WI4011-17 - 2023-24
Computational Fluid Dynamics
Final Project - Part I
Deadline - 23:59, June 23, 2024 (May 12, 2024 for early feedback)

Instructions and assessment criteria to keep in mind:

- A submission for Final Project - Part I and II is **required for passing** WI4011-17.
- This is a **group project**. By submitting your group report, you will be confirming that all group members have contributed equally its preparation. Each group member will receive the same grade for their submission.
- The report needs to be **typed in LaTeX or Word**; only one submission is needed per group. Your submission must also contain the code to any computer program that you used for numerical computations.
- The **deadline** for uploading your solutions to Final Project - Part I and II on Brightspace is 23:59, June 23, 2024 (May 12, 2024 for early feedback).
- In order to receive **formative (ungraded) feedback** for your solutions to Final Project - Part I, you must upload them on Brightspace by the deadline of 23:59, May 12, 2024.
- Late submissions will NOT be accepted.
- Provide **clear and motivated answers** to the questions. No/reduced points will be awarded if your solutions are unaccompanied by explanations.

1 Unsteady convection: Upwind and SUPG methods (10 points)

Consider the convective limit of one-dimensional unsteady convection-diffusion equation with no source, i.e.

$$\begin{aligned} \varphi_{,t} + u\varphi_{,x} &= 0, \quad x \in (0, L), \\ \varphi(x, 0) &= \varphi_0(x), \quad x \in (0, L), \\ \varphi(0, t) &= 0, \quad t > 0, \end{aligned} \tag{1}$$

where the convection speed $u > 0$ is a constant.

> In the lectures, you saw that the upwind discretization of the convective term produces nodally exact solutions for the steady convection diffusion equation with constant source in its convective limit, i.e., $\mathrm{Pe}_h \to \infty$. Here, you will examine the stability and accuracy properties of the upwind discretization in the *unsteady* case and compare it to the SUPG method.

We discretize the spatial domain uniformly as $x_j = jh$, $j = 0, \ldots, n$, $h = L/n$. The semi-discrete equations for the upwind method applied to system (1) are given as

$$\dot{c}_j + u\frac{c_j - c_{j-1}}{h} = 0, \quad j = 1, \ldots, n, \tag{2}$$

where c_j is the time-dependent nodal unknown at node x_j, $\dot{c}_j$ denotes its derivative with respect to t, and $c_0 = 0$.

1. Using von Neumann stability analysis, find any restriction that applies to the time step Δt to ensure stability of time integration of system (2) using the Crank-Nicolson method. 2 points
2. For applying the SUPG method to (1):
    - Show that a relevant stabilization parameter for problem (1) is defined by $\tau = \frac{\bar{\epsilon}}{u^2}$, with $\bar{\epsilon} = \frac{uh}{2}$. 1 point

- Using a basis of piecewise linear (hat) trial functions, derive the semi-discrete equations for the SUPG method. Specifically, without evaluating any integrals, write down the entries for the matrices $\mathbf{M}, \mathbf{K}$ in the following system obtained after SUPG discretization of (1): 2 points

$$\mathbf{M}\dot{\mathbf{c}} + \mathbf{K}\mathbf{c} = \mathbf{0} \tag{3}$$

3. Consider $u = 1$, $L = 4$ and the initial condition

$$\varphi_0(x) = \begin{cases} \frac{1+\cos(\pi(x-1))}{2} & |x-1| \leq 1, \\ 0 & |x-1| > 1. \end{cases} \tag{4}$$

- What behavior do you expect from the exact solution at $t = 1$? Provide an analytical expression if possible. 1 point
- Implement the Crank-Nicolson method for numerical time integration of systems (2) and (3). Specifically, use a *two-point Gaussian quadrature* to numerically integrate the spatial integrals in system (3). For three different Courant numbers, present the numerical results for both methods at $t = 1$. 3 points
- Compare and contrast the results of numerical time integration for the upwind and SUPG schemes. Which solution features are faithfully reproduced by either of the methods? 1 point

Finite element discretization of Stokes flow (15 points)

As discussed during the lectures, Stokes flow is a simplification of the Navier–Stokes problem obtained by ignoring all inertial terms. Thus, the usual numerical issues with convection-dominated flows are not present here. However, there might still be challenges due to the incompressibility constraint and the velocity-pressure coupling. You will examine some of these challenges via an implementation of finite element solver for Stokes flow.

Consider Stokes flow on a square domain $\Omega := [0,1]^2$, with Dirichlet boundary conditions,

$$\begin{aligned} -\nabla^2\mathbf{u} + \nabla p = \mathbf{f}\ , &\quad \text{on } \Omega\ , \\ \nabla \cdot \mathbf{u} = \mathbf{0}\ , &\quad \text{on } \Omega\ , \\ \mathbf{u} = \mathbf{u}_\partial\ , &\quad \text{on } \partial\Omega\ , \end{aligned} \tag{5}$$

Here, $\mathbf{u}_\partial$ is the known velocity on the boundary of Ω and $\mathbf{f}$ is a known source function.

Similarly to the previous assignments, the finite element space will be based on a uniform cartesian mesh with $(n \times n)$ elements of size $h = \frac{1}{n}$ in each direction. We will use the piecewise bilinear functions B_{ij} as our basis for both velocity components, u_1, u_2 as well as the pressure p, also known as Q1Q1 interpolation.

1. Write down the weak form system (5) with appropriately defined function spaces. 1 point
2. Use the Bubnov-Galerkin method to approximate the weak form and derive the corresponding linear system in the form

$$\begin{aligned} \mathbf{A}\mathbf{v} + \mathbf{G}\mathbf{p} = \mathbf{f}_h\ , \\ \mathbf{B}\mathbf{v} = \mathbf{g}_h\ . \end{aligned} \tag{6}$$

Specifically,

- Derive the stencils for both components of the momentum equation in system (5) by choosing the test functions $\boldsymbol{w}_k(x,y) = B_i^1(x)B_j^1(y)\boldsymbol{e}_x$, $k = n_x(j-1)+i$, and $\boldsymbol{w}_k(x,y) = B_i^1(x)B_j^1(y)\boldsymbol{e}_y$, $k = n_x(j-1)+i+n_xn_y$, with n_x, n_y the number of vertices in the x- and y-directions, respectively. 2 points
- Derive the stencil for the incompressibility constraint in system (5) choosing the test functions $q = B_iB_j$. 1 point
- Explain why $\mathbf{G} = \mathbf{B}^T$. 1 point

3. Consider the following Stokes problem with an analytical solution.

- Derive an expression for $\mathbf{f}$ such that the exact solution to the system (5) is given as 1 point

$$\begin{aligned} u_1(x,y) &= x^2(1-x)^2(2y - 6y^2 + 4y^3), \\ u_2(x,y) &= -y^2(1-y)^2(2x - 6x^2 + 4x^3), \\ p(x,y) &= x(1-x). \end{aligned} \tag{7}$$

This approach of finding an analytical solution for a PDE, as discussed in the lectures, is known as the *method of manufactured solutions*.

- Implement the finite element system (6) in a computer code to numerically solve the above problem and compare the numerical solution with the analytical solution. Present plots for velocity streamlines and pressure for $n = 8, 16, 32, 64$. Discuss any numerical challenges and their causes. 5 points
- Replace the incompressibility constraint in your finite element approximation with the penalty formulation, i.e.,

$$\nabla \cdot \mathbf{u} = -\frac{p}{\lambda}, \tag{8}$$

whose discretization is obtained in the form

$$\mathbf{B}\mathbf{v} + \frac{1}{\lambda}\mathbf{M}\mathbf{p} = \mathbf{g}_h, \tag{9}$$

where λ is a large parameter that penalizes the deviation from incompressibility. Discuss the results in contrast to the fully incompressible formulation in the previous step. 3 points
- Plot $\nabla \cdot \mathbf{v}$ for your finite element velocity solutions for $n = 64$. Examine the influence of λ on this plot. 1 point