傅里叶变换笔记

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Contents

Chapter 1

连续傅里叶

1.1 经典定义

1.1.1 傅里叶变换

对$f(t+1)=f(t)$

$$\begin{aligned}
f(t) &= \frac{a_0}{2}+\sum_{k=1}^{n}A_k\cos(2\pi kt+\phi_k)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos(2\pi kt+\phi_k)+b_k\sin(2\pi kt)) \\
&= \sum_{k=1}^{n}\frac{1}{2}A_k e^{-i\phi_k}e^{-i2\pi kt}+\frac{a_0}{2}+\sum_{k=1}^{n}\frac{1}{2}A_k e^{i\phi_k}e^{i2\pi kt} \\
&= \sum_{k=-n}^{n}c_k e^{2\pi ikt} \\
c_k &= \int_0^1 e^{-2\pi ikt}f(t)dt
\end{aligned}$$

对$f(t+T)=f(t)$

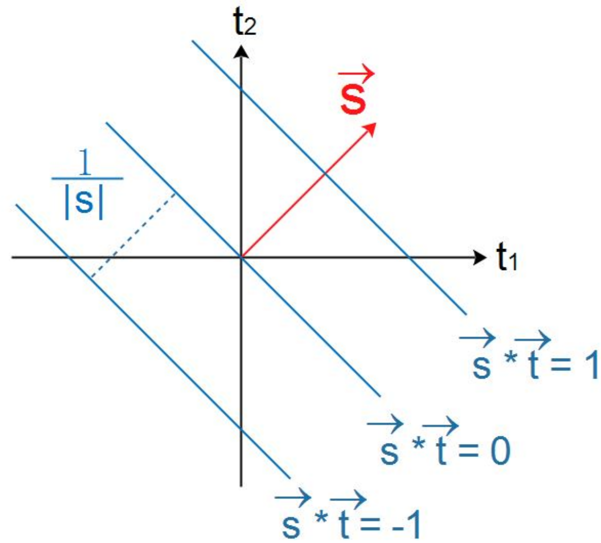
$$\begin{aligned}
f(t) &= \sum_{k=-\infty}^{+\infty}c_k e^{2\pi i\frac{k}{T}t}\frac{1}{T} \\
c_k &= \int_{-\frac{T}{2}}^{\frac{T}{2}}f(t)e^{-2\pi i\frac{k}{T}t}dt
\end{aligned}$$

对$f(t+\infty)=f(t)$

$$\begin{aligned}
f(t) &= \int_{-\infty}^{+\infty}F(s)e^{2\pi ist}ds &&= \mathcal{F}^{-1}[F(s)](t) \\
F(s) &= \int_{-\infty}^{+\infty}f(t)e^{-2\pi ist}dt &&= \mathcal{F}[f(t)](s)
\end{aligned}$$

对$f(\vec{t}+\infty)=f(\vec{t})$

$$\begin{aligned}
f(\vec{t}) &= \int_{-\infty}^{+\infty}F(\vec{s})e^{2\pi i\vec{s}\vec{t}}d\vec{s} &&= \mathcal{F}^{-1}[F(\vec{s})](\vec{t}) \\
F(\vec{s}) &= \int_{-\infty}^{+\infty}f(\vec{t})e^{-2\pi i\vec{s}\vec{t}}d\vec{t} &&= \mathcal{F}[f(\vec{t})](\vec{s})
\end{aligned}$$



1.1.2 特殊坐标系

1.1.2.1 极坐标(r,θ)

对径向函数$f(r_t)$有

$$F(r_s,\theta_s)=\int_0^{+\infty}\int_0^{2\pi}f(r_t)e^{-2\pi i r_s r_t\cos(\theta_s-\theta_t)}r_t dr_t d\theta_t=\int_0^{+\infty}f(r_t)J_0(2\pi r_t r_s r_t dr_t)dr_t$$

（其中J_0为零阶贝塞尔函数）

1.1.2.2 球坐标系(r,θ,ϕ)

径向角向

$$\begin{cases}\phi_i(\vec{r}) & = & f_i(r)Y_{l_i,m_i}(\hat{r})\\ \Phi_i(\vec{g}) & = & i^{-l_i}F_i(g)\tilde{Y}_{l_i,m_i}(\hat{g})\end{cases}$$

傅里叶变换

$$\begin{cases}\phi_i(\vec{r}) & = & \mathcal{F}^{-1}[\Phi_i] & = & \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3g\ \Phi_i(\vec{g})e^{i\vec{g}\cdot\vec{r}}\\ \Phi_i(\vec{g}) & = & \mathcal{F}[\phi_i] & = & \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3r\ \phi_i(\vec{r})e^{-i\vec{g}\cdot\vec{r}}\end{cases}\qquad\begin{cases}f(r)_{\underset{i}{l}} & = & \mathcal{F}_l^{-1}[F_i] & = & \sqrt{\frac{2}{\pi}}\int_0^\infty dg\ j_l(gr)F_i(g)g^2\\ F_{\underset{i}{l}}(g) & = & \mathcal{F}_l[f_i] & = & \sqrt{\frac{2}{\pi}}\int_0^\infty dr\ j_l(gr)f_i(r)r^2\end{cases}$$

1.2 现代定义

1.2.1 速降函数ϕ定义

- $\phi(x)$ 为光滑函数
- 对$\forall m,n\geq 0$，$\lim\limits_{x\to\infty}|x|^m|\frac{d^n}{dx^n}\phi(x)|\to 0$

（Porseval/Radey等式（速降函数性质）：$\int\limits_{-\infty}^{+\infty}|\mathcal{F}[\phi]|^2ds=\int\limits_{-\infty}^{+\infty}|\phi|^2dt$）

1.2.2 内积定义

$$\langle f, \phi \rangle = \int_{-\infty}^{+\infty} f(t)\phi(t)dt$$

1.2.3 傅里叶变换定义

$$\begin{cases} \langle \mathcal{F}[f], \phi \rangle = \langle f, \mathcal{F}[\phi] \rangle & (\text{即} \int_{-\infty}^{+\infty} \mathcal{F}[f(t)](s)\phi(s)ds = \int_{-\infty}^{+\infty} f(t)\mathcal{F}[\phi(s)](t)dt) \\ \langle \mathcal{F}^{-1}[F], \phi \rangle = \langle F, \mathcal{F}^{-1}[\phi] \rangle & (\text{即} \int_{-\infty}^{+\infty} \mathcal{F}^{-1}[F(s)](t)\phi(t)dt = \int_{-\infty}^{+\infty} F(s)\mathcal{F}^{-1}[\phi(t)](s)ds) \end{cases}$$

1.2.4 内积性质

$$\langle f', \phi \rangle = -\langle f, \phi' \rangle$$

$$\langle fg, \phi \rangle = \langle g, f\phi \rangle \text{（}f\phi\text{需满足速降函数定义）}$$

泊松求和公式：

$$\sum_{n=-\infty}^{+\infty} \phi(n) = \sum_{m=-\infty}^{+\infty} \mathcal{F}[\phi](m)$$

1.3 性质

1.3.1

$$\mathcal{F}[f(-t)] = \mathcal{F}^{-1}[f(t)]$$

$$\mathcal{F}[\mathcal{F}[f(t)]] = \mathcal{F}^{-1}[\mathcal{F}^{-1}[f(t)]] = f(-t)$$

1.3.2

$$\mathcal{F}[f(\vec{t} \pm \vec{b})](\vec{s}) = e^{\pm 2\pi i \vec{s}\vec{b}} \mathcal{F}[f(\vec{t})](\vec{s})$$

1.3.3

标量a，矩阵A

$$\begin{aligned} \mathcal{F}[f(at)](t) &= \frac{1}{|a|}\mathcal{F}[f(t)](\frac{s}{a}) \\ \mathcal{F}[f(A\vec{t})](\vec{s}) &= \frac{1}{|det(A)|}\mathcal{F}[f(\vec{t})](A^{-1T}\vec{s}) \end{aligned}$$

1.3.4

$$\mathcal{F}[f^{(n)}(t)](s) = (2\pi i s)^n \mathcal{F}[f(t)](s)$$

$$(\mathcal{F}[f(t)](s))^{(n)} = \mathcal{F}[(-2\pi i t)^n f(t)](s)$$

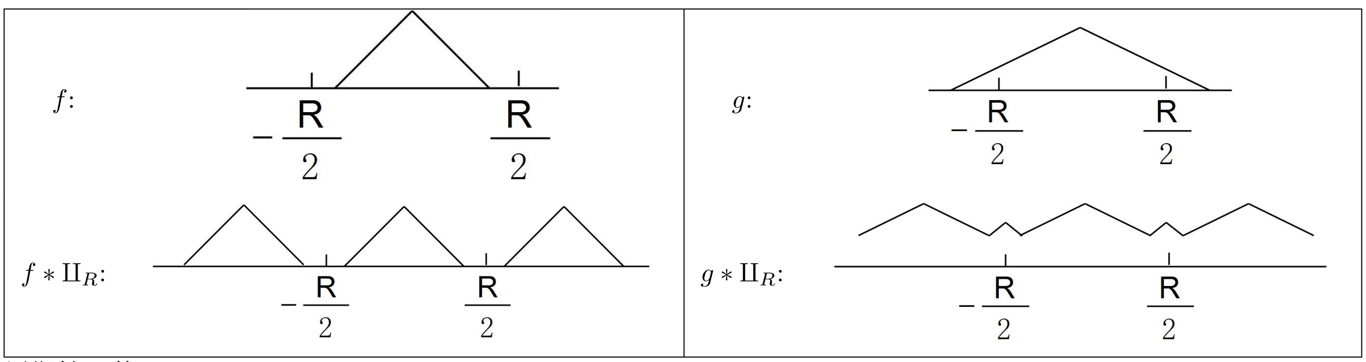
1.3.5

若$f(\vec{t}) = f_1(t_1)\cdots f_n(t_n)$，则$F(\vec{s}) = \mathcal{F}[f(\vec{t})](\vec{s}) = \mathcal{F}[f_1(t_1)](s_1)\cdots\mathcal{F}[f_n(t_n)](s_n)$

Chapter 2

离散傅里叶

2.1 香农采样定理



周期性函数

$$V_R(t) = \sum_{n=-\infty}^{+\infty} V(r - nR) = (V * \mathrm{II}_R)(t)$$

香农采样定理：滤去$\mathcal{F}[f]$中大于$|\frac{R}{2}|$的频率，内插重现原函数（抽样速率R）

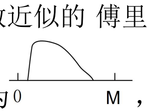
$$\mathcal{F}[f] = \prod_R(\mathcal{F}[f] * \mathrm{II}_R) \qquad \mathcal{F}[g] \neq \prod_R(\mathcal{F}[g] * \mathrm{II}_R)$$

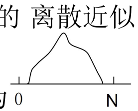
$$\Downarrow \qquad\qquad \Downarrow$$

$$f(t) = \sum_{n=-\infty}^{+\infty} f(\tfrac{n}{R}) sinc(R(t - \tfrac{n}{R})) \qquad \text{无法还原，高频被破坏}$$

2.2 离散近似

f的 离散近似的 傅里叶变换的 离散近似（f与F都必须周期延拓（故可1 MN或0 MN-1等等））

设$f(t)$为

，$F(s)$为

则

$$\begin{aligned} f(t) &= \sum_{n=1}^{MN} f(\tfrac{n}{N})\delta(t - \tfrac{n}{N}) \\ &\Downarrow \\ F(s) &= \sum_{m=1}^{MN}\sum_{n=1}^{MN} f(\tfrac{n}{N}) e^{-2\pi i \frac{mn}{MN}} \delta(s - \tfrac{m}{M}) \end{aligned}$$

或

$$\begin{array}{rcll} \hat{f}(\frac{n}{N}) & & & (n=1,\cdots,MN) \\ \Downarrow & & & \\ \hat{F}(\frac{m}{M}) & = & \sum_{n=1}^{MN} \hat{f}(\frac{n}{N}) e^{-2\pi i \frac{mn}{MN}} & (m=1,\cdots,MN) \end{array}$$

则

$$\begin{array}{rcl} F(s) & = & \sum_{m=1}^{MN} F(\frac{m}{M}) \delta(s-\frac{m}{M}) \\ \Downarrow & & \\ f(t) & = & \sum_{m=1}^{MN} \sum_{n=1}^{MN} F(\frac{m}{M}) e^{2\pi i \frac{mn}{MN}} \delta(t-\frac{n}{N}) \end{array}$$

或

$$\begin{array}{rcll} \hat{F}(\frac{m}{M}) & & & (m=1,\cdots,MN) \\ \Downarrow & & & \\ \hat{f}(\frac{n}{N}) & = & \frac{1}{MN} \sum_{m=1}^{MN} \hat{F}(\frac{m}{M}) e^{2\pi i \frac{mn}{MN}} & (n=1,\cdots,MN) \end{array}$$

2.3 矩阵形式

令$w \overset{def}{=} e^{2\pi i \frac{1}{MN}}$，则

$$\begin{array}{rcl} \hat{F}_m & = & \sum_{n=1}^{MN} w^{-mn} \hat{f}_n \\ \hat{f}_n & = & \frac{1}{MN} \sum_{m=1}^{MN} w^{mn} \hat{F}_m \end{array}$$

$$\begin{bmatrix} \hat{F}_1 \\ \vdots \\ \hat{F}_m \\ \vdots \\ \hat{F}_{MN} \end{bmatrix} = \begin{bmatrix} w^{-1} & \cdots & w^{-n} & \cdots & w^{-MN} \\ \vdots & & \vdots & & \vdots \\ w^{-m} & \cdots & w^{-mn} & \cdots & \\ \vdots & & \vdots & & \vdots \\ w^{-MN} & \cdots & & \cdots & w^{-M^2N^2} \end{bmatrix} \begin{bmatrix} \hat{f}_1 \\ \vdots \\ \hat{f}_n \\ \vdots \\ \hat{f}_{MN} \end{bmatrix}$$

$$\begin{bmatrix} \hat{f}_1 \\ \vdots \\ \hat{f}_n \\ \vdots \\ \hat{f}_{MN} \end{bmatrix} = \frac{1}{MN} \begin{bmatrix} w^{1} & \cdots & w^{m} & \cdots & w^{MN} \\ \vdots & & \vdots & & \vdots \\ w^{n} & \cdots & w^{mn} & \cdots & \\ \vdots & & \vdots & & \vdots \\ w^{MN} & \cdots & & \cdots & w^{M^2N^2} \end{bmatrix} \begin{bmatrix} \hat{F}_1 \\ \vdots \\ \hat{F}_m \\ \vdots \\ \hat{F}_{MN} \end{bmatrix}$$

$$\hat{f} = \mathcal{F}^{-1} \hat{F} = \frac{1}{MN} \mathcal{F}^{+} \hat{F}$$

2.4 FFT

对$m = 0, \cdots, \frac{MN}{2} - 1$，令$n' = \frac{n}{2}$

$$\begin{array}{rcl} F(\frac{m}{M}) & = & \sum_{n'=0}^{\frac{MN}{2}-1} f(\frac{2n'}{N}) e^{-2\pi i \frac{mn'}{\frac{MN}{2}}} + e^{-2\pi i \frac{m}{MN}} \sum_{n'=0}^{\frac{MN}{2}-1} f(\frac{2n'+1}{N}) e^{-2\pi i \frac{mn'}{\frac{MN}{2}}} \\ F(\frac{m+\frac{MN}{2}}{M}) & = & \sum_{n'=0}^{\frac{MN}{2}-1} f(\frac{2n'}{N}) e^{-2\pi i \frac{mn'}{\frac{MN}{2}}} - e^{-2\pi i \frac{m}{MN}} \sum_{n'=0}^{\frac{MN}{2}-1} f(\frac{2n'+1}{N}) e^{-2\pi i \frac{mn'}{\frac{MN}{2}}} \end{array}$$

Chapter 3

其他

3.1 δ

3.1.1 δ经典定义

- $\delta(x) = \begin{cases} 0 & , \quad x \neq 0 \\ \infty & , \quad x = 0 \end{cases}$
- $\int\limits_{-\infty}^{+\infty} \delta(x) dx = 1$
- $\int\limits_{-\infty}^{+\infty} \delta(x) f(x) dx = f(0)$

3.1.2 性质

3.1.2.1

$$\delta(a-x)f(x) = \delta(a-x)f(a)$$
$$(\delta * f)(a) = \int\limits_{-\infty}^{+\infty} \delta(a-x)f(x)dx = f(a)$$
$$(\delta(x-a) * f(x))(y) = f(y-a)$$

3.1.2.2

$$\delta(A\vec{x}) = \frac{1}{|det(A)|}\delta(x)$$

3.2 Ш

$$Ш_R(x) \overset{def}{=} \sum_{n=-\infty}^{+\infty} \delta(x - nR)$$
$$\mathcal{F}[Ш_R(x)](s) = Ш_1(Rs) = \frac{1}{R}\,Ш_{\frac{1}{R}}\,Ш_{\frac{1}{R}}(s)$$
晶格矢量A，倒格矢A^{-1T}，$\mathcal{F}[Ш_A](\vec{s}) = \frac{1}{\text{体积}(A)}\,Ш_{A^{-1T}}(\vec{s})$

3.3 卷积

$(g*f)(u)\overset{def}{=}\int\limits_{-\infty}^{+\infty}g(u-x)f(x)dx$

$\mathcal{F}[f(t)](s)\mathcal{F}[g(t)](s)=\mathcal{F}[(f*g)(u)](s)$

若$x_1\cdots x_n$独立，则$\int\cdots\int_{x_1+\cdots+x_n\leq t}p_1(x_1)\cdots p_n(x_n)dx_1\cdots dx_n=\int\limits_{-\infty}^{+\infty}(p_1*\cdots*p_n)(v)dv$

$(f*g)'=f'*g$（只要f可微，则$f*g$可微）

一般而言，$f*g$比f、g都光滑