Aide à la décision/ Decision aid

Souhila KACI

Partie 4/Part 4
Décision collective : Théorie du choix social/
Collective decision: Social choice theory

Social choice theory

What is it?

It is the study of decision problems in which a group of agents has to choose among different alternatives.

Domains of interest

Economy, Political science, Applied mathematics, Operational research

Deep results

Two Nobel distinctions: Kenneth J. Arrow (1963), Amartya Sen (1976)

Application domains

Political elections, Other elections, Multiple criteria decision, Artificial intelligence (Multiagent systems)

Notations

- Let $X = \{a, b, c, \ldots\}$ be a set of alternatives.
- A subset of X is called an agenda.
- Every individual i has a preference relation $\succeq_i$ over the set of alternatives. $\succeq_i$ is a total order (This is a simplifying hypothesis).
- A set of individuals $N = \{1, \ldots, n\}$ is called a profile. The preference relation over X associated to N is denoted by $\succeq$.

Two problems

- Determine the chosen alternative for a given profile.
- Determine the preference relation $\succeq$ for a given profile.

Our Focus: Political elections

- Alternatives $\Longrightarrow$ Candidates
- Individuals $\Longrightarrow$ Voters
- Profile $\Longrightarrow$ Society

The problem

Study election situations in which a society has to take a decision regarding several candidates.

Intuition

Democracy $\Longrightarrow$ Elections $\Longrightarrow$ Majority

Majority with two candidates: We choose the candidate who got the majority of voters' preferences

$X = \{a, b\}$. The candidate a is elected iff
$\mid \{i,\ a \succ_i b\} \mid > \mid \{i,\ b \succ_i a\} \mid$.

Majority with more than two candidates

- There are different ways to extend the majority principle.
- These extensions are not equivalent.
- Sometimes they lead to undesirable results.

Elections typology

Two criteria

1. Ballots
   - single chosen candidate in the ballots
   - rank-ordering of all candidates
   - others (determine acceptable candidates)
2. Organizing election methods and counting votes

Consequences

- Many kinds of possible elections methods
- Many of these election methods are used in practice

Plurality voting: UK (1)

Rule

- A single round
- A single chosen candidate in the ballots
- The candidate who gets the majority of votes is elected

Ex-aequo winners

- The queen decides the winner
- Older candidate
- Random choice

Plurality voting: UK (2)

Example

- 3 candidates
- 21 voters

10 voters	:	$a \succ b \succ c$
6 voters	:	$b \succ c \succ a$
5 voters	:	$c \succ b \succ a$

Results

$a : 10$	$b : 6$	$c : 5$

- a is elected
- however a majority of voters (11/21) prefers all other candidates to the elected one

Plurality voting

- Problems arise when we have more than two candidates.
- A system based on the idea of majority may violate the desire of the majority of voters.

Vote with two rounds (1)

Rules

- A single chosen candidate in the ballots
- First round
  - the candidate who gets the majority of votes is elected if she gets more than 50% of votes
  - else organize a second round
- Second round
  - consider the two candidates who got the more of votes in the first round
  - apply plurality voting

Vote with two rounds (2)

Example continued

- 3 candidates, 21 voters

10 voters	:	$a \succ b \succ c$
6 voters	:	$b \succ c \succ a$
5 voters	:	$c \succ b \succ a$

Results

$a : 10$	$b : 6$	$c : 5$

- Absolute majority: $\lceil 21/2 \rceil = 11$ votes
- a and b are considered for a second round

$a : 10$	$b : 11$

- b is elected
- no candidate is preferred to b by a majority of voters

Vote with two rounds (3)

Example with 4 candidates

- 4 candidates $\{a, b, c, d\}$, 21 voters

10 voters	:	$b \succ a \succ c \succ d$
6 voters	:	$c \succ a \succ d \succ b$
5 voters	:	$a \succ d \succ b \succ c$

Results 1st round

$a : 5$	$b : 10$	$c : 6$	$d : 0$

- Absolute majority: $\lceil 21/2 \rceil = 11$ votes
- b and c are considered for a second round

Results 2nd round

$b : 15$	$c : 6$

- b is elected $15/21$
- A majority $11/21$ prefers a to b

Vote with two rounds: Observations

Vote with one vs two rounds

- The French system just does little better than the British system.
- What about truthfulness of votes?

Vote with two rounds: Manipulability

Example with 4 candidates

- 4 candidates $\{a, b, c, d\}$, 21 voters

10 voters	:	$b \succ a \succ c \succ d$
6 voters	:	$c \succ a \succ d \succ b$
5 voters	:	$a \succ d \succ b \succ c$

Non truthful votes

- The 6 candidates for which $c \succ a \succ d \succ b$ vote as if their preferences were $a \succ c \succ d \succ b$.

Results

- a is elected in the first round 11/21.
- This is profitable for the six manipulator voters for whom $a \succ b$.

Manipulable voting rules

Definition

A voting rule is manipulable if it is possible for certain voters to better give non truthful votes.

Problems

- Elections are no longer a means to reveal preferences
  - Manipulation and counter-manipulation
  - Equilibrium
- Bonus to "astute" voters

Vote with two rounds: Monotonicity (1)

Example: before the election compaign

- 3 candidates $\{a, b, c\}$, 17 voters

6 voters	:	$a \succ b \succ c$
5 voters	:	$c \succ a \succ b$
4 voters	:	$b \succ c \succ a$
2 voters	:	$b \succ a \succ c$

Problems

- Absolute majority: $\lceil 17/2 \rceil = 9$ voters

$a : 6$	$b : 6$	$c : 5$
$a : 11$	$b : 6$	

- a is elected

Vote with two rounds: Monotonicity (2)

- a gets more money to compaign against b
- Two voters $b \succ a \succ c$ change their vote in favor of a
- Their new preference is $a \succ b \succ c$

$$\begin{array}{lcl} 6 \text{ voters} & : & a \succ b \succ c \\ 5 \text{ voters} & : & c \succ a \succ b \\ 4 \text{ voters} & : & b \succ c \succ a \\ 2 \text{ voters} & : & b \succ a \succ c \end{array} \quad \rightarrow \quad \begin{array}{lcl} 8 \text{ voters} & : & a \succ b \succ c \\ 5 \text{ voters} & : & c \succ a \succ b \\ 4 \text{ voters} & : & b \succ c \succ a \end{array}$$

$a:8$	$b:4$	$c:5$
$a:8$	$c:9$	

- c is elected. The nice compaign of a has been fatal for her.
- The two rounds majority voting is non-monotonic: increasing the position of a candidate in individual preferences may lead to a decreasing of his position at the end of the vote.

Vote with two rounds: Participation (1)

Example

- 3 candidates $\{a, b, c\}$, 11 voters

4 voters	:	$a \succ b \succ c$
4 voters	:	$c \succ b \succ a$
3 voters	:	$b \succ c \succ a$

Problems

- Absolute majority $= \lceil 11/2 \rceil = 6$ voters

$a : 4$	$b : 3$	$c : 4$
$a : 4$	$c : 7$	

- c is elected.
- The result is not satisfactory for the first 4 voters.
- Two voters among them will go fishing and abstain (for the two rounds).

Vote with two rounds: Participation (2)

Before

4 voters	:	$a \succ b \succ c$
4 voters	:	$c \succ b \succ a$
3 voters	:	$b \succ c \succ a$

After

2 voters	:	$a \succ b \succ c$
4 voters	:	$c \succ b \succ a$
3 voters	:	$b \succ c \succ a$

Results

- Absolute majority $= \lceil 11/2 \rceil = 6$ voters.

$a : 2$	$b : 3$	$c : 4$
$b : 5$	$c : 4$	

- b is elected.
- Abstention of 2 voters for whom $b \succ c$ has been rational.

Vote with two rounds: Separability (1)

Results

- 3 candidates $\{a, b, c\}$
- 26 voters in two districts

District 1

4 voters	:	$a \succ b \succ c$
3 voters	:	$b \succ a \succ c$
3 voters	:	$c \succ a \succ b$
3 voters	:	$c \succ b \succ a$
$a:4$	$b:3$	$c:6$
$a:7$	$c:6$	

- a is elected (7/13)

District 2

4 voters	:	$a \succ b \succ c$
3 voters	:	$c \succ a \succ b$
3 voters	:	$b \succ c \succ a$
3 voters	:	$b \succ a \succ c$
$a:4$	$b:6$	$c:3$
$a:7$	$b:6$	

- a is elected (7/13)

Vote with two rounds: Separability (2)

National level

4 voters	:	$a \succ b \succ c$
3 voters	:	$b \succ a \succ c$
3 voters	:	$c \succ a \succ b$
3 voters	:	$c \succ b \succ a$

4 voters	:	$a \succ b \succ c$
3 voters	:	$c \succ a \succ b$
3 voters	:	$b \succ c \succ a$
3 voters	:	$b \succ a \succ c$

$$a : 8 \quad b : 9 \quad c : 9$$

- a is eliminated in the 1st round.
- The voting method is not separable.

Summary: French voting system vs British voting system

- Voting with two rounds does just a little better than voting with one round from democracy point of view.
- However it has several problems:
  - Manipulable
  - Non-monotonic
  - No incentive for participation
  - Non separable
- Do better systems exist?
- Conventional wisdom "choose in the first round and eliminate in the second round"?

Condorcet

Principles

- Compare all pairwise candidates.
- *a* is "socially preferred" to *b* if there are strictly more voters who prefer *a* to *b* (indifference in case of equality).
- Condorcet principle: if a candidate is preferred to all other candidates then she must be elected.
- The Condorcet winner (CW) must be unique.

Remarks

- Voting systems with one or two rounds violate the Condorcet principle.
- Voting system with one round may elect a Condorcet looser.
- Condorcet principle does not resolve the problem of "majority dictature".
- The CW is not necessarily "well rank-ordered" by voters.

Condorcet & One round voting system

Example

- 3 candidates $\{a, b, c\}$
- 21 voters

10 voters	:	$a \succ b \succ c$
6 voters	:	$b \succ c \succ a$
5 voters	:	$c \succ b \succ a$

- a is winner of one round voting.
- a is Condorcet looser.
- b is Condorcet winner
  - b beats a (11/21)
  - b beats c (16/21)

Condorcet & Two rounds voting system

Example

- 4 candidates $\{a, b, c, d\}$
- 21 voters

10 voters	:	$b \succ a \succ c \succ d$
6 voters	:	$c \succ a \succ d \succ b$
5 voters	:	$a \succ d \succ b \succ c$

- b is winner of two rounds voting (against c in the second round).
- a is Condorcet winner
  - a beats b (11/21)
  - a beats c (15/21)
  - a beats d (21/21)

Condorcet & Rankings (1)

Example

- 5 candidates $\{a, b, c, d, e\}$
- 50 voters

10 voters	:	$a \succ b \succ c \succ d \succ e$
10 voters	:	$b \succ c \succ e \succ d \succ a$
10 voters	:	$e \succ a \succ b \succ c \succ d$
10 voters	:	$a \succ b \succ d \succ e \succ c$
10 voters	:	$b \succ d \succ c \succ a \succ e$

- a is Condorcet winner; it beats all other candidates (30/20)

Example

- 26 candidates $\{a, b, c, \ldots, z\}$
- 100 voters

51 voters	:	$a \succ b \succ c \succ \ldots \succ y \succ z$
49 voters	:	$z \succ b \succ c \succ \ldots \succ y \succ a$

- a is Condorcet winner
- b could be a "reasonable" choice

Consistent Condorcet rules

Definition

They are systems that elect Condorcet winner when it exists.

- Generalization of Condorcet rule

Copeland rule

Associate to each candidate a the following score: for each candidate $b \neq a$, $+1$ if a majority prefers a to b, -1 if a majority prefers b to a, 0 otherwise. The candidate who gets the higher Copeland score is elected.

Kramer-Simpson rule

Associate to each candidate a the following score: for each candidate $b \neq a$, compute $N(a, b)$ the number of voters who prefer a to b. The Simpson score is the lowest $N(a, b)$. The candidate with higher $N(a, b)$ is elected.

Example: Copeland & Kramer-Simpson rules

4 candidates $\{a, b, c, d\}$, 12 voters, 5 voters : $a \succ b \succ c \succ d$, 4 voters : $b \succ c \succ d \succ a$, 3 voters : $d \succ c \succ a \succ b$.

Copeland

Cop(a)	=		+1	-1	-1	=	-1
Cop(b)	=	-1		+1	+1	=	+1
Cop(c)	=	+1	-1		+1	=	+1
Cop(d)	=	+1	-1	-1		=	-1

Candidates b and c are elected.

Kramer-Simpson

Sim(a)	=		8	5	5	=	5
Sim(b)	=	4		9	9	=	4
Sim(c)	=	7	3		9	=	3
Sim(d)	=	7	3	3		=	3

Consistent Condorcet rules

- Copeland & Kramer-Simpson rules are monotonic.
- No consistent Condorcet rule satisfies separability.
- No consistent Condorcet rule satisfies participation.

Scoring voting rules

Definition

- Consider m candidates.
- Let $s_0 \leq \ldots \leq \ s_{m-1}$, with $s_0 < s_{m-1}$, be a non-decreasing sequence of integer numbers.
- Each voter provides a total order over candidates.
- For each voter, s_0 is associated to the worst rank-ordered candidate, ..., s_{m-1} is associated to the best rank-ordered candidate.
- The score of a candidate is the sum of all scores given by voters.
- The candidate who gets the higher score is elected.

Particular cases

- $s_0 = s_1 = \ldots s_{m-2} < \ s_{m-1}$ is the majority voting rule.
- $s_0 = 0$, $s_1 = 1$, ..., $s_{m-1} = m - 1$ is Borda rule.

Borda rule (1)

Example

4 candidates $\{a, b, c, d\}$, 3 voters

1 voter	:	$a \succ c \succ d \succ b$
2 voters	:	$b \succ a \succ c \succ d$

Borda scores

a	b	c	d
7	6	4	1

Results

- a is elected following Borda rule.
- b is the Condorcet winner.

Borda rule (2)

Properties

- No scoring voting rule is Condorcet consistent.
- Separable, monotonic, participation.