Some Integrals Involving Bessel Functions

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Abstract

A number of new definite integrals involving Bessel functions are presented. These have been derived by finding new integral representations for the product of two Bessel functions of different order and argument in terms of the generalized hypergeometric function with subsequent reduction to special cases. Connection is made with Weber's second exponential integral and Laplace transforms of products of three Bessel functions.

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1. Introduction

The aim of this work is to derive a number of infinite integrals involving Bessel functions which appear to be new. The approach is, beginning with an expression for the product of two Bessel functions as a sum of Gauss functions, to integrate and perform a resummation to obtain other hypergeometric functions, and then to reduce these to more familiar form. In particular we exploit the relation between Bessel functions and the function $_0F_3[1]$. Finally, we generalize Weber's second exponential integral by expressing the Laplace transform of a product of three Bessel functions as an infinite series of products of modified Bessel functions. We have not aimed at complete rigor or generality; operations such as interchange of limits and Hankel inversion are carried out formally, i.e. without verifying that the conditions stated ensure their validity. In the majority of cases approriate conditions can be supplied by appeal to convergence and analytic continuation.

2. On the product $J_\mu(ax)J_\nu(bx)$

Our starting point is the familiar expansion [2]

$$\Gamma(\mu + 1)\Gamma(\nu + 1)J_\mu(ax)J_\nu(bx) =$$

$$(\frac{1}{2}ax)^\mu(\frac{1}{2}bx)^\nu \sum_{m=0}^\infty \frac{(-1)^m(\frac{1}{2}ax)^{2m}}{m!(\mu + 1)_m} \, _2F_1(-m, -\mu - m; \nu + 1; \frac{b^2}{a^2}) \qquad (2.1)$$

By making use of the standard transformation[3], we have

$$_2F_1(-m, -\mu - m; \nu + 1; \frac{b^2}{a^2}) =$$

$$= (1 - \frac{b^2}{a^2})^{\mu+\nu+2m+1} \, _2F_1(\nu + m + 1, \mu + \nu + m + 1; \nu + 1; \frac{b^2}{a^2}) \qquad (2.2)$$

Next, we recall that [4]

$$_2F_1(\nu + m + 1, \nu + \mu + m + 1; \nu + 1; \frac{b^2}{a^2}) =$$

$$= \frac{4(a/b)^\nu}{\Gamma(\mu + \nu + 1)(\nu + 1)_m(\mu + \nu + 1)_m} \cdot$$

$$\cdot \int_0^\infty t^{\mu+\nu+2m+1} K_\mu(2t) I_\nu(2\frac{b}{a}t) dt \qquad (2.3)$$

$$m = 0, 1, 2, \ldots, Re\,\nu > -1, \quad Re\,(\mu + \nu) > -1, \quad |Re(\frac{b}{a})| < 1.$$

By inserting (2.2) and (2.3) into (2.1), we get

$$\Gamma(\nu + 1)\Gamma(\mu + 1)\Gamma(\mu + \nu + 1) J_\mu(ax) J_\nu(bx) = 4(\frac{1}{2}ax)^\mu(\frac{1}{2}bx)^\nu(1 - \frac{b^2}{a^2})^{\mu+\nu+1} \cdot$$

$$\cdot (\frac{a}{b})^\nu \int_0^\infty t^{\mu+\nu+1} \, _0F_3(\mu + 1, \nu + 1, \mu + \nu + 1; -z^2t^2) K_\mu(2t) I_\nu(2\frac{b}{a}t) dt \quad (2.4)$$

$$Re\,\nu > -1, \quad Re\,(\mu + \nu) > -1, \quad |Re(\frac{b}{a})| < 1,$$

where $z = \frac{1}{2}ax(1 - \frac{b^2}{a^2})$.

It is interesting to observe that eq. (2.4) enables us to derive the Mellin transform of $J_\mu(ax) J_\nu(bx)$, i.e. the Weber-Schafheitlin integral [5], in a simple way. Indeed from [6],

$$\int_0^\infty x^{\mu+\nu-s} \, _0F_3(\mu + 1, \nu + 1, \mu + \nu + 1; -z^2t^2) dx =$$

$$= \frac{1}{2}[\frac{1}{2}a(1 - \frac{b^2}{a^2})t]^{s-\mu-\nu-1}\frac{\Gamma(\frac{\mu+\nu-s+1}{2})\Gamma(\mu + 1)\Gamma(\nu + 1)\Gamma(\mu + \nu + 1)}{\Gamma(\frac{\mu-\nu+s+1}{2})\Gamma(\frac{\nu-\mu+s+1}{2})\Gamma(\frac{\mu+\nu+s+1}{2})}, \quad (2.5)$$

$$Re\,(\mu + \nu - s) > -1,$$

and [4]

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$$\int_0^\infty t^s K_\mu(2t) I_\nu(2\frac{b}{a}t)dt = \frac{(b/a)^\nu}{4\Gamma(\nu+1)}\Gamma(\frac{\mu+\nu+s+1}{2})\Gamma(\frac{\nu-\mu+s+1}{2}).$$

$$_2F_1(\frac{\nu+\mu+s+1}{2},\frac{\nu-\mu+s+1}{2};\nu+1;\frac{b^2}{a^2}) \qquad (2.6)$$

$$Re(\nu\pm\mu+s) > -1, \quad |Re(\frac{b}{a})| < 1$$

we immediately obtain

$$\int_0^\infty x^{-s} J_\mu(ax) J_\nu(bx)dx = 2^{-s}b^\nu a^{s-\nu-1}\frac{\Gamma(\frac{\mu+\nu-s+1}{2})}{\Gamma(\nu+1)\Gamma(\frac{\mu-\nu+s+1}{2})}.$$

$$(1-\frac{b^2}{a^2})^s {}_2F_1(\frac{\mu+\nu+s+1}{2},\frac{\nu-\mu+s+1}{2};\nu+1;\frac{b^2}{a^2})$$

$$= 2^{-s}b^\nu a^{s-\nu-1}\frac{\Gamma(\frac{\mu+\nu-s+1}{2})}{\Gamma(\nu+1)\Gamma(\frac{\mu-\nu+s+1}{2})} {}_2F_1(\frac{\nu-\mu-s+1}{2},\frac{\nu+\mu-s+1}{2};\nu+1;\frac{b^2}{a^2})$$
$$(2.7)$$

$$Re(\mu+\nu-s) > -1, \qquad 0 < b < a$$

With $b \to ib$ $(b > 0)$, and x positive, eq. (2.4) becomes

$$\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\mu+\nu+1)J_\mu(ax)I_\nu(bx) =$$

$$(\frac{1}{4}ax)^\mu(\frac{1}{4}bx)^\nu(1+\frac{b^2}{a^2})^{\mu+\nu+1}(\frac{a}{b})^\nu.$$

$$\int_0^\infty t^{\mu+\nu+1} {}_0F_3(\mu+1,\nu+1,\mu+\nu+1;-\frac{1}{16}a^2x^2(1+\frac{b^2}{a^2})^2t^2)K_\mu(t)J_\nu(\frac{b}{a}t)dt \quad (2.8)$$

$$Re\ \nu > -1, \quad Re(\mu+\nu) > -1$$

or, by writing b=ay and x=a/4

$$\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\mu+\nu+1)J_\mu(\frac{1}{4}a^2)I_\nu(\frac{1}{4}a^2y) = (\frac{a^2}{16})^{\mu+\nu}(1+y^2)^{\mu+\nu+1}.$$

$$\cdot\int_0^\infty t^{\mu+\nu+1} {}_0F_3(\mu+1,\nu+1,\mu+\nu+1;-\frac{a^4}{256}(1+y^2)^2t^2)K_\mu(t)J_\nu(yt)dt \quad (2.9)$$

$$Re\ \nu > -1, \quad Re(\mu+\nu) > -1.$$

Also, with $a^2 \to \frac{16a}{1+y^2}$,

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$$\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\mu+\nu+1)J_\mu(\frac{4a}{1+y^2})I_\nu(\frac{4ay}{1+y^2}) = (1+y^2)a^{\mu+\nu}.$$

$$\cdot \int_0^\infty t^{\mu+\nu+1} \, {}_0F_3(\mu+1,\nu+1,\mu+\nu+1;-a^2t^2)K_\mu(t)J_\nu(yt)dt \qquad (2.10)$$

$$Re\,\nu > -1, \quad Re(\mu+\nu) > -1, a, y > 0.$$

For $Re\nu \le -1/2$ we may use the Hankel inversion formula [7], and eq. (2.10) gives

$$(at)^{\mu+\nu} \, {}_0F_3(\mu+1,\nu+1,\mu+\nu+1;-a^2t^2)K_\mu(t) = \Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\mu+\nu+1)\cdot$$

$$\cdot \int_0^\infty \frac{y}{1+y^2}J_\nu(ty)J_\mu(\frac{4a}{1+y^2})I_\nu(\frac{4ay}{1+y^2})dy \qquad (2.11)$$

This is an interesting addition to the class of Sonine-Gegenbauer integrals [8]. In particular, by dividing by $a^{\mu+\nu}$ and taking the limit $a \to 0$, which may be taken under the integral sign, we have

$$(\frac{t}{2})^{\mu+\nu}K_\mu(t) = \Gamma(\mu+\nu+1)\int_0^\infty \frac{y^{\nu+1}}{(1+y^2)^{\mu+\nu+1}}J_\nu(ty)dy \qquad (2.12)$$

The four particular cases

$$\mu = 0, \;\; \nu = -\frac{1}{2}; \;\; \mu = 0, \;\; \nu = \frac{1}{2}; \;\; \nu = 0, \;\; \mu = -\frac{1}{2}; \;\; \nu = 0, \;\; \mu = \frac{1}{2}$$

of eq.(2.9) are of some interest. By using

$$\,_0F_3(\frac{1}{2},\frac{1}{2},1;-\frac{x^4}{256}) = ber(x) \qquad (2.13)$$

and

$$\frac{x^2}{4} \, {}_0F_3(\frac{3}{2},\frac{3}{2},1:-\frac{x^4}{256}) = bei(x) \qquad (2.14)$$

we obtain the four integral representations

$$\frac{\pi}{2}(1+y^2)^{-1/2}J_0(\frac{a^2}{4})\cosh(\frac{a^2y}{4}) = \int_0^\infty K_0(t)ber(a\sqrt{(1+y^2)t})\cos(yt)dt \qquad (2.15)$$

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$$\frac{\pi}{2}(1+y^2)^{-1/2}J_0(\frac{a^2}{4})\sinh(\frac{a^2y}{4}) = \int_0^\infty K_0(t)bei(a\sqrt{(1+y^2)t})\sin(yt)dt \quad (2.16)$$

$$(1+y^2)^{-1/2}I_0(\frac{a^2y}{4})\cos(\frac{a^2}{4}) = \int_0^\infty e^{-t}ber(a\sqrt{(1+y^2)t})J_0(yt)dt \quad (2.17)$$

$$(1+y^2)^{-1/2}I_0(\frac{a^2y}{4})\sin(\frac{a^2}{4}) = \int_0^\infty e^{-t}bei(a\sqrt{(1+y^2)t})J_0(yt)dt \quad (2.18).$$

Therefore,

$$K_0(t)ber(a\sqrt{t}) = \int_0^\infty (1+y^2)^{-1/2}J_0(\frac{1}{4}\frac{a^2}{1+y^2})\cosh(\frac{1}{4}\frac{a^2y}{1+y^2})\cos(ty)dy$$
$$(2.19)$$

$$K_0(t)bei(a\sqrt{t}) = \int_0^\infty (1+y^2)^{-1/2}J_0(\frac{1}{4}\frac{a^2}{1+y^2})\sinh(\frac{1}{4}\frac{a^2y}{1+y^2})\sin(ty)dy \quad (2.20)$$

$$\frac{1}{t}e^{-t}ber(a\sqrt{t}) = \int_0^\infty y(1+y^2)^{-1/2}I_0(\frac{1}{4}\frac{a^2y}{1+y^2})\cos(\frac{1}{4}\frac{a^2}{1+y^2})J_0(ty)dy \quad (2.21)$$

$$\frac{1}{t}e^{-t}bei(a\sqrt{t}) = \int_0^\infty y(1+y^2)^{-1/2}I_0(\frac{1}{4}\frac{a^2y}{1+y^2})\sin(\frac{1}{4}\frac{a^2}{1+y^2})J_0(ty)dy \quad (2.22)$$

A curious formula arising from (2.4) may be mentioned. We set $\mu = \frac{1}{2}$, replace $I_\nu(2\frac{b}{a}t)$ by Poisson's integral [9]

$$I_\nu(2\frac{b}{a}t) = \frac{1}{\Gamma(\nu+\frac{1}{2})}(\frac{bt}{a})^\nu \int_{-1}^1 e^{-2\frac{b}{a}tu}(1-u^2)^{\nu-\frac{1}{2}}du \quad (2.23)$$

$$Re\ \nu > -\frac{1}{2}$$

interchange the order of integration, and then use the Laplace transform

$$\int_0^\infty e^{-\beta t}t^{2\nu+1}\ {}_0F_3(\frac{3}{2}, \nu+1, \nu+\frac{3}{2}; -\alpha t^2)dt = \frac{\Gamma(2\nu+2)}{4\sqrt{\alpha}}\beta^{-(2\nu+1)}\sin(4\frac{\sqrt{\alpha}}{\beta})$$
$$(2.24)$$

$$Re\ \nu > -1, \quad Re\ \beta > 0$$

which is readily established by expressing the ${}_0F_3$ as its power series and integrating term-by-term. This leads to

$$\sin(ax)J_\nu(bx) =$$

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$$= \frac{1}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}(\frac{1}{2}bx)^\nu(a^2 - b^2)^{\nu+\frac{1}{2}} \int_{-1}^{1} \frac{(1 - u^2)^{\nu-\frac{1}{2}}}{(a + bu)^{2\nu+1}} \sin(\frac{a^2 - b^2}{a + bu}x)du \quad (2.25)$$

$$Re\ \nu > -\frac{1}{2}, \quad |a| > |b| > 0.$$

Therefore, with $x = \frac{\pi}{2a}$ and $b = ay$

$$J_\nu(\frac{\pi y}{2})$$

$$= \frac{1}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}(\frac{1}{4}\pi y)^\nu(1 - y^2)^{\nu+\frac{1}{2}} \int_{-1}^{1} \frac{(1 - u^2)^{\nu-1/2}}{(1 + uy)^{2\nu+1}} \sin(\frac{\pi}{2}\frac{1 - y^2}{1 + uy})du \quad (2.26)$$

$$Re\ \nu > -\frac{1}{2}, \quad |y| < 1.$$

By taking advantage of other known reductions for the $_0F_3$ functions in (2.4) many additional new integrals can be derived. For example from [23] we obtain

$$\int_0^\infty e^{-x}I_\nu(x\sin\ \theta)[\sin(\frac{3\pi\nu}{2})ber_{2\nu}(2\cos\ \theta\sqrt{ux})+\cos(\frac{3\pi\nu}{2})bei_{2\nu}(2\cos\ \theta\sqrt{ux})]dx$$

$$= sec\ \theta\sin\ u\ J_\nu(u\sin\ \theta),$$

and in particular

$$\int_0^\infty e^{-x}I_{2n}(x\sin\ \theta)bei_{4n}(2\cos\ \theta\sqrt{ux})dx = (-1)^n sec\ \theta\ \sin\ u\ J_{2n}(u\sin\ \theta)$$

$$\int_0^\infty e^{-x}I_{2n+1}(x\sin\ \theta)ber_{4n+2}(2\cos\ \theta\sqrt{ux})dx = (-1)^{n+1} sec\ \theta\ \sin\ u\ J_{2n+1}(u\sin\ \theta).$$

We conclude this section by deriving a further integral representation for the product $J_\nu(ax)J_\nu(bx)$ in terms of a $_0F_3$, but different from (2.4).

Let us consider eq.(2.1) with $\mu = \nu$. According to the quadratic transformation [10]

$$_2F_1(-m, -\nu - m; \nu + 1; \frac{b^2}{a^2}) = (1 + \frac{b^2}{a^2})^m\ _2F_1(-\frac{m}{2}, \frac{1 - m}{2}; \nu + 1; (\frac{2ab}{a^2 + b^2})^2) =$$

$$= (1 + \frac{b^2}{a^2})^m m! \sum_{r=0}^{[\frac{m}{2}]} \frac{1}{(m - 2r)!r!(\nu + 1)_r}(\frac{ab}{a^2 + b^2})^{2r} \quad (2.27)$$

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and (2.1) becomes

$$J_\nu(ax)J_\nu(bx) = (\frac{1}{4}abx^2)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}x\sqrt{a^2+b^2})^{2m}}{\Gamma(\nu+m+1)} \cdot$$

$$\cdot \sum_{r=0}^{[\frac{m}{2}]} \frac{(\frac{ab}{a^2+b^2})^{2r}}{(m-2r)!r!\Gamma(\nu+r+1)}. \tag{2.28}$$

By using [11]

$$\sum_{m=0}^{\infty} \sum_{r=0}^{[\frac{m}{2}]} c(m,r) = \sum_{m,r=0}^{\infty} c(m+2r,r) \tag{2.29}$$

it follows that

$$J_\nu(ax)J_\nu(bx) =$$

$$\sum_{r=0}^{\infty} \frac{1}{r!\Gamma(\nu+r+1)} (\frac{abx}{2\sqrt{a^2+b^2}})^{\nu+2r} \sum_{m=0}^{\infty} \frac{(-1)^m(\frac{1}{2}x\sqrt{a^2+b^2})^{\nu+2r+2m}}{m!\Gamma(\nu+2r+m+1)} =$$

$$= (\frac{1}{2}\frac{abx}{\sqrt{a^2+b^2}})^\nu \sum_{r=0}^{\infty} \frac{(\frac{abx}{2\sqrt{a^2+b^2}})^{2r}}{r!\Gamma(\nu+r+1)} J_{\nu+2r}(x\sqrt{a^2+b^2}) \tag{2.30}$$

We note, in passing, that (2.30) provides a quick derivation of Weber's second integral [12]. Indeed, by using

$$\int_0^\infty e^{-px^2} x^{\nu+2r+1} J_{\nu+2r}(x\sqrt{a^2+b^2})dx = \frac{(a^2+b^2)^{\nu/2+r}}{(2p)^{\nu+2r+1}} e^{-\frac{a^2+b^2}{4p}} \tag{2.31}$$

we have

$$\int_0^\infty xe^{-px^2} J_\nu(ax)J_\nu(bx)dx = \frac{1}{2p} e^{-\frac{a^2+b^2}{4p}} \sum_{r=0}^{\infty} \frac{(ab/4p)^{\nu+2r}}{r!\Gamma(\nu+r+1)} =$$

$$= \frac{1}{2p} e^{-\frac{a^2+b^2}{4p}} I_\nu(\frac{ab}{2p}) \tag{2.32}$$

We also observe that (2.30) can be obtained from the formula [13]

$$_2F_1(a,b;c;x)\,_2F_1(a,b;c;y) =$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c-a)_r(c-b)_r}{r!(c)_r(c)_{2r}} (xy)^r \,_2F_1(a+r,b+r;c+2r;x+y-xy) \tag{2.33}$$

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by applying the confluence principle [14] twice. We first replace x by x/b, y by y/b, and let b→ ∞. This gives

$$_1F_1(a;c;x)_1F_1(a;c;y) = \sum_{r=0}^{\infty} \frac{(a)_r(c-a)_r}{r!(c)_r(c)_{2r}}(-xy)^r \, _1F_1(a+r;c+2r;x+y)$$

(2.34).

Next, we replace x by x/a, y by y/a and let a→ ∞; then

$$_0F_1(c;x)\,_0F_1(c;x) = \sum_{r=0}^{\infty} \frac{(xy)^r}{r!(c)_r(c)_{2r}}\,_0F_1(c+2r;x+y)$$

(2.35)

which is equivalent to the desired result. If on the right hand side of (2.30) we write [9]

$$J_{\nu+2r}(x\sqrt{a^2+b^2}) =$$

$$\frac{2}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \frac{((1/2)x\sqrt{a^2+b^2})^{\nu+2r}}{(\frac{\nu}{2}+\frac{1}{4})_r(\frac{\nu}{2}+\frac{3}{4})_r 2^{2r}} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}+2r} \cos(xt\sqrt{a^2+b^2})dt,$$

(2.36)

we get $(Re\ \nu > -\frac{1}{2})$

$$J_\nu(ax)J_\nu(bx) = \frac{2}{\pi\Gamma(2\nu+1)}(abx^2)^\nu \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(xt\sqrt{a^2+b^2})\cdot$$

$$\cdot \, _0F_3(\nu+1,\frac{\nu}{2}+\frac{1}{4},\frac{\nu}{2}+\frac{3}{4};\frac{1}{64}a^2b^2x^4(1-t^2)^2)dt$$

(2.37)

and with $b=1$, $x=a^{-1/2}$, and $a = \frac{1}{4}(\sqrt{u^2+2}+\sqrt{u^2-2})^2$,

$$J_\nu(\frac{\sqrt{u^2+2}+\sqrt{u^2-2}}{2})J_\nu(\frac{\sqrt{u^2+2}-\sqrt{u^2-2}}{2}) =$$

$$\frac{2}{\pi\Gamma(2\nu+1)} \int_0^1 (1-t^2)^{\nu-1/2} \cos(ut)\cdot$$

$$\cdot \, _0F_3(\nu+1,\frac{\nu}{2}+\frac{1}{4},\frac{\nu}{2}+\frac{3}{4};\frac{1}{64}(1-t^2)^2)dt$$

(2.38)

Finally, with $\nu = 1/2$ (2.37) yields [23]

$$\int_0^1 \cos(ut\sqrt{\frac{a^2+b^2}{2ab}})[I_1(u\sqrt{1-t^2})+J_1(u\sqrt{1-t^2})]\frac{dt}{\sqrt{1-t^2}}$$

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$$= (2/u)\sin(u\sqrt{a/2b})\sin(u\sqrt{b/2a}).$$

Further, more complex, evaluations are possible by the same procedure.

3. The integral $\int_0^\infty e^{-\alpha x} J_0(\beta_1\sqrt{x}) J_0(\beta_2\sqrt{x}) J_0(\beta_3\sqrt{x}) dx$.

The integral in this section heading, which we denote $I(\beta_1, \beta_2, \beta_3)$ is an extension of the $\nu = 0$ case of Weber's second integral, eq. (2.32), to which it reduces when one of the three parameters vanishes.

By writing (see eq.(2.1))

$$J_0(\beta_2\sqrt{x}) J_0(\beta_3\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}\beta_3\sqrt{x})^{2n}}{n!^2} \, {}_2F_1(-n, -n; 1; \frac{\beta_2^2}{\beta_3^2}) \qquad (3.1)$$

and [15]

$$\int_0^\infty e^{-\alpha x} x^n J_0(\beta_1\sqrt{x}) dx = \frac{n!}{\alpha^{n+1}} e^{-\beta_1^2/4\alpha} L_n(\frac{\beta_1^2}{4\alpha}), \qquad (3.2)$$

$L_n(x)$ being a Laguerre polynomial, we first have

$$I(\beta_1, \beta_2, \beta_3) = \frac{1}{\alpha} e^{-\frac{\beta_1^2}{4\alpha}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\frac{\beta_3^2}{4\alpha})^n \, {}_2F_1(-n, -n; 1; \frac{\beta_2^2}{\beta_3^2}) L_n(\frac{\beta_1^2}{4\alpha}) \qquad (3.3)$$

Now [16]

$$_2F_1(-n, -n; 1; \frac{\beta_2^2}{\beta_3^2}) = (1 - \frac{\beta_2^2}{\beta_3^2})^n \, {}_2F_1(-n, n+1; 1; \frac{\beta_2^2}{\beta_2^2 - \beta_3^2})$$

$$= (1 - \frac{\beta_2^2}{\beta_3^2})^n P_n(\frac{\beta_3^2 + \beta_2^2}{\beta_3^2 - \beta_2^2}) = \frac{1}{\pi\beta_3^{2n}} \int_0^\pi (\beta_2^2 + \beta_3^2 - 2\beta_2\beta_3\cos\,\theta)^n d\theta. \qquad (3.4)$$

Therefore, in terms of a well known generating function[17], (3.3) becomes

$$I(\beta_1, \beta_2, \beta_3) = \frac{e^{-\frac{\beta_1^2}{4\alpha}}}{\pi\alpha} \int_0^\pi \sum_{n=0}^{\infty} \frac{(1/4\alpha)^n}{n!} (\beta_2^2 + \beta_3^2 - 2\beta_2\beta_3\cos\,\theta)^n L_n(\frac{\beta_1^2}{4\alpha}) d\theta$$

$$= \frac{e^{-\beta_1^2/4\alpha}}{\pi\alpha} \int_0^\pi e^{-R^2/4\alpha} I_0(\frac{\beta_1 R}{2\alpha}) d\theta \qquad (3.5)$$

where $R = (\beta_2^2 + \beta_3^2 - 2\beta_2\beta_3\cos\,\theta)^{1/2}$. Next, by using Graf's addition theorem [18]

$$I_0(\frac{\beta_1 R}{2\alpha}) = \sum_{n=0}^{\infty} (2 - \delta_{no}) I_n(\frac{\beta_1\beta_2}{2\alpha}) I_n(\frac{\beta_1\beta_2}{2\alpha}) \cos(n\theta) \qquad (3.6)$$

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and the familiar integral representation

$$\int_0^\pi e^{\frac{\beta_1\beta_2}{2\alpha}\cos\theta}\cos(n\theta)d\theta = \pi I_n(\frac{\beta_1\beta_2}{2\alpha}),$$ (3.7)

we finally have

$$\int_0^\infty e^{-\alpha x}J_0(\beta_1\sqrt{x})J_0(\beta_2\sqrt{x})J_0(\beta_3\sqrt{x})dx$$

$$= \frac{1}{\alpha}e^{-\frac{1}{4\alpha}(\beta_1^2+\beta_2^2+\beta_3^2)}\sum_{n=0}^\infty (2-\delta_{0n})F_n(\beta_1,\beta_2,\beta_3)$$ (3.8)

where

$$F_n(\beta_1,\beta_2,\beta_3) = I_n(\frac{\beta_1\beta_2}{2\alpha})I_n(\frac{\beta_1\beta_3}{2\alpha})I_n(\frac{\beta_2\beta_3}{2\alpha})$$ (3.9)

We conclude by sketching the evaluation of the more general integral

$$I_m(\beta_1,\beta_2,\beta_3) = \int_0^\infty e^{-\alpha x}J_0(\beta_1\sqrt{x})J_m(\beta_2\sqrt{x})J_m(\beta_3\sqrt{x})dx$$ (3.10)

where m is a positive integer. In the first place, (3.4) takes the form

$$I_m(\beta_1,\beta_2,\beta_3) = \frac{1}{m!\alpha}(\frac{\beta_2\beta_3}{4\alpha})^m e^{-\frac{\beta_1^2}{4\alpha}}.$$

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!}(\frac{\beta_3^2}{4\alpha})^n {}_2F_1(-n,-n-m;m+1;\frac{\beta_2^2}{\beta_3^2})L_{m+n}(\frac{\beta_1^2}{4\alpha})$$ (3.11)

By observing that [19]

$${}_2F_1(-n,-n-m;m+1;\frac{\beta_2^2}{\beta_3^2}) = (1-\frac{\beta_2^2}{\beta_3^2})^n\frac{n!(2m)!}{(2m+n)!}C_n^{m+1/2}(\frac{\beta_3^2+\beta_2^2}{\beta_3^2-\beta_2^2})$$

$$= \frac{2^{2m}m!^2}{(2m)!\pi\beta_3^{2n}}\int_0^\pi \sin^{2m}\theta\,(\beta_2^2+\beta_3^2-2\beta_2\beta_3\cos\theta)^n d\theta$$ (3.12)

eq. (3.11) becomes

$$I_m(\beta_1,\beta_2,\beta_3) = \frac{1}{\pi\alpha}\frac{m!}{(2m)!}(\frac{\beta_2\beta_3}{\alpha})^m e^{-\beta_1^2/4\alpha}\int_0^\pi \sin^{2m}\theta\sum_{n=0}^\infty\frac{(-1/4\alpha)^n}{n!}.$$

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$$(\beta_2^2 + \beta_3^2 - 2\beta_2\beta_3 \cos \theta)^n L_{m+n}(\frac{\beta_1^2}{4\alpha})d\theta. \tag{3.13}$$

Now, from [20],

$$L_{m+n}(x) = \frac{n!}{(m+n)!}e^x(\frac{d}{dx})^m e^{-x} x^m L_n^m(x) \tag{3.14}$$

and [17]

$$\sum_{n=0}^{\infty} \frac{t^n}{(m+n)!}L_n^m(x) = (xt)^{-m/2}e^t J_m(2\sqrt{xt}) \tag{3.15}$$

it follows that (where R has the same meaning as before)

$$\sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{R^2}{4\alpha})^n L_{m+n}(\frac{\beta_1^2}{4\alpha}) = e^{(\beta_1^2-R^2)/4\alpha}2^m(\frac{d}{dx})^m e^{-x} x^m.$$

$$(R\sqrt{\frac{x}{\alpha}})^{-m}I_m(R\sqrt{\frac{x}{\alpha}})|_{x=\beta_1^2/4\alpha}. \tag{3.16}$$

By inserting (3.16) into (3.13) and using [21]

$$(R\sqrt{\frac{x}{\alpha}})^{-m}I_m(R\sqrt{\frac{x}{\alpha}}) = (\frac{\beta_2\beta_3 x}{2\alpha})^{-m}(m-1)!\sum_{n=0}^{\infty}(m+n)C_n^m(\cos \theta)\cdot$$

$$I_{m+n}(\beta_2\sqrt{\frac{x}{\alpha}})I_{m+n}(\beta_3\sqrt{\frac{x}{\alpha}}) \tag{3.18}$$

and [22]

$$\int_0^\pi \sin^{2m} \theta \ e^{\frac{\beta_2\beta_3}{2\alpha} \cos \theta}C_n^m(\cos \theta)d\theta$$

$$= \frac{\pi 2^{1-m}(2m+n-1)!}{n!(m-1)!}(\frac{2\alpha}{\beta_2\beta_3})^m I_{m+n}(\frac{\beta_2\beta_3}{2\alpha}), \tag{3.18}$$

we finally obtain

$$\int_0^\infty e^{-\alpha x}J_0(\beta_1\sqrt{x})J_m(\beta_2\sqrt{x})J_m(\beta_3\sqrt{x})dx$$

$$= \frac{1}{\alpha}\frac{2^{2m+1}m!}{(2m)!}(\frac{\alpha}{\beta_2\beta_3})^m e^{-\frac{\beta_2^2+\beta_3^2}{4\alpha}}(\frac{d}{dx})^m e^{-x}. \tag{3.19}$$

$$\sum_{n=0}^{\infty} \frac{(n+m)(2m+n-1)!}{n!}I_{m+n}(\beta_2\sqrt{\frac{x}{\alpha}})I_{m+n}(\beta_3\sqrt{\frac{x}{\alpha}})I_{m+n}(\frac{\beta_2\beta_3}{2\alpha})|_{x=\beta_1^2/4\alpha}.$$

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In particular, by letting $\beta_3 \to 0$, after dividing by β_3^m,

$$\int_0^\infty e^{-\alpha x} J_0(\beta_1\sqrt{x}) J_m(\beta_2\sqrt{x}) x^{m/2} dx$$

$$= \frac{1}{\alpha} e^{-\beta_2^2/4\alpha} (\frac{d}{dx})^m e^{-x} (\frac{x}{\alpha})^{m/2} I_m(\beta_2\sqrt{\frac{x}{\alpha}})|_{x=\beta_1^2/4\alpha} \qquad (3.20)$$

$$= \frac{1}{\alpha} (\frac{\beta_2}{2\alpha})^m e^{-\frac{\beta_1^2+\beta_2^2}{4\alpha}} \sum_{n=0}^m (-1)^n \binom{m}{n} (\frac{\beta_1}{\beta_2})^n I_n(\frac{\beta_1\beta_2}{2\alpha}).$$

which is of interest in connection with formulas (39)-(42) on page 186 of reference [15].

In conclusion, we point out that the derivation of (2.24) can be extended to give

$$\int_0^\infty e^{-\beta x} {}_0F_3(\mu, \nu, \nu + \frac{1}{2}; -a^2 x^2) dx = (2a)^{1-\mu} \Gamma(\mu) \Gamma(2\nu) \beta^{\mu-2\nu-1} J_{\mu-1}(4a/\beta)$$
$$(3.21)$$

Therefore, the results of section 2, for example, are capable of extension in a variety of directions. We leave this for the future and merely quote one example, interesting because it contains each of the four types of Bessel functions:

$$\int_0^\infty x J_1(ax) I_1(ax) Y_0(x) K_0(x) \, dx = -(2\pi a^2)^{-1} \ln(1 - a^4) \qquad (3.22)$$

where $0 < a < 1$.

References

[1] T.J. Lardner, Siam Review <u>11</u>, 69 (1969).

[2] Bateman Manuscript Project, Higher Transcendental Functions (McGraw-Hill, 1953), Vol. 2, p.11, eq. (47).

[3] Ibid. Vol. 1, p.64, eq. (23).

[4] Ibid. Vol. 2, p.52, eq. (31).

[5] Ibid. Vol. 2, p.51, eq. (29).

[6] L.J. Slater, Generalized Hypergeometric Functions (Cambridge 1966), Sec. 4.8.

[7] I.N. Sneddon, The Use of Integral Transforms (McGraw-Hill 1972), Chap. 5, p.309.

[8] See Ref.[2] Vol. 2, p.96, eq. (59).

[9] Ibid. Vol. 2, p.81, eq. (10).

[10] See Ref. [1], p. 110, eq. (1).

[11] H.M. Srivastava and H.L. Manocha, A treatise on Generating Functions (John Wiley 1983), p. 101, eq. (6).

[12] See Ref. [2], Vol. 2, p.50, eq. (25).

[13] J.L. Burchnall and T.W. Chaundy, Quart. J. Math. Oxford Ser. 11, 249-270 (1940).

[14] See Ref. [11], p.36.

[15] Bateman Manuscript Project, Tables of Integral Transforms (McGraw-Hill 1953), Vol. 2, p. 30, eq. (13).

[16] See Ref. [1], p. 157, eq. (15).

[17] See Ref. [2], p. 189, eq. (18).

[18] Ibid. Vol. 2, p.44, eq.(5).

[19] Ibid. Vol. 2, p.177, eq.(31).

[20] Ibid. Vol. 2, p.190, eq.(28).

[21] Ibid. Vol. 2, p.101, eq.(30).

[22] See Ref. [15], p.281, eq.(7).

[23] A.P. Prudnikov, Ya. A. Brychkov and O.I. Marichev, Integrals and Series (Gordon and Breach, N.Y. 1989) Vol. 3 Sec. 7.16.3 Eq. (6).

[24] Ibid. Sec. 7.16.2 Eq.(15).