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# Riesz means associated with certain product type convex domain ☆

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ABSTRACT

In this paper we study the maximal operators $T_*^\delta$ and the convolution operators $T^\delta$ associated with multipliers of the form

$$\left(1 - \max\left\{|\xi_0|, |\xi_1|, \ldots, |\xi_{n-2}|\right\}\right)_+^\delta, \quad (\xi_0, \xi_1, \ldots, \xi_{n-2}) \in \mathbb{R}^2 \times \mathbb{R}^{n-2},\ n \geqslant 3.$$

We prove that $T_*^\delta$ satisfies the sharp weak type $(p, p)$ inequality on $H^p(\mathbb{R}^n)$ when $\frac{2(n-1)}{2n-1} < p < 1$ and $\delta = n(\frac{1}{p} - 1) + \frac{1}{2}$, or when $p = \frac{2(n-1)}{2n-1}$ and $\delta > n(\frac{1}{p} - 1) + \frac{1}{2}$. We also obtain that $T^\delta$ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $\delta > \max\{2|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0\}$ and $1 \leqslant p < \infty$. The indicated ranges of parameters $p$ and $\delta$ cannot be improved.



## 1. Introduction

For a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$ we denote by $\widehat{f}$ the Fourier transform of $f$. For $n \geqslant 3$ we define a distance function $\varrho$ as

$$\varrho(\xi) = \max\left\{|\xi_0|, |\xi_1|, \ldots, |\xi_{n-2}|\right\}, \quad \xi = (\xi_0, \xi_1, \ldots, \xi_{n-2}) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}.$$

For $\delta > 0$ we consider a convolution operators $T_\epsilon^\delta$ by

$$\widehat{T_\epsilon^\delta f}(\xi) = \left(1 - \frac{\varrho(\xi)}{\epsilon}\right)_+^\delta \widehat{f}(\xi) \tag{1.1}$$

and the maximal operators $T_*^\delta$ by

$$T_*^\delta f(x) = \sup_{\epsilon > 0}\left|T_\epsilon^\delta f(x)\right|, \quad x = (x_0, x_1, \ldots, x_{n-2}) \in \mathbb{R}^2 \times \mathbb{R}^{n-2},\ n \geqslant 3.$$

Here $t_+^\delta = t^\delta$ for $t > 0$ and zero otherwise. In this paper we consider the multiplier $m$ defined by

$$m(\xi) = \left(1 - \max\left\{|\xi_0|, |\xi_1|, \ldots, |\xi_{n-2}|\right\}\right)_+^\delta,$$

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which is supported in the product type convex domain $D \times I \times \cdots \times I$, where $D$ is the unit disc in $\mathbb{R}^2$ centered at the origin and $I = [-1, 1]$. In this paper we are aiming to establish sharp weak type estimates for the maximal operator $T_*^\delta$ and strong type estimates for the convolution operators $T^\delta$ associated with a multiplier $m$. One of the important parts of doing this is to analyze the kernel function which is the inverse Fourier transform of the multiplier $m$. Especially when $p < 1$ decay estimates of the kernel immediately give us the desired estimates if we combine them with standard arguments developed by Stein, Taibleson and Weiss in [16]. The decay of the kernel is related with the differentiability of the multiplier so delicate analysis near points where the differentiability of the multiplier breaks down is necessary for what we desire to obtain. Even though estimates for the complementary case carry a different story, understanding the singular set of the multiplier enables us to successfully perform appropriate dyadic decompositions of the kernel. One can put together optimal estimates for each dyadic piece to obtain results for the original kernel without any loss. Hence it is worth noticing that the differentiability of the multiplier breaks down on the singular set

$$\mathcal{R} = \{\xi\colon |\xi_i| = |\xi_j|\} \cup \{\xi\colon \varrho(\xi) = 1\}. \tag{1.2}$$

As we see that $\mathcal{R}$ is a union of two sets. Intuitively, the differentiability of the multiplier on the set $\{\xi\colon \varrho(\xi) = 1\}$ can be improved when we take bigger $\delta$. However the size of $\delta$ does not affect the differentiability of the multiplier on the complementary set $\{\xi\colon |\xi_i| = |\xi_j|\}$. These roughly indicate that the singular set $\{\xi\colon \varrho(\xi) = 1\}$ will give the restriction of $\delta$ and the set $\{\xi\colon |\xi_i| = |\xi_j|\}$ will give the other restriction, which turns out to be the restriction on the range of $p$.

One more useful observation to comprehend the multiplier is as follows:

The distance function $\varrho(\xi)$ is equal to lower dimensional Euclidean distance function $|\xi_\ell|$ in conical regions $\varrho(\xi) = |\xi_\ell|$. So in each conical region the multiplier behaves like lower dimensional Bochner–Riesz multiplier. For example, we consider the case $n = 3$. In this case $\varrho(\xi) = \max\{|\xi_0|, |\xi_1|\}$ where $\xi = (\xi_0, \xi_1) \in \mathbb{R}^2 \times \mathbb{R}$ and the multiplier is equal to $\omega(\xi) = (1 - \max\{|\xi_0|, |\xi_1|\})_+^\delta$. If $|\xi_0| > |\xi_1|$, $\omega(\xi) = (1 - |\xi_0|)_+^\delta$ which is the multiplier for 2-dimensional spherical Bochner–Riesz means. If $|\xi_0| < |\xi_1|$, $\omega(\xi) = (1 - |\xi_1|)_+^\delta$ which is the multiplier for one-dimensional Bochner–Riesz means. When $|\xi_0| = |\xi_1|$, the multiplier $\omega$ is not smooth on the cone $\{(\xi_0, \xi_1)\colon |\xi_0| = |\xi_1|\}$. So the operator is a combination of 1 and 2-dimensional Bochner Riesz means and the cone multipliers in three-dimensional Euclidean space.

H. Luers has investigated this type of operator in a different setting in [8]. He obtained $L^p$ estimates of convolution operators $S^\delta$ associated with a cylinder multiplier defined by

$$\widehat{S^\delta f}(\xi) = \big(1 - \max\big\{|\xi_0|, |\xi_1|\big\}\big)_+^\delta \widehat{f}(\xi), \quad \xi = (\xi_0, \xi_1) \in \mathbb{R}^k \times \mathbb{R},$$

where $k \geqslant 2$. We note that when $k \geqslant 3$ the multiplier is related with higher dimensional Bochner–Riesz multiplier. His results are not sharp and it is conjectured that $S^\delta$ is bounded for all $\delta > \delta(p) := \max\{k|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0\}$ and $\delta(p) < 1$. P. Taylor recently improved the Luers' result in [17] by adapting the arguments in [1,9,10,18].

As for $p < 1$, it has been obtained for $k = 2$ that the maximal cylinder operator is of weak type $(p, p)$ on $H^p(\mathbb{R}^3)$ when $\frac{4}{5} < p < 1$ and $\delta = 3(\frac{1}{p} - 1) + \frac{1}{2}$, or when $p = \frac{4}{5}$ and $\delta > 3(\frac{1}{p} - 1) + \frac{1}{2}$ in [7]. It has been also shown that the above estimates are sharp. For higher dimensional cases $k \geqslant 3$, the range of $p$ for the boundedness of $S^\delta$ does not intersect with $p < 1$. In these case the kernels are not even integrable for any admissible $\delta$ (see [8]). Hence one cannot expect any result in the range $p < 1$ when $k \geqslant 3$. Even for $p \geqslant 1$, any estimate for higher dimensional Bochner–Riesz means and cone multipliers has not been sharp.

One of the main purposes of this paper is to investigate the relations between the geometry of singular sets and restrictions on the range of both $p$ and $\delta$ via careful analysis. We are interested in taking a look at both cases: $p \geqslant 1$ and $p < 1$. These have led us to take account of the case $k = 2$, that is, the case where the multiplier is associated with a product of a disc and intervals.

When the operator is defined by the multiplier associated with a product of intervals, similar restrictions on $p$ and $\delta$ have been observed by P. Oswald in [12,13]. However the existence of a disc in the product causes a more interesting feature of the operator due to the non-vanishing curvature of the boundary of the disc. The results in [7,17] are on the multiplier associated with a product of a disc and one interval and those in [12,13] are on the multipliers associated with a product of only intervals. Even though operators of the first type conveys property caused by the curvature, operators of the second type also have interesting mapping property because of the interactions between intervals. It would be interesting to see all phenomena at once.

In terms of method we develop ideas in [7,17]. However our situation is much more complicated and general than that in [7,17]. We need more systematic approach to obtain the kernel estimates. Once we obtain optimal kernel estimates, the integrability of the kernel is immediate. This helps us to obtain $L^p$ estimates when $p \geqslant 1$. If we just follow the idea in [7], then we might be in trouble because there are too many cases which we have to deal with and this will make things to be complicated. To conquer this difficulty we have to figure out essential role of each procedure made to obtain kernel estimates in [7]. As soon as one completely analyze arguments for the kernel estimates in [7], one can simplify the argument enough to deal with much more general case. This is one of the points which distinguishes our results from those in [7,17].

Now we describe how the arguments will be processed to realize all mentioned before. In view of the dilation property of the kernel of $T_\epsilon^\delta$ it suffices to consider the kernel of $T_1^\delta$. For the sake of notational convenience we set $T^\delta = T_1^\delta$ in (1.1). To obtain the sharp decay of the kernel we decompose the multiplier. Firstly, we decompose the multiplier into two parts: the one supported near the origin and the other supported near the boundary of the unit ball of the distance function $\varrho$. For each part we perform further decomposition in a dyadic way. Before doing this we subtract by the lower dimensional Bochner–Riesz multiplier of the maximal component in space variables to obtain better decay in the maximal component. Dyadic decompositions are done based on the observation of the geometry of the singular set $\mathcal{R}$. More precisely we dyadically decompose the multiplier in a way that each dyadic piece has a positive distance from the singular set and the distance is of the same size as the side-length of the piece. In this way we can obtain precise decay estimates for each piece of the kernels by using careful integration by parts and change of variables. Finally we put together the estimates for each dyadic pieces to obtain estimates for the original one. Necessity will show that this process is so efficient that there is no loss of information of the kernel in our doing this procedure.

As soon as we obtain the kernel estimates, we develop the ideas in [16] to establish weak type estimates for $p < 1$.

As for the $L^p$-bound for $T^\delta$ ($1 \leqslant p < \infty$) we obtain an $L^4$ estimate, by combining square function arguments for the Nikodym maximal function with light rays. We develop the famous idea of C. Fefferman for the square function estimates. For $L^\infty$ estimate we use the decays of the kernel to show that it is integrable (see Section 2). Owing to the complex interpolation between the results on $L^4(\mathbb{R}^n)$ for $\delta > 0$ and on $L^\infty(\mathbb{R}^n)$ for $\delta > \frac{1}{2}$, we obtain the desired bound for $\delta > \max\{2|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0\}$ and $4 \leqslant p < \infty$. The remaining ranges are obtained from duality. From this it is pointed out that the critical index $\delta(p) = 2|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$ of $T^\delta$ associated with $(1 - \max\{|\xi_0|, |\xi_1|, \dots, |\xi_{n-2}|\})_+^\delta$ on $\mathbb{R}^{2+(n-2)}$ ($n \geqslant 3$) is the same as that of a cylinder operator associated with $(1 - \max\{|\xi_0|, |\xi_1|\})_+^\delta$ on $\mathbb{R}^{2+1}$.

For $p > 1$ since the critical indexes of one- and two-dimensional Bochner–Riesz means are 0 and $2|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$ respectively, that of $T^\delta$ is $\delta(p) = \{2|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}\} + \{0 + \cdots + 0\}$. For $p < 1$ we shall show that the critical index is $\delta(p) = n(\frac{1}{p} - 1) + \frac{1}{2}$. This is due to the summation of the critical index $\frac{1}{p} - 1$ of 1-dimensional Bochner–Riesz means $(n-2)$ times and the critical index $2(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$ of 2-dimensional Bochner–Riesz means, that is,

$$\delta(p) = \left\{2\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}\right\} + \underbrace{\left\{\left(\frac{1}{p} - 1\right) + \cdots + \left(\frac{1}{p} - 1\right)\right\}}_{(n-2)\text{ times}}.$$

We shall prove the following.

**Theorem 1.** *The maximal operator $T_*^\delta$ is bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, that is,*

$$\left\| T_*^\delta f \right\|_{L^p(\mathbb{R}^n)} \leqslant C \| f \|_{H^p(\mathbb{R}^n)} \tag{1.3}$$

*if and only if $\frac{2(n-1)}{2n-1} < p < 1$ and $\delta > n(\frac{1}{p} - 1) + \frac{1}{2}$. The constant $C$ does not depend on $f$.*

**Theorem 2.** *If $\frac{2(n-1)}{2n-1} < p < 1$ and $\delta = n(\frac{1}{p} - 1) + \frac{1}{2}$ or $p = \frac{2(n-1)}{2n-1}$ and $\delta > n(\frac{1}{p} - 1) + \frac{1}{2}$, then $T_*^\delta$ maps $H^p(\mathbb{R}^n)$ boundedly into weak-$L^p(\mathbb{R}^n)$, that is,*

$$\left| \left\{ x \in \mathbb{R}^n \colon\ T_*^\delta f(x) > \alpha \right\} \right| \leqslant C\alpha^{-p} \| f \|_{H^p(\mathbb{R}^n)}^p,$$

*where the constant $C$ does not depend on $\alpha$ or $f$, and $|E|$ denotes the Lebesgue measure of $E$.*

Here $H^p$ are the standard real Hardy space as defined in E.M. Stein [15].

**Remark 1.** (i) Let $\delta(p) = n(\frac{1}{p} - 1) + \frac{1}{2}$ be the critical index. If $\delta \leqslant \delta(\frac{2(n-1)}{2n-1}) = \frac{2n-1}{2(n-1)}$ or $p \leqslant \frac{2(n-1)}{2n-1}$, one can find that $T^\delta$ fails to be bounded on $L^p(\mathbb{R}^n)$ in Section 4. We also show that $T^\delta$ is unbounded from $H^p(\mathbb{R}^n)$ to weak-$L^p(\mathbb{R}^n)$ for $p = \frac{2(n-1)}{2n-1}$ and $\delta(\frac{2(n-1)}{2n-1}) = \frac{2n-1}{2(n-1)}$, and $p = 1$ and $\delta(1) = \frac{1}{2}$. Thus we note that the indicated ranges of parameters $p$ and $\delta$ cannot be improved (see Fig. 1).

(ii) For the case $\varrho(\xi) = \max\{|\xi_0|, |\xi_1|, \dots, |\xi_{n-k}|\}$ with $\xi_0 \in \mathbb{R}^k$ and $(\xi_1, \dots, \xi_{n-k}) \in \mathbb{R}^{n-k}$ ($3 \leqslant k < n$), we can show that a critical index in the above sense does not exist, because the inverse Fourier transform of $\varrho$ is not integrable independently of $\delta$ near the set $\{x_0 \in \mathbb{R}^k,\ (x_1, \dots, x_{n-k}) \in \mathbb{R}^{n-k}\ (3 \leqslant k < n)\colon\ |x_0| = \cdots = |x_{n-k}|\}$ (see Remark 3 in Section 4).

**Theorem 3.** *The convolution operator $T^\delta$ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if and only if $\delta > \max\{2|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0\}$ and $1 \leqslant p < \infty$.*

This paper is organized as follows. In Section 2 we decompose the multiplier to obtain the decay estimates of the kernel. In Section 3 we prove Theorem 2 by using an atomic decomposition of Hardy spaces and the decays of the kernel. In Section 4 we obtain the asymptotic expansion of the operator to prove Theorem 1 and explain that the results stated above are sharp. In Section 5 in order to prove Theorem 3, we obtain an $L^4$ estimate for the convolution operator $T^\delta$ with $\delta > 0$ by using a square function arguments for the Nikodym maximal function with light rays.

We shall use $A \approx B$ if $cB \leqslant A \leqslant CB$ for some constants $c, C > 0$.

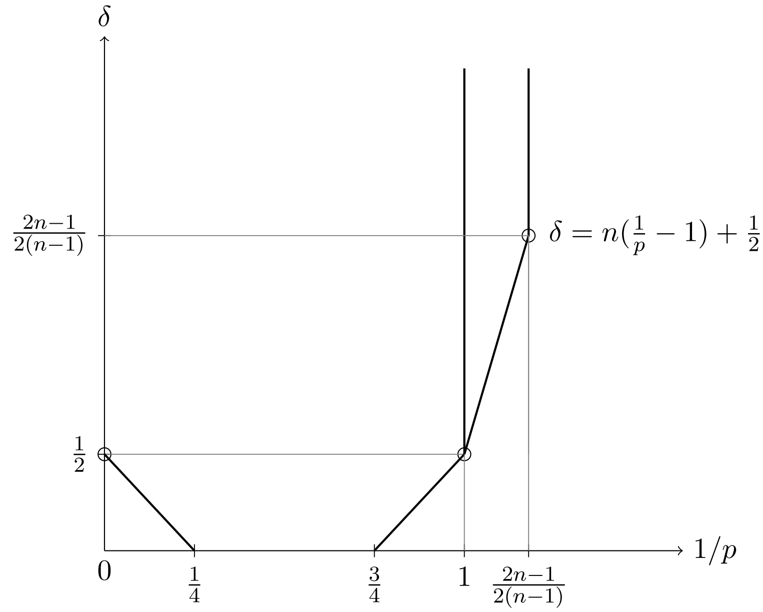


**Fig. 1.** Boundedness of $T^\delta$ when $n \geqslant 3$.

## 2. Kernel estimates

In this section we shall estimate the kernels of the multipliers associated with the distance function $\varrho$. We write

$$T^\delta_\epsilon f(x_0, \dots, x_{n-2}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} K_\epsilon(x_0 - y_0, \dots, x_{n-2} - y_{n-2}) f(y_0, \dots, y_{n-2})\, dy_0 \cdots dy_{n-2},$$

where

$$K_\epsilon(x_0, \dots, x_{n-2}) = \int_{\mathbb{R}^n} e^{i(\langle x_0, \xi_0\rangle + \sum_{j=1}^{n-2} x_j \xi_j)} \left(1 - \frac{\varrho(\xi)}{\epsilon}\right)_+^\delta d\xi_0 \cdots d\xi_{n-2}.$$

For each $\epsilon$, the kernel $K_\epsilon$ has the dilation property

$$K_\epsilon(\cdot, \dots, \cdot) = \epsilon^n K_1(\epsilon \cdot, \dots, \epsilon \cdot). \tag{2.1}$$

To simplify notations we set $K = K_1$. Let $\phi \in C_0^\infty(\mathbb{R})$ be supported in $[1/2, 2]$ such that $\sum_{k=2}^\infty \phi(2^k s) = 1$ for $s \in (0, 1/4)$ and set $\phi_1(s) = \chi_{[0,1)}(s) - \sum_{k=2}^\infty \phi(2^k(1-s))$. Let $\varphi_\circ \in C_0^\infty(\mathbb{R}^4)$ be supported in $B(0,1)$ with $\varphi_\circ(\xi) = 1$ for $\xi \in B(0, 1/2)$. We set

$$m(\xi) = \big(1 - \varrho(\xi)\big)_+^\delta$$

and decompose the multiplier $m$ into

$$\begin{aligned} m(\xi) &= m(\xi)\varphi_\circ(\xi) + m(\xi)\big(1 - \varphi_\circ(\xi)\big) \\ &= m(\xi)\varphi_\circ(\xi) + m(\xi)\varphi_\partial(\xi) \\ &= m^\circ(\xi) + m^\partial(\xi), \end{aligned} \tag{2.2}$$

where $\xi = (\xi_0, \dots, \xi_{n-2}) \in \mathbb{R}^n$.

We write

$$K^\circ = \mathcal{F}^{-1}\big[m^\circ\big], \qquad K^\partial = \mathcal{F}^{-1}\big[m^\partial\big] \quad \text{and} \quad K = K^\circ + K^\partial,$$

where $\mathcal{F}^{-1}[f]$ is denoted by the inverse Fourier transform of $f$.

Now since the multiplier has symmetry with respect to $\xi_0$ the kernel can be expressed by the Bessel function $J_\mu$ of order $\mu > -\frac{1}{2}$ defined by

$$\text{(i)}\quad J_\mu(t) = A_\mu t^\mu \int_{-1}^{1} e^{it\sigma}\big(1-\sigma^2\big)^{\mu-\frac{1}{2}}\,d\sigma \quad \text{where } A_\mu = \big[2^\mu \Gamma(2\mu+1)\Gamma(1/2)\big]^{-1},$$

$$\text{(ii)}\quad \frac{d}{dt}\big\{t^{-\mu}J_\mu(t)\big\} = -t\big\{t^{-(\mu+1)}J_{\mu+1}(t)\big\}. \tag{2.3}$$

See also [15] for more detailed properties of $J_\mu$.

For further decomposition of the kernel we shall use the idea of Müller and Seeger in [11]. They used dyadic decomposition of Bessel function to prove local smoothing conjecture for spherically symmetric initial data including endpoint results. Let $\eta \in C_0^\infty(\mathbb{R})$ be supported in $(-1/2, 2)$ and equal to 1 in $(-1/4, 1/4)$. For $m = 0, 1, 2, \ldots,$ we set

$$\eta_m(\sigma, \nu) = \begin{cases} \eta(\nu(1-\sigma^2)) & \text{if } m = 0,\\ \eta(2^{-m}\nu(1-\sigma^2)) - \eta(2^{-m+1}\nu(1-\sigma^2)) & \text{if } m > 0, \end{cases}$$

and

$$J_\mu^m(u\nu) = A_\mu(u\nu)^\mu \int_{-1}^{1} e^{i(u\nu)\sigma}\big(1-\sigma^2\big)^{\mu-1/2}\eta_m(\sigma,\nu)\,d\sigma.$$

For a positive integer $M$ we define

$$\phi_{m\nu}(\sigma) = \begin{cases} (1-\sigma^2)^{\mu-1/2}\eta_m(\sigma,\nu) & \text{if } m = 0,\\ (\frac{1}{iu\nu})^M(\frac{d}{d\sigma})^M[\eta_m(\sigma,\nu)(1-\sigma^2)^{\mu-1/2}] & \text{if } m > 0. \end{cases}$$

Then by integration by parts if $m > 0$ we have

$$J_\mu^m(u\nu) = A_\mu(u\nu)^\mu \int_{-1}^{1} e^{i(u\nu)\sigma}\phi_{m\nu}(\sigma)\,d\sigma. \tag{2.4}$$

We note that the integrand in (2.4) has the following upper bound:

$$\big|\phi_{m\nu}(\sigma)\big| \leqslant Cu^{-M}2^{-mM}\big(2^m\nu^{-1}\big)^{\mu-1/2} \tag{2.5}$$

and that $\phi_{m\nu}$ vanishes unless either $1-\sigma^2 \approx 2^m\nu^{-1}$ for $m > 0$, or $1-\sigma^2 \leqslant \nu^{-1}$ for $m = 0$. So if $\sigma$ is in the support of $\phi_{m\nu}$ then either $|\nu - \nu\sigma| \leqslant 2^m$ or $|\nu + \nu\sigma| \leqslant 2^m$.

We now give optimal kernel estimates. Let $A_\ell$ be a subset of $\mathbb{R}^n$ defined by

$$A_\ell = \big\{(x_0, \ldots, x_{n-2})\colon |x_\ell| = \max\big\{|x_0|, \ldots, |x_{n-2}|\big\}\big\}$$

and let $\chi_A$ be the characteristic function of $A$. We note that $\mathbb{R}^n$ has a non-overlapping decomposition of $A_\ell$'s. We write the kernel $K$ as

$$K(x_0, \ldots, x_{n-2}) = \sum_{\ell=0}^{n-2}\mathcal{K}_\ell(x_0, \ldots, x_{n-2}),$$

where $\mathcal{K}_\ell = \chi_{A_\ell}K$. We define functions $V_\ell$ by

$$V_\ell(x_0, \ldots, x_{n-1}) = \big(1+|x_\ell|\big)\left(\sum_{k=0}^{n-2}\big(1+|x_k|\big) - \big(1+|x_\ell|\big)\right)\prod_{j=0}^{n-2}\big(1+|x_j|\big)^{-1}.$$

We set $\mathcal{P} = \{(a_1, \ldots, a_{n-2})\colon a_k = \pm 1 \text{ for } k = 0, \ldots, n-2\}$. For constants $c$ and $b$ we shall often use the notation

$$\left(1+c\left||x_0| + \sum_{k=1}^{n-2}\pm x_k\right|\right)^b = \sum_{(a_1,\ldots,a_{n-2})\in\mathcal{P}}\left(1+c\left||x_0| + \sum_{k=1}^{n-2}a_k x_k\right|\right)^b.$$

**Proposition 1.** *There are estimates as follows*:

$$\big|\mathcal{K}_0(x_0, \ldots, x_{n-2})\big| \lesssim V_0(x_0, \ldots, x_{n-2})\Bigg[\big(1+|x_0|\big)^{-\frac{5}{2}}\left(1+\left||x_0| + \sum_{k=1}^{n-2}\pm x_k\right|\right)^{-\min\{2,\delta\}}$$
$$+ \big(1+|x_0|\big)^{-(\delta+\frac{3}{2})}\left(1+\left||x_0| + \sum_{k=1}^{n-2}\pm x_k\right|\right)^{-1}\Bigg]\chi_{A_0}(x_0, \ldots, x_{n-2})$$

*and for* $\ell = 1, \dots, n-2$

$$\left|\mathcal{K}_\ell(x_0,\dots,x_{n-2})\right| \lesssim V_\ell(x_0,\dots,x_{n-2})\left[\left(1+|x_\ell|\right)^{-2}\left(1+\left||x_0|+\sum_{k=1}^{n-2}\pm x_k\right|\right)^{-\min\{2,\delta\}} + \left(1+|x_\ell|\right)^{-(\delta+1)}\left(1+\left||x_0|+\sum_{k=1}^{n-2}\pm x_k\right|\right)^{-1}\right]\left(1+|x_0|\right)^{-\frac{1}{2}}\chi_{A_\ell}(x_0,\dots,x_{n-2}). \tag{2.6}$$

**Proof.** We first consider $\mathcal{K}_\ell$ for $\ell = 1, \dots, n-2$. Since the arguments for $\mathcal{K}_\ell$'s are exactly same for each $\ell$'s if $\ell \geqslant 1$, we only treat $\mathcal{K}_1$. By the definition of $\mathcal{K}_1$

$$|x_1| = \max\{|x_0|, \dots, |x_{n-2}|\} \tag{2.7}$$

in the support of $\mathcal{K}_1$. Hence we obtain the estimates for the kernel $K$ in the domain $A_1$.

Since $K = K^\circ + K^\partial$, we separately consider two operators $K^\circ$ and $K^\partial$. We take into account of $K^\circ$. We write the multiplier $m^\circ$ of $K^\circ$ as

$$m^\circ(\xi) = \left(\left(1-\varrho(\xi)\right)_+^\delta - \left(1-|\xi_1|\right)_+^\delta\right)\varphi_\circ(\xi) + \left(1-|\xi_1|\right)_+^\delta\varphi_\circ(\xi) = m^{\circ,1}(\xi) + m^{\circ,2}(\xi). \tag{2.8}$$

The reason why we subtract the original multiplier $m^\circ(\xi)$ by the one-dimensional Bochner–Riesz multiplier $(1-|\xi_1|)_+^\delta$ is because we eventually make use of the cancellation to gain extra decay of the maximal component $|x_1|$ of space variables. Now we pick up the maximal component of frequency variables. We note that $\mathcal{F}^{-1}(m^{\circ,2})$ has a decay in (2.6). So it suffices to treat $\mathcal{F}^{-1}(m^{\circ,1})$. Here we denote $\phi_\sigma(\cdot) = \phi(2^\sigma\cdot)$. By the definition of $\varrho$, we have $m^{\circ,1}(\xi) = 0$ if $\varrho(\xi) = |\xi_1|$. Hence the support of $m^{\circ,1}$ can be written as a union of non-overlapping domains, namely $\bigcup_{j'\neq 1} B_{j'}$, where

$$B_{j'} = \left\{(\xi_0,\dots,\xi_{n-2}) \in \operatorname{supp}(\varphi_\circ)\colon\ |\xi_{j'}| = \max\{|\xi_0|,\dots,|\xi_{n-2}|\}\right\}.$$

If $\xi = (\xi_0,\dots,\xi_{n-2}) \in B_{j'}$ for $j' \neq 1$, then $\varrho(\xi) = |\xi_{j'}|$ in (2.8). We consider the case where $j' = 0$, that is, $\varrho(\xi) = |\xi_0|$. The other cases $j' = 2,\dots,n-2$ can be treated in the same way. Since $|\xi_0| - |\xi_j| \geqslant 0$ for all $j = 1,\dots,n-2$, we decompose $m^{\circ,1}$ as

$$m^{\circ,1}(\xi) = \sum_{\ell_0,\dots,\ell_{n-2}} \phi^\circ_{\ell_0}(\xi_0)\prod_{j=1}^{n-2}\phi_{\ell_j}\left(|\xi_0|-|\xi_j|\right)m^{\circ,1}(\xi) = \sum_{\ell_0,\dots,\ell_{n-2}} m^{\circ,1}_{\ell_0,\dots,\ell_{n-2}}(\xi), \tag{2.9}$$

where $\phi^\circ_{\ell_0}(\xi_0) = \phi_{\ell_0}(|\xi_0|)$.

We set

$$K^{\circ,1}_{\ell_0,\dots,\ell_{n-2}} = \mathcal{F}^{-1}\left[m^{\circ,1}_{\ell_0,\dots,\ell_{n-2}}\right],$$

where

$$K^{\circ,1}_{\ell_0,\dots,\ell_{n-2}}(x_0,\dots,x_{n-2}) = C\int_{\mathbb{R}^n} e^{i(\langle x_0,\xi_0\rangle + \sum_{k=1}^{n-2}x_k\xi_k)} m^{\circ,1}_{\ell_0,\dots,\ell_{n-2}}(\xi_0,\dots,\xi_{n-2})\,d\xi.$$

We apply Bochner's formula in [14] to write

$$K^{\circ,1}_{\ell_0,\dots,\ell_{n-2}}(x_0,\dots,x_{n-2}) = \int_{\mathbb{R}^{n-2}}\int_0^\infty J_0\left(|x_0|s\right)\phi^\circ_{\ell_0}(s)\prod_{j=1}^{n-2}\phi_{\ell_j}\left(s-|\xi_j|\right) \times \left[(1-s)_+^\delta - \left(1-|\xi_1|\right)_+^\delta\right]\Phi_\circ\left(s,\xi'\right)e^{i\sum_{k=1}^{n-2}x_k\xi_k}s\,ds\,d\xi_1\cdots d\xi_{n-2}.$$

Thanks to the idea of Müller and Seeger in (2.4), we are able to carry out further decomposition of $K^{\circ,1}_{\ell_0,\dots,\ell_{n-2}}$ as

$$K^{\circ,1}_{\ell_0,\dots,\ell_{n-2}} = \sum_{m=0}^\infty K^{\circ,1,m}_{\ell_0,\dots,\ell_{n-2}},$$

where

$$K^{\circ,1,m}_{\ell_0,\ldots,\ell_{n-2}}(x_0,\ldots,x_{n-2}) = \int_{\mathbb{R}^{n-2}} \int_0^1 J_0^m\big(|x_0|s\big)\phi^{\circ}_{\ell_0}(s)\prod_{j=1}^{n-2}\phi_{\ell_j}\big(s-|\xi_j|\big)\Phi_{\circ}\big(s,\xi'\big) \times \big[(1-s)_+^{\delta} - \big(1-|\xi_1|\big)_+^{\delta}\big] e^{i\sum_{k=1}^{n-2} x_k\xi_k} s\,ds\,d\xi_1\cdots d\xi_{n-2}$$

$$= A_0 \int_{\mathbb{R}^{n-2}} \int_{-1}^1 \phi_{m|x_0|}(\sigma)\int_0^1 \phi^{\circ}_{\ell_0}(s)\prod_{j=1}^{n-2}\phi_{\ell_j}\big(s-|\xi_j|\big)\Phi_{\circ}\big(s,\xi'\big) \times \big(s-|\xi_1|\big)\int_0^1 \big(1-(1-t)s-t|\xi_1|\big)^{\delta-1}\,dt \times e^{i(|x_0|s\sigma+\sum_{k=1}^{n-2}x_k\xi_k)} s\,ds\,d\sigma\,d\xi_1\cdots d\xi_{n-2}.$$

We make a change of variables: $\zeta_0 = s$ and $\zeta_j = s - |\xi_j|$ to write

$$K^{\circ,1,m}_{\ell_0,\ldots,\ell_{n-2}}(x_0,\ldots,x_{n-2}) = A_0 \int_{\mathbb{R}^{n-2}} \int_{-1}^1 \phi_{m|x_0|}(\sigma)\int_0^1 \phi^{\circ}_{\ell_0}(\zeta_0)\prod_{j=1}^{n-2}\phi_{\ell_j}(\zeta_j)\widetilde{\Phi}_{\circ}(\zeta) \times \zeta_1\zeta_0\int_0^1 \big(1-t\zeta_0-(1-t)(\zeta_0-\zeta_1)\big)^{\delta-1}\,dt \times e^{i\{(|x_0|\sigma+\sum_{k=1}^{n-2}\pm x_k)\zeta_0-\sum_{k=1}^{n-2}x_k\zeta_k\}}\,d\zeta_0\,d\sigma\,d\zeta_1\cdots d\zeta_{n-2}. \tag{2.10}$$

We use

$$\zeta_1\zeta_0\int_0^1 \big(1-t\zeta_0-(1-t)(\zeta_0-\zeta_1)\big)^{\delta-1}\,dt \lesssim 2^{-\ell_0}2^{-\ell_1} \tag{2.11}$$

and integrate by parts to obtain

$$\big|K^{\circ,1,m}_{\ell_0,\ldots,\ell_{n-2}}(x_0,\ldots,x_{n-2})\big| \leqslant C2^{-\ell_0}2^{-\ell_1}\int_{\mathbb{R}^{n-2}}\int_{-1}^1\int_0^1 \big|\phi_{m|x_0|}(\sigma)\big|\frac{1}{(1+2^{-\ell_0}||x_0|\sigma+\sum_{k=1}^{n-2}\pm x_k|)^{N_0}} \times \prod_{j=1}^{n-2}\frac{1}{(1+2^{-\ell_j}|x_j|)^{N_j}}\left|\partial_\zeta^N\left(\phi^{\circ}_{\ell_0}(\zeta_0)\prod_{j=1}^{n-2}\phi_{\ell_j}(\zeta_j)\widetilde{\Phi}_{\circ}(\zeta)\right)\right|d\zeta_0\,d\sigma\,d\zeta_1\cdots d\zeta_{n-2},$$

where $N = (N_0, N_1, \ldots, N_{n-2})$. Then in view of (2.5) and the size of the domain of the integration we obtain

$$\big|K^{\circ,1,m}_{\ell_0,\ldots,\ell_{n-2}}(x_0,\ldots,x_{n-2})\big| \lesssim 2^{m(1/2+N_0-M)}\big(1+|x_0|\big)^{-1/2}2^{-\ell_0}2^{-\ell_1} \times \frac{2^{-\ell_0}}{(1+2^{-\ell_0}||x_0|+\sum_{k=1}^{n-2}\pm x_k|)^{N_0}}\prod_{j=1}^{n-2}\frac{2^{-\ell_j}}{(1+2^{-\ell_j}|x_j|)^{N_j}}.$$

We take $M$ large enough so that $M > N_0 + 1/2$ and take summation over $m$ to obtain

$$\big|K^{\circ,1}_{\ell_0,\ldots,\ell_{n-2}}(x_0,\ldots,x_{n-2})\big| \lesssim \big(1+|x_0|\big)^{-1/2}2^{-\ell_1}\frac{2^{-2\ell_0}}{(1+2^{-\ell_0}||x_0|+\sum_{k=1}^{n-2}\pm x_1|)^{N_0}}\prod_{j=1}^{n-2}\frac{2^{-\ell_j}}{(1+2^{-\ell_j}|x_j|)^{N_j}}. \tag{2.12}$$

By combining this with (2.12) we obtain

$$\big|K^{\circ,1}(x_0,\ldots,x_{n-2})\big| \leqslant \sum_{\ell_0,\ldots,\ell_{n-2}}\big|K^{\circ,1}_{\ell_0,\ldots,\ell_{n-2}}(x_0,\ldots,x_{n-2})\big| \lesssim \big(1+|x_0|\big)\big(1+|x_1|\big)\prod_{j=0}^{n-2}\big(1+|x_j|\big)^{-1}\big(1+|x_1|\big)^{-2}\left(1+\left||x_0|+\sum_{j=1}^{n-2}\pm x_j\right|\right)^{-2}\big(1+|x_0|\big)^{-\frac{1}{2}}.$$

By considering the other cases where $j' = 2, \dots, n-2$ and taking the summation we obtain that if $(x_0, \dots, x_{n-2}) \in A_1$, then

$$\left|K^{\circ,1}(x_0, \dots, x_{n-2})\right| \lesssim V_1(x_0, \dots, x_{n-2})\bigl(1 + |x_1|\bigr)^{-2}\left(1 + \left||x_0| + \sum_{k=1}^{n-2} \pm x_k\right|\right)^{-2}\bigl(1 + |x_0|\bigr)^{-\frac{1}{2}}.$$

For $K^{\partial,1}$ we manipulate the above arguments replacing $\phi^{\circ}_{\ell_0}(\xi_0)$ with $\phi^{\partial}_{\ell_0}(\xi_0) = \phi_{\ell_0}(1 - |\xi_0|)$ in (2.9) and using

$$\zeta_1\zeta_0 \int_0^1 \bigl(1 - t\zeta_0 - (1-t)(\zeta_0 - \zeta_1)\bigr)^{\delta-1}\, dt \lesssim 2^{-\ell_1} 2^{-\min\{\ell_0,\ell_1\}(\delta-1)}$$

in (2.11) to obtain that if $(x_0, \dots, x_{n-2}) \in A_1$, then

$$\begin{aligned}\left|K^{\partial,1}(x_0, \dots, x_{n-2})\right| &\lesssim V_1(x_0, \dots, x_{n-2})\bigl(1 + |x_1|\bigr)^{-(\delta+1)}\left(1 + \left||x_0| + \sum_{k=1}^{n-2} \pm x_k\right|\right)^{-1}\bigl(1 + |x_0|\bigr)^{-\frac{1}{2}} \\ &\quad + V_1(x_0, \dots, x_{n-2})\bigl(1 + |x_1|\bigr)^{-2}\left(1 + \left||x_0| + \sum_{k=1}^{n-2} \pm x_k\right|\right)^{-\delta}\bigl(1 + |x_0|\bigr)^{-\frac{1}{2}}.\end{aligned}$$

By combining the estimates for $K^{\circ,1}$ and $K^{\partial,1}$ when the space variables are in $A_1$ we obtain the desired estimates for $\mathcal{K}_1$. Arguments for the other $\mathcal{K}_\ell$'s are same as above so we leave the detailed proof to interested readers. This completes the proof. □

**Remark 2.** The estimates for the derivatives of the kernel can be obtained by the fact that $\sum_{|\gamma|=N}(K^{\circ})^{(\gamma)} = \sum_{|\gamma|=N}\Psi_1^{(\gamma)} * K^{\circ}$ and $\sum_{|\gamma|=N}(K^{\partial})^{(\gamma)} = \sum_{|\gamma|=N}\Psi_2^{(\gamma)} * K^{\partial}$ for some Schwartz functions $\Psi_1, \Psi_2 \in \mathcal{S}(\mathbb{R}^n)$. Thus, it is easily seen that the decays of the derivatives of the kernels are the same with those of kernels in Proposition 1 with different constants by using (2.3).

## 3. Weak types $(p, p)$ estimates

In this section we shall prove Theorem 2.

**Definition 1.** Let $0 < p \leqslant 1$ and $s$ be an integer that satisfies $s \geqslant n(\frac{1}{p} - 1)$. Let $Q$ be a cube in $\mathbb{R}^n$. We say that $\mathfrak{a}$ is a $(p, s)$-atom associated with $Q$ if $\mathfrak{a}$ is supported on $Q \subset \mathbb{R}^n$ and satisfies

(i) $\|\mathfrak{a}\|_{L^\infty(\mathbb{R}^n)} \leqslant |Q|^{-1/p}$;

(ii) $\displaystyle\int_{\mathbb{R}^n} \mathfrak{a}(x)x^\beta\, dx = 0$,

where $\beta = (\beta_1, \dots, \beta_n)$ is an $n$-tuple of non-negative integers satisfying $|\beta| \leqslant \beta_1 + \dots + \beta_n \leqslant s$, and $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.

If $\{\mathfrak{a}_j\}$ is a collection of $(p, s)$-atoms and $\{c_j\}$ is a sequence of complex numbers with $\sum_{j=1}^\infty |c_j|^p < \infty$, then the series $f = \sum_{j=1}^\infty c_j\mathfrak{a}_j$ converges in the sense of distributions, and its sum belongs to $H^p$ with the quasinorm (see [15])

$$\|f\|_{H^p} = \inf_{\sum_{j=1}^\infty c_j\mathfrak{a}_j = f}\left(\sum_{j=1}^\infty |c_j|^p\right)^{1/p}.$$

To prove Theorem 2 we shall need a lemma by Stein, Taibleson and Weiss [16].

**Lemma 1.** *Suppose* $0 < p < 1$ *and* $\{f_j\}$ *is a sequence of measurable functions such that*

$$\left|\left\{x\colon\ \left|f_j(x)\right| > \alpha > 0\right\}\right| \leqslant \alpha^{-p}$$

*for* $j = 1, 2, 3, \dots$. *If* $\sum_{j=1}^\infty |c_j|^p \leqslant 1$, *then*

$$\left|\left\{x\colon\ \left|\sum_{j=1}^\infty c_jf_j(x)\right| > \alpha\right\}\right| \leqslant \frac{2-p}{1-p}\alpha^{-p}.$$

**Proof.** See Lemma 1.8 in [16]. □

We shall use the following elementary lemma to obtain weak type estimates in the proof of Proposition 2. For $|x_\ell| > 2$ $(0 \leqslant \ell \leqslant n-2)$ we set

$$\mathcal{D}_1 = \{x\colon |x_0| \geqslant |x_1| \geqslant \cdots \geqslant |x_{n-2}|\},$$
$$\mathcal{D}_2 = \{x\colon |x_\ell| \geqslant |x_0| \geqslant \cdots \geqslant |\widehat{x_\ell}| \geqslant \cdots \geqslant |x_{n-2}|\}$$

and

$$\mathcal{F}(x) = |x_0|^{-(\delta+\frac{3}{2})}|x_1|^{-1}\cdots|x_{n-2}|^{-1}| \cdot \chi_{\mathcal{D}_1},$$
$$\mathcal{G}(x) = |x_\ell|^{-(\delta+1)}|x_0|^{-\frac{3}{2}}|x_1|^{-1}\cdots|\widehat{x_\ell}|^{-1}\cdots|x_{n-2}|^{-1} \cdot \chi_{\mathcal{D}_2},$$
$$\mathcal{H}(x) = |x_0|^{-\frac{5}{2}}|x_1|^{-1}\cdots|x_{n-2}|^{-1} \cdot \chi_{\mathcal{D}_1},$$

where we denote $\widehat{x_\ell}$, provided that $x_\ell$ component is missing.

**Lemma 2.** *There are estimates as follows*:

(1) *If* $\frac{2(n-1)}{2n-1} < p < 1$ *and* $\delta = n(\frac{1}{p}-1)+\frac{1}{2}$, *then*

$$\text{(i)} \quad \left|\{x \in \mathcal{D}_1\colon \mathcal{F}(x) > \alpha/C\}\right| \lesssim \alpha^{-p},$$
$$\text{(ii)} \quad \left|\{x \in \mathcal{D}_2\colon \mathcal{G}(x) > \alpha/C\}\right| \lesssim \alpha^{-p}.$$

(2) *If* $p = \frac{2(n-1)}{2n-1}$ *and* $\delta > n(\frac{1}{p}-1)+\frac{1}{2}$, *then*

$$\left|\{x \in \mathcal{D}_1\colon \mathcal{H}(x) > \alpha/C\}\right| \lesssim \alpha^{-p}.$$

**Proof.** We consider (1). We note that the sum of each exponent in $\mathcal{F}$ and $\mathcal{G}$ is $\delta + \frac{3}{2} + (n-2) = \frac{n}{p}$ if $\delta = n(\frac{1}{p}-1)+\frac{1}{2}$.

To prove (i), let us denote

$$\mathcal{D}_{11} = \{x \in \mathcal{D}_1\colon \mathcal{F}(x) > \alpha/C,\ 2 < |x_0| \leqslant \alpha^{-\frac{p}{n}}\} \quad \text{and}$$
$$\mathcal{D}_{12} = \{x \in \mathcal{D}_1\colon \mathcal{F}(x) > \alpha/C,\ |x_0| > \alpha^{-\frac{p}{n}}\}.$$

The weak type set in the left-hand side of (i) is bounded by

$$\int\limits_{\{x\in\mathcal{D}_{11}\}} + \int\limits_{\{x\in\mathcal{D}_{12}\}} dx_{n-2}\cdots dx_1\,dx_0 \lesssim \int\limits_{\{x\colon 2<|x_{n-2}|\leqslant\cdots\leqslant|x_1|\leqslant|x_0|,\ 2<|x_0|\leqslant\alpha^{-\frac{p}{n}}\}} dx_{n-2}\cdots dx_1\,dx_0 + \int\limits_{\substack{\{x\colon 2<|x_{n-2}|\leqslant\alpha^{-1}|x_{n-3}|^{-\{\delta+\frac{3}{2}+(n-3)\}} \\ 2<|x_{n-3}|\leqslant\cdots\leqslant|x_1|\leqslant|x_0|,\ |x_0|>\alpha^{-\frac{p}{n}}\}}} dx_{n-2}\cdots dx_1\,dx_0.$$

Then by using polar coordinates with $|x_0| = r$,

$$\begin{aligned}\left|\{x \in \mathcal{D}_1\colon \mathcal{F}(x) > \alpha/C\}\right| &\lesssim \int\limits_{\{2<|x_0|\leqslant\alpha^{-\frac{p}{n}}\}} |x_0|^{n-2}\,dx_0 + \alpha^{-1}\int\limits_{\{|x_0|>\alpha^{-\frac{p}{n}}\}} |x_0|^{(n-3)-\{\delta+\frac{3}{2}+(n-3)\}}\,dx_0 \\ &= \int_0^{2\pi}\int\limits_{\{r\colon 2<r\leqslant\alpha^{-\frac{p}{n}}\}} r^{n-1}\,dr\,d\theta + \alpha^{-1}\int_0^{2\pi}\int\limits_{\{r>\alpha^{-\frac{p}{n}}\}} r^{(n-2)-\{\delta+\frac{3}{2}+(n-3)\}}\,dr\,d\theta \\ &\lesssim \alpha^{-p}.\end{aligned}$$

For the proof of (ii) let us denote

$$\mathcal{D}_{21} = \{x \in \mathcal{D}_2\colon \mathcal{G}(x) > \alpha/C,\ 2 < |x_\ell| \leqslant \alpha^{-\frac{p}{n}}\} \quad \text{and}$$
$$\mathcal{D}_{22} = \{x \in \mathcal{D}_2\colon \mathcal{G}(x) > \alpha/C,\ |x_\ell| > \alpha^{-\frac{p}{n}}\}.$$

Likewise (i), we have

$$\begin{aligned}
\left|\left\{x\in\mathcal{D}_2\colon \mathcal{G}(x)>\alpha/C\right\}\right| &\lesssim \int\limits_{\{x\colon 2<|x_{n-2}|\leqslant\cdots\leqslant|x_1|\leqslant|x_0|\leqslant|x_\ell|,\ 2<|x_\ell|\leqslant\alpha^{-\frac{p}{n}}\}} dx_{n-2}\cdots d\widehat{x_\ell}\cdots dx_1\,dx_0\,dx_\ell \\
&\quad+ \int\limits_{\substack{\{x\colon 2<|x_{n-2}|\leqslant\alpha^{-1}\ |x_{n-3}|^{-\{\delta+\frac{3}{2}+(n-3)\}} \\ 2<|x_{n-3}|\leqslant\cdots\leqslant|x_0|\leqslant|x_\ell|,\ |x_\ell|>\alpha^{-\frac{p}{n}}\}}} dx_{n-2}\cdots d\widehat{x_\ell}\cdots dx_1\,dx_0\,dx_\ell \\
&\lesssim \int\limits_{\{2<r\leqslant|x_\ell|,\ 2<|x_\ell|\leqslant\alpha^{-\frac{p}{n}}\}} r^{n-3} r\,dr\,d\theta\,dx_\ell \\
&\quad+\alpha^{-1}\int\limits_{\{2<r\leqslant|x_\ell|,|x_\ell|>\alpha^{-\frac{p}{n}}\}} r^{(n-4)-\{\delta+\frac{3}{2}+(n-4)\}} r\,dr\,d\theta|x_\ell|^{-1}\,dx_\ell \\
&\lesssim \int\limits_{\{2<|x_\ell|\leqslant\alpha^{-\frac{p}{n}}\}} |x_\ell|^{n-1}\,dx_\ell+\alpha^{-1}\int\limits_{\{|x_\ell|>\alpha^{-\frac{p}{n}}\}} |x_\ell|^{(n-2)-\{\delta+\frac{3}{2}+(n-3)\}}\,dx_\ell \\
&\lesssim \alpha^{-p}.
\end{aligned}$$

Next, we prove (2). If $p=\frac{2(n-1)}{2n-1}$ and $\delta>\frac{2n-1}{2(n-1)}$, we put $\delta=\frac{2n-1}{2(n-1)}+(n-2)\epsilon_0$ for some $\epsilon_0>0$ and for all $n\geqslant 3$. Then for $x\in\mathcal{D}_1$

$$\mathcal{H}(x)\lesssim|x_0|^{-\frac{2n-1}{n-1}}\prod_{\ell=1}^{n-2}|x_\ell|^{-(\frac{2n-1}{2(n-1)}+\epsilon_0)},$$

since $\frac{5}{2}=\frac{2n-1}{n-1}+\frac{n-3}{2(n-1)}$. Using polar coordinates, we obtain

$$\begin{aligned}
\left|\left\{x\in\mathcal{D}_3\colon \mathcal{H}(x)>\alpha/C\right\}\right| &\lesssim \int\limits_{\{x\in\mathcal{D}_3\colon |x_0|^{-\frac{2n-1}{n-1}}\prod_{\ell=1}^{n-2}|x_\ell|^{-(\frac{2n-1}{2(n-1)}+\epsilon_0)}>\alpha\}} dx_0\,dx_1\cdots dx_{n-2} \\
&\lesssim \alpha^{-\frac{2(n-1)}{2n-1}}\int_2^\infty\prod_{\ell=1}^{n-2}|x_\ell|^{-\frac{2(n-1)}{2n-1}(\frac{2n-1}{2(n-1)}+\epsilon_0)}\,dx_1\cdots dx_{n-2} \\
&\lesssim \alpha^{-\frac{2(n-1)}{2n-1}}. \qquad\square
\end{aligned}$$

To prove Theorem 2, we shall need uniform weak type estimates for $T_*^\delta$ with a $(p,N)$-atom ($N\geqslant n(\frac{1}{p}-1)$).

**Proposition 2.** *Let $n\geqslant 3$. Suppose $f$ is a $(p,N)$-atom ($N\geqslant n(\frac{1}{p}-1)$) on $\mathbb{R}^n$. Suppose that $\frac{2(n-1)}{2n-1}<p<1$ and $\delta=n(\frac{1}{p}-1)+\frac{1}{2}$ or $p=\frac{2(n-1)}{2n-1}$ and $\delta>n(\frac{1}{p}-1)+\frac{1}{2}$. Then there exists a constant $C=C(p)$ such that*

$$\left|\left\{x\in\mathbb{R}^n\colon T_*^\delta f(x)>\alpha\right\}\right|\leqslant C\alpha^{-p} \tag{3.1}$$

*for all $\alpha>0$.*

**Proof.** Let $f$ be supported in a cube $Q_0$ of diameter 1 centered at the origin. We first consider the case $x\in Q_0^*$ which is the cube centered at the origin with diameter 4. In view of (2.1) and Proposition 1 we can easily see that $K_\epsilon$ is integrable and its $L^1$ norm is independent of $\epsilon$. Thus we have

$$\left|T_\epsilon^\delta f(x)\right|\leqslant\|K_\epsilon\|_1\|f\|_\infty\leqslant\|K_\epsilon\|_1\left|Q_0^*\right|^{-1/p}.$$

Therefore

$$T_*^\delta f(x)=\sup_{\epsilon>0}\left|T_\epsilon^\delta f(x)\right|\leqslant C\left|Q_0^*\right|^{-1/p}$$

for all $x\in Q_0^*$, and which implies that for $\alpha>0$

$$\left|\left\{x\in Q_0^*\colon T_*^\delta f(x)>\alpha/C\right\}\right|\leqslant C\alpha^{-p}. \tag{3.2}$$

Hence it suffices to show that for $\alpha > 0$

$$\left|\left\{x \in \mathbb{R}^n \setminus Q_0^*:\ T_*^\delta f(x) > \alpha/C\right\}\right| \leqslant C\alpha^{-p}.$$

We denote $T_\epsilon^\delta f = K_\epsilon * f$. For the notational convenience we set $K_1 = K$. We note that $A_\ell$ is a subset of $\mathbb{R}^n$ defined by

$$A_\ell = \left\{(x_0, \ldots, x_{n-2}):\ |x_\ell| = \max\left\{|x_0|, \ldots, |x_{n-2}|\right\}\right\}$$

and denote

$$V_{\epsilon,\ell}(x_0, \ldots, x_{n-2}) = \left(1 + \epsilon|x_\ell|\right)\left(\sum_{k=0}^{n-2}\left(1 + \epsilon|x_k|\right) - \left(1 + \epsilon|x_\ell|\right)\right)\prod_{j=0}^{n-2}\left(1 + \epsilon|x_j|\right)^{-1}.$$

We fix $\epsilon \geqslant 1$ and use the fact that $f$ is supported in $Q_0$ and (2.1) to write

$$T_\epsilon^\delta f(x) = \epsilon^n \int_{Q_0} f(y_0, \ldots, y_{n-2}) K\left(\epsilon(x_0 - y_0), \ldots, \epsilon(x_{n-2} - y_{n-2})\right) dy_0 \cdots dy_{n-2} := \sum_{\ell=0}^{n-2} \mathcal{T}_{\epsilon,\ell}^\delta f(x),$$

where

$$\mathcal{T}_{\epsilon,\ell}^\delta f(x) = \epsilon^n \int_{Q_0} f(y_0, \ldots, y_{n-2}) \mathcal{K}_\ell\left(\epsilon(x_0 - y_0), \ldots, \epsilon(x_{n-2} - y_{n-2})\right) dy_0 \cdots dy_{n-2}$$

with $\mathcal{K}_\ell = \chi_{A_\ell} K$. By the kernel estimates in Proposition 1, we write

$$\begin{aligned}\left|\mathcal{T}_{\epsilon,0}^\delta f(x)\right| \lesssim \epsilon^n \int_{Q_0} \Bigg[&\left(1 + \epsilon|x_0 - y_0|\right)^{-\frac{5}{2}}\left(1 + \epsilon\left||x_0 - y_0| \pm \sum_{k=1}^{n-2}(x_k - y_k)\right|\right)^{-\min\{\delta,2\}} \\ &+ \left(1 + \epsilon|x_0 - y_0|\right)^{-(\delta+\frac{3}{2})}\left(1 + \epsilon\left||x_0 - y_0| \pm \sum_{k=1}^{n-2}(x_k - y_k)\right|\right)^{-1}\Bigg] V_{\epsilon,0}(x)\chi_{A_0}(x)\,dy\end{aligned}$$

and for $\ell = 1, \ldots, n-2$

$$\begin{aligned}\left|\mathcal{T}_{\epsilon,\ell}^\delta f(x)\right| \leqslant \epsilon^n \int_{Q_0} \left(1 + \epsilon|x_0 - y_0|\right)^{-\frac{1}{2}} \Bigg[&\left(1 + \epsilon|x_\ell - y_\ell|\right)^{-2}\left(1 + \epsilon\left||x_0 - y_0| \pm \sum_{k=1}^{n-2}(x_k - y_k)\right|\right)^{-\min\{\delta,2\}} \\ &+ \left(1 + \epsilon|x_\ell - y_\ell|\right)^{-(\delta+1)}\left(1 + \epsilon\left||x_0 - y_0| \pm \sum_{k=1}^{n-2}(x_k - y_k)\right|\right)^{-1}\Bigg] V_{\epsilon,\ell}(x)\chi_{A_\ell}(x)\,dy.\end{aligned}$$

Since $(x_0, x_1, \ldots, x_{n-2}) \in (Q_0^*)^c$, we have

$$\begin{aligned}\left|\mathcal{T}_{\epsilon,0}^\delta f(x)\right| \lesssim \epsilon^{n-\min\{\delta,2\}-(n-1)-\frac{1}{2}} \Bigg[&\left(1 + |x_0|\right)^{-\frac{5}{2}}\left(1 + \left||x_0| \pm \sum_{k=1}^{n-2} x_k\right|\right)^{-\min\{\delta,2\}} \\ &+ \left(1 + |x_0|\right)^{-(\delta+\frac{3}{2})}\left(1 + \left||x_0| \pm \sum_{k=1}^{n-2} x_k\right|\right)^{-1}\Bigg] V_{1,0}(x)\chi_{A_0}(x)\end{aligned} \tag{3.3}$$

and for $\ell = 1, \ldots, n-2$

$$\begin{aligned}\left|\mathcal{T}_{\epsilon,\ell}^\delta f(x)\right| \lesssim \epsilon^{n-\min\{\delta,2\}-(n-1)-\frac{1}{2}}\left(1 + |x_0|\right)^{-\frac{1}{2}} \Bigg[&\left(1 + |x_\ell|\right)^{-2}\left(1 + \left||x_0| \pm \sum_{k=1}^{n-2} x_k\right|\right)^{-\min\{\delta,2\}} \\ &+ \left(1 + |x_\ell|\right)^{-(\delta+1)}\left(1 + \left||x_0| \pm \sum_{k=1}^{n-2} x_k\right|\right)^{-1}\Bigg] V_{1,\ell}(x)\chi_{A_\ell}(x).\end{aligned} \tag{3.4}$$

Then we apply Chebyshev's inequality for (3.3), (3.4) and use Lemma 2 to obtain

$$\left|\left\{x \in \mathbb{R}^n \setminus Q_0^*:\ \sup_{\epsilon \geqslant 1}\left|\sum_{\ell=0}^{n-2} \mathcal{T}_{\epsilon,\ell}^\delta f(x)\right| > \alpha/C\right\}\right| \leqslant C\alpha^{-p}. \tag{3.5}$$

Now we consider the complementary case. Fix $0 < \epsilon < 1$.

Suppose that $\delta = n(\frac{1}{p} - 1) + \frac{1}{2}$ and $\frac{2(n-1)}{2n-1} < p < 1$. Let $P_N$ be the $N$-th order Taylor polynomial of the function $y_{n-2} \to \epsilon^n K(\epsilon(x_0 - y_0), \dots, \epsilon(x_{n-2} - y_{n-2}))$ expanded about the origin. Then by using the moment conditions on $f$ in Definition 1(ii) and integrating with respect to $y_{n-2}$ first, we write

$$T_\epsilon^\delta f(x) = \int_{Q_0} f(y)\big[\epsilon^n K\big(\epsilon(x_0 - y_0), \dots, \epsilon(x_{n-2} - y_{n-2})\big) - P_N(y_{n-2})\big]\, dy_{n-2} \cdots dy_1\, dy_0.$$

By using the integral version of the mean value theorem, we obtain

$$\begin{aligned} &\big|\epsilon^n K\big(\epsilon(x_0 - y_0), \dots, \epsilon(x_{n-2} - y_{n-2})\big) - P_N(y_{n-2})\big| \\ &\quad \lesssim \epsilon^n \epsilon^{(N+1)} \int_{[0,1]^{N+1}} \left| \frac{\partial^{N+1} K_1}{\partial y_{n-2}^{N+1}} \big(\epsilon(x_0 - y_0), \dots, \epsilon(x_{n-2} - \hat{u} y_{n-2})\big) \right| du_1 \cdots du_{N+1}, \end{aligned}$$

where $\hat{u} = \prod_{i=1}^{N+1} u_i$. Since we gain $\epsilon^{(N+1)}$ and the kernel $K_1$ has the same decay after taking derivatives, we see that the exponent of $\epsilon$ is

$$N + 1 + \big(n - \min\{\delta, 2\} - (n-1) - 1/2\big) > 0$$

with $N \geqslant n(\frac{1}{p} - 1)$.

Suppose now that $\delta > n(\frac{1}{p} - 1) + \frac{1}{2}$ and $p = \frac{2(n-1)}{2n-1}$. Then we take Taylor polynomial $P_{N+[\delta]}$ of $(N + [\delta])$-th order and repeat the same argument as above. Then we can show that $N + [\delta] + 1 + (n - \min\{\delta, 2\} - (n-1) - 1/2) > 0$.

Finally, if we use Chebyshev's inequality for (3.3), (3.4) and Lemma 2 for $\epsilon < 1$, which imply

$$\left| \left\{ x \in \mathbb{R}^n \setminus Q_0^*\colon \sup_{0<\epsilon<1} \left| \sum_{\ell=0}^{n-2} \mathcal{T}_{\epsilon,\ell}^\delta f(x) \right| > \alpha/C \right\} \right| \leqslant C\alpha^{-p}. \tag{3.6}$$

Together with (3.5) and (3.6), we see that

$$\left| \left\{ x \in \big(Q_0^*\big)^c\colon \sup_{\epsilon>0} \big|T_\epsilon^\delta f(x)\big| > \alpha/C \right\} \right| \leqslant C\alpha^{-p}.$$

Thus, together with (3.2) it follows that (3.1) holds for the cube $Q_0$ of diameter 1.

Now we suppose that $f$ is a $(p, N)$-atom ($N \geqslant n(\frac{1}{p} - 1)$), supported in a cube $Q$ of diameter $R$ centered at $({x_0}^Q, \underline{x}^Q) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$. By translation invariance we can assume $({x_0}^Q, \underline{x}^Q)$ is the origin. Let $h(x_0, \underline{x}) = R^{n/p} f(Rx_0, R\underline{x})$. Then $h$ is an atom supported in the cube $Q_0$ centered at the origin and we write

$$T_\epsilon^\delta f(x_0, \underline{x}) = \int_{\mathbb{R}^n} R^{-n/p} h\big(R^{-1}(x_0 - y_0), R^{-1}(\underline{x} - \underline{y})\big) K_\epsilon(y_0, \underline{y})\, dy_0\, d\underline{y} = R^{-n/p} T_{\epsilon R}^\delta h\big(R^{-1}x_0, R^{-1}\underline{x}\big),$$

which implies

$$\sup_{\epsilon>0} \big|T_\epsilon^\delta f(x_0, \underline{x})\big| = R^{-n/p} \sup_{\epsilon>0} \big|T_{\epsilon R}^\delta h\big(R^{-1}x_0, R^{-1}\underline{x}\big)\big|.$$

We therefore have

$$\begin{aligned} \big|\big\{(x_0, \underline{x}) \in \mathbb{R} \times \mathbb{R}^{n-2}\colon T_*^\delta f(x_0, \underline{x}) > \alpha/C\big\}\big| &= \big|\big\{(x_0, \underline{x}) \in \mathbb{R} \times \mathbb{R}^{n-2}\colon T_*^\delta h\big(R^{-1}x_0, R^{-1}\underline{x}\big) > R^{n/p}\alpha/C\big\}\big| \\ &\leqslant C\big(R^{n/p}\alpha\big)^{-p} R^n = C\alpha^{-p}. \end{aligned}$$

This completes the proof. □

## 4. The proof of Theorem 1 and sharpness

In this section we prove Theorem 1 and it cannot be improved in the sense that there exists a function $f$ in $H^p$ space such that if $\delta \leqslant \delta_p = n(\frac{1}{p} - 1) + \frac{1}{2}$, then

$$\big\|T_1^\delta f\big\|_{L^p} = \infty$$

and if $p \leqslant \frac{2(n-1)}{2n-1}$, then

$$\big\|T_1^\delta f\big\|_{L^p} = \infty.$$

**Proof of Theorem 1.** We first show the sufficiency. Let $f$ be a $(p,N)$-atom ($N \geqslant n(\frac{1}{p}-1)$) on $\mathbb{R}^n$ and supported in a cube $Q$. Due to the translation invariance and maximality of $T_*^\delta$ as above we may assume that $Q$ is centered at the origin with diameter 1. In view of atomic decomposition, it suffices to show that there exists a $C$ independent of $f$ such that

$$\Bigl\|\sup_{\epsilon>0}\bigl|T_\epsilon^\delta f\bigr|\Bigr\|_{L^p(\mathbb{R}^n)} \leqslant C|Q|^{-1/p}.$$

By the integrability of the kernel, it is easy to see that

$$\Bigl\|\sup_{\epsilon>0}\bigl|T_\epsilon^\delta f\bigr|\Bigr\|_{L^p(Q^*)}^p \leqslant C\int_{Q^*}|Q|^{-1}\,dx_0\,dx_1\cdots dx_{n-2} \leqslant C.$$

For the complementary case we take the $p$-th power on both sides in (3.3), (3.4) and integrate them if $\frac{2(n-1)}{2n-1} < p < 1$ and $\delta > n(\frac{1}{p}-1)+\frac{1}{2}$. Then we can obtain the $L^p$ boundedness (1.3) as desired.

We now show the necessity. Let us denote

$$\mathcal{U} = \left\{\xi = (\xi_0,\xi_1,\dots,\xi_{n-2}) \;\middle|\; \frac{1}{2} < |\xi_0| < 1,\ \frac{1}{2} < \xi_1,\dots,\xi_{n-2} < 1\right\}. \tag{4.1}$$

We take compactly supported smooth function $\varphi(\xi) = \phi(|\xi_0|)\phi(\xi_1)\cdots\phi(\xi_{n-1})$, where $\phi$ is a smooth cut-off function which is supported in a small neighborhood of 1 and identically 1 near 1. We choose suitable cut-off functions $\varphi_\alpha$ supported in $\mathbb{R}^n\setminus\mathcal{U}$ and constants $c_\alpha$ such that

$$\int_{\mathbb{R}^n}\Bigl[\widehat{\varphi}(x) + \sum_{|\alpha|\leqslant N_p} c_\alpha\widehat{\varphi_\alpha}(x)\Bigr]x^\alpha\,dx = 0,$$

where $\alpha = (\alpha_1,\dots,\alpha_n)$ is an $n$-tuple of non-negative integers satisfying $|\alpha| \leqslant \alpha_1+\alpha_2+\cdots+\alpha_n \leqslant s$, and $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$. If we set $f(x) = \widehat{\varphi}(x) + \sum_{|\alpha|\leqslant N_p} c_\alpha\widehat{\varphi_\alpha}(x)$, then $f$ is in $H^p(\mathbb{R}^n)$ (see Lemma 4.1 in [6]). We therefore have for $\underline{\xi} = (\xi_1,\dots,\xi_{n-2})$

$$\widehat{T_1^\delta f}(\xi) = \bigl(1-\max\bigl\{|\xi_0|,|\xi_1|,\cdots,|\xi_{n-2}|\bigr\}\bigr)_+^\delta\varphi(\xi), \quad (\xi_0,\underline{\xi}) \in \mathbb{R}^2\times\mathbb{R}^{n-2}.$$

Let us denote $\underline{\widehat{\xi_\ell}} = (\xi_1,\dots,\widehat{\xi_\ell},\dots,\xi_{n-2})$, provided that $\xi_\ell$ component is missing. We consider

$$\begin{aligned}
T_1^\delta f(x) &= \int_0^1\int_0^{|\xi_0|} e^{i\langle x_0,\xi_0\rangle + i(x_1\xi_1+\cdots+x_{n-2}\xi_{n-2})}\bigl(1-|\xi_0|\bigr)_+^\delta\phi\bigl(|\xi_0|\bigr)\prod_{k=1}^{n-2}\phi(\xi_k)\,d\underline{\xi_\ell}\,d\xi_0\\
&\quad + \sum_{\ell=1}^{n-2}\int_0^1\int_0^{\xi_\ell} e^{i\langle x_0,\xi_0\rangle + i(x_1\xi_1+\cdots+x_{n-2}\xi_{n-2})}(1-\xi_\ell)_+^\delta\phi\bigl(|\xi_0|\bigr)\prod_{k=1}^{n-2}\phi(\xi_k)\,d\xi_0\,d\underline{\widehat{\xi_\ell}}\,d\xi_\ell\\
&= \mathcal{I}' + \sum_{\ell=1}^{n-2}\mathcal{I}_\ell.
\end{aligned}$$

We treat $\mathcal{I}'$. By integrating by parts with respect to $\xi_1,\dots,\xi_{n-2}$ variables in $\mathcal{I}'$, we obtain

$$\int_0^{|\xi_0|} e^{ix_\ell\xi_\ell}\phi(\xi_\ell)\,d\xi_\ell = \frac{e^{ix_\ell|\xi_0|}}{ix_\ell}\phi\bigl(|\xi_0|\bigr) - \frac{e^{ix_\ell|\xi_0|}}{(ix_\ell)^2}\phi'\bigl(|\xi_0|\bigr) + \frac{1}{(ix_\ell)^2}\int_0^{|\xi_0|} e^{ix_\ell\xi_\ell}\phi''(\xi_\ell)\,d\xi_\ell,$$

where $\ell = 1,\dots,n-2$. Thus we see that the first each term $\frac{e^{ix_\ell|\xi_0|}}{ix_\ell}\phi(|\xi_0|)$ is a leading term in the above integrals, respectively. With those terms we rewrite the integral about $\xi_0$ to have

$$\mathcal{I}'_A = \frac{1}{\prod_{\ell=1}^{n-2}(ix_\ell)}\int_0^1 e^{i\langle x_0,\xi_0\rangle + i(x_1+\cdots+x_{n-2})|\xi_0|}\bigl(1-|\xi_0|\bigr)_+^\delta\phi\bigl(|\xi_0|\bigr)^{n-1}\,d\xi_0.$$

Now let $m = \frac{n}{2}-1$. For a function $f(r)$ in $(0,\infty)$ we define the Hankel transform $\widehat{f}(r)$ by

$$\widehat{f}(r) = \int_0^\infty f(s)J_m(rs)(rs)^{-m}s^{2m+1}\,ds, \tag{4.2}$$

where $J_m$ is the Bessel function of order $m$. It is also well known (see [15]) that for $r \geqslant 1$

$$J_m(r) = e^{ir} r^{-1/2}\left(\sum_{j=0}^{N} a_j r^{-j} + A_N(r)\right) + e^{-ir} r^{-1/2}\left(\sum_{j=0}^{N} b_j r^{-j} + B_N(r)\right), \tag{4.3}$$

where $A_N$ and $B_N$ satisfy

$$\left|\left(\frac{d}{dr}\right)^k A_N(r)\right| \leqslant C r^{-k-N}, \qquad \left|\left(\frac{d}{dr}\right)^k B_N(r)\right| \leqslant C r^{-k-N}. \tag{4.4}$$

By applying (4.2) and (4.3) in the integral $\mathcal{I}'_A$, we rewrite

$$\begin{aligned}
\mathcal{I}'_A &= \frac{1}{\prod_{\ell=1}^{n-2}(ix_\ell)} \int_0^1 e^{i(x_1+\cdots+x_{n-2})s}(1-s)^\delta J_0\big(|x_0|s\big)\phi(s)^{n-1} s\,ds \\
&= \frac{1}{|x_0|^{1/2}\prod_{\ell=1}^{n-2}(ix_\ell)}\left(\sum_{j=0}^{N} a_j |x_0|^{-j} \int_0^1 e^{i(x_1+\cdots+x_{n-2}+|x_0|)s}(1-s)^\delta \phi(s)^{n-1} s^{1/2}\,ds\right. \\
&\quad \left. + \sum_{j=0}^{N} b_j |x_0|^{-j} \int_0^1 e^{i(x_1+\cdots+x_{n-2}-|x_0|)s}(1-s)^\delta \phi(s)^{n-1} s^{1/2}\,ds\right) + R_N(x_0, x_1, \ldots, x_{n-2}) \\
&:= \mathcal{I}_{A(1)} + R_N(x_0, x_1, \ldots, x_{n-2}),
\end{aligned}$$

where $R_N(x_0, x_1, \ldots, x_{n-2})$ is a constant multiple of

$$|x_0|^{-1/2} \int_0^1 \big(A_N\big(|x_0|s\big)e^{i(x_1+\cdots+x_{n-2}+|x_0|)s} + B_N\big(|x_0|s\big)e^{i(x_1+\cdots+x_{n-2}-|x_0|)s}\big)(1-s)^\delta \phi(s)^{n-1} s^{1/2}\,ds.$$

Using (4.4) and $\operatorname{supp}\phi \subset (1/2, 1)$, we obtain that

$$\big|R_N(x_0, x_1, \ldots, x_{n-2})\big| \leqslant C|x_0|^{-N}\big\{\big(x_1+\cdots+x_{n-2}+|x_0|\big)^{-M} + \big(x_1+\cdots+x_{n-2}-|x_0|\big)^{-M}\big\}.$$

For the estimates of $\mathcal{I}_{A(1)}$ we apply the asymptotic expansion in [4, pp. 46–51] to obtain

$$\int_0^1 e^{i(x_1+\cdots+x_{n-2}+|x_0|)s}(1-s)^\delta \big\{\phi(s)^{n-1}\big\} s^{1/2}\,ds = \mathcal{A}_{N_0}(x_0, x_1, \ldots, x_{n-2}) + O\big(\big(x_1+\cdots+x_{n-2}+|x_0|\big)^{-N_0}\big), \tag{4.5}$$

where

$$\mathcal{A}_{N_0}(x_0, x_1, \ldots, x_{n-2}) = \sum_{\mu=0}^{N_0-1} \frac{\Gamma(\mu+\delta+1)}{\mu!}\big[\phi(s)^{n-1}s^{1/2}\big]^{(\mu)}(1)\frac{e^{i(x_1+\cdots+x_{n-2}+|x_0|)}}{(x_1+\cdots+x_{n-2}+|x_0|)^{\mu+\delta+1}},$$

and $\{\phi(s)\}^{n-1}s^{1/2}$ are $C^\infty$-functions. Then we write

$$\begin{aligned}
\mathcal{I}_{A(1)} &= \frac{C}{|x_0|^{1/2}\prod_{\ell=1}^{n-2}(ix_\ell)}\left[\left\{\frac{e^{i(x_1+\cdots+x_{n-2}+|x_0|)}}{(x_1+\cdots+x_{n-2}+|x_0|)^{\delta+1}} + \frac{e^{i(x_1+\cdots+x_{n-2}-|x_0|)}}{(x_1+\cdots+x_{n-2}-|x_0|)^{\delta+1}}\right\}\right. \\
&\quad + \sum_{j=1}^{N} a_j|x_0|^{-j}\sum_{\mu=1}^{j_0-1}\frac{C_\mu}{(x_1+\cdots+x_{n-2}+|x_0|)^{\mu+\delta+1}}e^{i(x_1+\cdots+x_{n-2}+|x_0|)} \\
&\quad \left. + \sum_{j=1}^{N} b_j|x_0|^{-j}\sum_{\mu=1}^{j_0-1}\frac{C_\mu}{(x_1+\cdots+x_{n-2}-|x_0|)^{\mu+\delta+1}}e^{i(x_1+\cdots+x_{n-2}-|x_0|)}\right] \\
&= \mathcal{I}_{A(11)} + \mathcal{I}_{A(12)} + \mathcal{I}_{A(13)}.
\end{aligned}$$

Since $\mathcal{I}_{A(12)}$ and $\mathcal{I}_{A(13)}$ have a nice decay such as

$$|\mathcal{I}_{A(12)}| + |\mathcal{I}_{A(13)}| \leqslant C\frac{1}{|x_0|^{3/2}\prod_{\ell=1}^{n-2}|x_\ell|}\left\{\frac{1}{(x_1+\cdots+x_{n-2}+|x_0|)^{\delta+2}} + \frac{1}{(x_1+\cdots+x_{n-2}-|x_0|)^{\delta+2}}\right\},$$

we have that $\mathcal{I}_{A(11)}$ is a leading term, where

$$\mathcal{I}_{A(11)} = \frac{B_1}{|x_0|^{1/2}\prod_{\ell=1}^{n-2} x_\ell}\left\{\frac{e^{i(x_1+\cdots+x_{n-2}+|x_0|)}}{(x_1+\cdots+x_{n-2}+|x_0|)^{\delta+1}} + \frac{e^{i(x_1+\cdots+x_{n-2}-|x_0|)}}{(x_1+\cdots+x_{n-2}-|x_0|)^{\delta+1}}\right\}. \tag{4.6}$$

If we apply the same argument as above for each $\mathcal{I}_\ell$, the leading term is

$$\frac{B_2}{\widehat{X}_\ell |x_0|^{3/2}}\left\{\frac{e^{i(x_1+\cdots+x_{n-2}+|x_0|)}}{(x_1+\cdots+x_{n-2}+|x_0|)^{\delta+1}} + \frac{e^{i(x_1+\cdots+x_{n-2}-|x_0|)}}{(x_1+\cdots+x_{n-2}-|x_0|)^{\delta+1}}\right\}, \tag{4.7}$$

where $\widehat{X}_\ell = x_1 x_2 \cdots \widehat{x}_\ell \cdots x_{n-2}$. Thus from (4.6) and (4.7) we obtain that

$$\begin{aligned}\left|T_1^\delta f(x)\right| &\geqslant C\frac{1}{|x_0|^{1/2}\,|x_1+\cdots+x_{n-2}\pm|x_0||^{\delta+1}}\left|\frac{1}{\prod_{\ell=1}^{n-2}x_\ell} + \sum_{\ell=1}^{n-2}\frac{1}{\widehat{X}_\ell|x_0|}\right| \\ &\geqslant C\frac{1}{|x_0|^{3/2}\prod_{\ell=1}^{n-2}|x_\ell|}\,\frac{1}{|x_1+\cdots+x_{n-2}\pm|x_0||^{\delta}}.\end{aligned} \tag{4.8}$$

If $\delta \leqslant n(\frac{1}{p}-1)+\frac{1}{2}$ and $0<p<1$, it follows that

$$\begin{aligned}\left\|T_1^\delta f\right\|_p^p &\geqslant \int\limits_{\{|x_0|>2,\ |x_\ell|>2,\ 1\leqslant\ell\leqslant n-2\}} \frac{dx_{n-2}\cdots dx_1\,dx_0}{(|x_0|^{\frac{3}{2}}\prod_{\ell=1}^{n-2}|x_\ell|)^p\,|x_1+\cdots+x_{n-2}+|x_0||^{\delta p}} \\ &\geqslant C\int\limits_{\{|x_0|>|x_1|>\cdots>|x_{n-2}|>2\}}\prod_{\ell=1}^{n-2}|x_\ell|^{-p}|x_0|^{-(\frac{3}{2}+\delta)p}\,dx_{n-2}\cdots dx_1\,dx_0 \\ &\geqslant C\int_0^{2\pi}\int_2^{\infty} r^{(n-2)-(\frac{2n-1}{2}+\delta)p}\,r\,dr\,d\theta = +\infty.\end{aligned}$$

If $0<p\leqslant\frac{2(n-1)}{2n-1}$, we have

$$\begin{aligned}\left\|T_1^\delta f\right\|_p^p &\geqslant \int\limits_{\{|x_0|>2,\ |x_\ell|>2,\ 1\leqslant\ell\leqslant n-2\}} \frac{dx_{n-2}\cdots dx_1\,dx_0}{(|x_0|^{\frac{3}{2}}\prod_{\ell=1}^{n-2}|x_\ell|)^p\,|x_1+\cdots+x_{n-2}-|x_0||^{\delta p}} \\ &\geqslant C\int\limits_{\{|x_0|>|x_1|\approx\cdots\approx|x_{n-2}|,\ 1/2<|x_1+\cdots+x_{n-2}-|x_0||<2\}} \frac{dx_{n-2}\cdots dx_1\,dx_0}{(|x_0|^{\frac{3}{2}}\prod_{\ell=1}^{n-2}|x_\ell|)^p} \\ &\geqslant C\int\limits_{\{|x_0|>2\}}|x_0|^{(n-3)-\{(n-2)+\frac{3}{2}\}p}\,dx_0 \\ &= C\int_0^{2\pi}\int_2^{\infty} r^{(n-3)-\{(n-2)+\frac{3}{2}\}p}\,r\,dr\,d\theta = +\infty,\end{aligned}$$

for all $n \geqslant 3$. $\square$

**Remark 3.** Let us consider the integrability for the case $\varrho(\xi) = \max\{|\xi_0|, |\xi_1|, \ldots, |\xi_{n-k}|\}$ with

$$\xi_0 \in \mathbb{R}^k, \qquad (\xi_1,\ldots,\xi_{n-k}) \in \mathbb{R}^{n-k} \quad (3 \leqslant k < n).$$

If we replace the roles of $\mathcal{U}$ in (4.1) by

$$\mathcal{U} = \left\{\xi = (\xi_0,\xi_1,\ldots,\xi_{n-k}) \;\Big|\; \frac{1}{2} < |\xi_0| < 1,\ \frac{1}{2} < |\xi_1|,\ldots,|\xi_{n-k}| < 1\right\}$$

and integrate over this region in $\mathcal{I}'$, the leading term is bounded below by

$$\left|T_1^\delta f(x)\right| \geqslant C\frac{1}{|x_0|^{\frac{k+1}{2}}\prod_{\ell=1}^{n-k}|x_\ell|}\,\frac{1}{||x_1|\pm\cdots\pm|x_{n-k}|\pm|x_0||^{\delta}}$$

by using the similar arguments used for the above. Then it is clear that the convolution operator $T_1^\delta$ is not integrable for any $\delta$ where $(x_0, \underline{x})$ with $x_0 \in \mathbb{R}^k$ and $\underline{x} \in \mathbb{R}^{n-k}$ $(3 \leqslant k < n)$ independent of $\delta$ near the set $\{x_0 \in \mathbb{R}^k,\ (x_1,\ldots,x_{n-k}) \in \mathbb{R}^{n-k}$ $(3 \leqslant k < n)\colon\ |x_0| = \cdots = |x_{n-k}|\}$.

### 4.1. The negative result of Theorem 2

Finally, we consider the negative result of Theorem 2. Let $p=\frac{2(n-1)}{2n-1}$ and $\delta\geqslant\delta(p)=\frac{2n-1}{2(n-1)}$ on the region $\mathcal{D}=\{x\colon |x_0|\approx|x_\ell|\geqslant 2,\ \ell=1,\dots,n-2\}$ in (4.8).

If $p=\frac{2(n-1)}{2n-1}$ and $\delta>\frac{2n-1}{2(n-1)}$, we put $\delta=\frac{2n-1}{2(n-1)}+(n-2)\epsilon_0$ for some $\epsilon_0>0$ and for all $n\geqslant 3$. By a change of variable $x_1+\cdots+x_{n-2}+|x_0|=\tilde{x}_{n-2}$, we have

$$\begin{aligned}
\left|\left\{x\colon \left|T_1^\delta f(x)\right|>\alpha\right\}\right| &\geqslant \int\limits_{\{x\in\mathcal{D}\colon |x_0|^{-\frac{5}{2}}\prod_{\ell=1}^{n-3}|x_\ell|^{-1}|x_1+\cdots+x_{n-2}+|x_0||^{-\delta}>\alpha\}} dx \\
&\geqslant C\int\limits_{\{x\in\mathcal{D}\colon |x_0|^{-\frac{2n-1}{n-1}}\prod_{\ell=1}^{n-3}|x_\ell|^{-(\frac{2n-1}{2(n-1)}+\epsilon_0)}|\tilde{x}_{n-2}|^{-(\frac{2n-1}{2(n-1)}+\epsilon_0)}>\alpha\}} dx_0\,dx_1\cdots dx_{n-3}\,d\tilde{x}_{n-2} \\
&\geqslant C\alpha^{-\frac{2(n-1)}{2n-1}}\int_2^\infty\prod_{\ell=1}^{n-3}|x_\ell|^{-\frac{2(n-1)}{2n-1}(\frac{2n-1}{2(n-1)}+\epsilon_0)}|\tilde{x}_{n-2}|^{-\frac{2(n-1)}{2n-1}(\frac{2n-1}{2(n-1)}+\epsilon_0)}\,dx_1\cdots dx_{n-3}\,d\tilde{x}_{n-2} \\
&\geqslant C\alpha^{-\frac{2(n-1)}{2n-1}},
\end{aligned}$$

since $\frac{5}{2}=\frac{2n-1}{n-1}+\frac{n-3}{2(n-1)}$.

On the other hand if $p=\frac{2(n-1)}{2n-1}$ and $\delta=\frac{2n-1}{2(n-1)}$, then

$$|x_0|^{-\frac{5}{2}}\prod_{\ell=1}^{n-3}|x_\ell|^{-1}\left|x_1+\cdots+x_{n-2}+|x_0|\right|^{-\delta}\approx|x_0|^{-\frac{2n-1}{n-1}}\prod_{\ell=1}^{n-2}|x_\ell|^{-\frac{2n-1}{2(n-1)}}$$

on $\mathcal{D}$. Thus, it follows that

$$\begin{aligned}
\left|\left\{x\colon \left|T_1^\delta f(x)\right|>\alpha\right\}\right| &\geqslant \int\limits_{\{x\in\mathcal{D}\colon |x_0|^{-\frac{2n-1}{n-1}}\prod_{\ell=1}^{n-2}|x_\ell|^{-\frac{2n-1}{2(n-1)}}>\alpha\}} dx \\
&\geqslant C\int\limits_{\{x\colon |x_{n-2}|\leqslant\cdots\leqslant|x_1|,\ |x_1|\leqslant|x_0|,\ 2\leqslant|x_0|\leqslant\alpha^{-\frac{2(n-1)}{(2n-1)n}}\}} dx \\
&\quad + C\int\limits_{\{x\colon |x_{n-2}|\leqslant\cdots\leqslant|x_1|,\ |x_1|\leqslant\alpha^{-\frac{2(n-1)}{(2n-1)(n-2)}}|x_0|^{-\frac{2}{n-2}},\ |x_0|>\alpha^{-\frac{2(n-1)}{(2n-1)n}}\}} dx \\
&= C\int\limits_{\{2<|x_0|\leqslant\alpha^{-\frac{2(n-1)}{(2n-1)n}}\}}|x_0|^{n-2}\,dx_0+C\alpha^{-\frac{2(n-1)}{(2n-1)}}\int\limits_{\{|x_0|>\alpha^{-\frac{2(n-1)}{(2n-1)n}}\}}\frac{1}{|x_0|^2}\,dx_0 \\
&= C\int_0^{2\pi}\int_2^{\alpha^{-\frac{2(n-1)}{(2n-1)n}}} r^{n-2}r\,dr\,d\theta+C\alpha^{-\frac{2(n-1)}{(2n-1)}}\int_0^{2\pi}\int_{\alpha^{-\frac{2(n-1)}{(2n-1)n}}}^{\infty} r^{-2}r\,dr\,d\theta \\
&\geqslant C\alpha^{-\frac{2(n-1)}{(2n-1)}}\bigl(1+\ln(1/\alpha)\bigr).
\end{aligned}$$

We now consider the case $p=1$ and $\delta>\frac{1}{2}$. For $x\in\mathcal{D}$ we have

$$|x_0|^{-\frac{3}{2}}\left|x_1+\cdots+x_{n-2}-|x_0|\right|^{-\delta}\prod_{\ell=1}^{n-2}|x_\ell|^{-1}\approx|x_0|^{-(\delta+\frac{3}{2})}\prod_{\ell=1}^{n-2}|x_\ell|^{-1}.$$

From $\delta>\frac{1}{2}$, we put $\delta=\frac{1}{2}+(n-2)\epsilon_0$ for some $\epsilon_0>0$. Thus it follows that

$$\begin{aligned}\left|\left\{x:\ \left|T_1^\delta f(x)\right|>\alpha\right\}\right| &\geqslant \int\limits_{\{x\in\mathcal{D}:\ |x_0|^{-2}\prod_{\ell=1}^{n-2}|x_\ell|^{-(1+\epsilon_0)}>\alpha\}} dx_0\,dx_1\cdots dx_{n-2}\\ &\geqslant C \int\limits_{\{x\in\mathcal{D}:\ |x_0|<\alpha^{-1/2}\prod_{\ell=1}^{n-2}|x_\ell|^{-\frac{1+\epsilon_0}{2}}\}} dx_0\,dx_1\cdots dx_{n-2}\\ &\geqslant C\alpha^{-1}.\end{aligned}$$

Finally if $p=1$ and $\delta=\frac{1}{2}$, then it follows that

$$\begin{aligned}\left|\left\{x:\ \left|T_1^\delta f(x)\right|>\alpha\right\}\right| &\geqslant \int\limits_{\{x\in\mathcal{D}:\ |x_0|^{-2}\prod_{\ell=1}^{n-2}|x_\ell|^{-1}>\alpha\}} dx\\ &\geqslant C \int\limits_{\{x:\ |x_{n-2}|\leqslant\cdots\leqslant|x_1|,\ |x_1|\leqslant|x_0|,\ 2\leqslant|x_0|\leqslant\alpha^{-\frac{1}{n}}\}} dx\\ &\quad + C \int\limits_{\{x:\ |x_{n-2}|\leqslant\cdots\leqslant|x_1|,\ |x_1|\leqslant\alpha^{-\frac{1}{n-2}}|x_0|^{-\frac{2}{n-2}},\ |x_0|>\alpha^{-\frac{1}{n}}\}} dx\\ &= C \int\limits_{\{2<|x_0|\leqslant\alpha^{-\frac{1}{n}}\}} |x_0|^{n-2}\,dx_0 + C\alpha^{-1} \int\limits_{\{|x_0|>\alpha^{-\frac{1}{n}}\}} \frac{1}{|x_0|^2}\,dx_0\\ &= C\int_0^{2\pi}\int_2^{\alpha^{-\frac{1}{n}}} r^{n-2}r\,dr\,d\theta + C\alpha^{-1}\int_0^{2\pi}\int_{\alpha^{-\frac{1}{n}}}^{\infty} r^{-2}r\,dr\,d\theta\\ &\geqslant C\alpha^{-1}\bigl(1+\ln(1/\alpha)\bigr).\end{aligned}$$

## 5. $L^p$ estimate for $1\leqslant p<\infty$

In this section we shall prove Theorem 3. That is, $T^\delta$ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $\delta>\delta(p)=\max\{2|\frac{1}{p}-\frac{1}{2}|-\frac{1}{2},0\}$ and $1\leqslant p<\infty$. For the case $p=1$, the operator $T^\delta$ is bounded on $L^1(\mathbb{R}^n)$ since the kernel of $T^\delta$ with $\delta>\delta(1)=1/2$ is integrable (see Section 2). If the $L^{4/3}(\mathbb{R}^n)$ boundedness of $T^\delta$ with $\delta>\delta(4/3)=0$ is proved, the interpolation yields the $L^p(\mathbb{R}^n)$ boundedness for the case $1\leqslant p\leqslant 4/3$. This range also covers its conjugate part $4\leqslant p<\infty$. The estimate for the remaining range $4/3<p<4$ is obtained from the interpolation of $p=4/3$ and $p=4$. Now it suffices to prove the case $p=4$ which is the Hölder conjugate of $p=4/3$.

### 5.1. Sufficiency

We consider that the operator $T^\delta$ is defined by

$$\widehat{T^\delta f}(\xi)=m(\xi)\widehat{f}(\xi),$$

where $m=m^\circ+m^\partial$. We denote $\|m\|_{M^p}$ by the operator norm of $f\to\mathcal{F}^{-1}[m\widehat{f}(\xi)]$. Then we have

$$\|m\|_{M^p}\leqslant\left\|m^\circ\right\|_{M^p}+\left\|m^\partial\right\|_{M^p}.$$

We define two operators $T^\circ$ and $T^\partial$ by

$$\widehat{T^\circ f}(\xi)=m^\circ(\xi)\widehat{f}(\xi)\quad\text{and}\quad\widehat{T^\partial f}(\xi)=m^\partial(\xi)\widehat{f}(\xi).$$

Since the kernel $\mathcal{F}^{-1}[m^\circ]$ is integrable for $\delta>1/2$ in Proposition 1, from Young's inequality we have

$$\begin{aligned}\left\|T^\circ f\right\|_{L^p(\mathbb{R}^n)} &= \left\|\mathcal{F}^{-1}\left[m^\circ\right]*f\right\|_{L^p(\mathbb{R}^n)}\\ &\leqslant \left\|\mathcal{F}^{-1}\left[m^\circ\right]\right\|_{L^1(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}\\ &\lesssim \|f\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

From the fact $\|m^\circ\|_{M^p} = \|T^\circ f\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}$, it follows that

$$\|m^\circ\|_{M^p} \leqslant C \quad \text{for } 1 \leqslant p \leqslant \infty.$$

We now proceed to the estimate of $\|m^\partial\|_{M^p}$. Let $\Sigma$ be the set of permutations defined by

$$\Sigma = \bigl\{\sigma \bigm| \sigma\colon \{1,\dots,n-2\} \to \{1,\dots,n-2\} \text{ is a permutation}\bigr\}.$$

To each element $\sigma \in \Sigma$, we assign a set on the frequency space $\mathbb{R}^2 \times \mathbb{R}^{n-2}$,

$$A_\sigma = \bigl\{(\xi_0,\xi_1,\dots,\xi_{n-2})\colon |\xi_{\sigma(1)}| \geqslant |\xi_{\sigma(2)}| \geqslant \cdots \geqslant |\xi_{\sigma(n-2)}|\bigr\}.$$

By symmetry, it suffices to work with the identity permutation such that $\sigma(i) = i$. So we let

$$A = \bigl\{(\xi_0,\xi_1,\dots,\xi_{n-2})\colon |\xi_1| \geqslant |\xi_2| \geqslant \cdots \geqslant |\xi_{n-2}|\bigr\},$$

and deal with

$$m_A(\xi) = \bigl(1-\varrho(\xi)\bigr)_+^\delta \chi_A(\xi)\varphi_\partial(\xi).$$

We split this into

$$m_A(\xi) = \bigl(\bigl(1-\varrho(\xi)\bigr)_+^\delta - \bigl(1-|\xi_1|\bigr)_+^\delta\bigr)\chi_A(\xi)\varphi_\partial(\xi) + \bigl(1-|\xi_1|\bigr)_+^\delta\chi_A(\xi)\varphi_\partial(\xi) = m_0(\xi) + m_1(\xi).$$

The multiplier for the second term $m_1(\xi)$ is the composition of directional Hilbert transform and one-dimensional Bochner–Riesz means. Thus the operator corresponding to the multiplier $m_1(\xi)$ is bounded on $L^p$ for $1 < p < \infty$. In the first term $m_0(\xi)$, we can observe from the cutoff function $\chi_A$ that

- $\varrho(\xi) = |\xi_0|$ if $|\xi_0| \geqslant |\xi_1|$,
- $\varrho(\xi) = |\xi_1|$ if $|\xi_0| \leqslant |\xi_1|$.

So the first term $m_0(\xi)$ can be rewritten as

$$m_0(\xi) = \bigl(\bigl(1-\tilde\varrho(\xi)\bigr)_+^\delta - \bigl(1-|\xi_1|\bigr)_+^\delta\bigr)\chi_A(\xi)\varphi_\partial(\xi),$$

where

$$\tilde\varrho(\xi) = \max\bigl\{|\xi_0|, |\xi_1|\bigr\}.$$

Since the multiplier operator corresponding to $\chi_A$ is essentially the directional Hilbert transform, which is bounded in $L^p(\mathbb{R}^n)$. Thus our problem is reduced to the cylinder case for $n = 3$ such that

$$m_0(\xi) = \bigl(\bigl(1-\tilde\varrho(\xi)\bigr)_+^\delta - \bigl(1-|\xi_1|\bigr)_+^\delta\bigr)\varphi_\partial(\xi).$$

We decompose

$$m_0(\xi) = \sum_{j_0>0,\, j_1>0} \bigl(\bigl(1-\tilde\varrho(\xi)\bigr)^\delta - \bigl(1-|\xi_1|\bigr)^\delta\bigr)\phi_{j_0}\bigl(1-|\xi_0|\bigr)\phi_{j_1}\bigl(1-|\xi_1|\bigr).$$

We next split the above term as

$$\sum_{|j_0-j_1|\geqslant 2} + \sum_{|j_0-j_1|\leqslant 1} \bigl(\bigl(1-\tilde\varrho(\xi)\bigr)^\delta - \bigl(1-|\xi_1|\bigr)^\delta\bigr)\phi_{j_0}\bigl(1-|\xi_0|\bigr)\phi_{j_1}\bigl(1-|\xi_1|\bigr).$$

The first term also is reduced to one or two-dimensional Bochner–Riesz means. It suffices to work with $|j_0 - j_1| \leqslant 1$. We consider only the case $j_0 = j_1 = j$, where

$$m_j(\xi_0,\xi_1) = \bigl(\bigl(1-\tilde\varrho(\xi)\bigr)^\delta - \bigl(1-|\xi_1|\bigr)^\delta\bigr)\phi_j\bigl(1-|\xi_0|\bigr)\phi_j\bigl(1-|\xi_1|\bigr).$$

We see that

$$\bigl(1-\tilde\varrho(\xi)\bigr)^\delta - \bigl(1-|\xi_1|\bigr)^\delta = 0 \quad \text{if } |\xi_1| \geqslant |\xi_0|,$$

since $\tilde\varrho(\xi) = |\xi_1|$. Thus we have only to deal with $|\xi_0| - |\xi_1| \geqslant 0$. Thus, by decomposing $|\xi_0| - |\xi_1| > 0$, we have

$$m_j(\xi) = \sum_{k\geqslant j}\bigl(\bigl(1-|\xi_0|\bigr)^\delta - \bigl(1-|\xi_1|\bigr)^\delta\bigr)\phi_j\bigl(1-|\xi_0|\bigr)\phi_j\bigl(1-|\xi_1|\bigr)\phi_k\bigl(|\xi_0|-|\xi_1|\bigr) = \sum_{k\geqslant j} m_{j,k}(\xi).$$

It suffices to show that the $L^4$ norm for the multiplier operator $M_{m_{j,k}}$ associated with $m_{j,k}$ is controlled by $O(2^{-c(k-j)}2^{-j\delta})$ for some $c > 0$ such that

$$\|M_{m_{j,k}}\|_{L^p(\mathbb{R}^3)\to L^p(\mathbb{R}^3)} \leqslant O\bigl(2^{-c(k-j)}2^{-j\delta}\bigr).$$

From now on, we fix $j$ and $k$. We decompose the support of $\xi_1$, which is the interval $[1-2^{-j}, 1-2^{-j-1}]$ of length $\approx 2^{-j}$ into $2^{k-j}$ intervals of the length $\approx 2^k$ by using the cutoff function

$$\phi\big(2^k\big(\xi_1-\xi_1^\tau\big)\big) \quad \text{where } \xi_1^\tau = 1-2^{-j}+\tau 2^{-k}$$

with $\tau = 1, \dots, 2^{k-j}$. Let $\psi$ be a $C_0^\infty(\mathbb{R})$ function supported in $(-1/2, 1/2)$. We also decompose the support of $\xi_0$ to $2^{k/2}$ as equal angular sectors of angle $\approx 2^{-k/2}$ by using

$$\psi\left(2^{k/2}\left(\frac{\xi_0}{|\xi_0|}-\xi_0^\nu\right)\right),$$

where $\nu = 1, \dots, 2^{k/2}$ and $\xi_0^\nu = (\cos(2\pi 2^{-k/2}\nu), \sin(2\pi 2^{-k/2}\nu))$. Then we decompose

$$m_{j,k} = \sum_{\tau=1}^{2^{k-j}} \sum_{\nu=1}^{2^{k/2}} m_{j,k}^{\tau,\nu},$$

where $m_{j,k}^{\tau,\nu}(\xi)$ is

$$\big(\big(1-|\xi_0|\big)^\delta - \big(1-|\xi_1|\big)^\delta\big)\phi_j\big(1-|\xi_0|\big)\phi_j\big(1-|\xi_1|\big)\phi_k\big(|\xi_0|-|\xi_1|\big)\varphi\big(2^k\big(\xi_1-\xi_1^\tau\big)\big)\psi\left(2^{k/2}\left(\frac{\xi_0}{|\xi_0|}-\xi_0^\nu\right)\right).$$

If the operator $T$ defined by

$$T: f = (f_\tau) \to T(f) = \sum_\tau f_\tau$$

maps from $\ell^2(L^2(\mathbb{R}^n))$ to $L^2(\mathbb{R}^n)$ and from $\ell^1(L^\infty(\mathbb{R}^n))$ to $L^\infty(\mathbb{R}^n)$, then $T$ maps from $\ell^{p'}(L^p(\mathbb{R}^n))$ to $L^p(\mathbb{R}^n)$. Thus by regarding $f_\tau = \sum_{\nu=1}^{2^{k/2}} M_{m_{j,k}^{\tau,\nu}}(f)$, we have

$$\left\| \sum_{\tau=1}^{2^{k-j}} \left( \sum_{\nu=1}^{2^{k/2}} M_{m_{j,k}^{\tau,\nu}}(f) \right) \right\|_{L^{p_0}(\mathbb{R}^3)} \leqslant \left( \sum_{\tau=1}^{2^{k-j}} \left\| \left( \sum_{\nu=1}^{2^{k/2}} M_{m_{j,k}^{\tau,\nu}}(f) \right) \right\|_{L^{p_0}(\mathbb{R}^3)}^{p_0'} \right)^{1/p_0'} \leqslant 2^{(k-j)/p_0'} \sup_\tau \left\| \sum_{\nu=1}^{2^{k/2}} M_{m_{j,k}^{\tau,\nu}}(f) \right\|_{L^{p_0}(\mathbb{R}^3)}.$$

We shall show the following.

**Proposition 3.** *For each fixed $\tau$,*

$$\left\| \sum_{\nu=1}^{2^{k/2}} M_{m_{j,k}^{\tau,\nu}}(f) \right\|_{L^4(\mathbb{R}^3)} \leqslant C 2^{-(k-j)} 2^{-j\delta} 2^{\frac{\epsilon k}{2}} \tag{5.1}$$

*for sufficiently small $\epsilon$.*

**Proof of (5.1).** Fix $j, k$ and $\tau$ and let us write for each $\nu$,

$$M_\nu(f) = M_{m_{j,k}^{\tau,\nu}}(f) = \big(m_{j,k}^{\tau,\nu}\big)^\vee * f \tag{5.2}$$

for simplicity. Let $B_\nu$ be the support of the multiplier $m_{j,k}^{\tau,\nu}$. Then by using the $L^4$ square function arguments of C. Fefferman and A. Cordoba in [2,3,5] (we also refer to pages 418–420 of [15] for a brief description of their $L^4$ estimates)

$$\begin{aligned}
\left\| \sum_\nu M_\nu(f) \right\|_{L^4(\mathbb{R}^3)}^4 &= \int \left| \sum_\nu \big(M_\nu(f)\big)^\vee * \sum_{\nu'} \big(\overline{M_{\nu'}(f)}\big)^\vee(\xi) \right|^2 d\xi \\
&= \int \left| \sum_\nu \big(M_\nu(f)\big)^\vee * \sum_{\nu'} \big(\overline{M_{\nu'}(f)}\big)^\vee(\xi)\chi_{B_\nu+B_{\nu'}}(\xi) \right|^2 d\xi \\
&\leqslant \int \sum_{\nu,\nu'} \big| \big(M_\nu(f)\big)^\vee * \big(\overline{M_{\nu'}(f)}\big)^\vee(\xi) \big|^2 \left( \sum_{\nu,\nu'} \big|\chi_{B_\nu+B_{\nu'}}(\xi)\big| \right)^2 d\xi \\
&\leqslant C \int \sum_{\nu,\nu'} \big| \big(M_\nu(f)\big)^\vee * \big(\overline{M_{\nu'}(f)}\big)^\vee(\xi) \big|^2 d\xi \\
&= C \int \sum_{\nu,\nu'} \big| M_\nu(f)(x)\overline{M_{\nu'}(f)}(x) \big|^2 dx = C \left\| \left( \sum_\nu \big|M_\nu(f)\big|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^3)}^4.
\end{aligned} \tag{5.3}$$

In the above we applied the Plancherel theorem for obtaining first and third equality and used the Schwartz inequality for obtaining the first inequality. For the second inequality, we use the inequality $\sum_{\nu,\nu'} \chi_{B_\nu+B_{\nu'}}(\xi) \leqslant C$ which was proved by the finite overlapping properties for the sums of two angular regions in [2,3,5] where we freeze $\xi_1$ in $(\xi_0, \xi_1)$.

We next need to observe that

$$\begin{aligned}\left|M_\nu(f)(x)\right|^2 &= \left|\left(m_{j,k}^{\tau,\nu}\right)^\vee * (\chi_{B_\nu})^\vee * f(x)\right|^2\\ &\leqslant \left\|\left(m_{j,k}^{\tau,\nu}\right)^\vee\right\|_{L^1}\left|\left(m_{j,k}^{\tau,\nu}\right)^\vee\right| * \left|(\chi_{B_\nu})^\vee * f\right|^2(x)\\ &= C(j,k)\left|\left(m_{j,k}^{\tau,\nu}\right)^\vee\right| * \left|(\chi_{B_\nu})^\vee * f\right|^2(x),\end{aligned}$$

where

$$C(j,k) = \sup_{\nu=1,\dots,2^{k/2}} \left\|\left(m_{j,k}^{\tau,\nu}\right)^\vee\right\|_{L^1}.$$

Using this, we obtain that

$$\begin{aligned}\left\|\left(\sum_\nu \left|M_\nu(f)\right|^2\right)^{1/2}\right\|_{L^4(\mathbb{R}^3)}^4 &\leqslant \left\|\sum_\nu C(j,k)\left|\left(m_{j,k}^{\tau,\nu}\right)^\vee\right| * \left|(\chi_{B_\nu})^\vee * f\right|^2\right\|_{L^2(\mathbb{R}^3)}^2\\ &= C(j,k)^2 \sup_{\|g\|_{L^2}\leqslant 1}\left(\int \sum_\nu \left|\left(m_{j,k}^{\tau,\nu}\right)^\vee\right| * \left|(\chi_{B_\nu})^\vee * f(x)\right|^2 g(x)\,dx\right)^2\\ &= C(j,k)^2 \sup_{\|g\|_{L^2}\leqslant 1}\left(\int \sum_\nu \left|(\chi_{B_\nu})^\vee * f(x)\right|^2\left|\left(m_{j,k}^{\tau,\nu}\right)^\vee\right| * g(x)\,dx\right)^2\\ &\leqslant C(j,k)^2 \sup_{\|g\|_{L^2}\leqslant 1}\left(\int \sum_\nu \left|(\chi_{B_\nu})^\vee * f(x)\right|^2 \mathcal{M}g(x)\,dx\right)^2\\ &\leqslant C(j,k)^2 \sup_{\|g\|_{L^2}\leqslant 1}\left\|\left(\sum_\nu \left|(\chi_{B_\nu})^\vee * f(x)\right|^2\right)^{1/2}\right\|_{L^4(\mathbb{R}^3)}^4 \|\mathcal{M}g\|_{L^2(\mathbb{R}^3)}^2,\end{aligned} \tag{5.4}$$

where

$$\mathcal{M}g(x) = \sup_{\nu=1,\dots,2^{k/2}} \left|\left(m_{j,k}^{\tau,\nu}\right)^\vee\right| * |g|(x). \tag{5.5}$$

By using the Littlewood–Paley inequality associated with the frequency decomposition of the same size intervals,

$$\left\|\left(\sum_\nu \left|(\chi_{B_\nu})^\vee * f\right|^2\right)^{1/2}\right\|_{L^4(\mathbb{R}^3)} \leqslant C\|f\|_{L^4(\mathbb{R}^3)} \quad \text{for } 2 \leqslant p < \infty, \tag{5.6}$$

see page 420 of [15] for the proof of (5.6). By (5.2), (5.3), (5.4) and (5.6), we have

$$\left\|\sum_{\nu=1}^{2^{k/2}} M_{m_{j,k}^{\tau,\nu}}(f)\right\|_{L^4(\mathbb{R}^3)} \leqslant C(j,k)^{1/2}\|f\|_{L^4(\mathbb{R}^3)}\|\mathcal{M}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)}^{1/2}.$$

Therefore, (5.1) follows from Lemma 3 below. □

**Lemma 3.** *The inverse Fourier transform of $m_{j,k}^{\tau,\nu}$ is also majorized by $2^{-k}2^{-j(\delta-1)}$ times some integrable function which is essentially supported on a tube embedded on a thin slab containing $x_1x_2$ plane. We have*

$$\left|\left(m_{j,k}^{\tau,\nu}\right)^\vee(x_1,x_2,x_3)\right| \leqslant C2^{-k}2^{-j(\delta-1)}\frac{2^{-k/2}}{(1+\frac{|(x_1,x_2)\cdot(-\sin\theta_\nu,\cos\theta_\nu)|}{2^{k/2}})^N}\frac{2^{-k}}{(1+\frac{|(x_1,x_2)\cdot(\cos\theta_\nu,\sin\theta_\nu)|}{2^k})^N}\frac{2^{-k}}{(1+\frac{|x_3|}{2^k})^N}, \tag{5.7}$$

*which implies that*

$$C(j,k) \leqslant C2^{-k}2^{-j(\delta-1)}.$$

*In* (5.5)*, we have*

$$\|\mathcal{M}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \leqslant C2^{-(k-j)}2^{\epsilon k}2^{-j\delta}. \tag{5.8}$$

**Proof of Lemma 3.** We first show (5.7). Let us define three directions

- $e_1(\nu) = (\sin\theta_\nu, -\cos\theta_\nu, 0)$,
- $e_2(\nu) = (\cos\theta_\nu, \sin\theta_\nu, -1)$,
- $e_3(\nu) = (\cos\theta_\nu, \sin\theta_\nu, 1)$.

Then we see that (5.7) is equivalent to the estimate:

$$\left|\left(m_{j,k}^{\tau,\nu}\right)^\vee(x_1, x_2, x_3)\right| \leqslant C2^{-k}2^{-j(\delta-1)}\frac{2^{-k/2}2^{-k}2^{-k}}{(1 + \frac{|x\cdot e_1(\nu)|}{2^{k/2}} + \frac{|x\cdot e_2(\nu)|}{2^k} + \frac{|x\cdot e_3(\nu)|}{2^k})^N}. \tag{5.9}$$

**Proof of (5.9).** Let $\xi_0 = (\xi_0^1, \xi_0^2)$. Define for $s = 1, 2, 3$,

$$D_{e_s(\nu)}u = e_s(\nu)\cdot\nabla_\xi u = e_s(\nu)\cdot\left(\frac{\partial u}{\partial\xi_0^1}, \frac{\partial u}{\partial\xi_0^2}, \frac{\partial u}{\partial\xi_1}\right).$$

We let $\Psi(\xi_0, \xi_1)$ be the amplitude given by

$$\left(\left(1-|\xi_0|\right)^\delta - \left(1-|\xi_1|\right)^\delta\right)\phi_j\left(1-|\xi_0|\right)\phi_j\left(1-|\xi_1|\right)\phi_k\left(|\xi_0|-|\xi_1|\right)\varphi\left(2^k\left(\xi_1-\xi_1^\tau\right)\right)\psi\left(2^{k/2}\left(\frac{\xi_0}{|\xi_0|}-\xi_0^\nu\right)\right).$$

For the integral

$$\left(m_{j,k}^{\tau,\nu}\right)^\vee(x_1, x_2, x_3) = \int\limits_{\mathbb{R}^3} e^{i(x_1,x_2,x_3)\cdot(\xi_0^1,\xi_0^2,\xi_1)}\Psi(\xi_0, \xi_1)\,d\xi_0^1\,d\xi_0^2\,d\xi_1,$$

we apply the directional integration by parts $N$ times with respect to $D_{e_s(\nu)}$ for $s = 1, 2, 3$. Then the integral

$$\int\limits_{\mathbb{R}^3}\frac{D^N_{e_s(\nu)}[e^{i(x_1,x_2,x_3)\cdot(\xi_0^1,\xi_0^2,\xi_1)}]}{(ix\cdot e_s(\nu))^N}\Psi(\xi_0, \xi_1)\,d\xi_0^1\,d\xi_0^2\,d\xi_1 = (-1)^N\int\limits_{\mathbb{R}^3} e^{i(x_1,x_2,x_3)\cdot(\xi_0^1,\xi_0^2,\xi_1)}\frac{D_{e_s(\nu)}[\Psi(\xi_0, \xi_1)]}{(ix\cdot e_s(\nu))^N}\,d\xi_0^1\,d\xi_0^2\,d\xi_1$$

yields the desired bound. □

We next show (5.8). By (5.7), we can observe that

$$\mathcal{M}g(x) = \sup_{\nu=1,\ldots,2^{k/2}} M_\nu(g)(x) \leqslant C2^{-(k-j)}2^{-j\delta}\mathcal{M}_3\circ\mathcal{M}_{R(2^k\times2^{k/2})}g(x). \tag{5.10}$$

Here $\mathcal{M}_3$ is the directional maximal operator along the $x_3$ axis and $\mathcal{M}_{R(2^k\times2^{k/2})}$ is the classical Nikodym maximal operator on the plane with the third variable freezed

$$\mathcal{M}_3 f(x) = \sup_{r>0}\frac{1}{2r}\int\limits_{-r}^{r}\left|f(x_1, x_2, x_3-y_3)\right|dy_3,$$

$$\mathcal{M}_{R(2^k\times2^{k/2})}f(x) = \sup_{\nu=1,\ldots,2^{k/2}}\frac{1}{|R(\nu)|}\int\limits_{(y_1,y_2)\in R(\nu)}\left|f(x_1-y_1, x_2-y_2, x_3)\right|dy_1\,dy_2, \tag{5.11}$$

where $R(\nu)$ is the rectangle whose dimensions are $2^k$ and $2^{k/2}$ with the angle between the long axis of this rectangle and the $x$ axis being about $\theta_\nu = 2^{-k/2}\nu$. From the well known $\sqrt{\log 2^k}$ bound of the $L^2$ norm for the classical Nikodym maximal operator associated with the two-dimensional plane in [2], we have

$$\|\mathcal{M}_{R(2^k\times2^{k/2})}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \leqslant O(\sqrt{k}) \leqslant 2^{\epsilon k}.$$

By this combined with the finite $L^p$ bound of $\mathcal{M}_3$ in (5.11) for $p > 1$, we have in view of (5.10)

$$\|\mathcal{M}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \leqslant C\,2^{-(k-j)}\,2^{-j\delta}\,2^{\epsilon k}$$

which is (5.8). □

*5.2. Necessity*

We take compactly supported smooth function $\varphi(\xi)=\phi(|\xi_0|)\phi(|\xi_1|)\cdots\phi(|\xi_{n-2}|)$, where $\phi$ is a smooth cut-off function which is identically equal to 1 in

$$\left\{(\xi_0,\ldots,\xi_{n-2})\ \Big|\ \frac{1}{2}<|\xi_j|<1,\ j=0,\ldots,n-2\right\}.$$

We then have $T^\delta(F^{-1}[\varphi])=K$ where $F^{-1}$ is the inverse Fourier transform and $K$ is the kernel of $T^\delta$. Since $\delta\leqslant\delta^*(p)=2|\frac{1}{p}-\frac{1}{2}|-\frac{1}{2}$ and $p\leqslant 2$ implies that $p\leqslant\frac{4}{3+2\delta}$, we claim that $\|K\|_p^p=+\infty$ for $p\leqslant\frac{4}{3+2\delta}$. If we apply the same arguments used in Section 4, we obtain that

$$\begin{aligned}\left|K(x)\right| &\geqslant C\frac{1}{|x_0|^{\frac{1}{2}}||x_1|+\cdots+|x_{n-2}|\pm|x_0||^{\delta+1}}\left|\frac{1}{\prod_{\ell=1}^{n-2}|x_\ell|}+\sum_{\ell=1}^{n-2}\frac{1}{\widehat{X}_\ell|x_0|}\right|\\ &\geqslant C\frac{1}{|x_0|^{\frac{3}{2}}\prod_{\ell=1}^{n-2}|x_\ell|}\frac{1}{||x_1|+\cdots+|x_{n-2}|\pm|x_0||^{\delta}}.\end{aligned}$$

If $p\leqslant\frac{4}{3+2\delta}$, then we have

$$\begin{aligned}\|K\|_p^p &\geqslant \int\limits_{\{|x_0|>2,\ |x_\ell|>2,\ 1\leqslant\ell\leqslant n-2\}}\frac{dx_{n-2}\cdots dx_1\,dx_0}{(|x_0|^{\frac{3}{2}}\prod_{\ell=1}^{n-2}|x_\ell|)^p||x_1|+\cdots+|x_{n-2}|+|x_0||^{\delta p}}\\ &\geqslant C\Bigg(\int\limits_{\{|x_0|>2\}}|x_0|^{-(\frac{3}{2}+\delta)p}\,dx_0\Bigg)\prod_{\ell=1}^{n-2}\Bigg(\int\limits_{\{|x_\ell|>2\}}|x_\ell|^{-p}\,dx_\ell\Bigg)\\ &\geqslant C\int\limits_0^{2\pi}\int\limits_2^{\infty}r^{1-(\frac{3}{2}+\delta)p}\,dr\,d\theta=+\infty.\end{aligned}$$

This completes the proof of Theorem 3.

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