Ejercicio 1

Equipo

2024-03-26

1. Regresión a través del origen.

$$y_i = \beta x_i + \xi_i \quad i = 1 \dots n$$

donde $\xi_1, \dots, \xi_n$ son v.a.i. talque $\xi_i \sim N\left(0, \frac{\sigma^2}{w_i}\right)$

$$\forall i = 1 \dots n$$

Suponiendo σ^2 conocida y $\omega_i = \frac{1}{x_i^2} \quad i = 1, \dots, n$ I) Como las ξ_i son normales, entonces $y_i \sim N\left(\beta x_i, x_i^2 \sigma^2\right)$ y son independientes, entonces la funcion de verosimilitud nos queda:

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi} x_i \sigma} e^{-\frac{(y_i - \beta x_i)^2}{2x_i^2 \sigma^2}}$$

es decir:

$$\frac{1}{(2\pi)^{n/2} x_i^n \sigma^n} e^{\sum_{i=1}^{n} \frac{-(y_i - \beta x_i)^2}{2x_i^2 \sigma^2}}$$

Aplicando logaritmo:

$$\ln(1) - \frac{n}{2} \ln(2\pi) - n \ln(x_i) - n \ln(\sigma) + \sum_{i=1}^{n} \frac{-\left(y_i - \beta x_i\right)^2}{2x_i^2 \sigma^2}$$

derivando e igualando a cero obtenemos:

$$\frac{d}{d\beta} \ln(f) = -\sum_{i=1}^{n} \frac{(y_i - \beta x_i)}{x_i^2 \sigma^2} (-x_i) = 0$$
$$\rightarrow \sum_{i=1}^{n} \frac{(y_i - \beta x_i)}{x_i \sigma^2} = 0 \rightarrow \sum_{i=1}^{n} \frac{y_i}{x_i \sigma^2} - \sum_{i=1}^{n} \frac{\beta}{\sigma^2} = 0$$

Asi

$$\sum_{i=1}^{n} \frac{y_i}{x_i \sigma^2} = \frac{n\beta}{\sigma^2} \rightarrow \hat{\beta} = \sum_{i=1}^{n} \frac{y_i}{x_i n}$$

II)

$$\begin{aligned}
&\operatorname{Var}(\hat{\beta})\\
&= \operatorname{Var}\left(\sum_{i=1}^{n} \frac{y_i}{x_i n}\right) = \frac{1}{n^2}\cdot \operatorname{Var}\left(\sum_{i=1}^{n} \frac{y_i}{x_i}\right)\\
&= \frac{1}{n^2}\sum_{i=1}^{n} \operatorname{Var}\left(\frac{y_i}{x_i}\right) = \frac{1}{n^2}\sum_{i=1}^{n} \frac{1}{{x_i}^2} \operatorname{Var}\left(y_i\right)\\
&= \frac{1}{n^2}\sum_{i=1}^{n} \frac{1^2}{x_i^2}\left(x_i^2\sigma^2\right) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{\sigma^2}{n}\\
&\therefore \operatorname{Var}(\hat{\beta}) = \frac{\sigma^2}{n}
\end{aligned}$$

III) Tenemos la funcion de verosimilitud

$$\frac{1}{\sigma^n (2\pi)^{n/2}\prod_{i=1}^{n} x_i} e^{\sum_{i=1}^{n} \frac{-(y_i-\beta x_i)^2}{2x_i^2\sigma^2}}$$

de donde

$$\begin{aligned}
&e^{\sum \frac{-(x_i-\beta x_i)^2}{2x_i^2\sigma^2}} = e^{\frac{1}{2\sigma^2}\sum - \frac{(y_i-\beta x_i)^2}{x_i^2}}\\
&= e^{\frac{1}{2\sigma^2}\sum - \frac{x_i^2+2\beta x_i y_i-\beta^2 x_i^2}{x_i^2}}\\
&= e^{\frac{1}{2\sigma^2}\sum \frac{-y_i^2}{x_i^2}+\sum \frac{2\beta y_i}{x_i}-\sum \beta^2}\\
&= e^{\frac{-\sum \beta^2}{2\sigma^2}}\cdot e^{\sum \frac{-y_i^2}{x_i^2}+2\beta\sum \frac{y_i}{x_i}}
\end{aligned}$$

Asi

$$\begin{array}{ll}
a(\gamma) = e^{\frac{-\sum \beta^2}{2\sigma^2}} & b(x) = \frac{1}{\sigma^n (2\pi)^{n/2}\prod x_i}\\
c_1(\gamma) = -1 & d_1(x) = \sum y_i^2/x_i^2\\
c_2(\gamma) = 2\beta & d_2(x) = \sum y_i/x_i
\end{array}$$

Por lo que forma parte de la fam. exponencial. Observanos: $\sum \frac{y_i}{x_i}$ es una estadistica suficiente y completa y $\hat{\beta} = \sum \frac{y_i}{x_i n}$ es funcion de $\sum y_i/x_i$ Ademas

$$\begin{aligned}
\mathbb{E}(\hat{\beta}) &= \mathbb{E}\left(\sum_{i=1}^{n}\frac{y_i}{x_i n}\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(\frac{y_i}{x_i}\right)\\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\mathbb{E}\left(y_i\right) = \frac{1}{n}\sum_{i=1}^{n}\beta = \beta
\end{aligned}$$

Por el teorema de Lehman-Scheffe $\hat{\beta}$ es el UMVUE de β