CALCULUS OF GENERALIZED RIESZ PRODUCTS

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ABSTRACT. In this paper we discuss generalized Riesz products bringing into consideration $\mathbb{H}^p$ theory, the notion of Mahler measure, and the zeros of polynomials appearing in the generalized Riesz product. Formula for Radon-Nikodym derivative between two generalized Riesz product is established under suitable conditions. This is then used to formulate a Dichotomy theorem and prove a conditional version of it. A discussion involving flat polynomials is given.

1. INTRODUCTION.

Generalized Riesz products considered in this paper are the ones defined in [13],[17] where one of the aims was to describe the spectrum of a non-singular rank one map as a generalized Riesz product. Generalized Riesz product has remained only in the state of definition although much deep work has appeared over last two decades on special generalized Riesz products arising in the spectral study of measure preserving rank one maps of ergodic theory.

The purpose of this paper is to set forth some basic facts of generalized Riesz products and bring into play Hardy class theory to discuss some of the problems arising in the subject. It is surprising that one can garner so much information simply from the fact that L^2 norm of the trigonometric polynomials appearing in a generalized Riesz product is one. These facts are discussed in section 2 and 3 of the paper. Section 4 gives a formula for Radon-Nikodym derivative of two generalized Riesz products. In section 5 we discuss a conditional dichotomy result. Connection with flat polynomials is discussed in section 6 and a result on zeros of polynomials under consideration is given in section 7.

In the rest of this section we give some background material.

1

Riesz Products. Consider a trigonometric series

$$\sum_{n=-\infty}^{+\infty} a_n z^n, \ z \in S^1,$$

where $S^1 = \{z \ : \ |z| = 1\}$, the circle group. If we ignore those terms whose coefficients are zero, then we can write the trigonometric series as

$$\sum_{k=-\infty}^{+\infty} a_{n_k} z^{n_k}, \ z \in S^1.$$

Now if $\left|\frac{n_{k+1}}{n_k}\right| > q$, $\forall k$, for some $q > 1$, then the series is said to be lacunary. The convergence questions for a lacunary trigonometric are answered by:

Theorem 1.1 ([23, p.203]). *A lacunary trigonometric series converges on a set of positive Lebesgue measure if and only if its coefficients form an ℓ^2 sequence. If the coefficients are not square summable then the lacunary trigonometric series is not a Fourier series (of any $L^1(S^1, dz)$ function).*

Next we need the notion of dissociated polynomials. Consider the following two products:

(1.1) $$(1+z)(1+z) = 1 + z + z + z^2 = 1 + 2z + z^2,$$
$$(1+z)(1+z^2) = 1 + z + z^2 + z^3.$$

In the first case we group terms with the same power of z, while in the second case all the powers of z in the formal expansion are distinct. In the second case we say that the polynomials $1+z$ and $1+z^2$ are dissociated. More generally we say that a set of trigonometric polynomials is dissociated if in the formal expansion of product of any finitely many of them, the powers of z in the non-zero terms are all distinct. (see section 2 for a detailed definition).

Now consider the infinite Product due to F. Riesz: ([23, p.208])

$$\prod_{k=1}^{+\infty} \left(1 + a_k \cos(n_k x)\right), \ -1 \le a_k \le 1, \ \frac{n_{k+1}}{n_k} \ge 3.$$

Each term of this product is non-negative and integrates to 1 with respect to the normalized Lebesgue measure on the circle group. We rewrite this product as

$$\prod_{k=1}^{+\infty} \left(1 + \frac{a_k}{2}\left(z^{n_k} + z^{-n_k}\right)\right).$$

Because of the Lacunary nature of the n_k's, the polynomials

$$1 + \frac{a_k}{2}\left(z^{n_k} + z^{-n_k}\right), \ k = 1, 2, \cdots$$

are dissociated. If we expand the finite product $\prod_{k=1}^{L}\left(1+\frac{a_k}{2}\left(z^{n_k}+z^{-n_k}\right)\right)$, we get a finite sum of the type

$$1+\sum_{\substack{k\neq 0 \\ k=-M}}^{M}\gamma_k z^{m_k},$$

and for the infinite product we get the series

$$\sum_{k=-\infty}^{+\infty}\gamma_k z^{m_k},$$

both sums being formal expansions of the corresponding products. Since the finite products are non-negative and integrate to 1, they are probability densities and the corresponding probability measures converge weakly to a probability measure, say μ, whose Fourier-Stieltjes series is the formal expansion of the infinite product. The main theorem about Riesz products is

Theorem 1.2 ([23, p.209]). *The Riesz product $\prod_{k=1}^{+\infty}\left(1+\frac{a_k}{2}\left(z^{n_k}+z^{-n_k}\right)\right)$, $-1\leq a_k\leq 1$, $\frac{n_{k+1}}{n_k}\geq 3,\forall k$. represents a continuous measure μ on S^1 which is absolutely continuous or singular with respect to the Lebesgue measure on S^1 according as the sequence $a_k, k=1,2,\cdots$ is in ℓ^2 or not. The finite products*

$$\prod_{k=1}^{L}\left(1+\frac{a_k}{2}\left(z^{n_k}+z^{-n_k}\right)\right),\ L=1,2,\cdots$$

converge to $\frac{d\mu}{dz}$ a.e. (dz) as $L\longrightarrow +\infty$.

We will improve this theorem using Hardy class theory later.

The above account about Riesz products is based on parts of chapter V of Zygmund [23]. The original four page paper of F. Riesz appeared nearly 95 years ago in 1918 [22], and has led to much further work. The aim of the paper was to give a continuous function of bounded variation whose Fourier-Stieltjes coefficients do not tend to zero.

Connection with Ergodic Theory. In [16] F. Ledrappier showed that Riesz products appear as maximal spectral type of a class of measure preserving transformation. We will assume that the reader is familiar with the stacking method of constructing measure preserving transformations. Consider a sequence stacks Σ_n, $n=1,2,\cdots$ of pairwise disjoint intervals, beginning with the unit interval as Σ_1. Each stack comes equipped with the usual linear maps among its element. Let h_n be the height of Σ_n, $n=1,2,\cdots$. For each $n\geq 2$, the stack Σ_n is obtained from Σ_{n-1} by dividing Σ_{n-1} into two equal parts and adding a finite number, say a_{n-1}, of spacers on the left subcolumn. The top of the left subcolumn (after adding spacers) is mapped linearly onto the bottom of the right subcolumn, and the resulting

new stack is Σ_n. If T is the measure preserving transformation given by this system of stacks, then, as shown by Ledrappier, the associated unitary operator U_T has simple spectrum whose maximal spectral type,(except possibly for some discrete part), is given by the weak∗ limit of probability measures

$$\mu_L = \prod_{n=1}^{L} \left| \frac{1}{\sqrt{2}} \left(1 + z^{h_n+a_n}\right) \right|^2 dz = \prod_{n=1}^{L} \left|1 + \frac{1}{2}\left(z^{h_n+a_n} + z^{-h_n-a_n}\right)\right|^2 dz, L = 1, 2, \cdots$$

as $L \longrightarrow +\infty$. We write this measure formally as

$$\mu = \prod_{n=1}^{+\infty} \left| \frac{1}{\sqrt{2}} \left(1 + z^{h_n+a_n}\right) \right|^2 \qquad (A).$$

More generally, consider a rank one measure preserving transformation made of a sequence of stacks Σ_n, $n = 1, 2, \cdots$ with h_n as the height of Σ_n, and Σ_1 being the unit interval. For $n \geq 2$, Σ_n is obtained from Σ_{n-1} by dividing Σ_{n-1} into m_{n-1} equal parts, and placing a certain number $a_j^{(n-1)}$ of spacers on the jth subcolumn, $1 \leq j \leq m_{n-1} - 1$. Let T denote the resulting measure preserving transformation and U_T the associated unitary operator. Here again the operator U_T has simple spectrum and the maximal spectral type (except possibly some discrete part) is given by the weak∗ limit of the probability measures [13],[17],[15]:

$$\mu_n = \prod_{k=1}^{n} \frac{1}{m_k} \left| \left(1 + \sum_{j=1}^{m_k-1} z^{jh_k + \sum_{i=1}^{j} a_i^{(k)}}\right) \right|^2 dz, \ n = 1, 2, 3, \cdots$$

We denote this weak∗ limit μ by the infinite product:

$$\mu = \prod_{k=1}^{+\infty} \frac{1}{m_k} \left| \left(1 + \sum_{j=1}^{m_k-1} z^{jh_k + \sum_{i=1}^{j} a_i^{(k)}}\right)^2 \right| \qquad (B)$$

Ornstein [20] has constructed a family of rank one measure preserving maps which are mixing. Bourgain [8] has shown that almost all of these rank one map have singular spectrum. It is not known if there exists a rank one measure preserving map whose maximal spectral type has a part which is absolutely continuous with respect to the Lebesgue measure on S^1. This question is naturally related to a question of Banach (in The Scottish Book) which asks if there is a measure preserving transformation T on the Real line (with Lebesgue measure) which admits a function f such that $f \circ T^n$, $n = 1, 2, \cdots$ are orthogonal and span $L^2(\mathbb{R})$.

Let Σ_n, $n = 1, 2, \cdots$ and T be as above, and let ϕ be a function of absolute value one which is constant on interval of Σ_n except the top piece, $n = 1, 2, \cdots$. It is known that the unitary operator $V_\phi = \phi \cdot U_T$ also has multiplicity one and its maximal spectral type is continuous whenever ϕ is not a coboundary. It is given by the weak∗ limit of a sequence of probability measures given by:

$$\mu_L = \prod_{n=1}^{L} \left| P_k(z) \right|^2 dz,$$

where P_k's are polynomials of the type:

$$P_k(z) = c_0^{(k)} + \sum_{j=1}^{m_k-1} c_j^{(k)} z^{jh_k + \sum_{i=1}^{j} a_i^{(k)}}, \quad \sum_{j=0}^{m_k-1} |c_j^{(k)}|^2 = 1.$$

The constants $c_j^{(k)}$ are determined by the m_k's and the function ϕ. We may write this weak$*$ limit as

$$\mu_\phi = \prod_{n=1}^{+\infty} \left| P_k(z) \right|^2 \tag{C}.$$

Note that in all the products (A), (B), and (C) there is no lacunarity condition imposed on the powers of z from outside. The gap between two consecutive nonzero terms of the polynomials are determined by parameters of construction, and need not be lacunary. In the rest of this paper we will, for most part, dispense with the dynamical origin of the measures of the type (A), (B) and (C) and discuss a larger class of measures, called generalized Riesz product, which include these measures.

2. GENERALIZED RIESZ PRODUCTS AND THEIR WEAK DICHOTOMY.

In this section we introduce generalized Riesz products and derive a weak dichotomy result about infinite product of polynomials associated to it. This also yields conditions for absolute continuity of the generalized Riesz product.

Definition 2.1. Let $P_1, P_2, \cdots$, be a sequence of trigonometric polynomials such that

(i) for any finite sequence $i_1 < i_2 < \cdots < i_k$ of natural numbers

$$\int_{S^1} \left| (P_{i_1} P_{i_2} \cdots P_{i_k})(z) \right|^2 dz = 1,$$

where S^1 denotes the circle group and dz the normalized Lebesgue measure on S^1,

(ii) for any infinite sequence $i_1 < i_2 < \cdots$ of natural numbers the weak limit of the measures $| (P_{i_1} P_{i_2} \cdots P_{i_k})(z) |^2 dz, k = 1, 2, \cdots$ as $k \to \infty$ exists,

then the measure μ given by the weak limit of $| (P_1 P_2 \cdots P_k)(z) |^2 dz$ as $k \to \infty$ is called generalized Riesz product of the polynomials $| P_1 |^2, | P_2 |^2, \cdots$ and denoted by

$$\mu = \prod_{j=1}^{\infty} |P_j|^2 \tag{1.1}.$$

For an increasing sequence $k_1 < k_2 < \cdots$ of natural numbers the product $\prod_{j=1}^{\infty} |P_{k_j}|^2$ makes sense as the weak limit of probability measures $|(P_{k_1} P_{k_2} \cdots P_{k_n})(z)|^2 dz$ as $n \to \infty$. It depends on the sequence $k_1 < k_2 \cdots$, and called a subproduct of the given generalized Riesz product.

Since the object under consideration is the generalized Riesz product $\prod_{j=1}^{\infty} |P_j|^2$, without loss of generality we assume that the polynomials $P_j, j = 1, 2, \cdots$ are analytic with positive constant term. Their domain of definition will mainly be the circle group, but with option to look at then as functions on the complex plane. Since $\int_{S^1} |P_j|^2(z) dz = 1$, the sum of the squares of the absolute values of coefficients of P_j is one, and so each coefficient of P_j is at most one in absolute value. Let $a_0^{(j)}$ denote the constant term of P_j, which is positive by assumption. The sequence of products $\prod_{j=1}^{n} a_0^{(j)}, n = 1, 2, \cdots$ is non-increasing, and so has a limit which is either zero or some positive constant which can be at most 1. The case when this constant is one is obviously the trivial case when each P_j is the constant 1.

Consider the sequence of polynomials $S_n \overset{\text{def}}{=} \prod_{j=1}^{n} P_j, n = 1, 2, \cdots$ (without the absolute value squared). For each n, let $b_0^{(n)} \overset{\text{def}}{=} \prod_{j=1}^{n} a_0^{(j)}$ denote the constant term of S_n. Write $b = \lim_{n \to \infty} b_0^{(n)}$. We have the following weak dichotomy theorem for generalized Riesz products.

Theorem 2.2. *If $b = 0$, the sequence of polynomials $S_n = \prod_{i=1}^{n} P_i, i = 1, 2, \cdots$ converges to zero weakly in $L^2(S^1, dz)$. If b is positive it converges in $L^1(S^1, dz)$ (and in $\mathbb{H}^1$) norm to a non-zero function f which is also in $\mathbb{H}^2$ with $\mathbb{H}^2$ norm at most 1, $\log(|f|)$ has finite integral.*

Proof. Assume that $b = 0$. We show that the sequence $S_n, n = 1, 2, \cdots$ has zero weak limit as functions in $L^2(S^1, dz)$. Assume that a subsequence $S_{k_n}, n = 1, 2, \cdots$ converges weakly $f \in L^2(S^1, dz)$. We show that f is the zero function. By choosing a further subsequence if necessary we can assume without any loss that the constant term of $\frac{S_{k_{n+1}}}{S_{k_n}}, n = 1, 2, \cdots$ goes to zero as $n \to \infty$. Since $b = 0$, the zeroth Fourier coefficient of f is zero. Since each S_n is an analytic trigonometric polynomial, the negative Fourier coefficients of f are all zero. Assume now that for $0 \le j < l$, $b_j = \int_{S^1} z^{-j} f(z) dz = 0$. Then, given ϵ, for large enough m, $\left| \int_{S^1} z^{-j} S_{k_m} dz \right| < \epsilon$, for $0 \le j < l$, and moreover the constant term of $\frac{S_{k_{m+1}}}{S_{k_m}}$ is less than ϵ. For $n > m$,

$$\prod_{j=1}^{k_n} P_i = \prod_{j=1}^{k_m} P_j \prod_{j=k_m+1}^{k_n} P_j.$$

Since P_j's are one sided trigonometric polynomials, it is easy to see from this that $\left| \int_{S^1} z^{-l} S_{k_n}(z) dz \right| \leq (l+1)\epsilon$. Since this holds for all $n > m$ we see that $\int_{S^1} z^{-l} f(z) dz = 0$. Induction completes the proof.

Assume now that $b > 0$. Then $a_0^{(n)}$ as well as $\prod_{j=m+1}^{n} a_0^{(j)}, m < n$, converge to 1 as $m, n \to \infty$. Since $L^2(S^1, dz)$ norm of all the finite products is one,

(2.1) $$P_n, \frac{S_n}{S_m} \to 1 \text{ in } L^2(S^1, dz), \ m < n, \ \text{ as } m, n \to \infty.$$

Moreover by Cauchy-Schwarz inequality

$$\left\| S_n - S_m \right\|_1 = \left\| S_m \left(1 - \prod_{m+1}^{n} P_i \right) \right\|_1 \leq \left\| S_m \right\|_2 \left\| 1 - \prod_{j=m+1}^{n} P_j \right\|_2$$

$$\to 0 \text{ as } m, n \to \infty.$$

Thus the sequence of analytic polynomials $S_n, n = 1, 2, \cdots$ converges in $L^1(S^1, dz)$ to a function which we denote by f, and view it also as a function in the Hardy class $\mathbb{H}^1$. A subsequence of $S_n, n = 1, 2, \cdots$ converges to f a.e (with respect to the Lebesgue measure of S^1), whence, over the same subsequence $S_n^2, |S_n|^2, n = 1, 2, \cdots$ converge to f^2 and $|f|^2$, respectively. Since $\|S_n\|_2 = 1, n = 1, 2, \cdots$, by Fatou's lemma we conclude that f is square integrable with $L^2(S^1, dz)$ norm at most 1. Thus f is in $\mathbb{H}^2$, and $\log |f|$ has finite integral.

$\square$

We do not know if the $L^2(S^1, dz)$ norm of f is 1, equivalently, if $S_n, n = 1, 2, \cdots$ converges to f in $L^2(S^1, dz)$. We give some sufficient conditions under which this holds. Let $S_n = \sum_{j=0}^{m_n} b_j^{(n)} z^j$, where m_n is the degree of the trigonometric polynomial S_n. Now

$$b_j^{(n)} = \int_{S^1} z^{-j} S_n(z) dz \to \int_{S^1} z^{-j} f(z) dz \stackrel{\text{def}}{=} b_j.$$

The series $\sum_{j=0}^{\infty} b_j z^j$ is the Fourier series of f and we call this series the formal expansion of $\prod_{j=1}^{\infty} P_i$. Since b is positive, the infinite product $\prod_{j=n+1}^{\infty} a_0^{(j)}$ is also positive, so the infinite product $\prod_{j=n+1}^{\infty} P_j$ has a formal expansion which we denote by $\sum_{j=0}^{\infty} c_j^{(n)} z^j$. Note that $c_0^{(n)} = \prod_{j=n+1}^{\infty} a_0^{(j)} \longrightarrow 1$ as $n \longrightarrow \infty$, as a result $\sum_{j=1}^{\infty} |c_j^{(n)}|^2 \leq 1 - (c_0^{(n)})^2 \longrightarrow 0$ as $n \to \infty$.

At this point, let us recall the following important notion in the Riesz product theory.

Definition 2.3. Finitely many trigonometric polynomials $q_0, q_1, \cdots, q_n$, $q_j = \sum_{i=-N_j}^{N_j} d_i^{(j)} z^i, j = 0, 1, 2, \cdots, n$ are said to be dissociated if in their product $q_0(z) q_1(z) \cdots q_n(z)$, (when expanded formally, i.e., without grouping terms or

canceling identical terms with opposite signs), the powers $i_0 + i_1 + \cdots + i_n$ of z in non-zero terms

$$d_{i_0}^{(0)} d_{i_1}^{(1)} \cdots d_{i_n}^{(n)} z^{i_0 + i_1 + \cdots + i_n}$$

are all distinct.

A sequence $q_0, q_1, \cdots$, of trigonometric polynomials is said to be dissociated if for each n the polynomials $q_0, q_1, \cdots, q_n$ are dissociated.

Suppose now that the polynomials $P_1, P_2, \cdots$ (without the squares) appearing in generalized Riesz product 1.1 are dissociated. Then, whenever $b_j^{(n)}$ is a non-zero coefficient in the expansion of S_n, $b_j^{(l)} = b_j^{(n)} \frac{b_0^{(l)}}{b_0^{(n)}}$ for all $l \geq n$. Thus, if the polynomials $P_j, j = 1, 2, \cdots$ are dissociated, then we see on letting $l \to \infty$ that $b_j = b_j^{(n)} c_0^{(n)}$, provided $b_j^{(n)} \neq 0$. We therefore have for any n

$$\sum_{j=0}^{m_n} |b_j|^2 \geq \sum_{j=1}^{m_n} |b_j^{(n)}|^2 (c_0^{(n)})^2 = (c_0^{(n)})^2 \longrightarrow 1$$

as $n \to \infty$. Thus f has $L^2(S_1, dz)$ norm 1. We have proved:

Theorem 2.4. *If the polynomials $P_n, n = 1, 2, \cdots$ are dissociated and b is positive then the partial products $S_n, n = 1, 2, \cdots$ converge in $\mathbb{H}^2$ to a non-zero function f and the generalized product $\prod_{j=1}^{\infty} |P_j|^2$ is the measure $|f|^2 dz$. Further,* $\int_{S^1} \log(|f(z)|)dz$ *is finite.*

If we replace the condition that the polynomials $P_n, n = 1, 2, \cdots$ are dissociated by the condition that coefficients of the polynomials $P_n, n = 1, 2, \cdots$ are all non-negative, then it is easy to verify that for $0 \leq k \leq m_n$,

$$b_k \geq c_0^{(n)} b_k^{(n)} + b_0^{(n)} c_k^{(n)},$$

whence

$$\sum_{k=0}^{\infty} |b_k|^2 \geq (\sum_{k=0}^{m_n} |b_k^{(n)}|^2) |c_0^{(n)}|^2 = 1 \cdot |c_0^{(n)}|^2 \to 1,$$

as $n \to \infty$. Thus, if the coefficients of all the $P_n, n = 1, 2, \cdots$ are non-negative, and if $b = b_0 > 0$, we necessarily have convergence of $S_n, n = 1, 2, \cdots$ in $\mathbb{H}^2$.

We continue with the assumption that b is positive, but no more assume that the polynomials $P_n, n = 1, 2, \cdots$ are dissociated or have non-negative coefficients. Fix n, and let $1 \leq j \leq m_n$, then

$$\sum_{j=0}^{\infty} b_j z^j = \left(\sum_{i=0}^{m_n} b_i^{(n)} z^i \right) \left(\sum_{k=0}^{\infty} c_k^{(n)} z^k \right).$$

This gives any $j \geq 0$,

$$b_j = b_j^{(n)} c_0^{(n)} + \sum_{i=0}^{j-1} b_i^{(n)} c_{j-i}^{(n)}.$$

Hence, for any $j \geq 1$,

$$\mid b_j - b_j^{(n)} c_0^{(n)} \mid^2 \leq \left(\sum_{i=0}^{j-1} \mid b_i^{(n)} \mid^2 \right) \left(\sum_{i=0}^{j-1} \mid c_{j-i}^{(n)} \mid^2 \right) \leq \sum_{j=1}^{\infty} \mid c_j^{(n)} \mid^2$$

$$\leq 1 - (c_0^{(n)})^2 \longrightarrow 0 \text{ as } n \to \infty$$

Assume now that $m_n(1 - c_0^{(n)}) \longrightarrow 0$ as $n \to \infty$ Then $\sum_{j=0}^{m_n} \mid b_j - b_j^{(n)} c_0^{(n)} \mid^2 \to 0$ as $n \to \infty$. Another use of the assumption that $m_n(1 - c_0^{(n)}) \longrightarrow 0$ as $n \to 0$ allows us to conclude that $\sum_{j=0}^{m_n} |b_j - b_j^{(n)}|^2 \longrightarrow 0$ as $n \to \infty$. Since $\sum_{j=1}^{m_n} |b_j^{(n)}|^2 = 1$, we conclude that $\sum_{j=1}^{\infty} |b_j|^2 = 1$, so that $L^2(S^1, dz)$ norm of f is one and $S_n, n = 1, 2, \cdots$ converges to f in $\mathbb{H}^2$. We have proved:

Theorem 2.5. *If b is positive and $m_n(1 - c_0^{(n)}) \longrightarrow 0$ as $n \to \infty$ then $S_n \to f$ in $\mathbb{H}^2$ and $|f|^2 dz$ is the generalized Riesz product $\prod_{j=1}^{\infty} |P_j|^2$. Moreover $\log(|f|)$ has finite integral.*

Our calculus can be interpreted as follows. Put

$$B = (b_j)_{j=0}^{+\infty}, \ B^{(n)} = (b_0^{(n)}, b_1^{(n)}, \cdots, b_{m_n}^{(n)}, 0, 0, \cdots) \text{ and } C^{(n)} = (c_j^{(n)})_{j=0}^{+\infty},$$

then

$$\widehat{B}(z) = \sum_{j=0}^{\infty} b_j z^j = \left(\sum_{i=0}^{m_n} b_i^{(n)} z^i \right) \left(\sum_{k=0}^{\infty} c_k^{(n)} z^k \right) = \widehat{B^{(n)}}(z) \widehat{C^{(n)}}(z) = \widehat{B^{(n)} * C^{(n)}}(z).$$

Therefore

$$\left\| B - B^{(n)} \right\|_2 = \left\| B^{(n)} * C^{(n)} - B^{(n)} \right\|_2 = \left\| B^{(n)} * \left(C^{(n)} - (1, 0, 0, \cdots) \right) \right\|_2.$$

Hence

$$\left\| B - B^{(n)} \right\|_2 \leq \left\| B^{(n)} \right\|_1 \left\| C^{(n)} - (1, 0, 0, \cdots) \right\|_2.$$

It follows under our condition $(m_n(1 - c_0^{(n)}) \longrightarrow 0$ as $n \to \infty)$ that

$$\left\| B^{(n)} \right\|_1 \left\| C^{(n)} - (1, 0, 0, \cdots) \right\|_2 \longrightarrow 0.$$

Corollary 2.6. *If $b > 0$ then there is always a subproduct $\prod_{k=1}^{\infty} P_{n_k}$ for which the condition of the above theorem is satisfied, so that if $b > 0$ holds, then a subproduct $\prod_{k=1}^{\infty} | P_k(z) |^2$ has the same null sets as Lebesgue measure.*

Proof. Put $k_1 = 1$ and $P_{k_1} = P_1$. Let m_1 be the degree of P_{k_1}. Since $b > 0$, $c_0^{(n)} \to 1$ as $n \to \infty$. Choose $k_2 > k_1$ such that $m_1(1 - c_0^{(k_2)}) \leq \frac{1}{2}$. Consider $P_{k_1} \cdot P_{k_2+1}$. Suppose its degree is m_2. Choose $k_3 > k_2$ such that $m_2(1 - c_0^{(k_3)}) \leq \frac{1}{4}$. Assume

that we have chosen $k_1 < k_2 < \cdots < k_{l-1}$ such that for any $i, 1 \leq i \leq l-2$ if m_i is the degree of $P_{k_1} P_{k_2+1} \cdots P_{k_i+1}$, then

$$m_i(1 - c_0^{(k_{(i+1)})}) \leq \frac{1}{2^i}.$$

Choose $k_l > k_{1-1}$ such that

$$m_{l-1}(1 - c_0^{(k_l)}) \leq \frac{1}{2^{l-1}}.$$

Thus we have inductively chosen a sequence $k_1 < k_2 < k_3 < \cdots < k_i < \cdots$. Write $J_1 = P_{k_1}, J_2 = P_{k_2+1}, \cdots, J_n = P_{k_n+1}, \cdots$ and $R = \prod_{i=0}^{\infty} | J_i(z) |^2$. If γ_n denotes the constant term of $\prod_{i=n+1}^{\infty} J_i$, then it is easy to see that $\gamma_n > c_0^{(k_n)}$ so that $p_n(1 - \gamma_{n+1}) \leq \frac{1}{2^n}$, where p_n is the degree of $\prod_{i=1}^{n} J_i$. By the theorem above the Riesz product $R = \prod_{i=1}^{\infty} | J_i(z) |^2$ is equivalent to the Lebesgue measure. This completes the proof of the corollary. $\qquad\square$

Assume that b is positive. Consider $L(z) = \prod_{j=m+1}^{n} |P_j(z)|^2$. If $d_k(m,n) = d_j$ is the coefficient of z^k in $\prod_{j=m+1}^{n} P_j$, then for $k > 0$, the coefficient of z^k in $L(z)$ is in absolute value

$$\left| \sum_{j \geq k} d_j \overline{d_{j-k}} \right| \leq \left(\sum_{j \geq k} | d_j |^2 \right)^{\frac{1}{2}} \leq (1 - d_0^2)^{\frac{1}{2}}.$$

Under the assumption that b is positive we can make this coefficient (which depends on m and n) as small as we please by choosing m large. We have proved:

Theorem 2.7. *If b is positive, the generalized Riesz products $\mu_n \overset{def}{=} \prod_{n+1}^{\infty} | P_i |^2$, $n = 1, 2, \cdots$ converge weakly to the Lebesgue measure on S^1 as $n \to \infty$.*

We do not know if the conclusion of Theorem 2.7 always holds when b is zero, but such generalized Riesz products form an important class of measures and will be discussed in section 5.

Remark 2.8. The weak dichotomy theorem (Theorem 2.2) is rather weak in the sense that no information can be garnered about μ, such as absolute continuity or singularity, when b is zero. Consider the classical Riesz product

$$\mu = \prod_{j=1}^{\infty} | \cos(\theta_j) + \sin(\theta_j) z^{n_j} |^2, \ \frac{n_{j+1}}{n_j} \geq 3, \ 0 < \theta_j < \frac{\pi}{2}, \ j = 1, 2, \cdots$$

It is known to be absolutely continuous if $\sum_{j=1}^{\infty} \cos^2(\theta_j) \sin^2(\theta_j)$ is finite and singular otherwise. Clearly the condition for absolute continuity is satisfied with $\cos(\theta_j) = \frac{1}{j}, j = 1, 2, \cdots$ and also with $\cos(\theta_j) = \sqrt{1 - \left(\frac{1}{j}\right)^2}$, $j = 1, 2, \cdots$. In the first case the product of the constant terms is zero, while in the second case it is positive. This defect is rectified if we replace the polynomials P_j with their outer parts, as discussed in the next section.

3. Outer Polynomials and Mahler Measure.

Let

$$\mu = \prod_{j=1}^{\infty} \mid P_j(z) \mid^2 \tag{1}$$

be a generalized Riesz product. Let μ_a denote the part of μ absolutely continuous with respect to dz. We write $\frac{d\mu}{dz}$, to mean $\frac{d\mu_a}{dz}$. In this section we use the classical prediction theoretic ideas to evaluate $\exp\left(\int_{S^1} \log\left(\frac{d\mu}{dz}\right)dz\right)$ a quantity which we call the Mahler measure of μ with respect to the Lebesgue measure.

We will prove:

Theorem 3.1.

$$\int_{S^1} \log \frac{d\mu_a}{dz}dz = \lim_{n\to\infty} \int_{S^1} \log \prod_{j=1}^{n} \mid P_j(z) \mid^2 dz.$$

Note that the theorem is false if we drop the log on both sides of the equation, for then the righthand side is always one, while the left hand side is zero for μ singular to Lebesgue measure. For the proof we begin by recalling Beurling's inner and outer factors for the case of polynomials and the expression for one step'prediction error', namely the quantity:

$$\inf_{q\in\mathcal{Q}} \int_{S^1} \mid 1 - q(z) \mid^2 \mid P(z) \mid^2 dz,$$

where $P(z)$ is an analytic trigonometric polynomial with $L^2(S^1, dz)$ norm 1 and non-zero constant term. $\mathcal{Q}$ is the class of all analytic trigonometric polynomials with zero constant term. To this end we have to bring into consideration the zeros of polynomials $P_j, j = 1, 2, \cdots$. Consider the kth polynomial of the generalized Riesz product product $\prod_{k=1}^{\infty} |P_k|^2$. Suppressing the index k, it is of the type:

$$P(z) = a_0 + a_1 z + \cdots + a_m z^m.$$

assuming that it is of degree m. Let

$$A = \{a : P(a) = 0, \mid a \mid < 1\},$$
$$B = \{b : P(b) = 0, \mid b \mid = 1\},$$
$$C = \{c : P(c) = 0, \mid c \mid > 1\}.$$

Then

$$P(z) = a_m \prod_{a\in A}(z - a) \prod_{b\in B}(z - b) \prod_{c\in C}(z - c)$$

$$= \prod_{a\in A}\frac{(z - a)}{(1 - \overline{a}z)}a_m \prod_{a\in A}(1 - \overline{a}z) \prod_{b\in B}(z - b) \prod_{c\in C}(z - c).$$

Write

$$I(z) = \overline{\gamma} \prod_{a \in A} \frac{(z-a)}{(1 - \overline{a}z)},$$

$$O(z) = \gamma a_m \prod_{a \in A} (1 - \overline{a}z) \prod_{b \in B} (z - b) \prod_{c \in C} (z - c).$$

where γ is a constant of absolute value 1 such that the constant term of $O(z)$ is positive, while $\overline{\gamma}$ is the complex conjugate of γ. We have,

$$P(z) = I(z)O(z).$$

Note that for $z \in S^1$, $\mid I(z) \mid = 1$, $\mid P(z) \mid = \mid O(z) \mid$. The function $O(z)$ is non-vanishing inside the unit disc. The functions I and O are Beurling's inner and outer parts of the polynomial P. Note that, since constant term of P is non-zero, the degree of O is same as that of P and that $O(0) = $ constant term of $O \geq P(0) = a_0$. Recall that outer functions in $\mathbb{H}^2$ are precisely those functions f in $\mathbb{H}^2$ for which the functions $z^n f, n \geq 0$ span $\mathbb{H}^2$ in the closed linear sense. Hence, if f is an outer function in $\mathbb{H}^2$, then the closed linear span of $\{z^n f, n \geq 1\}$ is $z\mathbb{H}^2$. The orthogonal projection of $O(z)$ on $z\mathbb{H}^2$ is $O(z) - O(0)$ where $O(0)$ is the constant term of $O(z)$ which we denote by α. Note that

$$|\alpha| = \left| a_m \prod_{b \in B} b \prod_{c \in C} c \right| = |a_m| \prod_{c \in C} |c|.$$

We have

$$|\alpha|^2 = \int_{S^1} \left| \alpha \right|^2 dz = \int_{S^1} \left| \left(O(z) - (O(z) - \alpha) \right) \right|^2 dz$$

$$= \int_{S^1} \left| 1 - \frac{(O(z) - \alpha)}{O(z)} \right|^2 \left| O(z) \right|^2 dz$$

$$= \inf_{q \in \mathcal{Q}} \int_{S^1} \left| 1 - q(z) \right|^2 \left| O(z) \right|^2 dz,$$

$$= \inf_{q \in \mathcal{Q}} \int_{S^1} \left| 1 - q(z) \right|^2 \left| P(z) \right|^2 dz$$

where the infimum is taken over the class $\mathcal{Q}$ of all analytic trigonometric polynomials q with zero constant term. Thus $\frac{O(z) - \alpha}{O(z)}$ is the orthogonal projection of the constant function 1 on the closed linear span of $\{z^n, n \geq 1\}$ in $L^2(S^1, |P(z)|^2 dz)$.

Lemma 3.2. *If λ is a probability measure on S^1 such that $d\nu =\mid P(z)\mid^2 d\lambda$ is again a probability measure then*

$$\mid \alpha \mid^2 \geq \inf_{q \in \mathcal{Q}} \int_{S^1} \mid 1 - q(z) \mid^2 d\nu.$$

Proof. If $O(z)$ has no zeros on the unit circle then $\dfrac{O(z) - \alpha}{O(z)}$ is analytic on the closed unit disk. The partial sums of the power series of this function converge to it uniformly on the unit circle. Let $q_k, k = 1, 2, \cdots$ be the sequence of these partial sums. Then

$$\mid \alpha \mid^2 = \int_{S^1} \left| 1 - \frac{O(z) - \alpha}{O(z)} \right|^2 \left| O(z) \right|^2 d\lambda$$

$$= \int_{S^1} \left| 1 - \frac{O(z) - \alpha}{O(z)} \right|^2 d\nu$$

$$= \lim_{k \to \infty} \int_{S^1} \left| 1 - q_k \right|^2 d\nu \geq \inf_{q \in \mathcal{Q}} \int_{S^1} \left| 1 - q(z) \right|^2 d\nu$$

This conclusion remains valid even if $O(z)$ has zeros on the circle but the proof is slightly different. For fixed r, $0 \leq r < 1$, the function $\frac{O(z)-\alpha}{O(rz)}$ is analytic on the closed unit disk, so the partial sums of its power series converge to it uniformly on S^1. Now for any fixed real θ, $\dfrac{z - e^{i\theta}}{rz - e^{i\theta}}$ remains bounded by 2 for $z \in S^1$ and $0 \leq r < 1$, and converges to 1 as $r \to 1$, for $z \neq e^{i\theta}$. Therefore $z \neq \theta$, $\dfrac{O(z)}{O(rz)} \to 1$ boundedly as $r \to 1$, whence

$$\left| 1 - \frac{O(z) - \alpha}{O(rz)} \right|^2 \left| O(z) \right|^2 \to |\alpha|^2$$

boundedly as $r \to 1$. It is easy to see from this that

$$\inf_{q \in \mathcal{Q}} \int_{S^1} \left| 1 - q \right|^2 d\nu \leq \lim_{r \to 1} \int_{S^1} \left| 1 - \frac{O(z) - \alpha}{O(rz)} \right|^2 d\nu = \mid \alpha \mid^2.$$

This proves the lemma. $\qquad\square$

Consider now the polynomials P_k, $k = 1, 2, \cdots$ and the associated finite products $\prod_{k=1}^n P_k$, $n = 1, 2, 3, \cdots$. Let $A_k, B_k, C_k, I_k, O_k, \alpha_0^{(k)}$ have the obvious meaning: they are for P_k what A, B, C, I, O, α are for P. Note that the inner and outer parts of $\prod_{k=1}^n P_k$ are $\prod_{k=1}^n I_k$ and $\prod_{k=1}^n O_k$ respectively and the constant term of the outer part is $\prod_{k=1}^n \alpha_0^{(k)}$. Note that $\prod_{k=1}^\infty \alpha_0^{(k)} \stackrel{\text{def}}{=} \beta \geq \prod_{k=1}^\infty a_0^{(k)} = b$, so if b is positive, then so is β. On the other hand, the positivity of β does not in general imply positivity of b as shown by the case of classical Riesz product (see remark 3.4 below).

To prove Theorem 3.1 We apply the lemma above to

$$P = \prod_{k=1}^{n} P_k(z)$$

and

$$\lambda = \prod_{k=n+1}^{\infty} \mid P_k(z) \mid^2,$$

Note that

$$d\nu = \Big(\prod_{k=1}^{n} \mid P_k(z) \mid^2\Big) d\lambda = d\mu.$$

We see that for any n

$$\inf_{q \in \mathcal{Q}} \int_{S^1} \Big|1 - q\Big|^2 d\mu \le \prod_{k=1}^{n} \mid \alpha_0^{(k)} \mid^2,$$

whence

$$\inf_{q \in \mathcal{Q}} \int_{S^1} \Big|1 - q\Big|^2 d\mu \le \prod_{k=1}^{\infty} \mid \alpha_0^{(k)} \mid^2 .$$

To prove the above inequality in the reverse direction we note that by Szegö's theorem as generalized by Kolmogorov and Krein (K. H. Hoffman [12]) that

$$\exp\Big\{ \int_{S^1} \log\Big(\frac{d\mu_a}{dz}\Big) dz \Big\} = \inf_{q \in \mathcal{Q}} \int_{S^1} \mid 1 - q(z) \mid^2 d\mu.$$

Denote this infimum by l. Then, given $\epsilon > 0$, there is a polynomial $q_0 \in \mathcal{Q}$ such that

$$l \le \int_{S^1} \mid 1 - q_0 \mid^2 d\mu < l + \epsilon,$$

whence for large enough n

$$\int_{S^1} \Big|1 - q_0\Big|^2 \prod_{k=1}^{n} |P_k|^2 \, dz < l + \epsilon.$$

Since

$$\Big| \prod_{k=1}^{n} \alpha_0^{(k)} \mid^2 = \inf_{q \in \mathcal{Q}} \int_{S^1} \mid 1 - q \mid^2 \prod_{k=1}^{n} \mid P_k(z) \mid^2 \, dz,$$

we see that $\mid \alpha_0^{(1)} \cdot \alpha_0^{(2)} \cdots \alpha_0^{(n)} \mid^2 \le l + \epsilon$. Since ϵ is arbitrary positive real number, and $\mid \alpha_0^{(k)} \mid < 1$ for all k, we have

$$\prod_{k=1}^{\infty} \mid \alpha_0^{(k)} \mid^2 \le l.$$

We also note that

$$\prod_{j=1}^{n} \mid \alpha_0^{(j)} \mid = \exp\left\{\int_{S^1} \log\left(\mid \prod_{j=1}^{n} P_j(z) \mid\right) dz\right\}$$

Thus we have proved (see [18]):

(3.1) $$\prod_{k=1}^{\infty} \mid \alpha_0^{(k)} \mid^2 = \exp\left\{\int_{S^1} \log\left(\frac{d\mu_a}{dz}\right) dz\right\}$$

$$= \lim_{n \to \infty} \exp\left\{\int_{S^1} \log\left(\prod_{j=1}^{n} \mid P_j(z) \mid^2\right) dz\right\}.$$

which is indeed theorem 3.1 with some additional information.

Corollary 3.3. If each $P_i, i = 1, 2, \cdots$ is outer, then $\log\left(\frac{d\mu}{dz}\right)$ has finite integral if and only if β is positive.

Remark 3.4. i) For the trigonometric polynomial $P(z) = \cos\theta_j + \sin\theta_j z^{n_j}$ appearing in the classical Riesz product of remark 2.8, its outer part has the constant term $\max\{\cos\theta_j, \sin\theta_j\}$ and the condition $\sum_{j=1}^{\infty} \cos^2\theta_j \sin^2\theta_j < \infty$ is equivalent to the condition $\prod_{j=1}^{\infty} \max\{\cos\theta_j, \sin\theta_j\}$ is positive. The additional information we have now is that in case μ is absolutely continuous with respect to dz, $\log\frac{d\mu}{dz}$ has finite integral.

ii) Using a deep result of Bourgain [8], the first author has shown recently [3] that if the cutting parameter $m_n, n = 1, 2, \cdots$ of a rank one transformation T satisfies $\frac{m_n}{n^\beta} < K$ for all n for some constant K and some $\beta \in (0, 1]$, then the Mahler measure of the spectrum of T is zero.

iii) It is to be noted that if each P_n is outer and the product $\Pi_{k=1}^{\infty} P_k(0)$ is non-zero, then formal expansion f of $\Pi_{k=1}^{\infty} P_k(z)$ is an outer function. Indeed the Mahler measure of $\mid f \mid^2$ is $\mid f(0) \mid^2$, so f can not admit a non-trivial inner factor. Also the measure $1 \cdot dz$ can be expressed as a generalized Riesz product only by choosing each $P_n = 1$, for if any of the P_n is not the constant equal to 1, then its normalized outer part will have constant term less than one, which will force the Mahler measure of $1 \cdot dz$ to be less than 1, which is false. It is not known if the measure $cdz + d\delta_1, c, d > 0, c + d = 1$ can be expressed as a generalized Riesz product, where δ_1 denotes the Dirac measure at 1.

iv) Let μ be a probability measure on S^1, and let q be a natural number. We contract the measure to the arc $A = \{z : z = \exp\{i\theta\}, 0 \le \theta < \frac{2\pi}{q}\}$, namely we consider the measure ν_1 supported on this arc given by $\nu_1(B) = \mu(z^q : z \in B), B \subset A$. We write similarly $\nu_i(C) = \nu_1(\exp\{-\frac{2\pi i}{q}\}C), C \subset \exp\{\frac{2\pi i}{q}\}A$. Let $\Pi_q(\mu) = \frac{1}{q}\sum_{i=1}^{q} \nu_i$. It can be verified that if $\mu = \mid p(z) \mid^2 dz$, then

$$\Pi_q(\mu) = \mid P(z^q) \mid^2 dz,$$

from which we conclude that if $\mu = \Pi_{k=1}^\infty \mid P_k(z) \mid^2$ then

$$\Pi_q(\mu) = \Pi_{k=1}^\infty \mid P_k(z^q) \mid^2 .$$

We see immediately that the Mahler measure of a generalized Riesz product is invariant under the application of Π_q for any q.

v)   Let us point out that one can obtain in the similar way a formula for the $(m+1)$-step prediction error. Precisely, for any $m \in \mathbb{N}^*$, we have

$$\inf_{q \in z^{m+1}\mathbb{H}^2} \int |1-q|^2 \prod_{j=1}^N |P_j|^2 dz \xrightarrow[N \to +\infty]{} \inf_{q \in z^{m+1}\mathbb{H}^2} \|1-q\|_{L^2(\mu)}.$$

4. A Formula for Radon Nikodym Derivative.

Consider two generalized Riesz products μ and ν based on polynomials $P_j, j = 1, 2, \cdots$ and $Q_j, j = 1, 2, \cdots$ where ν is continuous except for a possible mass at 1. Under suitable assumptions we prove the formula:

$$\frac{d\mu}{d\nu} = \lim_{n \to \infty} \frac{\prod_{j=1}^n \mid P_j \mid^2}{\prod_{j=1}^n \mid Q_j \mid^2},$$

in the sense of $L^1(S^1, \nu)$ convergence.

Let σ and τ be two measures on the circle. Then, by Lebesgue decomposition of σ with respect to τ, we have

$$\sigma = \frac{d\sigma}{d\tau} d\tau + \sigma_s,$$

where σ_s is singular to τ and $\frac{d\sigma}{d\tau}$ is the Radon-Nikodym derivative. In the case of two Riesz products $\mu = \prod_{j=1}^\infty \mid P_j \mid^2$ and $\nu = \prod_{j=1}^\infty \mid Q_j \mid^2$, we are able to see that their affinities, namely the ratios $\frac{\prod_{j=1}^n |P_j|}{\prod_{j=1}^n |Q_j|}, k = 1, 2, \cdots$, converge in L^1 to $\sqrt{\frac{d\mu}{d\nu}}$, assuming that ν has no point masses except possibly at 1. This result extends a theorem of G. Brown and W. Moran [6]. Let δ_1 denote the unit mass at one. We have (see [3])

Theorem 4.1. *Let $\mu = \prod_{j=0}^\infty \mid P_j \mid^2$, $\nu = \prod_{j=0}^\infty \mid Q_j \mid^2$ be two generalized Riesz products. Let*

$$\mu_n = \prod_{j=n+1}^\infty \mid P_j \mid^2, \ \nu_n = \prod_{j=n+1}^\infty \mid Q_j \mid^2$$

Assume that

(1) $\nu = \nu' + b\delta_1$, ν' is continuous measure, $0 \le b < 1$.
(2) $\prod_{j=0}^n \mid P_j \mid^2 d\nu_n \longrightarrow \mu$ weakly as $n \longrightarrow \infty$
(3) $\prod_{j=0}^n \mid Q_j \mid^2 d\mu_n \longrightarrow \nu$ weakly as $n \longrightarrow \infty$

Then the finite products $R_n = \prod_{k=1}^{n} \left| \frac{P_k(z)}{Q_k(z)} \right|, n = 1, 2, \cdots$ *converge in* $L^1(S^1, \nu)$ *to* $\sqrt{\frac{d\mu}{d\nu}}$.

To prove this we need the following proposition.

Proposition 4.2. The sequence $\prod_{j=0}^{n} \left| \frac{P_j(z)}{Q_j(z)} \right|, n = 1, 2, \cdots$ converges weakly in $L^2(S^1, \nu)$ to $\sqrt{\frac{d\mu}{d\nu}}$.

Proof. Put $f = \sqrt{\frac{d\mu}{d\nu}}$ and let n be a positive integer. Now

$$\int_{S^1} R_n^2 d\nu = \int_{S^1} \prod_{j=1}^{n} |P_j|^2 d\nu_n \to \int_{S^1} d\mu = 1$$

by assumption (2). Hence $\int_{S^1} R_n^2 d\nu, n = 1, 2, \cdots$ remain bounded. Thus, the weak closure of $R_n(z), n = 1, 2, \cdots$ in $L^2(S^1, \nu)$ is not empty.

We show that this weak closure has only one point, namely, $\sqrt{\frac{d\mu}{d\nu}}$. Indeed, let g be a weak subsequential limit, say, of $R_{n_j}(z), j = 1, 2, \cdots$. Then, for any continuous positive function h, we have, by judicious applications of Cauchy-Schwarz inequality,

$$\left(\int_{S^1} f(z)h(z)d\nu(z) \right)^2 = \left(\int_{S^1} h(z)R_{n_j}(z)\frac{1}{R_{n_j}}\sqrt{\frac{d\mu}{d\nu}}d\nu(z) \right)^2$$

$$\leq \left(\int_{S^1} h(z)R_{n_j}(z)d\nu(z) \right) \left(\int_{S_1} h(z)R_{n_j}(z)\frac{1}{R_{n_j}^2(z)}\frac{d\mu}{d\nu}d\nu(z) \right)$$

$$\leq \left(\int_{S^1} h(z)R_{n_j}(z)d\nu(z) \right) \left(\int_{S^1} h(z)\frac{1}{R_{n_j}(z)}d\mu \right)$$

$$\leq \int_{S^1} h(z)R_{n_j}(z)d\nu(z) \left(\int_{S^1} h(z)d\mu \right)^{\frac{1}{2}} \left(\int_{S^1} h(z)\frac{d\mu}{R_{n_j}^2(z)} \right)^{\frac{1}{2}}$$

$$\leq \left(\int_{S^1} h(z)R_{n_j}(z)d\nu(z) \right) \left(\int_{S^1} h(z)d\mu \right)^{\frac{1}{2}} \left(\int_{S^1} h(z) \mid \prod_{k=1}^{n_j} Q_k \mid^2 d\mu_{n_j} \right)^{\frac{1}{2}}$$

Letting $j \to +\infty$, from our assumption (3), we get

$$\left(\int_{S^1} fhd\nu \right)^2 \leq \left(\int_{S^1} hgd\nu \right) \left(\int_{S^1} hd\mu \right)^{\frac{1}{2}} \left(\int_{S^1} hd\nu \right)^{\frac{1}{2}} \tag{2}.$$

But, since the space of continuous functions is dense in $L^2(\mu + \nu)$, we deduce from (2) that, for any Borel set B,

$$\left(\int_B f d\nu\right)^2 \leq \left(\int_B g d\nu\right)\left(\int_B d\mu\right)^{\frac{1}{2}}\left(\int_B d\nu\right)^{\frac{1}{2}}.$$

By taking a Borel set E such that $\mu_s(E) = 0$ and $\nu(E) = 1$, we thus get, for any $B \subset E$,

$$\left(\int_B f d\nu\right)^2 \leq \left(\int_B g d\nu\right)\left(\int_B f^2(z) d\nu\right)^{\frac{1}{2}}\left(\int_B d\nu\right)^{\frac{1}{2}}.$$

It follows from Martingale convergence theorem that:

$$f(z) \leq g(z) \text{ for almost all } z \text{ with respect to } \nu.$$

Indeed, let $\mathcal{P}_n = \{A_{n,1}, A_{n,2} \cdots, A_{n,k_n}\}$, $n = 1, 2, \cdots$, be a refining sequence of finite partitions of E into Borel sets such that such that they tend to the partition of singletons. If $\{x\} = \cap_{n=1}^{\infty} A_{n,j_n}$,

$$\left(\frac{1}{\mu(A_{n,j_n})}\int_B f d\nu\right)^2 \leq$$

$$\left(\frac{1}{\mu(A_{n,j_n})}\int_{A_{n,j_n}} g d\nu\right)\left(\frac{1}{\mu(A_{n,j_n})}\int_{A_{n,j_n}} f^2(z) d\nu\right)^{\frac{1}{2}}\left(\frac{1}{\mu(A_{n,j_n})}\int_{A_{n,j_n}} d\nu\right)^{\frac{1}{2}}.$$

Letting $n \to \infty$ we have, by Martingale convergence theorem as applied to the theory of derivatives, for a.e $x \in E$ w.r.t. ν,

$$(f(x))^2 \leq g(x)f(x), \text{ whence } f(x) \leq g(x)$$

For the converse note that for any continuous positive function h we have

$$\int_{S^1} g h d\nu = \lim_{j \to +\infty} \int_{S^1} h(z) R_{n_j} d\nu$$

$$\leq \lim_{j \to \infty} \left(\int h R_{n_j}^2 d\nu\right)^{\frac{1}{2}}\left(\int_{S^1} h d\nu\right)^{\frac{1}{2}}$$

$$\leq \left(\int_{S^1} h d\mu\right)^{\frac{1}{2}}\left(\int h d\nu\right)^{\frac{1}{2}}.$$

As before we deduce that $g(z) \leq f(z)$ for almost all z with respect to ν. Consequently, we have proved that $g = f$ for almost all z with respect to ν and this complete the proof of the proposition. $\qquad\square$

Proof of Theorem 4.1. We will show that $\beta_n \stackrel{\text{def}}{=} \int_{S^1} \mid R_n - f \mid d\nu \to 0$ as $n \to \infty$, where $f = \sqrt{\frac{d\mu}{d\nu}}$. Now,

$$\frac{d\mu}{d\nu} = R_n^2(z)\frac{d\mu_n}{d\nu_n} \text{ and } \sqrt{\frac{d\mu}{d\nu}} = R_n(z)\sqrt{\frac{d\mu_n}{d\nu_n}}$$

Put

$$f_n^2 = \frac{d\mu_n}{d\nu_n},$$

Then,

$$\int_{S^1} f_n^2 d\nu = \int_{S^1} \prod_{k=1}^n \mid Q_k \mid^2 d\mu_n \to \int_{S^1} d\nu = 1,$$

by assumption (3). The functions $f_n, n = 1, 2, \cdots$ are therefore bounded in $L^2(S^1, \nu)$. Hence, there exists a subsequence $f_{n_j} = \sqrt{\frac{d\mu_{n_j}}{d\nu_{n_j}}}, j = 1, 2, \cdots$ which converges weakly to some $L^2(S^1, \nu)$-function ϕ. We show that $0 \leq \phi \leq 1$ a.e (ν). For any continuous positive function h, we have

$$\left(\int_{S^1} hf_{n_j} d\nu\right)^2 \leq \left(\int_{S^1} hd\nu\right)\left(\int_{S^1} hf_{n_j}^2 d\nu\right)$$

$$\leq \left(\int_{S^1} hd\nu\right)\left(\int_{S^1} h\frac{d\mu_{n_j}}{d\nu_{n_j}}d\nu\right).$$

Hence, by letting j go to infinity combined with our assumption (3), we deduce that

$$\int_{S^1} h(z)\phi(z)d\nu \leq \int_{S^1} h(z)d\nu.$$

Since this hold for all continuous positive functions h, we conclude that $0 \leq \phi \leq 1$ for almost all z with respect to ν. Thus any subsequential limit of the sequence $f_n, n = 1, 2, \cdots$ assumes values between 0 and 1. Now, for any subsequence $n_j, j = 1, 2, \cdots$ over which $f_{n_j}, j = 1, 2, \cdots$ has a weak limit , from our assumption (2) combined with Cauchy-Schwarz inequality, we have

$$\left(\int_{S^1} |R_{n_j} - f|d\nu\right)^2 = \left(\int_{S^1} |R_{n_j} - R_{n_j}f_{n_j}|d\nu\right)^2$$

$$= \left(\int_{S^1} R_{n_j}|1 - f_{n_j}|d\nu\right)^2$$

$$\leq \left(\int_{S^1} R_{n_j} |1 - f_{n_j}|^2 d\nu \right) \left(\int_{S^1} R_{n_j} d\nu \right)$$

$$\leq \left(\int_{S^1} R_{n_j} d\nu - 2 \int_{S^1} R_{n_j} f_{n_j} d\nu + \int_{S^1} R_{n_j} (f_{n_j})^2 d\nu \right) \left(\int_{S^1} R_{n_j} d\nu \right)$$

$$\leq \left(\int_{S^1} R_{n_j} d\nu - 2 \int_{S^1} f d\nu + \int_{S^1} R_{n_j} f_{n_j} . f_{n_j} d\nu \right) \left(\int_{S^1} R_{n_j} d\nu \right)$$

$$\leq \left(\int_{S^1} R_{n_j} d\nu - 2 \int_{S^1} f d\nu + \int_{S^1} f . f_{n_j} d\nu \right) \left(\int_{S^1} R_{n_j} d\nu \right)$$

Hence, letting j go to infinity,

$$\left(\lim_{j \to \infty} \int_{S^1} | R_{n_j} - f | d\nu \right)^2$$

$$\leq \int_{S^1} f d\nu - 2 \int_{S^1} f d\nu + \int_{S^1} f . \phi d\nu$$

$$\leq \int_{S^1} (\phi(z) - 1) f(z) d\nu(z).$$

$$\leq 0,$$

and this implies that $R_{n_j}, j = 1, 2, \cdots$ converges to f in $L^1(S^1, \nu)$ and the proof of the theorem is achieved. $\qquad\square$

Remark 4.3. Notice that $\int_{S^1} \frac{d\mu}{d\nu} d\nu = 1$, implies the convergence of $\prod_{j=0}^{N} |R_j|$ to $\sqrt{\frac{d\mu}{dz}}$ in $L^2(d\nu)$, by virtue of the classical results on "when weak convergence implies strong convergence".

We further have [17]

Corollary 4.4. Two generalized Riesz products $\mu = \prod_{j=1}^{\infty} |P_j|^2$, $\nu = \prod_{j=1}^{\infty} |Q_j|^2$ satisfying the conditions of Theorem 4.1 are mutually singular if and only if

$$\int_{S^1} \prod_{j=0}^{n} \left| \frac{P_j}{Q_j} \right| d\nu \to 0 \text{ as } n \to \infty.$$

5. A Conditional Strong Dichotomy and Other Discussion.

An important class of generalized Riesz products is the one arising in the study of rank one transformations of ergodic theory [17]. Indeed much of the recent work on generalized Riesz products (including the present contribution) is motivated by or focussed on the question whether these generalized Riesz products are always singular to Lebesgue measure. For in the contrary case, a counter example to this belief, will in all probability yield an affirmative answer to an old problem of Banach as to whether there exists a measure preserving transformation on an atom free measure space with simple Lebesgue spectrum.

The kth polynomial in the generalized Riesz product arising in the study of measure preserving rank one transformation is of the type

$$P_k(z) = \frac{1}{\sqrt{m_k}}(1 + \sum_{j=1}^{m_k-1} z^{jh_{k-1}+a_1^{(k)}+\cdots+a_j^{(k)}}),$$

$$h_{k-1} = m_{k-2}h_{k-2} + \sum_{j=1}^{m_{k-2}} a_j^{(k)}, h_0 = 1,$$

where m_k's are the cutting parameter and $a_j^{(k)}$'s are the spacers of the rank one transformation under consideration. It is easy to see that the partial products $\prod_{j=1}^{k} P_j, j = 1, 2, \cdots$ converge weakly to zero in $L^2(S^1, dz)$. These generalized Riesz products $\prod_{j=1}^{\infty} |P_j|^2$ have the property that the sequence of their tails $\prod_{j=n+1}^{\infty} |P_j|^2$, $n = 1, 2, \cdots$ converges weakly to the Lebesgue measure. In the rest of this section we will assume that the generalized Riesz products have this additional property, although is not assumed that they arise from rank one transformations as above.

Definition 5.1. A generalized Riesz product $\mu = \prod_{j=1}^{\infty} |P_j|^2$ is said to be of class (L) if for each sequence $k_1 < k_2 < \cdots$ of natural numbers, the tail measures $\prod_{j=n+1}^{\infty} | P_{k_j} |^2, n = 1, 2, \cdots$ converge weakly to Lebesgue measure.

Proposition 5.2. If the generalized Riesz product $\mu = \prod_{j=1}^{\infty} | P_j |^2$ is of class (L) then the partial products $\prod_{j=1}^{n} | P_j |, n = 1, 2, \cdots$ converge in $L^1(S^1, dz)$ to $\sqrt{\frac{d\mu}{dz}}$, and the convergence is almost everywhere (w.r.t dz) over a subsequence.

Proof. In Theorem 4.1 we put $Q_j(z) = 1$ for all j, so that ν is the Lebesgue measure on S^1. The first conclusion follows from theorem 4.1. The second conclusion follows since L^1 convergence implies convergence a.e over a subsequence. $\square$

The following formula follows immediately from this:

Corollary 5.3. Let a generalized Riesz product μ be of class (L). Let $\mathcal{K}_1, \mathcal{K}_2$ be two disjoint subsets of natural numbers and let $\mathcal{K}_0$ be their union. Let μ_1, μ_2 and μ_0 be the generalized Riesz subproducts of μ over $\mathcal{K}_1, \mathcal{K}_2$, and $\mathcal{K}_0$ respectively. Then we have:

$$\frac{d\mu_0}{dz} = \frac{d\mu_1}{dz}\frac{d\mu_2}{dz}, \tag{1}$$

where equality is a.e. with respect to the measure dz.

Let $\mu = \prod_{j=1}^{\infty} \mid P_j \mid^2$ be a generalized Riesz product of class (L). We assume that the polynomials $P_j, j = 1, 2, \cdots$ are outer. Write $S_n = \prod_{j=1}^{n} P_j$, and let $\phi_n = \frac{S_n}{|S_n|}$, a function of absolute value one. The functions $\phi_n, n = 1, 2, \cdots$ admit weak$\ast$ limits as functions in $L^{\infty}(S^1, dz)$. By Theorem 4.1 if β is positive then there is a unique nowhere vanishing weak star limit $\frac{f}{|f|}$ which is indeed also a limit in $L^1(S^1, dz)$. On the other hand consider the simplest classical Riesz product given by

$$\mu = \prod_{j=1}^{\infty} \frac{1}{\sqrt{2}} \Big| 1 + z^{n_j} \Big|^2, \ \frac{n_j}{n_{j-1}} \geq 3,$$

which is singular to the Lebesgue measure on S^1. Since $1 + e^{it} = \mid 1 + e^{it} \mid e^{i\frac{t}{2}}$, we see that

$$\phi_k(e^{it}) = e^{i\left(\sum_{j=1}^{k} n_j\right)\frac{t}{2}} \longrightarrow 0$$

in the weak$\ast$ topology as $k \to \infty$, by virtue of the Riemann-Lebesgue lemma. However the following conditional strong dichotomy holds.

Theorem 5.4. *If the functions $\phi_n, n = 1, 2, \cdots$ admit a weak star limit ϕ in $L^{\infty}(S^1, dz)$ which is non-vanishing a.e. (dz) on the set $\{z : \frac{d\mu}{dz} > 0\}$, then μ is either singular to Lebesgue measure, or its absolutely continuous part has positive Mahler measure.*

Proof. Let $f = \sqrt{\frac{d\mu}{dz}}$. Fix an integer k, then

$$\left| \int_{S^1} z^k S_n(z) - z^k \phi_n(z) f(z) dz \right| = \left| \int_{S^1} z^k \phi_n(z)(|S_n(z)| - f(z)) dz \right|$$

$$\leq \int_{S^1} ||S_n(z)| - f(z)| \, dz \to 0 \text{ as } n \to \infty$$

by Theorem 4.1. On the other hand, by assumption, since $f \in L^1(S^1, dz)$,

$$\int_{S^1} z^k \phi_n(z) f(z) dz \to \int_{S^1} z^k \phi(z) f(z) dz$$

Now for $k < 0$, $\int_{S^1} z^k S_n(z) dz = 0$, so for $k < 0$,

$$\int_{S^1} z^k \phi(z) f(z) dz = 0$$

By F and M Riesz theorem ϕf is either the zero function or a non-zero function in $\mathbb{H}^1$. In the first case f is the zero function a.e dz, since ϕ is assumed to be non-vanishing a.e. (dz) on the set where f is positive. In the second case $\mid \phi f \mid$ has an integrable log, which implies that f has an integrable log. Thus f is either the zero function or has an integrable log. $\square$

Remark 5.5. The proof of Theorem 5.4 in fact shows that any weak limit ϕ of ϕ_n's either never vanishes or vanishes on the set where f does not vanish. Suppose S_n has degree m_n and let $z_1, z_2, \cdots, z_{m_n}$ be the zeros of S_n, counting multiplicity. Since S_n is outer, $\mid z_j \mid \geq 1, j = 1, 2, \cdots, m_n$, whence $\mid \frac{1}{z_j} \mid \leq 1, j = 1, 2, \cdots, m_n$, so the function $(1 - \frac{z}{z_j}), j = 1, 2, \cdots, m_n$ has continuous arguments except when z_j has absolute value 1, in which case z_j is the only point where the argument is not defined, and a continuous argument can be defined at all other points. Thus the polynomials $B_n \stackrel{\text{def}}{=} \prod_{j=1}^{m_n}(1 - \frac{z}{z_j})$ admits a continuous argument, denoted by A_n, except at points z_j with $\mid z_j \mid = 1$. If $\lim_{n \to \infty} A_n(z)$ exists at almost every point where $\frac{d\mu}{dz}$ is positive, then it is clear that $\phi_n, n = 1, 2, \cdots$ admit a weak limit not vanishing a.e. on the set $\{z : \frac{d\mu}{dz} > 0\}$. Theorem 5.4 is a soft version in the context of generalized Riesz product of similar results in the context of lacunary series (see Theorem 1.1 and Theorem 6.4 in [23, T1, p.202]).).

View the functions $S_n(z), n = 1, 2, \cdots$ as outer analytic functions on the open unit disk. From weak dichotomy theorem 1.1, we immediately see that $S_n, n = 1, 2, \cdots$ converge uniformly on every compact subset of the open unit disk to a function which is non-zero and in $\mathbb{H}^1$ if β is positive and the identically zero function if β is zero. We have, using notation from $\mathbb{H}^p$ theory, with $0 \leq r < 1$

$$\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \left| S_n(re^{i\theta}) \right| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left| S_n(e^{i\theta}) \right| d\theta$$

We prefer to write this in our notation as

$$\lim_{r \to 1} \int_{S^1} |S_n(rz)| \, dz = \int_{S^1} |S_n(z)| \, dz$$

Letting $n \to \infty$ we get

$$\lim_{n \to \infty} \left(\lim_{r \to 1} \int_{S^1} | S_n(rz) | \, dz \right) = \lim_{n \to \infty} \left(\int_{S^1} | S_n(z) | \, dz \right) = \int \sqrt{\frac{d\mu}{dz}} \, dz.$$

However it is not clear if we can interchange the order of taking limits and write this as

$$= \lim_{r \to 1} \lim_{n \to \infty} \int_{S^1} | S_n(rz) | \, dz,$$

for that would immediately establish the singularity of μ with respect to the Lebesgue measure when $\beta = 0$.

6. Flat Polynomials

Consider the class of (B) of all polynomials of the type

$$P(z) = \frac{1}{\sqrt{m+1}}(1 + z^{n_1} + z^{n_2} + \cdots + z^{n_m}),$$

where $0 < n_1 < n_2 < \cdots < n_m$. Note that such polynomials can be seen as a polynomial appearing in the description of the spectrum of a rank one transformation. To see this, write

$$h = \min\{n_1, n_2 - n_1, n_3 - n_2 \cdots, n_m - n_{m-1}\},$$

and consider a stack of height h, which is divided into m columns. Now we add $n_1 - h$ spacers on the first column, $n_2 - n_1 - h$ spacers on the second column, and in general $n_j - n_{j-1} - h$ spacers on the jth column, $j \leq m$. The polynomial associated to this tower is precisely the P written above.

Since $L^2(S^1, dz)$ norm of a P in the class (B) is one, its $L^1(S^1, dz)$-norm is at most one. J. Bourgain has raised the question if the supremum of the $L^1(S^1, dz)$-norms of elements in the class (B) can be 1, see [8].

This is similar to a question of J. E. Littlewood on non-trivial trigonometric polynomials with coefficients of absolute value 1. Let (U) denote the class of all unimodular polynomials, i.e., polynomials of the type

$$\left\{\sum_{j=0}^n a_j z^j :| a_j |= 1, 0 \leq j \leq n, a_n = 1, n \geq 1\right\}.$$

Note that $\|\frac{1}{\sqrt{n+1}}P\|_2 = 1$ for any polynomial in the class (U) of degree n. A question of Littlewood, answered in the affirmative by J-P. Kahane [14], [21], asks if there is a sequence $P_j, j = 1, 2, \cdots$ of polynomials in the class (U) such that $\frac{1}{\sqrt{d_j+1}}|P_j|$, $j = 1, 2, \cdots$ converges to the constant function 1 uniformly on S^1, where d_j is the degree of the polynomial $P_j, j = 1, 2, \cdots$. Littlewood problem remains open if we require that the coefficients of P_j be either -1 or 1, for all j. Let $P_j, j = 1, 2, \cdots$ be a sequence Kahane polynomials. Clearly then $\int_{S^1} \log|P_j(z)|dz \to 0$ as $j \to \infty$, which in turn implies that the Mahler measure of P_j converges to 1 as $j \to \infty$. If O_j denotes the outer part of P_j, normalized so that the constant term $b_0^{(j)}$ of O_j is positive, then $b_0^{(j)} \to 1$ as $j \to \infty$. Replacing the sequence by a subsequence if necessary we can assume that $\prod_{j=1}^\infty b_0^{(j)}$ is positive. We also note that for any positive integer n,

$$\sup_{z \in S^1} || P_j(z) | -1 |= \sup_{z \in S^1} || P_j(z^n) | -1 |$$

$$= \sup_{z \in S^1} || O_j(z) | -1 |= \sup_{z \in S^1} || O_j(z^n) | -1 | .$$

Also, since $z \to z^n$ preserves the Lebesgue measure on S^1, we see that Mahler measures of $O_j(z), O_j(z^n)$ agree for all n and j. Finally, inductively, we can find

a sequence $n_j, j = 1, 2, \cdots$ of positive integers such the trigonometric polynomials $\mid O_j(z^{n_j}) \mid^2, j = 1, 2, \cdots$ are dissociated, so that the generalized Riesz product

$$\mu = \prod_{j=1}^{\infty} \mid P_j(z^{n_j}) \mid^2$$

is mutually absolutely continuous with respect to dz and has positive Mahler measure, (by Theorem 2.4). We have proved:

Theorem 6.1. *Kahane polynomials give rise to generalized Riesz products absolutely continuous with respect to the Lebesgue measure and with positive Mahler measure.*

Consider the quantity $\sup_{P \in (B)} \int_{S^1} \log \mid P(z) \mid dz$, for which we ask if it is strictly negative of equal to zero. If it is zero, then, following the method of Theorem 4.1, we can exhibit a measure preserving rank one transformation with simple Lebesgue spectrum.

Since product of the zeros of polynomials appearing in any of the classes (B), or (U) is of absolute value 1, we note that a polynomial in any of this class can be outer if and only if all its zeros are of absolute value 1.

Polynomials appearing in a Kahane sequence are often called ultraflat. In the rest of this section we will discuss the notion of flatness for polynomials with $L^2(S^1, dz)$ norm one, but with no other restriction on the coefficients. We mention that in [11] M. Guenais has already used ultraflat polynomials with coefficients -1 or $+1$ on a countable discrete torsion Abelian group to produce a group action with a simple Lebesgue spectrum on the orthocomplement of the eigenfunctions (see also [2]).

Definition 6.2. Call a trigonometric polynomial P with $L^2(S^1, dz)$ norm one ϵ−short if its $L^1(S^1, dz)$ norm is less than $1 - \epsilon$, it is called ϵ-flat otherwise. A sequence $P_j, j = 1, 2, \cdots$ with each P_j of $L^2(S^1, dz)$ norm 1 is said to be ϵ-short if all finite products $\prod_{j=1}^{n} P_{i_j}$ are ϵ-short. It is called a flat sequence if $\int_{S^1} \mid P_j(z) \mid dz \to 1$ as $j \to \infty$.

An open question about flat sequence $P_j, j = 1, 2, \cdots$ of polynomials is whether the Mahler measure of these polynomials converges to one. In [3] it is shown that this is true under the assumption that $\int_{S^1} \frac{1}{\mid P_j(z) \mid} dz < K, j = 1, 2, \cdots$ where K is a constant independent of j.

Remark 6.3. If P is ϵ-short then so is $P(z^n)$ for every positive integer n, since the map $z \to z^n$ preserves the Lebesgue measure.

Bourgain's question is equivalent to asking if there is a flat sequence of polynomials from the class (B).

Consider now a generalized Riesz product $\nu = \prod_{j=1}^{\infty} \mid P_j \mid^2$ of class (L). We know from Theorem 4.1 that partial products $Q_n = \prod_{j=1}^{n} \mid P_j \mid$ converge in $L^1(S^1, dz)$ to $\sqrt{\frac{d\mu}{dz}}$, hence over a subsequence, say $n_k, k = 1, 2, \cdots$, they converge a,e. (dz)

to $\sqrt{\frac{d\mu}{dz}}$. Write $R_k = Q_{n_k}, k = 1, 2, \cdots$. $T_k = R_k/R_{k-1}, k = 1, 2, \cdots$. Then the functions $R_k, k = 1, 2, \cdots$ converge a.e. (dz) to $\sqrt{\frac{d\mu}{dz}(z)}$, and so $T_k, k = 1, 2, \cdots$ converges a.e. (dz) to 1 on the set $E = \{z : \frac{d\mu}{dz}(z) > 0\}$. By Fatou's lemma

$$\int_{S^1} 1_E(z)dz = \int_{S^1} \lim_{k\to\infty} T_k(z)dz \leq \liminf_{k\to\infty} \int_{S^1} T_k(z)dz \leq 1$$

We conclude that if $dz(E) = 1$, then sequence of polynomials $\prod_{j=n_{k-1}+1}^{n_k} P_j, k = 1, 2, \cdots$ is flat.

In particular if the generalized Riesz product

$$\mu = \prod_{k=1}^{\infty} \left| \frac{1}{\sqrt{m_k}} \left(1 + \sum_{j=0}^{m_k-1} z^{jh_{k-1}+a_1^{(k)}+\cdots+a_j^{(k)}} \right) \right|^2,$$

describing the spectrum of a rank one transformation is such that $\frac{d\mu}{dz} > 0$ a.e.(dz), then there is a flat sequence of polynomials in the class (B), and we will have an answer to Bourgain's question.

Concerning Bourgain's question, it is known $\sup_{P\in R} \int_{S^1} \mid P(z) \mid dz \geq \frac{\sqrt{\pi}}{2}$. Indeed, let $P_n, n = 1, 2, \cdots$ be the polynomials as in the generalized Riesz product associated to the rank one map. Put $X_{n,j}(z) = z^{jh_n+a_{1,n}+\cdots+a_{n,j}}$. $X_{n,j}$ is a random variables. Since $\|P_n\|_2 = 1$, the random variables P_n are uniformly integrable. In [1], [4] and [5], the authors, proved that there is a subclass of $P_n, n = 1, 2, \cdots$, for which $P_n, n = 1, 2, \cdots$ converges in distribution to the complex Gaussian measure $\mathcal{N}_{\mathbb{C}}(0, 1)$ on $\mathbb{C}$, that is,

$$dz\{P_n \in A\} \xrightarrow[n\to+\infty]{} \int_A \frac{1}{\pi} e^{-|z|^2} dz.$$

Denote by $\mathcal{D}(P_n)$ the distribution of P_n. It follows from the Standard Moment Theorem [10, pp.100] that

$$\|P_n\|_1 = \int |P_n|dz = \int |w|d\mathcal{D}(P_n)(w) \xrightarrow[n\to\infty]{} \int |w|\frac{1}{\pi} e^{-|w|^2} dw = \frac{\sqrt{\pi}}{2},$$

dw is the usual Lebesgue measure on $\mathbb{C}$, that is, $dx \cdot dy = rdrd\theta$.

Assume now that the polynomials $P_j, j = 1, 2, \cdots$ arising in μ form an ϵ-short sequence. Then clearly $dz(E) < 1$. We will give an upper estimate of $dz(E)$, which will, of course, be less than 1. Let $a \overset{\text{def}}{=} \sup \|P_{i_1} P_{i_2} \cdots P_{i_l}\|_1$, where the supremum is taken over all finite sequences of increasing natural numbers $i_1 < i_2 < \cdots < i_l$. Clearly $a \leq 1 - \epsilon$ and by Fatou's lemma we know that $\left\| \sqrt{\frac{d\mu}{dz}} \right\|_1 \leq a$. Take an infinite subset $\mathcal{K}_1$ of natural numbers such that its complement $\mathcal{K}_2$ within natural numbers is also infinite. Let μ_1 and μ_2 be the Riesz subproducts of μ over $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Then by corollary 5.3

$$\left(\frac{d\mu}{dz}\right)^{\frac{1}{4}} = \left(\frac{d\mu_1}{dz}\right)^{\frac{1}{4}} \left(\frac{d\mu_2}{dz}\right)^{\frac{1}{4}}$$

By Cauchy-Schwarz inequality we get

$$\int_{S^1} \left(\frac{d\mu}{dz}\right)^{\frac{1}{4}} dz \leq \sqrt{a}\sqrt{a} = a$$

In general, by iterating,

$$\int_{S^1} \left(\frac{d\mu}{dz}\right)^{\frac{1}{2^n}} dz \leq a \tag{2}$$

Letting $n \to \infty$, we see that $dz(E) \leq a \leq 1 - \epsilon$.

Let

$$d \stackrel{\text{def}}{=} \liminf \left\{ \int_{S^1} |P_{i_1} P_{i_2} \cdots P_{i_k}| : i_1 < i_2 < \cdots i_k, k = 1, 2, \cdots \right\}$$

Now for any $\eta > 0$, considerations leading to equation (2) above can be applied to a suitable subproduct, say μ_1, over a set $\mathcal{K}_1$ of natural numbers, so that

$$dz\left\{ z : \frac{d\mu_1}{dz}(z) > 0 \right\} \leq d + \eta.$$

By formula (1) we see that $dz(E) \leq d + \eta$. Since η is arbitrary, we have $dz(E) \leq d$. We have proved:

Theorem 6.4. Let $\mu = \prod_{j=1}^{\infty} |P_i|^2$ be of class (L). Let $E = \{z : \frac{d\mu}{dz}(z) > 0\}$. If $dz(E) = 1$ then there is a flat sequence of finite subproducts of P_j's. If $dz(E)$ is less than 1, then $dz(E) \leq d$, where d is the liminf of $L^1(S^1, dz)$ norms of all finite subproducts of P_j's.

In connection with the discussion above, we have the following:

Theorem 6.5. Let $\mu = \prod_{j=1}^{\infty} |P_j|^2$ be of class (L) and assume that

(1) $\|P_j\|_1 \xrightarrow[j \to +\infty]{} c, c \in [0, 1[$ and,

(2) for any continuous function g on $\mathbb{T}$, we have

$$\int g|P_j|dz \xrightarrow[j \to +\infty]{} c \int g dz.$$

   Then μ is singular.

Proof. The sequence $||P_j| - 1|$ is bounded in $L^2(dz)$. It follows that there exists a function ϕ in $L^2(dz)$ such that $||P_j| - 1|$ converge weakly over a subsequence, say $n_j, j = 1, 2, \cdots$, to ϕ (without loss of generality we assume that $n_j = j, j = 1, 2, \cdots$). It is shown in [1] that the measure $\phi(z)dz$ is singular with respect to μ. According

to our assumptions, we further have that $\phi(z)dz$ is equivalent to Lebesgue measure. Indeed, for any nonnegative continuous function g on $\mathbb{T}$, we have

$$\int g||P_j| - 1|dz \geq \int gdz - \int g|P_j|dz.$$

Hence, by taking the limit combined with our assumptions, we get

$$\int g\phi dz \geq (1-c)\int gdz,$$

which finish the proof of the theorem.

$$\square$$

In [11] M. Guenais has shown that if the polynomials $P_n, n = 1, 2, \cdots$ form a flat sequence then a subproduct of μ is not singular to the Lebesgue measure on S^1.

7. Zeros of Polynomials.

Consider the polynomial of the type

$$P(z) = 1 + z^{h+a_1} + z^{2h+a_1+a_2} + \cdots + z^{(m-1)h+a_1+a_2+\cdots+a_{m-1}}, \qquad (R)$$

which appears in the generalized Riesz product connected with rank one measure preserving transformation.

It is easy to see that zeros of these polynomials cluster near the unit circle as h tends to ∞. We prove a quantitative result, namely, if w is a zero of this polynomial then

$$\left(\frac{1}{2}\right)^{\frac{1}{h}} \leq |w| \leq (2)^{\frac{1}{h}} \qquad (3)$$

To see this we write $|w| = a$. Assume first that $a \leq 1$. Then, since w is a zero of P,

$$a^h + a^{2h} + \cdots + a^{(m-1)h} \geq 1.$$

Equivalently,

$$a^h \frac{(1 - a^{(m-1)h})}{1 - a^h} \geq 1.$$
$$a^h - a^{mh} \geq 1 - a^h$$
$$2a^h \geq 1 + a^{mh} \geq 1$$

which proves the result when $|w| \leq 1$. To prove the second half of the inequality we note that if $|w|$ is greater than 1 then $\frac{1}{|w|} \leq 1$ and $\frac{1}{w}$ is a zero of $P(\frac{1}{z})$ so the second half follows from the first half. A slight improvement of the inequality is possible. If $m = 2$ then all the zeros of P lie on the unit circle. It is easy to show that if $m > 2$ then the equation $x^m - 2x + 1$ has a zero, say b_m, in the open interval $\frac{1}{2} < x < 1$. and one can show that

$$(b_m)^{\frac{1}{h}} \leq \mid w \mid \leq \left(\frac{1}{b_m}\right)^{\frac{1}{h}}.$$

However, it is not a very big improvement since one can show that $b_m \to \frac{1}{2}$ as $m \to \infty$.

This simple result tells us that if each P_k has less than ch_{k-1} zeros bigger than 1 in absolute value where c is a positive constant less than one, then $\prod_{k=1}^{\infty} |\alpha_k| = 0$.

We mention that M. Odlyzko and B. Poonen in [19] proved that the zeros of the polynomials with coefficients in $\{0,1\}$ are contained in the annulus $\frac{1}{\phi} < |z| < \phi$ where ϕ is the golden ratio.

Zeros of polynomials with restricted coefficients has deep and extensive literature. We mention only a result in a recent paper. (P. Brown, T. Erdélyi, F. Littmann [7]). Let

$$K_n = \left\{ \sum_{k=0}^{n} a_k z^k : \mid a_0 \mid = \mid a_n \mid = 1, \mid a_k \mid \leq 1 \right\},$$

and let n be so large that $\delta_n = 33\pi \frac{log(n)}{\sqrt{n}} < 1$, then any polynomial in K_n admits at least $8\sqrt{n} \log n$ zeros in δ_n neighborhood of any point of the unit the circle. Thus the derived set, i.e., the set of limit points of the zeros of the polynomials appearing in the generalized Riesz product (R) is the full unit circle.

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