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On Viète-like formulas

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Dedicated to Felipe Zó, with admiration, respect and affection.

Abstract

The very first recorded use of an infinite product in mathematics is the so-called Viète's formula, in which each of its factors contains nested square roots of 2 with plus signs inside. Concretely, it reads

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots,$$

and it can be proved by iterating the double angle formula $\sin 2x = 2\cos x \sin x$, thus obtaining the infinite product $2/\pi = \prod_{n=2}^{\infty}\cos(\pi/2^n)$.

This paper focuses, first, on the wide variety of iterations that the identity $\cos x = 2\cos((\pi + 2x)/4)\cos((\pi - 2x)/4)$ admits; next, on the infinite products of cosines derived from these iterations and finally, on how these infinite products of cosines give rise to striking formulas.

Keywords: Viète's infinite product; Nested square roots

1. Introduction

The first recorded use of an infinite product in mathematics is the so-called *Viète's formula*, which reads

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots. \tag{1}$$

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Viète derived his formula by first obtaining the ratio of the areas of the regular polygons with n and $2n$ sides, and next doing a telescopic product of these ratios (see the Appendix below and references therein). With some minor changes, the geometric proof given by Viète can be adapted to fulfill the current rigor requirements. Nonetheless, the now standard proof was found more than a century later by Euler, who showed how (1) came from the iteration of the double angle formula $\sin 2x = 2\cos x \sin x$, from which the infinite product $2/\pi = \prod_{n=2}^{\infty} \cos(\pi/2^n)$ can be derived (thus, the only point remaining is to rewrite the factors $\cos(\pi/2^n)$ as the corresponding nested square roots of 2 by using the half angle formula $\cos(x/2) = \sqrt{2 + 2\cos x}/2$, valid for $x \in [(4k-1)\pi, (4k+1)\pi]$ with $k \in \mathbb{Z}$).

A main motivation for this research comes from one of the last (and, by the way, very amusing) contributions of Professor Nevai [13], in which he reviewed the book of S. Khrushchev on Orthogonal Polynomials and Continued Fractions; undoubtedly, this review is a must read (in fact, it led us to buy the book). In one of the many homework proposals in his review, Professor Nevai asked for proving (1). After putting us on notice that this is neither obvious nor an easy task, so we should be aware of the possibility of either cheat or defeat in doing this homework, he said (and we quote): *"Of course, it is quite possible that you will prove it on your own, in which case, hats off to you"*. We simply could not resist the temptation to attempt to gain this honor.

A second motivation comes from the reading of a recent and very exciting paper by A. Levin [9] concerning infinite products which can be obtained on the basis of the existence of a meromorphic function F satisfying $F(\alpha z) = g(F(z))$, for some complex α (with $|\alpha| > 1$) and some meromorphic function g (with $g(1) = 1$). Roughly speaking, Levin's idea consists of generalizing the double angle formula $\cos 2x = 2\cos^2 x - 1$ in order to obtain infinite products that generalize Viète's formula (1). In the last section of his paper, Levin said (again, we quote): *"It is sincerely hoped that the products presented here will inspire further discoveries of similar formulas..... In fact, the number* 2 *has been somewhat ubiquitous in the above products. One wonders what other constants have nice representations of this form"*. Well, it is a fact that Levin has inspired us and, as a consequence, we have obtained a family of constants with representations similar to the ones appearing in his paper.

The purpose of this paper is first to exploit, by successive cycles of iterations, the identity $\cos x = 2\cos((\pi + 2x)/4)\cos((\pi - 2x)/4)$ in order to get infinite products of cosines that seem to be new; next, and with the aid of a formula that we have obtained recently in [10], we transform the previous infinite products of cosines into infinite products of nested square roots of 2, having inside signs that follow a pattern.

Now we give some of the highlights from later on, picking as attention-grabber some candidates to illustrate the direction we are going and also the power of the method. For example, we establish (Example 5) that for all complex numbers x,

$$\frac{2\sqrt{2}\cos x}{\sqrt{5+\sqrt{5}}} = \prod_{n=0}^{\infty} \left(2\cos\left(\frac{c_{n+1}\pi}{4^{n+1}} + \frac{(-1)^{n+1}2x}{4^{n+1}}\right) 2\cos\left(\frac{c_{n+2}\pi}{2\cdot 4^{n+1}} + \frac{(-1)^{n+2}x}{4^{n+1}}\right)\right),$$

where $c_j = (4^j - (-1)^j)/5$ for each nonnegative integer j. From this relation we get (Example 2) the new Viète-like formula

$$\frac{2\sqrt{2}}{\sqrt{5+\sqrt{5}}} = \left(\sqrt{2}\sqrt{2-\sqrt{2}}\right)\left(\sqrt{2+\sqrt{2-\sqrt{2}}}\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}\right)\cdots.$$

We must also mention something about $\sin(\pi/7)$: this is a very striking example of what we have achieved in this paper (and, probably, it might be music to Professor Levin's ears). As well-known, $\sin(\pi/1) = 0$, $\sin(\pi/2) = 1$, $\sin(\pi/3) = \sqrt{3}/2$, $\sin(\pi/4) = 1/\sqrt{2}$, $\sin(\pi/5) = \sqrt{5-\sqrt{5}}/(2\sqrt{2})$, $\sin(\pi/6) = 1/2$. Nevertheless, it is not possible to express $\sin(\pi/7)$ in terms of sums, products, and finite root extractions of real rational numbers (see Example 3, its previous comments, and references therein). This is so because 7 is not a product of a power of 2 and distinct Fermat primes. By this reason, it may be of interest that $\sin(\pi/7)$ can be given in terms of an infinite product of nested square roots of 2 (see the last formula in Example 3).

It is worth mentioning that one of the referees has pointed out the strong connection of some of the formulas that we give in this paper with some kind of transformation of the limit

$$2\sin\left(\left(1+\sum_{n=1}^{\infty}\frac{\varepsilon_1\varepsilon_2\cdots\varepsilon_n}{2^n}\right)\frac{\pi}{4}\right) = \lim_{n\to\infty}\sqrt{2+\varepsilon_1\sqrt{2+\cdots+\varepsilon_n\sqrt{2}}},$$

which is a well-known result (see [16, pp. 221–222] and also [2]), and which can also be obtained as a limit version of our Theorem 3, in the particular case $\xi = 0$. The trick consists on multiplying the nested square roots by its algebraic conjugate recursively in order to get

$$\frac{1}{\sqrt{2}}\gamma_n\widetilde{\gamma_n} = \frac{1}{\prod_{k=0}^{n-2}\sqrt{2+\varepsilon_{n-k}\sqrt{2+\cdots+\varepsilon_n\sqrt{2}}}},$$

where $\gamma_n = \sqrt{2+\varepsilon_1\sqrt{2+\cdots+\varepsilon_n\sqrt{2}}}$ and $\widetilde{\gamma_n} = \sqrt{2-\varepsilon_1\sqrt{2+\cdots+\varepsilon_n\sqrt{2}}}$. Next, if we let the sequence $(\varepsilon_n)_{n=1}^{\infty}$ to be periodic with period p, i.e. $\varepsilon_{kp+i} = \varepsilon_i$ for $i = 1, 2, \ldots, p$, and we consider $\lim_{k\to\infty}(\gamma_{pk}\widetilde{\gamma_{pk}}/2)$, the corresponding Viète-like infinite product is obtained. In fact, some months before receiving the referee report, we had developed a very similar idea in [5,11].

Finally, let us say that from the formulas that we derive in the current paper, striking Viète-like infinite products which links Fibonacci and Lucas numbers can be obtained (see [6]).

Structure of the paper. The paper is organized as follows: In Section 2 we give the main result, namely, a general Viète-like formula, and we give a detailed explanation of how it works with some illustrative particular cases. Section 3 is devoted to develop all the machinery which will be needed to prove our main result. Roughly speaking, by iterating the formula $\cos x = 2\cos((\pi+2x)/4)\cos((\pi-2x)/4)$ in the many ways it can be done, we deduce a formula for expanding $\cos x$ as an infinite product of cosines whose arguments are certain linear functions of x; after that, and using our recent result in [10] about how to transform the cosines into nested square roots, we transform the infinite products of cosines into the so-called Viète-like formulas. In Section 4 we give the final details to complete the proof of the main result. An extension of the original very classical Viète's formula is analyzed, as a spurious case, in Section 5. When studying Viète's original paper, we reach some conclusions that we would like to share; this is why we include a last section as an Appendix, in which we revise how Viète deduced his famous infinite product, how easy and elegant his idea is, how easily Viète's deduction can be made rigorous, and also how Viète made a naive mistake which, to the best of our knowledge, has not been pointed out in the literature.

Conventions and terminologies. In the sequel, the variables x, z are assumed to be complex. Also, the symbol $\sqrt{\cdot}$ will stand for the principal value of the complex square root function. Thus, for a nonnegative real x, $\sqrt{x}$ stands for its nonnegative square root, and for a complex $z \neq 0$, $\sqrt{z} = \sqrt{|z|}e^{i\frac{\mathrm{Arg}(z)}{2}}$, where $|\cdot|$ is the modulus function and $\mathrm{Arg}(\cdot)$ is the principal value of the argument function, which takes values in the interval $(-\pi, \pi]$. The overline symbol, $\bar{\cdot}$ will not denote complex conjugation: it must be understood as in Definition 1 (see Section 3.1). As usual, $\mathbb{C}$ denotes the set of complex numbers and $m \bmod n$ will stand for the remainder on division of m by n.

In closing, let us say that this paper can be considered as a continuation of our previous work [12].

2. Main result

Our aim is to provide an "à la carte" result for obtaining Viète-like formulas:

- You choose, first, a positive integer N.
- Next you choose a particular sequence of N signs b_i (either 1 or -1) with only one restriction: there must be at least one sign -1 in your choice.
- Then our result evaluates some constants depending on the chosen sequence of signs you have made. After that, our result constructs an infinite sequence of signs $\widehat{b_i}$ by extending periodically the chosen finite sequence of signs b_i, namely

$$(\widehat{b_n})_{n=1}^{\infty} = (\underbrace{b_1, b_2, \ldots, b_N}, \underbrace{b_1, b_2, \ldots, b_N}, \ldots, \underbrace{b_1, b_2, \ldots, b_N}, \ldots).$$

Finally, and for each complex number z, you get a closed form expression for the infinite product

$$\sqrt{2 \pm z}\sqrt{2 \pm \sqrt{2 \pm z}}\sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm z}}} \cdots, \tag{2}$$

in which

(i) the nth factor is the nested square root

$$\underbrace{\sqrt{2 \pm \sqrt{2 \pm \sqrt{2 + \cdots \pm \sqrt{2 \pm z}}}},}_{n \text{ square roots}}$$

such that the signs inside, written from right to left, are the first n signs in the sequence $(\widehat{b_n})_{n=1}^{\infty}$ described above. (For example, for $N = 3$ and $(b_1, b_2, b_3) = (1, 1, -1)$, the fourth factor is $\sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 + z}}}}$.)

(ii) for convergence purposes, and starting from the left, we must group each sequence of N consecutive factors in the infinite product (2).

Now we give the result, illustrating it with some examples.

Theorem 1. *Let N be a positive integer and let $(b_1, b_2, \ldots, b_N)$ be a sequence such that each b_i equals either 1 or -1 and with not all the b_is equal to 1. Define*

$$\alpha = \sum_{i=1}^{N} \left(\prod_{j=1}^{i-1} b_{N-j+1} \right) 2^{N-i} \qquad \left(\textit{with the understanding that } \prod_{j=1}^{0} = 1 \right),$$

$$\sigma = \prod_{j=1}^{N} b_{N-j+1},$$
$$\widehat{b}_k = b_{1+(k-1) \bmod N}, \quad k = 1, 2, \ldots.$$

If for each complex number z we recursively define

$$\rho_0(z) = z, \qquad \rho_i(z) = \sqrt{2 + \widehat{b}_i \rho_{i-1}(z)}, \quad i \geq 1,$$

then we have

$$\frac{\sqrt{1 - \frac{z^2}{4}}}{\cos\left(\frac{\alpha\,\sigma}{2^N\sigma - 1}\,\frac{\pi}{2}\right)} = \prod_{j=0}^{\infty}\left(\prod_{i=jN+1}^{(j+1)N} \rho_i(z)\right). \tag{3}$$

Note that $\sigma = \pm 1$, i.e. $\sigma^2 = |\sigma| = 1$. Now we illustrate how Theorem 1 works by giving some examples. The simplest case of Theorem 1 corresponds to a sequence with only the minus sign:

Example 1. With the choice $N = 1$, we must define $b_1 = -1$. This gives $\alpha = 1$, $\sigma = -1$ and, accordingly,

$$\cos\left(\frac{\alpha\,\sigma}{2^N\sigma - 1}\,\frac{\pi}{2}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

In addition, the sequence of signs that will appear inside the nested square roots (written from the right to the left) is:

$$(\widehat{b}_k)_{k=1}^{\infty} = (-1, -1, -1, \ldots).$$

Taking into account that

$$\prod_{j=0}^{\infty}\left(\prod_{i=jN+1}^{(j+1)N} \rho_i(z)\right) = \prod_{j=0}^{\infty}\left(\prod_{i=j+1}^{j+1} \rho_i(z)\right) = \prod_{j=1}^{\infty} \rho_j(z),$$

then Theorem 1 reads

$$\sqrt{\frac{4 - z^2}{3}} = \sqrt{2 - z}\sqrt{2 - \sqrt{2 - z}}\sqrt{2 - \sqrt{2 - \sqrt{2 - z}}}\cdots. \tag{4}$$

Picking $z = 0$ in formula above, one gets the remarkable Viète-like formula

$$\sqrt{3} = 2\left(\sqrt{2}\sqrt{2 - \sqrt{2}}\sqrt{2 - \sqrt{2 - \sqrt{2}}}\cdots\right)^{-1}. \tag{5}$$

Formula (4) is not new (neither is (5)). Indeed, it has been recently obtained by A. Levin [9, formulas (48) and (50)] with a different proof. Nevertheless, we believe that an additional advantage of our method is that, with very few extra effort, we can give the *remainder factor* for all the Viète-like formulas that we are going to establish in the sequel. In particular, the truncated version of the infinite product expansion (4) reads

$$\sqrt{\frac{4-z^2}{3}} = \sqrt{2-z}\sqrt{2-\sqrt{2-z}}\sqrt{2-\sqrt{2-\sqrt{2-z}}}\cdots$$
$$\times \underbrace{\sqrt{2-\sqrt{2-\sqrt{2-\cdots\sqrt{2-z}}}}}_{n \text{ square roots}} \underbrace{\left(\frac{1}{\sqrt{3}}\underbrace{\sqrt{2+\sqrt{2-\sqrt{2-\cdots\sqrt{2-z}}}}}_{n \text{ square roots}}\right)}_{\text{remainder factor}}.$$

Although the Viète-like formulas with remainder factors are not the central aim of this work, let us say that they can be obtained from Propositions 1 and 3, using also Theorem 3 to write the factor $\langle v^k \rangle$ in (8) as a nested square root.

If we now choose $N = 2$ in Theorem 1, the two associated sequences are $(b_1, b_2) = (1, -1)$ and $(b_1, b_2) = (-1, 1)$. (The case $(b_1, b_2) = (-1, -1)$ can be obtained from the case $(b_1) = (-1)$ analyzed in Example 1, by grouping each pair of consecutive factors in the infinite expansion (4).)

Example 2. Now consider $N = 2$ and $(b_1, b_2) = (1, -1)$. In such a case, $\alpha = 1$, $\sigma = -1$ and thus

$$\cos\left(\frac{\alpha\,\sigma}{2^N\sigma - 1}\,\frac{\pi}{2}\right) = \cos\frac{\pi}{10} = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}.$$

In this case, the signs to be placed from right to left inside the nested square roots are given by

$$(\widehat{b_k})_{k=1}^{\infty} = (\underbrace{1, -1}, \underbrace{1, -1}, \underbrace{1, -1}, \ldots),$$

so noting that

$$\prod_{j=0}^{\infty}\left(\prod_{i=jN+1}^{(j+1)N} \rho_i(z)\right) = \prod_{j=0}^{\infty}\left(\prod_{i=2j+1}^{2j+2} \rho_i(z)\right) = \prod_{j=1}^{\infty}\left(\rho_{2j-1}(z)\,\rho_{2j}(z)\right),$$

we can establish, from Theorem 1, a Viète-like formula which seems to have appeared for the first time in the literature in [12, Section 4]:

$$\frac{\sqrt{2(4-z^2)}}{\sqrt{5+\sqrt{5}}}$$
$$= \left(\sqrt{2+z}\sqrt{2-\sqrt{2+z}}\right)\left(\sqrt{2+\sqrt{2-\sqrt{2+z}}}\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+z}}}}\right)$$
$$\times\left(\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+z}}}}}\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+z}}}}}}\right)\cdots.$$

We can follow in the same manner for increasing values for N. But it may happen that the denominator in the left hand side of (3), namely $\cos(\alpha\sigma\pi/((2^N\sigma-1)\pi))$, does not have a closed form expression. We recall that $\sin(\pi/n)$ can be expressed in terms of sums, products, and finite root extractions on real rational numbers if and only if n is a product of a power of 2 and distinct Fermat primes 3, 5, 17, 257, ... (see [7,20]). For this reason we think that it can be of some interest to write down the following particular case of Theorem 1, which give a closed form expression for the sin value of the first "problematic angle" $\pi/7$ in terms of nested square roots of 2:

Example 3. Set $N=3$, and consider $(b_1,b_2,b_3)=(-1,-1,1)$. Clearly $\alpha=5$, $\sigma=1$, and

$$\cos\left(\frac{\alpha\,\sigma}{2^N\sigma-1}\,\frac{\pi}{2}\right)=\cos\left(\frac{5\pi}{14}\right)=\sin\frac{\pi}{7}.$$

Noting that

$$(\widehat{b_k})_{k=1}^{\infty}=(\underbrace{-1,-1,1},\underbrace{-1,-1,1},\underbrace{-1,-1,1},\ldots),$$

from Theorem 1, with the choice $z=0$, we deduce (cf. [21,3])

$$\sin\frac{\pi}{7}=\left(\left(\sqrt{2}\sqrt{2-\sqrt{2}}\sqrt{2+\sqrt{2-\sqrt{2}}}\right)\left(\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}\right.\right.$$
$$\left.\left.\times\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}}\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}}}\right)\cdots\right)^{-1}.$$

In closing this section, let us recall the legendary famous episode about a young Gauss and his discovery that the regular heptadecagon (17-gon) could be constructed with straight lines and circles. At his nineteen years, and by this reason, he was so pleased that he decided to dedicate his life to Mathematics instead of dedicating it to the study of languages, his second alternative. Gauss' proof relies on the possibility of expressing $\sin(\pi/17)$ in terms of sums, products, and finite root extractions of real rational numbers (see [7,22] for the explicit formulas). We invite the interested reader to find a "more regular" formula for $\sin(\pi/17)$ in terms of nested square roots of 2 (Hint: consider $N=4$, $(b_1,b_2,b_3,b_4)=(-1,1,1,1)$ and $z=0$ in Theorem 1).

3. Two key tools towards the main result

This section is divided in four blocks. In the first part we introduce the notation and terminology that will be necessary to develop a description of the variety of possibilities when iterating the formula $\cos x=2\cos((\pi+2x)/4)\cos((\pi-2x)/4)$. The second part is devoted to deduce a formula for expanding $\cos x$ as (a constant times) an infinite product of cosines whose arguments are certain linear functions of x. In the third part we give, without proof, a result that we have recently obtained and which will be necessary for the fourth part. Finally, the last part is concerned with the technique to transform the cosines into nested square roots, which will be the way for obtaining, from infinite products of cosines, infinite products of nested square roots (the so-called Viète-like formulas).

3.1. The iterative scheme

Our aim consists of analyzing and describing the variety of possibilities that exist when iterating the well-known formula

$$\cos x = 2\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right). \tag{6}$$

Roughly speaking, the distinct expressions obtained after k iterations of formula above come from the particular choice that can be done in each step of the procedure. This means that in each step one decides which of the functions $\cos(\pi/4 + w_j(x)/2)$ or $\cos(\pi/4 - w_j(x)/2)$ will be replaced, after applying formula (6), by two times the product of two new cosines $\cos(\pi/4 + w_{j+1}(x)/2)$ and $\cos(\pi/4 - w_{j+1}(x)/2)$. For example, if one decides that after each iteration the factor $\cos(\pi/4 - \cdot/2)$ is the one to be used for running the next iteration, the first three steps of the process give us

$$\begin{aligned}\cos x &= \left(2\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)\\ &= \left(2^2\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{3\pi}{8}-\frac{x}{4}\right)\right)\cos\left(\frac{\pi}{8}+\frac{x}{4}\right)\\ &= \left(2^3\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{3\pi}{8}-\frac{x}{4}\right)\cos\left(\frac{5\pi}{16}+\frac{x}{8}\right)\right)\cos\left(\frac{3\pi}{16}-\frac{x}{8}\right).\end{aligned}$$

For the sake of clarity and simplicity we introduce the following definitions and notations:

Definition 1. Let $b \in \{-1, 1\}$. The basic function $f_b : \mathbb{C} \to \mathbb{C}$ is defined by means of

$$f_b(x) = \frac{\pi}{4} + b\,\frac{x}{2}. \tag{7}$$

The conjugate basic function $\overline{f_b} : \mathbb{C} \to \mathbb{C}$ is the basic function defined by $\overline{f_b} = f_{-b}$.

Definition 2. Let N be a positive integer and let $\{u_i\}_{i=1}^N$ be a sequence of basic functions. Fixed $x \in \mathbb{R}$ we define

$$\langle u_1u_2\cdots u_N\rangle = \cos\left(u_1(u_2(\cdots u_{N-1}(u_N(x))\cdots))\right).$$

In the same fashion, if v stands for a composition of a finite number of basic functions, and k is a nonnegative integer, we define

$$\langle u_1u_2\cdots u_N v^k\rangle = \cos\big(u_1(u_2(\cdots u_N(\underbrace{v(v(\cdots v}_{k\text{ times}}(x)\cdots)))\cdots))\big).$$

Now we analyze in detail how to develop the different iterative schemes for (6). Before running multiple iterations over (6) we must choose a sequence $(b_1, b_2, \ldots, b_N)$ for which each b_i equals either -1 or 1. This sequence acts as a pattern of signs for selecting which one of the two cosines will be kept fixed and which one will be used to run the next iteration. To be precise:

Proposition 1. *For a fixed positive integer N, we choose a sequence $(b_1, b_2, \ldots, b_N)$ such that $b_i \in \{-1, 1\}$ for $i = 1, 2, \ldots, N$. If $(u_1, u_2, \ldots, u_N)$ is the sequence of basic functions in which $u_i = f_{b_{N-i+1}}$ for $i = 1, 2, \ldots, N$, then for each positive integer k*

$$\cos x = 2^{kN} \left(\prod_{j=0}^{k-1} \left(\prod_{i=1}^{N} \langle v_i^j \rangle \right) \right) \langle v^k \rangle, \tag{8}$$

where $v = u_1 \circ u_2 \circ \cdots \circ u_N$ and $\langle v_i^j \rangle = \langle \overline{u_{N-i+1}}\, u_{N-i+2}\, u_{N-i+3} \cdots u_N\, v^j \rangle$, with $1 \le i \le N$ and $j \ge 0$.

Proof. The point is that each iteration of formula (6) transforms a factor $\cos(\cdot)$ into the factors $2\cos(u(\cdot))\cos(\overline{u}(\cdot))$, u being a basic function. Since (6) reads $\cos x = 2\cos(\overline{u_N}(x))\cos(u_N(x))$, we start with $\cos(u_N(x))$ for running the first iteration. Note that u_N is the basic function associated with b_1. Among the two factors $\cos(\overline{u_{N-1}}(u_N(x)))$ and $\cos(u_{N-1}(u_N(x)))$ that the first iteration generates, we pick $\cos(u_{N-1}(\cdot))$ to run the second iteration (this selection is done because of the connection of u_{N-1} with b_2). We proceed in like manner, choosing in the ith step the factor $\cos(u_{N-i+1}(\cdot))$, $i = 1, 2, \ldots, N$. After N iterations we will have completed a cycle, and the process starts again. When $2N$ iterations are done, a second cycle will have been completed, and thus the process reinitializes $\cdots$ and so on. This means that for each positive integer i, the ith step of the process consists of running formula (6) with the factor $\cos(u_{N-((i-1) \bmod N)}(\cdot))$, associated with $b_{1+((i-1) \bmod N)}$. (As usual $m \bmod n$ will stand for the remainder on division of m by n.)

The proof is by induction on k.

Since u_N is a basic function, (6) can be rewritten as

$$\cos x = 2\cos(\overline{u_N}(x))\cos(u_N(x)) = (2\langle \overline{u_N} \rangle)\, \langle u_N \rangle. \tag{9}$$

Now note that u_{N-1} is again a basic function. Thus, (6) yields

$$\cos(u_N(x)) = 2\cos(\overline{u_{N-1}}(u_N(x)))\cos(u_{N-1}(u_N(x))). \tag{10}$$

By (10), formula (9) transforms to

$$\begin{aligned} \cos x &= 2^2 \cos(\overline{u_N}(x))\cos(\overline{u_{N-1}}(u_N(x)))\cos(u_{N-1}(u_N(x))) \\ &= \left(2^2 \langle \overline{u_N} \rangle \langle \overline{u_{N-1}}\, u_N \rangle\right) \langle u_{N-1}\, u_N \rangle. \end{aligned} \tag{11}$$

Once more

$$\cos(u_{N-1}(u_N(x))) = 2\cos(\overline{u_{N-2}}(u_{N-1}(u_N(x))))\cos(u_{N-2}(u_{N-1}(u_N(x)))), \tag{12}$$

so replacing (12) in (11) we get

$$\begin{aligned} \cos x &= 2^3 \cos(\overline{u_N}(x))\cos(\overline{u_{N-1}}(u_N(x)))\cos(\overline{u_{N-2}}(u_{N-1}(u_N(x)))) \\ &\quad \times \cos(u_{N-2}(u_{N-1}(u_N(x)))) \\ &= \left(2^3 \langle \overline{u_N} \rangle \langle \overline{u_{N-1}}\, u_N \rangle \langle \overline{u_{N-2}}\, u_{N-1}\, u_N \rangle\right) \langle u_{N-2}\, u_{N-1}\, u_N \rangle. \end{aligned}$$

Following in the same manner, after N iterations we have

$$\begin{aligned}\cos x &= \left(2^N \langle \overline{u_N}\rangle \langle \overline{u_{N-1}}\, u_N\rangle \cdots \langle \overline{u_1}\, u_2 \cdots u_N\rangle\right) \langle u_1\, u_2 \cdots u_N\rangle \\ &= \left(2^N \langle \overline{u_N}\rangle \langle \overline{u_{N-1}}\, u_N\rangle \cdots \langle \overline{u_1}\, u_2 \cdots u_N\rangle\right) \langle v\rangle \\ &= 2^N \left(\prod_{i=1}^{N} \langle v_i^0\rangle\right) \langle v\rangle, \end{aligned} \tag{13}$$

and so (8) holds for $k = 1$.

Assuming (8) to hold for $k = p$, we will prove it for $p + 1$.

Suppose that

$$\cos x = 2^{pN} \left(\prod_{j=0}^{p-1} \left(\prod_{i=1}^{N} \langle v_i^j\rangle\right)\right) \langle v^p\rangle$$

holds for some positive integer p. Following a similar procedure as above, we get

$$\begin{aligned}\cos x &= 2^{pN+1} \left(\left(\prod_{j=0}^{p-1} \left(\prod_{i=1}^{N} \langle v_i^j\rangle\right)\right) \langle \overline{u_N}\, v^p\rangle\right) \langle u_N\, v^p\rangle \\ &= 2^{pN+2} \left(\left(\prod_{j=0}^{p-1} \left(\prod_{i=1}^{N} \langle v_i^j\rangle\right)\right) \langle \overline{u_N}\, v^p\rangle \langle \overline{u_{N-1}}\, u_N\, v^p\rangle\right) \langle u_{N-1} u_N\, v^p\rangle \\ &= \cdots \\ &= 2^{pN+N} \left(\left(\prod_{j=0}^{p-1} \left(\prod_{i=1}^{N} \langle v_i^j\rangle\right)\right) \langle \overline{u_N}\, v^p\rangle \langle \overline{u_{N-1}}\, u_N\, v^p\rangle \cdots \langle \overline{u_1}\, u_2 \cdots u_N\, v^p\rangle\right) \\ &\quad \times \langle u_1\, u_2 \cdots u_N\, v^p\rangle \\ &= 2^{(p+1)N} \left(\left(\prod_{j=0}^{p-1} \left(\prod_{i=1}^{N} \langle v_i^j\rangle\right)\right) \prod_{i=1}^{N} \langle v_i^p\rangle\right) \langle v^{p+1}\rangle \\ &= 2^{(p+1)N} \left(\prod_{j=0}^{p} \left(\prod_{i=1}^{N} \langle v_i^j\rangle\right)\right) \langle v^{p+1}\rangle, \end{aligned}$$

which completes the proof. □

3.2. *First key tool: infinite products of cosines*

Now our purpose is to expand $\cos x$ as certain infinite products of cosines. For this to be done we might take the limit as k tends to infinite in formula (8), this limit being assumed to exist. Therefore, we are left with the task of determining if an explicit expression of $\langle v^k\rangle$ can be given in such a way that $\lim_{k\to\infty} \langle v^k\rangle$ may be computable. In an attempt to do this, we first establish:

Lemma 1. *Let N be a positive integer and let $(b_1, b_2, \ldots, b_N)$ be a sequence such that $b_i \in \{-1, 1\}$ for $i = 1, 2, \ldots, N$. Let u_i denote the basic function defined by means of*

$u_i(x) = f_{b_{N-i+1}}(x)$ ($i = 1, 2, \ldots, N$), and set $v = u_1 \circ u_2 \circ \cdots \circ u_N$. We have

$$v(x) = \left(\sum_{i=1}^{N}\left(\prod_{j=1}^{i-1} b_{N-j+1}\right)2^{N-i}\right)\frac{\pi}{2^{N+1}} + \left(\prod_{j=1}^{N} b_{N-j+1}\right)\frac{x}{2^N}, \tag{14}$$

with the understanding that the empty product $\prod_{j=1}^{0}$ is equal to 1.

Proof. The proof goes by induction on N.

First, the base case: we need to verify that (14) holds for $N = 1$. To this end, let $b_1^{(1)}$ be equal to either -1 or 1, and define $u_1^{(1)} = f_{b_1^{(1)}}$. In such a case (14) yields

$$\begin{aligned} v^{(1)}(x) &= \left(\sum_{i=1}^{1}\left(\prod_{j=1}^{i-1} b_{1-j+1}^{(1)}\right)2^{1-i}\right)\frac{\pi}{2^2} + \left(\prod_{j=1}^{1} b_{1-j+1}^{(1)}\right)\frac{x}{2} \\ &= \frac{\pi}{4} + b_1^{(1)}\frac{x}{2} = u_1^{(1)}(x), \end{aligned}$$

which is our first claim.

Second, the induction step: let M be a positive integer and assume that (14) holds for $N = M$, i.e., that for any sequence $(b_1^{(M)}, b_2^{(M)}, \ldots, b_M^{(M)})$ such that $b_i^{(M)} \in \{-1, 1\}$ for $i = 1, 2, \ldots, M$, if $u_i^{(M)}$ stands for the basic function $f_{b_{M-i+1}^{(M)}}$ ($i = 1, 2, \ldots, M$), and $v^{(M)} = u_1^{(M)} \circ u_2^{(M)} \circ \cdots \circ u_M^{(M)}$, then

$$v^{(M)}(x) = \left(\sum_{i=1}^{M}\left(\prod_{j=1}^{i-1} b_{M-j+1}^{(M)}\right)2^{M-i}\right)\frac{\pi}{2^{M+1}} + \left(\prod_{j=1}^{M} b_{M-j+1}^{(M)}\right)\frac{x}{2^M}. \tag{15}$$

We need to show that (14) holds for $N = M + 1$. To this purpose, we pick any sequence $(b_1^{(M+1)}, b_2^{(M+1)}, \ldots, b_{M+1}^{(M+1)})$ in which each $b_i^{(M+1)}$ equals either -1 or 1 for $i = 1, 2, \ldots, M+1$, and we define

$$v^{(M+1)} = u_1^{(M+1)} \circ u_2^{(M+1)} \circ \cdots \circ u_{M+1}^{(M+1)}$$

where $u_i^{(M+1)} = f_{b_{M-i+2}^{(M+1)}}$ for each $i = 1, 2, \ldots, M + 1$. Now, for the induction hypothesis (15) to be used, we consider the particular sequence $(b_1^{(M)}, b_2^{(M)}, \ldots, b_M^{(M)})$ for which $b_i^{(M)} = b_{i+1}^{(M+1)}$ ($i = 1, 2, \ldots, M$). With this choice one has that $u_i^{(M)} = f_{b_{M-i+1}^{(M)}} = f_{b_{M-i+2}^{(M+1)}} = u_i^{(M+1)}$ for each $i = 1, 2, \ldots, M$. Therefore,

$$\begin{aligned} v^{(M+1)}(x) &= \left(u_1^{(M+1)} \circ u_2^{(M+1)} \circ \cdots \circ u_M^{(M+1)}\right)\left(u_{M+1}^{(M+1)}(x)\right) \\ &= \left(\sum_{i=1}^{M}\left(\prod_{j=1}^{i-1} b_{M-j+2}^{(M+1)}\right)2^{M-i}\right)\frac{\pi}{2^{M+1}} + \left(\prod_{j=1}^{M} b_{M-j+2}^{(M+1)}\right)\frac{u_{M+1}^{(M+1)}(x)}{2^M} \\ &= \left(\sum_{i=1}^{M}\left(\prod_{j=1}^{i-1} b_{M-j+2}^{(M+1)}\right)2^{M-i}\right)\frac{\pi}{2^{M+1}} + \left(\prod_{j=1}^{M} b_{M-j+2}^{(M+1)}\right)\frac{\pi}{2^{M+2}} \\ &\quad + \left(\prod_{j=1}^{M} b_{M-j+2}^{(M+1)}\right)b_1^{(M+1)}\frac{x}{2^{M+1}} \end{aligned}$$

$$= \left(\sum_{i=1}^{M+1} \left(\prod_{j=1}^{i-1} b_{M-j+2}^{(M+1)} \right) 2^{M+1-i} \right) \frac{\pi}{2^{M+2}} + \left(\prod_{j=1}^{M+1} b_{M-j+2}^{(M+1)} \right) \frac{x}{2^{M+1}},$$

so we are done. □

The second step consists of providing an explicit expression for $v \circ v \circ \cdots \circ v$, where v stands for the composition $u_1 \circ u_2 \circ \cdots \circ u_N$, as introduced in Lemma 1.

Lemma 2. *Under the assumptions of Lemma 1, if moreover k is a positive integer and we define*

$$v^{\widehat{k}} = \underbrace{v \circ v \circ \cdots \circ v}_{k\ times},$$

we have

$$v^{\widehat{k}}(x) = \frac{\alpha \gamma_k \pi + 2\sigma^k x}{2^{kN+1}}, \tag{16}$$

where

$$\alpha = \sum_{i=1}^{N} \left(\prod_{j=1}^{i-1} b_{N-j+1} \right) 2^{N-i}, \tag{17}$$

$$\sigma = \prod_{j=1}^{N} b_{N-j+1}, \tag{18}$$

$$\gamma_k = \sum_{i=1}^{k} \left(\sigma^{i+1} 2^{(k-i)N} \right) = \frac{2^{kN}\sigma - \sigma^{k+1}}{2^N \sigma - 1}. \tag{19}$$

Proof. We proceed by induction on k.

A trivial verification shows that (16) holds for $k = 1$. Now assuming our claim to hold for $k = p$, we will prove it for $p + 1$.

We first observe that

$$v^{\widehat{p+1}}(x) = v^{\widehat{p}}(v(x)) = \frac{\alpha \gamma_p \pi + 2\sigma^p \left(\frac{\alpha\pi + 2\sigma x}{2^{N+1}} \right)}{2^{pN+1}} = \frac{\alpha(2^N \gamma_p + \sigma^p)\pi + 2\sigma^{p+1} x}{2^{(p+1)N+1}}.$$

But now note that $\sigma^2 = 1$. Therefore

$$2^N \gamma_p + \sigma^p = \left(2^N \sum_{i=1}^{p} \left(\sigma^{i+1} 2^{(p-i)N} \right) \right) + \sigma^p = \left(\sum_{i=1}^{p} \left(\sigma^{i+1} 2^{(p+1-i)N} \right) \right) + \sigma^{p+2}$$
$$= \sum_{i=1}^{p+1} \left(\sigma^{i+1} 2^{(p+1-i)N} \right) = \gamma_{p+1},$$

which yields

$$v^{\widehat{p+1}}(x) = \frac{\alpha \gamma_{p+1} \pi + 2\sigma^{p+1} x}{2^{(p+1)N+1}};$$

hence (16) is proved for $k = p + 1$.

Finally, we must verify the sum (19). It is fairly easy to verify the case $k = 1$ (just observe again that σ^2 equals 1). For $k \geq 2$ we use that $\sigma = \sigma^{-1}$ and we apply the standard technique for geometric series:

$$
\begin{aligned}
(2^N\sigma - 1)\gamma_k &= 2^N\sigma^{-1}\gamma_k - \gamma_k = \sum_{i=1}^{k}\left(\sigma^i 2^{(k+1-i)N}\right) - \sum_{i=1}^{k}\left(\sigma^{i+1}2^{(k-i)N}\right) \\
&= 2^{kN}\sigma + \sum_{i=1}^{k-1}\left(\sigma^{i+1}2^{(k-i)N}\right) - \sum_{i=1}^{k-1}\left(\sigma^{i+1}2^{(k-i)N}\right) - \sigma^{k+1} \\
&= 2^{kN}\sigma - \sigma^{k+1}. \quad \square
\end{aligned}
$$

Having disposed of these two preliminary lemmas, the task is now to compute $\lim_{k\to\infty}\langle v^k\rangle = \lim_{k\to\infty}\cos\left(\widehat{v^k}(x)\right) = \lim_{k\to\infty}\cos(\underbrace{v(v(\cdots v}_{k\text{ times}}(x)\cdots)))$.

Proposition 2. *Under the assumptions of Lemmas 1 and 2 it may be concluded that*

$$
\lim_{k\to\infty}\langle v^k\rangle = \cos\left(\alpha\,\frac{\sigma}{2^N\sigma - 1}\,\frac{\pi}{2}\right). \tag{20}
$$

Furthermore, $\lim_{k\to\infty}\langle v^k\rangle = 0$ *if and only if* $b_1 = b_2 = \cdots = b_N = 1$.

Proof. Our proof starts with the observation that $|\sigma| = 1$. According to this remark, we have

$$
\begin{aligned}
\lim_{k\to\infty}\left(\widehat{v^k}(x)\right) &= \lim_{k\to\infty}\frac{\alpha\frac{2^{kN}\sigma - \sigma^{k+1}}{2^N\sigma - 1}\pi + 2\sigma^k x}{2^{kN+1}} = \alpha\left(\lim_{k\to\infty}\frac{2^{kN}\sigma - \sigma^{k+1}}{(2^N\sigma - 1)2^{kN+1}}\right)\pi \\
&= \alpha\frac{\sigma}{(2^N\sigma - 1)2}\pi.
\end{aligned}
$$

Now, assertion (20) is nothing but the statement above, together with the continuity of the cosine function.

For the second part of the proof, we begin by noting that $1 \leq \alpha \leq 2^N - 1$. Moreover, $\alpha = 2^N - 1$ if and only if $b_N = b_{N-1} = \cdots = b_2 = 1$. Let us consider two cases:

(i) If $\alpha = 2^N - 1$ (and thus $b_N = b_{N-1} = \cdots = b_2 = 1$) it may happen that
- Either $b_1 = 1$, so $\sigma = 1$, which yields
$$\alpha\,\frac{\sigma}{2^N\sigma - 1}\,\frac{\pi}{2} = \frac{\pi}{2}.$$
- Or $b_1 = -1$, which gives $\sigma = -1$ and therefore
$$\alpha\,\frac{\sigma}{2^N\sigma - 1}\,\frac{\pi}{2} = \frac{2^N - 1}{2^N + 1}\,\frac{\pi}{2} \in (\pi/6, \pi/2).$$

(ii) In the case $1 \leq \alpha < 2^N - 1$ (in which at least one of the numbers $b_N, b_{N-1}, \ldots, b_2$ equals -1), we analyze the two possibilities for σ:
- First, if $\sigma = 1$ then
$$\alpha\,\frac{\sigma}{2^N\sigma - 1}\,\frac{\pi}{2} = \frac{\alpha}{2^N - 1}\,\frac{\pi}{2} \in (0, \pi/2).$$

- Second, when $\sigma = -1$ we have

$$\alpha \frac{\sigma}{2^N\sigma - 1}\frac{\pi}{2} = \frac{\alpha}{2^N + 1}\frac{\pi}{2} \in (0, \pi/2).$$

Therefore, the first bullet of case (i), which corresponds to $b_1 = b_2 = \cdots = b_N = 1$, is the only case for which $\cos\big((\alpha\,\sigma\,\pi)/(2(2^N\sigma - 1))\big) = 0$. $\square$

Propositions 1 and 2 combined give the desired expansions of $\cos x$ in terms of infinite cosine products. In order to get this asymptotic result we have to impose one further restriction on $(b_1, \ldots, b_N)$. That is the content of the following theorem, which provides the first key tool for obtaining our main result.

Theorem 2. *Let N be a positive integer and let $(b_1, b_2, \ldots, b_N)$ be a sequence such that $b_i \in \{-1, 1\}$ for $i = 1, 2, \ldots, N$ and such that at least one b_i equals -1. Define $u_i = f_{b_{N-i+1}}$ (see (7)) for $i = 1, 2, \ldots, N$ and set $v = u_1 \circ u_2 \circ \cdots \circ u_N$. We have*

$$\frac{\cos x}{\cos\left(\frac{\alpha\,\sigma}{2^N\sigma-1}\,\frac{\pi}{2}\right)} = \prod_{j=0}^{\infty}\left(\prod_{i=1}^{N}\left(2\cos\left(\overline{u_{N-i+1}}(u_{N-i+2}(u_{N-i+3}\cdots u_N(\underbrace{v(v\cdots v}_{j\ times}(x)\cdots))\cdots))\right)\right)\right),$$

where α and σ are defined in (17) and (18), respectively.

Proof. Taking limits in both sides of (8) we obtain

$$\cos x = \lim_{k\to\infty}(\cos x) = \lim_{k\to\infty}\left(2^{kN}\left(\prod_{j=0}^{k-1}\left(\prod_{i=1}^{N}\langle v_i^j\rangle\right)\right)\langle v^k\rangle\right). \tag{21}$$

Since not all the b_i's are equal to 1, by Proposition 2 we can assert that $\lim_{k\to\infty}\langle v^k\rangle \neq 0$, and thus

$$\lim_{k\to\infty}\frac{1}{\langle v^k\rangle} = \frac{1}{\cos\left(\frac{\alpha\,\sigma}{2^N\sigma-1}\,\frac{\pi}{2}\right)}. \tag{22}$$

Combining (21) with (22) we see that

$$\begin{aligned}\frac{\cos x}{\cos\left(\frac{\alpha\,\sigma}{2^N\sigma-1}\,\frac{\pi}{2}\right)} &= \cos x\lim_{k\to\infty}\frac{1}{\langle v^k\rangle}\\ &= \lim_{k\to\infty}\left(2^{kN}\left(\prod_{j=0}^{k-1}\left(\prod_{i=1}^{N}\langle v_i^j\rangle\right)\right)\langle v^k\rangle\right)\lim_{k\to\infty}\frac{1}{\langle v^k\rangle}\\ &= \lim_{k\to\infty}\left(\prod_{j=0}^{k-1}\left(\prod_{i=1}^{N}\left(2\langle v_i^j\rangle\right)\right)\right) = \prod_{j=0}^{\infty}\left(\prod_{i=1}^{N}\left(2\langle v_i^j\rangle\right)\right),\end{aligned}$$

and by Definition 2, this is the desired conclusion. $\square$

In closing we pick two simple examples, which may help to illustrate Theorem 2. We leave it to the reader to verify the details (also, see [12]).

Example 4 (*See* Example 1). Choose $N = 1$ and take $b_1 = -1$. This clearly forces $\alpha = 1$, $\sigma = -1$ and $\cos(\alpha\sigma\pi/((2^N\sigma - 1)\pi)) = \sqrt{3}/2$. Therefore, Theorem 2 becomes

$$\frac{2}{\sqrt{3}}\cos x = \prod_{n=0}^{\infty} 2\cos\left(\frac{a_{n+2}\pi}{2^{n+2}} + \frac{(-1)^{n+2}x}{2^{n+1}}\right),$$

where $a_j = (2^j - (-1)^j)/3$ denotes the jth Jacobsthal number [18] ($a_0 = 0$, $a_1 = 1$ and $a_{n+2} = a_{n+1} + 2a_n$ for $n \geq 0$).

Example 5 (*See* Example 2). Now consider $N = 2$ and $(b_1, b_2) = (1, -1)$. Thus, $(u_1, u_2) = (f_{-1}, f_1)$, $\alpha = 1$, $\sigma = -1$ and $\cos(\alpha\sigma\pi/((2^N\sigma - 1)\pi)) = \sqrt{5 + \sqrt{5}}/(2\sqrt{2})$. Moreover, an easy computation shows that, in this case, Theorem 2 takes the form

$$\frac{2\sqrt{2}\cos x}{\sqrt{5+\sqrt{5}}} = \prod_{n=0}^{\infty}\left(2\cos\left(\frac{c_{n+1}\pi}{4^{n+1}} + \frac{(-1)^{n+1}2x}{4^{n+1}}\right)2\cos\left(\frac{c_{n+2}\pi}{2\cdot 4^{n+1}} + \frac{(-1)^{n+2}x}{4^{n+1}}\right)\right),$$

where $c_j = (4^j - (-1)^j)/5$ for each positive integer j.

It is worth noting that if M is again taken to be equal to 2, but now with the choice $(b_1, b_2) = (-1, 1)$, then Theorem 2 reads

$$\frac{2\sqrt{2}\cos x}{\sqrt{5-\sqrt{5}}} = \prod_{n=0}^{\infty}\left(2\cos\left(\frac{d_{n+1}\pi}{4^{n+1}} + \frac{(-1)^{n}2x}{4^{n+1}}\right)2\cos\left(\frac{d_{n+1}\pi}{2\cdot 4^{n+1}} + \frac{(-1)^{n}x}{4^{n+1}}\right)\right),$$

where $d_j = (2 \cdot 4^j + 3(-1)^j)/5$ for j a positive integer, and this is essentially entry (d115) at the list [8] by R.W. Gosper who can be considered (we quote a referee of our paper) "the master of formulas like the ones in this paper".

3.3. Extension of Servi's and Nyblom's formulas

Our second key tool, to be developed in the next subsection, will consist of identifying each one of the factors

$$\cos\big(\overline{u_{N-i+1}}(u_{N-i+2}(u_{N-i+3}\cdots u_N(\underbrace{v(v\cdots v}_{j\text{ times}}(x)\cdots))\cdots))\big)$$

with an appropriate nested square root of 2. This will be a necessary step in order to transform our infinite products of cosines in Theorem 2 into the corresponding Viète-like formulas. In order to do this in full generality (namely, in the complex setting) we give here a technical result that generalizes the formulas by Servi [17] and by Nyblom [14] (see [10] for further details).

Theorem 3 (*Theorem 4 in [10]*). *Let ξ be an arbitrary complex number and let k be a positive integer. If $\beta_l \in \{1, -1\}$ for $l = 1, 2, \ldots, k$, then*

$$\beta_k\sqrt{2 + \beta_{k-1}\sqrt{2 + \beta_{k-2}\sqrt{2 + \cdots + \beta_1\sqrt{2 + 2\xi}}}}$$

$$= 2\cos\left(\left(\frac{1}{2} - \frac{\beta_k}{2^2} - \frac{\beta_k\beta_{k-1}}{2^3} - \cdots - \frac{\beta_k\beta_{k-1}\cdots\beta_1\beta_0}{2^{k+2}}\right)\pi\right)$$

$$= 2\cos\left(\left(2^{-1} - \sum_{j=1}^{k+1}\left(2^{-(j+1)}\prod_{i=0}^{j-1}\beta_{k-i}\right)\right)\pi\right) \tag{23}$$

where

$$\beta_0 = \frac{4}{\pi}\operatorname{Arcsin}\xi = \frac{4}{\pi}\operatorname{Arg}\left(i\xi + \sqrt{1-\xi^2}\right) - i\frac{4}{\pi}\ln|i\xi + \sqrt{1-\xi^2}|. \tag{24}$$

3.4. *Second key tool: the factors $\langle v_i^j\rangle$ as nested square roots of 2*

Now we proceed to show that it is still possible to make our first key tool go further. We mean that we already are in a position to rewrite the infinite product of cosines in Theorem 2 as an infinite product of certain nested square roots. The point is that we can use Theorem 3 to express each one of the cosine factors

$$\langle v_i^j\rangle = \cos\left(\overline{u_{N-i+1}}\left(u_{N-i+2}(u_{N-i+3}\cdots u_N(\underbrace{v(v\cdots v}_{j \text{ times}}(x)\cdots))\cdots)\right)\right), \tag{25}$$

appearing both in (8) and in Theorem 2, as the nested square roots

$$\frac{1}{2}\sqrt{2 - \widehat{b}_{jN+i}\sqrt{2 + \widehat{b}_{jN+i-1}\sqrt{2 + \widehat{b}_{jN+i-2}\sqrt{2 + \cdots + \widehat{b_2}\sqrt{2 + 2\widehat{b_1}\sin x}}}}},$$

where each $\widehat{b_k}$ carries a value of -1 or 1 and is related with the corresponding kth basic function (counting from right to left) appearing in the argument of (25).

For simplicity of notation, we will write $\surd(z)$ instead of $\sqrt{z}$, namely, the principal value of the complex square root of z. Therefore, throughout the proof below, the nested square root

$$b_n\sqrt{2 + b_{n-1}\sqrt{2 + b_{n-2}\sqrt{2 + \cdots + b_2\sqrt{2 + 2b_1 z}}}}$$

will be written as

$$b_n\surd(2 + b_{n-1}\surd(2 + b_{n-2}\surd(2 + \cdots + b_2\surd(2 + 2b_1 z)\cdots))).$$

Proposition 3. *Under the same hypotheses as in Proposition 1,*

$$\begin{aligned} 2\langle v_i^j\rangle = {} & \surd(2 + b_i\surd(2 + b_{i-1}\surd(2 + \cdots + b_1\,\underbrace{\surd(2 + b_N\surd(2 + b_{N-1}\surd(2 + \cdots + b_1}\\ & \times\ \underbrace{\surd(2 + b_N\surd(2 + b_{N-1}\surd(2 + \cdots + b_1}\cdots\\ & \times\ \underbrace{\surd(2 + b_N\surd(2 + \cdots + b_2\surd(2 + b_1}\,2\sin x)\cdots))\cdots)))\cdots)))\cdots))), \end{aligned} \tag{26}$$

where the structure marked by horizontal curly brackets occurs j times.

Proof. First note that

$$2\langle v_i^j\rangle = 2\cos\big(\overline{u_{N-i+1}}(u_{N-i+2}(u_{N-i+3}\cdots u_N(\underbrace{v(v\cdots v}_{j\text{ times}}(x)\cdots))\cdots))\big)$$

$$= 2\cos\Big(\overline{u_{N-i+1}}(u_{N-i+2}\cdots u_N(\underbrace{u_1(u_2\cdots u_N}(\underbrace{u_1(u_2\cdots u_N}(\cdots$$

$$\cdots\underbrace{u_1(u_2\cdots u_N}(x)\cdots)\cdots)\cdots))\cdots))\cdots)\Big),$$

where in the last equality, the structure $u_1(u_2(\cdots u_N(\cdot)\cdots))$, with horizontal curly brackets below, occurs j times. The task is now to use Lemma 1 for replacing the composition of basic functions $v(x) = \left(\widehat{u}_1 \circ \widehat{u}_2 \circ \cdots \circ \widehat{u}_{jN+i}\right)(x)$ given by

$$\overline{u_{N-i+1}}(u_{N-i+2}\cdots u_N(\underbrace{\underbrace{u_1(u_2\cdots u_N}(\cdots\underbrace{u_1(u_2\cdots u_N}}_{j\text{ times}}(x)\cdots)\cdots)\cdots))\cdots) \tag{27}$$

by a linear function in x. Thus, the goal is to find the sequence $(\widehat{b}_1, \widehat{b}_2, \ldots, \widehat{b}_{jN+i})$ associated to the composition (27), where the requirement on the $\widehat{b}_k$'s is that $\widehat{u}_k = f_{\widehat{b}_{jN+i-k+1}}$ for each $1 \le k \le jN+i$. But note that we must work under the hypothesis that $u_k = f_{b_{N-k+1}}$ for each $1 \le k \le N$ (this means that $(b_1, b_2, \ldots, b_N)$ is the sequence associated with $(u_1, u_2, \ldots, u_N)$). Hence we check at once that for $1 \le k \le jN+i-1$

$$f_{\widehat{b}_k} = \widehat{u}_{jN+i-k+1} = u_{N-(k-1) \bmod N} = f_{b_{1+(k-1) \bmod N}},$$

and also

$$f_{\widehat{b}_{jN+i}} = \widehat{u}_1 = \overline{u_{N-i+1}} = \overline{f_{b_i}} = f_{-b_i}.$$

Therefore, $\widehat{b}_k = b_{1+(k-1) \bmod N}$ for $1 \le k \le jN+i-1$, and $\widehat{b}_{jN+i} = -b_i$, which yields

$$(\underbrace{\underbrace{\widehat{b}_1, \widehat{b}_2, \ldots, \widehat{b}_N}, \underbrace{\widehat{b}_{N+1}, \ldots, \widehat{b}_{2N}}, \ldots, \underbrace{\widehat{b}_{(j-1)N+1}, \ldots, \widehat{b}_{jN}}}_{j\text{ times}}, \widehat{b}_{jN+1}, \ldots, \widehat{b}_{jN+i})$$

$$= (\underbrace{\underbrace{b_1, b_2, \ldots, b_N}, \underbrace{b_1, b_2, \ldots, b_N}, \ldots, \underbrace{b_1, b_2, \ldots, b_N}}_{j\text{ times}}, b_1, b_2, \ldots, b_{i-1}, -b_i). \tag{28}$$

Next, use will be made of Lemma 1 to write $2\langle v_i^j\rangle$ as a cosine whose argument is a linear function in x which depends on the $\widehat{b}_k$'s. On account of (14), being (28) the considered sequence whose elements are either -1 or 1, we get

$$2\langle v_i^j\rangle = 2\cos\left(\left(\sum_{r=1}^{jN+i}\left(\prod_{s=1}^{r-1}\widehat{b}_{jN+i-s+1}\right)2^{jN+i-r}\right)\frac{\pi}{2^{jN+i+1}}\right.$$

$$\left.+\left(\prod_{s=1}^{jN+i}\widehat{b}_{jN+i-s+1}\right)\frac{x}{2^{jN+i}}\right)$$

$$= 2\cos\left(\left(\sum_{r=1}^{jN+i}\left(\prod_{s=1}^{r-1}\widehat{b}_{jN+i-s+1}\right)\frac{\pi}{2^{r+1}}\right) + \left(\prod_{s=1}^{jN+i}\widehat{b}_{jN+i-s+1}\right)\frac{4x}{\pi}\frac{\pi}{2^{jN+i+2}}\right)$$

$$= 2\cos\left(\left(\left(\frac{1}{2^2} + \frac{\widehat{b}_{jN+i}}{2^3} + \frac{\widehat{b}_{jN+i}\widehat{b}_{jN+i-1}}{2^4} + \cdots + \frac{\widehat{b}_{jN+i}\widehat{b}_{jN+i-1}\cdots\widehat{b}_2}{2^{jN+i+1}} \right.\right.\right.$$
$$\left.\left. + \frac{\widehat{b}_{jN+i}\widehat{b}_{jN+i-1}\cdots\widehat{b}_2\widehat{b}_1}{2^{jN+i+2}}\frac{4x}{\pi}\right)\pi\right)$$
$$= 2\cos\left(\left(\frac{1}{2} - \frac{1}{2^2} - \frac{(-\widehat{b}_{jN+i})}{2^3} - \frac{(-\widehat{b}_{jN+i})\widehat{b}_{jN+i-1}}{2^4} \right.\right.$$
$$\left.\left. - \cdots - \frac{(-\widehat{b}_{jN+i})\widehat{b}_{jN+i-1}\cdots\widehat{b}_2}{2^{jN+i+1}} - \frac{(-\widehat{b}_{jN+i})\widehat{b}_{jN+i-1}\cdots\widehat{b}_2\widehat{b}_1}{2^{jN+i+2}}\frac{4x}{\pi}\right)\pi\right).$$

Therefore, if we define

$$\alpha_k = \begin{cases} 4x/\pi, & \text{if } k = 1; \\ 1, & \text{if } 2 \le k \le jN+i-1; \\ -1, & \text{if } k = jN+i, \end{cases}$$

and we also define $\alpha_{jN+i+1} = \widehat{b}_{jN+i+1} = 1$, relation above becomes

$$2\langle v_i^j\rangle = 2\cos\left(\left(2^{-1} - \sum_{r=1}^{jN+i+1}\left(\prod_{s=0}^{r-1}\alpha_{jN+i+1-s}\widehat{b}_{jN+i+1-s}\right)2^{-(r+1)}\right)\pi\right). \tag{29}$$

Now we can use Theorem 3 with the replacements $k = jN+i$, and $\beta_s = \alpha_{s+1}\widehat{b}_{s+1}$ for $s = 0, 1, \ldots, jN+i$. Thus, if we take into account that

$$\xi = \sin\left(\frac{\beta_0\pi}{4}\right) = \sin\left(\frac{\alpha_1\widehat{b}_1\pi}{4}\right) = \sin(\widehat{b}_1 x) = \widehat{b}_1\sin x = b_1\sin x,$$

then the sequence of variables to be replaced on the nested radical expression in Theorem 3 reads

$$\begin{aligned}
&(\beta_{jN+i}, \beta_{jN+i-1}, \beta_{jN+i-2}, \ldots, \beta_2, \beta_1, \xi) \\
&\quad = (\alpha_{jN+i+1}\widehat{b}_{jN+i+1}, \alpha_{jN+i}\widehat{b}_{jN+i}, \alpha_{jN+i-1}\widehat{b}_{jN+i-1}, \ldots, \alpha_3\widehat{b}_3, \alpha_2\widehat{b}_2, \widehat{b}_1\sin x) \\
&\quad = (1, -\widehat{b}_{jN+i}, \widehat{b}_{jN+i-1}, \widehat{b}_{jN+i-2}, \ldots, \widehat{b}_3, \widehat{b}_2, \widehat{b}_1\sin x) \\
&\quad = (1, b_i, b_{i-1}, \ldots, b_1, \\
&\qquad \underbrace{\underbrace{b_N, b_{N-1}, \ldots, b_1}, \underbrace{b_N, b_{N-1}, \ldots, b_1}, \ldots, \underbrace{b_N, b_{N-1}, \ldots, b_1}}_{j \text{ times}}\sin x),
\end{aligned}$$

and (26) is proved. □

4. Proof of the main result

Roughly speaking, the main idea in the proof of Theorem 1 is to replace Proposition 3 into Theorem 2, and the only point remaining concerns a change of variables.

Proof of Theorem 1. As easy to check, the vertical strip $|\mathrm{Re}(x)| < \pi/2$ is mapped by the function $z(x) = 2\sin x$ one-to-one onto the complex z-plane with two slits $(-\infty, -2]$ and $[2, \infty)$. Note that if we denote $z = u + iv$ and $x = a + ib$, then

$$z(x) = u(x) + iv(x) = 2\sin(a + ib) = 2(\sin a\cosh b + i\cos a\sinh b),$$

from which we deduce that:

(i) The image of each vertical line $x = \lambda$, with $0 < \lambda < \pi/2$, is the right branch of the hyperbola $u^2/(2\sin\lambda)^2 - v^2/(2\cos\lambda)^2 = 1$ with foci at $z = -2$ and $z = 2$.
(ii) The image of the vertical line $x = \lambda$, with $\pi/2 < \lambda < 0$, is the left branch of the hyperbola $u^2/(2\sin\lambda)^2 - v^2/(2\cos\lambda)^2 = 1$, with the same foci as above.
(iii) The vertical axis $a = 0$ in the complex x-plane is mapped onto the vertical axis $u = 0$ in the complex z-plane.
(iv) The vertical ray $x = \pi/2 + i\lambda$ with $\lambda \le 0$ is mapped onto the slit $\{u + iv = 2\cosh\lambda : \lambda \le 0\} = [2, \infty)$, and the vertical ray $x = -\pi/2 + i\lambda$ with $\lambda \ge 0$ is mapped onto the other slit $\{u + iv = -2\cosh\lambda : \lambda \le 0\} = (\infty, -2]$.

Therefore, if D stands for the set containing the vertical strip $\{|\mathrm{Re}(x)| < \pi/2\}$ together with $\{\mathrm{Re}(x) = \pi/2, \mathrm{Im}(x) \le 0\}$ and $\{\mathrm{Re}(x) = -\pi/2, \mathrm{Im}(x) \ge 0\}$, then the function $z(\cdot) = 2\sin(\cdot)$ is a bijective map from D to $\mathbb{C}$, so it admits an inverse. Furthermore, the inverse map $x(\cdot) = \mathrm{Arcsin}(\cdot/2) : \mathbb{C} \to D$ can be defined by means of [4, p. 32]

$$x(z) = -i\mathrm{Log}\left(i\frac{z}{2} + \sqrt{1 - \frac{z^2}{4}}\right), \quad z \in \mathbb{C}, \tag{30}$$

where, as stated before, $\sqrt{\cdot}$ is the principal value of the square root function. Also, it is a standard exercise in Complex Analysis to verify that for each $z \in \mathbb{C}$,

$$\cos\left(\mathrm{Arcsin}\left(\frac{z}{2}\right)\right) = \cos\left(-i\mathrm{Log}\left(i\frac{z}{2} + \sqrt{1 - \frac{z^2}{4}}\right)\right) = \sqrt{1 - \frac{z^2}{4}}. \tag{31}$$

With this in mind, with the requirement that x lives in D, if we replace Proposition 3 into Theorem 2, then the change of variables $x = \mathrm{Arcsin}(z/2)$ yields

$$\frac{\cos\left(\mathrm{Arcsin}(z/2)\right)}{\cos\left(\frac{\alpha\,\sigma}{2^N\sigma - 1}\,\frac{\pi}{2}\right)} = \prod_{j=0}^{\infty}\left(\prod_{i=1}^{N}\left(2\langle v_i^j\rangle_z\right)\right), \tag{32}$$

where we have used the symbol $2\langle v_i^j\rangle_z$ to denote the nested square root (26) with the factor $2\sin x$ interchanged by z.

In closing, if we define $\widehat{b_k} = b_{1+(k-1) \bmod N}$ $(k = 1, 2, \ldots)$ and we recursively define $\rho_0(z) = z$ and $\rho_k(z) = \sqrt{2 + \widehat{b_k}\rho_{k-1}(z)}$ for $k \ge 1$, it is straightforward to check that

$$2\langle v_i^j\rangle_z = \rho_{jN+i}(z), \quad j = 0, 1, 2, \ldots, i = 1, 2, \ldots, N. \tag{33}$$

In closing, the replacement of (31) and (33) into (32) gives

$$\frac{\sqrt{1 - \frac{z^2}{4}}}{\cos\left(\frac{\alpha\,\sigma}{2^N\sigma - 1}\,\frac{\pi}{2}\right)} = \prod_{j=0}^{\infty}\left(\prod_{i=1}^{N}\rho_{jN+i}(z)\right) = \prod_{j=0}^{\infty}\left(\prod_{i=jN+1}^{(j+1)N}\rho_i(z)\right),$$

so we are done. □

5. The spurious case

Our final move consists of analyzing the "forbidden case" $N = 1$ and $b_1 = 1$ in Theorem 1. (Let us note that the other problematic cases, namely $N = 2$ with $(b_1, b_2) = (1, 1)$, $N = 3$ with $(b_1, b_2, b_3) = (1, 1, 1)$, and so on, are essentially the same and coincide with the simplest one $N = 1$, $b_1 = 1$.) We will show how a simple trick led us to obtain, also in this case, a Viète-like formula, which in fact can be considered as the genuine generalization of the original Viète's formula.

Applying Proposition 1 to the case $N = 1$ with $b_1 = 1$, so that $v = u_1 = f_1$, it follows that for each positive integer k

$$\cos x = 2^k \left(\prod_{j=0}^{k-1} \langle v_1^j \rangle \right) \langle v^k \rangle, \tag{34}$$

where

$$\langle v_1^j \rangle = \cos(f_{-1}(\underbrace{f_1(f_1(\ldots(f_1}_{j \text{ times}}(x))\ldots)))), \tag{35}$$

$$\langle v^k \rangle = \cos(\underbrace{f_1(f_1(\ldots(f_1}_{k \text{ times}}(x))\ldots))). \tag{36}$$

By Lemma 2, and using that $\alpha = \sigma = 1$ and $\gamma_k = 2^k - 1$, we can rewrite (35) and (36) as

$$\begin{aligned}\langle v_1^j \rangle &= \cos(f_{-1}(\widehat{f_1^j}(x))) = \cos\left(\frac{\pi}{4} - \frac{1}{2}\left(\frac{(2^j - 1)\pi + 2x}{2^{j+1}}\right)\right)\\ &= \cos\left(\frac{\pi}{2^{j+2}} - \frac{x}{2^{j+1}}\right),\end{aligned}$$

$$\langle v^k \rangle = \cos(\widehat{f_1^k}(x)) = \cos\left(\frac{(2^k - 1)\pi + 2x}{2^{k+1}}\right) = \sin\left(\frac{\pi}{2^{k+1}} - \frac{x}{2^k}\right).$$

Now consider $x \neq \pi/2$. If we first replace the above expressions into (34), and then we divide both members by $\pi/2 - x$, we obtain

$$\frac{\cos x}{\frac{\pi}{2} - x} = \left(\prod_{j=0}^{k-1} \cos\left(\frac{\pi}{2^{j+2}} - \frac{x}{2^{j+1}}\right)\right) \left(\frac{\sin\left(\frac{\pi}{2^{k+1}} - \frac{x}{2^k}\right)}{\frac{\pi}{2^{k+1}} - \frac{x}{2^k}} \right),$$

and thus

$$\frac{\cos x}{\frac{\pi}{2} - x} = \prod_{k=1}^{\infty} \cos\left(\frac{\pi}{2^{k+1}} - \frac{x}{2^k}\right), \quad x \neq \pi/2. \tag{37}$$

The next step consists of substituting each cosine factor in the right hand side of (37) into its corresponding nested square root. In order to introduce the change of variables $x = \mathrm{Arcsin}(z/2)$, and similarly as we did in Section 4, the requirement on x is that it satisfies one of the following conditions:

(i) $|\mathrm{Re}(x)| < \pi/2$,
(ii) $x = \pi/2 - iy$, with $y > 0$ ($x = \pi/2$ must be excluded in formula (37)),
(iii) $x = -\pi/2 + iy$ with $y \geq 0$.

Having disposed this preliminary assumption, the procedure is to apply Theorem 3 with the choice $\beta_k = \beta_{k-1} = \cdots = \beta_1 = 1$ and to define a new complex variable z in terms of the complex variable x by means of $x = \mathrm{Arcsin}(z/2)$. In such a case we have, for k a positive integer,

$$\cos\left(\frac{\pi}{2^{k+1}} - \frac{x}{2^k}\right) = \cos\left(\left(\frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} - \cdots - \frac{1}{2^{k+1}} - \frac{(4/\pi)\mathrm{Arcsin}(z/2)}{2^{k+2}}\right)\pi\right)$$
$$= \frac{1}{2}\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+z}}}}}_{k \text{ square roots}}. \tag{38}$$

Finally, using (38), and taking into account (31) and $\pi/2 - \mathrm{Arcsin}(z/2) = \mathrm{Arccos}(z/2)$ (as easy to check), Eq. (37) transforms to

$$\frac{\sqrt{1-\frac{z^2}{4}}}{\mathrm{Arccos}\left(\frac{z}{2}\right)} = \frac{\sqrt{2+z}}{2}\cdot\frac{\sqrt{2+\sqrt{2+z}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2+z}}}}{2}\cdots, \quad z \neq 2. \tag{39}$$

To the best of our knowledge, formula above must be credited to A. Levin [9, formula (24)], who obtained it with a different method (see also [15, Theorem 1] for a weaker formulation). The very classical Viète's formula (1) is the case $z = 0$ of (39). And maybe at this point in time, Professor Nevai should take his hat off. Or perhaps he should decide not to remove it. Anyway, with Professor Nevai either hatted or un-hatted (of course all this is just a joke), our desire has been to make a contribution that we hope can enrich the legendary famous Viète's formula.

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Appendix. Lights and shadows in Viète's original derivation

The technique with which in 1593 the French amateur mathematician François Viète derived formula above was based on the following geometric considerations (see Fig. 1):

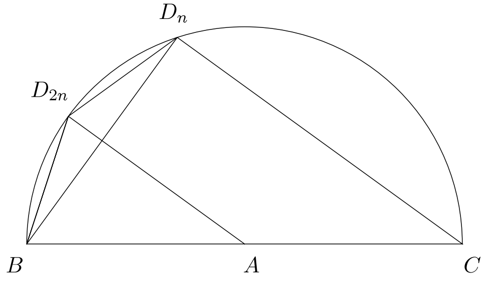

Fig. 1. Viète's geometric construction.

Proposition 4 (*[19, Propositio I] and [1, pp. 754–755]*). *If A_n stands for the area of the regular polygon with n sides (of length BD_n) and A_{2n} stands for the area of the regular polygon with $2n$ sides (of length BD_{2n}), then $A_n/A_{2n} = a_n/d$, being $a_n = D_nC$ the so-called apotome of the side BD_n and d the diameter of the circle.*

With the use of a telescopic product of the formula in Proposition 4, Viète established:

Proposition 5 (*[19, Propositio II] and [1, pp. 755–756]*). *For each nonnegative integer k,*

$$\frac{A_n}{A_{2^{k+1}n}} = \frac{a_n a_{2n} a_{2^2 n} \cdots a_{2^k n}}{d^{k+1}}, \tag{40}$$

where for each $j = 0, 1, \ldots, k$, $a_{2^j n} = D_{2^j n}C$ is the apotome of the side $BD_{2^j n}$.

Starting with $n = 4$, that is, with the square inscribed in the circle, and passing to the limit when k tends to infinite, Viète obtained:

Corollary 1 (*[19, Corollarivm] and [1, pp. 756–757]*). *If A_∞ denotes the area of the circle, which can be considered as the regular polygon with infinitely many sides, then*

$$\frac{A_4}{A_\infty} = \frac{a_4}{d}\frac{a_8}{d}\frac{a_{16}}{d}\cdots = \prod_{i=0}^{\infty}\frac{a_{2^{i+2}}}{d}. \tag{41}$$

In Corollary 1 we have used modern terminology, avoiding the problematic "highest power of the diameter", d^∞, used by Viète. Let us say also that Viète's original *Corollarivm* consists in a paraphrased version of formula

$$\frac{A_4}{A_\infty} = \frac{l_4}{\frac{d^\infty}{\prod_{i=1}^{\infty} a_{2^{i+2}}}}, \tag{42}$$

where $l_4 = a_4$ is the side of the square.

After Corollary 1 Viète wrote without additional explanation that if $d = 2$, then $a_4 = \sqrt{2}$, $a_8 = \sqrt{2+\sqrt{2}}$, $a_{16} = \sqrt{2+\sqrt{2+\sqrt{2}}}$ and "so on in that progression". This comment should lead, after using formula (41), to formula (1). As easy to deduce, it is clear that for each $n \geq 3$

$$a_{2n} = \sqrt{\frac{d^2}{2} + \frac{d}{2}a_n}, \tag{43}$$

which should yield to the values for $a_8, a_{16}, \ldots$ given by Viète.

The next step in Viète's paper consists of applying formula (42) in the particular case $d = 1$. This is the only case in which the factor d^∞ occurring in the denominator in (42) is no longer problematic. Although being very shrewd in avoiding the problem with d^∞, Viète made a naive mistake. He paraphrased the wrong formula

$$\frac{A_4}{A_\infty} = \frac{\sqrt{1/2}}{\frac{1}{\sqrt{1/2+\sqrt{1/2}}\sqrt{1/2+\sqrt{1/2+\sqrt{1/2}}}\cdots}}, \tag{44}$$

in which factors $1/2$ before each one of the interior square root symbols should have been written to give the correct formula

$$\frac{2}{\pi} = \frac{\sqrt{1/2}}{\frac{1}{\sqrt{1/2+(1/2)\sqrt{1/2}}\sqrt{1/2+(1/2)\sqrt{1/2+(1/2)\sqrt{1/2}}}\cdots}}$$
$$= \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots,$$

which is a slight variation of (1). Viète's mistake (44) for sure was not a typo. In fact, lines below in his paper he did not consider these same interior factors $1/2$ when analyzing the general case of a circle with diameter X (see [19, p. 400] and [1, p. 757]).

References

[1] L. Berggren, J. Borwein, P. Borwein, Pi: A Source Book, third ed., Springer-Verlag, New York, 2004.
[2] C.J. Efthimiou, A class of periodic continued radicals, Amer. Math. Monthly 119 (2012) 52–58.
[3] http://functions.wolfram.com/ElementaryFunctions/Sin/03/02/0011/.
[4] T.W. Gamelin, Complex Analysis, Springer-Verlag, New York, 2001.
[5] E.M. García Caballero, S.G. Moreno, M.P. Prophet, The golden ratio and Viète's formula, submitted for publication.
[6] E.M. García Caballero, S.G. Moreno, M.P. Prophet, New Viète-like infinite products of nested radicals with Fibonacci and Lucas numbers, Fibonacci Quart. (2013) in press.
[7] C.F. Gauss, Disquisitiones Arithmeticae, Springer-Verlag, New York, 1986.
[8] R.W. Gosper, http://www.tweedledum.com/rwg/idents.htm.
[9] A. Levin, A new class of infinite products generalizing Viète's product formula for π, Ramanujan J. 10 (2005) 305–324.
[10] S.G. Moreno, E.M. García Caballero, Chebyshev polynomials and nested square roots, J. Math. Anal. Appl. 394 (2012) 61–73.
[11] S.G. Moreno, E.M. García Caballero, The unruly $\sin \pi/7$, submitted for publication.
[12] S.G. Moreno, E.M. García Caballero, New infinite products of cosines and Viète-like formulae, Math. Mag. 86 (2013) 15–25.
[13] P. Nevai, Book review, J. Approx. Theory 164 (2012) 353–365.
[14] M.A. Nyblom, More nested square roots of 2, Amer. Math. Monthly 112 (2005) 822–825.
[15] M.A. Nyblom, Some closed-form evaluations of infinite products involving nested radicals, Rocky Mountain J. Math. 42 (2012) 751–758.
[16] G. Pólya, G. Szegő, Problems and Theorems in Analysis I, Springer-Verlag, Berlin, 1978.
[17] L.D. Servi, Nested square roots of 2, Amer. Math. Monthly 110 (2003) 326–330.
[18] N.J.A. Sloane, The on-line encyclopedia of integer sequences, 2010. http://oeis.org/A001045.
[19] F. Viète, Variorum de rebus mathematicis responsorum liber VIII, Capvt XVIII (1593) 398–400. (See also [1] for both the original paper in Latin in pp. 53–55, and for a translated version in pp. 754–757, in which line 16 from above in p. 757 must be deleted, and the symbol + must be inserted after $1/2X^2$ in line 20 from above at the same page.).
[20] M.L. Wantzel, Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas, J. Math. Pures Appl. 1 (1836) 366–372.
[21] E.W. Weisstein, "Trigonometry Angles–Pi/7", From MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/TrigonometryAnglesPi7.html.
[22] E.W. Weisstein, "Trigonometry Angles–Pi/17", From MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/TrigonometryAnglesPi17.html.