Ejercicio 1

Equipo

2024-03-26

1. Regresión a través del origen.

$$y_i = \beta x_i + \xi_i \quad i : 1, \ldots, n$$

donde $\xi_1, \ldots, \xi_n$ son v.a.i talque $\xi_i \sim N\left(0, \frac{\sigma^2}{w_i}\right)$ $\forall i = 1, \ldots, n$

Suponiendo σ^2 conocida y $w_i = \frac{1}{x_i^2} \quad i = 1, \ldots, n$

I)

$$\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} (y_i - \beta x_i)^2$$

$$\frac{d}{d\beta} = 2\sum_{i=1}^{n} (y_i - \beta x_i)(-x_i) = 0$$

$$\to \sum_{i=1}^{n} (y_i - \beta x_i)(-x_i) = 0 \to \sum_{i=1}^{n} y_i(-x_i) + \sum_{i=1}^{n} \beta x_i^2 = 0$$

$$\therefore \hat{\beta} = \frac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2}$$

II) $\mathrm{Var}(\hat{\beta})$

$$\begin{aligned}\mathrm{Var}(\hat{\beta}) &= \mathrm{Var}\left(\frac{\sum Y_i X_i}{\sum {X_i}^2}\right) \\ &= \mathrm{Var}\left(\frac{\sum X_i (\beta X_i + \xi_i)}{\sum {X_i}^2}\right)\end{aligned}$$

$$\mathrm{Var}(\hat{\beta}) = \mathrm{Var}\left(\frac{\sum X_i^2 \beta + \sum X_i \xi_i}{\sum X_i^2}\right)$$

$$= \mathrm{Var}\left(\beta + \frac{\sum X_i \xi_i}{\sum X_i^2}\right) = \mathrm{Var}\left(\frac{\sum X_i \xi_i}{\sum {X_i}^2}\right)$$

$$\text{sea } C_i = \frac{X_i}{\sum X_i^2} \longrightarrow \mathrm{Var}(\hat{\beta}) = \sum C_i \xi_i$$

$$\mathrm{Var}(\hat{\beta}) = \left(\sum C_i\right)^2 \mathrm{Var}(\xi_i) = \left(\sum C_i\right)^2 \frac{\sigma^2}{\omega_i^2}$$

$$\therefore \mathrm{Var}(\hat{\beta}) = \frac{\left(\sum X_i\right)^2 \sigma^2}{\left(\sum X_i^2\right)^3}$$

III) Veamos que $\hat{\beta}$ es estimador lineal

$$\hat{B} = \frac{\sum X_i Y_i}{\sum_{i=1}^n X_i^2} \quad \text{Sea} \quad c_i = \frac{X_i}{\sum {X_i}^2}$$

$$\rightarrow \hat{B} = \sum_{i=1}^n C_i Y_i \quad \therefore \text{ es estimador lineal}$$

Ademas

$$\begin{aligned} \mathbb{E}(\hat{\beta}) &= \mathbb{E}\left(\frac{\Sigma X_i^2 \beta + \sum X_i \xi_i}{\sum X_i^2}\right) \\ &= \mathbb{E}(\beta) + \mathbb{E}\left(\frac{\sum X_i \xi_i}{\Sigma X_i^2}\right) = \beta + \frac{\sum X_i \mathbb{E}\left(\xi_i\right)}{\Sigma X_i^2} \end{aligned}$$

Como $\varepsilon_i \sim N\left(0, \frac{\sigma^2}{w_i}\right)$ entrances $\mathbb{E}(\hat{\beta}) = \beta$ De esta forma como $\hat{\beta}$ es estimador lineal y ademas es insesgado, por el teorema Gauss - Markov $\hat{\beta}$ es el UMVUE