(Complete) Search Algorithms

Speed-up Backtracking

- look-back (reasoning on failure)
- look-ahead (constraint propagation)
- variable ordering
- value ordering

Look back
(reasoning on failure)

- Foncer sur les échecs (BT-like), réfléchir ensuite

  C’est à dire :

- Tirer parti des informations implicites contenues dans un échec pour :

  1. sélectionner un meilleur point de retour arrière

  2. ne pas retomber sur la même raison d’échec

Backjumping

- Quand domaine vide : remonter à la variable la plus basse « impliquée dans l'échec »

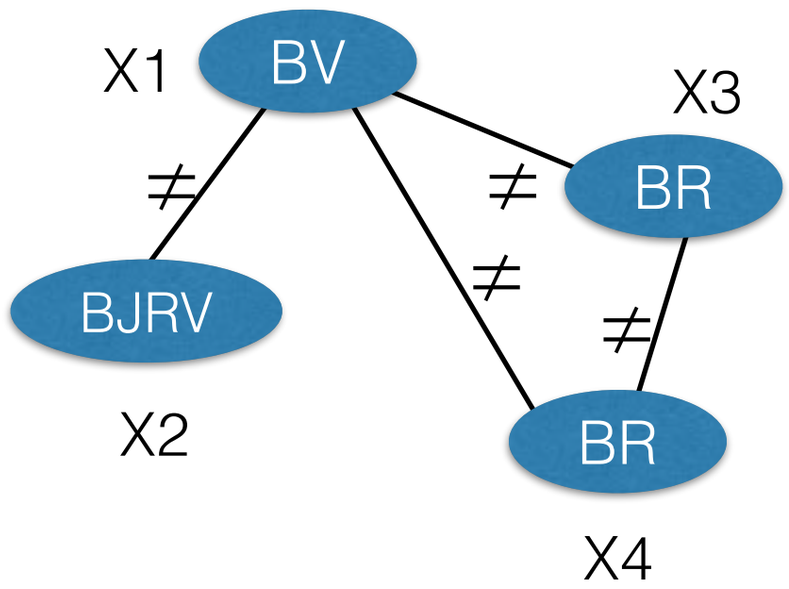
CBJ on running example

X1

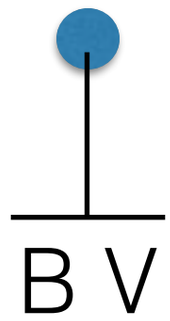
X2

X3

X4



CBJ on running example

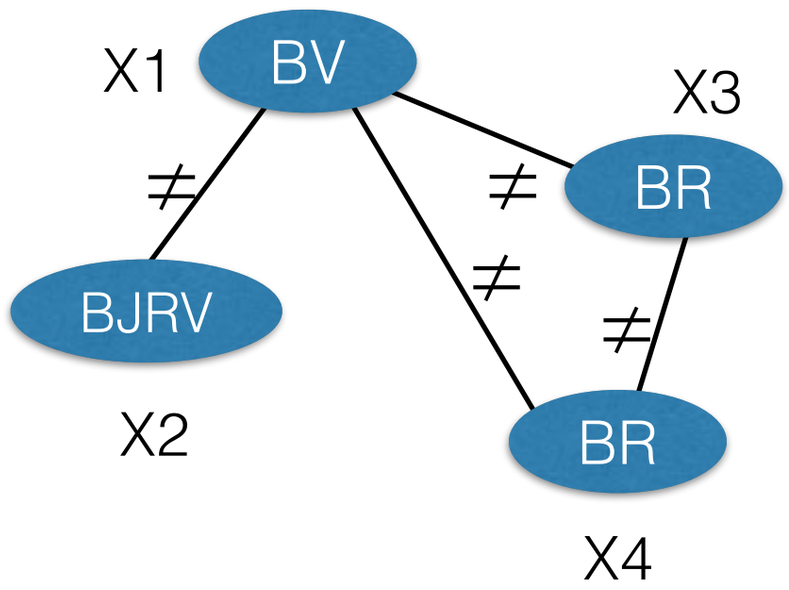


X1

X2

X3

X4



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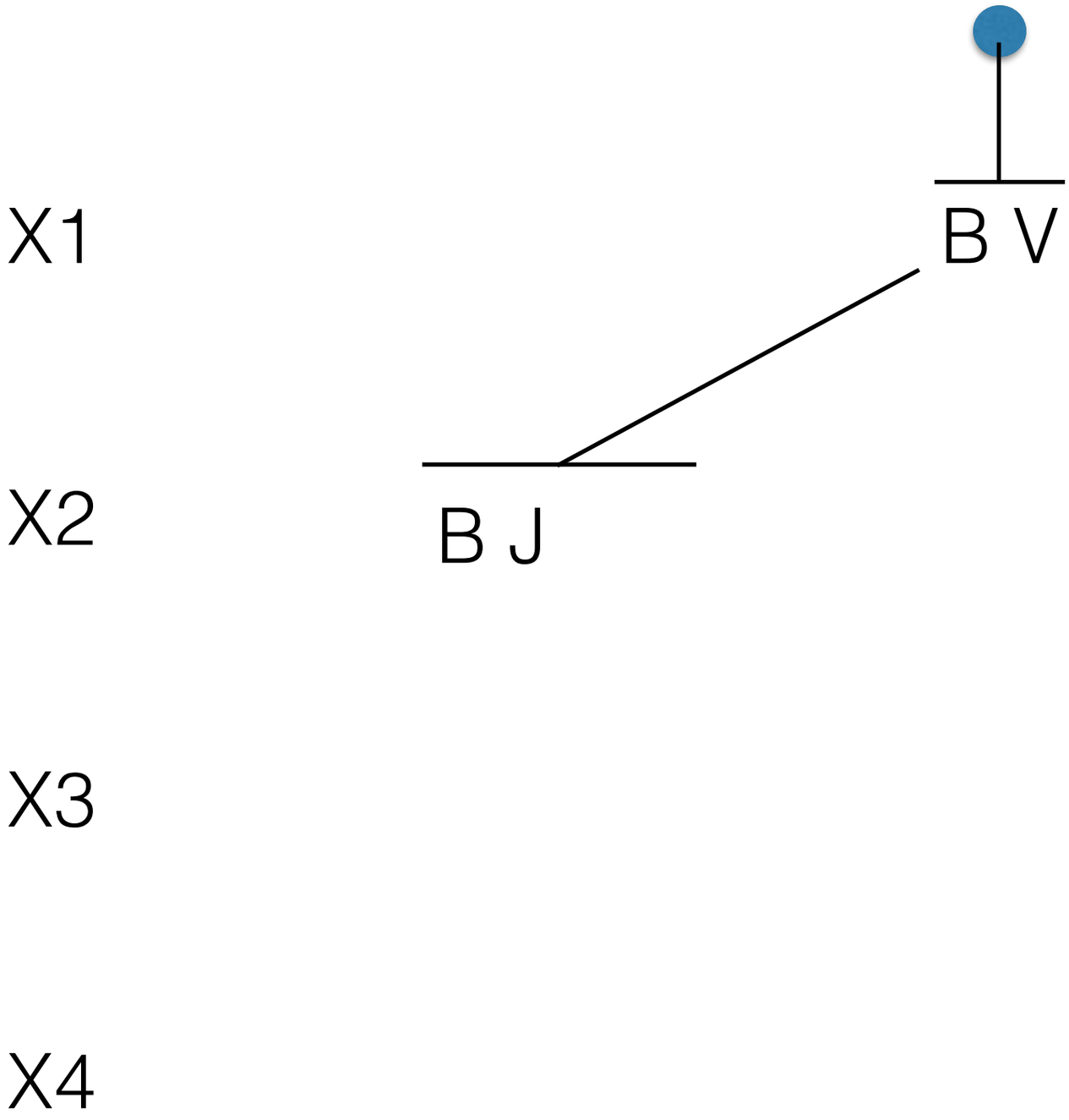

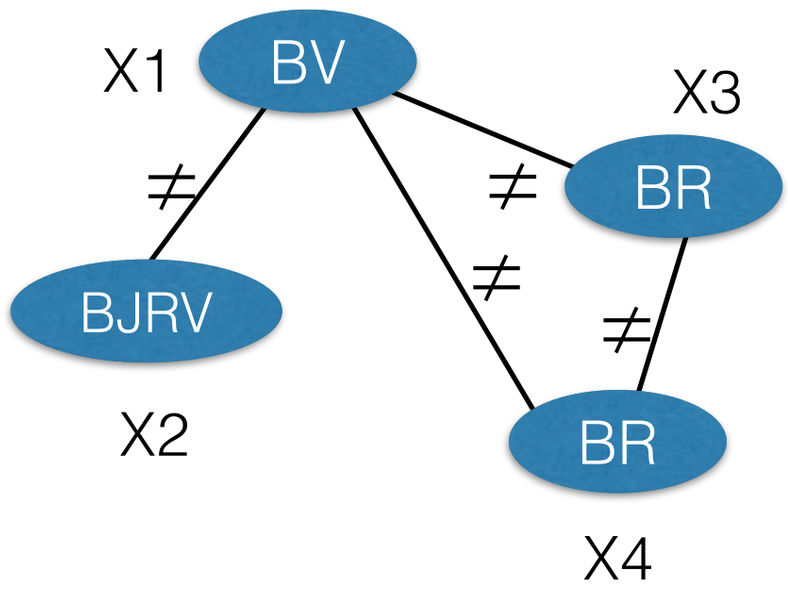

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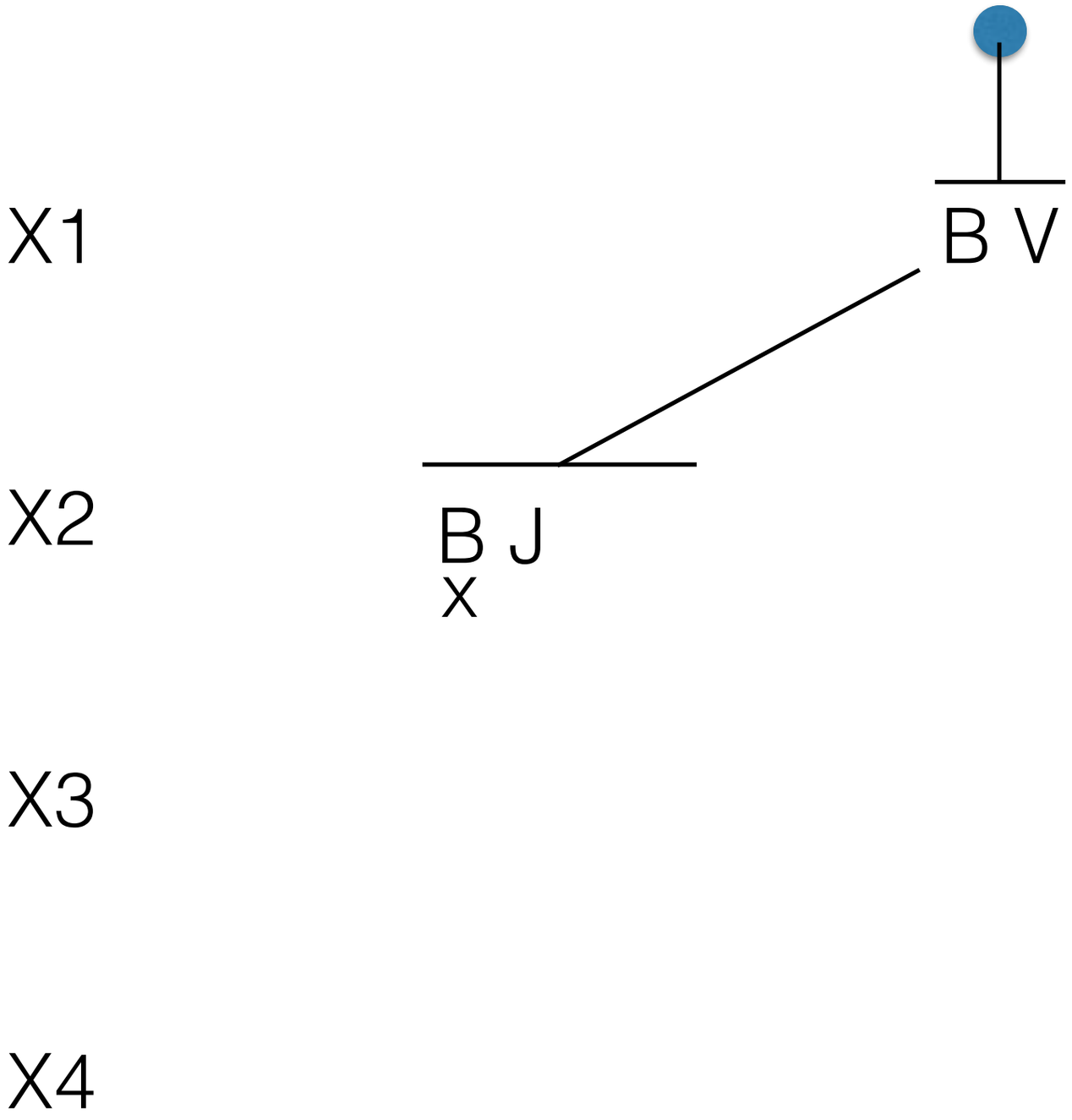

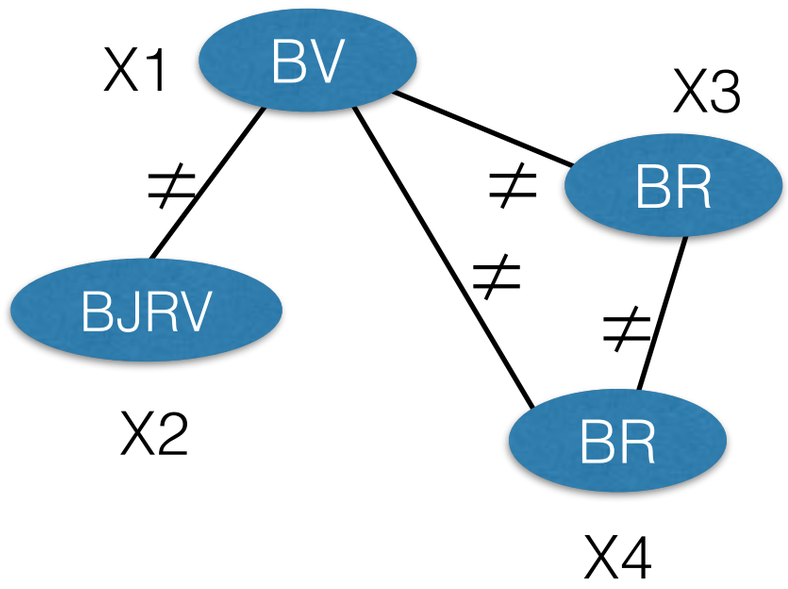

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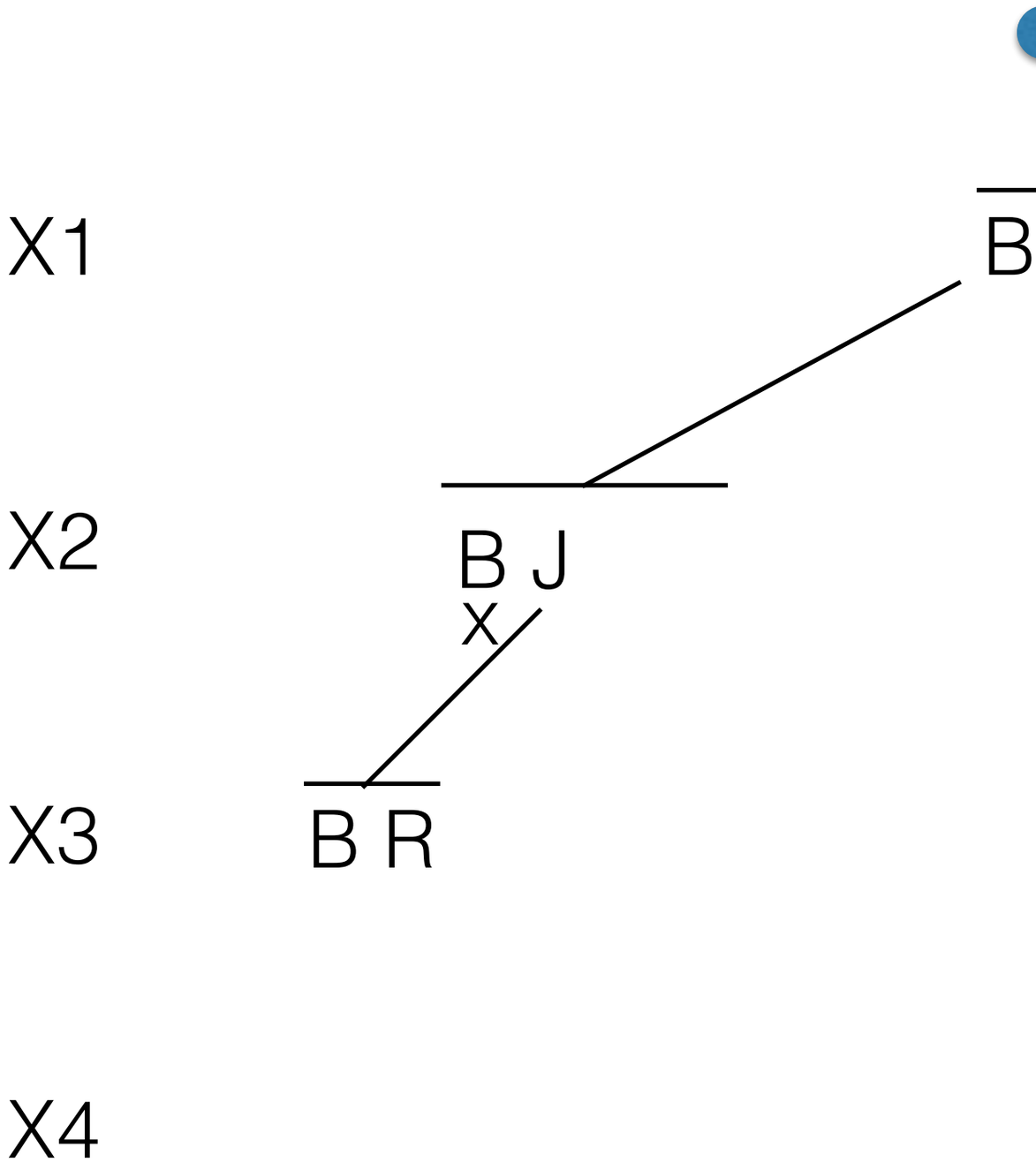

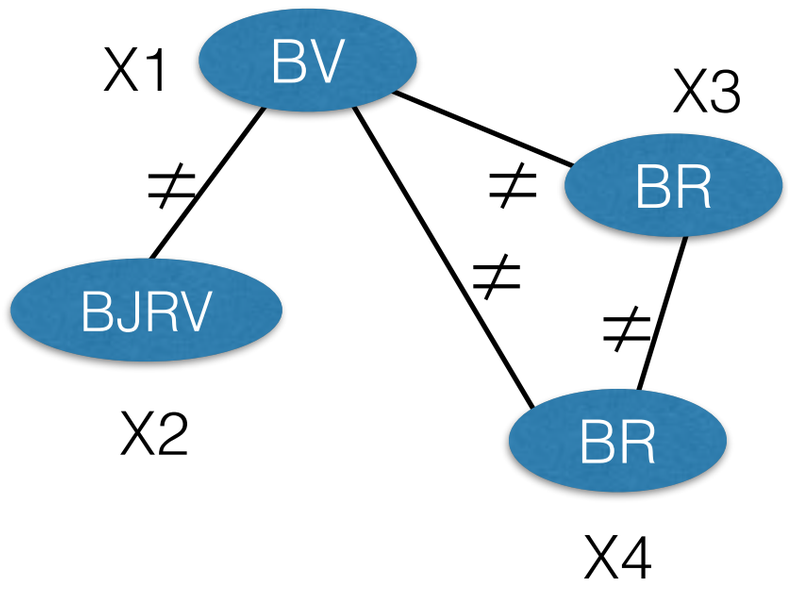

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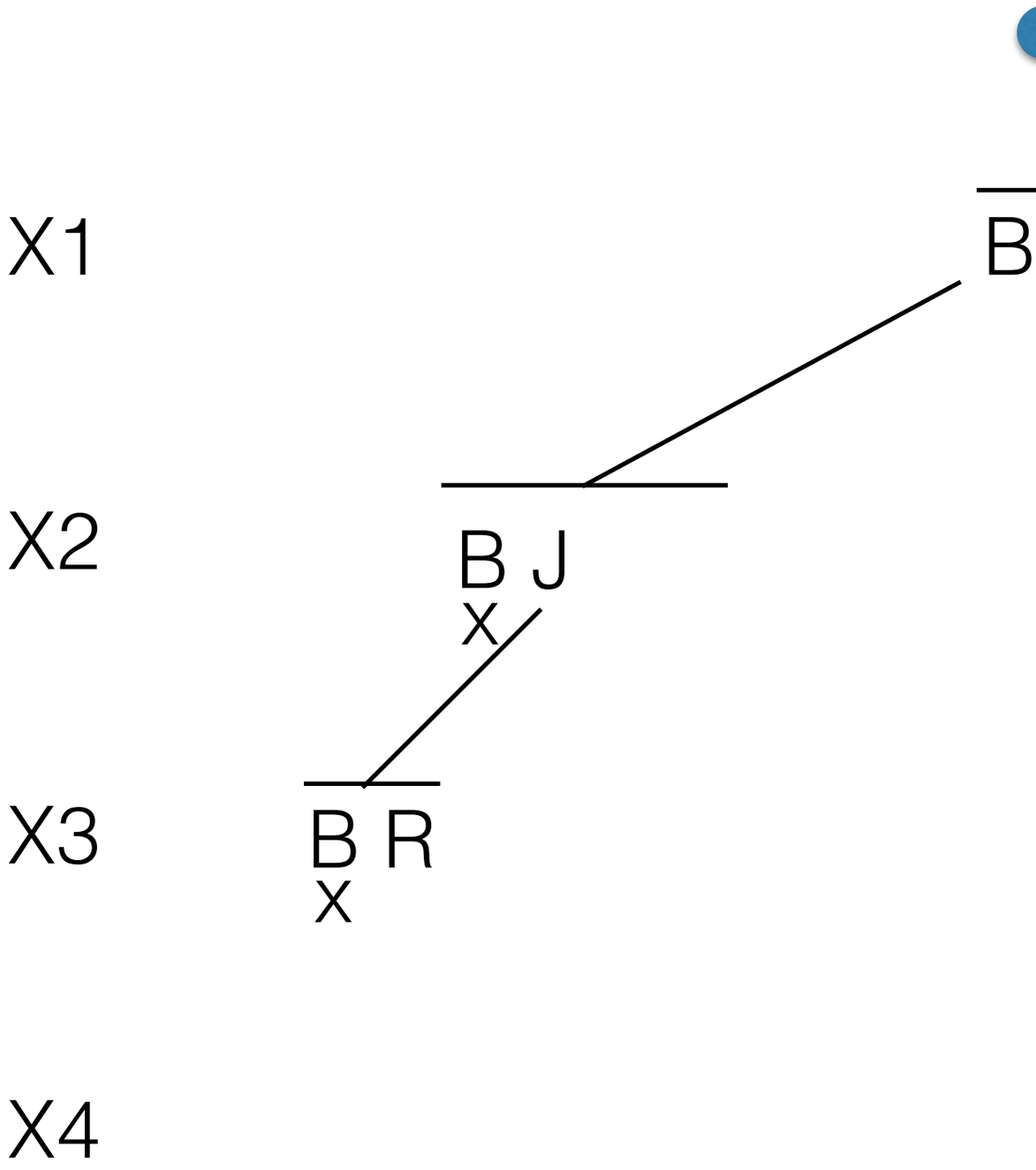

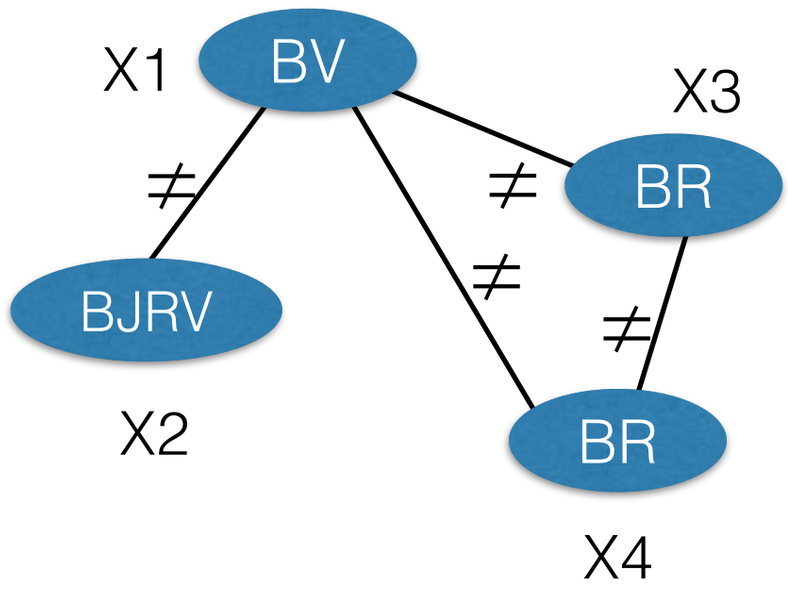

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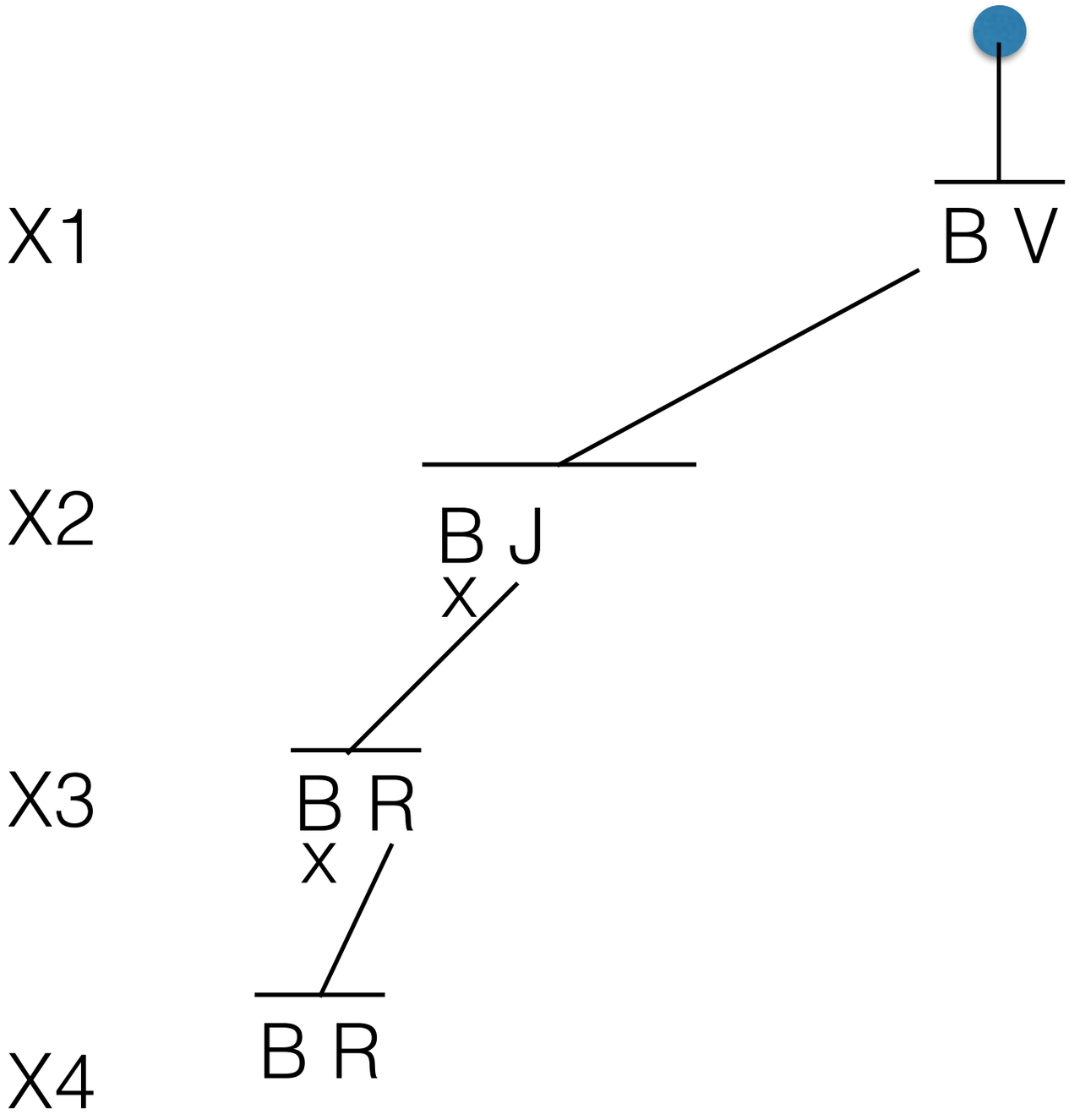

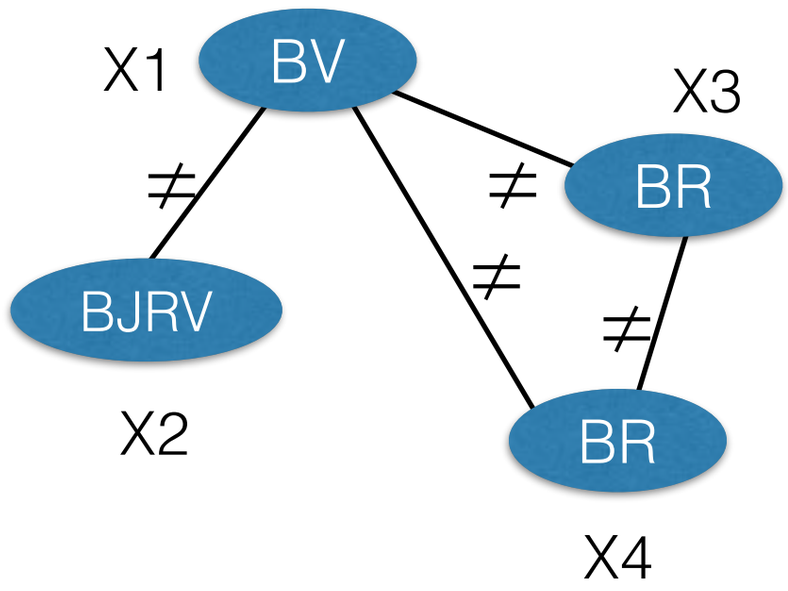

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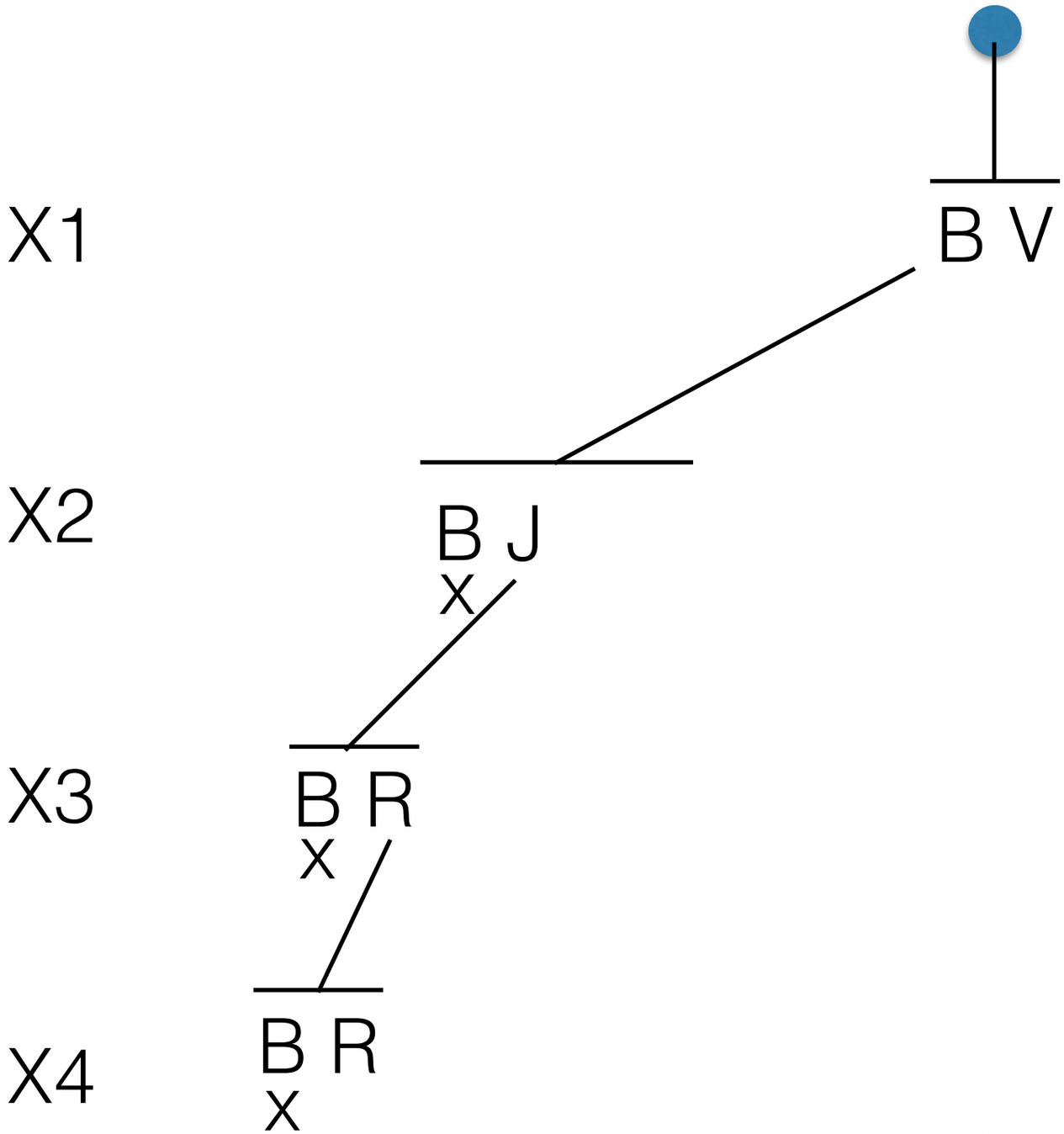

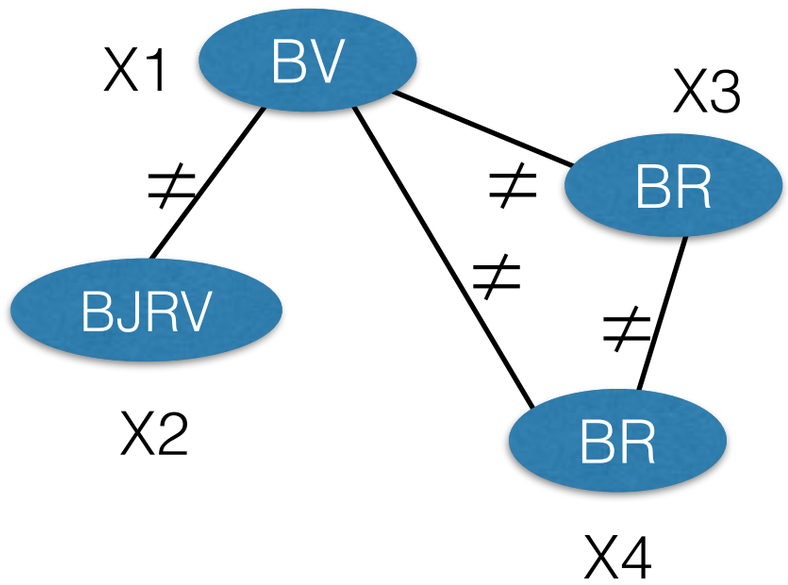

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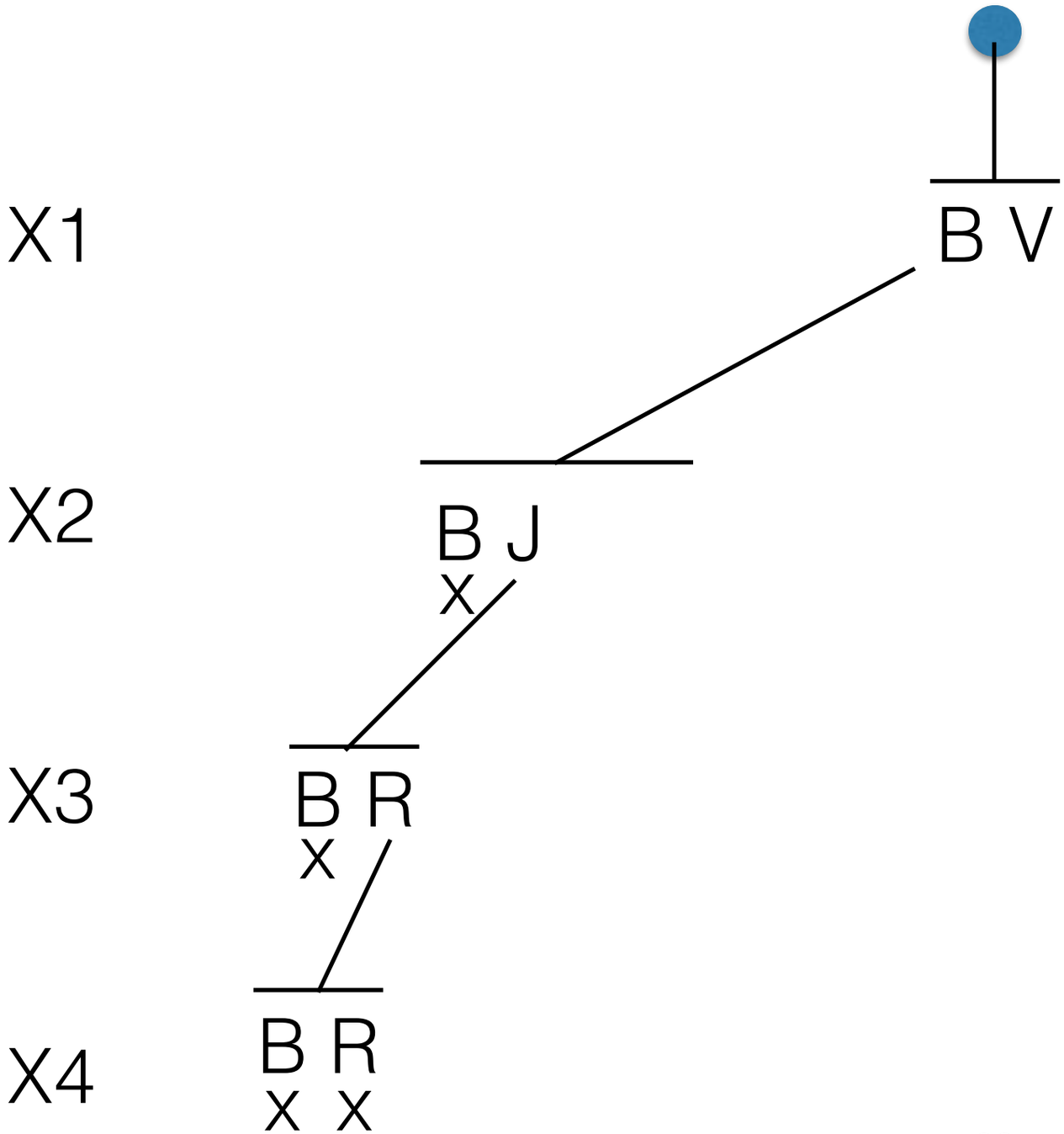

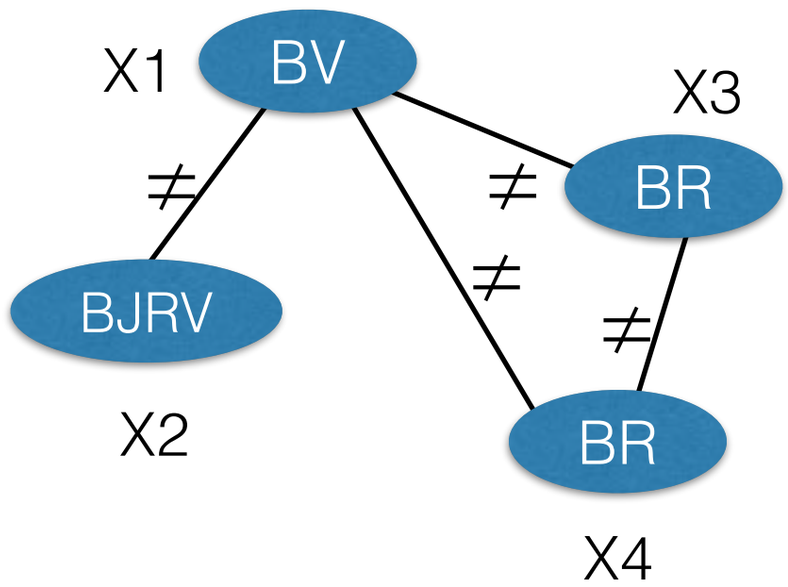

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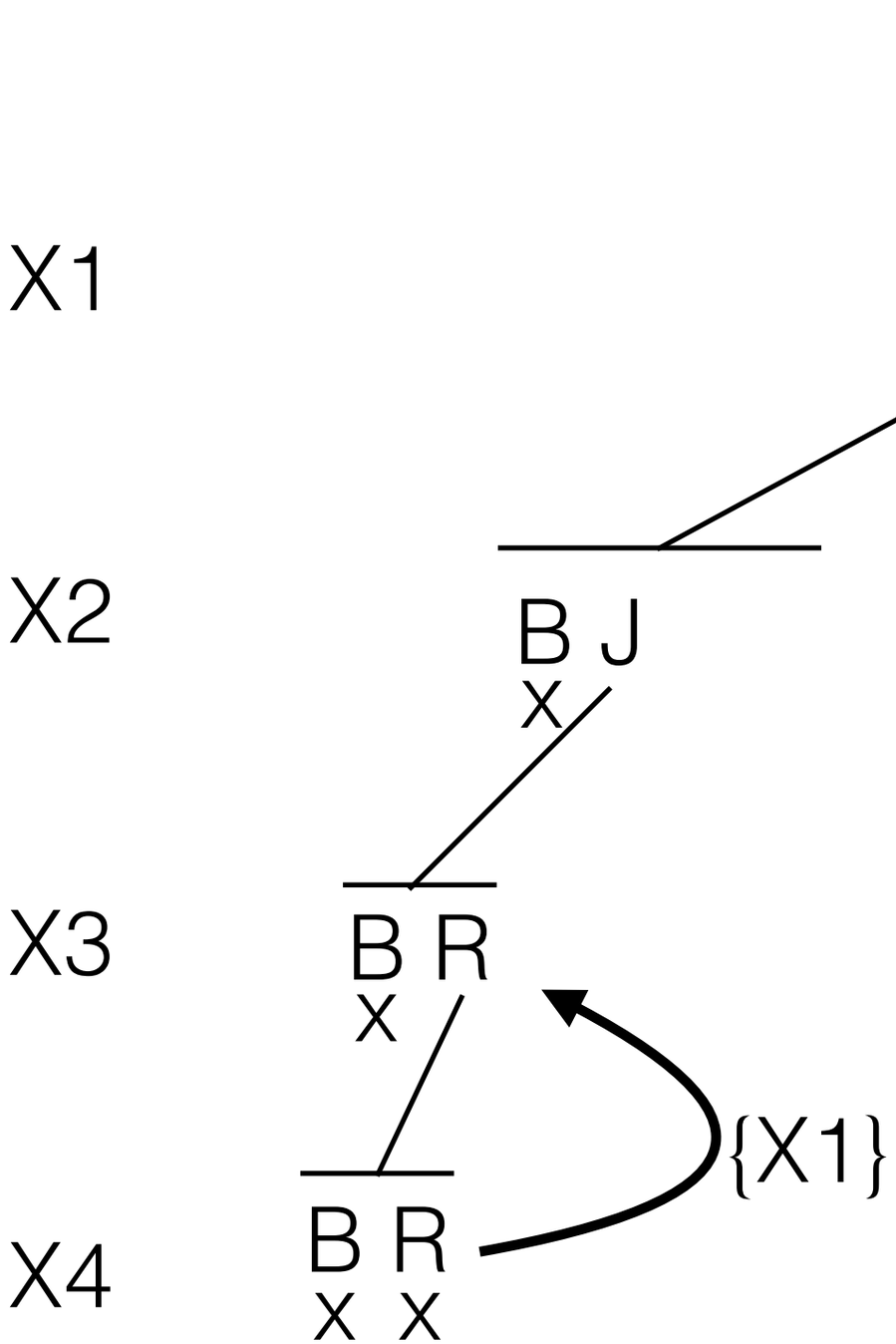

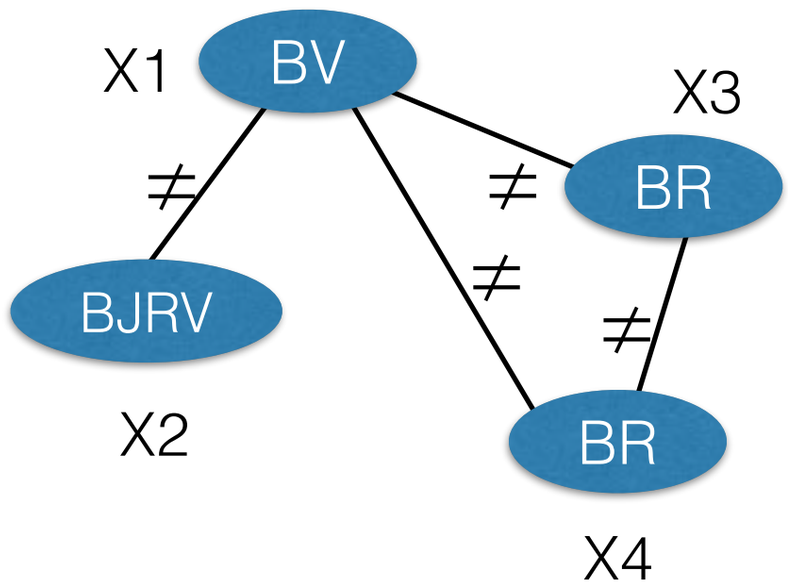

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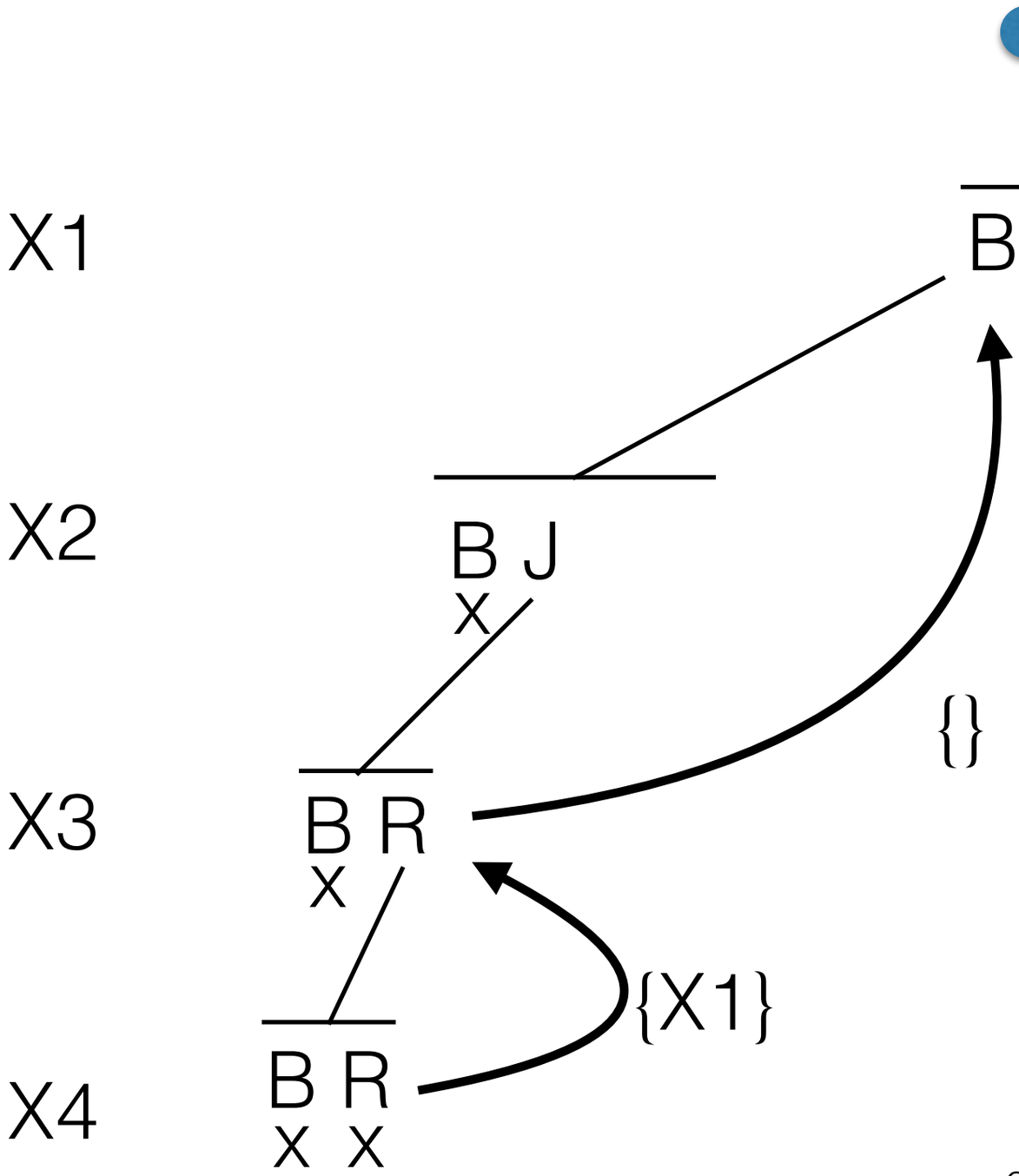

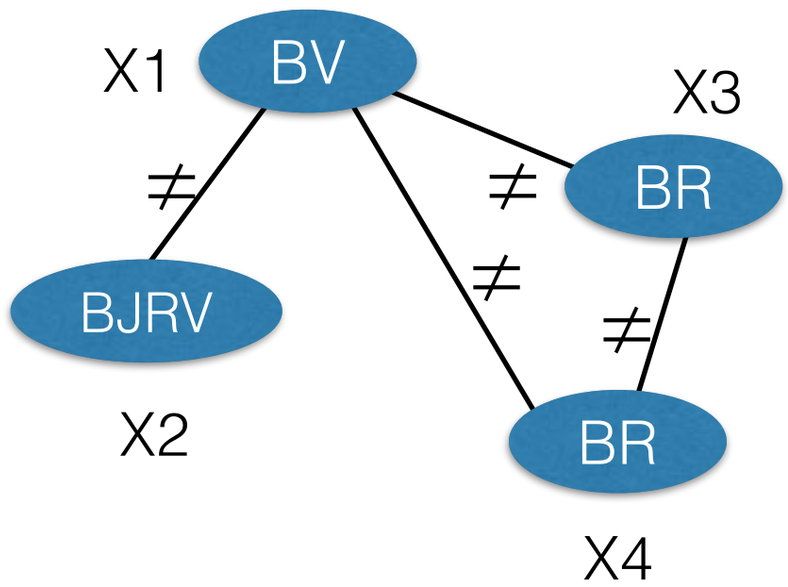

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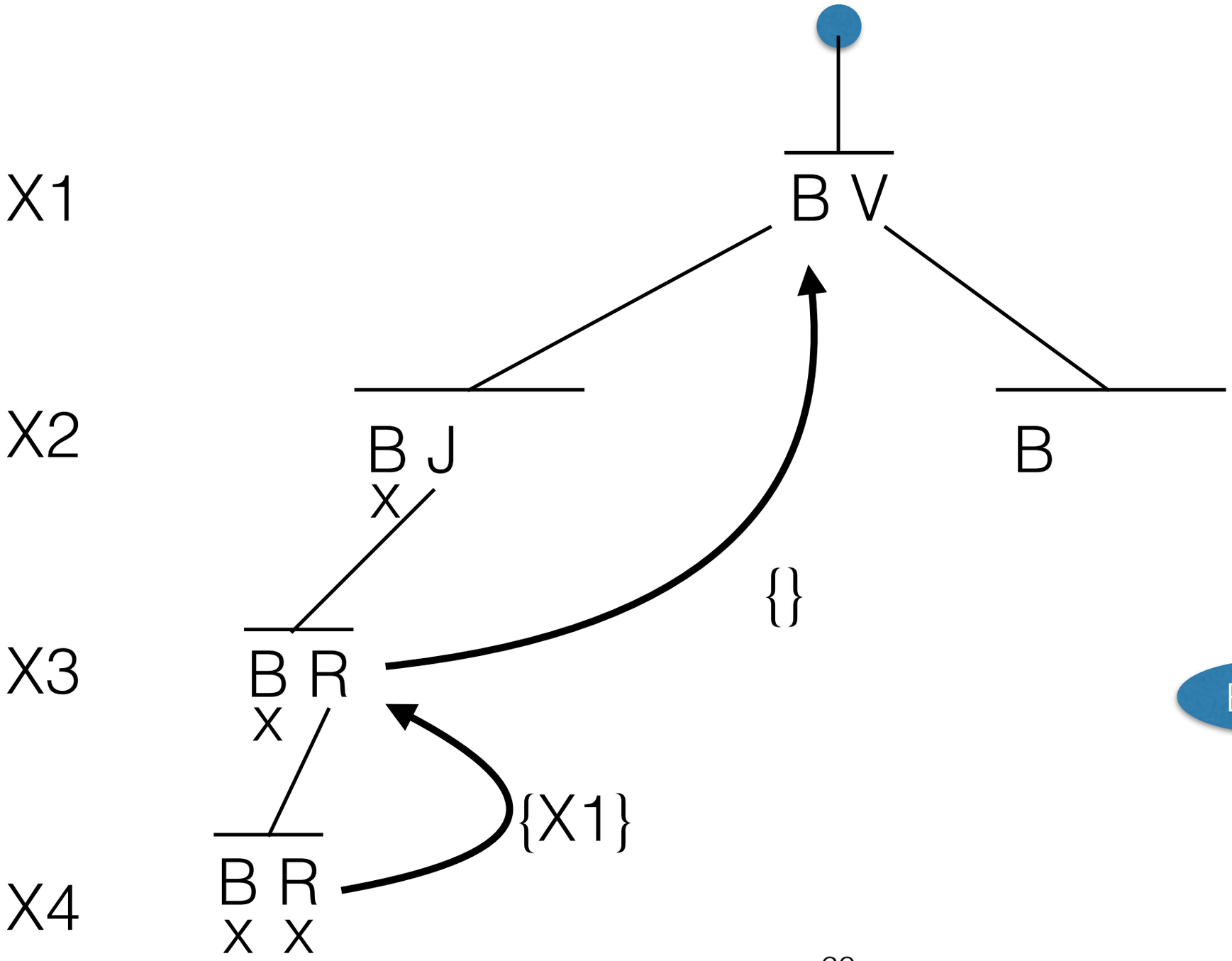

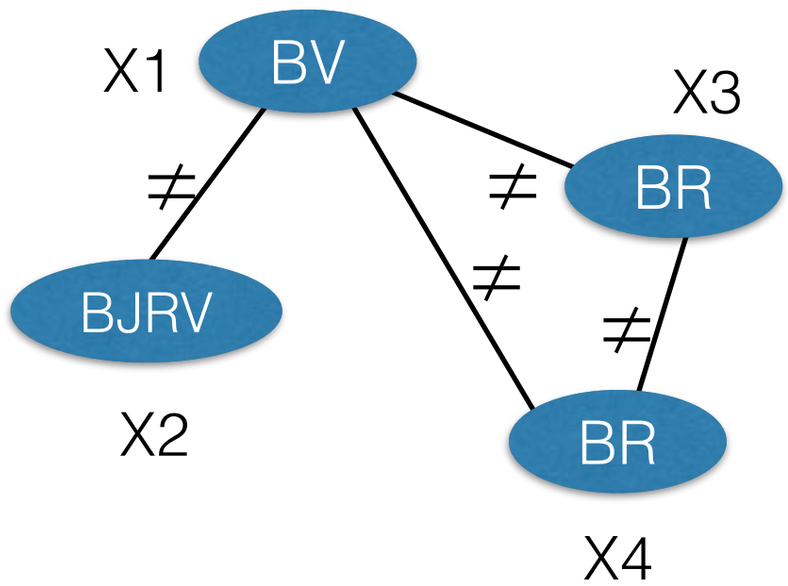

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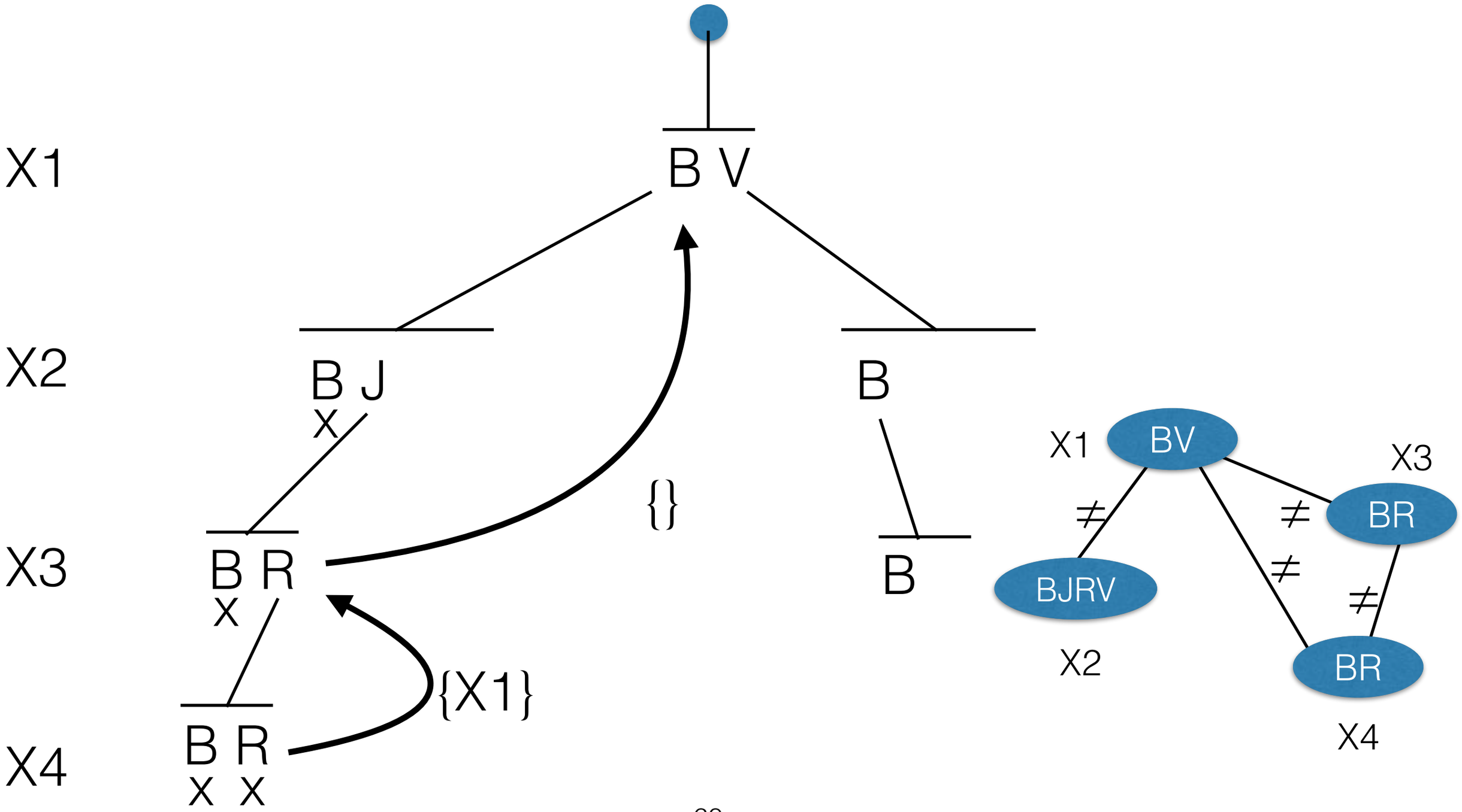

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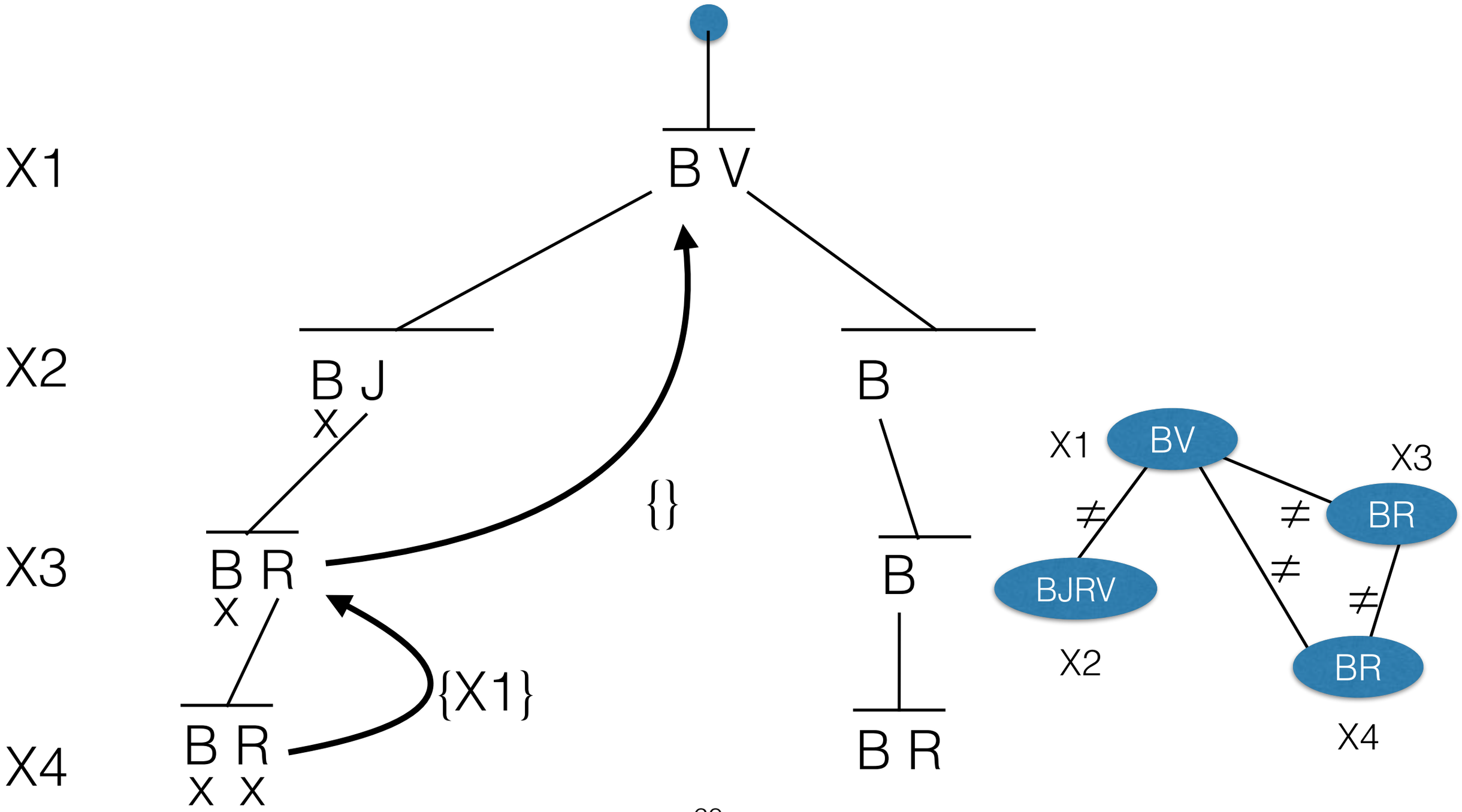

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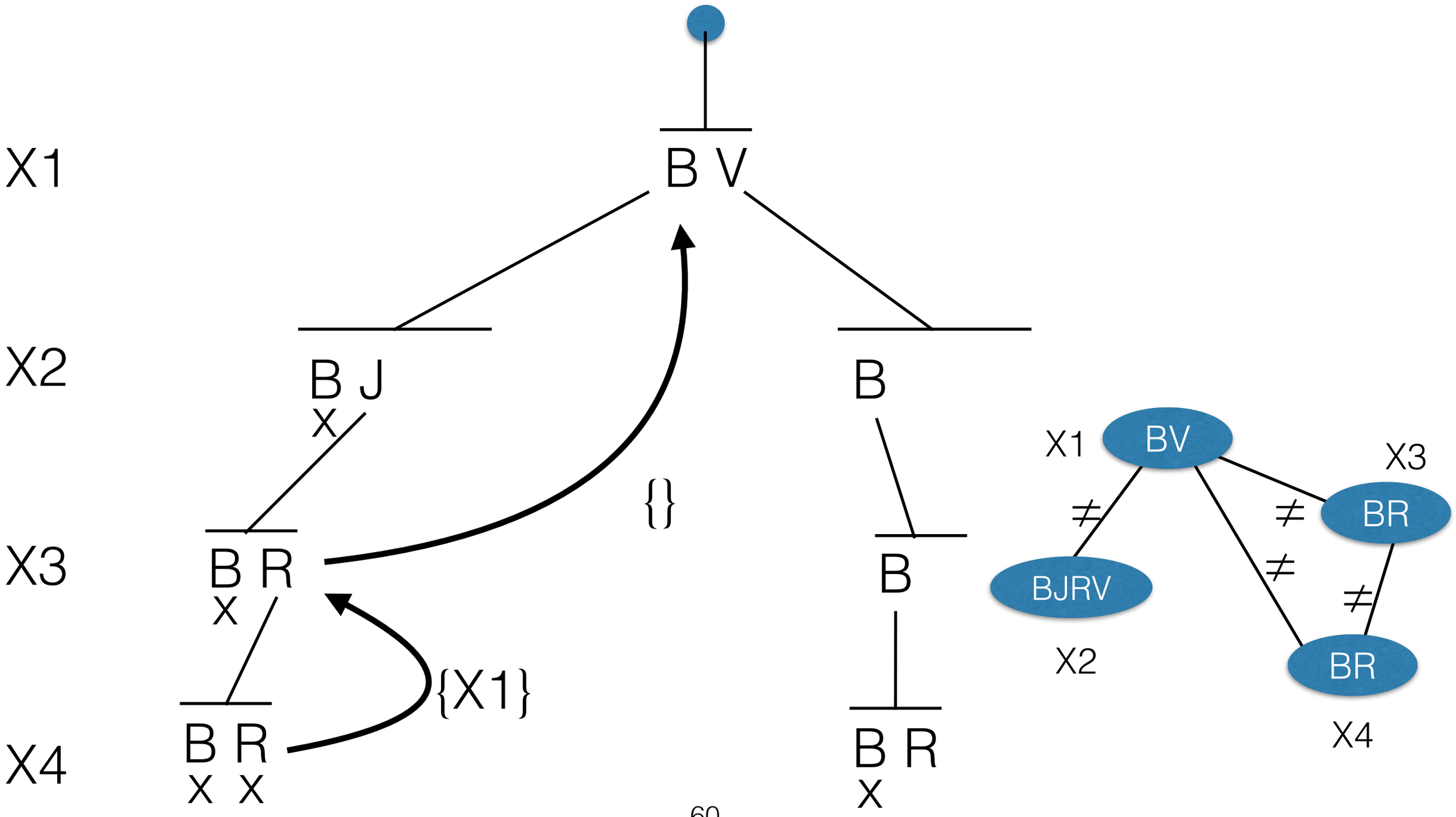

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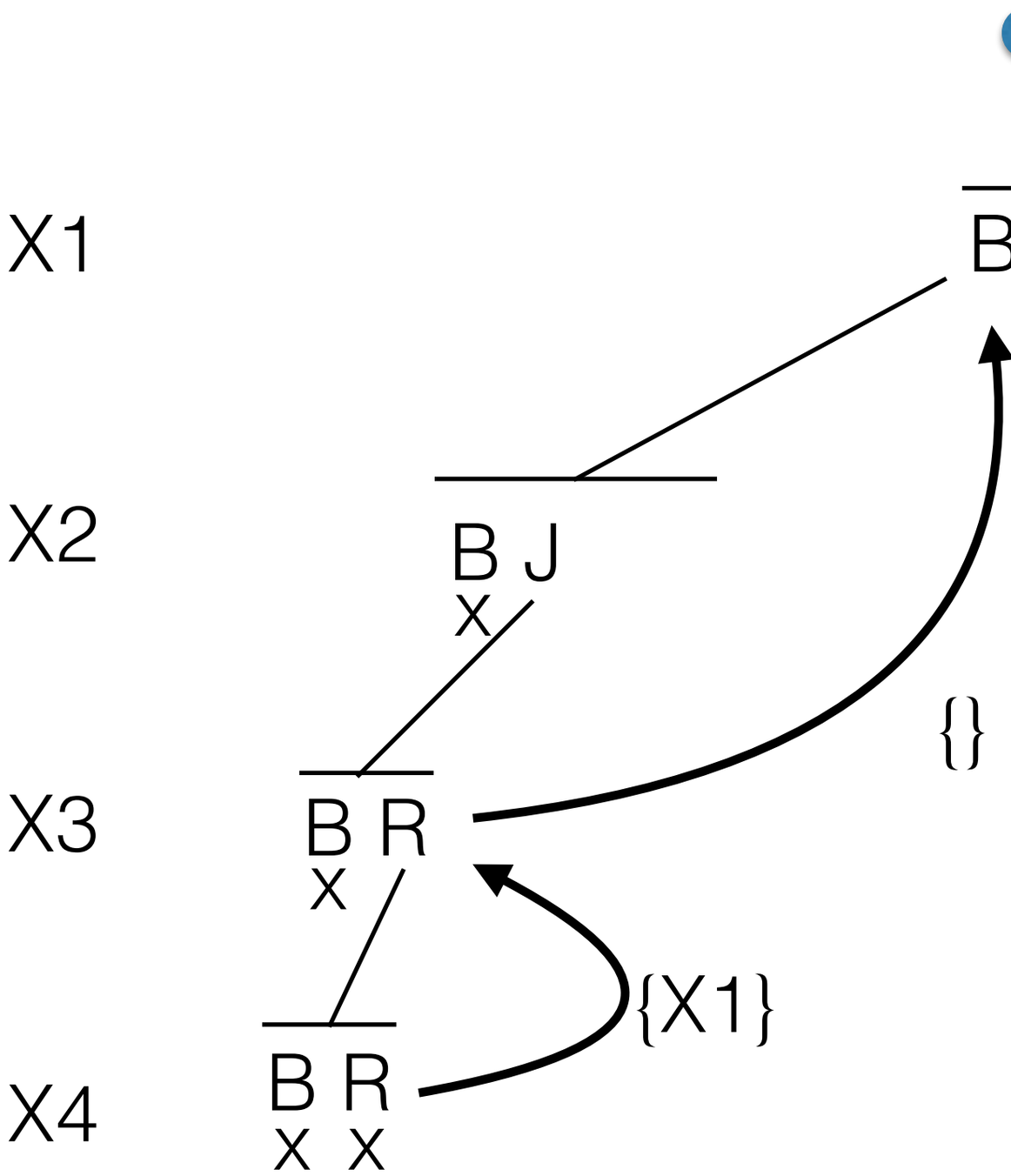

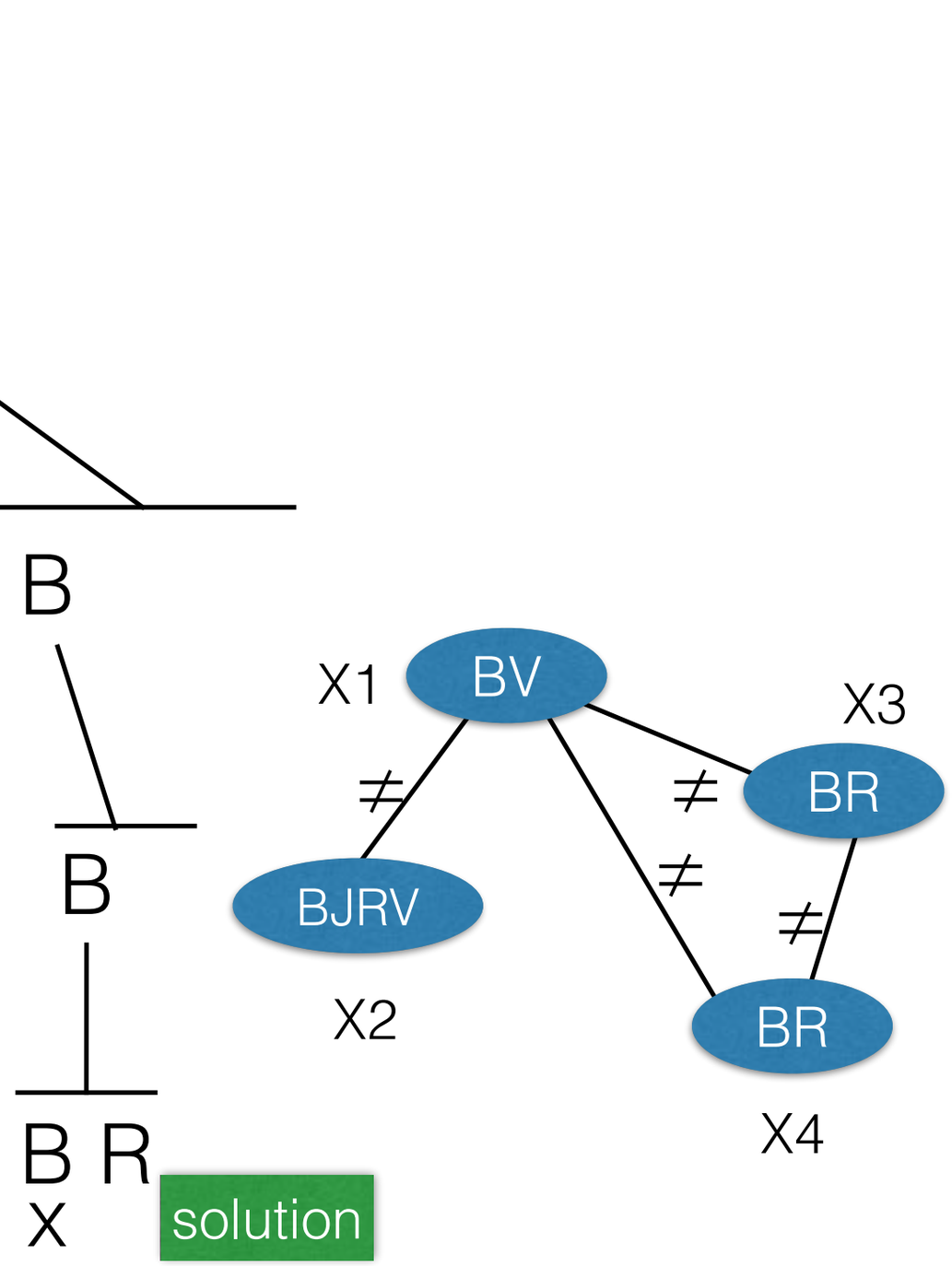

Nogood learning

- A chaque domaine vide, on mémorise une raison de l'échec (nogood), par exemple : [$X_i \neq 2$]v[$X_j \neq 3$]

- Approche proposée dans les 80s sur les CSPs
  - lourd, peu efficace

- Récupéré par SAT en 1997 puis gros succès à partir de 2001 (1UIP et restart) —> solveurs CDCL

- Revenu dans les CSPs depuis 2007 (*lazy clause generation*). Nogoods plus expressifs : [$X_i \leq 6$]v[$X_j \neq 3$].

Look-ahead (constraint propagation)

- Réfléchir d'abord où ne pas descendre, c'est à dire:
- Enlever ***des*** branches qui conduisent à un échec
  - ✤ Instancier X_i
  - ✤ Pour tout $k > i$ supprimer de D(X_k) ***des*** valeurs incompatibles avec X_1.. X_i
  - ✤ Si $D(X_k)=\varnothing$, arrêter cette branche

Forward checking (FC)

function FC(N : network ; I : instantiation) : **Boolean**
begin
  if $|I| = n$ **then** return **true**
  choose a variable x_i not in I
  for each $v_i \in D(x_i)$ **do**
    $D(x_i) \leftarrow \{v_i\}$
    remove all (x_j, v_j) inconsistent with $I \leftarrow I \cup (x_i, v_i)$
    if not empty domain **then**
      if FC(N, $I \cup (x_i, v_i)$) **then** return **true**
    restore all values pruned because of $X_i = v_i$
  return **false**
end

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FC on running example

X1

X2

X3

X4

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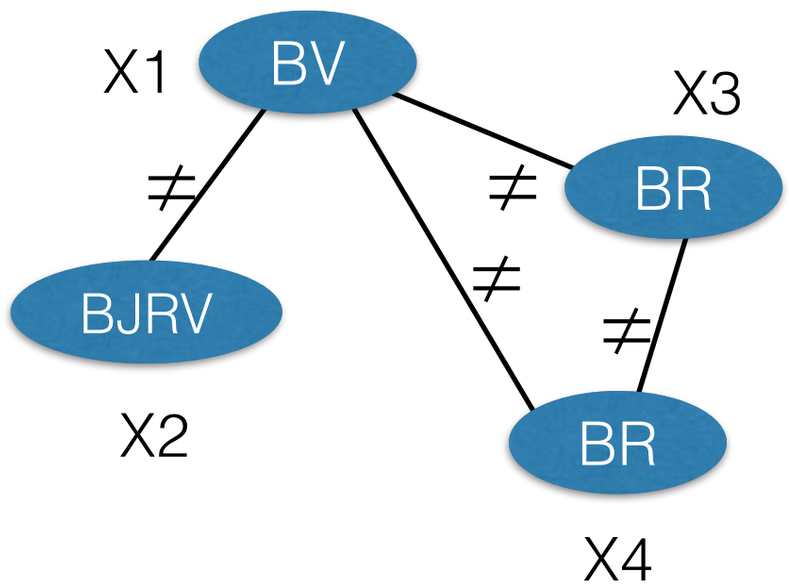
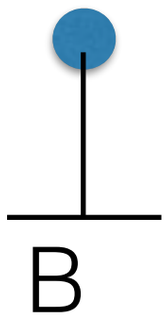


FC on running example

X1



X2

X3

X4

```
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begin
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   choose a variable x_i not in I
   for each v_i ∈ D(x_i) do
      D(x_i) ← {v_i}
      remove all (x_j, v_j) inconsistent with I ← I ∪ (x_i, v_i)
      if not empty domain then
         if FC(N, I ∪ (x_i, v_i)) then return true
      restore all values pruned because of X_i = v_i
   return false
end
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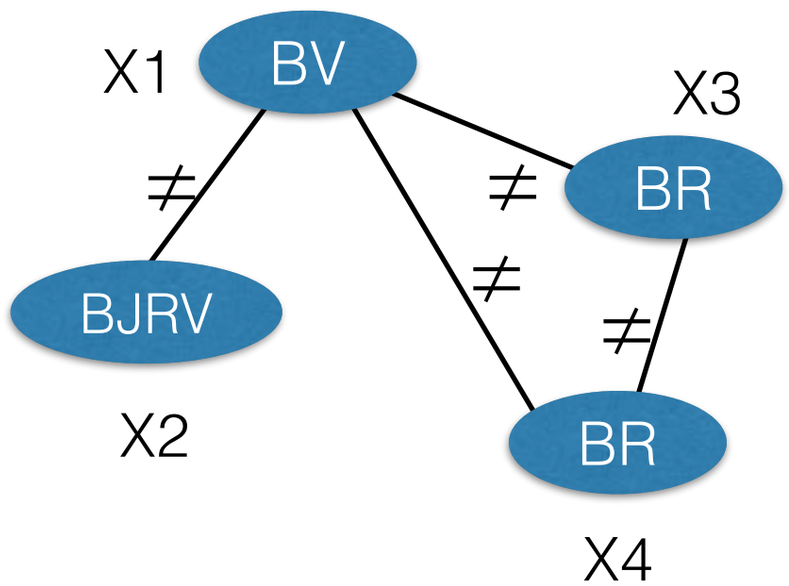

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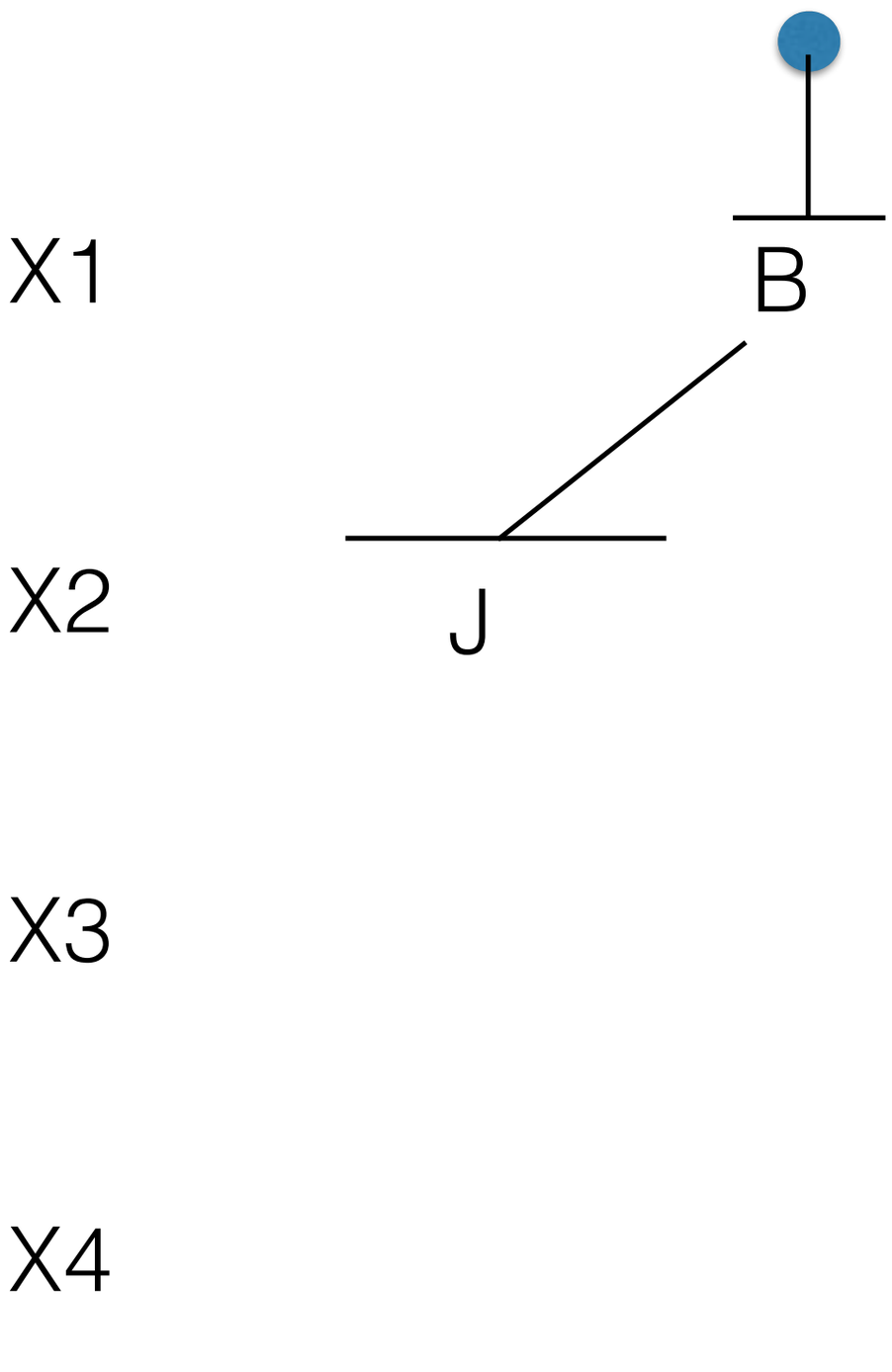

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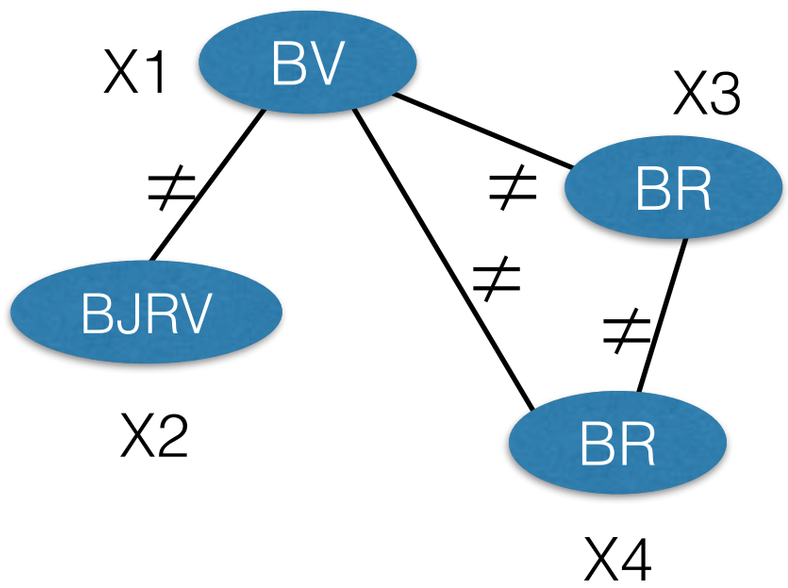

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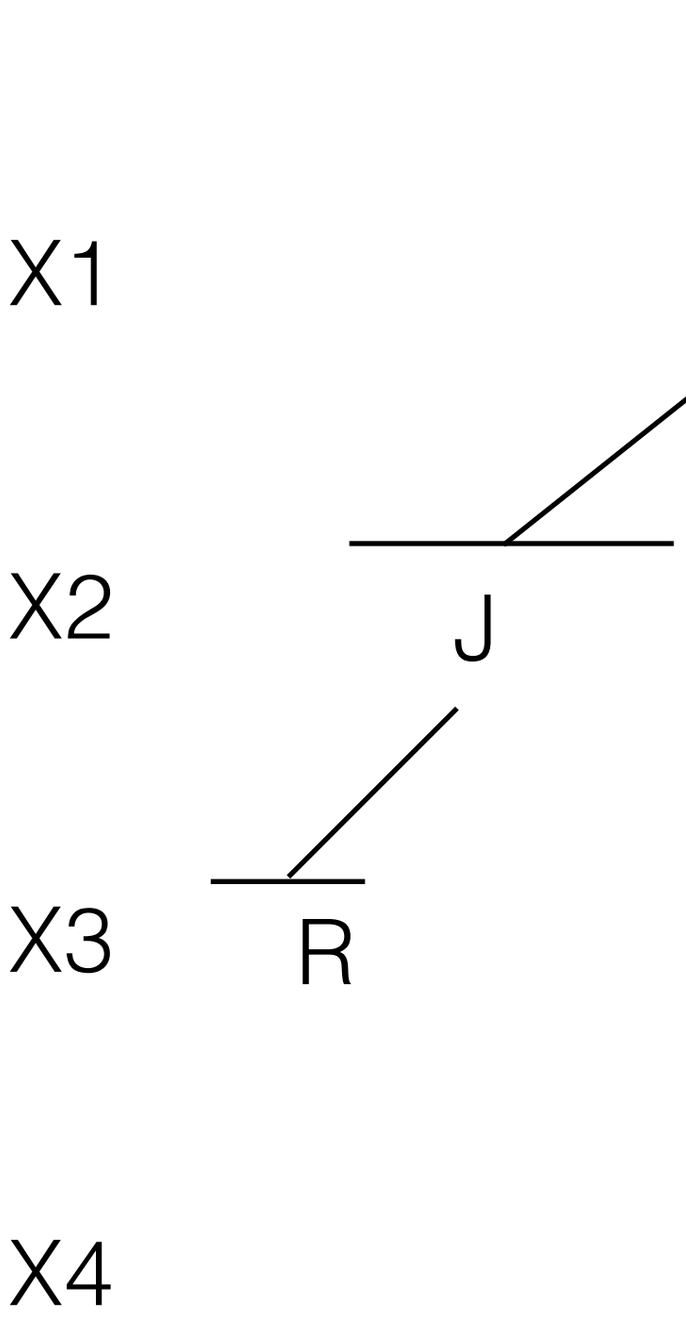

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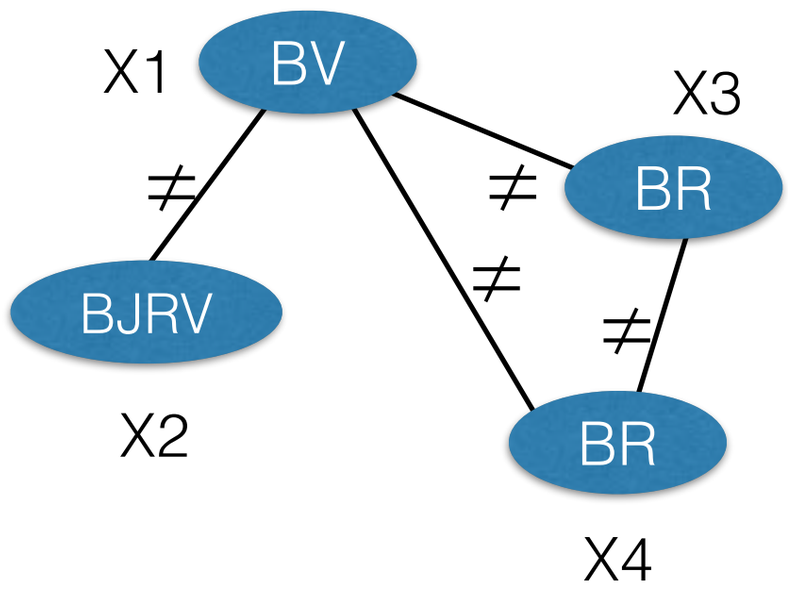

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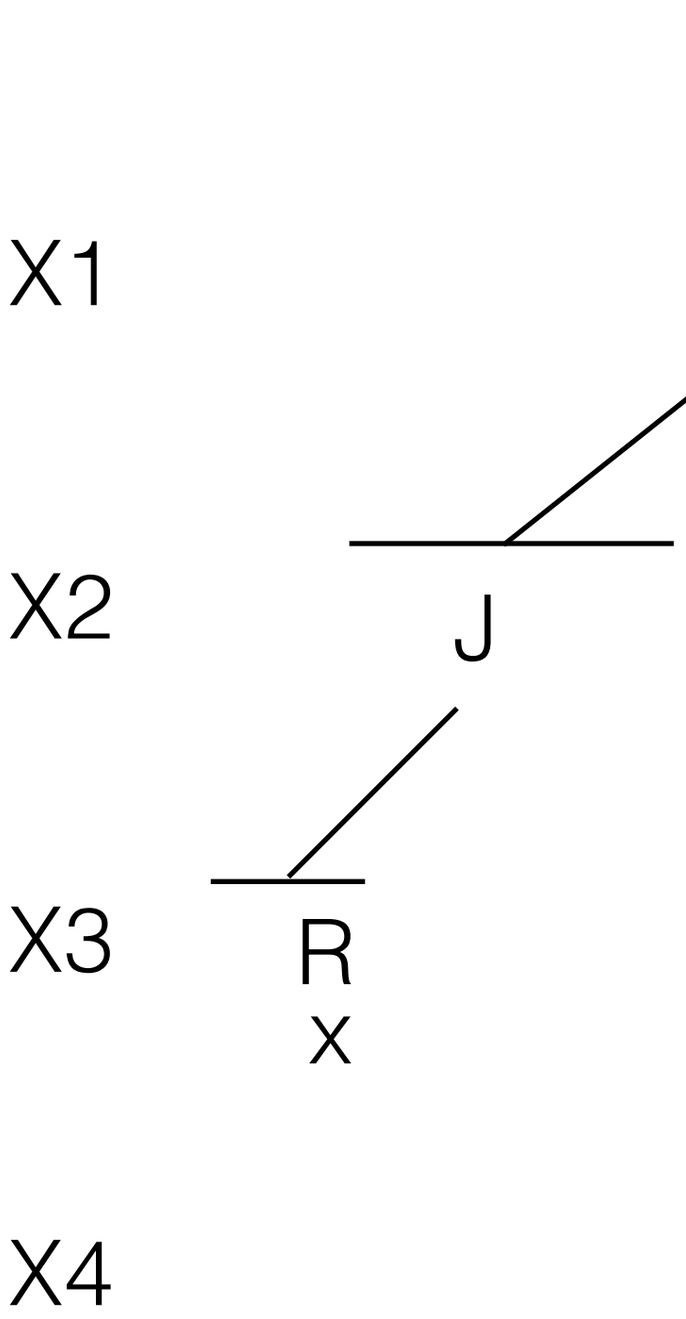

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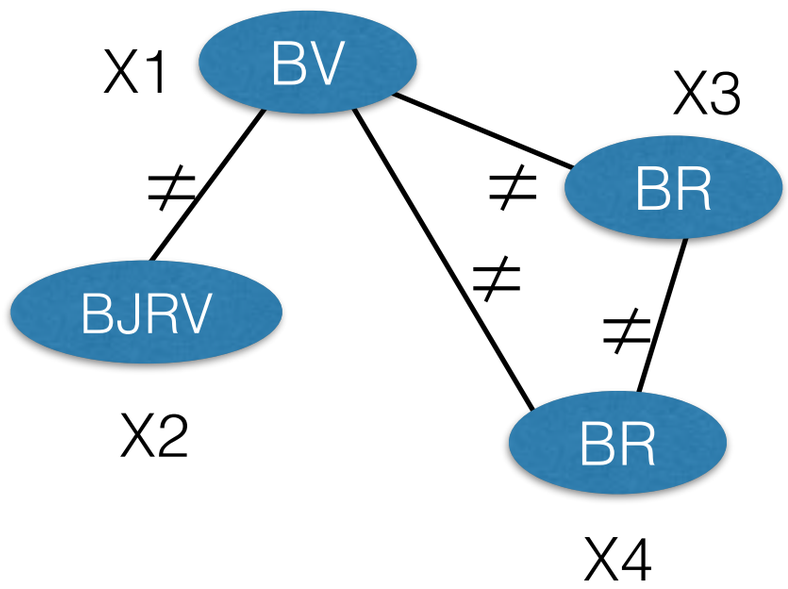

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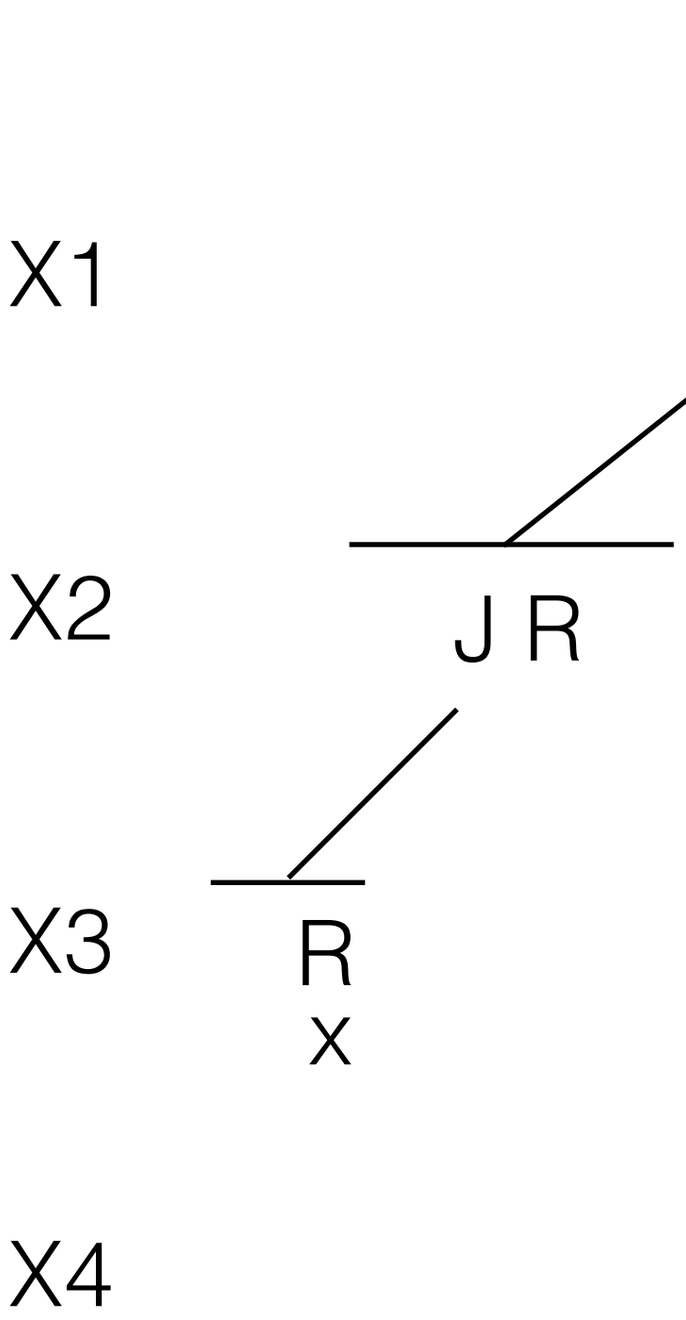

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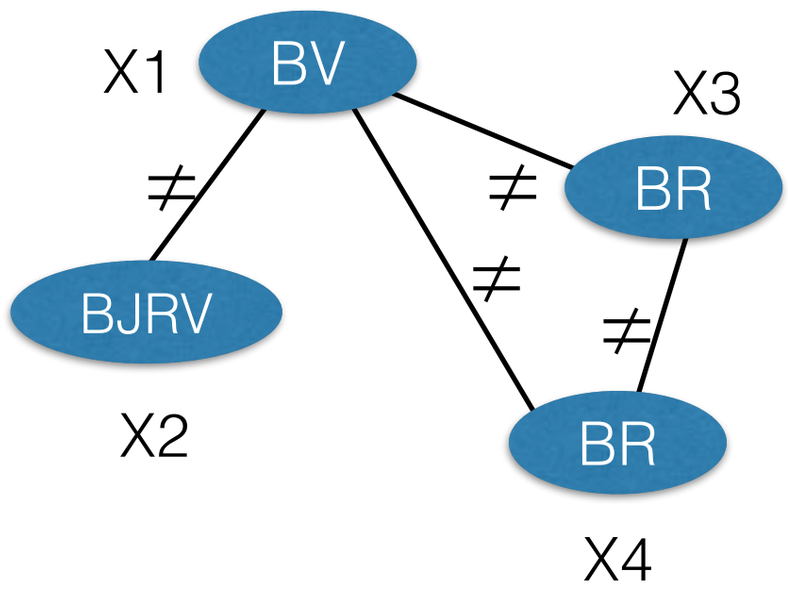

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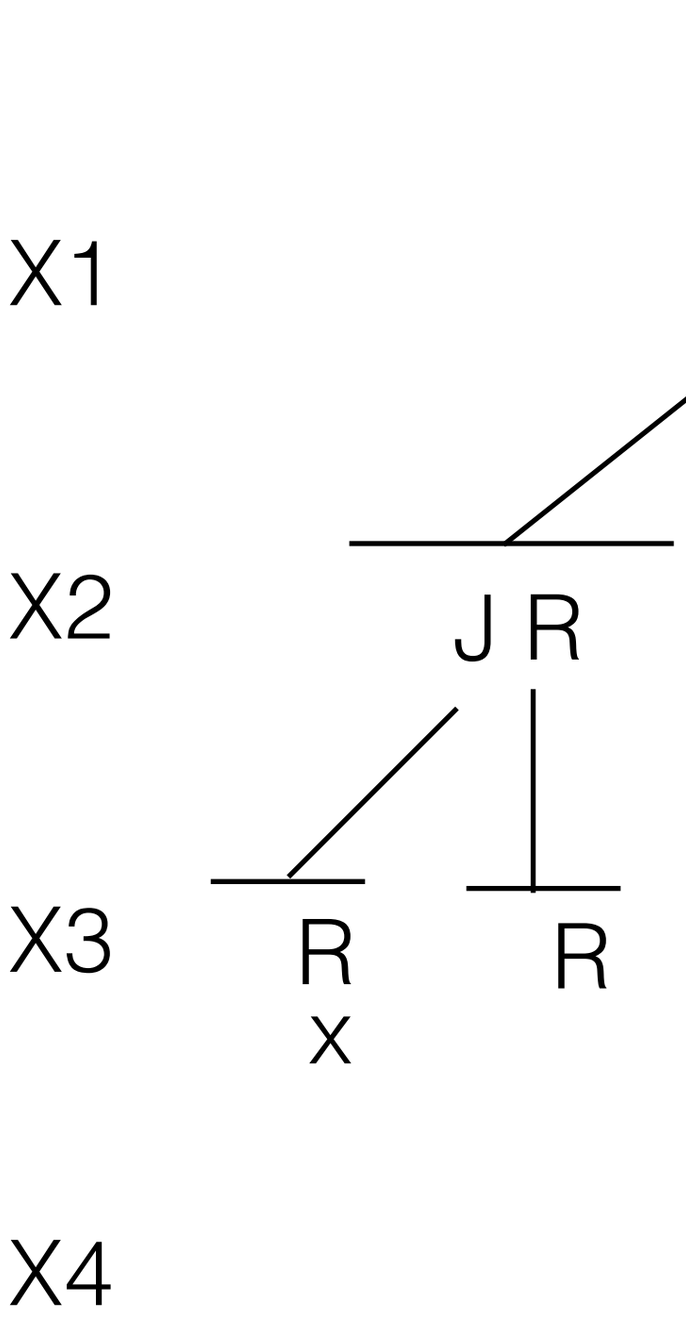

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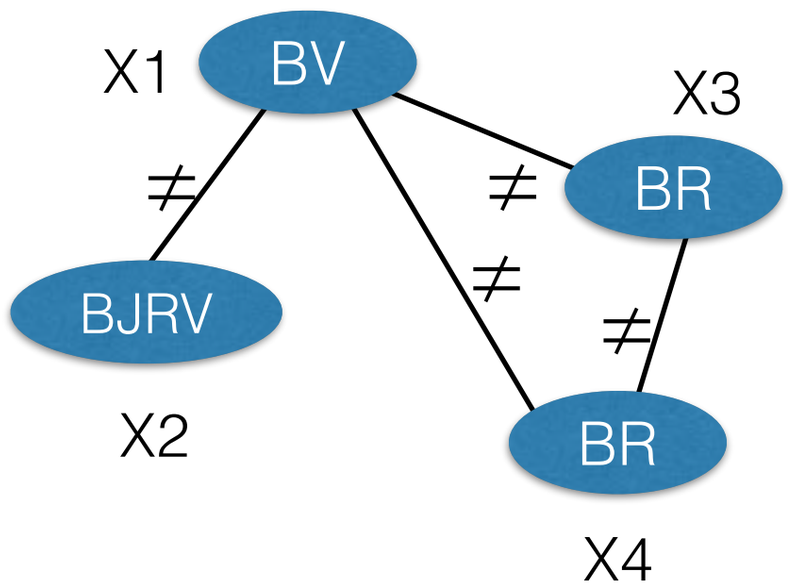
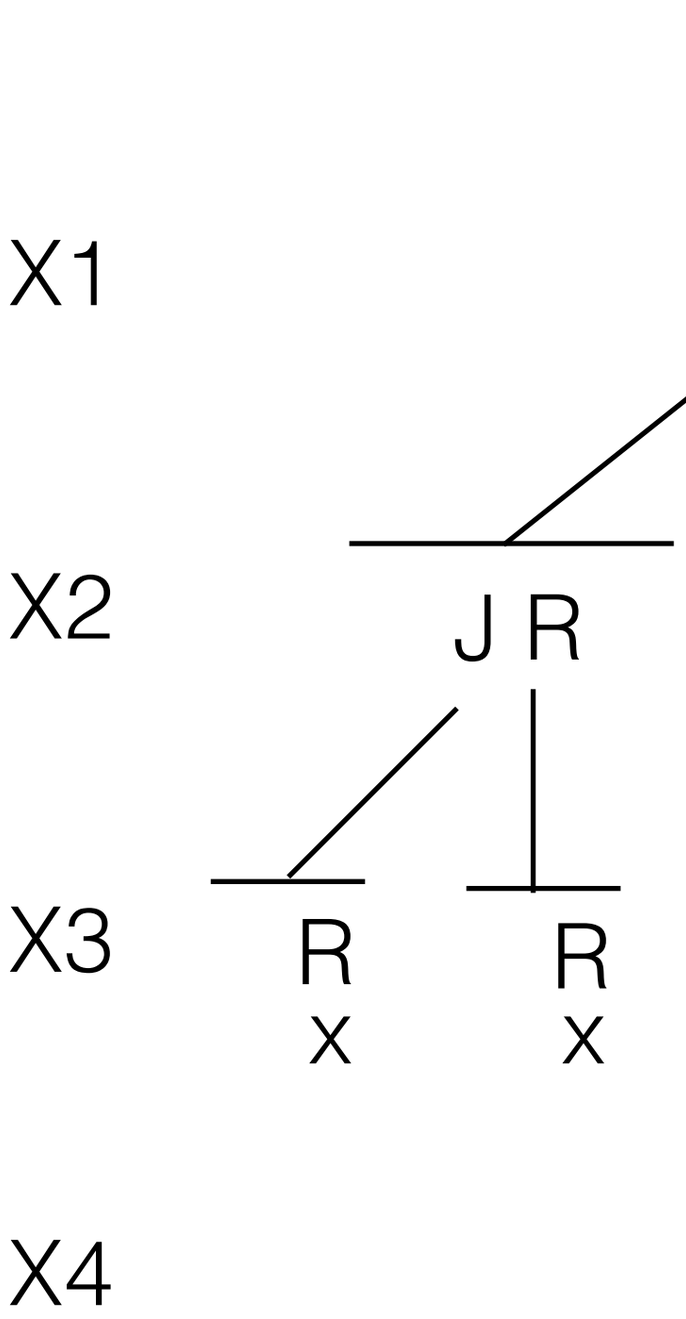


FC on running example



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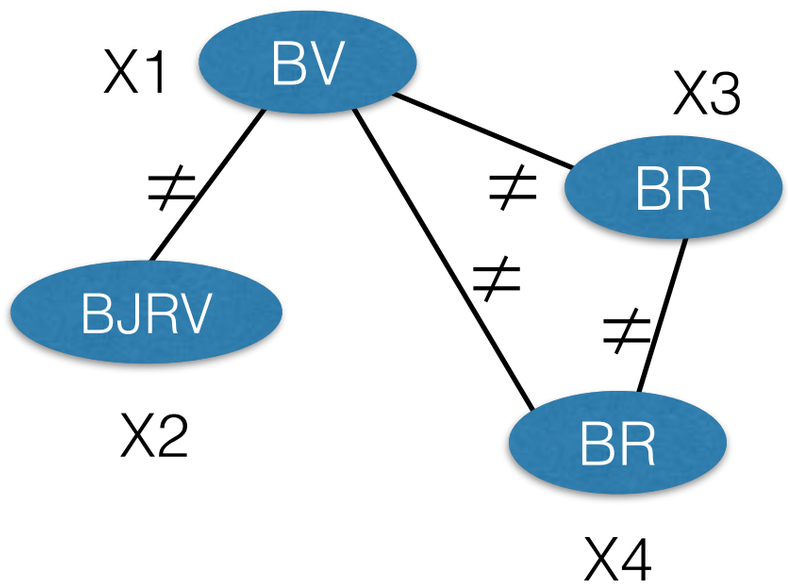
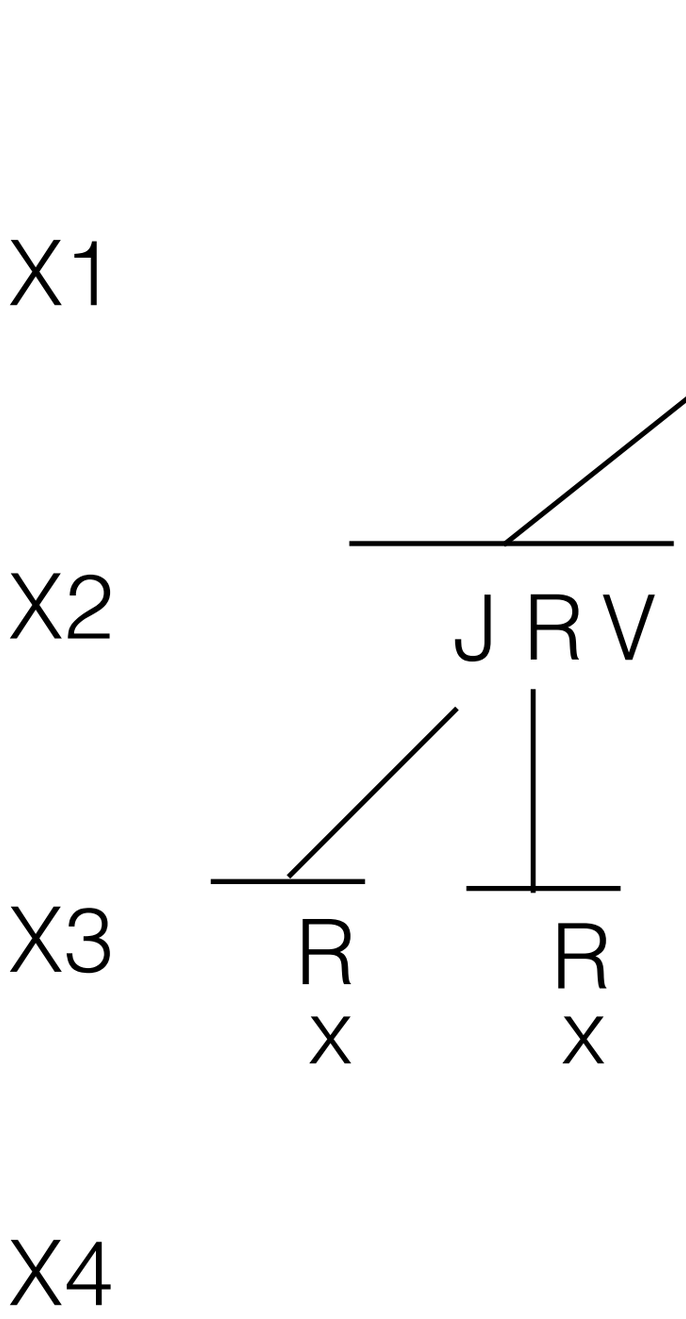


FC on running example



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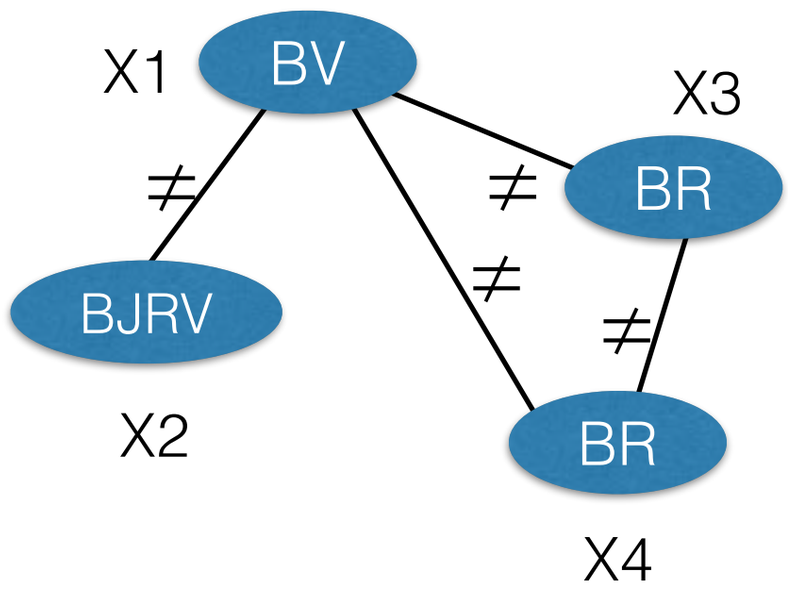

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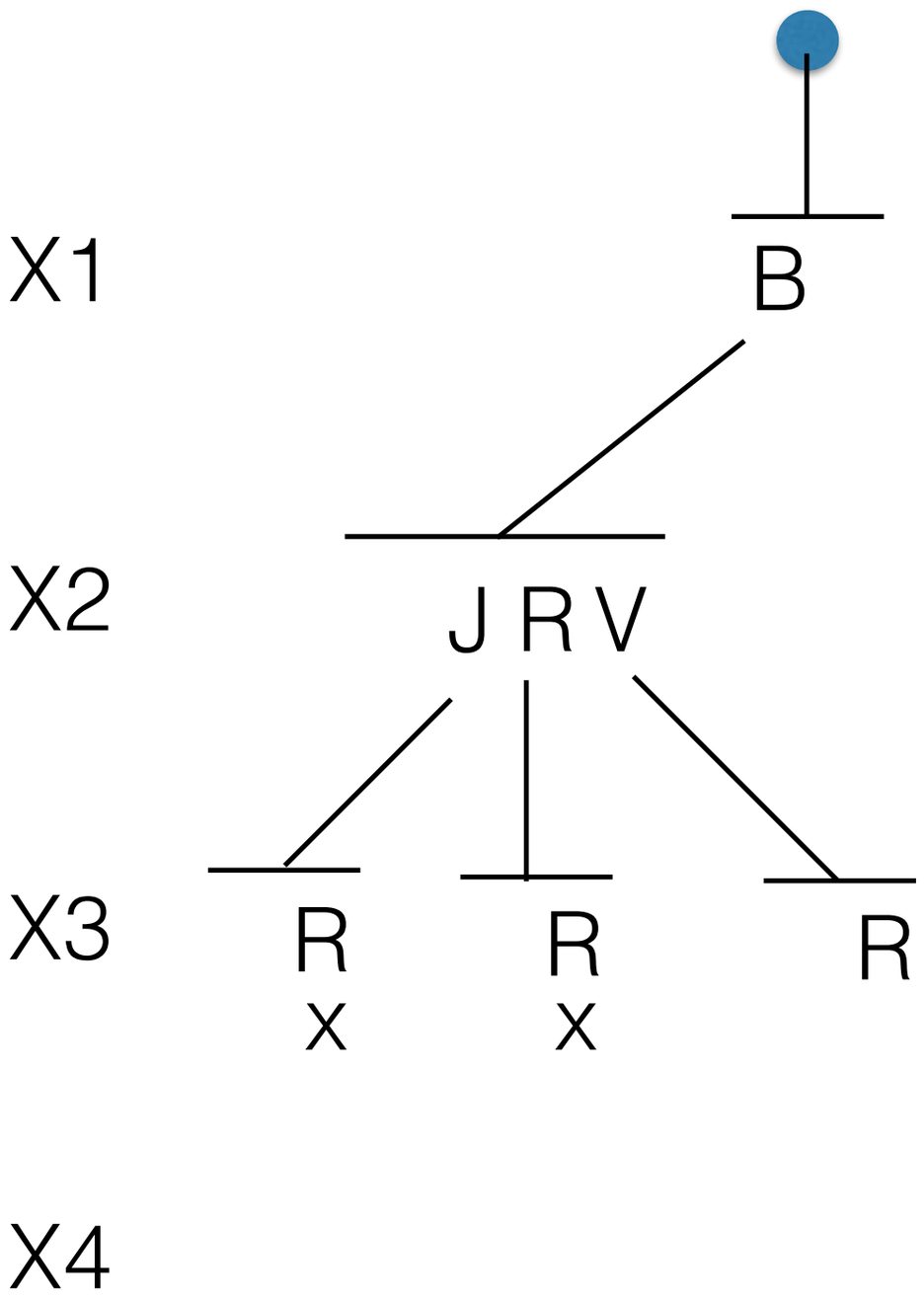

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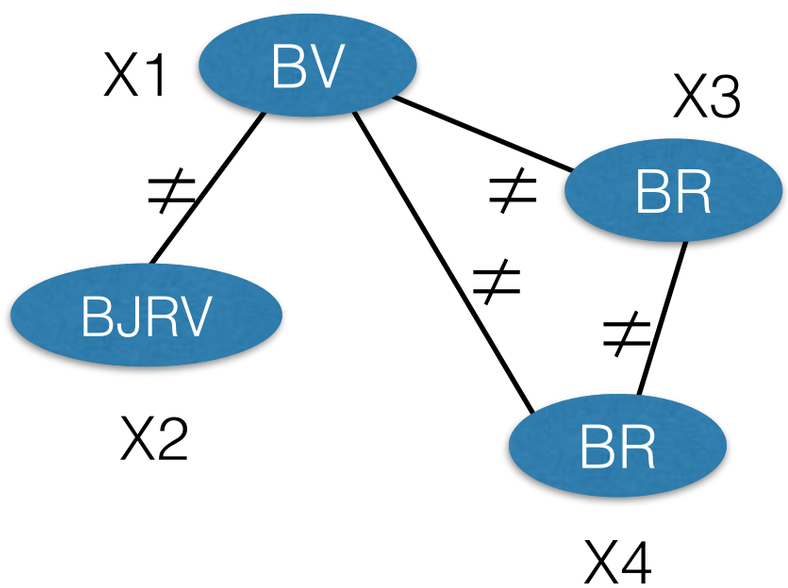
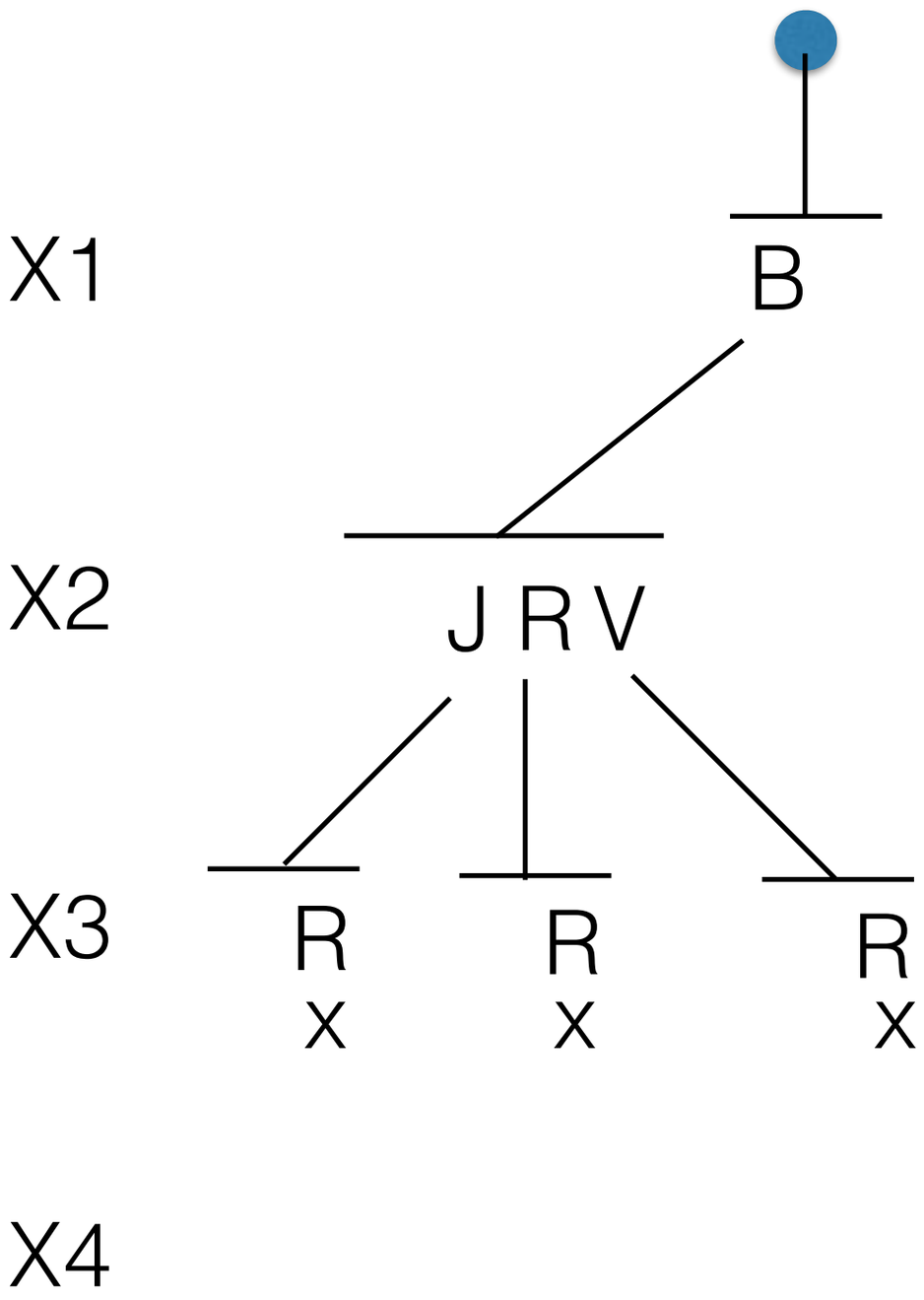


FC on running example



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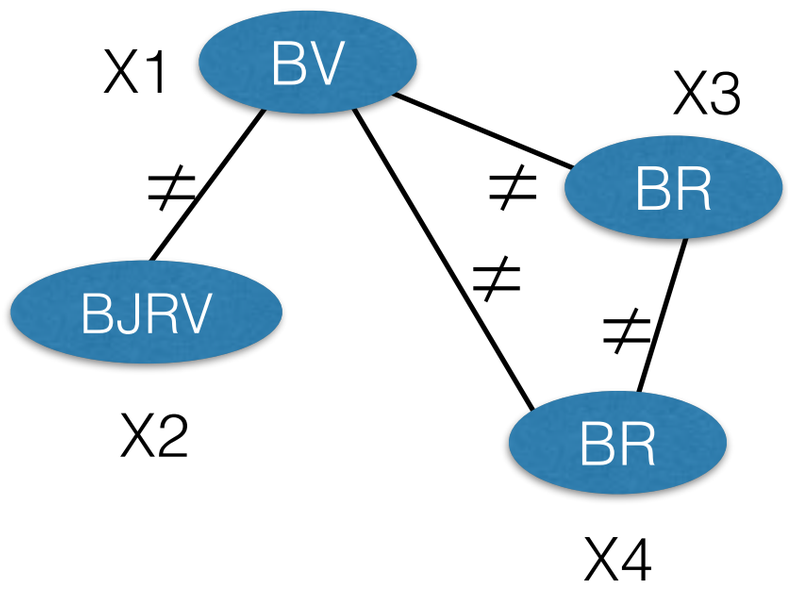

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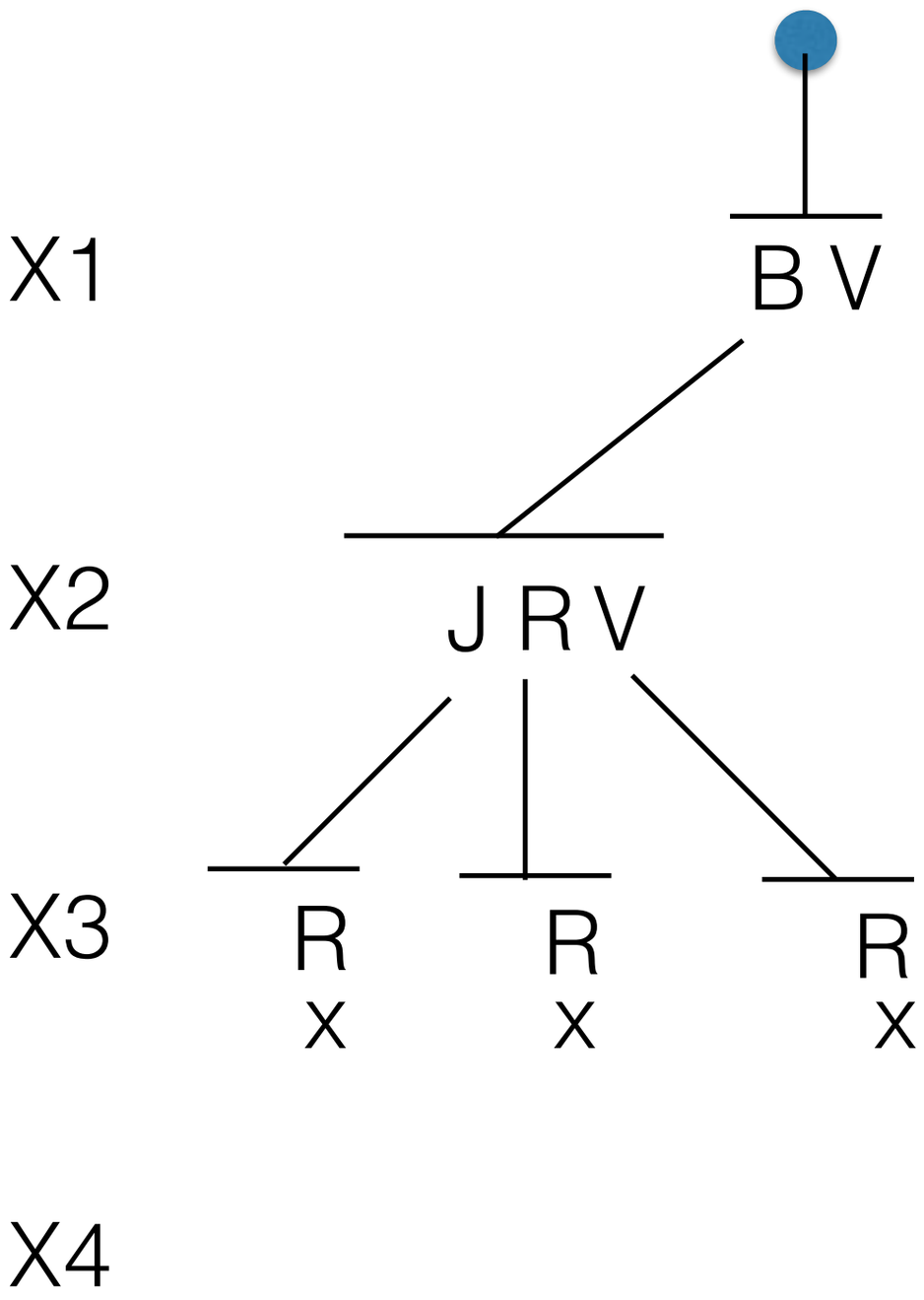

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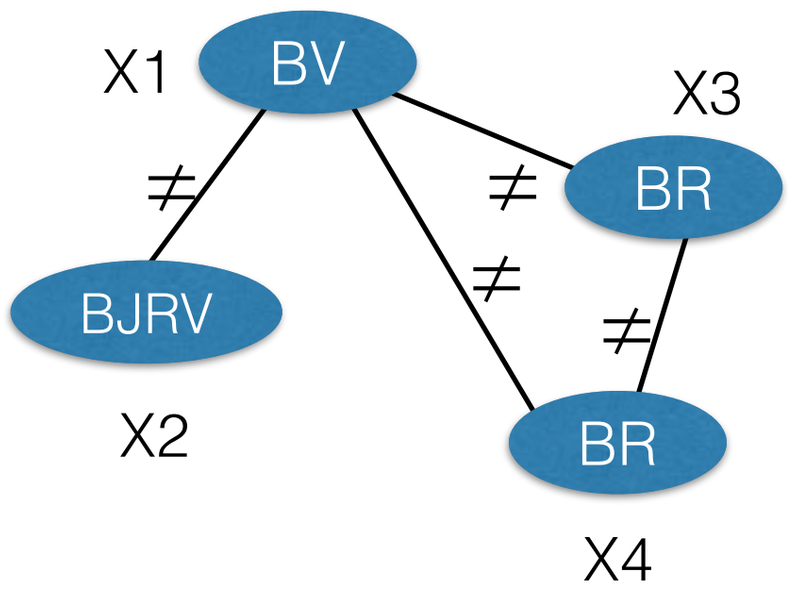

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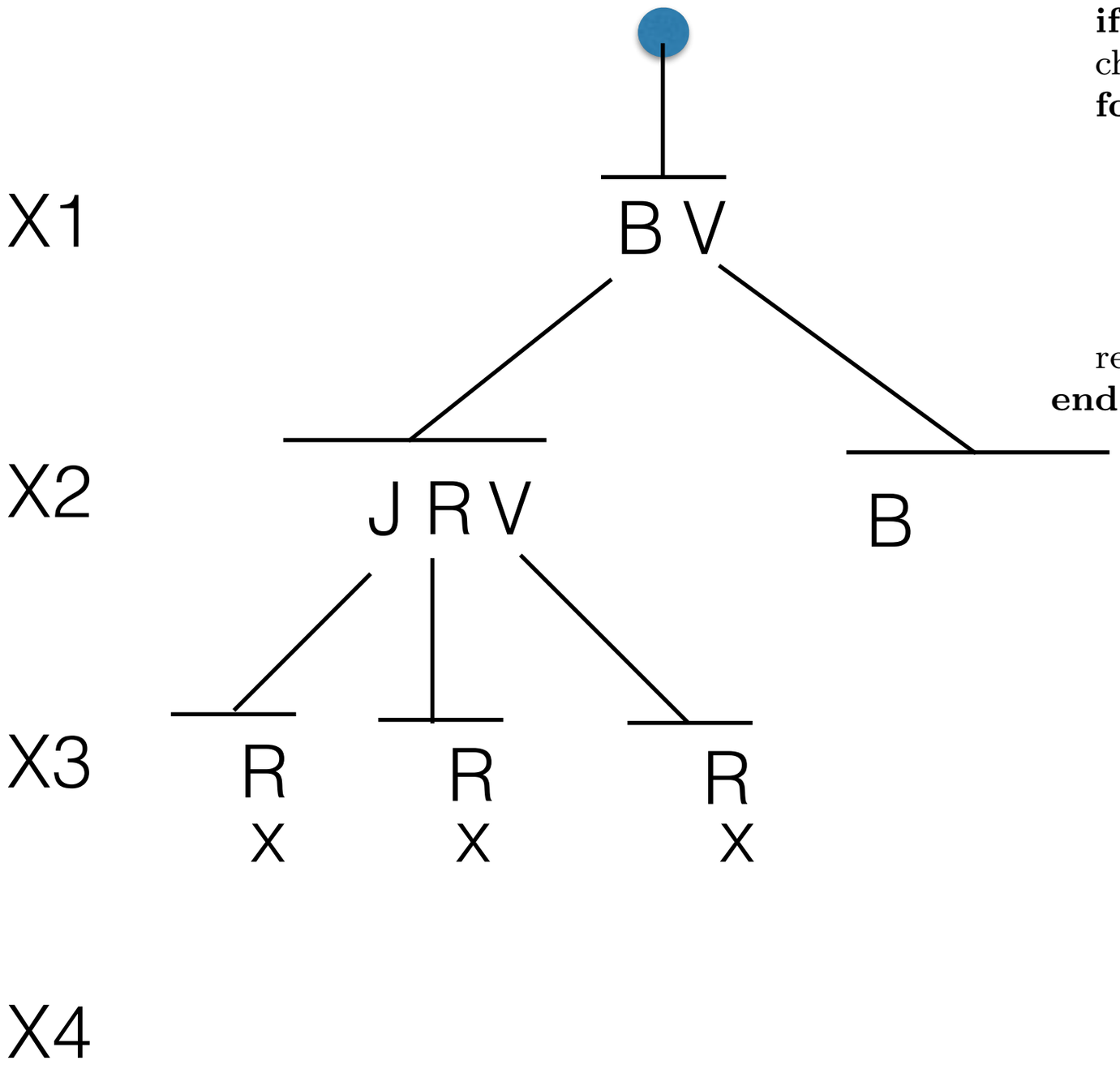

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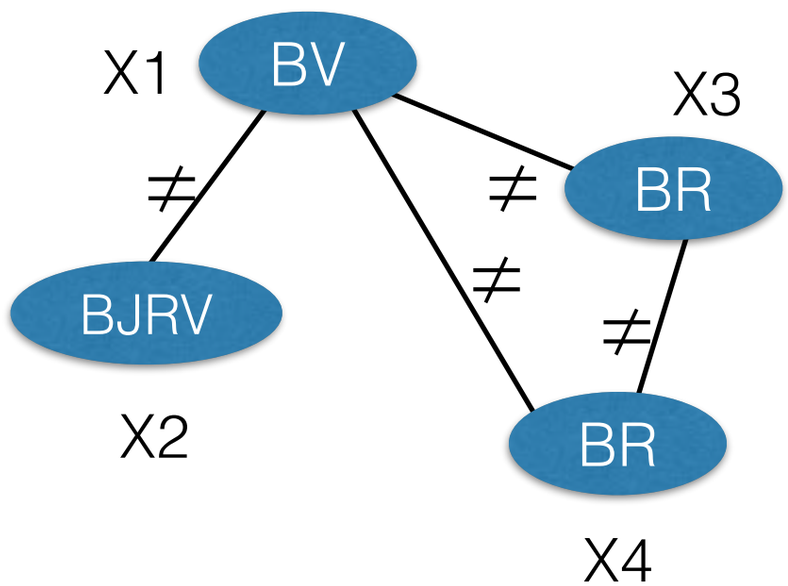
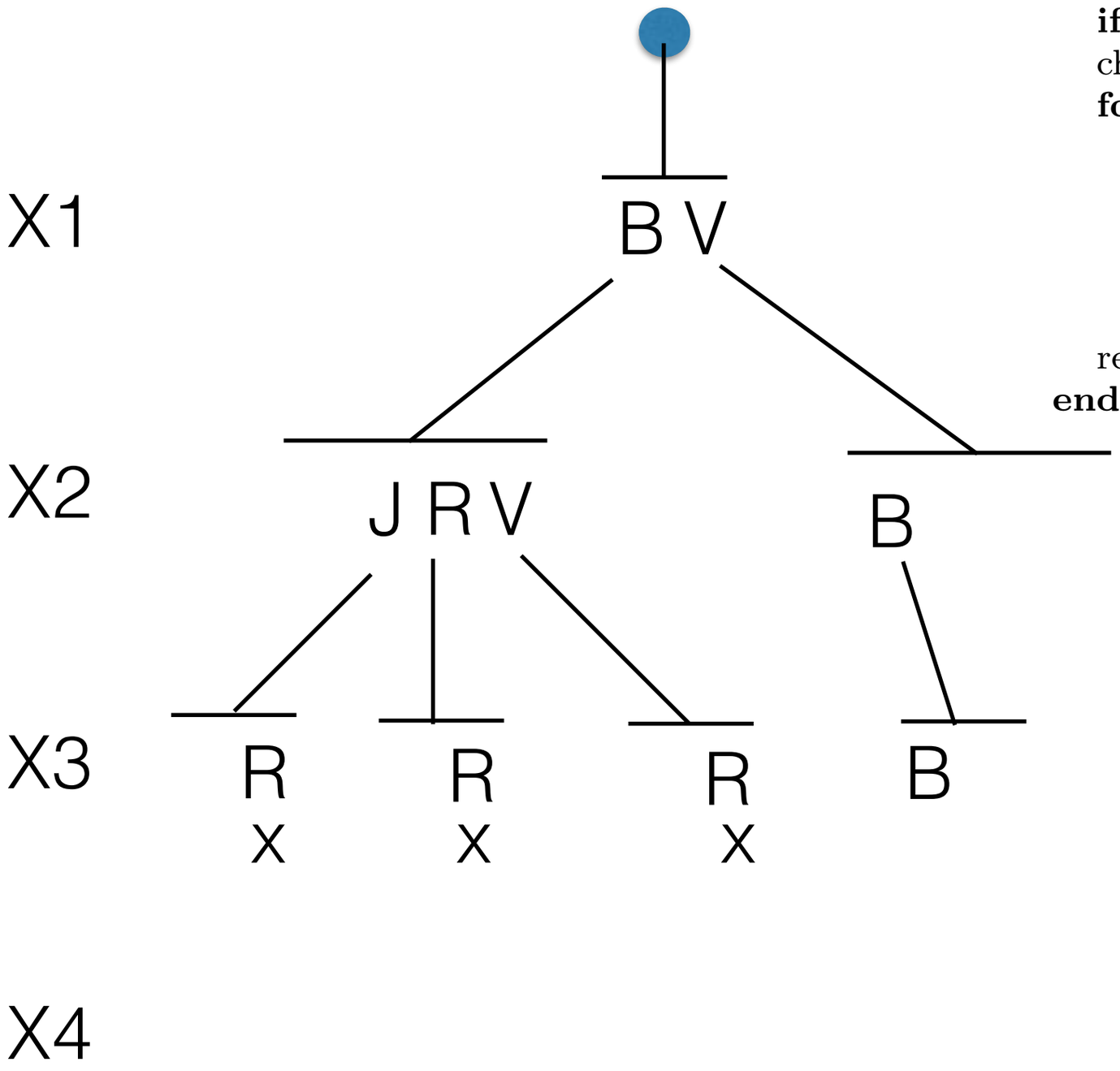


FC on running example



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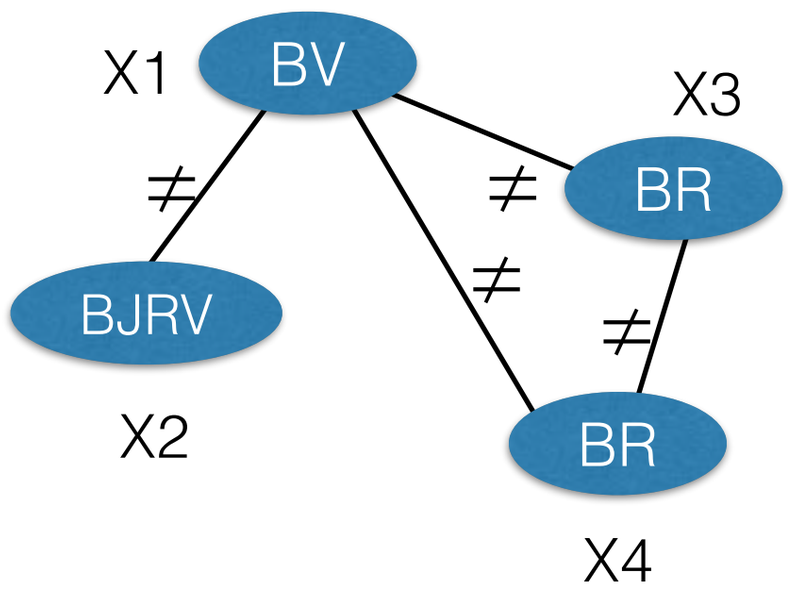

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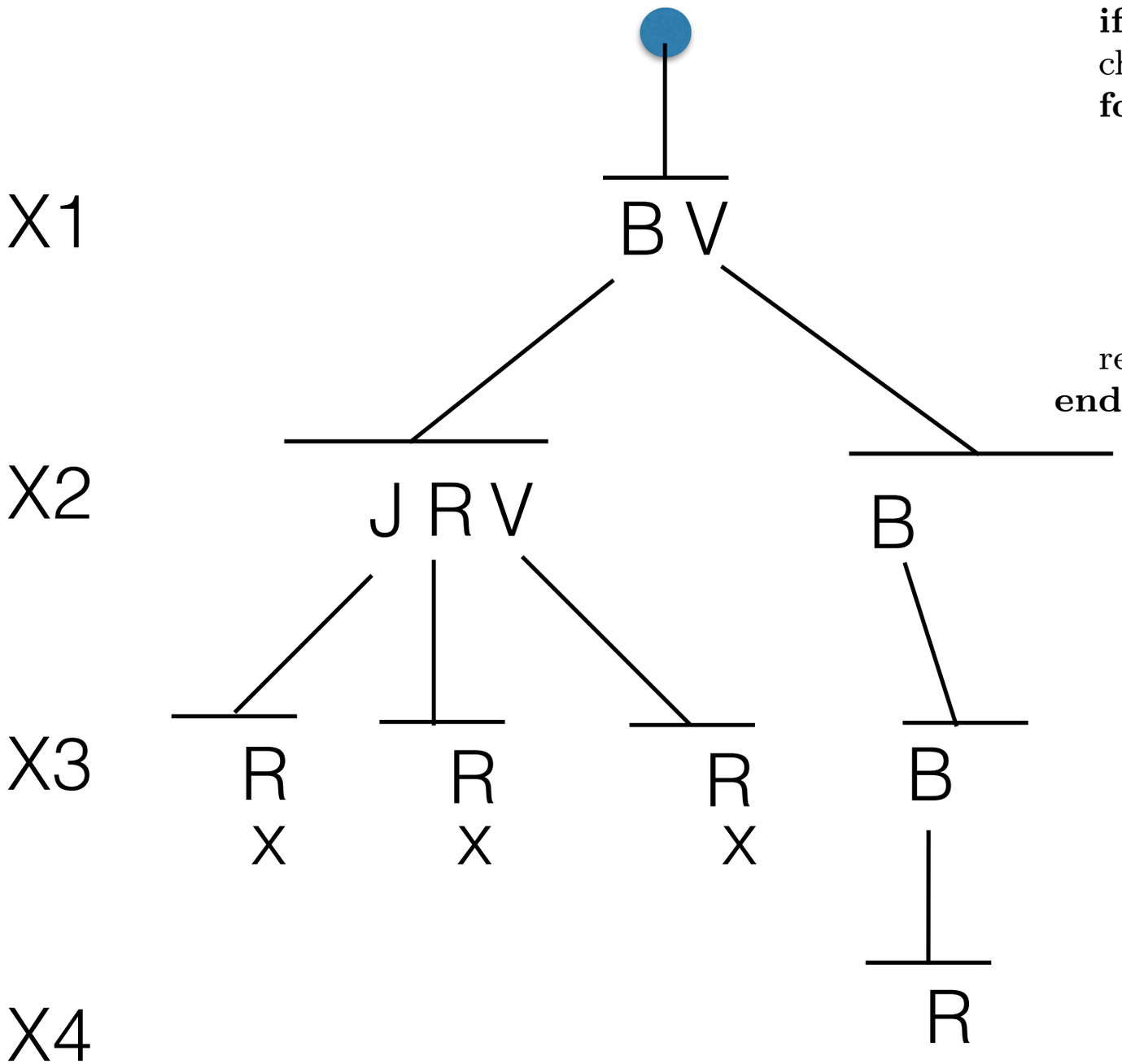

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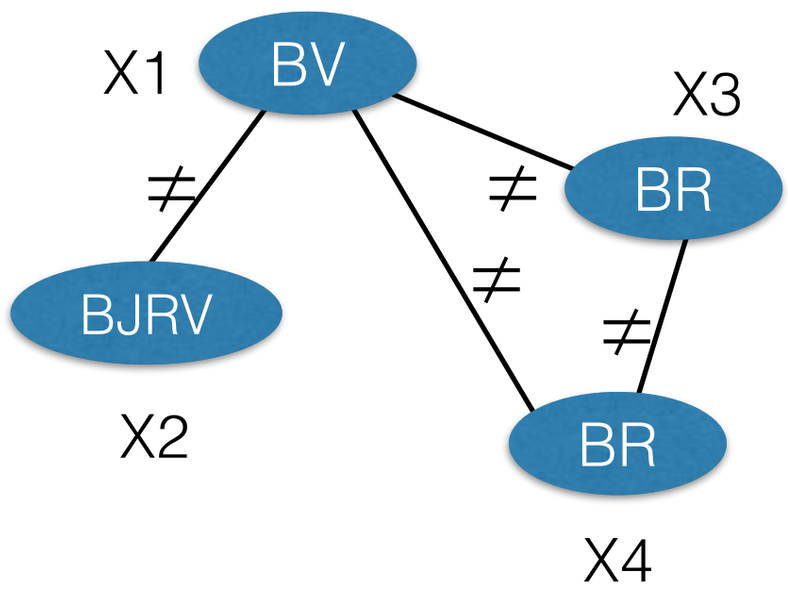
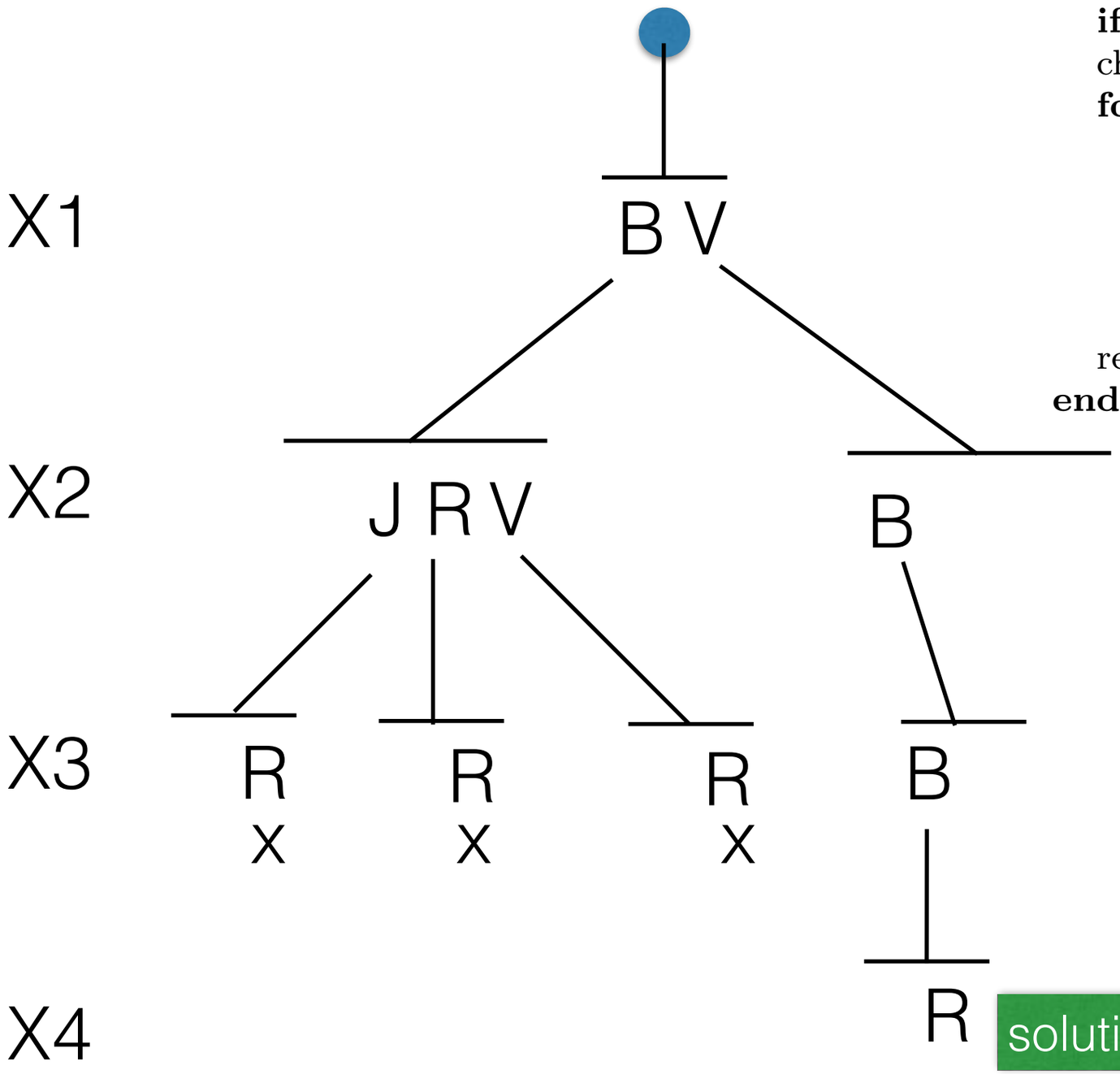


FC on running example



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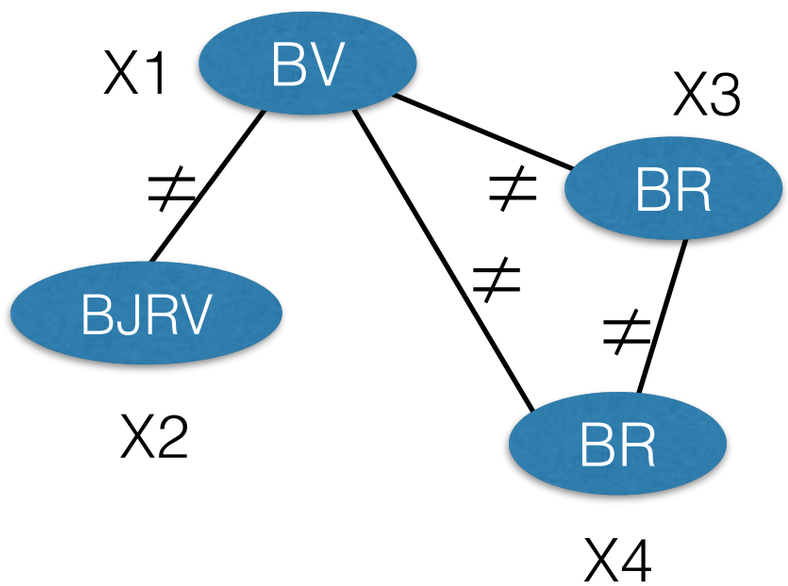
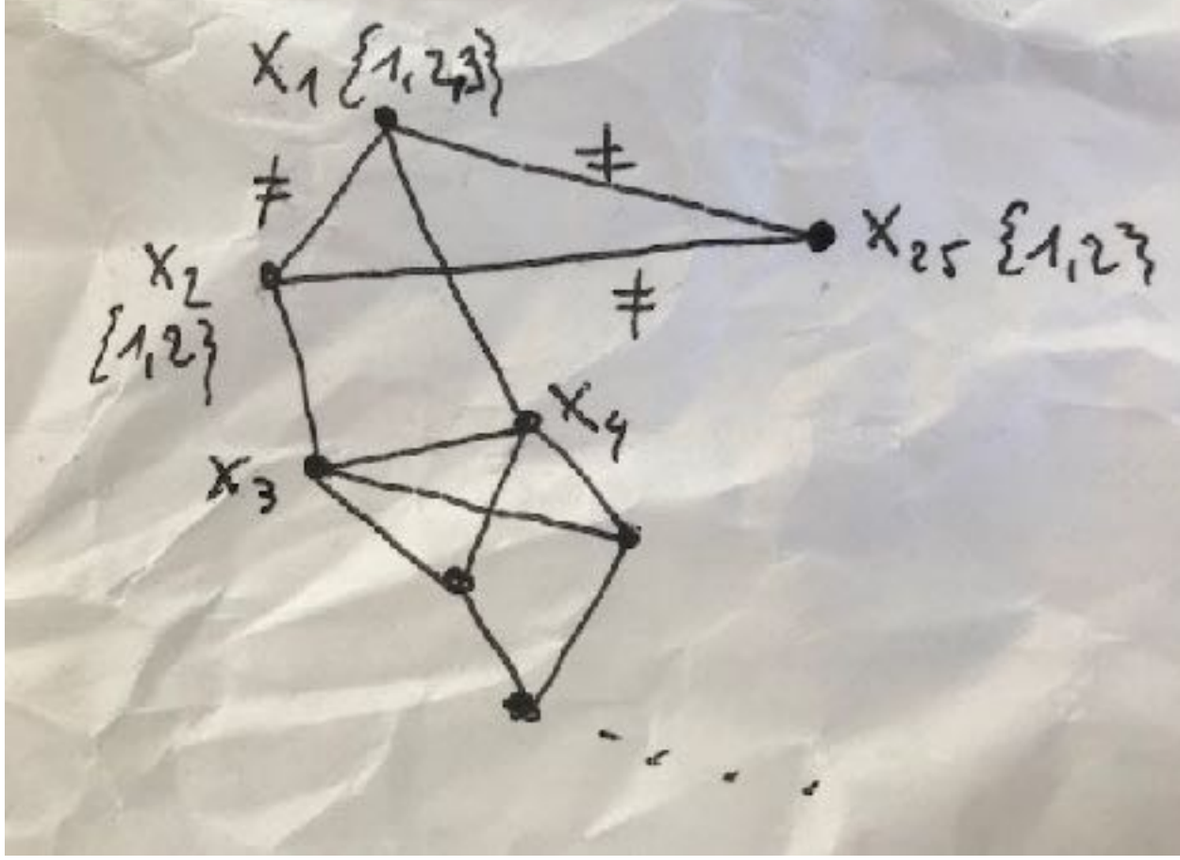
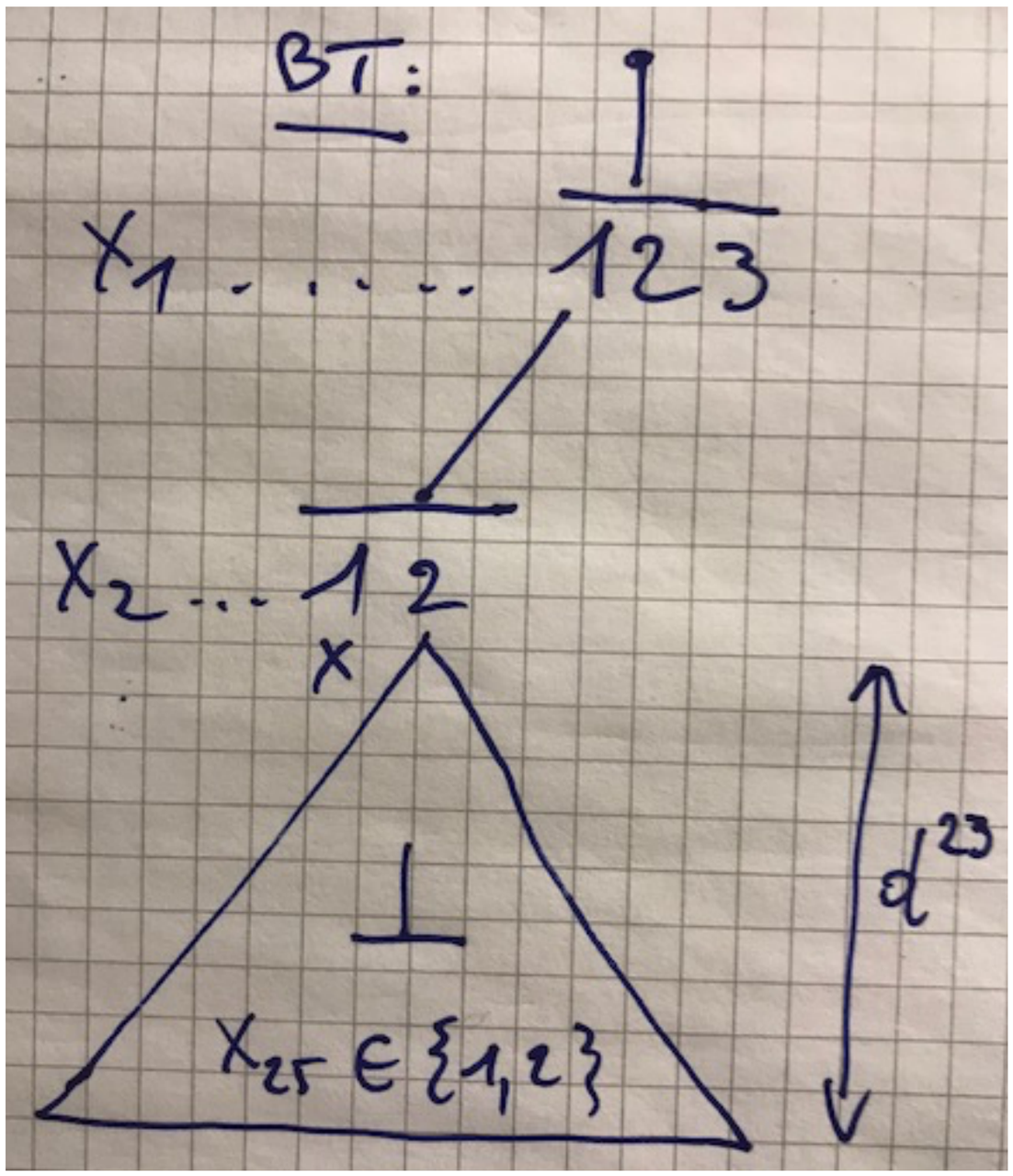
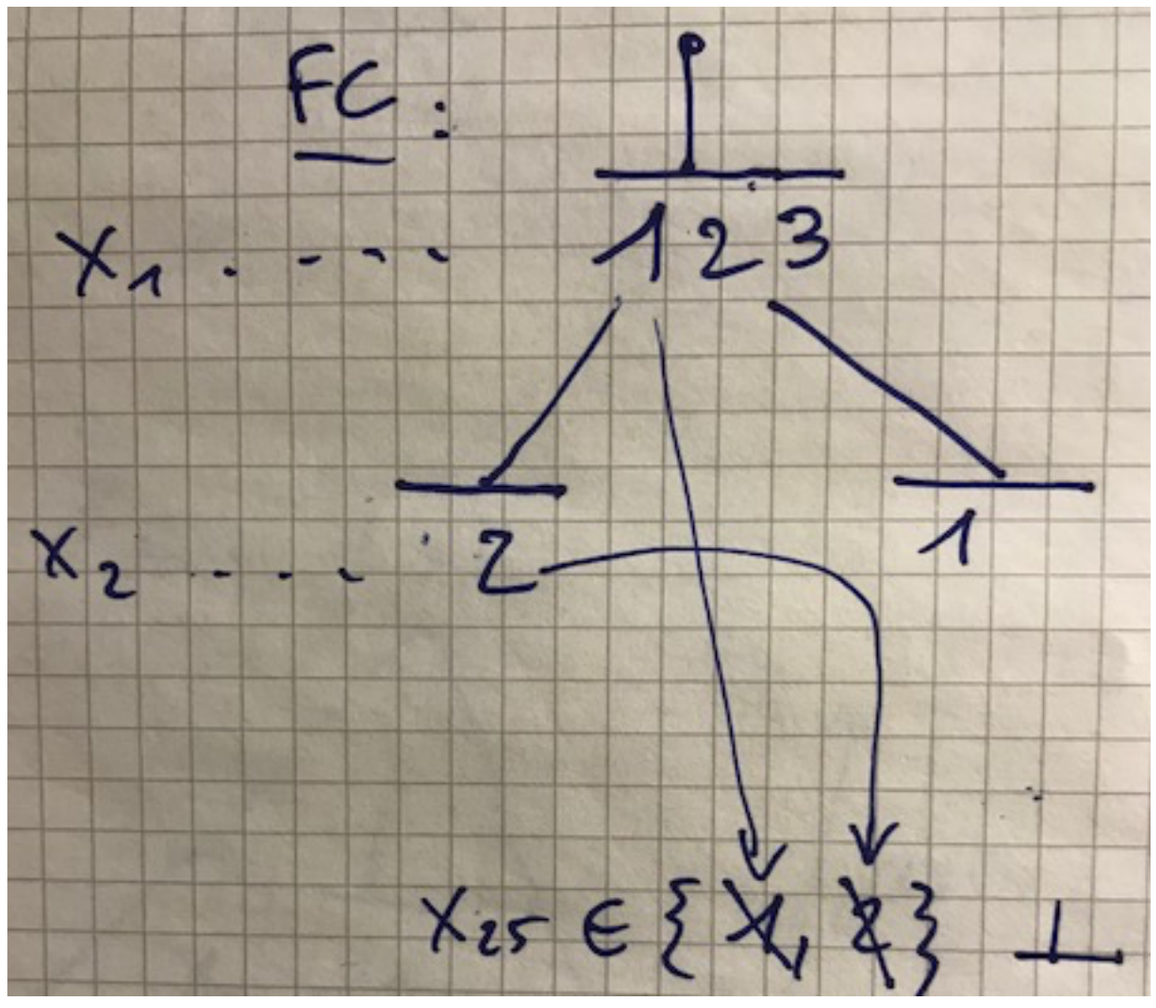
solution



Forward checking







Maintaining Arc Consistency (MAC)

```
Function MAC(N)
    AC(N)
    if empty domain then return 0
    if N fully instantiated then return 1
    select a non singleton variable Xi
    select a value vi ∈ D(Xi)
    return MAC(N+{Xi=vi}) or MAC(N+{Xi≠vi})
```

MAC on running example

X1

X2

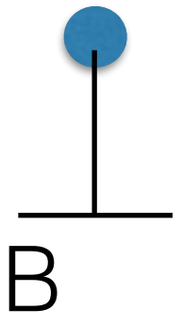
X3

X4

X1	X2	X3	X4
B	B	B	B
	J		
	R	R	R
V	V		

MAC on running example

X1



X2

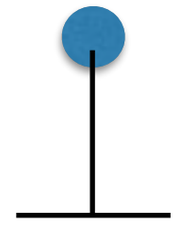
X3

X4

X1	X2	X3	X4
B	B	B	B
	J		
	R	R	R
V	V		

MAC on running example

X1



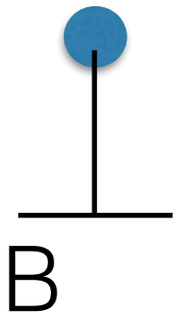
X2

X3

X4

X1	X2	X3	X4
B	B	B	B
	J		
	R	R	R
V	V		

MAC on running example



X1

B

X1	X2	X3	X4
B	~~B~~	~~B~~	~~B~~
	J		
	R	R	R
V	V		

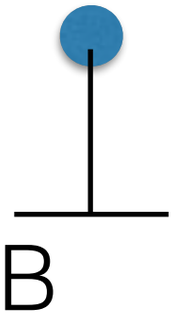
X2

X3

X4

MAC on running example

X1



X2

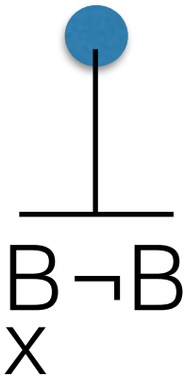
X3

X4

X1	X2	X3	X4
B	B	B	B
	J		
	R	R	R
V	V		

MAC on running example

X1



X1	X2	X3	X4
B	B	B	B
	J		
	R	R	R
V	V		

X2

X3

X4

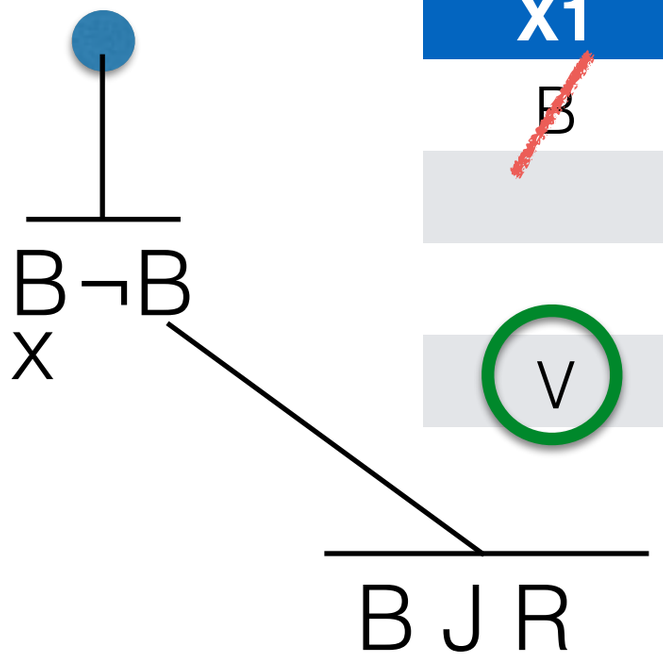
MAC on running example

X1

X2

X3

X4



X1	X2	X3	X4
B	B	B	B
	J		
	R	R	R
V	V		

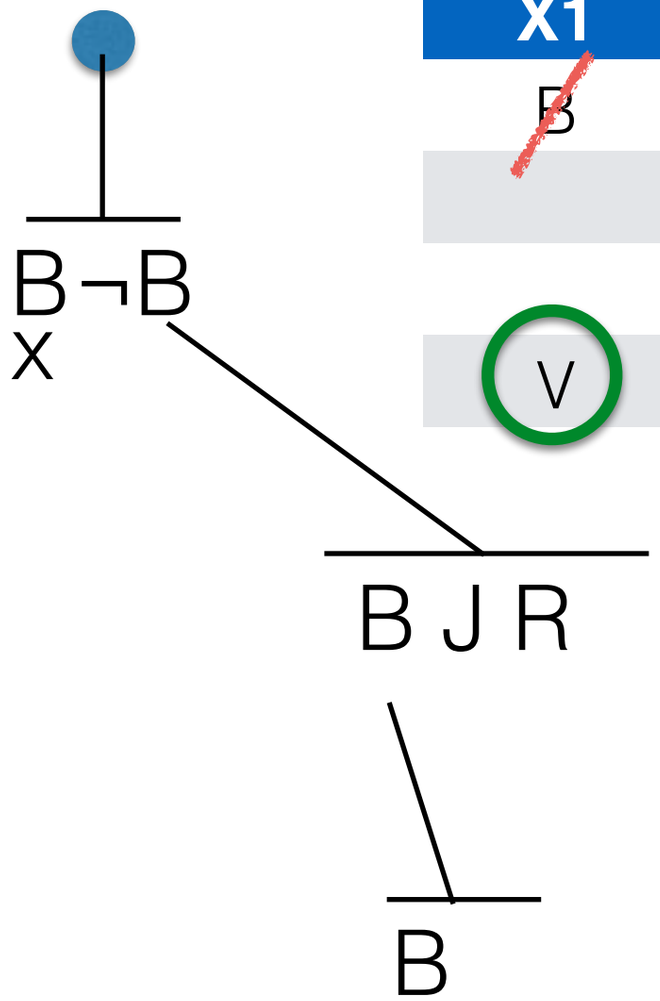
MAC on running example

X1

X2

X3

X4



X1	X2	X3	X4
B	B	B	B
	J		
	R	R	R
V	V		

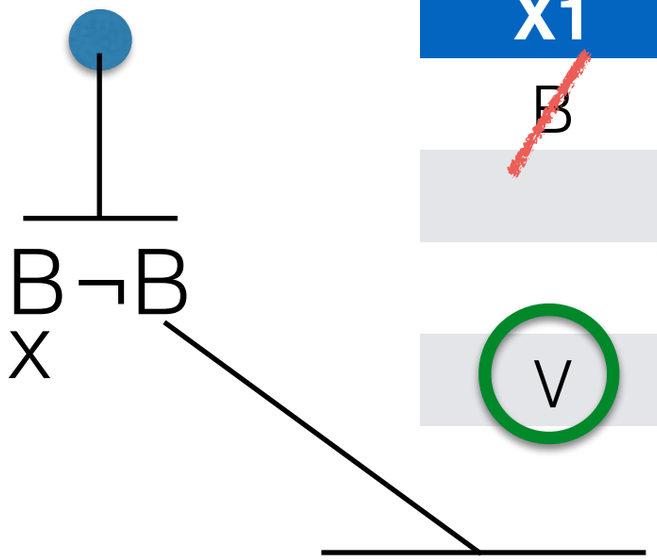
MAC on running example

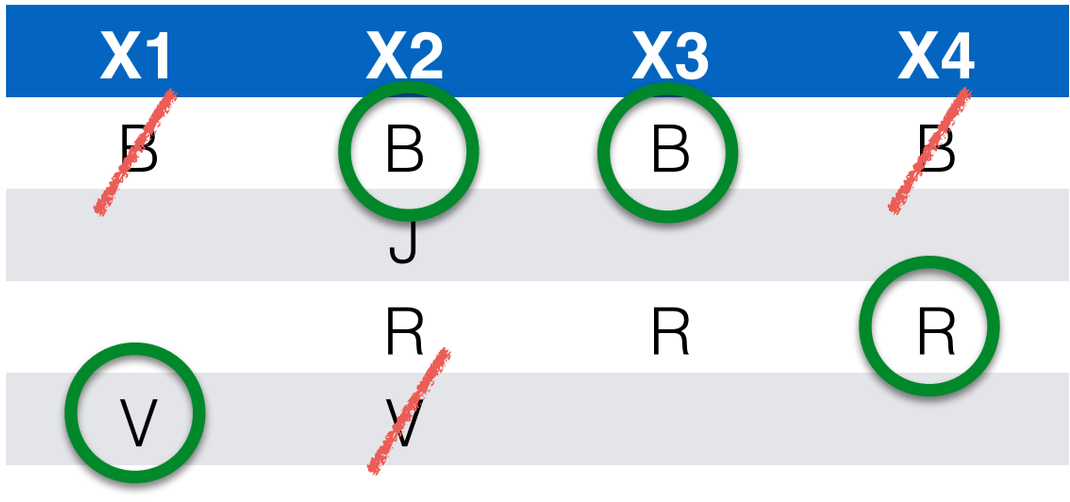
X1

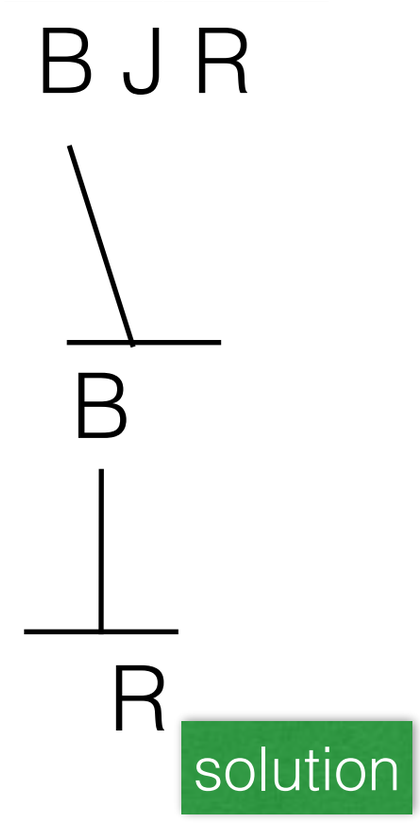
X2

X3

X4







solution

Parenthèse sur les expérimentations

Comment comparer différents algorithmes ?

- Analyse en *O(…)* —> pas adapté à la pratique
- Comparer les performances sur la résolution de problèmes
- Mais quels problèmes ?
    - années 80/90: n-reines (poly!), zebra problem (25 variables), random problems

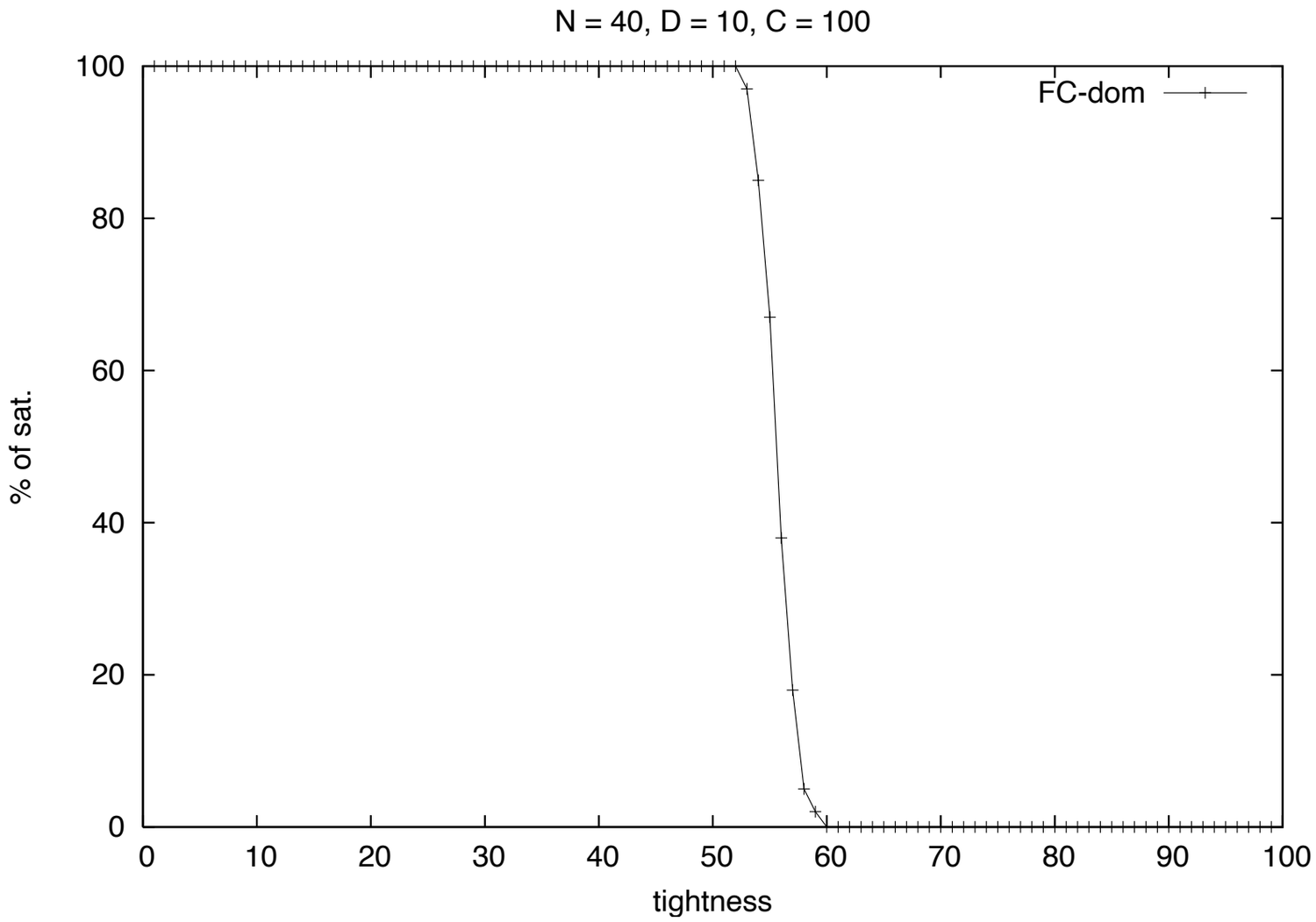
Random problems (binary)

- *n*: number of variables
- *d*: size of domains
- *p1*: proportion of constraints
  - number of constraints = $p1 \times n(n-1)/2$
- *p2*: tightness of constraints
  - number of forbidden tuples = $p2 \times d^2$

Cheeseman, Kanefsky, Taylor's conjecture

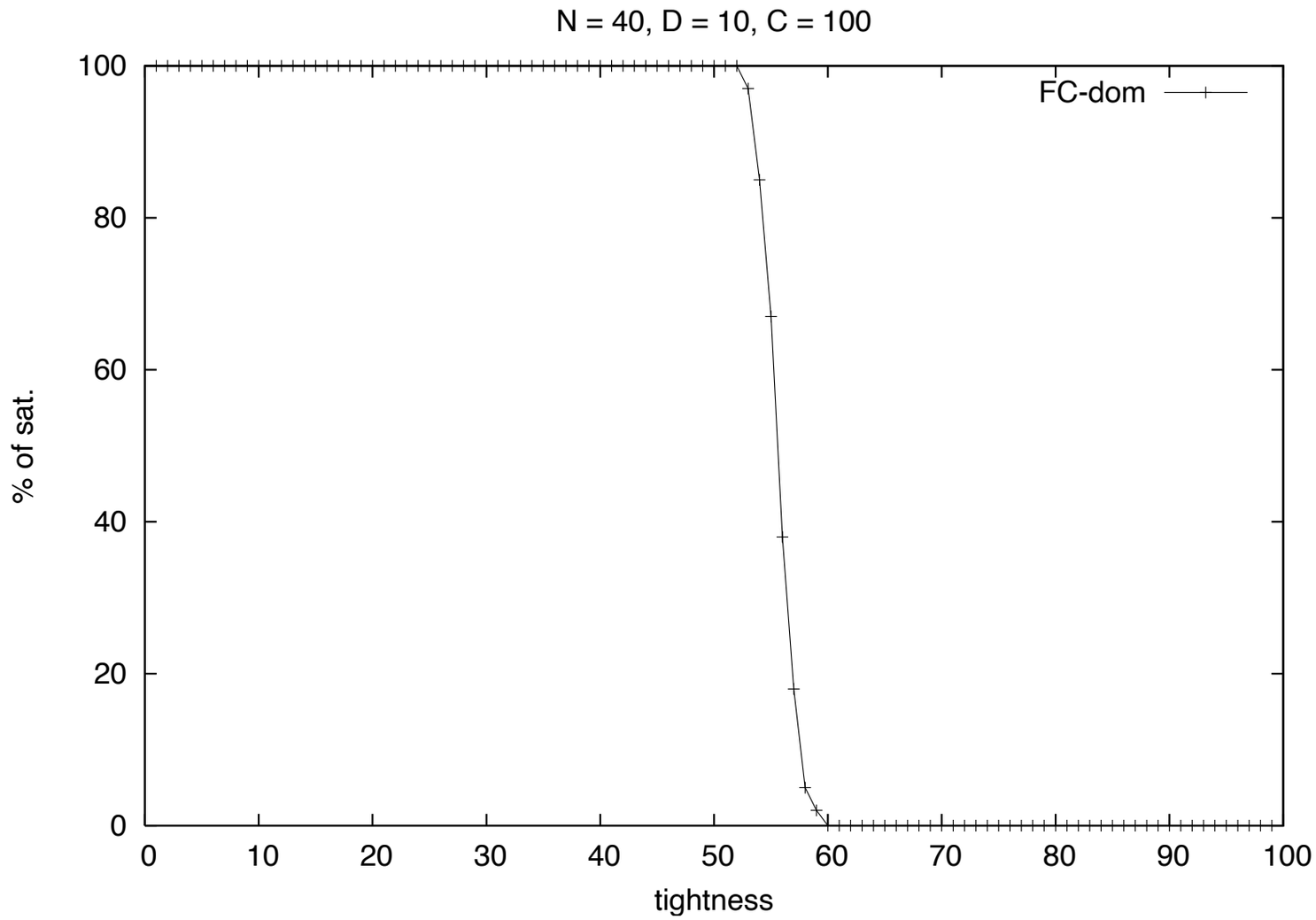
- Take any NP-complete problem
- Characterize it by *k* parameters
  - 3SAT: #atoms, #clauses
  - 3COL: #nodes, #edges
  - 2CSP: #variables, #values, #constraints, #tuples
- Fix *k-1* parameters and vary the free one
- Generate 100 instances for each value of *k*

Phase transition

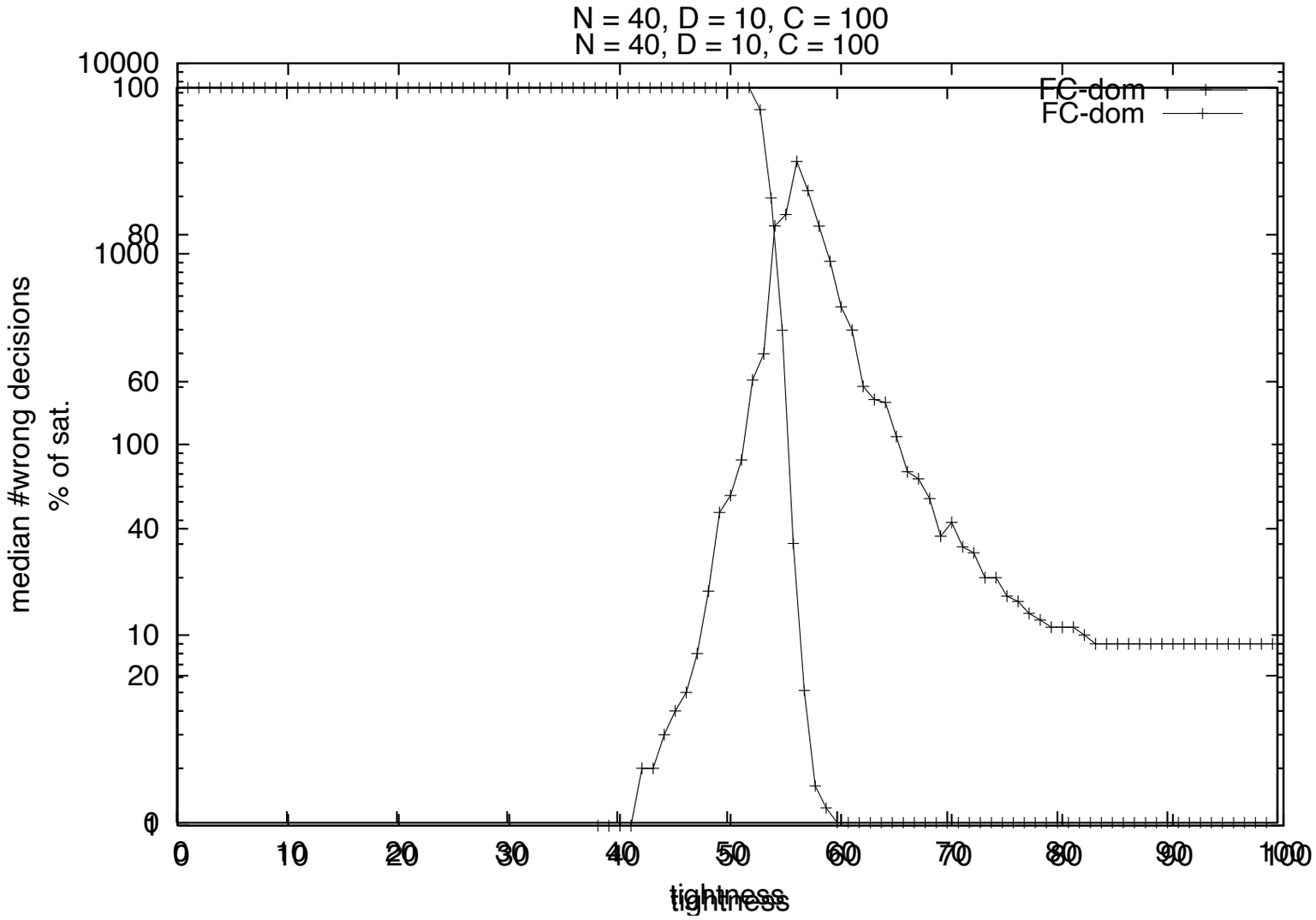


We observe a sudden *phase transition* from ‘all satisfiable’ to ‘all unsatisfiable’ while *k* (in/de)creases

Solving these instances



Solving these instances



We observe a sudden increase of hardness at the phase transition

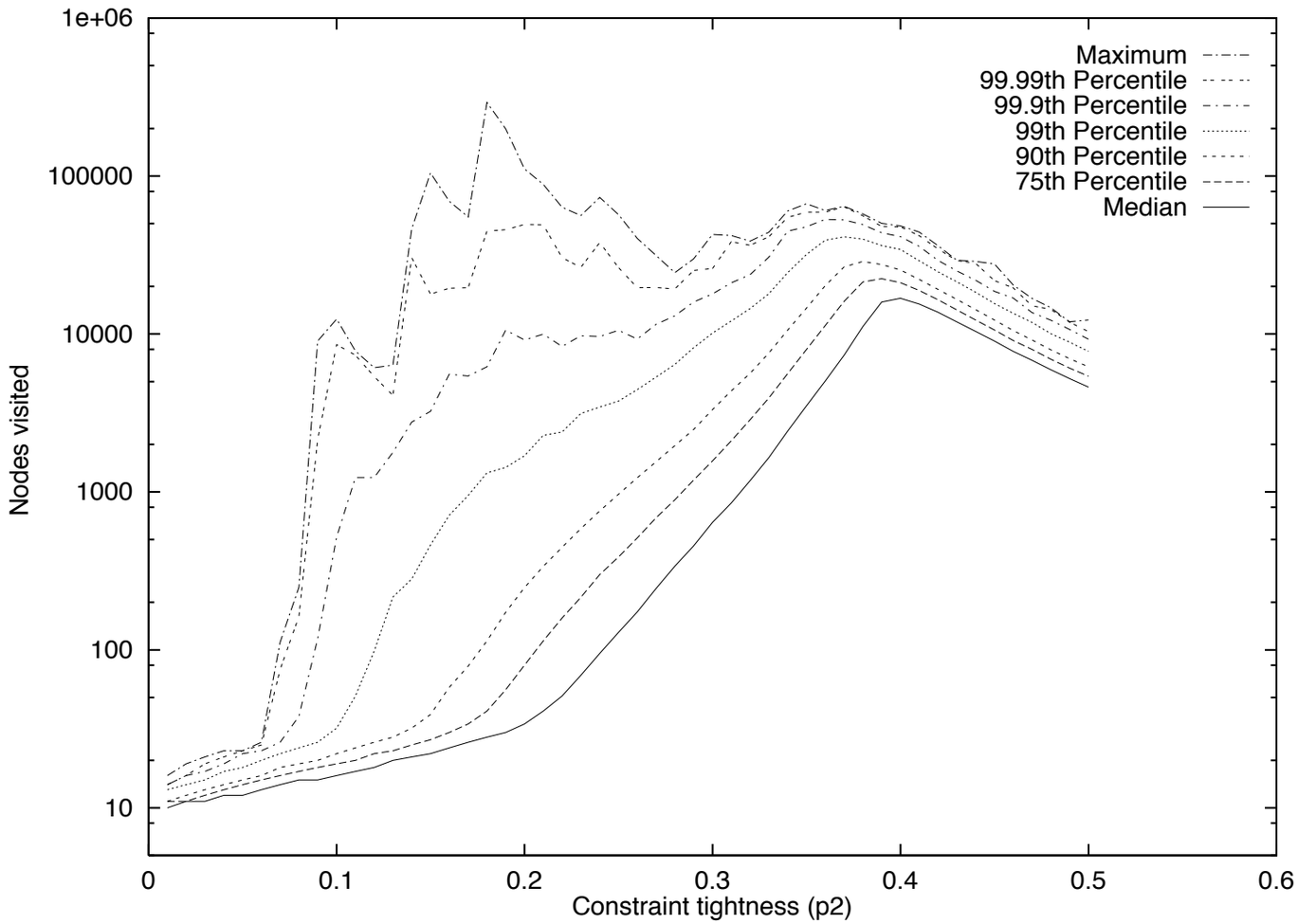
Median versus mean

- When plotting the **mean** we observe a ‘stochastic’ behavior in the easy satisfiable region

- **➜ exceptionally hard problems (ehps)**

  - in the easy sat region

  - orders of magnitude harder than hard instances (in the phase transition)



What are these EHPs?

- They are algorithm-dependent!!
- They are as easy as the other instances in the same class **but** the algorithm made a huge mistake up in the tree

  ➜ restart strategy!

Conclusion on phase transition

- Allowed us to get rid of inefficient algorithms (FC vs MAC)
- Fortunate that real problems led to the same conclusions!

Exploration Heuristics

Variable ordering heuristics

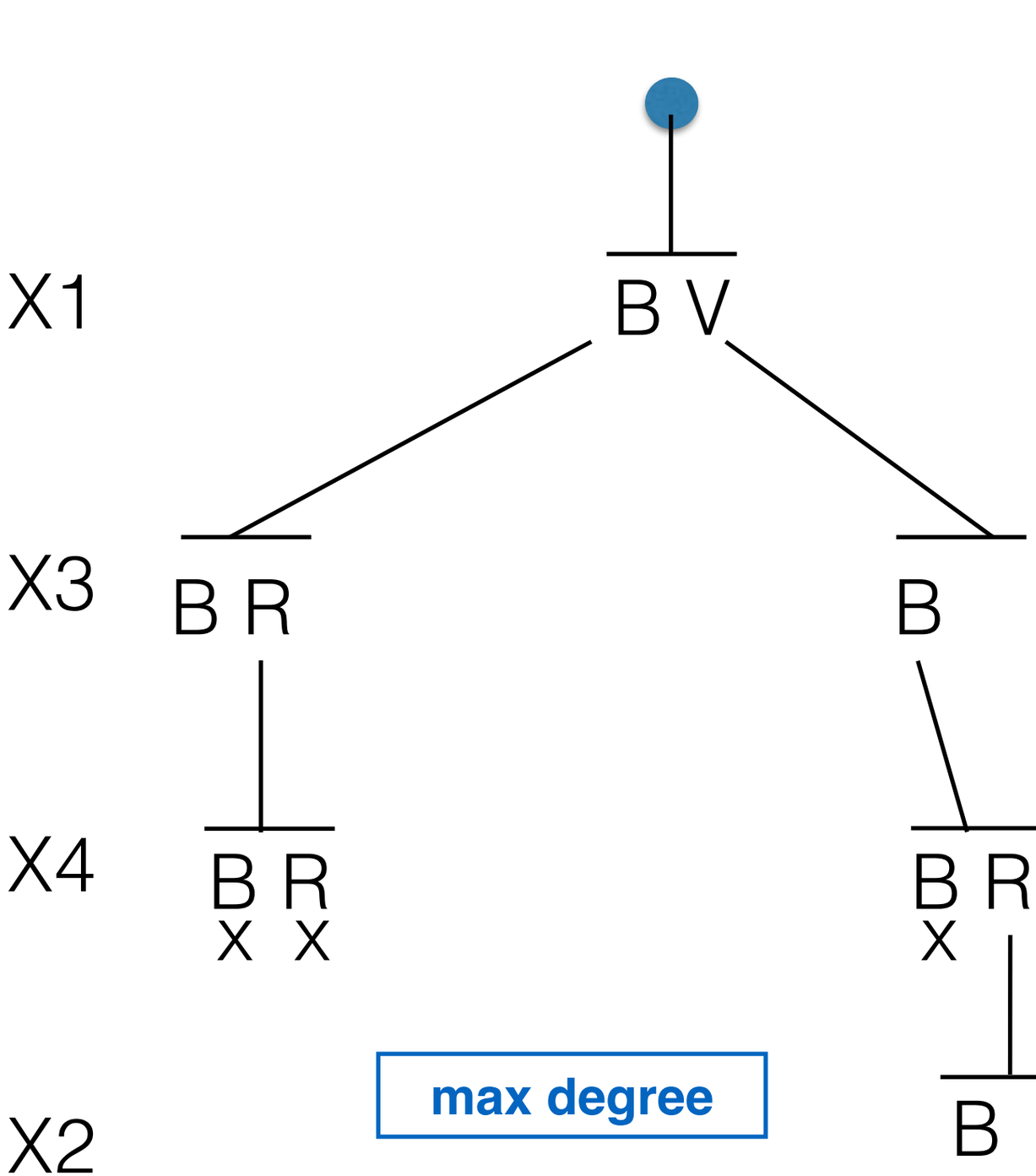
Variable ordering heuristics

- We must instantiate **all** variables
  - ➜ start by the ‘most difficult’ variables
  - ➜ reduce depth of search tree (called *fail first principle*)

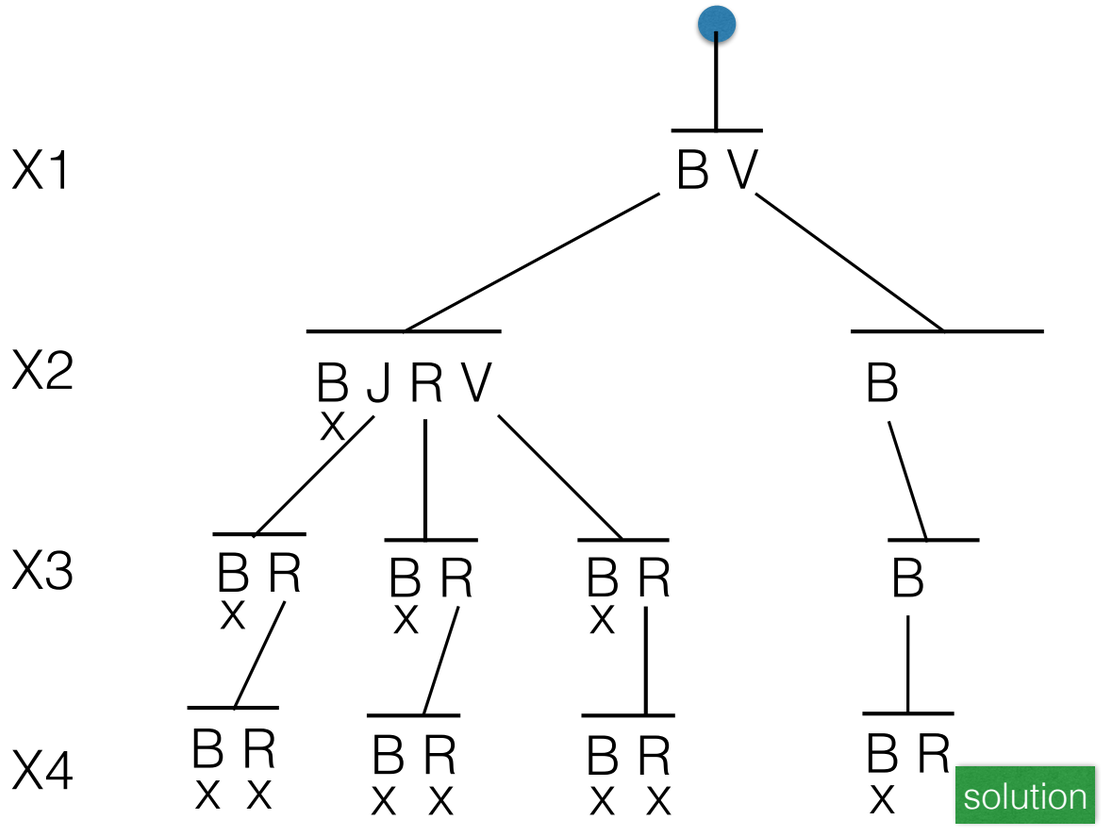
Static variable ordering (SVO)

- **deg**: select the variable with greatest degree

BT-deg



max degree



lexico

solution

Variable ordering heuristics

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Variable ordering heuristics

- We must instantiate **all** variables
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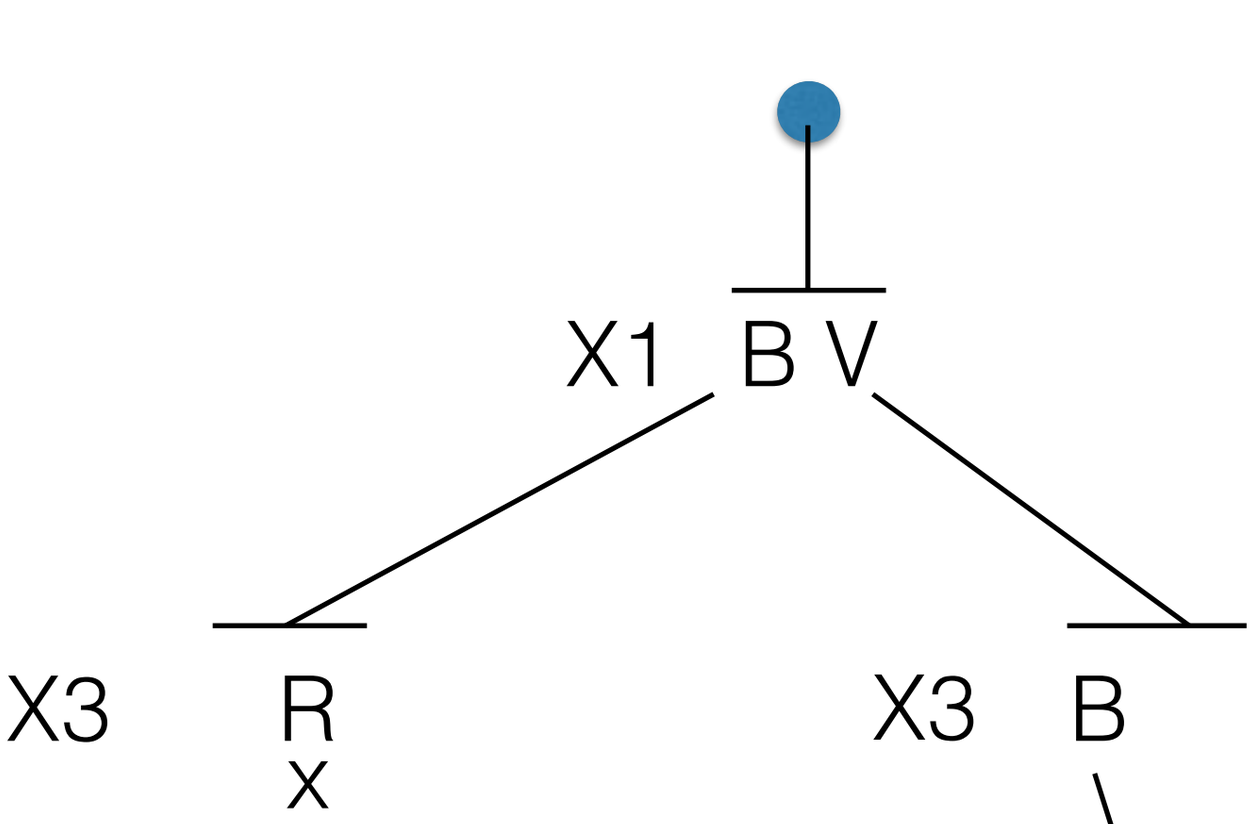
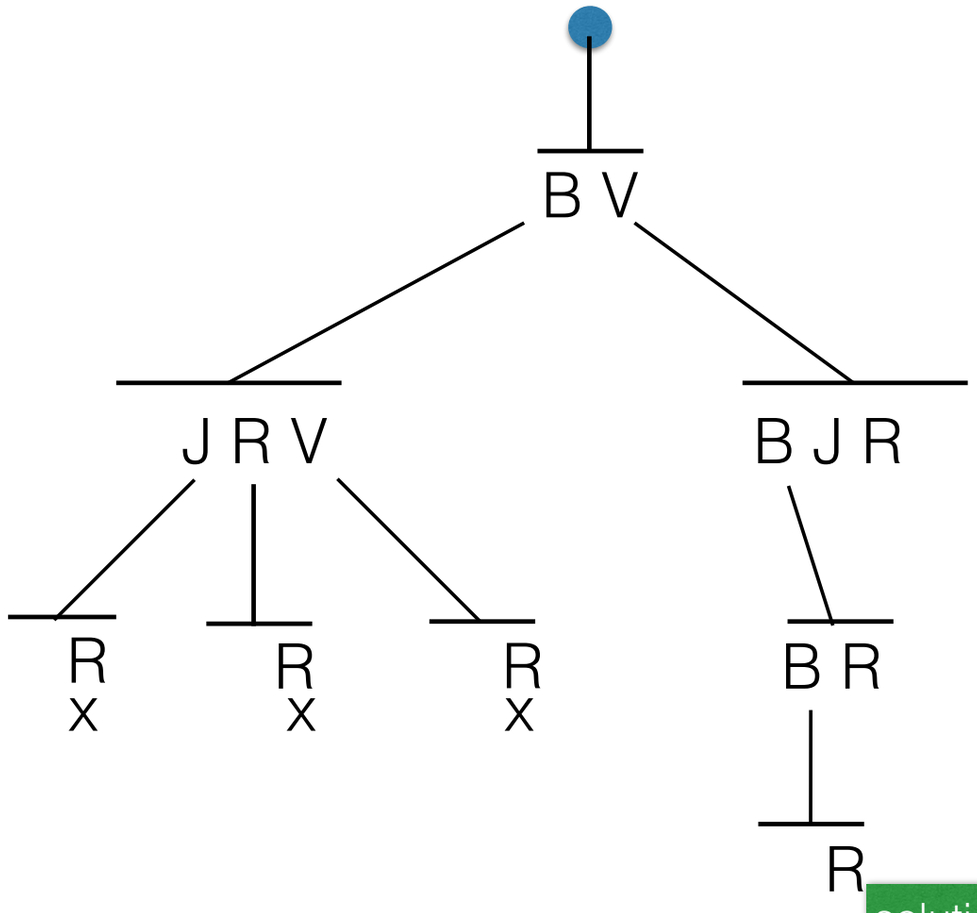
- We may need to try all values of the selected variable
  - ➜ start by small domains
  - ➜ reduce branching factor

Dynamic variable ordering (DVO)

- **dom**: select the variable with smallest domain
- dom does **not** follow the fail first principle
- …but it works well

FC-dom





solution

solution

deg versus dom

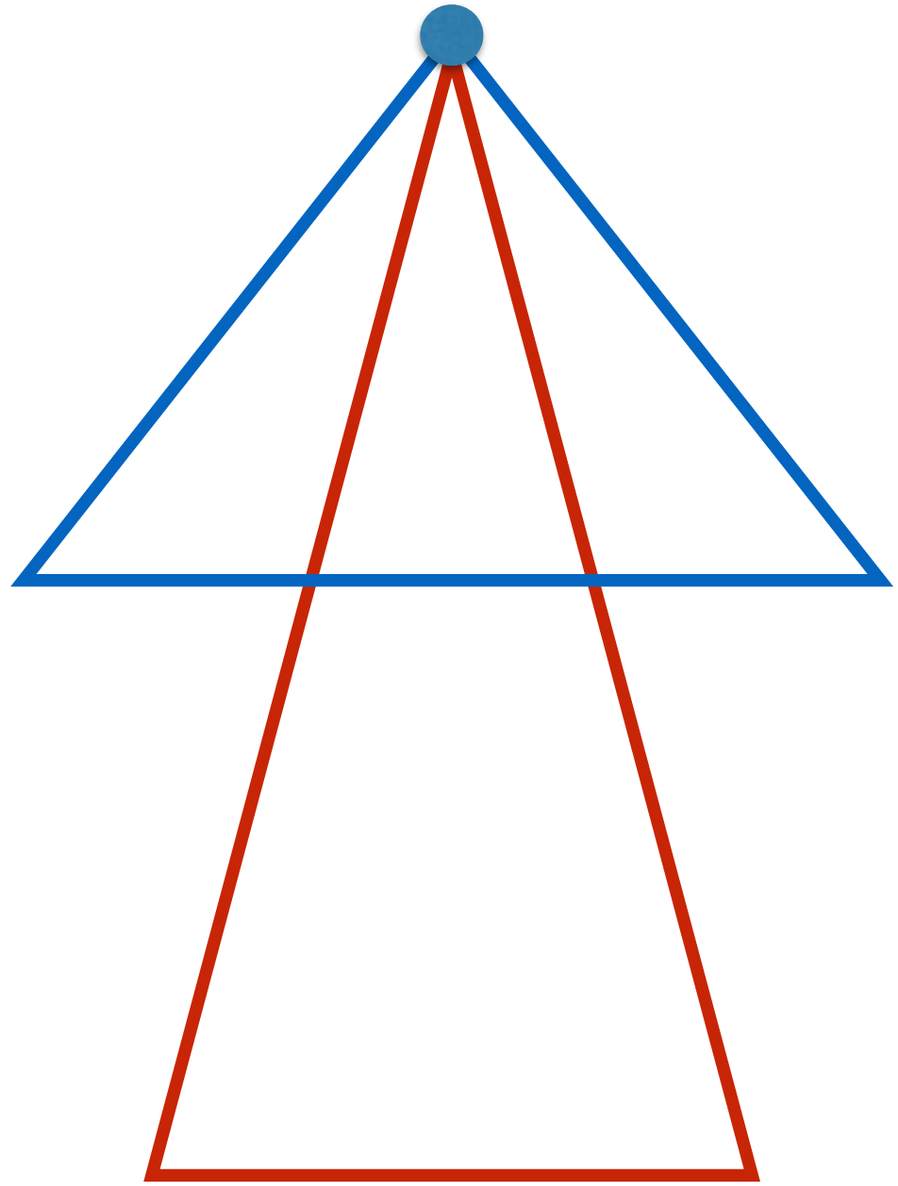
- deg reduces the ***depth*** of the search tree
- dom reduces the ***width*** of the search tree (branching factor)

- dom is good on dense networks, deg on sparse ones

Search tree shape

- difficult variables reduces the ***depth*** of the search tree

- small domains reduces the ***width*** of the search tree



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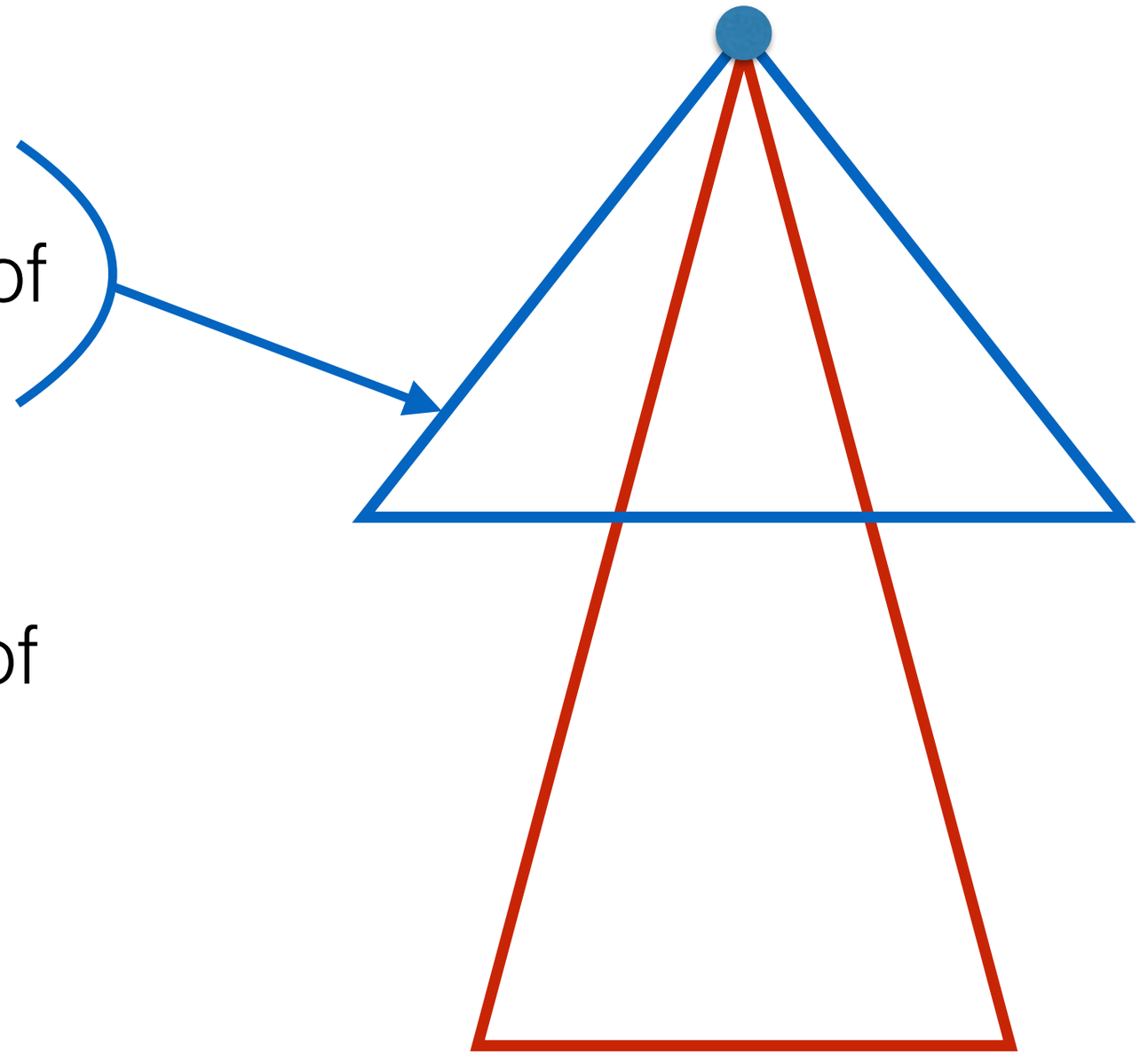

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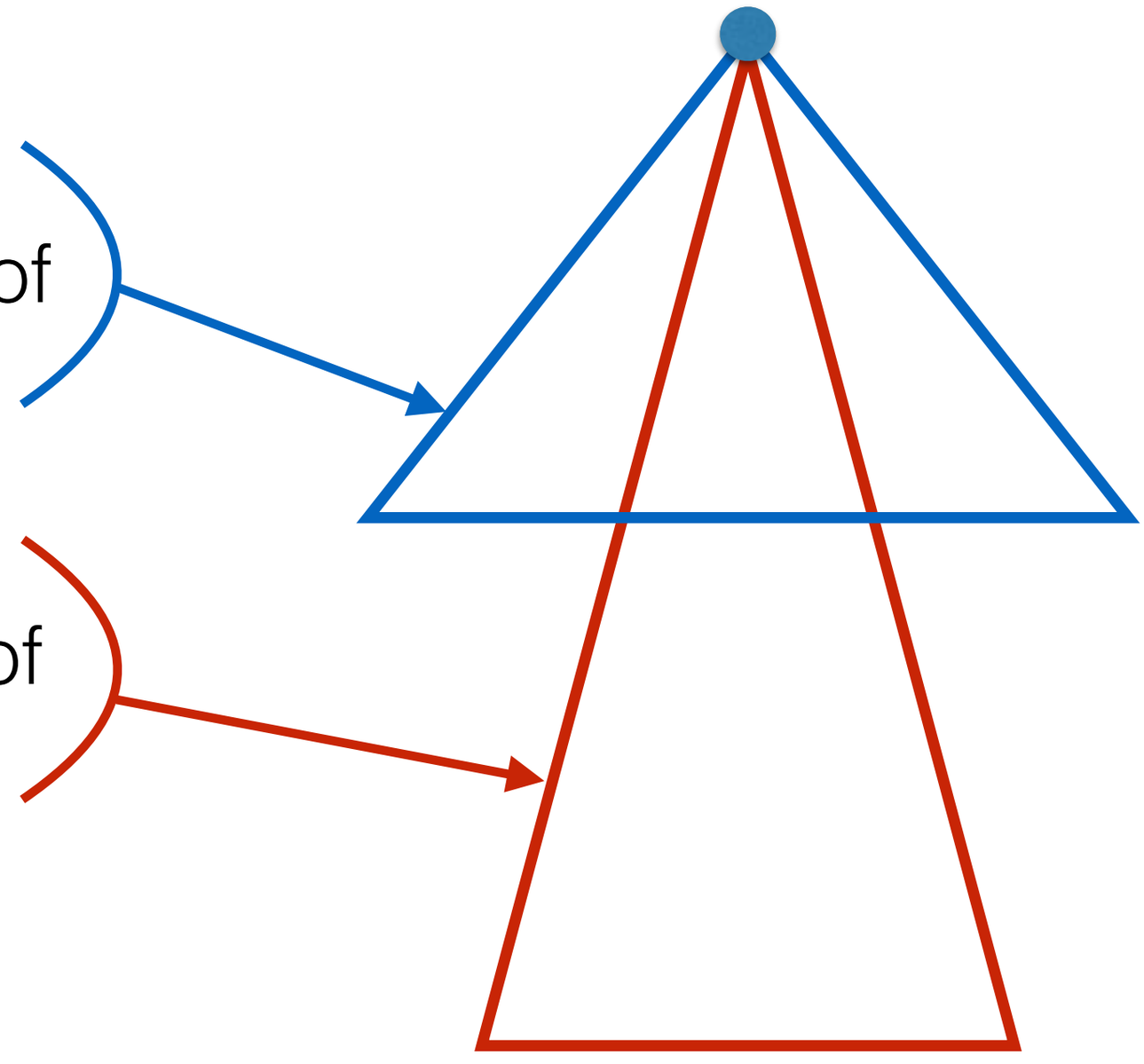
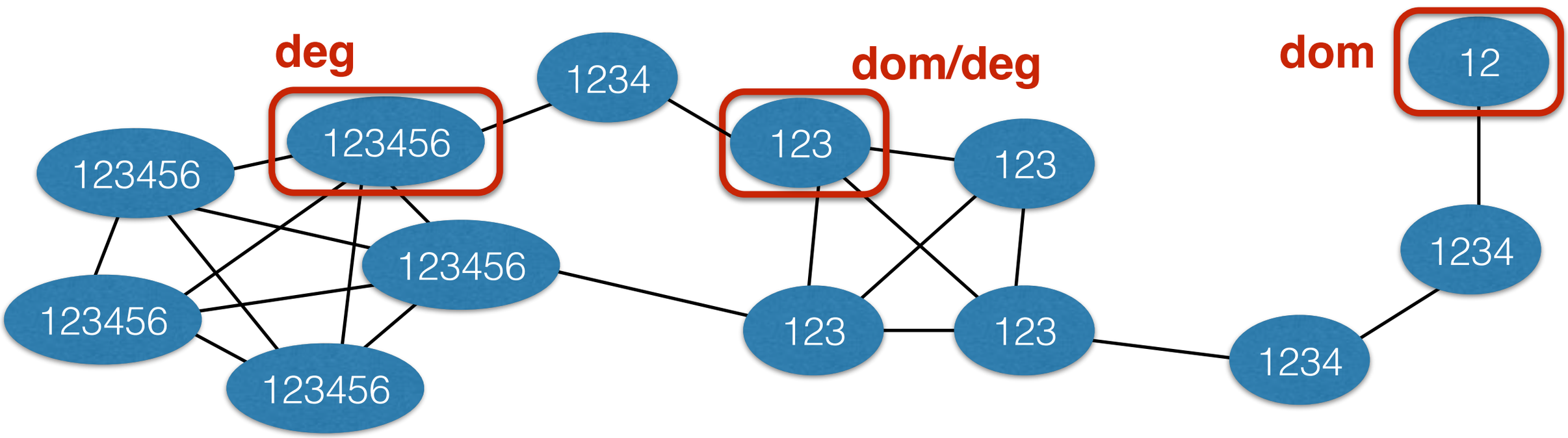


Combining dom and deg

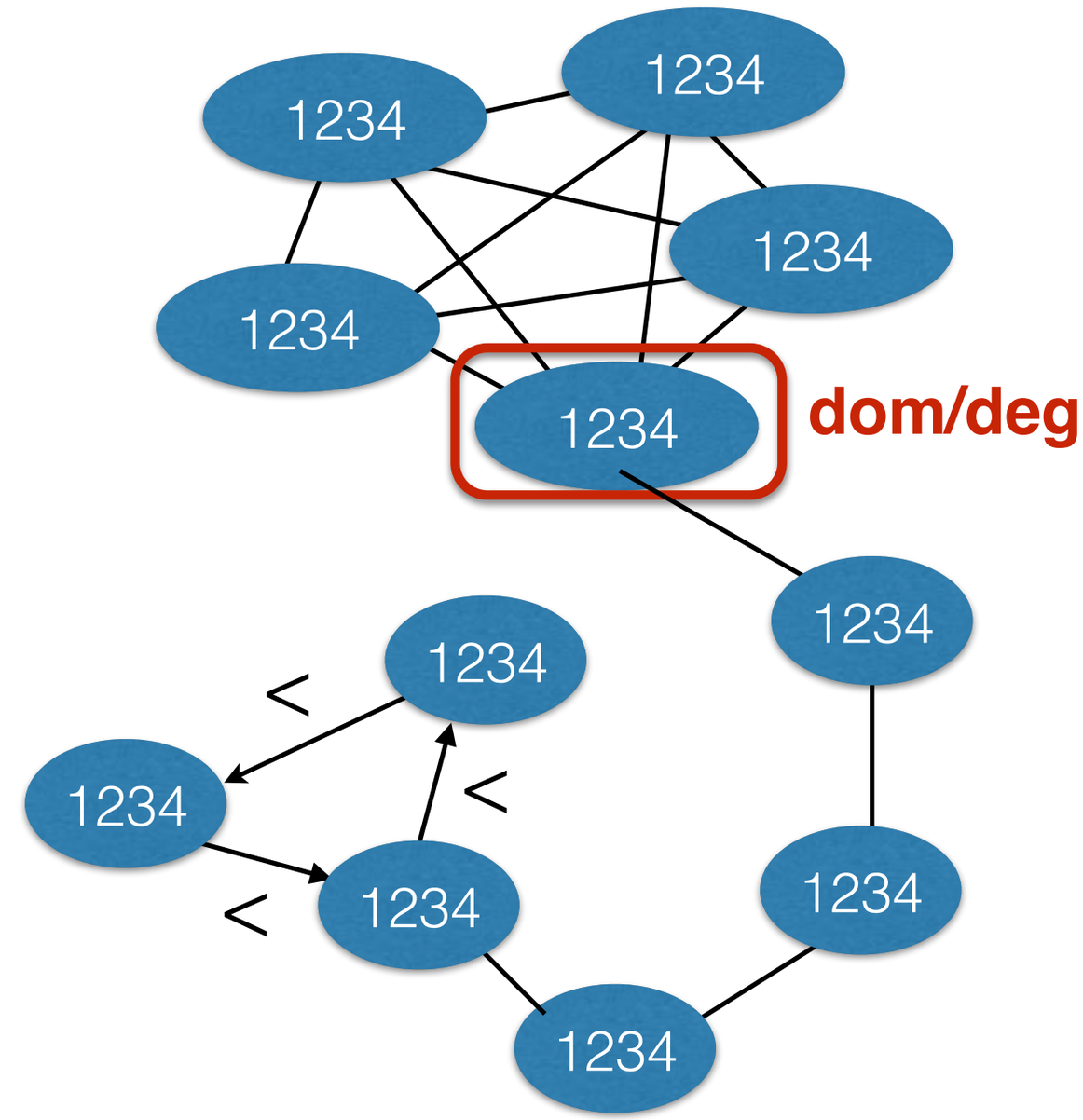
- **dom/deg**: select the variable X_i with smallest ratio: $\frac{|D(X_i)|}{|\Gamma(X_i)|}$



Reasoning with conflicts

- **dom/wdeg**:
  - Each constraint c_j is attached a weight w_j initialized to 1
  - At each fail caused by Revise(c_j), then w_j ++
  - select the variable X_i with smallest ratio

$$\frac{|D(X_i)|}{\sum_{c_j \in \Gamma(X_i)} w_j}$$



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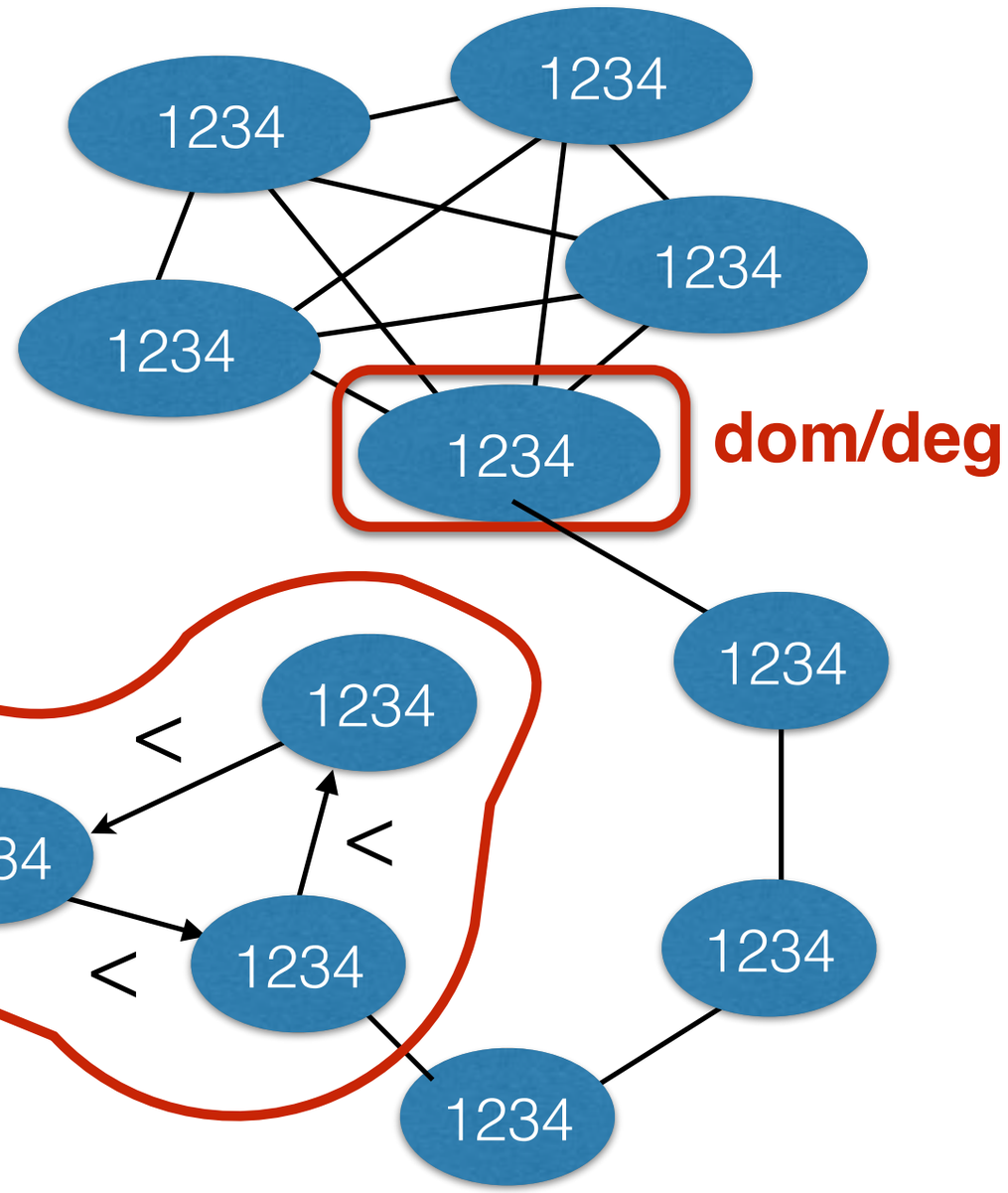

Value ordering heuristics

- We may not need to try all values of a variable to find a solution
  - ➜ start by the most promising

- Choosing a promising value is expensive (except if expert knowledge)

- On inconsistent problems value ordering is useless (Now, 95% of the time to solve hard problems is spent on inconsistent subtrees)

- ➜ Use only if ***expert knowledge available***

To take home

- Two essential components
  - Constraint propagation to reduce the size of the search tree
  - Variable ordering heuristics to pull failures as high as possible in the search tree

Domain-based
consistencies stronger than AC

Singleton Arc Consistency (SAC)

Definition 20 (Singleton arc consistency) *A network* $N = (X, D, C)$ *is* singleton arc consistent *iff for all* $X_i \in X$*, for all* $v_i \in D(X_i)$*, the network* $N + \{X_i = v_i\}$ *is not arc inconsistent (i.e.,* $AC(N + \{X_i = v_i\}) \neq \emptyset$*).*