density Functions As Visualizations

Lebesgue probability density functions quantify probability distributions by integration; for example on a real line the area under the curve defined by a probability density function corresponds to interval probabilities. If we train ourselves to "integrate by eye", qualitatively mapping intervals to areas under a given curve, then we can extract a wealth of information by visually examining a Lebesgue probability density function.

For example consider two intervals I_1 and I_2 of the same width but at different positions on a real line. If $\pi(x)$ is larger for all points in I_1 than it is for all points in I_2 then the probability allocated to the first interval will be larger than the probability allocated to the second interval (Figure 9a)

$$\pi(\mathsf{I}_1) > \pi(\mathsf{I}_2).$$

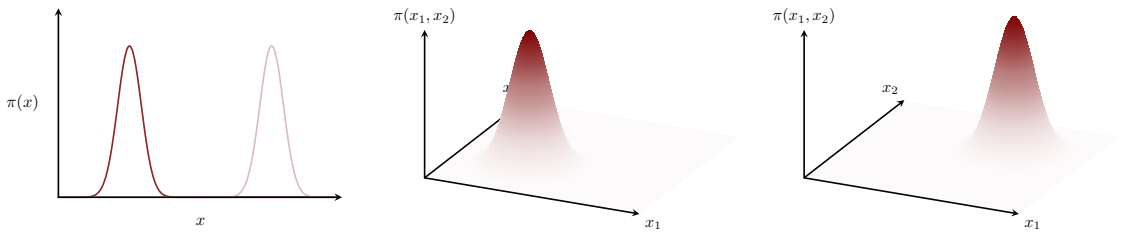


(a) (b)

Figure 9: Integrating a probability density function by eye is not always straightforward. In particular bounds between probability densities do not always imply bounds between interval probabilities. (a) Here largest probability density in the first interval, π_1, is smaller than the smallest probability density in the second interval, π_2. Because the two intervals are the same length the probability allocated to the second interval must be larger than the probability allocated to the second interval. (b) In this case we still have the largest probability density in the first interval smaller than the smallest probability density in the second interval, $\pi_1 < \pi_2$. Because the two intervals are not the same length, however, this does not imply that $\pi(\mathsf{I}_1) < \pi(\mathsf{I}_2)$. Indeed $\pi(\mathsf{I}_1)$ is almost twice as large as $\pi(\mathsf{I}_2)$!

That said we have to take care when visually comparing intervals of different lengths. A narrow interval can be allocated negligible probability even if the probability density function is extremely large everywhere within it. Similarly intervals where the probability density function is everywhere small can still be allocated appreciable probabilities if the interval is large enough (Figure 9b).

With care these visual comparisons can convey a wealth of qualitative insights. For example if a probability density function peaks at a single point then intervals containing the peak will tend to be allocated larger probabilities than intervals that don't contain the peak. In other words the probability distribution implicitly defined by that probability density function will concentrate in the neighborhood of that peak (Figure 10).



(a) (b)

Figure 10: The peaks of a Lebesgue probability density function qualify where the corresponding probability density function concentrates. (a) The dark red probability density function defines a probability distribution that concentrates at smaller values of x than the probability distribution defined by the light red probability density function. (b) Similarly the two-dimensional probability density function on the left defines a probability distribution over $\mathbb{R}^2$ that concentrates at values where both x_1 and x_2 are small, while the one on the right defines a probability distribution that concentrates at values where both x_1 and x_2 are large.

If a probability density function exhibits multiple peaks then the probability distribution will concentrate locally in the neighborhood of each. Subsets falling into the gaps between these local modes will be allocated much less probability.

The behavior of a probability density function *around* a peak qualifies how the implied probability distribution concentrates (Figure 11). For example if the probability density function is wide then the concentration of probability will be weak and if the probability density function is narrow then the concentration will be strong. The shape of the probability density function away from the peak qualifies the relative probability allocated to neighborhoods around the peak compared to those away from it. Similarly is the probability density function is asymmetric then the concentration will be skewed.

All of this said we have to be careful to not misinterpret the point-wise behavior of a Lebesgue probability density function. For one point-wise behavior is formally ambiguous because

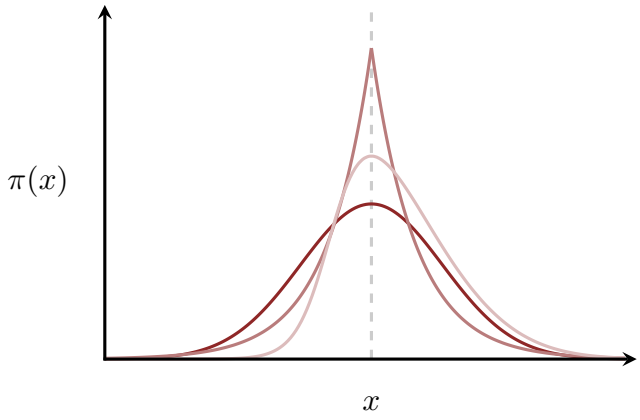


Figure 11: The decay of a probability density function around a peak determines how the implied probability distribution concentrates. Here the dark red probability density function decays symmetrically. In comparison the red probability density function also decays symmetrically but the decay is more nuanced, falling off quickly at first but then settling into a much slower decay as we move away from the peak. Finally the light red probability denstiy function decays asymmetricaly, with the implied probability distribution allocating more probability to larger values than smaller values.

Radon-Nikodym derivatives are defined only up to null subsets. More importantly point-wise evaluations of a probability density function don't correspond to any well-defined expectation values. Probability density functions exist to be integrated, and visualizations of probability density functions are useful only when they qualitatively inform how certain integrals behave.

Finally the visualization of probability density functions is limited to only one and two-dimensional real spaces. In higher dimensions we cannot plot how a Lebesgue probability density function changes in every direction at the same time.

We can plot how a probability density function varies along one and two-dimensional cross sections of the ambient space. For example when working on $\mathbb{R}^3$ we can partically visualize a probability density function $\pi(x_1, x_2, x_3)$ by plotting the slices $\pi(x_1, x_2, \tilde{x}_3)$, $\pi(x_1, \tilde{x}_2, x_3)$, and $\pi(\tilde{x}_1, x_2, x_3)$. Properly interpreting these slices, however, can be tricky. In **Chapter Eight** we'll learn how to interpret these slices as *conditional probability density functions*.

A more effective approach in practice is to not try to visualize an entire probability density function at once but rather investigate its behavior when projected to lower-dimensional summaries spaces. We'll learn how to construct these projections in the next chapter.

4.4 Lebesgue Probability Densities As Limiting Interval Probabilities

Lebesgue probability density functions can be integrated to compute the probability allocated to intervals. We can also use certain limiting interval probabilities to compute Lebesgue probability densities.

To set up this latter construction let's consider a real line $X = \mathbb{R}$ and the interval

$$\mathsf{I} = (x_1, x_1 + L]$$

of length $L > 0$. We can then decompose this initial interval into N disjoint subintervals

$$\mathsf{I}_n = (x_1 + n\,\epsilon, x_1 + (n+1)\,\epsilon],$$

each of length

$$\epsilon = \frac{L}{N}.$$

By countable additivity the probability that a probability distribution π allocates to the interval I is the same as the sum of the probabilities allocated to each subinterval I_n,

$$\pi(\,(x_1, x_1 + L]\,) = \sum_{n=0}^{N} \pi(\,(x_1 + n\,\epsilon, x_1 + (n+1)\,\epsilon]\,).$$

Multiplying and dividing by the subinterval length ϵ this becomes

$$\pi(\,(x_1, x_1 + L]\,) = \sum_{n=0}^{N} \epsilon \cdot \frac{\pi(\,(x_1 + n\,\epsilon, x_1 + (n+1)\,\epsilon]\,)}{\epsilon}.$$

This, however, is exactly the setup for a Riemann integral. As the number of subintervals increases, $N \to \infty$, and the length of each subinterval decreases, $\epsilon \to 0$, the left hand side will remain the same but the sum on the right hand side will converge to a Riemann integral,

$$\begin{aligned} \pi(\,(x_1, x_1 + L]\,) &= \lim_{N \to \infty} \sum_{n=0}^{N} \pi\left((x_1 + n\,\frac{L}{N}, x_1 + (n+1)\,\frac{L}{N}] \right) \\ &= \int_{x_1}^{x_1 + L} \mathrm{d}x\, p(x), \end{aligned}$$

with the integrand

$$p(x) = \lim_{\epsilon \to 0} \frac{\pi(\,(x, x + \epsilon]\,)}{\epsilon}.$$

At the same time we can also use a probability density function between π and the Lebesgue measure λ to compute the same interval probability,

$$\pi(\,(x_1, x_1 + L]\,) = \int_{x_1}^{x_1 + L} \mathrm{d}x\, \frac{\mathrm{d}\pi}{\mathrm{d}\lambda}(x).$$

Comparing these two results we can identify the Riemann integrand with the Lebesgue probability density function,

$$\frac{\mathrm{d}\pi}{\mathrm{d}\lambda}(x) \overset{\lambda}{=} \lim_{\epsilon \to 0} \frac{\pi(\,(x, x + \epsilon]\,)}{\epsilon}.$$

Note that this result relies on the choice of metric. The convergence of a scaled interval probability to a probability density function depends on the metric we use to define interval lengths. Different metrics will result in different limiting values, consistent with the fact that different metrics define different Lebesgue measures and hence different Lebesgue probability density functions.

This result helps to explain why we have to take care with interpreting probability densities. Lebesgue probability densities don't correspond to interval probabilities but rather how quickly interval probabilities change as we scan across the ambient real line; they encode *differential* information about probability allocations. Equivalently probability density functions are endowed with units of probability over length, not units of probability.

One practical corollary of this relationship is that we can use properly scaled interval probabilities to *approximate* probability density functions. For small but finite ϵ the quantity

$$\frac{\pi(\,(x, x + \epsilon]\,)}{\epsilon}$$

approximates the Lebesgue probability density

$$\frac{\mathrm{d}\pi}{\mathrm{d}\lambda}(x).$$

Consequently a histogram where the bin heights are scaled by the inverse bin widths,

$$\frac{\pi(\,(x_i, x_{i+1}]\,)}{x_{i+1} - x_i},$$

approximately visualizes a probability density function. As the bins become narrower and narrower the scaled histogram becomes a more and more accurate representation of the Lebesgue probability density function (Figure 12).

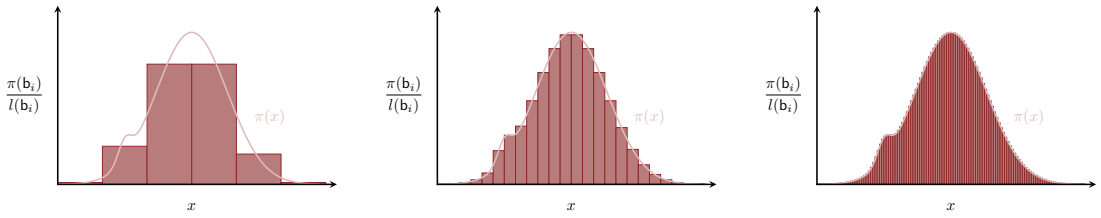


Figure 12: When we scale the bin probabilities $\pi(\mathsf{b}_i) = \pi(\,(x_i, x_{i+1}]\,)$ by the bin widths $l(\mathsf{b}_i) = x_{i+1} - x_i$ a histogram approximately visualizes a probability density function. As the bin widths are decreased the approximation improves.

This approximate visualization is particularly useful when we can compute interval probabilities but we can't evaluate the Lebesgue probability density function. As we'll discover in the next chapter this exact circumstance often arises when we try to project a probability distribution from a higher-dimensional space to a lower-dimensional space.

4.5 The Normal Family of Probability Density Functions

The most convenient Lebesgue probability density functions are those that facilitate analytic Riemann integration as much as possible. The two-parameter **normal** family of Lebesgue probability density functions

$$\text{normal}(x; \mu, \sigma) = \frac{1}{\sqrt{2\,\pi}\sigma} \exp\left(-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right),$$

where $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, is particularly convenient.

The two parameters μ and σ directly determine the basic shape of each normal probability density function (Figure 13). A normal probability density function peaks at μ, which is referred to generally as a **location parameter**. The second parameter σ determines how quickly the probability density function decays as we move away from the peak; the smaller σ is the narrower the density function will be. It is known as a **scale parameter**.

Each $\text{normal}(x; \mu, \sigma)$ is referred to as a **normal probability density function**, or just **normal density function** for short. We might be tempted to refer to the probability distribution

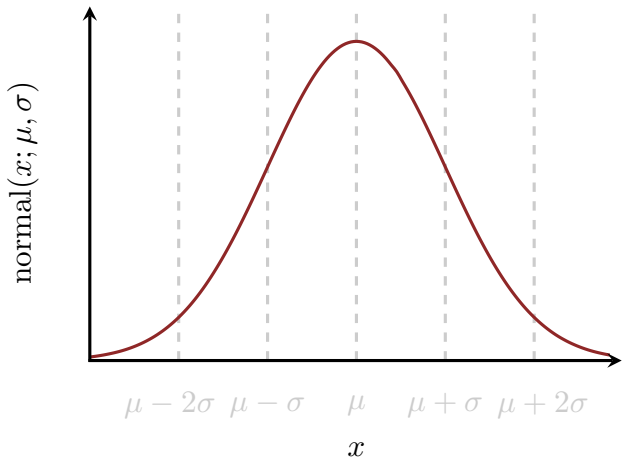


Figure 13: Every normal probability density function peaks at the location parameter μ, with the scale parameter σ controlling how quickly it decays as we move away from the peak. The implied probability distributions allocate most of their probability, over 99%, to the interval $(\mu - 3\,\sigma, \mu + 3\,\sigma)$.

defined by a normal density function as a **normal distribution**, but this association is well-defined only once we fix a real line and hence a particular Lebesgue measure. Because the same normal density function will define different probability distributions on different real lines I try to avoid terms like "normal distribution".

The conventions I've used here are common, but no means standard. For example in some fields this family is described as "Gaussian" in honor of Carl Friedrich Gauss who first introduced it. Moreover there are multiple, equivalent ways to parameterize the family including

$$\text{normal}(x; \mu, v) = \frac{1}{\sqrt{2\,\pi\,v}} \exp\left(-\frac{1}{2\,v}\,(x - \mu)^2\right),$$
$$\text{normal}(x; \mu, \tau) = \sqrt{\frac{\tau}{2\,\pi}} \exp\left(-\frac{\tau}{2}\,(x - \mu)^2\right),$$

and even

$$\text{normal}(x; \eta_1, \eta_2) = \sqrt{\frac{-\eta_2}{\pi}} \exp\left(\eta_1\,x + \eta_2\,x^2 + \frac{\eta_1^2}{4\,\eta_2^2}\right).$$

Each of these parameterizations can be convenient in certain circumstances, but the initial parameterization defined above tends to the most useful for practical applications

The integrals

$$\int_{-\infty}^{+\infty} \mathrm{d}x\,\text{normal}(x; \mu, \sigma)\,f(x)$$

that define expectation values are particularly nice, at least as far as integrals go. That isn't to say that the integrals are easy to evaluate but rather that they many of them actually admit closed-form solutions, which is pretty miraculous when it comes to integrals. For those twisted individuals who fancy a good integral calculation, myself included, I've included those calculations in the Appendix. Everyone else can take these results at face value.

For example we can verify that each probability density function in the normal family is properly normalized with the integral

$$\begin{aligned}
\text{normal}(\mathbb{R}; \mu, \sigma) &= \int_{-\infty}^{+\infty} \mathrm{d}x \, \text{normal}(x; \mu, \sigma) \\
&= \int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{1}{\sqrt{2\,\pi}\sigma} \exp\left(-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right) \\
&= 1.
\end{aligned}$$

Similarly we can compute the mean of the probability distribution implied by each normal density function,

$$\begin{aligned}
\mathbb{M}(\mu, \sigma) &= \mathbb{E}_{\text{normal}}[\iota; \mu, \sigma] \\
&= \int_{-\infty}^{+\infty} \mathrm{d}x \, \text{normal}(x; \mu, \sigma) \, x \\
&= \int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{1}{\sqrt{2\,\pi}\sigma} \exp\left(-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right) \, x \\
&= \mu.
\end{aligned}$$

The parameter μ determines not just the peak of each normal probability density function but also the mean of each corresponding probability distribution. Because of this μ is sometimes referred to as a **mean parameter**.

With even more mathematical elbow grease we can show that the variance is given by

$$\begin{aligned}
\mathbb{V}(\mu, \sigma) &= \mathbb{E}_{\text{normal}}[(\iota - \mathbb{M}(\mu, \sigma))^2; \mu, \sigma] \\
&= \mathbb{E}_{\text{normal}}[(\iota - \mu)^2; \mu, \sigma] \\
&= \int_{-\infty}^{+\infty} \mathrm{d}x \, \text{normal}(x; \mu, \sigma) \, (x - \mu)^2 \\
&= \int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{1}{\sqrt{2\,\pi}\sigma} \exp\left(-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right) \, (x - \mu)^2 \\
&= \sigma^2.
\end{aligned}$$

As we increase σ the normal probability density functions widen and the variance of the implied probability distributions increases.

Normal probability density functions can also be used to evaluate the cumulative distribution function corresponding to the implicitly-defined probability distributions,

$$\begin{aligned}
\Pi_{\text{normal}}(x; \mu, \sigma) &= \text{normal}(\, (-\infty, x]; \mu, \sigma \,) \\
&= \mathbb{E}_{\text{normal}}[I_{(-\infty, x]}; \mu, \sigma] \\
&= \int_{-\infty}^{+\infty} \mathrm{d}x' \, \text{normal}(x'; \mu, \sigma) \, I_{(-\infty, x]}(x') \\
&= \int_{-\infty}^{x} \mathrm{d}x' \, \text{normal}(x'; \mu, \sigma) \\
&= \frac{1}{2} + \frac{1}{2} \, \text{erf}\left(\frac{x - \mu}{\sqrt{2}\,\sigma} \right),
\end{aligned}$$

where

$$\text{erf}(x) \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{d}t \, \exp\left(-t^2\right)$$

is known as the **error function**.

Conveniently the error function, if not the normal cumulative distribution functions themselves, are available in most programming languages. This allows us directly compute interval probabilities by subtracting cumulative probabilities (Figure 14),

$$\text{normal}(\, (x_1, x_2] \,; \mu, \sigma) = \frac{1}{2} \left(\text{erf}\left(\frac{x_2 - \mu}{\sqrt{2}\,\sigma} \right) - \text{erf}\left(\frac{x_1 - \mu}{\sqrt{2}\,\sigma} \right) \right).$$

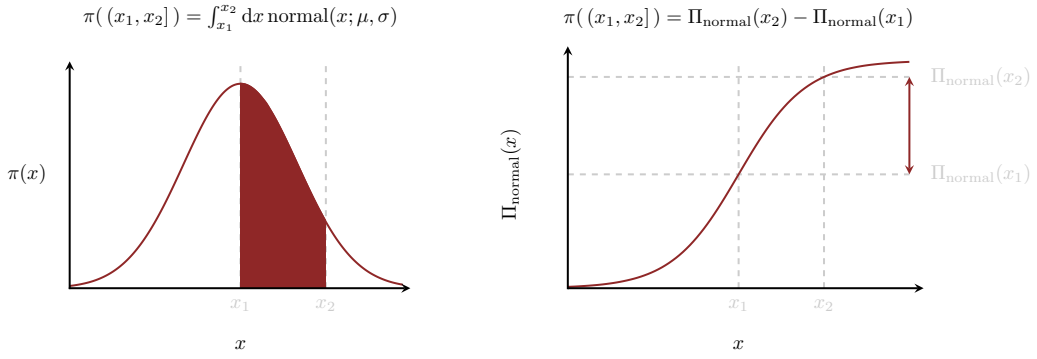


Figure 14: The normal cumulative distribution functions $\Pi_{\text{normal}}(x; \mu, \sigma)$ provide another way to compute interval probabilities. In addition to integrating under the curve defined by a normal probability density function we can also subtract the cumulative probabilities at the interval boundaries.

5 Other Useful Probability Density Functions

Most of applications of probability theory that we will tackle in this book will use probability distributions that are absolutely continuous with respect to a counting measure or a Lebesgue measure and implemented with appropriate probability density functions. There are a few exceptions, however, that we will occasionally need to accommodate.

5.1 Singular Probability Density Functions

On any measurable space where the atomic subsets are measurable we can always define a *singular* probability distribution $\delta_{x'}$ that concentrates all probability on a single point $x' \in X$,

$$\delta_x(\mathsf{x}) = \begin{cases} 1, & x' \in \mathsf{x} \\ 0, & x' \notin \mathsf{x} \end{cases}.$$

More generally because only a single point contributes we can define expectation values by the point-wise evaluation of the expectand,

$$\mathbb{E}_{\delta_x}[f] = f(x').$$

These probability distributions are known as **Dirac distributions**.

Because $\delta_{x'}(\{x'\}) = 1$ the atomic subset $\{x'\}$ is *not* a null subset with respect to $\delta_{x'}$. Consequently Dirac distributions are not absolutely continuous with respect to any reference measure that allocates vanishing probability to the atomic subsets. In particular singular probability distributions on real lines are not absolutely continuous with respect to the Lebesgue measure, and we *cannot represent them with ordinary probability density functions*!

For example if a probability density function existed then we should be able to engineer a function $\delta(x - x')$ such that

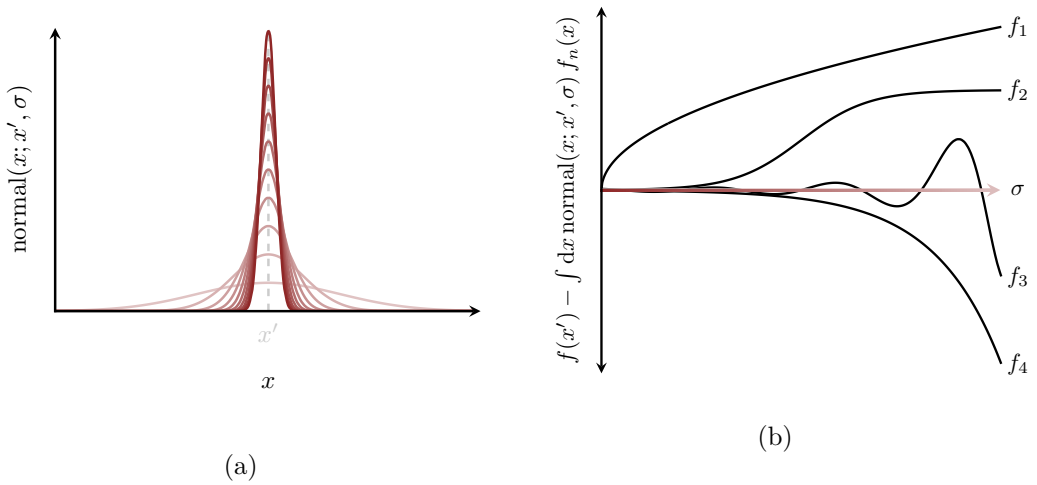
$$f(x') = \mathbb{E}_{\delta_{x'}}[f] = \mathbb{I}_\lambda[\delta(\cdot - x') \cdot f] = \int_{-\infty}^{\infty} \mathrm{d}x \, \delta(x - x') \, f(x)$$

for any expectand $f : \mathbb{R} \to \mathbb{R}$. How could we achieve this behavior?

Well if the hypothetical density function $\delta(x - x')$ concentrated around x' then the integrals would also concentrate around $f(x')$. For example integrals of normal probability density functions centered at x' approximate the desired behavior better and better as the scale becomes smaller and smaller (Figure 15).

In the limit $\sigma \to 0$ the normal probability density functions reduce to an infinitely high spike at the mean,

$$\lim_{\sigma \to 0} \frac{1}{\sqrt{2\,\pi}\sigma} \exp\left(-\frac{1}{2} \left(\frac{x' - x}{\sigma} \right)^2 \right) = \begin{cases} \infty, & x' = x \\ 0, & x' \neq x \end{cases}.$$



(a) (b)

Figure 15: As they become narrower and narrower normal probability density functions start to behave like a hypothetical singular density function. (a) In the limit $\sigma \to 0$ the normal probability density functions centered at $\mu = x'$ converge to an infinitely narrow spike at x'. (b) At the same the expectation values of all expectands f coverge to the point evaluations $f(x')$.

Perhaps we can we define $\delta(x - x')$ by this spike?

Unfortunately this intuition doesn't quite work out. Remember that probability density functions can be modified at a null subset of input points without affecting their integrals. Because $\{x'\}$ is a Lebesgue null subset this implies that our infinite spike is equivalent to a function z that returns zero for *all* inputs,

$$\mathbb{I}_\lambda[\delta(\cdot - x') \cdot f] = \mathbb{I}_\lambda[z \cdot f] = \mathbb{I}_\lambda[z].$$

Unfortunately this results in an ill-defined integral,

$$\mathbb{I}_\lambda[z] = \int_{-\infty}^{+\infty} \mathrm{d}x \, 0 = 0 \cdot \infty.$$

that contradicts the desired behavior

$$\mathbb{I}_\lambda[\delta(\cdot - x') \cdot f] = f(x').$$

We cannot define a function that can scale a Lebesgue measure into a singular Dirac distribution. Of course this is what the failure of absolutely continuity was trying to tell us in the first place!

Because the expectation values are trivial to compute, working with a singular probability distribution directly is straightforward. The lack of a well-defined singular density function,

however, can be awkward when we're exclusively using probability density functions to specify every other probability distribution of interest.

One way to get around this issue is to just *define* an object δ called the **Dirac delta function** that satisfies

$$f(0) = \int_{-\infty}^{\infty} \mathrm{d}x \, \delta(x) \, f(x)$$

for all functions $f : \mathbb{R} \to \mathbb{R}$. Frustratingly contrary to the name, δ *is not a function* but rather what mathematicians call a **generalized function**. Fortunately these technicalities don't matter so long as we only ever use the formal integral definition in calculations.

For example consider an application where we want to *inflate* the probability allocated to the atomic subset $\{x'\}$ from 0 to $0 < \gamma \leq 1$, breaking absolute continuity with respect to the Lebesgue measure in the process. We can achieve this with a **mixture probability distribution** that combines a Dirac distribution $\delta_{x'}$ with the initial probability distribution π that is absolutely continuous with respect to the Lebesgue measure,

$$\rho = \gamma \, \delta_{x'} + (1 - \gamma) \, \pi.$$

This mixture distribution defines the subset allocations

$$\begin{aligned} \rho(\mathsf{x}) &= \gamma \, \delta_{x'}(\mathsf{x}) + (1 - \gamma) \, \pi(\mathsf{x}) \\ &= \begin{cases} \gamma + (1 - \gamma) \, \pi(\mathsf{x}), & x' \in \mathsf{x} \\ (1 - \gamma) \, \pi(\mathsf{x}), & x' \notin \mathsf{x} \end{cases} \end{aligned}$$

and the expectation values

$$\begin{aligned} \mathbb{E}_{\rho}[f] &= \gamma \, \mathbb{E}_{\delta_{x'}}[f] + (1 - \gamma) \, \mathbb{E}_{\pi}[f] \\ &= \gamma \, f(x') + (1 - \gamma) \, \mathbb{E}_{\pi}[f]. \end{aligned}$$

Using the Dirac delta function we can *heuristicaly* represent this mixture distribution as

$$\rho(x) = \gamma \, \delta(x - x') + (1 - \gamma) \, \pi(x)$$

where all expectation values are calculated as

$$\begin{aligned} \mathbb{E}_{\rho}[f] &= \int_{-\infty}^{\infty} \mathrm{d}x \, \rho(x) \, f(x) \\ &= \int_{-\infty}^{\infty} \mathrm{d}x \, \left(\gamma \, \delta(x - x') + (1 - \gamma) \, \pi(x) \right) \, f(x) \\ &= \gamma \int_{-\infty}^{\infty} \mathrm{d}x \, \delta(x - x') \, f(x) + (1 - \gamma) \int_{-\infty}^{\infty} \mathrm{d}x \, \pi(x) \, f(x) \\ &= \gamma \, f(x') + (1 - \gamma) \int_{-\infty}^{\infty} \mathrm{d}x \, \pi(x) \, f(x) \\ &= \gamma \, f(x') + (1 - \gamma) \, \mathbb{E}_{\pi}[f]. \end{aligned}$$

If we restrict consideration to continuous probability density functions then we can visualize this mixture density function $\rho(x)$ as a continuous base density function $\pi(x)$ with a single discontinuity at x' (Figure 16). Without this restriction, however, visualizations like this are ambiguous because $\pi(x)$ is defined only up to subsets of null Lebesgue measure.

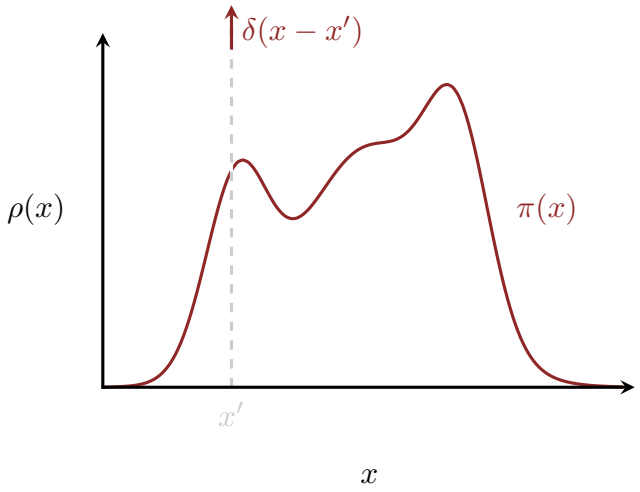


Figure 16: One way to visually represent the mixture of a base probability distribution with a singular probability distribution is to inflate the base probability density function with an infinite spike. This visualization is well-defined, however, only if we assume a continuous probability density function for the base probability distribution.

Because of all of this delicate mathematical baggage the Dirac delta "function" requires care when using in practice. That said the compact, dare I say elegant, probability density function descriptions it enables is often worth the added subtlety.

5.2 Geometric Probability Density Functions

Real spaces adequately model many phenomena that arise in practical applications, but by no means all of them. In some cases we will need to consider continuous spaces that look like a real spaces *locally* but exhibit different shapes *globally* (Figure 18). These include for example spheres, torii, and even more foreign spaces. Mathematically these spaces, along with real spaces, are collectively known as **manifolds**.

One nice feature of manifolds is that we can always equip them with consistent metric structures. A chosen metric then allows us to define a compatible uniform measure that emulates many of the features of the Lebesgue measure on real spaces. In particular these uniform measures serve as natural reference measures on which we can build many useful probability distributions.

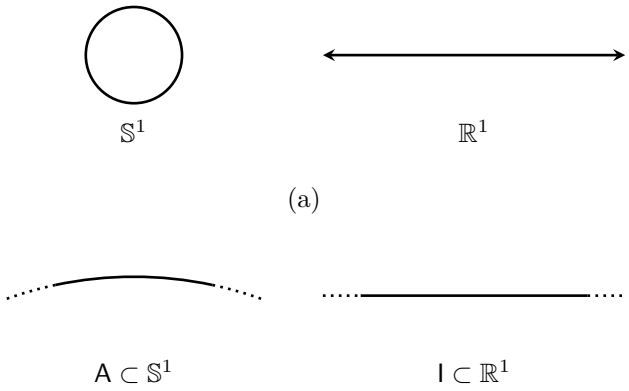


Figure 17: A circle $\mathbb{S}^1$ is an example of a manifold. (a) Globally the space defined by a circle is distinct from the space defined by a real line. (b) If we zoom in, however, the local behavior of the circle is equivalent to the local behavior of a real line.

For example we can equip a circle S^1 with a metric that endows angular intervals with a notion of length (Figure 18a). We can then define a uniform measure that allocates the same measure to every angular interval of the same length (Figure 18b).

Another important consequence of this construction is that, similar to the Lebesgue measure, these uniform measures allocate zero measure to every atomic subset. Consequently any probability distribution over S^1 that also allocates vanishing probability to the atomic subsets will be absolutely continuous to these uniform measures. This allows us to define **circular probability density functions**

$$\pi : \mathbb{S}^1 \to \mathbb{R}^+$$

to represent each of these absolutely continuous probability distributions (Figure 19).

We have to take care, however, not to confuse these circular probability density functions with Lebesgue probability density functions. In particular expectation values

$$\mathbb{E}_\pi[f] = \mathbb{I}_\nu[\pi \cdot f]$$

are not implemented with classic Riemann integration but rather a more general **manifold integration** that cannot be implemented in the same way.

That said sometimes there are work arounds. For example removing a point $x' \in \mathbb{S}^1$ from the circle defines a new space $\mathbb{S}^1 \setminus x'$. Circular probability distributions, circular probability density functions, and circular expectands $f : \mathbb{S} \to \mathbb{R}$ on the circle all define corresponding objects on this excised space. Once we've removed the point we can then unroll and stretch out $\mathbb{S}^1 \setminus x'$ into a real line $\mathbb{R}$, taking all of the probabilistic objects along with us (Figure 20).

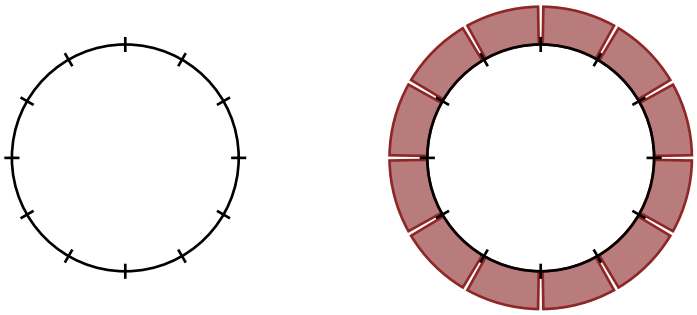


Figure 18: Manifolds like the circle can be equipped with metrics, and uniform measures compatible with that metric structure. (a) A metric endows each angular interval with a length. (b) We can then define a uniform measure that allocates to each angular interval a measure equal to its length.

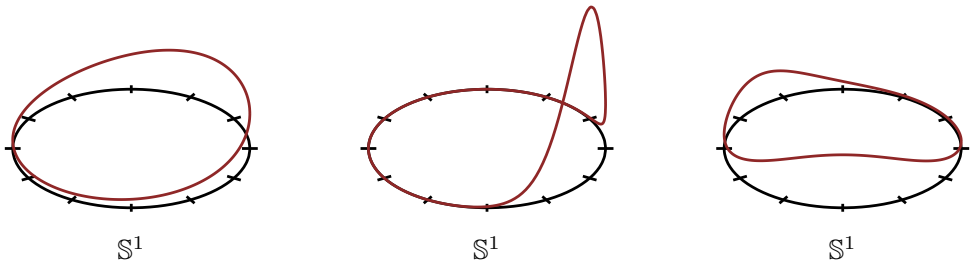


Figure 19: Sufficiently nice probability distributions on a circle $\mathbb{S}^1$ can be represented by circular probability density functions $\pi : \mathbb{S}^1 \to \mathbb{R}^+$. Similar to Lebesgue probability density functions these circular probability density functions visualize a host of qualitative behaviors, such as where and how the implied probability distributions concentrate.

In general expectation values on the initial circular probability space will be different from the expectation values on the excised probability space. If $\{x'\}$ is a null subset, however, then they will be the same,

$$\mathbb{E}_{\pi}[f] = \mathbb{I}_{\nu}[\pi \cdot f] = \mathbb{I}_{\nu'}[\pi' \cdot f'] = \mathbb{I}_{\nu''}[\pi'' \cdot f''],$$

where no ticks denotes objects on $\mathbb{S}^1$, one tick denotes objects on $\mathbb{S}^1 \setminus \mathbb{R}$, and two ticks denotes objects on $\mathbb{R}$.

To summarize, by removing a point from the circle we can implement circular expectation values with classic Riemann integrals. Not every cut point, however, will be as convenient as others. Cutting the circle away from where the initial probability density function concentrates results in a well-behaved, unimodal Lebesgue probability density function (Figure 20a). On the other hand cutting near the centrality of the probability density function results in a Lebesgue probability density function with two peaks at relatively extreme values, which can easily frustrate integral calculations (Figure 20b).

If we don't know where the initial circular probability density function concentrates then we won't know where to make a good cut, and we can end up with a difficult Lebesgue probability density function that doesn't actually get us any closer to a completed calculation. The subtle relationship between circular probability density functions, let alone other, more complicated manifold probability density functions, and Lebesgue probability density functions is dark and full of terrors. To avoid computational problems, or worse corrupting the target expectation values, we need to be very careful when working on these more sophisticated spaces.

6 Conclusion

Because they provide a straightforward way to implement probability distributions in practice, probability density functions are absolutely critical in transitioning probability theory from abstract mathematics to a viable tool for applied practice. That said because they define probability distributions only in the context of integrals informed by a given reference measure they are not without their subtleties. When we ignore this context we become prone to incorrect interpretations and implementations which then results in inconsistent applications of the underlying probability theory.

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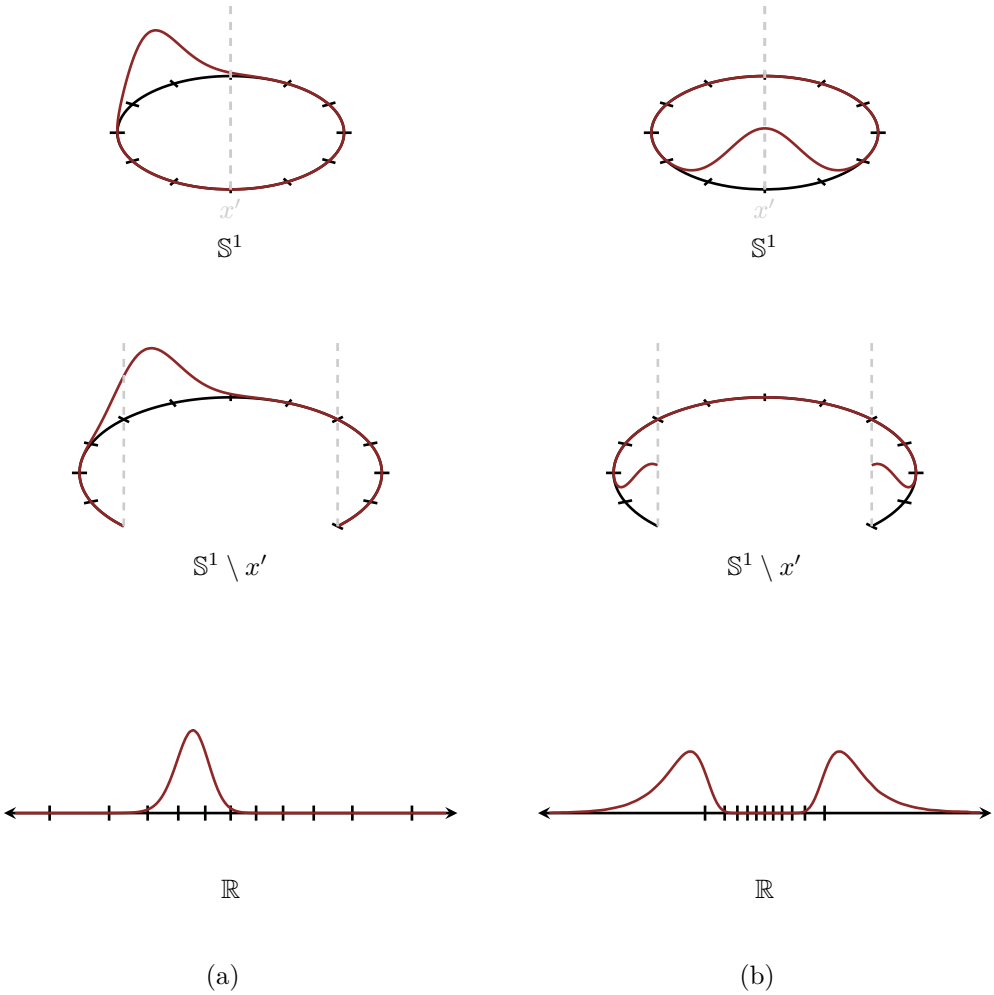

Figure 20: Probabilistic computations on a circle can be transformed into integrals on a real line with a little bit of surgery. Removing a point x' from a circle $\mathbb{S}^1$ gives the space $\mathbb{S}^1 \setminus x'$ which can be unrolled and stretched out into a real line $\mathbb{R}$. Because its removes only a null subset this surgery doesn't affect integrals, allowing us to compute circular expectation values with Riemann integrals on $\mathbb{R}$. Which point we remove, however, can have a strong influence no how difficult those Riemann integrals are. (a) Cutting the circle at a point around which little probability is allocated results in a well-behaved Lebesgue probability density function that facilitates integration. (b) On the other hand cutting at a point around which a substantial amount of probability is allocated results in a more pathological Lebesgue probability density function that frustrates integration.

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Appendix: Sums and Integrals

Probability density functions allow us to compute expectation values using explicit mathematical operations, in particular summation when working with a counting reference measure and Riemann integration when working with a Lebesgue reference measure. Just because these operations are explicit, however, doesn't mean that they're straightforward.

Deriving analytic results for even the most convenient summations and integrals can require a substantial amount of experience with mathematical techniques, and in many cases tricks that

seem to come out of nowhere. For those who are curious about these techniques this appendix gathers full calculations for the Poisson and normal results that we used in this chapter.

Note that these calculations will absolutely not be necessary for keeping up with future chapters. Indeed in most applications we will take advantage of other computational tools that don't require these kinds of onerous calculations at all.

6.1 Poisson Summations

Let's warm up with some summations.

The normalization of each Poisson probability mass function is given by

$$\begin{aligned}
\text{Poisson}(X; \lambda) &= \mathbb{E}_{\text{Poisson}}[I_X; \lambda] \\
&= \mathbb{I}_{\chi}[\text{Poisson}(; \lambda) \cdot I_X] \\
&= \sum_{n=0}^{\infty} \text{Poisson}(n; \lambda) \cdot 1 \\
&= \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}.
\end{aligned}$$

This summation, however, is just the power series definition for the exponential function,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Consequently

$$\begin{aligned}
\text{Poisson}(X; \lambda) &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \\
&= e^{-\lambda} e^{\lambda} \\
&= 1,
\end{aligned}$$

as required.

Similarly the mean for each Poisson probability distribution is given by

$$\begin{aligned}
\mathbb{M}(\lambda) &= \mathbb{E}_{\text{Poisson}}[\iota; \lambda] \\
&= \mathbb{I}_{\chi}[\text{Poisson}(; \lambda) \cdot \iota] \\
&= \sum_{n=0}^{\infty} \text{Poisson}(n; \lambda) \cdot \iota(n) \\
&= \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \cdot n \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{(n-1)!}.
\end{aligned}$$

To evaluate this sum we need to shift the summation index from n to $m = n-1$ which gives

$$\begin{aligned}
\mathbb{M}(\lambda) &= \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{(n-1)!} \\
&= \sum_{m=1}^{\infty} \frac{\lambda^{m+1} e^{-\lambda}}{m!} \\
&= \lambda \sum_{m=1}^{\infty} \frac{\lambda^m e^{-\lambda}}{m!}.
\end{aligned}$$

Conveniently the remaining sum is exactly the normalization that we showed above is equal to one. Substituting this gives

$$\begin{aligned}
\mathbb{M}(\lambda) &= \lambda \sum_{m=1}^{\infty} \frac{\lambda^m e^{-\lambda}}{m!} \\
&= \lambda.
\end{aligned}$$

To compute the variances we'll first need the second-order moment,

$$\begin{aligned}
\mathbb{M}_2(\lambda) &= \mathbb{E}_{\text{Poisson}}[\iota^2; \lambda] \\
&= \mathbb{I}_{\chi}[\text{Poisson}(; \lambda) \cdot \iota^2] \\
&= \sum_{n=0}^{\infty} \text{Poisson}(n; \lambda) \cdot iota(n)^2 \\
&= \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \cdot n^2 \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{(n-1)!} n
\end{aligned}$$

Shift the summation index from n to $m = n - 1$ as we did above gives

$$\begin{aligned}
\mathbb{M}_2(\lambda) &= \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{(n-1)!} n \\
&= \sum_{m=1}^{\infty} \frac{\lambda^{m+1} e^{-\lambda}}{m!} (m+1) \\
&= \lambda \sum_{m=1}^{\infty} \frac{\lambda^m e^{-\lambda}}{m!} m + \lambda \sum_{m=1}^{\infty} \frac{\lambda^m e^{-\lambda}}{m!}.
\end{aligned}$$

Now the first summation is the Poisson mean while second summation is the normalization, both of which we've already computed. Substituting gives

$$\begin{aligned}
\mathbb{M}_2(\lambda) &= \lambda \sum_{m=1}^{\infty} \frac{\lambda^m e^{-\lambda}}{m!} m + \lambda \sum_{m=1}^{\infty} \frac{\lambda^m e^{-\lambda}}{m!} \\
&= \lambda \, \mathbb{M}(\lambda) + \lambda \, \text{Poisson}(X; \lambda) \\
&= \lambda \, \lambda + \lambda \, 1 \\
&= \lambda^2 + \lambda.
\end{aligned}$$

We can now construct the Poisson variances by

$$\begin{aligned}
\mathbb{C}_2 &= \mathbb{E}_{\text{Poisson}}[(\iota - \mathbb{M}(\lambda))^2; \lambda] \\
&= \mathbb{E}_{\text{Poisson}}[\iota^2 - 2 \, \mathbb{M}(\lambda) \, \iota + \mathbb{M}(\lambda)^2; \lambda] \\
&= \mathbb{E}_{\text{Poisson}}[\iota^2; \lambda] - 2 \, \mathbb{M}(\lambda) \, \mathbb{E}_{\text{Poisson}}[\iota; \lambda] + \mathbb{M}(\lambda) \\
&= (\lambda^2 + \lambda) - 2 \, \lambda \, \lambda + \lambda^2 \\
&= (2\lambda^2 - 2\lambda^2) + \lambda \\
&= \lambda.
\end{aligned}$$

6.2 Normal Integrals

Working with normal expectation values is substantially easier once we've established a few foundational results.

6.2.1 The Basic Normal Integral

First let's compute the so-called "normal integral"

$$H = \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-a \, x^2\right).$$

The trick to computing this integral is to work not with a *single* copy of this integral but rather *two* copies of this integral,

$$\begin{aligned} H^2 &= H \cdot H \\ &= \int_{-\infty}^{\infty} \mathrm{d}x_1 \, \exp\left(-a\, x_1^2\right) \int_{-\infty}^{\infty} \mathrm{d}x_2 \, \exp\left(-a\, x_2^2\right) \\ &= \int_{-\infty}^{\infty} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_2 \, \exp\left(-a\, x_1^2\right) \exp\left(-a\, x_2^2\right) \\ &= \int_{-\infty}^{\infty} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_2 \, \exp\left(-a\left(x_1^2 + x_2^2\right)\right). \end{aligned}$$

Using two copies of the integral remarkable results in a two-dimensional integral that is ripe for polar coordinates. Making the transformation

$$\begin{aligned} r &= x_1^2 + x_2^2 \\ \theta &= \arctan \frac{x_2}{x_1} \end{aligned}$$

with

$$\mathrm{d}x_1 \, \mathrm{d}x_2 = \mathrm{d}\theta \, \mathrm{d}r \, r$$

gives

$$\begin{aligned} H^2 &= \int_{-\infty}^{\infty} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_2 \, \exp\left(-a\left(x_1^2 + x_2^2\right)\right) \\ &= \int_0^{2\,\pi} \mathrm{d}\theta \int_0^{\infty} \mathrm{d}r \, r \exp\left(-a\, r^2\right) \\ &= \int_0^{\infty} \mathrm{d}r \, r \exp\left(-a\, r^2\right) \int_0^{2\,\pi} \mathrm{d}\theta \\ &= \int_0^{\infty} \mathrm{d}r \, r \exp\left(-a\, r^2\right) \, 2\,\pi \\ &= 2\,\pi \int_0^{\infty} \mathrm{d}r \, r \exp\left(-a\, r^2\right). \end{aligned}$$

At this point we make the substitution

$$u = -a\, r^2$$

with

$$\mathrm{d}u = \mathrm{d}r \, (-2\, a\, r)$$

which gives

$$\begin{aligned} H^2 &= 2\,\pi \int_0^{\infty} \mathrm{d}r\, r \exp\left(-a\, r^2\right) \\ &= 2\,\pi \int_0^{-\infty} \mathrm{d}u\, \frac{-1}{2\,a} \exp\left(u\right) \\ &= -\frac{\pi}{a} \int_0^{-\infty} \mathrm{d}u \exp\left(u\right) \\ &= -\frac{\pi}{a} \left[\exp\left(u\right)\right]_0^{-\infty} \\ &= -\frac{\pi}{a} \left[\exp\left(-\infty\right) - \exp\left(0\right)\right] \\ &= -\frac{\pi}{a} \left[0 - 1\right] \\ &= \frac{\pi}{a}. \end{aligned}$$

Consequently

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left(-a\, x^2\right) = \sqrt{H^2} = \sqrt{\frac{\pi}{a}}.$$

6.2.2 Higher-Order Normal Integrals

We can now use this basic normal integral to compute the slightly more sophisticated integral

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^k\, \exp\left(-a\, x^2\right).$$

One thing that we can note immediately is that when k is an odd integer the integrand will be an odd function, and the integral will vanish by symmetry. Consequently we really only need to compute the integrals

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^{2k}\, \exp\left(-a\, x^2\right)$$

for $k \in \mathbb{N}$.

We could hack away at this integral by integrating the basic normal integral by parts. For

example

$$\begin{aligned}
\sqrt{\frac{\pi}{a}} &= \int_{-\infty}^{\infty} \mathrm{d}x \, (1) \, (\exp(-a\,x^2) \,) \\
&= [x \, \exp(-a\,x^2)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \mathrm{d}x \, (x) \, ((-2\,a\,x) \, \exp(-a\,x^2) \,) \\
&= [0 - 0] + 2\,a \int_{-\infty}^{\infty} \mathrm{d}x \, x^2 \, \exp(-a\,x^2) \\
&= 2\,a \int_{-\infty}^{\infty} \mathrm{d}x \, x^2 \, \exp(-a\,x^2)
\end{aligned}$$

or

$$\int_{-\infty}^{\infty} \mathrm{d}x \, x^2 \, \exp(-a\,x^2) = \frac{1}{2\,a} \sqrt{\frac{\pi}{a}} = \sqrt{\pi} \, \frac{1}{2} \, a^{-3/2}.$$

Higher-order integrals can then be derived by repeating this process over and over again.

That said we can also compute these integrals more efficiently with the help of a devious trick of mathematics. The idea is to start with our initial integral

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \exp(-a\,x^2) = \sqrt{\frac{\pi}{a}}$$

and differentiate both sides with respect to the parameter a,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a} \int_{-\infty}^{\infty} \mathrm{d}x \, \exp(-a\,x^2) &= \frac{\mathrm{d}}{\mathrm{d}a} \sqrt{\frac{\pi}{a}} \\
&= \sqrt{\pi} \left(-\frac{1}{2} \right) a^{-3/2}.
\end{aligned}$$

Because the integrand is sufficiently nice the derivative and integral operations here commute, allowing us to pull the derivative inside of the integral,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a} \int_{-\infty}^{\infty} \mathrm{d}x \, \exp(-a\,x^2) &= \sqrt{\pi} \left(-\frac{1}{2} \right) a^{-3/2} \\
\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{\mathrm{d}}{\mathrm{d}a} \exp(-a\,x^2) &= \sqrt{\pi} \left(-\frac{1}{2} \right) a^{-3/2} \\
\int_{-\infty}^{\infty} \mathrm{d}x \, (-x^2) \exp(-a\,x^2) &= \sqrt{\pi} \left(-\frac{1}{2} \right) a^{-3/2} \\
\int_{-\infty}^{\infty} \mathrm{d}x \, x^2 \exp(-a\,x^2) &= \sqrt{\pi} \, \frac{1}{2} \, a^{-3/2},
\end{aligned}$$

consistent with our earlier integration by parts calculation.

The beauty of this latter approach, however, is that it generalizes to larger k immediately with repeated differentiation,

$$\begin{aligned}
\frac{\mathrm{d}^k}{\mathrm{d}a^k} \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-a\, x^2\right) &= \frac{\mathrm{d}^k}{\mathrm{d}a^k} \sqrt{\frac{\pi}{a}} \\
\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{\mathrm{d}^k}{\mathrm{d}a^k} \exp\left(-a\, x^2\right) &= \frac{\mathrm{d}^k}{\mathrm{d}a^k} \sqrt{\frac{\pi}{a}} \\
\int_{-\infty}^{\infty} \mathrm{d}x \, (-1)^k x^{2k} \exp\left(-a\, x^2\right) &= \sqrt{\pi} \prod_{k'=1}^{k} \left(-\frac{1}{2} - (k'-1)\right) \, a^{-(1/2+k)} \\
\int_{-\infty}^{\infty} \mathrm{d}x \, (-1)^k x^{2k} \exp\left(-a\, x^2\right) &= \sqrt{\pi}\, (-1)^k \prod_{k'=1}^{k} \frac{2\,k'-1}{2} \, a^{-(1/2+k)} \\
\int_{-\infty}^{\infty} \mathrm{d}x \, x^{2k} \exp\left(-a\, x^2\right) &= \sqrt{\pi}\, \frac{(2\,k-1)!!}{2^k} \, a^{-(1/2+k)},
\end{aligned}$$

where

$$n!! = \prod_{k'=1}^{\frac{n+1}{2}} (2\,k'-1)$$

is the **double factorial**.

6.2.3 Normal Expectation Values

With the general result

$$\int_{-\infty}^{\infty} \mathrm{d}x \, x^{2k} \exp\left(-a\, x^2\right) = \sqrt{\pi} \frac{k!!}{2^k} a^{-(k+1/2)}$$

we can now go to town on the standard normal expectation values.

For example the normalization of each normal density function is given by

$$\begin{aligned}
\mathrm{normal}(\mathbb{R}; \mu, \sigma) &= \mathbb{E}_{\mathrm{normal}}[I_{\mathbb{R}}; \mu, \sigma] \\
&= \mathbb{I}_{\lambda}[\mathrm{normal}(\cdot; \mu, \sigma) \cdot I_{\mathbb{R}}] \\
&= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{\sqrt{2\,\pi}\,\sigma} \mathrm{normal}(x; \mu, \sigma) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{\sqrt{2\,\pi}\,\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \\
&= \frac{1}{\sqrt{2\,\pi}\,\sigma} \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right).
\end{aligned}$$

To proceed further we have to change variables to

$$u = \frac{x - \mu}{\sigma}$$

with

$$\mathrm{d}u = \mathrm{d}x \, \frac{1}{\sigma}$$

to give

$$\begin{aligned}
\mathrm{normal}(\mathbb{R}; \mu, \sigma) &= \frac{1}{\sqrt{2\,\pi}\,\sigma} \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \\
&= \frac{1}{\sqrt{2\,\pi}\,\sigma} \int_{-\infty}^{\infty} \mathrm{d}u \, \sigma \, \exp\left(-\frac{1}{2}u^2\right) \\
&= \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \mathrm{d}u \, \exp\left(-\frac{1}{2}u^2\right).
\end{aligned}$$

For $a = 1/2$ and $k = 0$ our general normal integral gives

$$\begin{aligned}
\mathrm{normal}(\mathbb{R}; \mu, \sigma) &= \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \mathrm{d}u \, \exp\left(-\frac{1}{2}u^2\right) \\
&= \frac{1}{\sqrt{2\,\pi}} \sqrt{2\,\pi} \\
&= 1,
\end{aligned}$$

as required.

Similarly the mean is given by

$$\begin{aligned}
\mathbb{M}(\mu, \sigma) &= \mathbb{E}_{\mathrm{normal}}[\iota; \mu, \sigma] \\
&= \mathbb{I}_{\lambda}[\mathrm{normal}(\cdot; \mu, \sigma) \cdot \iota] \\
&= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{\sqrt{2\,\pi}\,\sigma} \mathrm{normal}(x; \mu, \sigma) \, \iota(x) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{\sqrt{2\,\pi}\,\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \, x \\
&= \frac{1}{\sqrt{2\,\pi}\,\sigma} \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \, x.
\end{aligned}$$

Applying the same change of variables as above gives

$$\begin{aligned}
\mathbb{M}(\mu, \sigma) &= \frac{1}{\sqrt{2\,\pi}\,\sigma} \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \, x \\
&= \frac{1}{\sqrt{2\,\pi}\,\sigma} \int_{-\infty}^{\infty} \mathrm{d}u \, \sigma \, \exp\left(-\frac{1}{2}u^2\right) \, (\sigma\, u + \mu) \\
&= \frac{\sigma}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \mathrm{d}u \, \exp\left(-\frac{1}{2}u^2\right) \, u + \frac{\mu}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \mathrm{d}u \, \exp\left(-\frac{1}{2}u^2\right).
\end{aligned}$$

The first integral here vanishes by symmetry while the second integral is the same application of the general normal integral that we just used,

$$\begin{aligned}
\mathbb{M}(\mu, \sigma) &= \frac{\sigma}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \mathrm{d}u\, \exp\left(-\frac{1}{2} u^2\right)\, u + \frac{\mu}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \mathrm{d}u\, \exp\left(-\frac{1}{2} u^2\right) \\
&= 0 + \frac{\mu}{\sqrt{2\,\pi}} \sqrt{2\,\pi} \\
&= \mu.
\end{aligned}$$

We can now compute all of the central moments at once. The odd central moments all vanish by symmetry, whereas the even central moments are given by

$$\begin{aligned}
\mathbb{D}_{2\,k}(\mu, \sigma) &= \mathbb{E}_{\mathrm{normal}}[(\iota - \mathbb{M}(\mu, \sigma))^{2\,k}; \mu, \sigma] \\
&= \mathbb{E}_{\mathrm{normal}}[(\iota - \mu)^{2\,k}; \mu, \sigma] \\
&= \mathbb{I}_{\lambda}[\mathrm{normal}(\cdot; \mu, \sigma) \cdot (\iota - \mu)^{2\,k}] \\
&= \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{1}{\sqrt{2\,\pi}\,\sigma} \mathrm{normal}(x; \mu, \sigma)\, (\iota(x) - \mu)^{2\,k} \\
&= \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{1}{\sqrt{2\,\pi}\,\sigma} \exp\left(-\frac{1}{2} \left(\frac{x - \mu}{\sigma}\right)^2\right)\, (x - \mu)^{2\,k} \\
&= \frac{1}{\sqrt{2\,\pi}\,\sigma} \int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left(-\frac{1}{2} \left(\frac{x - \mu}{\sigma}\right)^2\right)\, (x - \mu)^{2\,k}.
\end{aligned}$$

After using the same change of variables *again* this becomes

$$\begin{aligned}
\mathbb{D}_{2k}(\mu, \sigma) &= \frac{1}{\sqrt{2\,\pi}\,\sigma} \int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left(-\frac{1}{2} \left(\frac{x - \mu}{\sigma}\right)^2\right)\, (x - \mu)^{2k} \\
&= \frac{1}{\sqrt{2\,\pi}\,\sigma} \int_{-\infty}^{\infty} \mathrm{d}u\, \sigma\, \exp\left(-\frac{1}{2} u^2\right)\, \sigma^{2\,k}\, u^{2k} \\
&= \frac{\sigma^{2\,k}}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \mathrm{d}u\, \exp\left(-\frac{1}{2} u^2\right)\, u^{2k}.
\end{aligned}$$

We already know, however, how to evaluate this integral.

$$\begin{aligned}
\mathbb{D}_{2k}(\mu,\sigma) &= \frac{\sigma^{2\,k}}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty}\mathrm{d}u\,\exp\left(-\frac{1}{2}u^2\right)\,u^{2k} \\
&= \frac{\sigma^{2\,k}}{\sqrt{2\,\pi}}\sqrt{\pi}\,\frac{(2\,k-1)!!}{2^k}\,\left(\frac{1}{2}\right)^{-(1/2+k)} \\
&= \frac{\sigma^{2\,k}}{\sqrt{2}}\,\frac{(2\,k-1)!!}{2^k}\,2^{1/2+k} \\
&= \frac{\sigma^{2\,k}}{\sqrt{2}}\,\frac{(2\,k-1)!!}{2^k}\,\sqrt{2}\,2^k \\
&= \sigma^{2\,k}\,(2\,k-1)!!.
\end{aligned}$$

In particular the variance is given by

$$\mathbb{C}_2(\mu,\sigma) = \mathbb{D}_2(\mu,\sigma) = \sigma^2.$$

Finally the normal cumulative distribution functions are given by

$$\begin{aligned}
\Pi_{\mathrm{normal}}(x;\mu,\sigma) &= \mathrm{normal}(\,(-\infty,x];\mu,\sigma\,) \\
&= \mathbb{E}_{\mathrm{normal}}[I_{(-\infty,x]};\mu,\sigma] \\
&= \int_{-\infty}^{+\infty}\mathrm{d}x'\,\mathrm{normal}(x';\mu,\sigma)\,I_{(-\infty,x]}(x') \\
&= \int_{-\infty}^{x}\mathrm{d}x'\,\mathrm{normal}(x';\mu,\sigma) \\
&= \int_{-\infty}^{x}\mathrm{d}x'\,\frac{1}{\sqrt{2\,\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{x'-\mu}{\sigma}\right)^2\right) \\
&= \frac{1}{\sqrt{2\,\pi}\sigma}\int_{-\infty}^{x}\mathrm{d}x'\,\exp\left(-\frac{1}{2}\left(\frac{x'-\mu}{\sigma}\right)^2\right).
\end{aligned}$$

After a change of variables to

$$t = \frac{x'-\mu}{\sqrt{2}\,\sigma}$$

this becomes

$$\begin{aligned}
\Pi(x; \mu, \sigma) &= \frac{1}{\sqrt{2\,\pi}\sigma} \int_{-\infty}^{x} \mathrm{d}x' \, \exp\left(-\frac{1}{2} \left(\frac{x' - \mu}{\sigma} \right)^2 \right). \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x-\mu}{\sqrt{2}\,\sigma}} \mathrm{d}t \, \exp\left(-t^2\right) \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} \mathrm{d}t \, \exp\left(-t^2\right) + \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x-\mu}{\sqrt{2}\,\sigma}} \mathrm{d}t \, \exp\left(-t^2\right) \\
&= \frac{1}{2} + \frac{1}{2} \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{x-\mu}{\sqrt{2}\,\sigma}} \mathrm{d}t \, \exp\left(-t^2\right) \\
&= \frac{1}{2} + \frac{1}{2} \, \mathrm{erf}\left(\frac{x - \mu}{\sqrt{2}\,\sigma} \right),
\end{aligned}$$

where

$$\mathrm{erf}(x) \frac{2}{\sqrt{\pi}} \int_{0}^{x} \mathrm{d}t \, \exp\left(-t^2\right)$$

is the **error function**.

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