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Mathematical Foundations of Quantum Field Theory

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TBD

1 Geometric Preliminaries

This chapter is aimed at giving an introduction to the geometric setting of Quantum Field Theory. While trying to be as self-contained as possible, we will have to assume that the reader is fluent in basic Differential Geometry. While there are many excellent textbooks on Differential Geometry, the textbook by Lee [1] offers a highly self-contained take on the topic. Note that in the following, the notion of fibre- and vector-bundles will be especially crucial. For an insightful treatment of the matter and a more compact introduction to Differential Geometry, see Will Merry's website [2].

The main sources for this section will be **Renormalization and Effective Field Theory** by K. Costello [3] and **Quantum Fields and Strings: A Course for Mathematicians** by various authors [4].

We will further make use of excerpts from **Introduction to Supergeometry** by A.S. Cattaneo and F. Schaetz [5], chapter 2.1 of **Algebraic Geometry** by R. Hartshorne [6], chapters 5.2, 5.3 and 6.1 of **Modern differential geometry for physicists** by C.J. Isham [7] and chapter 2 of A. Hatcher's **Algebraic Topology** [8]. Note that most of these texts are available online, respective links are provided in the References.

1.1 The Graded Setting

In the following section we will generalize the notion of Differential Geometry to a graded setting. To this end we start with the preliminary definitions of a graded structure and discuss some interesting examples.

Definition 1.1 (Graded Vector Space)
A **graded vector space** $V^\bullet$ is a collection $\{V^k\}_{k\in\mathbb{Z}}$ of vector spaces. The *realization* of $V^\bullet$ is the direct sum

$$V^\bullet = \bigoplus_{k\in\mathbb{Z}} V^k.$$

To extend the notion of a graded vector space to a full category, we now investigate the morphisms between them:

Definition 1.2 (Graded Morphisms)
Let $V^\bullet, W^\bullet$ be graded vector spaces. A **graded morphism** $\varphi : V^\bullet \to W^\bullet$ is a collection of linear maps $\varphi^k : V^k \to W^k \ \ \forall k \in \mathbb{Z}$. We extend this notion to that of a graded morphism of **degree** r by requiring that $\varphi^k : V^k \to W^{k+r}\ \forall k \in \mathbb{Z}$. In the case $V^\bullet = W^\bullet$ we call φ a **graded endomorphism** (of degree r, respectively).

Given a graded vector space $V^\bullet$ we call an element $v \in V^k$ an homogenous element of degree $k = |v|$.

Example 1.3

- The most common examples of graded vector spaces emerge from the tensor product. The k-fold tensor product of a vector space V denoted by $V^{\otimes k}$, the k-fold exterior algebra of V denoted by $\bigwedge^k V$ and the space of differential k-forms on a smooth manifold M denoted by $\Omega^k(M)$ each generate a graded vector space denoted by $T^\bullet(V)$, $\bigwedge^\bullet V$ and $\Omega^\bullet(M)$ respectively.
- For the space of differential forms on a smooth manifold M given by $\Omega^\bullet(M)$ we can define a graded morphism of degree 1 given by the *de Rham differential* $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$.
- The *contraction* of a differential form by a vector field defines another graded morphism on the space of differential forms $\Omega^\bullet(M)$. For $X \in \mathfrak{X}(M)$ we define

$$\begin{array}{rcl} \imath_X : \Omega^k(M) & \to & \Omega^{k-1}(M), \\ \omega & \mapsto & \imath_X\omega = \omega(X, \cdot, \dots, \cdot). \end{array}$$

  The contraction is a graded morphism of degree -1.
- Given $X \in \mathfrak{X}(M)$ the *Lie Derivative* $\mathcal{L}_X$ defines a graded morphism of degree 0 by

$$\begin{array}{rcl} \mathcal{L}_X : \Omega^k(M) & \to & \Omega^k(M), \\ \omega & \mapsto & \mathcal{L}_X\omega := \imath_X \circ d\omega + d \circ \imath_X\omega \end{array}$$

- Given two graded vector spaces $V^\bullet, W^\bullet$ we obtain another graded vector space through the set of graded morphisms

$$\mathrm{Hom}^\bullet(V^\bullet, W^\bullet) := \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}^k(V^\bullet, W^\bullet)$$

- Given a graded vector space $V^\bullet$ we define the dual graded vector space $(V^*)^\bullet$ via the collection $(V^*)_i = (V_{-i})^*$.[1]

Due to their importance for the forthcoming discussion, we classify certain special cases of graded endomorphisms.

Definition 1.4 Let φ be a graded endomorphism such that $\varphi \circ \varphi = 0$. We call φ a **boundary operator**, if φ is of degree -1. If φ is of degree $+1$, we call it a **coboundary operator**. A graded vector space $V^\bullet$ together with a coboundary operator is called a **complex**. If it is equipped with a boundary operator, we call it a **cocomplex**.

[1] Note that in the case where $V^\bullet = T^\bullet(W)$ where W is a classical vector space, $(V^*)_i$ becomes trivally zero for positive indices.

Note that a very important example for a complex is $\Omega^\bullet(M)$ equipped with the *de Rham differential*, well known from *de Rham cohomology*. To define further structure on graded vector spaces, we turn to algebraic relations defined on its realization.

Definition 1.5 (Graded Algebra)
A **graded algebra** is a graded vector space $A^\bullet$ together with a collection of bilinear maps called the *product* on $A^\bullet$

$$A^k \times A^l \to A^{k+l}, \qquad (a,b) \mapsto ab \quad \forall k,l \in \mathbb{Z}$$

- $A^\bullet$ is **associative**, if
$$(ab)c = a(bc) \quad \forall a,b,c \in A^\bullet$$
- $A^\bullet$ is **graded commutative**, if
$$ab = (-1)^{kl}\, ba \quad \forall a \in A^k,\ b \in A^l$$
- $A^\bullet$ is **graded skew-commutative**, if
$$ab = (-1)^{kl+1}\, ba \quad \forall a \in A^k,\ b \in A^l$$

Example 1.6

- The tensor algebra $T^\bullet(V)$ of a graded vector space $V^\bullet$ is a graded algebra with its product given by the tensor product $\otimes$. Note that this makes $T^\bullet(V)$ an associative graded algebra.
- The two spaces $(\bigwedge^\bullet V, \wedge)$ and $(\Omega^\bullet(M), \wedge)$ are associative commutative graded algebras as can be readily deduced from the definition of the wedge-product.
- Given a graded vector space $V^\bullet$ we can take the graded vector space of endomorphisms $\mathrm{End}^\bullet(V^\bullet)$ and elevate it to an associative graded algebra. To this end, we define the product of two endomorphisms $\varphi \in \mathrm{End}^r(V^\bullet)$ and $\psi \in \mathrm{End}^s(V^\bullet)$ as the graded endomorphism given by the collection of linear maps
$$\varphi\psi : \varphi^{k+s}\psi^k : V^k \to V^{k+r+s} \quad \forall k \in \mathbb{Z}$$
resulting in a new graded endomorphism of degree $(r+s)$. It is left as an easy exercise to prove that this does indeed define an associative graded algebra on $\mathrm{End}^\bullet(V^\bullet)$.
- The *graded symmetric algebra* $S^\bullet(V)$ over a graded vector space $V^\bullet$ is the quotient of the tensor algebra $T^\bullet(V)$ by the ideal generated by all elements of the form
$$v \otimes w - (-1)^{|v||w|} w \otimes v \quad \forall v,w \in V^\bullet$$
This graded algebra is graded commutative with respect to the tensor product, the proof is left as an exercise.

- The *algebra of polynomial functions* on $V^\bullet$ is the graded symmetric algebra $S^\bullet(V^*)$ over $(V^*)^\bullet$. This notion will be particularly useful when talking about graded manifolds later in the course.

Extending the notion of vector subspaces and closed subspaces of algebras, so called subalgebras, we go on to define the respective notions in the graded context:

Definition 1.7 A **graded subspace** $W^\bullet \subset V^\bullet$ of a graded vector space $V^\bullet$ is a collection of subspaces $\{W^k\}_{k\in\mathbb{Z}}$ such that $W^k \subset V^k \quad \forall k \in \mathbb{Z}$.
A **graded subalgebra** is a graded subspace that is closed under the product.

We go on to further classify certain graded morphisms by restricting to those satisfying an adapted *Leibniz Rule*. The idea is to generalize the notion of a derivation on an algebra, which is required to satisfy the classical *Leibniz Rule*, to general graded algebras. This will provide us with an excellent example of a graded subalgebra:

Definition 1.8 (Graded Derivations)
Let $V^\bullet$ be a graded algebra. A **graded derivation** of degree n is a graded endomorphism $\mathfrak{D}$ of degree n that satisfies the **graded Leibniz rule**

$$\mathfrak{D}(ab) = \mathfrak{D}(a)\ b + (-1)^{rn}\ a\ \mathfrak{D}(b) \qquad \forall a \in V^r,\ b \in V^s$$

Notice that thus $\mathfrak{D}(ab) \in V^{n+r+s}$.

To no surprise we encounter some regular customers that fit the definition of a graded derivation perfectly and historically inspired this more general definition:

Example 1.9

- The *de Rham differential* d is a graded derivation of degree $+1$
- The *contraction* $\imath_X$ is a graded derivation of degree -1
- The *Lie Derivative* $\mathcal{L}_X$ is a graded derivation of degree 0

Now as promised, we note that on a graded algebra $A^\bullet$ the set of graded derivations $\mathrm{Der}^\bullet(A^\bullet)$ which turns out to be a natural subspace of $\mathrm{End}^\bullet(A^\bullet)$ in the sense that $\mathrm{Der}^k(A^\bullet) \subset \mathrm{End}^k(A^\bullet)$, turns out to be a subalgebra of $\mathrm{End}^\bullet$. The proof is again left as an exercise to the reader.

Combining the different classifications for graded endomorphisms, we go on to define yet another utterly important subspecies:

Definition 1.10 A **differential** is a coboundary operator that is also a graded derivation.

Yet again, we find that the *de Rham differential* lives up to its name by qualifying as a *differential* in the sense of the above definition. We go on to define the notion of a graded *Lie Algebra*. The definition is easily anticipated from that of a Lie algebra

Definition 1.11 (Graded Lie Algebra)
A **graded Lie algebra** is a graded algebra $g^\bullet$ whose product is skew-commutative and satisfies the Jacobi-Identity.

As usual we provide some insightful examples to support the understanding of the above definition:

Example 1.12

- Take $(A^\bullet, [\cdot,\cdot])$ such that $A^\bullet$ is a graded associative algebra. We define for $a \in A^k, b \in A^l$:

$$[a,b] := ab - (-1)^{kl}\ ba$$

  This constitutes a graded Lie algebra as can be checked through tedious calculation.

- The set $(\mathrm{End}^\bullet(V^\bullet), [\cdot,\cdot])$ with the bracket defined as in the previous example is a graded Lie algebra seeing $\mathrm{End}^\bullet(V^\bullet)$ as a graded associative algebra with composition as its product.

***Remark* 1.13** We note that if both $a, b \in A^\bullet$ are odd, the bracket as defined in the above examples turns out to be the anticommutator

$$[a,b] = ab + ba$$

Thus, if a is odd, we obtain $[a,a] = 2a^2 \neq 0$.

Now again, we can induce the idea of a subspace on our newly defined construct. Note that we again use the algebraic structure it has to define them:

Definition 1.14 A **graded Lie subalgebra** is a graded subalgebra of a graded Lie algebra.

An example for a graded Lie subalgebra would be the space of Derivations as a subset of the space of endomorphisms, namely $\mathrm{Der}^\bullet(A^\bullet) \subset \mathrm{End}^\bullet(A^\bullet)$.

***Remark* 1.15** In all of our previous discussion, we used $\mathbb{Z}$ as the set that defines our graded setting. Now if we would like to generalize the graded setting ideas to more general spaces, we need some extra work:

Take a set G. We define a G-graded vector space $\{V^k\}_{k\in G}$. But in order to define further notions like that of graded morphisms etc. we previously used the specific structure of $\mathbb{Z}$ and thus need to generalize the required properties:

- We require a composition law $G \times G \to G$ to define graded morphisms.
- If we want to define graded associated algebras, G has to be a group (or a monoid) such that the order of composition does not matter.
- G has to be abelian (or a commutative monoid), if we want to define graded commutative algebras.

So far we did everything on $\mathbb{Z}$. Taking $\mathbb{Z}/2\mathbb{Z}$ as the underlying set, we obtain the notion of *super-linear algebra*. To prove the above claims is an insightful and interesting exercise!

After this small detour, we continue to define further structure in our graded setting. Next, we introduce the notion of a $\mathbb{Z}$-grading and discuss the so-called *Cartan Calculus* using d and $\imath$.

Proposition 1.16 The span over $\mathbb{R}$ of the set

$$d, \imath_X, \mathcal{L}_X : X \in \mathfrak{X}(M)$$

is a graded Lie subalgebra of $\mathrm{Der}^\bullet(\Omega^\bullet(M))$. Moreover it satisfies the following algebraic properties with regard to the usual bracket as defined before:

(1) $[d, d] = 0$ (2) $[d, \imath_X] = \mathcal{L}_X$ (3) $[d, \mathcal{L}_X] = 0$

(4) $[\imath_X, \imath_Y] = 0$ (5) $[\imath_X, \mathcal{L}_Y] = \imath_{[X,Y]}$ (6) $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$

Proof. We only need to prove the claims for functions and exact 1-forms as all of the above are local operators. First we note that $[d, d] = 0$ simply due to $d \circ d = 0$. For the second identity we note that for functions

$$\mathcal{L}_X(f) = X(f) = \imath_X df = [\imath_X, d]f$$

Now for exact 1-forms we obtain

$$[d, \imath_X]df = d \circ \imath_X df = d \circ \mathcal{L}_X f = \mathcal{L}_X df$$

Now the third claim is again easily obtained using the Cartan-Magic-Formula and the graded Jacobi-Identity:

$$[d, \mathcal{L}_X] = [d, [d, \imath_X]] = [[d, d], \imath_X] - [d, [d, \imath_X]] = -[d, \mathcal{L}_X]$$

Thus it trivialy becomes 0. For the fourth identity we note $[\imath_X, \imath_Y] = \imath_X \imath_Y + \imath_Y \imath_X$. Now if we define $I_{XY} := [\imath_X, \imath_Y]$, we can apply to the wedge product of two forms to obtain

$$I_{XY}(\alpha \wedge \beta) = (I_{XY}\alpha) \wedge \beta + \alpha \wedge (I_{XY}\beta)$$

Thus I_{XY} is defined by its action on 1-forms, for which we see $I_{XY}(\alpha) = 0$ thus $I_{XY} = 0$.

For the fifth identity we simply note

$$[\imath_X, \mathcal{L}_Y](f) = \imath_X \imath_Y(f), \qquad [\imath_X, \mathcal{L}_Y](df) = \imath_{[X,Y]} df$$

Now the only one left to prove is the sixth identity. First, for functions, we get

$$[\mathcal{L}_X, \mathcal{L}_Y](f) = [X, Y](f)$$

Now for 1-forms we obtain, using the Cartan-Magic-Formula and the Jacobi-Identity:

$$\mathcal{L}_{[X,Y]} = [d, \imath_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$$

□

We go on to find an interesting expression for the de Rham differential d. To this end, we note that for a closed form ω (thus $d\omega = 0$) we have

$$\mathcal{L}_X \omega = d \circ \imath_X \omega$$

Now we use some induction and the above observation to prove the following statement:

Lemma 1.17 Given $k + 1 \geq 2$ vector fields $X_0, ..., X_k \in \mathfrak{X}(M)$, the following identity holds:

$$[\imath_{X_k} ... \imath_{X_1}, \mathcal{L}_{X_0}] = \sum_{i=1}^{k} (-1)^{i+1} \imath_{X_k} ... \widehat{\imath_{X_i}} ... \imath_{X_1} \; \imath_{[X_i, X_0]}$$

Proof. We use induction on k. Since we have already shown the formula for $k = 1$, let the statement hold for some $(k - 1)$. Now for k we get

$$[\imath_{X_k} ... \imath_{X_1}, \mathcal{L}_{X_0}] = \imath_{X_k} ... \imath_{X_2} [\imath_{X_1}, \mathcal{L}_{X_0}] + [\imath_{X_k} ... \imath_{X_2}, \mathcal{L}_{X_0}] \imath_{X_1}$$

Using our assumption for $k - 1$, we obtain the required form which proves the statement. □

Now we use the previous lemma, to constitute a way of calculating the de Rham differential of a form.

Proposition 1.18 Let again $X_0, ..., X_k \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$. Then

$$\begin{aligned} d\omega(X_0, ..., X_k) = & \sum_{i=0}^{k} (-1)^i \; X_i(\omega(X_0, ..., \widehat{X_i}, ..., X_k)) \\ & + \sum_{0 \leq i,j \leq k} (-1)^{i+j} \omega([X_i, X_j], X_0, ..., \widehat{X_i}, ..., \widehat{X_j}, ..., X_k) \end{aligned}$$

Proof. We again use induction on k. For $k = 0$ we already know the formular holds. Now given for some $k-1$, we want to show that it also holds for k. To this end, note that

$$\begin{aligned}(d\imath_{X_0}\omega)(X_1,...,X_k) &= \sum_{i=1}^{k}(-1)^{i+1}\ X_i(\imath_{X_0}\omega(X_1,...,\widehat{X_i},...,X_k)) \\ &+ \sum_{1\leq i,j\leq k}(-1)^{i+j}\imath_{X_0}\omega([X_i,X_j],X_1,...,\widehat{X_i},...,\widehat{X_j},...,X_k) \\ &= \sum_{i=1}^{k}(-1)^{i+1}\ X_i(\omega(X_0,X_1,...,\widehat{X_i},...,X_k)) \\ &+ \sum_{1\leq i,j\leq k}(-1)^{i+j}\omega([X_i,X_j],X_0,X_1,...,\widehat{X_i},...,\widehat{X_j},...,X_k)\end{aligned}$$

Now we further note

$$\begin{aligned}d\circ\imath_{X_0}\ \omega(X_1,...,X_k) &= \imath_{X_k}...\imath_{X_1}\circ d\circ\imath_{X_0}\omega \\ &= -d\omega(X_k,X_0) + \sum_{i=1}^{k}(-1)^{i+1}\ \imath_{X_k}...\widetilde{\imath_{X_i}}...\imath_{X_1}\imath_{[X_i,X_0]} - X_0(\omega(X_1,...,X_k))\end{aligned}$$

Putting those two together proves the Proposition. □

1.2 Sheaves

Definition 1.19 (Presheaf)
Let X be a topological space. A **presheaf** $\mathcal{F}$ of rings on X consists of the following data:

(1) For every open subset $U \subset X$ there exists a ring $\mathcal{F}(U)$

(2) For every inclusion $V \subset U$ of open subsets of X, there exists a morphism of rings

$$\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

such that the following properties hold

a) $\mathcal{F}(\emptyset) = 0$

b) ρ_{UU} is the identity morphism

c) If $W \subset V \subset U$ then

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV}$$

Now we make our way to define the notion of a full sheaf:

Definition 1.20 (Sheaf)
A presheaf $\mathcal{F}$ on a topological space X is a **sheaf**, if it satisfies the following additional properties:

d) If U is an open subset of X and $\{V_i\}$ and open cover of U, $s \in \mathcal{F}(U)$ such that $\rho_{UV_i}(s) = 0 \quad \forall i$ then $s = 0$.

e) If U is an open subset of X and $\{V_i\}$ and open cover of U. Let $s_i \in \mathcal{F}(U_i)$ such that

$$\rho_{V_i \ V_i \cap V_j}(s_i) = \rho_{V_j \ V_j \cap V_i}(s_i) \quad \forall i, j$$

then there exists an $s \in \mathcal{F}(U)$ such that $\rho_{UV_i}(s) = s_i \quad \forall i$.

We again provide some interesting examples of sheaves and presheaves:

Example 1.21

- Let X be a topological space. For each open set $U \subseteq X$ let $C(U)$ be the ring of continuous functions with codomain $\mathbb{R}$. The maps $\rho_{UV} : C(U) \to C(V)$ are now given by the restrictions of these functions in the usual sense. This makes C a sheaf of rings on X.

- Let $f : X \to Y$ be a continuous map between topological spaces X, Y. Define

$$\mathcal{F}(U) := \{s : U \to Y,\ f \circ s = \mathrm{id}_U\}$$

This constitutes a sheaf on X. The maps s in $\mathcal{F}(U)$ are called **sections** of the sheaf.

- Let $X = \mathbb{R}$ and define $\mathcal{F}(U)$ to be the ring of bounded functions of an open subset $U \subseteq \mathbb{R}$. Now this is not a sheaf: If $U = \mathbb{R}$, we choose $V_i := \{x \in \mathbb{R} \; s.t. \; |x| < i\}$. Now let $\mathrm{id} : x \mapsto x$ denote the identity map. We note that $\mathrm{id} \in \mathcal{F}(V_i) \; \forall i \in \mathbb{N}$. Thus there cannot exist $s \in \mathcal{F}(\mathbb{R})$ s.t. $s|_{V_i} = \mathrm{id}$ which implies that $\mathrm{id} \notin \mathcal{F}(\mathbb{R})$ is not bounded. Thus we found a presheaf that is not a full sheaf.

The next natural step is to again identify the data needed for morphisms in the category of presheaves:

Definition 1.22 (Morphisms of Sheaves)
Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves (or sheaves) on a topological space X. A **morphism of (pre-)sheaves** consist of a morphisms of rings $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for all open sets $U \in X$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho'_{UV}"] \\
\mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V)
\end{tikzcd}$$

Which sums up to saying $\rho' \circ \varphi = \varphi \circ \rho$.

To further expand upon the agricultural language of our setting, we go on to discuss the notion of germs and stalks:

Definition 1.23 (Germs)
Let X be a topological spaxe and $x \in X$. A **germ** on X at x is an equivalence class of functions $f : X \to Y$ for a set Y where we define $f \sim g$ if there exists a neighbourhood $U \subseteq X$ of x such that $f|_U = g|_U$.

Now it is very reasonable to ask how a sheaf behaves at an arbitrary point of the base topological space X. To this end, we can look at its behaviour in a small neighbourhood of a chosen point and then implement a fitting limit to come arbitrarily close. To this end, we use the direct limit known from category theory.

Definition 1.24 (Stalks)
Let $\mathcal{F}$ be a presheaf on X, $p \in X$. The **stalk** $\mathcal{F}_p$ of $\mathcal{F}$ at p is the direct limit of the rings $\mathcal{F}(U)$ for all open sets U containing p through the restiction maps ρ.

Exercise 1.25 As an interesting exercise you can show that the space of continuous functions at a point $p \in X$ is the stalk at p of the sheaf of continuous functions on X.

Definition 1.26 A ring R is a **local ring** if it has a unique maximal left (or equivalently right) ideal.

Definition 1.27 (Ringed Space)
A **ringed space** (X, O_X) is a topological space X together with a sheaf of rings O_X on X. O_X is called the **structure sheaf** of X.

Definition 1.28 A **locally ringed space** is a ringed space such that all the stalk of O_X are local rings in the sense of 1.26.

In the following we will mostly use notion of locally ringed spaces. Before continuing, we again provide some examples for the above constructs:

Example 1.29

- Let X be a topological space. X is a locally ringed space if it is equipped with the sheaf of real valued continuous functions on X. The same holds for X a manifold equipped with the sheaf of smooth functions. As an exercise you can prove that X is indeed locally ringed. Hint: The unique maximal ideal consists of the germs for which the value at x is 0.

1.3 Graded Manifolds

While it may seem like some of the definitions were just loosely related, we gradualy closed in on the important definition of a graded manifold:

Definition 1.30 (Graded Manifold) A **graded manifold** M is a locally ringed space (M, O_M) where M is a manifold and O_M is a sheaf that is locally isomorphic to $(U, \mathcal{C}^\infty(U) \otimes S^\bullet(W^*))$ where $U \subseteq M$ is an arbitrary open subset and W is a graded vector space. Also recall that $S^\bullet(W^*)$ denotes the graded algebra of graded polynomial functions on $W^\bullet$.

The sections of the sheaf O_M are called **graded functions** on the graded manifold M. Note that the graded functions form a graded algebra.

***Remark* 1.31** The local isomorphism required above should preserve the grading of

$$\bigoplus_{k\geq 0} \mathcal{C}^\infty(U) \otimes S^k W^*$$

Note further that $|f \otimes v| = |v|$.

On a graded manifold we do have local coordinates. But aside from the usual coordinates defined using open coverings of M, we also have coordinates that do justice to the graded setting:

We denote the coordinates by $(x_i)^n_{i=1,\dots,m}$. Now coordinates of degree 0 are coordinates on the open sets of an open cover U_i. Meanwhile coordinates with $|x_i| \neq 0$ are coordinates on $S^\bullet(W^*)$.

Now we once again turn to the respective morphisms of graded manifolds. Looking at the definition of a graded manifold, we first need to define the morphisms of (locally) ringed spaces:

Definition 1.32 (Morphisms of Ringed Spaces)
A **morphism between ringed spaces** (X, O_X) and (Y, O_Y) is a pair (f, ψ) where $f : X \to Y$ is continuous and $\psi : O_Y \to f_*O_X$ is a morphism of sheaves (on Y). f_*O_X denotes the direct image of the structure sheaf of X defined for $U \subseteq Y$ as

$$(f_*O_X)(U) := O_X(f^{-1}(U))$$

Unpacking the above definition we note: We have $f : X \to Y$ continuous and a family of ring morphisms $\psi_U : O_Y(V) \to O_X(f^{-1}(V))$ for all $V \subseteq Y$ open. Now this makes the following diagram commute for some $V_1 \subseteq V_2$:

$$\begin{tikzcd} O_Y(V_2) \arrow[r, "\psi_{V_2}"] \arrow[d, "\rho^Y_{V_2 V_1}"'] & O_Y(f^{-1}(V_2)) \arrow[d, "\rho^X_{f^{-1}(V_2) f^{-1}(V_1)}"] \\ O_Y(V_1) \arrow[r, "\psi_{V_1}"] & O_Y(f^{-1}(V_1)) \end{tikzcd}$$

An algebraic version of the diagram can be formed as such: Let $s \in O_Y(V_2)$ be a section. Then $\psi(s) \in O_Y(f^{-1}(V_2))$ and $\rho^X(\psi(s)) \in O_X(f^{-1}(V_1))$. Using ρ^Y instead we finally obtain:

$$\rho^X \circ \psi = \psi \circ \rho^Y \tag{1}$$

Now in the case of a locally ringed space, we have an additional condition that must uphold. Namely the ring homomorphism induced by ψ between the stalks of Y and X has to be a local homomorphism i.e. it has to map maximal ideal to maximal ideal.

Definition 1.33 (The Algebra of Smooth Functions)
The **algebra of smooth functions** $\mathcal{C}^\infty(M)$ of a graded manifold (M, O_M) is the algebra of global sections of O_M. It automatically inherits a $\mathbb{Z}$ grading and thus forms a graded algebra.

Note that the graded symmetric algebra is the generalization of both the symmetric algebra and the exterior algebra. We use the previous definitions to pull the concept of a vector bundle to the graded setting:

Definition 1.34 (Graded Vector Bundles)
A **graded vector bundle** over a manifold M is a collection of ordinary vector bundles $(E_i)_{i\in\mathbb{Z}}$ over M.

We try to bring the matter closer to the reader by giving one examples:

Example 1.35 Given a manifold M and a vector bundle $E \longrightarrow M$, the sheaf of sections

$$U \mapsto \Gamma(U, S^\bullet(E^*|_U))$$

is a graded manifold denoted by E. As a rather complicated exercise, you can prove the above statement.

Example 1.36

- Let V be a vector bundle. We can construct a graded manifold as in Example 1.35 by taking $E := V[1]$.
- In the same manner we can take $T[1]M$ to get the graded manifold $\Omega^\bullet(M)$.

Theorem 1.37 (Serre-Swan Theorem)
Any graded manifold is isomorphic to a graded manifold associated to a graded vector-bundle.

This theorem tells us that all graded manifolds can be considered to be of the form seen in the aforementioned example regarding the sheaf of sections.

We go on to define *graded vector fields* to fit some anaylsis into our graded setting.

Definition 1.38 (Graded Vector Fields)
A **graded vector field** on a graded manifold M is a graded linear map

$$X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)[k]$$

satisfying the **graded Leibniz rule**

$$X(fg) = X(f)g + (-1)^{k|f|}\ f\ X(g) \qquad \forall f, g \in \mathcal{C}^\infty(M)$$

It is easy to show that the graded vector fields form a graded vector space.

Now we use the coordinates x^i, we previously defined on a graded manifold, to locally express graded vector fields as

$$X := X^i\ \frac{\partial}{\partial x^i}$$

Example 1.39 The *graded Euler vector field* is a graded vector field existing for any graded manifold M defined as

$$E(f) := |f|\ f$$

for f any homogenous function. As an exercise you can show that E is a graded derivation of degree 0.

In local coordinates, we can express the Euler vector field as

$$E = \sum_i |x^i| x^i\ \frac{\partial}{\partial x^i}$$

***Remark* 1.40** The **graded commutator** equips the graded vector space of graded vector fields with the structure of a graded Lie algebra:

$$[X, Y] := X \circ Y - (-1)^{kl}\ Y \circ X$$

It is an easy exercise to prove this statement.

Now we further investigate the graded Euler vector field and its remarkable properties.

Proposition 1.41 For any graded vector field X we have

$$[E, X] = deg(X)\ X$$

Proof.

$$\begin{aligned}[E, X](f) &= E(X(f)) - X(E(f)) = |X(f)|X(f) - X(|f|f) \\ &= (|X(f)| - |f|)X(f) = deg(X)\ X(f)\end{aligned}$$

□

To bring further notions of our usual geometric background to the graded setting, we turn towards the notion of homology as a powerful tool to classify manifolds using their innert topological properties:

Definition 1.42 (Cohomological Vector Fields)
A **cohomological vector field** is a graded vector field of degree $+1$ which commutes with itself, thus

$$[Q, Q] = 0 \quad \Rightarrow \quad 2Q \cdot Q = 0 \quad \Rightarrow \quad Q^2 = 0$$

Note that thus every cohomological vector field corresponds to a differential on the graded algebra of smooth functions $\mathcal{C}^\infty(M)$.

Again, we provide some interesting examples to further illuminate the definitions:

Example 1.43

- Consider $T[1]M \rightarrow$. The algebra of smooth functions on it is nothing but the algebra of differential forms on M, namely $\Omega^\bullet(M)$ using the de Rham differential. Note that M is just a normal manifold in this example.
  This shows that the de Rham differential is a cohomological vector field on $T[1]M$ seen as the graded manifold $(U, \mathcal{C}^\infty(U) \otimes S(T[1]M^*))$

- Let $\mathfrak{g}$ be a finite dimensional Lie algebra. We look at the graded manifold $\mathfrak{g}[1]$. This graded manifold carries a cohomological vector field Q corresponding to the Chivally-Eulenberg differential on $\bigwedge^\bullet \mathfrak{g}^* = \mathcal{C}^\infty(\mathfrak{g}[1])$. We use $(e_i)_{i=1}^n$ as a basis of $\mathfrak{g}$ and denote by f_{ij}^k the induced structure constants

  $$[e_i, e_j] = \sum_{k=1}^n f_{ij}^k e_k$$

  Now we can define Q to be

  $$Q := \frac{1}{2} \sum_{i,j=1}^n x^i x^j f_{ij}^k \frac{\partial}{\partial x^k}$$

  Where we denoted by $(x^i)_{i=1}^n$ the coordinates on $\mathfrak{g}[1]$ dual to the basis $(e_i)_{i=1}^n$. As an interesting exercise, prove that Q is indeed cohomological, i.e. $Q^2 = 0$.

Definition 1.44 (Differential Graded Manifolds)
A graded manifold together with a cohomological vector field is called a **differential graded manifold** or short dg-manifold. Notice that the morphisms of dg-manifolds are morphisms of graded manifolds with respect to which the respective cohomological vector fields are related.

We turn our heads towards integration in the graded setting. To this end, we investigate differential forms. The main problem will be the problematic transformation behaviour of differential forms that will dramatically increase the notion of integration. As a result, we will turn towards the symplectic setting which eliminates many of the upcoming difficulties.

Locally, we reconstruct the algebra of differential forms on M by adding a new set of coordinates $(dx^i)_{i=1}^n$ with degree $(|x^i| + 1)$. First note that generally $(dx^i)^2 \neq 0$. However we again go the other way and start with a global setting to then zoom into the local one. The rough outline of the procedure looks as follows:

1. Differential forms are reconstructed as a graded manifold starting from $T[1]M$.
2. The de Rham differential has already been shown to be a cohomological vector field on $\mathcal{C}^\infty(T[1]M)$.
   Now we interpret $T[1]M$ as a dg-manifold where Q is the de Rham differential. In coordinates, we have
$$Q = \sum_{i=1}^n dx^i \frac{\partial}{\partial x^i}$$

Definition 1.45 (Graded de Rham Complex)
The **graded de Rham complex** $(\Omega(M), d)$ is defined as $\mathcal{C}^\infty(T[1]M)$ equipped with the differential Q as defined above. Thus the elements of the space of this complex are differential forms on M.

Now our next goal is to extend the Cartan Calculus to this setting. The main point is thus to define the respective operators. First note that $\imath_{\frac{\partial}{\partial x^i}}$ has degree $(|x^i| - 1)$ and $\imath_{\frac{\partial}{\partial x^i}} dx^j = \delta_i^j$. Note further that
$$\mathcal{L}_X \omega := \imath_X \circ d\omega + (-1)^{|X|} d \circ \imath_X \omega$$

Now it would be convenient to pinpoint the degree of a differential form. To no surprise, the graded Euler vector field can be used to do just that:

$$\mathcal{L}_E x^i = E(x^i) = |x^i| x^i$$
$$\mathcal{L}_E dx^i = d \circ \mathcal{L}_E(x^i) = |x^i| dx^i$$

Thus we can unambigously define the degree of a differential form:

Definition 1.46 We define the degree $|\omega|$ of a differential form ω through the formula

$$\mathcal{L}_E \omega = |\omega| \omega$$

With this definition at hand, we are ready to introduce the symplectic setting.

Definition 1.47 (Graded Symplectic Forms)
A **graded symplectic form** of degree k is a two-form ω which has the following properties:

1. ω is homogenous of degree k
2. ω is closed wrt. the de Rham operator
3. ω is non-degenerate, i.e. the induced musical morphism

$$\begin{aligned} \omega^i : TM &\to T^*[k]M, \\ X &\mapsto \omega(X, \cdot) \end{aligned}$$

   is an isomorphism.

Definition 1.48 (Graded Symplectic Manifolds)
A **graded symplectic manifold of degree** k is a pair (M, ω) where M is a graded manifold and ω is a graded symplectic form on M of degree k.

Now this definition directly implies some interesting properties of graded symplectic manifolds:

Lemma 1.49 Let ω be the respective graded symplectic form of M and let $\deg(\omega) = k \neq 0$. This implies that ω is exact.

Proof. We easily show that

$$k \cdot \omega = \mathcal{L}_E \omega = d \circ \imath_E \omega \quad \Rightarrow \quad \omega = d \left(\frac{1}{k} \imath_E \omega \right)$$

□

We extend the notion of a symplectic form to vector fields:

Definition 1.50

- Let ω be a graded symplectic form on M. A vector field X is a **symplectic vector field** if

$$\mathcal{L}_X \omega = 0$$

- A vector field X is an **hamiltonian vector field** if the contraction of X and ω is an exact one-form, i.e. $\exists H \in \mathcal{C}^\infty(M)$ s.t.
$$\imath_X \omega = dH$$

Lemma 1.51 Let ω be a graded symplectic form on M of degree k and let X be a symplectic vector field of degree l. If $k + l \neq 0$, then X is hamiltonian.

Proof. First, we note that

$$[E, X] = lX, \qquad \mathcal{L}_X \omega = d \circ \imath_X \omega = 0, \qquad \mathcal{L}_E \omega = k\omega$$

Now we can define $H := \imath_E \imath_X \omega$ to obtain

$$\begin{aligned} dH &= d \circ \imath_E \imath_X \omega = \mathcal{L}_E \imath_X \omega - \imath_E \circ d\imath_X \omega \\ &= \mathcal{L}_E \imath_X \omega = \imath_{[E,X]}\omega + \imath_X \mathcal{L}_E \omega = l\imath_X \omega + \imath_X(k\omega) \\ &= l\imath_X \omega + k\imath_X \omega = (l+k)\imath_X \omega \end{aligned}$$

and thus
$$\imath_X \omega = \frac{dH}{k+l}$$
which proves the claim. □

We go on to extend dg-manifolds to the symplectic setting.

Definition 1.52 (Differential Graded Symplectic Manifolds)
A graded manifold with a graded symplectic form and a symplectic cohomological vector field is called a **differential graded symplectic manifold** and is denoted by (M, ω, Q).

Using the previous lemma we directly see that, given $\deg(\omega) \neq -1$, the cohomological vector field Q is necessarily hamiltonian. As an interesting analogy, we can define a structure similar to that of the well-known *Poisson-bracket* using the symplectic form of a differential graded symplectic manifold. To this end, define

$$\begin{aligned} \{\cdot,\cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \quad &\to \quad \mathcal{C}^\infty(M), \\ (f, g) \quad &\mapsto \quad \{f, g\} := (-1)^{|f|+1}\, X_f(g) \end{aligned}$$

where we denote by X_f the unique hamiltonian graded vector field corresponding to f, thus satisfying
$$\imath_{X_f} \omega = df$$

As an exercise you can show that the thus defined bracket satisfies relations strikingly similar to those of the Poisson-bracket.

Using the notion of a hamiltonian graded vector field, we take a differential graded symplectic manifold (M, ω, Q) and use the innert cohomological vector field Q to define the unique function S that satisfies (modulo sign)

$$\imath_Q \omega = dS$$

Now since $Q = \{S, \cdot\}$ we see

$$Q(f) = \{S, f\} = (-1)^{|S|+1} X_S(f) = X_S(f)$$

implying that $[Q, Q](f) = \{\{S, S\}, f\} = 0$ making $\{S, S\}$ a constant. Now since $\{S, S\}$ has degree $k + 2$, ω has degree k and S has degree $k + 1$ we see that $\{S, S\} = 0$ if $k \neq -2$. This equation is called the **Classical Master Equation** and will become increasingly important towards the end of the course.

1.4 Principal Bundles

***Remark* 1.53** In the following subsection, we consider everything to be smooth which makes a lot of subtle details easier.

We begin by giving the barebone definition of a Fibre bundle. Good additional material on this topic can be found in [1] and [2].

Definition 1.54 (Fibre Bundles and Sections)
Let M be a manifold. A **bundle** over M is a triple $E \xrightarrow{\pi} M$ where E is a smooth manifold and π a smooth surjective map $\pi : E \to M$.
A **fibre bundle** is a bundle for which all fibres, that is preimages of the form $\pi^{-1}(p)$ for $p \in M$, are diffeomorphic to one another.
Now a **section** (or cross section) of a bundle (E, π, M) is a smooth map $s : M \to E$ satisfying the section property $\pi \circ s = id_M$.

Now recall that a G**-space** X is a topological space X equipped with a Lie Group G and a smooth right (or left) action $\sigma : G \times X \to X$. We now use the notion of orbit spaces to form special fibre bundles using the quotient spaces under a given action:

Definition 1.55 A bundle (E, π, M) is a G**-bundle** if E is a right G-space and if (E, π, M) is isomorphic to the bundle $(E, \rho, E/G)$ where E/G denotes the quotient space of the respective action and ρ is the standard projection.

We introduce further classification of G-bundles by restricting to certain types of actions:

Definition 1.56 (Principal G-bundles)
A G-bundle (E, π, M) is a **principal** G**-bundle** of G acts freely on E and transitively on fibres.

Remember that an action σ is free, if for any $g \in G$ and $x \in X$ the equation $\sigma_g(x) = x$ implies $g = e_G$. Now a free action results in each orbit, and thus each fibre, being homomorphic to G itself and hence (E, π, M) is indeed a fibre bundle as implied before. We give some interesting examples to illuminate the definitions:

Example 1.57

- The **product bundle** $(M \times G, \pi_1, M)$ is a principal G-bundle under the action $(p, g_0)g = (p, g_0 g)$.
- Let again G be a Lie group and H be a closed Lie subgroup of G. Then H acts freely on G via the right action and we denote the orbit space by G/H. Now the triple $(G, \pi, G/H)$ where π is the standard projection to the orbit space is a principal H-bundle.
- Let M be an m-dimensional manifold. A linear frame b_p at $p \in M$ is an ordered set $(b_1, ..., b_m)$

of vectors that forms a basis of the tangent space T_pM. Now we define the set $B(M)$ to be the set of all frames at all point of M and the projection $\pi : B(M) \to M$ that maps $b_p \mapsto p$. Using the vector space structure of the tangent spaces, we can define a natural free action on $B(M)$ using the lie group $GL(m, \mathbb{R})$ via

$$(b_1, ..., b_m)g := \left(\sum_{j_1}^{m} b_{j_1} g_{j_1 1}, ..., \sum_{j_m}^{m} b_{j_m} g_{j_m 1} \right)$$

Now as an exercise $B(M)$ can be given the structure of a smooth manifold. This enables us to show that $(B(M), \pi, M)$ is a $GL(m, \mathbb{R})$-principal bundle.

Having defined a new set of objects, we again define the morphisms between them to enble comparison and transition.

Definition 1.58 (Principal Bundle Morphisms)
Let (P, π_1, M) be a principal G-bundle and (Q, π_2, N) be a principal H-bundle. A **principal bundle map** is a triple (u, φ, Λ) such that $u : P \to Q$ and $\varphi : M \to N$ are smooth maps and $\Lambda : G \to H$ is a group homomorphism and such that

$$\begin{aligned} &\pi_2 \circ u = \varphi \circ \pi_1 \\ &u(pg) = u(p)\Lambda(g) \quad \forall p \in P,\ g \in G \end{aligned}$$

Namely the following two diagrams should commute:

We thus arrive at an important conclusion regarding bundles over the same base manifold.

Theorem 1.59 Let (u, id_M, id_G) be a principal bundle map between the principal G-bundles (P, π_1, M) and (Q, π_2, M). Then u is an isomorphism.

Proof. This proof is left as an easy but important exercise left to the reader. □

We further classify special cases of principal G-bundles using the aforementioned concepts:

Definition 1.60 A principal G-bundle (P, π, M) is **trivial** if there is a principal bundle map to the product bundle $(M \times G, \pi_1, M)$.

An important property to classify trivial bundles is the existence of smooth global sections (which imply global frames):

Theorem 1.61 A principal G-bundle (P, π, M) is trivial iff it there exsist a smooth global section.

Proof. Let $\sigma : M \to P$ be a section such that $\pi \circ \sigma(x) = x$ for all $x \in M$. Now for all $p \in P$ there exists a $\chi_\sigma(p) \in G$ such that

$$p = \sigma(\pi(p))\chi_\sigma(p)$$

Now $\chi_\sigma : P \to G$ is uniquely defined since G acts freely on P. Furthermore we have

$$\chi_\sigma(pg) = \chi_\sigma(p)g$$

and thus

$$\begin{aligned} & \sigma(\pi(pg))\chi_\sigma(pg) = pg \\ \Rightarrow \quad & \sigma(\pi(p))\chi_\sigma(p)g = \sigma(\pi(p))\chi_\sigma(pg) \end{aligned}$$

Thus we can define a map

$$\begin{aligned} u_\sigma : P &\longrightarrow M \times G, \\ p &\longmapsto u_\sigma(p) := (\pi(p), \chi_\sigma(p)) \end{aligned}$$

Now note that $pr_1 \circ u_\sigma = \pi$ and further $u_\sigma(pg) = u_\sigma(p)g$. Thus u_σ is a principal bundle map and conversely we can define $h : M \times G \longrightarrow P$ as a principal bundle map via

$$\begin{aligned} \sigma_u : M &\longrightarrow P, \\ x &\longmapsto h(x, e_G) \end{aligned}$$

This proves the theorem. □

We take a step back and investigate a natural product defined on G-spaces. This will enable us to define associated bundles, a very important naturally occuring form of principal bundles.

Definition 1.62 Let X and Y be a pair of G-spaces. Then their **G-product** $X \times_G Y$ is the space of orbits of the G-action on the cartesian product $X \times Y$, namely $X \times_G Y = (X \times Y)/G$ where

$$(x, y) \sim (x', y') \quad \text{if} \quad \exists g \in G \; s.t. \; x' = xg, \; y' = yg$$

Definition 1.63 (Associated Bundles)
Let $\xi = (P, \pi, M)$ be a principal G-bundle an let F be a left G-space. Define

$$\begin{aligned} P_F &:= P \times_G F \\ (p, v)g &:= (pg, g^{-1}v) \end{aligned}$$

and further $\pi_F : P_F \to M$ via $\pi_F([p, v]) = \pi(p)$. Then $\xi[F] := (P_F, \pi_F, M)$ is a fibre bundle on M with fibre F. We call this bundle the **associated bundle** to the principal bundle ξ.

To show that the above is indeed a well-defined concept, we need to prove that the introduced map π_F does not care about the choice of the respresentant. Also we need to show that all fibres of the given bundle are indeed homeomorphic to one another:

Proof. This proof can be split in two steps:

(1) Let $(p_1, v_1) \sim (p_2, v_2)$ thus there exists $g \in G$ such that $(p_2, v_2) = (p_1 g, g^{-1} v_1)$. Now our goal is to show that $\pi_F(p_1, v_1) = \pi_F(p_2, v_2)$ and $\pi(p_1 g) = \pi(p_1)$. But since $\pi_F(p_1, v_1) = \pi(p_1)$, we easily see that

$$\pi_F(p_2, v_2) = \pi(p_2) = \pi(p_1 g) = \pi(p_1) = \pi_F(p_1, v_1)$$

(2) For each $x \in M$ the fibre $\pi_F^{-1}(\{x\})$ is homeomorphic to F. As an exercise, you can prove this claim.

□

Now we use associated bundles, to build a bridge to a very important type of bundle that is used heavily throughout physics and will play a major role in our course:

Theorem 1.64 (Associated Bundle Theorem)
Let V be any real, finite-dimensional vector space and let $GL(V, \mathbb{R})$ be the group of automorphisms of V. Then the associated bundle $\xi[V]$ is a **vector bundle** (note that here, ξ is a principal $GL(V, \mathbb{R})$-bundle). Conversely, every vector bundle is bundle-isomorphic to an associated bundle of this type. Note that this is not unique since every such principal bundle qualifies for this transition.

Proof. A proof of this claim can be found in [7]. □

The next important construction is that of a connection on a bundle and the induces notion of curvature both of which will play a major role when we talk about Yang-Mills-Theory later in the course.

Definition 1.65 (Vertical Subspace)
Let $P \xrightarrow{\pi} M$ be a principal G-bundle. The **vertical subspace** of T_pP is defined and denoted as

$$V_pP := \{\tau \in T_pP | \pi_* \tau = 0\}$$

Where we denote by π_* the pushforward $TP \to TM$.

Now defining the vertical subspace leaves us with the converse concept, a kind of horizontal space. This will be achieved by defining connections on a bundle:

Definition 1.66 (Connections)
A **connection** on a principal bundle $P \to M$ is choice of subspaces of TP denoted by HP defined for each $p \in P$ such that $H_pP \subset T_pP$ and such that

1. $T_pP \simeq H_pP \oplus V_pP \quad \forall p \in P$
2. $(\delta_g)_*(H_pP) = H_{pg}P \quad \forall p \in P,\ \forall g \in G$

where we denoted by $\delta_g(p) = pg$ the action of G.

The main point is that, given a connection on a principal bundle, we can take any $\tau \in T_pP$ and uniquely decompose it as $\tau = hor(\tau) + vert(\tau)$. The same is possible for $X \in \mathfrak{X}(P)$ where $X = hor(X) + ver(X)$ such that $hor(X)_p \in H_pP$ and $ver(X)_p \in V_pP$ for all $p \in P$. Note that these two vector fields are smooth. Thus a connection ultimately gives us a tool to uniquely split a vector field which allows .

Now in order to avoid working with spaces, we aim to give an equivalent notion using Lie algebras and some geometry. First recall that, given a Lie group G, we can identify the Lie algebra $\mathfrak{g} \simeq T_eG$. Now for $A \in \mathfrak{g}$ and a principal G-bundle P, we can define a vector field on TP for $f : P \to \mathbb{R}$ via

$$X_p^A(f) := \frac{d}{dt}\bigg|_{t=0} f(p\exp(tA))$$

where $t \in \mathbb{R}$. We denoted by exp the well-known exponential map

$$\begin{aligned} \exp : \mathfrak{g} &\longrightarrow G \\ x &\longmapsto \gamma(1) \end{aligned}$$

where $\gamma : \mathbb{R} \longrightarrow G$ is the unique 1-parameter subgroup with tangent vector x at the identity. The exponential map is particularly important because it allows us to prove the following theorem:

Theorem 1.67 Let P be a manifold on which G has a right action. Then the map

$$\begin{aligned} \psi : \mathfrak{g} &\longrightarrow \mathfrak{X}(P) \\ A &\longmapsto X^A \end{aligned}$$

is a homomorphism of $\mathfrak{g}$ into the infinite dimensional Lie algebra of all vector fields on P, i.e.

$$[X^A, X^B] = X^{[A,B]} \quad \forall A, B \in \mathfrak{g}$$

Proof. The proof is quite complicated and can be found on page 196 of [7]. □

Another result which further explains, why we care about these results in the context of vertical and horizontal vector fields is presented in the following proposition:

Proposition 1.68 For a principal bundle P, the map $\psi_p : A \to X_p^A$ is an isomorphism of $\mathfrak{g}$ onto V_pP. This can be seen by using $D\pi(X)(f) = 0$ due to $\pi(pg) = \pi(p)$ for any $g \in G$. Note that $\dim V_pP = \dim G = \dim \mathfrak{g}$

Hence we can denote the inverse

$$\psi_p^{-1} : V_pP \longrightarrow \mathfrak{g}$$

mapping the vertical space at p to the Lie algebra. Using the isomorphism at hand, we can define a uniquely determined one-form on a principal G-bundle which will turn out to be equivalent to a unique connection.

Definition 1.69 (Connection Form)
A **connection form** ω is a $\mathfrak{g}$-valued one-form on a principal G-bundle $P \to M$ such that

1. $\omega_p(X^A) = A \quad \forall p \in P, \ \forall A \in \mathfrak{g}$
2. $\delta_g(p) = pg \qquad (\delta_g)_*\omega = Ad_{g^{-1}}\omega \quad \forall g \in G$

where we denote by Ad the adjoint map defined as

$$\begin{aligned} Ad : G &\longrightarrow Aut(G) \\ g &\longmapsto \psi_g[h \in G \longmapsto ghg^{-1}] \end{aligned}$$

Note that we can also interpret Ad as a map to the automorphisms of $\mathfrak{g}$ since

$$\begin{aligned} Ad_g = (d\psi_g)_e &: T_eG \longrightarrow T_eG \\ \Rightarrow \quad Ad &: G \longrightarrow Aut(\mathfrak{g}) \end{aligned}$$

Now take a smooth map $\varphi : M \to N$ and look at the corresponding pullback map

$$\begin{aligned} \varphi^* : \Omega^k(N) &\longrightarrow \Omega^k(M) \\ \alpha &\longrightarrow \varphi^*\alpha \end{aligned}$$

Now if we take $\varphi = \delta_g : P \longrightarrow P$ that maps $p \mapsto pg$, we obtain a pullback map $\Omega^1(P) \longrightarrow \Omega^1(P)$. Using a connection form ω we see that $\delta_g^*\omega = Ad_{g^{-1}}\omega$ amounts to the claim

$$\omega_{pg}(D\delta_g(\tau)) = Ad_{g^{-1}}(\omega_p(\tau))$$

This leads us to the following theorem:

Theorem 1.70 A connection form on a principal bundle P uniquely defines a connection. Vice versa a connection on P uniquely defines a connection form. We can thus use the two interchangeably.

Proof. With some details left out, we give the main sketch of this proof and encourage the reader to fill in the details:

Let ω be a connection form on P. Define $H_pP := \ker \omega_p$. As an exercise one can prove that this definition of H_pP indeed satisfies the properties of a connection. Basically one needs to use that

ω_p maps elements of the horizontal space to zero by definition while for vertical vectors we obtain nonzero elements of the Lie algebra. Thus we obtain a way to identify vertical vectors. One then verifies, using the second property of connection forms, the second property of 1.66.

Starting with a connection, we can define

$$\omega_p(\tau) := \psi_p^{-1}(ver(\tau))$$

which clearly defines a one-form. As an exercise you can check that the thus defined form satisfies the properties of a connection form. For the first property, note that $\omega_p(X^A) = \psi_p^{-1}(\psi_p(A)) = A$. For the second, we remember $H_{pg}P = (\delta_g)_* H_p P$. □

We will now turn towards the notion of curvature: Given a connection on a principal G-bundle P we have the following splittings for bundles and their sections at hand:

- $TP = VP \oplus HP$
- $T^*P = V^*P \oplus H^*P$
- $\bigwedge^2 T^*P = \left(\bigwedge^2 V^*P\right) \oplus (V^*P \wedge H^*P) \oplus \left(\bigwedge^2 H^*P\right)$
- $\Omega^1(P) = \Omega^1_{vert}(P) \oplus \Omega^1_{hor}(P)$
- $\Omega^2(P) = \Omega^2_{vert}(P) \oplus \Omega^2_{mix}(P) \oplus \Omega^2_{hor}(P)$

Note that thus a connection form ω lies in $\Omega^1_{vert}(P) \otimes \mathfrak{g}$. Now we use the exterior derivative (de Rham differential) to state

$$d\omega \in \Omega^2(P) \otimes \mathfrak{g} = (\Omega^2_{vert}(P) \oplus \Omega^2_{mix}(P) \oplus \Omega^2_{hor}(P)) \otimes \mathfrak{g}$$

and thus

$$d\omega = d\omega_{vert} + d\omega_{mix} + d\omega_{hor}$$

which leads us to the following proposition:

Proposition 1.71

- $d\omega_{vert}(X, Y) = [\omega(X), \omega(Y)]$
- $d\omega_{mix} = 0$

Proof. The proof uses the previously presented properties of the action of forms on vector fields and the connection form. It is left as an exercise to the reader. □

We can now define the notion of curvature on a principal bundle using the differential of a connection form:

Definition 1.72 (Curvature)
Let P be a principal G-bundle and ω a connection form on P. The **curvature** of ω is the unique 2-form

$$F_\omega := d\omega_{hor} \in \Omega^2_{hor}(P) \otimes \mathfrak{g}$$

Keep in mind that this notion will be particularly useful in Yang-Mills-Theory. Now one simple case of curvature is trivial curvature:

Definition 1.73 A connection with zero curvature is called **flat**.

Definition 1.74 (Torsors)
A **Torsor** for a group G is a non-empty set X on which G acts freely and transitively.

Above we used an arbitrary set X. If we take G to be a Lie group and X a manifold, we need the action to be smooth to define a torsor of G on X. Thus note, that the definition of a torsor can vary depending on the context and is thus given in a rather general form above.

Unpacking the definition, we first note that we have an action

$$\begin{aligned} X \times G &\longrightarrow X \quad s.t. \\ x \cdot e &= x \qquad \forall x \in X \\ x(gh) &= (xg)h \quad \forall g, h \in G \end{aligned}$$

and such that the following map is an isomorphism of X

$$\begin{aligned} X \times G &\longrightarrow X \times X \\ (x, g) &\longrightarrow (x, xg) \end{aligned}$$

Thus X and G are isomorphic as sets or manifolds depending on the context. We present some insightful examples of a torsor:

Example 1.75

- A group G is a torso itself using group multiplication as action on itself.
- Take an affine space, thus a set A together with a vector space $\vec{A}$ with a free and transitive action of $\vec{A}$ on A. An affine space A underlying a vector space $\vec{A}$ is a torso for $\vec{A}$ acting as the additive group of translations.
- A principal bundle $P \xrightarrow{\pi} \{x\}$ over a single point as base space trivialy defines a torsor.

resource: Wiki principal homogenous space.

1.5 (Co-)chain Complexes

In this subsection, we will completely change topic and turn towards the notion of *chain complexes* and *cochain complexes* and thus *honology* and *cohomology*. We start by giving the definition of a (co-)chain complex:

Definition 1.76 A **Chain complex** is a sequence of homomorphisms of abelian groups denoted as

$$... \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow ... \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{N}$. In other words: $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$

Definition 1.77 We call elements of C_n that lie in $\ker \partial_n$ **cycles** and denote the set of cycled by $\mathcal{Z}_n$. We call elements of C_n that lie in the image of ∂_{n+1} **boundaries** and denote their set by $\mathcal{B}_n$.

We now use the cycles and boundaries to define homology groups as quotient groups:

Definition 1.78 (Homology groups)
Let $(C_\bullet, \partial_\bullet)$ be a chain complex. We define its **n-th homology group** to be the quotient group

$$\mathcal{H}_n := \ker \partial_n / \operatorname{im} \partial_{n+1} = \mathcal{Z}_N / \mathcal{B}_n$$

The equivalence classes that form $\mathcal{H}_n$ (thus cosets of $\mathcal{Z}_n$) are called **homology classes**.

Like for any introduced category, we are interested in the morphisms between them:

Definition 1.79 (Chain maps)
A **chain map** between two chain complexes $(C_{1\bullet}, \partial_{1\bullet})$ and $(C_{2\bullet}, \partial_{2\bullet})$ is a sequence of homomorphisms $f_n : C_{1n} \longrightarrow C_{2n}$ such that

$$\partial_{2n} \circ f_n = f_{n-1} \circ \partial_{1n}$$

We can pictorially denote a chain map in the following commuting diagram which of course only represents an excerpt of the full commuting diagram spanned using the chain map:

$$\begin{array}{ccccccccc}
... & \longrightarrow & C_{1\ n+1} & \xrightarrow{\partial_{1n+1}} & C_{1\ n} & \xrightarrow{\partial_{1n}} & C_{1\ n-1} & \longrightarrow & ... \\
 & & \big\downarrow{\scriptstyle f_{n+1}} & & \big\downarrow{\scriptstyle f_n} & & \big\downarrow{\scriptstyle f_{n-1}} & & \\
... & \longrightarrow & C_{2\ n+1} & \xrightarrow{\partial_{2n+1}} & C_{2\ n} & \xrightarrow{\partial_{2n}} & C_{2\ n-1} & \longrightarrow & ...
\end{array}$$

We use the above diagram to arrive at an interesting intuitive lemma connecting the homology groups of two chain complexes:

Lemma 1.80 Let $f_\bullet : (C_{1\bullet}, \partial_{1\bullet}) \longrightarrow (C_{2\bullet}, \partial_{2\bullet})$ be a chain map. The $f_\bullet$ induces an homomorphism of homology groups

$$\mathcal{H}_n(f_\bullet) : \mathcal{H}_n(C_{1\bullet}) \longrightarrow \mathcal{H}_N(C_{2\bullet})$$

for each $n \in \mathbb{N}$, defined by

$$[\mathcal{Z}_n] \longrightarrow [f_n(\mathcal{Z}_n)]$$

Proof. Our main goal is to show that $f_\bullet$ maps cycles to cycles and boundaries to boundaries. This is equivalent to showing that it is a homomorphism of homology groups.

We begin by fixing $n \in \mathbb{N}$. Note that $f_n(\mathcal{Z}_n(C_{1\bullet})) \subseteq \mathcal{Z}_n(C_{2\bullet})$ and let $z_n \in \mathcal{Z}_n(C_{1\bullet})$ which means $\partial_{1n} z_n = 0$. Now we calculate

$$\partial_{2n}(f_n(z_n)) = f_{n-1}(\partial_{1n}(z_n)) = 0$$

which shows that $f_\bullet$ maps cycles to cycles. For boundaries, we note $f_n(\mathcal{B}_n)(C_{1\bullet}) \subseteq \mathcal{B}_n(C_{2\bullet})$. Now choose $b \in \mathcal{B}_n(C_{1\bullet})$ i.e. $b = \partial_{1\ n+1}(a)$ for some $a \in C_{1\ n+1}$. Now we calculate

$$\begin{aligned} f_n(b) &= f_n(\partial_{1\ n+1}(a)) = \partial_{2\ n+1}(f_{n+1}(a)) \\ &\Rightarrow f_n(b) \in \operatorname{im} \partial_{2\ n+1} \end{aligned}$$

Thus f_n is constant on equivalence classes. Let now $[z_n^1]$ be an equivalence class and let x^1, x^2 be two representatives. Thus

$$x^1, x^2 \in \mathcal{Z}_n \qquad \textit{and} \qquad x^1 - x^2 = b \in \mathcal{B}_n$$

Which proves that the equivalence class of $f_n(x^1)$ is the same as that of $f_n(x^2)$. $\square$

We go on by classifying chain maps which will prove useful later in the course.

Definition 1.81 A chain map $f_\bullet : C_{1\bullet} \longrightarrow C_{2\bullet}$ is called a **quasi-isomorphism** if $\mathcal{H}_n(f_\bullet)$ is an isomorphism for all $n \in \mathbb{N}$.

Note that the name "quasi-isomorphism" makes sense since the isomorphisms only identify the respective abelian groups and not the entire complex. While it gives us a tool to identify their *homology groups* which will, in most cases, carry most of the important information, it does not tell us anything about the respective homomorphism sequences on the chain complexes. For some examples of chain complexes and their (quasi-)isomorphisms, which require far more work than the scope of this lecture allows, see [8].

Definition 1.82 A sequence of homomorphisms

$$\ldots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \ldots$$

is said to be an **exact sequence** if $\ker \alpha_n = \operatorname{im} \alpha_{n+1} \quad \forall n \in \mathbb{N}$. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

is called a **short exact sequence**. Note that for a short exact sequence $\ker(j) = \operatorname{im}(i)$ as well as

$$\operatorname{im}(j) = \ker(0) \qquad \Rightarrow j \text{ is surjective}$$
$$\operatorname{im}(0) = \ker(i) \qquad \Rightarrow i \text{ is injective}$$

We can use short exact sequences for many important proofs and definitions. Thus we provide some extra classification properties:

Definition 1.83 A short exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

is said to **split** if it satisfies one of the following equivalent properties:

1. There is a homomorphism
$$p : B \longrightarrow A$$
such that $p \circ i = 1_A : A \longrightarrow A$
2. There is a homomorphism
$$s : C \longrightarrow B$$
such that $j \circ s = 1_C : C \longrightarrow C$
3. There exists an isomorphism
$$\psi : B \longrightarrow A \oplus C$$
making the following diagram commute:

$$\begin{tikzcd}
 & & B \arrow[dr, "j"] & & \\
0 \arrow[r] & A \arrow[ur, "i"] \arrow[dr, "i'"'] & & C \arrow[r] & 0 \\
 & & A \oplus C \arrow[ur, "j'"'] & &
\end{tikzcd}$$

where $i'(a) = (a, 0)$ and $j'(a, c) = c$.

Lemma 1.84 The three properties above are indeed equivalent.

Proof.

$1 \to 3$ We have $p : B \longrightarrow A$. Now define $\psi(b) := (p(b), j(b))$ which can be proven to be an isomorphism and makes the above diagram commute.

$2 \to 3$ We use s, i and j: Define

$$\psi^{-1}(a, c) := i(a) + s(c)$$

which is an isomorphism and makes the above diagram commute. Note that $j(i(a)) + j(s(c)) = c$.

$3 \to 1/2$ Here we use ψ, i, j. Define $p(b) := pr_1 \circ \psi(b)$ and $s(c) := \psi^{-1} \circ (j')^{-1}(c)$. Now note that $p \circ i = id_A$ due to

$$p(i(a)) = pr_1(\psi(i(a))) = pr_1(i'(a)) = a$$

We further have $j \circ s = id_C$ due to

$$j \circ s(c) = j(\psi^{-1}(0, c)) = j'(0, c) = c$$

$2 \to 1$ The last thing we need to show uses s, i, j. We define $p(b) := i^{-1}(b - s \circ j(b))$. This

$\square$

We now move to the important concept of cochain complexes, which form the dual concept to chain complexes. We will thus encounter *cohomology groups* and the like:

Definition 1.85 A **cochain complex** is a sequence of homomorphisms of abelian groups

$$... \longleftarrow D^{n+1} \xleftarrow{\delta^n} D^n \xleftarrow{\delta^{n-1}} D^{n-1} \longleftarrow ...$$

such that $\delta^n \circ \delta^{n-1} = 0$ for each $n \in \mathbb{N}$, namely $\mathrm{im}(\delta^{n-1}) \subseteq \ker(\delta^n)$

A namescheme very similar to that of chain complexes is given to that of cochain complexes:

Definition 1.86 Elements of D^n are called **cocycles** if they lie in $\ker(\delta^n)$ and **coboundaries** if they lie in $\mathrm{im}(\delta^{n-1})$.

As promised this allows us to define the dual concept to homology groups:

Definition 1.87 (Cohomology groups)
Let $(D^\bullet, \delta^\bullet)$ be a cochain complex. We define the **n-th cohomology group** to be the quotient group

$$\mathcal{H}^n := \ker(\delta^n)/\,\mathrm{im}(\delta^{n-1})$$

Elements of $\mathcal{H}^n$ (thus cosets of $\ker(\delta^n)$) are called **cohomology classes**.

Now we aim to further illuminate the claim that cochaincomplexes is indeed the dual concept to chain complexes. To this end, let G be a group and $(C_\bullet, \partial_\bullet)$ be a chain complex. Now define its dual

$$D^{n+1} := (C_n)^* = \mathrm{Hom}(C_n, G)$$

as the group of homomorphisms $C_n \longrightarrow G$. Now remember that a **dual homomorphism** is defined as follows: Take a group homomorphism $\alpha : A \longrightarrow B$. One defines the dual homomorphism (sometimes called "pullback") via

$$\begin{aligned} \alpha^* : \mathrm{Hom}(A, B) &\longrightarrow \mathrm{Hom}(A, G), \\ \varphi &\longmapsto \alpha^*(\varphi) = \varphi \circ \alpha \end{aligned}$$

We apply the same concept to our chain complex $\partial_n : C_n \longrightarrow C_{n-1}$ which lets us define

$$\delta^n := \partial_n^* : \mathrm{Hom}(C_{n-1}, G) \longrightarrow \mathrm{Hom}(C_n, G)$$

These maps automatically satisfy $\delta^n \circ \delta^{n-1} = 0$ due to the corresponding equation for ∂_n. Now there exists a theorem, which is out of the scope of this lecture, but can be found in [8], that states that the cohomology groups of $C_\bullet$ under G denoted by $\mathcal{H}^n(C; G)$ are fully determined by G and the homology groups $\mathcal{H}_n(C_\bullet)$.

***Remark* 1.88** Cochain maps and all of the other notions defined for chain complexes carry over to cochains. For example the analogous equation for cochain maps is

$$f^n \circ \delta_1^{n-1} = \delta_2^{n-1} \circ f^{n-1}$$

for maps $f^n : D_1^n \longrightarrow D_2^n$. As an exercise, you can transport all of these results to the cochains.

We conclude this subchapter by bringing the notion of substructures to (co-)chain complexes.

Definition 1.89 A **subcomplex** $C'_\bullet$ of a chain complex $(C_\bullet, \partial_\bullet)$ is a sequence of subgroups C'_n of C_n for all $n \in \mathbb{N}$ such that $\partial_n(C'_n) \subseteq C'_{n-1}$ for all $n \in \mathbb{N}$.

***Remark* 1.90** Note that a subcomplex is indeed itself a chain complex. In this case the inclusion map acts as a chain complex map.

2 An Introduction to Classical Field Theories

In this section, we will explore classical field theories in order to give a solid foundation to their concepts, especially locality. While field theories come in many forms like thermodynamics, electrodynamics, general relativity, the standard model of physics and even string theory, we will mainly focus on classical Lagrangian field theories.

In Lagrangian mechanics, as taught in undergrad physics, one uses the notion of the action functional, being a measure for the "excitednes" of a system, to find physically favoured and thus realized trajectories. A common action in this context would be

$$S(q) = \int_{\mathbb{R}^+} \left(\frac{m}{2} \dot{q}^i(t) \dot{q}^i(t) - V(q(t)) \right) dt$$

For $q \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^n)$ and $V \colon \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^n) \to \mathbb{R}$. Now the condition of vanishing variation $\delta S \equiv 0$ imposes two conditions

$$\begin{aligned} \Rightarrow \qquad & \frac{m}{2} \ddot{q}^i(t) + \nabla^i V = 0 \quad \text{Euler-Lagrange-equations} \\ \Rightarrow \qquad & \delta q^i \Big|_{t=0} = 0 \quad \text{"Boundary conditions"} \end{aligned}$$

Now the **boundary term** $\frac{m}{2} \delta q^i(0) \dot{q}^i = \alpha$ can be thought of as a 1-form on $T^*\mathbb{R}^n$ such that the boundary term, using $p^i(0) = \frac{m}{2} \dot{q}^i$, can be writen as $\omega := \delta\alpha = \delta q^i(0) \delta p^i(0)$ which is a symplectic form.

This marks the starting point of a procedure called **canonical quantisation** of classical mechanics. As a Hilbert space we use $\mathcal{H} \colon = l^2(\mathbb{R}^n)$. Our main goal will be to generalize this procedure.

2.1 Spaces of fields and Locality

We will usually work on a fibre bundle $\pi \colon F \longrightarrow M$ over some smooth manifold M. For simplicity, we will assume that M is closed and without boundary, furthermore we assume that M is oriented and connected.

Definition 2.1 (Sections and local sections) A **section** of $\pi \colon F \longrightarrow M$ is a smooth map $\varphi \colon M \longrightarrow F$ such that $\pi \circ \varphi = \mathrm{id}_M$. We call φ a **local section** if it is only defined on an open subset $U \subseteq M$ such that $\pi \circ \varphi = \mathrm{id}_U$. We further denote the space of sections by

$$\Gamma(M, F) \colon = \{\varphi \in \mathcal{C}^\infty(M, F) | \pi \circ \varphi = \mathrm{id}_M\} \equiv \mathcal{F}$$

We often refer to $\mathcal{F}$ as the **space of fields**. Note that if we work with a vector bundle, $\mathcal{F}$ inherits a linear structure.

We are mainly interested in **locality**. Thus we work with equivalence classes of such sections that coincide in a neighbourhood of a point up to some arbitrary k-th derivative:

Definition 2.2 If $p \in M$ we denote by $\Gamma(p)$ the space of local sections whose domain contains p.

Using the thus defined local spaces allows us to define the utterly important notion of *Jets of sections*:

Definition 2.3 (k-Jets of sections) Let $\pi\colon F \longrightarrow M$ be a fibre bundle and k any integer. We say that two local sections of π at $p \in M$ **have the same k-th jet at** p if their partial derivatives agree at p up to k-th order in some chart around p. We denote by J^k_pF the set of such equivalence classes and use

$$j^k_p\varphi\colon = [(\varphi, p)]_k$$

to denote such equivalence classes.

***Remark* 2.4** As a rather abstract but interesting exercise, you can show that the above definition does not depend on the choice of coordinate charts. *Hint:* Introduce multiindices I such that if U^α is a chart for F, we look at

$$\frac{\partial^{|I|}}{\partial x^I}(u^\alpha \circ \varphi)\Big|_p$$

As we will see later, this will also introduce coordinates on the objects J^k_pF

Definition 2.5 (Jet bundles) Given a fibre bundle $\pi\colon F \longrightarrow M$ and an integer k we denote

$$J^kF\colon = \{j^k_p\varphi | p \in M, \varphi \in \Gamma(p)\}$$

and $J^0F \equiv F$ together with the maps

$$\begin{aligned} \pi_k\colon J^kF &\longrightarrow M, \quad \textbf{k-th Jet bundles} \\ j^k_p &\longmapsto p \\ \pi^l_k\colon J^kF &\longrightarrow J^lF, \quad 1 \leq l \leq k \\ j^k_p\varphi &\longmapsto j^l_p\varphi \end{aligned}$$

such that $\pi_k = \pi \circ \pi^0_k$, $\pi^l_k = \pi^l_m \circ \pi^m_k$ for $0 \leq m \leq l$. Moreover if φ is a section of our fibre bundle local in some $U \subseteq M$, we define the **Jet Prolongations**

$$\begin{aligned} j^k : \mathcal{F}_pU &\longrightarrow J^kF, \\ (\varphi, p) &\longmapsto j^k(\varphi)(p) := j^k_p(\varphi) \end{aligned}$$

such that the following diagram commutes for $i \geq k \geq l$:

$$\begin{tikzcd}
\mathcal{F}_p M \arrow[r, "j^k"] \arrow[drr, "j^i"'] & J^k F \arrow[rr, "\pi^l_k"] & & J^l F \arrow[r, "\pi_l"] & M \\
& & J^i F \arrow[ul, "\pi^k_i"] \arrow[ur, "\pi^l_i"'] \arrow[urr, "\pi_i"'] & &
\end{tikzcd}$$

The next proposition will go without its proof which can be found in (Saunders, Ch6).

Proposition 2.6 There exists a sequence of smooth fibre bundles

$$\ldots \longrightarrow J^k F \xrightarrow{\pi^{k-1}_k} J^{k-1} F \longrightarrow \ldots \longrightarrow F \longrightarrow M$$

for every k. Furthermore that maps π^{k-1}_k are *surjective* with *surjective tangent map* (sumbersions).

Now if $(U, \mathcal{U})$ is an adapted coordinate system for F such that $(x^i, u^\alpha) \equiv u$ and thus

$$\Rightarrow \qquad \begin{aligned} U^k &= \{j^k_p\varphi \colon \varphi(p) \in U\} \\ u^k &= (x^i, u^\alpha, u^\alpha_I) \\ u^\alpha_I(j^k_p\varphi) &= \frac{\partial^{|I|}(u^\alpha \circ \varphi)}{\partial x^I}\bigg|_p \end{aligned}$$

Now to make precise the statement "for all k" in the above proposition, we need to introduce some extra structures:

Definition 2.7 (Inverse/Projective systems) Define $Sys_x \colon = (\{x_i\}, f_{ij} | i, j \in I \subset \mathbb{N})$ with

- $\{x_i\}$ a collection of spaces (generalizes to objects in a category)
- $f_{ij} : x_i \longrightarrow x_j$ for all i, j s.t. $i \leq j$ and

$$f_{ik} = f_{ij} \circ f_{jk}$$

We call this construct an **inverse system** or **projective system**.

Now we denote by $\varprojlim x_i$ the subset of $\Pi_i x_i$ of elements $x \in \{x_i\}$ such that $x_i = f_{ij}(x_j) \quad \forall j \geq i$. We call $\varprojlim$ the **projective/inverse** limit of the inverse system.

Definition 2.8 The sequence $\{J^k F\}_{n \in \mathbb{N}}$ together with $\pi^k_l : J^l F \longrightarrow J^k F$ defines an inverse system (of fibre bundles). We thus use the inverse limit to define $J^\infty F := \varprojlim J^k F$.

To be precise: $J^\infty F$ is the space of equivalence classes of sections $\varphi : M \longrightarrow F$ such that two sections s_1 and s_2 are equivalent, if their partial derivatives agree at all orders. We denote these eqivalence classes by $j^\infty(\varphi)$. As an exercise you can show that the germs of functions surject over

$J^\infty \mathcal{R}$ where $\mathcal{R} = M \times \mathbb{R} \longrightarrow R$. You can also find a counterexample for the opposite statement.

Now the interesting question for our physical analysis will be if $J^\infty F$ can be given a smooth manifold structure. It will turn out to be very convenient to consider smooth functions on $J^\infty F$ first:

Definition 2.9 Let $\pi : F \longrightarrow M$ be a fibre budle and let $J^k F$ denote the space of k-jets. Since we are dealing with regular finite dimensional manifolds, we can consider $\mathcal{C}^\infty(J^k F, P)$ for some manifold P. For every $l \geq k$ there are connecting maps

$$\widetilde{\pi}^l_k : \mathcal{C}^\infty(J^k(E), P) \longrightarrow \mathcal{C}^\infty(J^l(E), P)$$

which can be constructed using precomposition, thus $(\pi^k_l)^* f := \widetilde{\pi}^l_k(f)$. Thus we obtain a so-called **direct system**

$$\mathcal{C}^\infty(F) \longrightarrow \mathcal{C}^\infty(J^1 F) \longrightarrow \mathcal{C}^\infty(J^2 F) \longrightarrow ...$$

with functions $f_{ij} : x_i \longrightarrow x_j$ for $i \leq j$. We further define the **direct limit**

$$\mathcal{C}^\infty(J^\infty F) := \varinjlim \mathcal{C}^\infty(J^k F) = \Pi_k \mathcal{C}^\infty(J^k F)/ \sim$$

where $g_i \sim g_l$ iff $\exists k \geq i, k \geq l$ s.t. $\widetilde{\pi}^k_i g_i = \widetilde{\pi}^k_l g_l$.

Note that by construction $f \in \mathcal{C}^\infty(J^\infty F)$ is fully represented by functions $\widehat{f_k} \in \mathcal{C}^\infty(J^k F)$ which only depend on a finite number of derivatives. This will be the essence of the notion of locality which we will unfold in the following pages. Thus if we consider $f \in \mathcal{C}^\infty(J^\infty F)$ represented by $\widehat{f}$ one some k-jet then on each coordinate neighbourhood $(\pi^\infty)^{-1}(U)$ and each point $\sigma = j^\infty(\varphi)(p) \in (\pi^\infty)^{-1}(U)$ we have

$$f(\sigma) = \widehat{f}(x^i, u^\alpha, u^\alpha_{i_1}, u^\alpha_{i_1 i_2}, ..., u^\alpha_{i_1 ... i_k})$$

From now on, we will, in a slight abuse of notation, not distinguish between functions on $J^\infty F$ and their representatives.

2.2 Fréchet Manifolds

Since field theory inherently works with infinite dimensional manifolds, we have a real need for a respective mathematical theory. We saw that if we look at such infinite dimensional manifolds in a local setting, they are "tame" in that we can consider many of their concepts using finite dimensional structures. This will bring us to the notion of a *Fréchet manifold*. While one can approach this concept from a very categorial direction, we will give a less abstract introduction. Note that this part will in no way be exhaustive and not even be enough to understand everything we will use it for later on. Its goal is rather to give a feeling for the formal background which is completely omitted in many introductions to the field due to its depth and level of abstraction.

Definition 2.10 (Seminorm) Let V be a vector space. A **seminorm** is a map $|\cdot| : V \longrightarrow \mathbb{R}$ that satisfies

1. $|v| \geq 0 \quad \forall v \in V$
2. $|v + w| \leq |v| + |w| \quad \forall v, w \in V$
3. $|a \cdot v| = |a| \cdot |v| \quad \forall v \in V, a \in \mathbb{R}$

A **family of seminorms** (for our purposes) is a set $\{|\cdot|_i\}_{i\in I}$ with $|\cdot|_i$ a seminorm for each $i \in I$.

Definition 2.11 (Locally convex topological vector spaces) A **locally convex topological vector space** is a vector space V together with a family of seminorms Γ. We denote a locally convex topological vector space with such a family of seminorms by (V, Γ).

Using families of seminorms, we can assign a unique topology to the vector space they live on:

Definition 2.12 A family of seminorms Γ on a vector space V defines a **unique topology** $\mathcal{T}_\Gamma$ compatible with the vector space structure. The neighbourhood base of $\mathcal{T}_\Gamma$ is given by the family

$$B_\Gamma := \{U^J_\varepsilon | \varepsilon > 0,\ J \subset I \text{ finite}\}$$

with $U^J_\varepsilon := \{v \in V \big| |v|_i < \varepsilon \ \forall i \in J\}$

Using the above notions, we arrive at a proposition that sheds further light onto such spaces.

Proposition 2.13 Let (V, Γ) be a locally convex topological vector space. Then the following statements hold:

1. $\mathcal{T}_\Gamma$ is the finest topology in which all included seminorms are continuous.
2. (V, Γ) is Hausdorff iff
$$v = 0 \quad \Leftrightarrow \quad (|v|_i = 0 \quad \forall i \in I)$$
3. If (V, Γ) is Hausdorff then it is metrizable iff the family Γ is countable (i.e. there exists a metric $d : V \times V \longrightarrow \mathbb{R}^+$ s.t. the topology induced by d is $\mathcal{T}_\Gamma$).
4. Convergence of sequences is controlled by the seminorms, i.e.:
$$(v_n)_{n\in\mathbb{N}} : v_n \longrightarrow v \quad \Leftrightarrow \quad |v_n - v|_i \longrightarrow 0 \ \ \forall i \in I$$
5. V is complete with respect to $\mathcal{T}_\Gamma$ iff every Cauchy sequence converges, i.e. iff every $(v_n)_{n\in\mathbb{N}}$ with $\lim\limits_{n,m\longrightarrow\infty} |v_n - v_m|_i = 0$ converges $\forall i \in I$.

Definition 2.14 (Fréchet spaces) A **Fréchet space** is a sequentially complete, Hausdorff, metrizable, locally convex vector space.

We give some popular examples to illuminate the above definition:

Example 2.15

- Every Banach space is a Fréchet space.
- $\mathbb{R}^\infty = \Pi_{n\in\mathbb{N}}\mathbb{R}^n$ with either the cartesian topology or the corresponding family of seminorms $\{p_n(x_1,...,x_n) = |x_1| + ... + |x_n|\}$ together with the metric $d(x,y) := \sum_i \frac{|x_i - y_i|}{2^i(1+|x_i-y_i|)}$ is a Fréchet space.
- The space of smooth sections on a vector bundle $V \longrightarrow M$ where (M,g) is a Riemannian manifold is Fréchet with $||f||_n = \sum_{i=0}^n \sup_x |\nabla^i f(x)|$ for $n \in \mathbb{N}$ with ∇ a covariant derivative and ∇^i denoting its i-th iteration.

Our next step is to generalize the notion of a derivative (differential) to infinite dimensional spaces. This will in turn enable us to define the idea of smoothness in this infinite dimensional setting.

Definition 2.16 (Gâteaux-differential) Let V, W be locally convex topological vector spaces, $U \subset V$ open and $F : V \longrightarrow W$ then $dF(u)(v)$, the **Gâteaux-differential** of F at $u \in U$ along $v \in V$, is defined as

$$dF(u)(v) := \lim_{\tau\to 0}\frac{F(u+\tau v) - F(u)}{\tau} = \frac{d}{d\tau}F(u+\tau v)\Big|_{\tau=0}$$

If the limit exists $\forall v \in V$, F is **Gâteaux differentiable** at $u \in U$.

Using this idea of a differential or derivative, we can define what it means for a homeomorphism to be smooth. Thus being able to use Fréchet spaces instead of $\mathbb{R}^n$ as our target space, we can define a corresponding manifold structure:

Definition 2.17 (Fréchet manifolds) A Hausdorff topological space M is a **Fréchet manifold** if it is equipped with an atlas of homeomorphisms to open sets U of a Fréchet space V such that the transition functions are smooth in the sense that the Gâteaux-derivations $D^{k+1}f : U \times ... \times U \longrightarrow V$ with

$$Df(u)v := \lim_{t\longrightarrow 0}\frac{f(u+tv) - f(u)}{t}$$

are continuous for all $k \in \mathbb{N}$.

Applying the above concept of infinite dimensional manifolds onto 2.7 leads us to the following idea:

Lemma 2.18 Inverse limits of normed spaces are Fréchet.

Proof. Consider the system $f_n^m : V_n \longrightarrow V_m$ with $(V_m, |\cdot|_n)$ a normed vector space $\forall n \in \mathbb{N}$. The inverse limit of the system, denoted by V, is endowed with the linear maps $f_\infty^n : V \longrightarrow V_n$. The norms $|\cdot|_n$ induce a family of seminorms on V via $||\cdot||_n := |\cdot|_n \circ f^n_\infty$, such that

1. $||v||_n = |f^n_\infty(v)|_n \geq 0 \quad \forall v \in V$
2. $||v+w||_n = ... \leq |v|_n + |w|_n$
3. $||av||_n = |f^n_\infty(a \cdot v)|_n = |a \cdot f^n_\infty(v)|_n = |a| \cdot ||v||_n$

As can be seen in *Schäfer A.2.3*, the space is also metrizable. Since

$$\{v = 0 \;\Leftrightarrow\; f^n_\infty(v) = 0 \;\forall n \in \mathbb{N} \;\Leftrightarrow\; |f^n_\infty(v)|_n = 0 \;\forall n \in \mathbb{N} \;\Leftrightarrow\; |v|_n = 0 \;\forall n \in \mathbb{N}\}$$

which proves that our space is indeed Hausdorff which proves the lemma. □

Now let $F \longrightarrow M$ be a fibre bundle with coordinate charts (U_a, u_a). Consider the induced cover $U^\infty = \{(\pi^0_\infty)^{-1}(U_a)\}$ and the induced maps $u^\infty_a : (\pi^0_\infty)^{-1}(U_a) \longrightarrow \mathbb{R}^\infty$. This defines an atlas as can be seen in *Saunders 7.2.4.*. We thus arrive at one of the marking stones of our infinite dimensional analysis:

Proposition 2.19 The infinite jet bundle $J^\infty F$ together with an atlas given by $(J^\infty F, \{u^\infty_a : (\pi^0_\infty)^{-1}(U_a)) \longrightarrow \mathbb{R}^\infty\}_{a\in A})$ is a Fréchet manifold.

Furthermore, we can state the following:

Proposition 2.20 Let $\pi : F \longrightarrow M$ be a smooth fibre bundle. For every $k \in \mathbb{N} \cup \{\infty\}$,

- $\pi^k_\infty : J^\infty F \longrightarrow J^k F$ is a smooth fibre bundle
- $\pi_\infty : J^\infty F \longrightarrow M$ is a smooth fibre bundle
- $j^\infty_\varphi : M \longrightarrow J^\infty F$ is smooth $\forall \varphi \in \mathcal{F}$

2.3 The Variational Bicomplex

Tangent Bundle: There are several alternative ways to progress further with differential geometry on Jet bundles:

- The tangent bundle $T_x J^\infty F$ at a point $x \in J^\infty F$ can be seen as the limit of the system $\{(T_{\pi^k_\infty(x)} J^k F, T\pi_k)\}$ such that the bundle $T(J^\infty F) = \bigcup_{x \in J^\infty F} T_x(J^\infty F)$ is modeled on the $T(J^k F)$ and the projection
$$pr_{J^\infty F} : T(J^\infty F) \longrightarrow J^\infty F$$
is represented by $\{pr^k_\infty = pr_{J^k F}\}$ where $pr_{J^k F} : T(J^k F) \longrightarrow J^k F$.
- Observe that if φ_t is a smooth 1-parameter family of sections, we can define $\dot\varphi_0 : M \longrightarrow TF$. But φ is a section such that $d\pi(\dot\varphi_0) = \frac{d}{dt}\big|_{t=0}(\pi(\varphi_t(x))) = 0_m$ and thus $\dot\varphi_0 \in \ker(d\pi)$. Hence

we can associate a tangent bundle to $J^\infty F$ in the context of Fréchet manifolds via

$$T(J^\infty F) := \mathcal{C}^\infty(\mathbb{R}, J^\infty F)/\sim$$

where $c \sim c'$ iff $c(0) = c'(0)$ and further

$$D(\varphi \circ c)(0,1) = D(\varphi \circ c')(0,1)$$

with φ any chart around $c(0) \in J^\infty F$ and D denoting the Gâteaux-derivative.

These two notions can be shown to be equivalent. Now in order to unambigously deal with vector fields and their relations to derivations of the respective algebra of functions, we still need to work a bit more. The following definitions can be carried over from the finite-dimensional setting:

Definition 2.21 (Complex of differential forms) Let $F \longrightarrow M$ be a smooth fibre bundle. We define the **complex of differential forms** $\Omega^\bullet(J^\infty F)$ to be the direct limit of the sequence

$$\Omega^\bullet(F) = \Omega^\bullet(J^0 F) \longrightarrow \Omega^\bullet(J^1 F) \longrightarrow ...$$

together with the morphism $d : \Omega^\bullet(J^\infty F) \longrightarrow \Omega^\bullet(J^\infty F)$ given by the collection $\{d^k : \Omega^\bullet(J^k F) \longrightarrow \Omega^\bullet(J^k F)\}$ such that $d \circ d = 0$.

We can now endow $\Omega^\bullet(J^\infty F)$ with the morphism

$$\wedge : \Omega^\bullet(J^\infty F) \otimes \Omega^\bullet(J^\infty F) \longrightarrow \Omega^\bullet(J^\infty F)$$

given by the collection of morphisms

$$\{\wedge_{k,l} : \Omega^\bullet(J^k F) \otimes \Omega^\bullet(J^l F) \longrightarrow \Omega^\bullet(J^{\max(k,l)} F)\}_{k,l}$$

This in turn makes the morphism d into a graded derivation of the wedge product. In the following we use the derivation to split the complex of differential forms into two parts. To this end, we first define certain derived differentials using d:

Definition 2.22 (Horizontal differential) Consider a local chart on $J^k F$ given by (x^i, u^α_I) for I running over all multiindices of length lesser than k. The concrete form of the maps is given by

$$x^i(j^k(\varphi, p)) := x^i(p)$$
$$u^\alpha_I(j^k(\varphi, p)) := \left.\frac{\partial^{|I|}(u^\alpha \circ \varphi)}{\partial x^I}\right|_p$$

Both are clearly smooth and uniquely determine the equivalence class $j^k(\varphi, p)$. We now define the

horizontal differential for any $f \in \mathcal{C}^\infty(J^\infty F)$ to be

$$d_H f := \frac{\partial f}{\partial x^i} dx^i + \frac{i_1!...i_m!}{k!} \frac{\partial f}{\partial u_I^\alpha} u_{I,i}^\alpha dx^i$$

where $k = |I|$, i_l denotes the number of occurences of l in I, $m = \dim(M)$ and $u_{I,i}^\alpha = \partial_{x^i} u_I^\alpha$.

Now for convenience, denote

$$\partial_\alpha^I := \frac{i_1!...i_m!}{k!} \frac{\partial}{\partial u_I^\alpha}$$
$$D_i := \frac{\partial}{\partial x^i} + u_{I,i}^\alpha + \partial_\alpha^I$$

We can thus denote $d_H f = D_i f dx^i$ and further define the **vertical differential**

$$d_V f = (d - d_H) f = \partial_\alpha^I f (d - d_H) u_I^\alpha = \partial_\alpha^I f d_V u_I^\alpha$$

We can also prove

$$(D_j f)(j^\infty \varphi) = \frac{\partial}{\partial x^i}(f(j^\infty \varphi))$$
$$d_H x^i = dx^i, \qquad d_H(u_I^\alpha) = u_{I,i}^\alpha dx^i$$

We follow our successful splitting of the operator d with a collection of useful definitions that will in turn lead us to a splitting of the complex $(\Omega^\bullet(J^\infty F), d)$:

Definition 2.23 (Vertical vector fields) Let $\pi_\infty : J^\infty F \longrightarrow M$ be the fibre bundle of infinite jets. We define the subbundle of **vertical vector fields** to be

$$V(J^\infty F) := \ker(T\pi_\infty), \text{ i.e. for } \chi \in T(J^\infty F)$$
$$V_\chi(J^\infty F) := \{x_\chi \in T_x(J^\infty F) \ | \ T\pi_k(T\pi_\infty^k x_\chi) = 0 \ \forall k\}$$

Using vertical vector fields, we can now define certain types of differential form that are classified by their behaviour when acting on vertical vector fields.

Definition 2.24

- The set of **horizontal** (p,s)**-forms** is defined as

$$\Omega_H^{(p,s)} := \{\omega \in \Omega^p(J^\infty F) : \omega_\chi(x_1, ..., x_{p-s+1}, \cdot, ..., \cdot) = 0 \ | \ x_i \in V_\chi(J^\infty F)\}$$

- A **contact form** is a differential form θ on $J^\infty F$ that is annihilated by all jets of the form $j^\infty \varphi : M \longrightarrow J^\infty F$ via pullback, i.e. $(j^\infty \varphi)^* \theta = 0$

- The set of **vertical** (p, r)**-forms** are defined as

$$\Omega_V^{(p,r)} := \{\omega \in \Omega^p(J^\infty F) : \theta_1 \wedge \theta_2 \wedge ... \wedge \theta_r \wedge \widetilde{\omega} \ \mid \ \theta_i \text{ a contact form}\}$$

For contact forms, we can attain a helpful extra property as well as a basis forming the space of contact forms:

Theorem 2.25 Contact forms generate a differential ideal denoted by $\mathcal{C}$. Its basis is given by $\theta_I^\alpha = du_I^\alpha - u_{I,i}^\alpha dx^i$.

Proof. Let θ be a contact form, thus $(j^\infty\varphi)^*\theta = 0$ for all $\varphi \in \mathcal{F}$. Then for all $\alpha \in \Omega^\bullet(J^\infty F)$ we have

$$\begin{aligned} &(j^\infty\varphi)^*(\theta \wedge \alpha) = (j^\infty\varphi)^*\theta \wedge (j^\infty\varphi)^*\alpha = 0 \\ \Rightarrow \quad &(j^\infty\varphi)^* d(\theta \wedge \alpha) = d(j^\infty\varphi)^*(\theta \wedge \alpha) = 0 \end{aligned}$$

Turning to the basis, we see that

$$\begin{aligned} (j^\infty\varphi)^* d_V u_I^\alpha &= (j^\infty\varphi)^*(du_I^\alpha - u_{I,i}^\alpha dx^i) \\ &= \frac{\partial(u_I^\alpha \circ \varphi)}{\partial x^i} dx^i - (u_{I,i}^\alpha \circ \varphi) dx^i \end{aligned}$$

Thus we have a basis of the given form, since locally any form θ splits to $\theta = d_H f + d_V f$. □

We can now use the ideal of contact forms to investigate local sections of the bundle π_∞:

Lemma 2.26 Let $F \longrightarrow M$ be a smooth fibre bundle. A local section ξ of $\pi_\infty : J^\infty F \longrightarrow M$ is holonomic, i.e. a ∞-prolongation of a section of F, if and only if $\xi^*\mathcal{C} = 0$.

Proof. Note that

$$\xi^* d_V u_I^\alpha = 0 \quad \text{iff} \quad \frac{\partial}{\partial x^i}(u_I^\alpha \circ \xi) = u_{I,i}^\alpha \circ \xi$$

for all indices I, i, α such that ξ can be constructed inductively from $u_i \circ \xi$. Thus define $\varphi := \pi_\infty^0 \circ \xi$ we see that $\xi = j^\infty\varphi$. □

Using the thus defined structures, we can approach the spliting of the complex $(\Omega^\bullet(J^\infty F), d)$ by defining forms of bi-degree:

Definition 2.27 (Forms of bi-degree) Let $F \longrightarrow M$ and $J^\infty F$ be as above. Define the space of **forms of bi-degree** (r, s) to be the intersection

$$\Omega^{r,s}(J^\infty F) := \Omega_V^{p,r}(J^\infty F) \cap \Omega_H^{s,s}(J^\infty F)$$

where $p := r + s$. A form $\omega \in \Omega^\bullet(J^\infty F)$ is of bi-degree (r, s) iff it is of the form

$$\omega^{I_1 \ldots I_r}_{\alpha_1 \ldots \alpha_r; i_1 \ldots i_s} \theta^{\alpha_1}_{I_1} \wedge \ldots \wedge \theta^{\alpha_r}_{I_r} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_s}$$

where all the functions $\omega^{(\cdot)}_{(\cdot)}$ are smooth in $\mathcal{C}^\infty(J^\infty F)$.

We are now in a position to "split" $(\Omega^\bullet(J^\infty F), d)$. First we define

$$\Omega^p(J^\infty F) := \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty F)$$

Since $d(dx^i) = 0$ we also have $d_H(dx^i) = 0$ and $d_V(dx^i) = 0$. Now we can also use $\theta^\alpha_I = du^\alpha_I - u^\alpha_{I,i} dx^j$ such that

$$d\theta^\alpha_I = -du^\alpha_{I,i} \wedge dx^i = -\theta^\alpha_{I,j} \wedge dx^j$$

and thus $d_V \theta^\alpha_I = 0$ and $d_H \theta^\alpha_I = -\theta^\alpha_{I,j} \wedge dx^j$. This brings us to the following theorem, which is the defining theorem for the *variational bicomplex*:

Theorem 2.28 (Variational bicomplex) The complex of differential forms on the infinite jet bundle $(\Omega^\bullet(J^\infty F), d)$ splits into the bicomplex $(\Omega^{\bullet,\bullet}(J^\infty F), d_H, d_V)$, i.e. $d_H^2 = d_V^2 = 0$ and further $d_H \circ d_V = -d_V \circ d_H$. We call this complex the **variational bicomplex**.

2.4 Local Lagrangian Field Theory

So far, we have been working on the infinite jet bundle. Our goal is to work on $\mathcal{F}$ in a *local* way. Thus we need to find some local calculus on $\mathcal{F} \times M$.

To this end, denote by $VF := \ker(d\pi)$ the vertical tangent bundle. Recall that if φ_t is a smooth curve in $\mathcal{F} = \Gamma(M, F)$, $\dot{\varphi}\big|_{t=0}$ is vertical and

$$0 = d\pi(\dot{\varphi}\big|_{t=0}) = \frac{d}{dt}(\pi(\varphi_t))\big|_{t=0}$$

Thus we can think of $\dot{\varphi}_0 : M \longrightarrow VF$ as a section covering $\varphi_0 = pr_F \circ \dot{\varphi}_0$. We can express this in the following diagram:

$$\begin{tikzcd}
VF \arrow[r, "pr_{\mathcal{F}}"] & F \arrow[dr, "\pi"] & \\
M \arrow[u, "\dot{\varphi}_0"] \arrow[r, "\mathrm{id}"] & M \arrow[u, "\varphi_0"] \arrow[r, "\mathrm{id}"] & M
\end{tikzcd}$$

Now we can think of sections $\Gamma(M, VF)_\varphi = \Gamma(M, \varphi^* VF)$. An element of $\Gamma(M, VF)_\varphi$ is a map that associates to every $x \in M$ a vector $v_{\varphi(x)} \in V_{\varphi(x)}F$. We can think of $v_\varphi \in \Gamma(M, \varphi^* VF)$.

Definition 2.29 We define the tangent bundles

$$T\mathcal{F} := \Gamma(M, VF), \qquad T(\mathcal{F} \times M) := \Gamma(M, VF) \times TM,$$
$$pr_{\mathcal{F}} \colon T\mathcal{F} \longrightarrow \mathcal{F} \qquad pr_{\mathcal{F}} \times pr_M \colon T(\mathcal{F} \times M) \longrightarrow \mathcal{F} \times M$$

An element of $T_\varphi \mathcal{F}$ is often called an **infinitesimal variation**.

Now let $E \longrightarrow M$ and $F \longrightarrow M$ be two fibre bundles with projections π_E and π_F respectively. Note that the following diagram commutes and thus defines the *product bundle*:

$$E \times F := \{(e, f) \in E \times F \big| \pi_E(e) = \pi_F(f)\}$$

such that the following diagram commutes:

$$\begin{tikzcd}
E \times_M F \arrow[r, "pr_F"] \arrow[d, "pr_E"'] & F \arrow[d, "\pi_F"] \\
E \arrow[r, "\pi_E"] & M
\end{tikzcd}$$

In general, this allows us to write $\Gamma(M, E \times_M F) \cong \mathcal{E} \times \mathcal{F}$ where we identify $\mathcal{E} = \Gamma(M, E)$ and $\mathcal{F} = \Gamma(M, F)$. We can also write

$$T(\mathcal{E} \times \mathcal{F}) \cong \Gamma(M, V(E \times_M F)) \cong \Gamma(M, VE \times_M VF) \cong T\mathcal{E} \times T\mathcal{F} \cong (T\mathcal{E} \times \mathcal{F}) \times_{\mathcal{E} \times \mathcal{F}} (\mathcal{E} \times T\mathcal{F})$$

As an exercise, you can prove the above chain of isomorphisms. Since we can further write $T(\mathcal{F} \times M) \cong (T\mathcal{F} \times M) \times_{\mathcal{F}\times M} (\mathcal{F} \times TM)$ we can for the above statements take a pair (φ, x) and denote

$$T_{(\varphi,x)}(\mathcal{F}_x M) \cong T_\varphi \mathcal{F} \times T_x M$$

We denote by $T_\varphi \mathcal{F}$ the **vertical tangent space** and by $T_x M$ the **horizontal tangent space**.

Taking the ∞-prolongation j^∞ and its tangent map we obtain the following commuting diagram:

$$\begin{tikzcd}
T(\mathcal{F} \times M) \arrow[r, "Tj^\infty"] \arrow[d, "pr_{\mathcal{F}} \times pr_M"'] & TJ^\infty F \arrow[d, "pr_{J^\infty F}"] \\
\mathcal{F} \times M \arrow[r, "j^\infty"] & J^\infty F
\end{tikzcd}$$

This leads us to the following interesting piece of information:

Proposition 2.30 The tangent map

$$T_{(\varphi,X)} j^k \colon T_\varphi \mathcal{F} \times T_x M \longrightarrow T_{j^k_x \varphi} J^k F$$

is given by

$$(T_{(\varphi,x)} j^k)(\xi_\varphi, v_x) := \sum_{|I|=0}^{k} \dot{u}^\alpha_I (j^k_x \xi_\varphi) \frac{\partial}{\partial u^\alpha_I} + v^i \left(\frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k} u^\alpha_{I,i} (j^{k+1}_x \varphi) \frac{\partial}{\partial u^\alpha_I} \right)$$

Proof. While the proof is left as an exercise to the reader, we give the following hint: To compute tangent maps in coordinates, compute the time derivative of a path $t \longmapsto (\varphi_t, x(t))$. For example

$$\frac{d}{dt} \left(u^\alpha_I (j^k(\varphi_t, x(t))) \right) \Big|_{t=0} = \frac{d}{dt} \left(\frac{\partial^{|I|}}{\partial x^I} \varphi^\alpha(t, x(t)) \right) \Bigg|_{t=0}$$

Now set $\xi_\varphi = \dot{\varphi}_0$ and $v_x = \dot{x}(0)$. □

***Remark* 2.31** The above proposition induces the following maps:

$$\begin{aligned} \tau_k \colon J^k(VF) &\longrightarrow TJ^k(F), \\ j^k_x \dot{\varphi}(0) &\longmapsto \frac{d}{dt}(j^k_x \varphi_t)_{t=0} \end{aligned} \qquad \begin{aligned} \sigma_k \colon J^{k+1}F \times TM &\longrightarrow TJ^k F, \\ (j^{k+1}_x \varphi, \dot{x}(0)) &\longmapsto \frac{d}{dt}(j^k_{x(t)} \varphi)_{t=0} \end{aligned}$$

Now using our results for $TJ^k F$ we obtain a splitting for $TJ^\infty F$:

Theorem 2.32 There exists a splitting of $TJ^\infty F$ that takes the form of

$$\begin{tikzcd}
(T\mathcal{F} \times M) \times_{\mathcal{F}\times M} (\mathcal{F} \times TM) \arrow[r, "\cong"] \arrow[d, "j^\infty_{T\mathcal{F}} \times (j^\infty_{\mathcal{F}} \times \mathrm{id})"'] & T(\mathcal{F} \times M) \arrow[d, "Tj^\infty"] \\
J^\infty(VF) \times_{J^\infty F} (J^\infty F \times_M TM) \arrow[r, "\cong"] & TJ^\infty F
\end{tikzcd}$$

One can read this diagram as the splitting into

$$\begin{aligned} J^\infty(VF) &\longrightarrow TJ^\infty F \quad \textbf{vertical tangent bundle} \\ J^\infty F \times_M TM &\longrightarrow TJ^\infty F \quad \textbf{horizontal tangent bundle} \end{aligned}$$

Corollary 2.33 The vector space of vector fields on $J^\infty F$ decomposes via $\mathfrak{X}(J^\infty F) = \mathfrak{X}_{vert} \oplus \mathfrak{X}_{hor}$ where

$$\begin{aligned} \mathfrak{X}_{vert} &\cong \Gamma(J^\infty F, J^\infty(VF)) \\ \mathfrak{X}_{hor} &\cong \mathrm{Hom}(J^\infty F, TM) \end{aligned}$$

Thus we can split any $V \in \mathfrak{X}(J^\infty F)$ into

$$V = V^i \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{\infty} V_I^\alpha \frac{\partial}{\partial u_I^\alpha}$$

where both V^i and V_I^α are in $\mathcal{C}^\infty(J^\infty F)$.

Now we can finally unambigously answer to the question *"What is Locality?"*. Essentially, locality is the requirement that we work with $(j^\infty)^*\Omega^{\bullet,\bullet}(J^\infty F)$.

***Remark* 2.34** Note that we always assume that our base manifold M is orientable. There exist generalizations for non-orientable manifolds (see Deligne/Freed).

Definition 2.35 (Local differential forms) A differential form on $\mathcal{F}_x M$ is called a **local differential form**, iff it is the pullback of a form on $J^\infty F$ by the ∞-jet prolongation $j^\infty \colon \mathcal{F}_x M \longrightarrow J^\infty F$. We define the **bicomplex of local forms** as

$$(\Omega^{\bullet,\bullet}_{loc}(\mathcal{F} \times M) := ((j^\infty)^*\Omega^{\bullet,\bullet}(J^\infty F), d, \delta)$$

such that

$$\delta((j^\infty)^*\alpha) := (j^\infty)^* d_V \alpha, \qquad d((j^\infty)^*\alpha) := (j^\infty)^* d_H \alpha$$

We can also extend this *local* setting to vector fields:

Definition 2.36 (Local vector fields) A **local vector field** χ on $\mathcal{F} \times M$ is a section of the tangent bundle $T(\mathcal{F}\times M)$ covered by a section $\chi^\infty \colon J^\infty F \longrightarrow TJ^\infty F$ and supported on $j^\infty(\mathcal{F}\times M)$.

In other words, the following diagram commutes:

$$\begin{tikzcd} T(\mathcal{F}\times M) \arrow[r, "Tj^\infty_F"] \arrow[d, "pr"] & TJ^\infty F \arrow[d, "pr"'] \\ \mathcal{F}\times M \arrow[r, "j^\infty_F"] \arrow[u, bend left, "\chi"] & J^\infty M \arrow[u, bend right, "\chi^\infty"'] \end{tikzcd}$$

Practically speaking, a vector field is **local**, if for any φ there exists an integer k such that the value of $\chi_\varphi \in \Gamma(M, \varphi^* VF)$ at $x \in M$ depends only on the k-th jet of φ at x. We denote the set of local vector fields by $\mathfrak{X}_{loc}(\mathcal{F}\times M)$. A local vector field in $\mathfrak{X}(\mathcal{F})$ is called **evolutionary**.

Notice that by definition, local functions $\mathcal{C}^\infty_{loc}(\mathcal{F}\times M)$ are represented by a pair (f, f^∞) meaning that they factor through some infinite jet as:

$$\begin{tikzcd} f\colon \mathcal{F}\times M \arrow[r] \arrow[d, "j^\infty"'] & \mathbb{R} \\ J^\infty F \arrow[ur, "f^\infty"'] & \end{tikzcd}$$

We also port the idea of a derivation to our newly defined setting by defining them on $\mathcal{C}^\infty_{loc}(\mathcal{F}\times M)$:

Definition 2.37 (Local derivations) A derivation D of $\mathcal{C}^\infty_{loc}(\mathcal{F}\times M)$ is called a **local derivation**, iff there exists a vector field $\chi^\infty\colon J^\infty F \longrightarrow TJ^\infty F$ such that

$$Df = (\chi^\infty f^\infty)\circ j^\infty$$

for every (f, f^∞). We denote a local derivation by (D, χ^∞).

A proposition coming directly from usual differential geometry is the correspondence between local vector fields and local derivations:

Proposition 2.38 Local vector fields are in 1:1 correspondence with local derivations of $\mathcal{C}^\infty_{loc}(\mathcal{F}\times M)$. Moreover they form a Lie subalgebra of $\mathrm{Der}(C^\infty_{loc})$.

Proof. The proof is left to the reader as it is fairly similar to the case in classical differential geometry. □

We go on with a short remark on local coordinates: If we identify

$$(j^\infty)^* x^i \equiv x^i, \qquad (j^\infty)^* u^\alpha_I \equiv u^\alpha_I$$

we can interpret $dx^i \in \Omega^{0,1}_{loc}(\mathcal{F}\times M)$ and $\delta u^\alpha_I \in \Omega^{1,0}_{loc}(\mathcal{F}\times M)$ as a basis for $\Omega^{\bullet,\bullet}_{loc}(\mathcal{F}\times M)$. This allows us to write a form $\omega \in \Omega^{p,q}_{loc}(\mathcal{F} \times M)$ as

$$\omega = \omega^{I_1 \dots I_p}_{\alpha_1 \dots \alpha_p i_1 \dots i_q} \delta u^{\alpha_1}_{I_1} \wedge \dots \wedge \delta u^{\alpha_p}_{I_p} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_q}$$

for $\omega^{I_1 \dots I_p}_{\alpha_1 \dots \alpha_p i_1 \dots i_q} \in \mathcal{C}^\infty_{loc}(\mathcal{F} \times M)$, $df = D_i f dx^i$ and $\delta f = \partial^I_\alpha f \delta u^\alpha_I$. Thus if we take $X \in \mathfrak{X}_{loc}(\mathcal{F})$ and $\omega \in \Omega^{\bullet,\bullet}_{loc}(\mathcal{F} \times M)$, we obtain a variational *Cartan Calculus*

$$\mathcal{L}_X \omega = \imath_X \circ \delta \omega + \delta \circ \imath_X \omega$$

This can be generalized to $\mathfrak{X}_{loc}(\mathcal{F} \times M)$.

2.5 Classical Field Theory

Definition 2.39 (Classical Lagrangian Field Theory) A **Classical Lagrangian Field Theory** is a triple (M, F, L) where $\pi : F \longrightarrow M$ is a smooth fibre bundle and $L \in \Omega^{0,top}_{loc}(\mathcal{F} \times M)$ called a **Lagrangian (density)**. We call the integral

$$S = \int_M L$$

the **action functional**.

This means

$$L = (j^\infty)^* L^\infty \quad \text{for} \quad L^\infty \in \Omega^{0,top}_{loc}(J^\infty F)$$

and thus

$$L^k \left(x^i, u^\alpha, u^\alpha_{i_1}, ..., u^\alpha_{i_1 ... i_n} \right) dx^1 ... dx^n$$

If we evaluate L at φ, we obtain

$$L(\varphi) = L \left(x^i, \varphi^\alpha, \frac{\partial \varphi^\alpha}{\partial x^i}, ..., \frac{\partial^k \varphi^\alpha}{\partial x^{i_1} ... \partial x^{i_n}} \right)$$

Definition 2.40 (Source forms) Let $\alpha \in \Omega^{1,top}_{loc}(\mathcal{F} \times M)$. α is called a **source form**, iff it only depends on the dx^i and δu^α, not the δu^α_I.

While this definition is more of a working version and does not really explain itself, the following theorem sheds some light onto the situation:

Theorem 2.41 Let (M, F, L) be a Lagrangian field theory. Then there exists a source form $EL \in \Omega^{1,top}_{loc}(\mathcal{F} \times M)$ and two "boundary" forms

$$\alpha \in \Omega^{1,top-1}_{loc}(\mathcal{F} \times M), \qquad \omega \in \Omega^{2,top-1}_{loc}(\mathcal{F} \times M)$$

such that

- $\delta L = EL - d\alpha$
- $\omega = \delta\alpha \quad \Rightarrow \quad \delta\omega = 0$
- $d\omega = -\delta EL$

This is the statement that by means of "integration by parts" we can write δL as

$$EL = E_\alpha \delta u^\alpha \wedge dx^1 \wedge ... \wedge dx^n$$

with $E_\alpha\colon J^kF \longrightarrow \mathbb{R}$ which defines the **Euler-Lagrange-equations** as

$$E_\alpha\left(x^i, \varphi^\beta, \frac{\partial\varphi^\beta}{\partial x^i}, ..., \frac{\partial^k\varphi^\beta}{\partial x^{i_1}...\partial x^{i_n}}\right)$$

To prove the above statement, we would need to reformulate integration by parts in "cohomological terms", thus in terms of operators on local forms, which does not fit into the scope of this course. Doing this, allows us to define the "Euler Operator" $\mathbb{E}$ such that $\mathbb{E}(L) \equiv EL$.

With the following statements, we introduce some notions necessary to make the concept *evolutionary vector fields* more rigid and well-defined:

Definition 2.42 (Strictly vertical/horizontal vector fields) A vector field $v \in \mathfrak{X}(J^\infty F)$ is called **strictly vertical** or **strictly horizontal**, iff $[\imath_v, d_V] = 0$ or $[\imath_v, d_H] = 0$ respectively.

We use this definition to formulate the following proposition:

Proposition 2.43 There are two commuting Cartan calculi: For ξ, ξ' strictly vertical and X, X' strictly horizontal we have:

$$[\imath_\xi, d_V] = \mathcal{L}_\xi, \qquad [\mathcal{L}_\xi, \imath_{\xi'}] = \imath_{[\xi,\xi']}, \qquad [\mathcal{L}_\xi, \mathcal{L}_{\xi'}] = \mathcal{L}_{[\xi,\xi']}$$
$$[d_V, d_V] = [\imath_\xi, \imath_{\xi'}] = [\mathcal{L}_\xi, d_V] = 0$$

The analogous equations for X, X' also hold.

Proof. Follows directly from representing them on finite jets together with the bicomplex structure and the definition of strictly vertical/horizontal vector fields 2.42. □

Lemma 2.44 A vector field $v \in \mathfrak{X}(J^\infty F)$ is strictly horizontal, iff $v = v^i D_i$ for $v^i \in \mathcal{C}^\infty(M)$.

Proof. First note that $[\imath_v, d_V] = 0$ iff it annihilates functions and 1-forms $\{d_H x^i, d_V u^\alpha_I\}$.

$$\begin{aligned}
\Rightarrow \quad & [i_v, d_V]f = i_v \frac{\partial f}{\partial u^\alpha_I} d_V u^\alpha_I = \frac{\partial f}{\partial u^\alpha_I}(v^\alpha_I - u^\alpha_{I,i} v^i) = 0 \\
\Leftrightarrow \quad & v^\alpha_I = u^\alpha_{I,i} v^i \\
\Leftrightarrow \quad & v = v^i \frac{\partial}{\partial x^i} + u^\alpha_{I,i} v^i \frac{\partial}{\partial u^\alpha_I}
\end{aligned}$$

Thus we can conclude by checking what happens on 1-forms. □

Lemma 2.45 $v \in \mathfrak{X}(J^\infty F)$ is strictly vertical iff it is of the form $v = (D_I v^\alpha)\frac{\partial}{\partial u^\alpha_I}$ for some $v^\alpha \in \mathcal{C}^\infty(J^\infty F)$.

Proof. The proof is left as an exercise to the reader. *Hint:* The proof can, as done in the previous proof, be concluded by checking the statement on the generators $\{x^i, u_I^\alpha\}$ and $\{dx^i, d_V u_I^\alpha\}$. $\square$

As mentioned above, the introduced terminology helps us better understand *evolutionary vector fields*. Namely if $\xi \in \mathfrak{X}_{loc}(\mathcal{F} \times M)$ it is called **evolutionary**, iff it covers a vector field $v \in \mathfrak{X}(J^\infty F)$ that is *strictly vertical*. This in turn allows us to start talking about symmetries leading us to Noethers first theorem:

Definition 2.46 (Local symmetries) An evolutionary vector field $\xi \in \mathfrak{X}_{loc}(\mathcal{F} \times M)$ is called a **local symmetry** of the Lagrangian field theory (M, F, L) iff $\exists A_\xi \in \Omega_{loc}^{0,top-1}(\mathcal{F} \times M)$ such that

$$\mathcal{L}_\xi L = \imath_\xi \delta L = dA_\xi$$

We denote a **symmetry** by the pair (ξ, A_ξ).

Showing the power of the presented formalism, we can now prove *Noether's first theorem*:

Theorem 2.47 (Noether's first) Let (M, F, L) be a Lagrangian field theory such that $\delta L = EL - d\alpha$ and let (ξ, A_ξ) be a local symmetry of the theory. Then

$$j := A_\xi - \imath_\xi \alpha \ \in \Omega_{loc}^{0,top-1}(\mathcal{F} \times M)$$

is called **on shell**, i.e.

$$dj = 0 \mod (EL)$$

Here, "on shell" means "on those configurations φ s.t. $EL_\varphi = 0$ that satisfy the Euler-Lagrange-equations".

Proof. To prove this, we can simply calculate:

$$\begin{aligned} dj &= d(A_\xi - \imath_\xi \alpha) = \mathcal{L}_\xi L - d \circ \imath_\xi \alpha = \mathcal{L}_\xi L + \imath_\xi \circ d\alpha \\ &= \imath_\xi(\delta L + d\alpha) = \imath_\xi EL = 0 \mod (EL) \end{aligned}$$

$\square$

Some terminology: j is usually called a **Noether current**, thus 2.47 is a conservation statement. One can introduce *Noether Charges* by integration of j over codimension 1 submanifolds (hypersurfaces).

If we allow dependency of currents on a vector bundle $W \longrightarrow M$ together with a linear map

$$\begin{aligned} \Gamma(M, W) &\longrightarrow \mathfrak{X}_W(\mathcal{F} \times M) \times \Omega_{loc}^{0,top-1}(\mathcal{F} \times M) \\ w &\longmapsto (\xi_w, A_{\xi_w}) \end{aligned}$$

such that

$$\mathcal{L}_{\xi_w} L = dA_{\xi_w} \quad \Rightarrow j_w = A_{\xi_w} - \imath_{\xi_w}\alpha$$

We move on to some examples of classical Lagrangian field theories:

Example 2.48 (**Classical Mechanics**) Let $F = \mathbb{R} \times M \longrightarrow \mathbb{R}$ such that $\mathcal{F} = \Gamma(\mathbb{R}, F)$ and take

$$L = \left(\frac{m}{2}\,\dot{q}_i\dot{q}^i - V(q)\right) dt$$

as a Lagrangian density. Note that L is defined on $J^1(M \times \mathbb{R}) \simeq TM \times \mathbb{R}$. We find the variation of L to be

$$\begin{aligned}\delta L &= (m\ \delta\dot{q}\ \dot{q} - \delta q \nabla V) \wedge dt \\ &= \frac{d}{dt}\,(m\ \delta q \cdot \dot{q}) \wedge dt - \delta q\,(m\ddot{q} + \nabla V) \wedge dt \\ &= d\,(m\ \delta q \cdot \dot{q}) - (m\ddot{q} + \nabla V)\,\delta q \wedge dt\end{aligned}$$

Thus we have $\alpha = m\dot{q}^i\delta q_i$ and $E_q = m\ddot{q} + \nabla V$. Now if we consider a time translation $t \longmapsto t + \varepsilon$ we find the induced evolutionary vector field $\xi = -\dot{q}^i \frac{\partial}{\partial q^i} - \ddot{q}^i \frac{\partial}{\partial q^i} \cdots$
Exercise: Show that j_ξ is an energy, i.e. $j = \frac{m}{2}\ \dot{q} \cdot \dot{q} + V(q)$.

2.6 Scalar Field Theory and Yang-Mills-Theory

Consider a smooth, compact manifold M equipped with a (pseudo-)Riemannian metric g. This metric defines a volume form on M given by $\mathrm{Vol}_g := \sqrt{|\det(g)|}dx^1 \wedge ... \wedge dx^n$.

Definition 2.49 (Hodge Duality) The **Hodge-Duality map** (Hodge-$*$) is the $\mathcal{C}^\infty(M)$-linear map

$$*\colon \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$$

uniquely determined by the equation

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle_g \mathrm{Vol}_g$$

The Hodge star acts on a local basis as

$$*(dx^1 \wedge ... \wedge dx^n) = \frac{\sqrt{|\det(g)|}}{(n-k)!} g^{i_1 j_1} ... g^{i_k j_k} \varepsilon_{j_1 ... j_k} dx^{j_k+1} ... dx^{j_n}$$

Using the Hodge star we can define the

Definition 2.50 (Codifferential) The **Codifferential** is the operator

$$\delta_g := \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$$

defined as

$$\delta_g := (-1)^{n(k-1)+1} \operatorname{sgn}(g) * d*$$

Using the codifferential, one can define the **Laplace-deRham operator**

$$\Delta_g := (d + \delta_g)^2 = d\delta_g + \delta_g d$$

Using the above definition, one can directly show some interesting properties of the codifferential:

Lemma 2.51 The following properties hold true:

- $* * \alpha = (-1)^{k(n-k)} \operatorname{sgn}(g)\alpha \quad \forall \alpha \in \Omega^k(M)$
- $*^{-1}\colon \eta \longmapsto (-1)^{k(n-k)} \operatorname{sgn}(g) * \eta$
- $(\delta_g)^2 = 0$

Definition 2.52 (Scalar Field Theory) Let (M, g) as above and $\mathcal{F} = \mathcal{C}^\infty(M)$ seen as the sections of a trivial $\mathbb{R}$-bundle. Then **Scalar field theory** on $\mathcal{F}$ is defined with the Lagrangian

$$L = \frac{1}{2} d\varphi \wedge *d\varphi \equiv g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \sqrt{|\det(g)|} d^n x$$

Exercise 2.53 Show that $\delta L = [\Delta_g \varphi]\delta\varphi \wedge \mathrm{Vol}_g + d(\delta\varphi \wedge *d\varphi)$.

Proof.

$$\begin{aligned}
\delta L &= \delta\left(\frac{1}{2}d\varphi \wedge *d\varphi\right) = \frac{1}{2}\left[\delta d\varphi \wedge *d\varphi + d\varphi \wedge \delta * d\varphi\right] \\
&= \frac{1}{2}\left[-d\delta\varphi \wedge *d\varphi - d\varphi \wedge *d\delta\varphi\right] = -\frac{1}{2}\left[d(\delta\varphi \wedge *d\varphi) - \delta\varphi \wedge d * d\varphi + d\varphi \wedge *d\delta\varphi\right] \\
&= -\frac{1}{2}\left[d(\delta\varphi \wedge *d\varphi) - \delta\varphi \wedge *\delta_g d\varphi + d\varphi \wedge *d\delta\varphi\right] \\
&= -\frac{1}{2}\left[d(\delta\varphi \wedge *d\varphi) - \delta\varphi \wedge *\delta_g d\varphi + d\delta\varphi \wedge *d\varphi\right] \\
&= -\frac{1}{2}\left[d(\delta\varphi \wedge *d\varphi) - \delta\varphi \wedge *\delta_g d\varphi + d(\delta\varphi \wedge *d\varphi) - \delta\varphi \wedge d * d\varphi\right] \\
&= -\frac{1}{2}\left[d(\delta\varphi \wedge *d\varphi) - \delta\varphi \wedge *\delta_g d\varphi + d(\delta\varphi \wedge *d\varphi) - \delta\varphi \wedge *\delta_g d\varphi\right] \\
&= -d(\delta\varphi \wedge *d\varphi) + \delta\varphi \wedge *\delta_g d\varphi = -d(\delta\varphi \wedge *d\varphi) + \delta\varphi \wedge (\Delta_g\varphi \cdot \mathrm{Vol}_g)
\end{aligned}$$

The last equation holds since $\delta_g\varphi = 0$ and thus $\delta_g d\varphi = (\delta_g d + d\delta_g)\varphi$. □

The equations of motion for scalar field theory are

$$\Delta_g \varphi = 0$$

***Remark* 2.54** One can add "potential terms" to the Lagrangian. A common example would be $V(\varphi) = \frac{m^2}{2}\varphi^2 + \frac{\beta}{4}\varphi^4$. Note furter that our theory *requires* a metric g on M and is thus a "Non-topological field theory".

Definition 2.55 (Pure, d-dimensional Yang-Mills-Theory) Let (M, g) be as above and $P \longrightarrow M$ a principal G-bundle where G is a compact Lie group ($U(n)$). Denote by A a connection 1-form and let F_A denote its curvature. In a local chart we can write it as

$$F_A = dA + \frac{1}{2}[A, A]$$

Assume for simplicity that $P \longrightarrow M$ is a trivial bundle and $A \in \Omega^1(P, y)$ descends to a form on M. Then a **Pure Yang-Mills-Theory** is the Lagrangian field theory given by $\mathcal{F} = \mathcal{A}$ (space of principal connections on the principal bundle) and the Lagrangian (density)

$$L_{YM} = \frac{1}{2}\,\mathrm{Tr}[F_A \wedge *F_A]$$

Exercise 2.56 As an exercise, you can show that the variation of L_{YM} is given by

$$\delta L_{YM} = \mathrm{Tr}[d_A * F_A \wedge \delta A + d(\delta A * F_A)]$$

Remember that $d_A\varphi = d\varphi + A\varphi$ for $\varphi \in \mathcal{C}^\infty(M, R)$ and R a representation of y. For any $\beta \in \Omega^1(M, y)$ one has $d_A\beta = d\beta + [A, \beta]$ where the latter term denotes the adjoint action/representation.

Definition 2.57 (Yang-Mills-Theory with scalar "matter") Working with the same setting as usual Yang-Mills-Theory, one takes the Lagrangian (density)

$$L_{YM}^{scalar} = \mathrm{Tr}[F_A \wedge *F_A] + \langle d_A\varphi, *d_A\varphi\rangle_R$$

for $\varphi \in \mathcal{C}^\infty(M, R)$ where R is a representation that comes with an inner product $\langle\cdot,\cdot\rangle_R$. One can then add

$$+\text{mass terms } \frac{m^2}{2}\langle\varphi,\varphi\rangle_R + \beta\varphi^4$$

to form the "Higgs-mechanism" Lagrangian that appears in the *Standard Modell of Particle Physics*. Here, φ plays the role of the "Higgs" field.

Due to its convenience in upcoming applications in Yang-Mills-Theory, we reformulate some parts of our setting. This induces the *First order formulation of YM-theory*. First note that

$$L_{YM} = \frac{1}{2}\mathrm{Tr}[F_A \wedge *F_A]$$

The trick is to introduce a field $B := *F_A$ allowing for the following desription:

$$\begin{aligned} \mathcal{F}^{1st} &:= \mathcal{A}_P \times \Omega^{n-2}(M, y) \\ L_{YM}^{1st} &:= \mathrm{Tr}[B \wedge F_A - \frac{1}{2}B \wedge *B] \end{aligned}$$

This results in the following variation

$$\delta L_{YM}^{1st} = (F_A - *B)\delta B + d_A B\delta A + d(B \wedge \delta A)$$

Further note that $F_A = *B$ together with $d_A B = 0$ is equivalent to $d_A * F_A = 0$. First and second order formulations are **classically equivalent**. This concludes our introduction to classical field theory.

3 Quantisation and Normalization Group Flow

This section will treat *scalar perturbative quantum field theory*. We will start by giving a general introduction to the matter and then talk about the *Renormalization group equation*. After introducting *Feynman graphs*, we will finally arrive at the first full definition of a *QFT* and discuss some immediate results. Finally we will discuss the *renormalization of QFT*.

3.1 Introduction

The main idea of Quantum mechanics is to define a deterministic evolution of a system as a superposition of possible evolutions where one weights them by how likely they are. Our ultimate goal is to describe Quantum mechanics and Classical mechanics in terms of the sets of objects, states and observables. The main idea is to take a state ω and an observable A such that the state associates a propability distribution to the observable on the real line in the form of $\omega_A(\lambda)$. To illuminate the concepts and systems we aim to unify, we present a short overview of both:

In Classical mechanics, states are nothing but normalized measures on the *phase space* (mind: a symplectic manifold). Meanwhile observables are functions on the phase space given by (p, q) such that $\omega = dp \wedge dq$. The propability distribution is then given by

$$\omega_A(\lambda) := \int_M \theta(\lambda - f_A(p, q)) d\mu_\omega$$

where θ denotes the step function, f_A an observable and $d\mu_\omega$ a measure corresponding to the state. The dynamics on the other hand are encoded in the *Poisson brackets* defined using the symplectic form ω as

$$\frac{df}{dt} = \{H, f\}$$

for some functions H which turns out to be the *Hamiltonian* of the system.

In Quantum mechanics, observables are linear operators on some complex *Hilbert spae* $\mathcal{H}$. States are positive trace-class operators with unit trace defined as $\operatorname{Tr} A = \sum_n \langle Ae_n, e_n\rangle$. The propability distribution for a state is defined by

$$\omega_A(\lambda) := \operatorname{Tr}(MP_A(\lambda))$$

where M is a trace-class (state) and $P_A(\lambda)$ is a projection (observable). Note that we are looking at the field from a mathematical perspective, thus the reader might encounter unfamiliar technical details in this and upcoming discussions.
The commutator of two states A, B is defined as

$$\{A, B\} = \frac{i}{\hbar}(AB - BA)$$

such that the time evolution, given the Hamiltonian operator H, is defined as

$$\frac{dA}{dt} = \{H, A\}$$

With these classifications of the two fields in mind, we arrive at a first sketch of a transition between them:

Definition 3.1 (Quantisation Scheme) A map passing from Classical mechanics to Quantum mechanics is called a **quantisation scheme**.

To illuminate this rather vague definition, we discuss a "toy example":

Example 3.2 (Weyl Quantisation) This quantisation treats a mechanical system with one degree of freedom. The classical system has $\mathbb{R}^2$ as its phase space with coordinates (p, q). The dynamics are given by the Poisson-brackets with the Hamiltonian function. Now we want to transition to a quantum description:

We take the Hilbert-space $\mathcal{H}$ defined as

$$\mathcal{H} = L^2(\mathbb{R}) := \left\{ \psi(q) \in C(\mathbb{R}) \;\Big|\; \int_{-\infty}^{\infty} |\psi(q)|^2 dq < \infty \right\}$$

Observables are now operators on $\mathcal{H}$. We define the two operators

$$(P\psi)(q) = \frac{\hbar}{i} \frac{d}{dq} \psi(q), \qquad (Q\psi)(q) = q\psi(q)$$

Now we can define products

$$(A\psi)(q) = \int_{-\infty}^{\infty} A(q, q')\psi(q')dq'$$

where $A(q, q')$ denotes the integral kernel of A defined as

$$A(q, q') = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} f\left(p, \frac{q + q'}{2}\right) e^{ip(q-q')/\hbar} dq$$

where the f are functions that represent A in classical mechanics. As an exercise you can show that with these definitions you obtain the previously defined operators P, Q if you set $f = p$ and $f = q$ respectively.

Next we need to define a product. Since generally $A_{fg} \neq A_f A_g$ we investigate

$$A_f A_g(q, q') = \int_{-\infty}^{\infty} A_f(q, q'')A_g(q'', q')dq''$$

and then define

$$f *_\hbar g = fg + \frac{i\hbar}{2}\{f,g\} + \mathcal{O}(\hbar^2) \quad \Rightarrow \quad \frac{i}{\hbar}(f *_\hbar g - g *_\hbar f) = \{f,g\} + \mathcal{O}(\hbar^2)$$

We now encode the dynamics via the time derivative using the Hamiltonian operator

$$\frac{dA(t)}{dt} = \{H, A(t)\}_\hbar, \qquad A(0) = A$$

where we define $A(t)$ as

$$A(t) = U^{-1}(t)AU(t), \qquad U(t) = e^{-iHt/\hbar}$$

As an exercise you can prove this in a formal setting. Note that we do not yet have a concrete way to define the exponential of an operator such as H. Thus, setting $\hbar = 1$ for simplicity, we investigate the integral kernel of $U = e^{-iH(t''-t')}$ in terms of $h(p,q)$. To this end let $\Delta \ll 1$ such that

$$U_\Delta := e^{-iH\Delta} \cong 1 - iH\Delta$$

Thus setting $t'' - t' = N\Delta$ we obtain $e^{-iH(t''-t')} = (U_\Delta)^N$. Now we calculate

$$\begin{aligned} U_\Delta(q,q') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dp e^{ip(q-q')}\left(1 - ih\left(p, \frac{q+q'}{2}\right)\Delta\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dp e^{ip(q-q') - ih\left(p,\frac{q+q'}{2}\right)\Delta} + \mathcal{O}(\Delta^2) \end{aligned}$$

Now setting $q = q_N, q' = q_0$ and q_j the $N-1$ intermediate variables, we arrive at the following form:

$$\begin{aligned} U(q,q',t''-t') = &\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{dp_1 dq_1}{2\pi} \dots \frac{dp_{N-1}dq_{N-1}}{2\pi}\frac{dp_N}{2\pi} \\ &\times e^{ip_N(q_N - q_{N-1}) + \dots + ip_1(q_1 - q_0)} \cdot e^{-ih\left(p_N, \frac{q_N + q_{N+1}}{2}\right) \dots - ih\left(p_1, \frac{q_1+q_0}{2}\right)} \end{aligned}$$

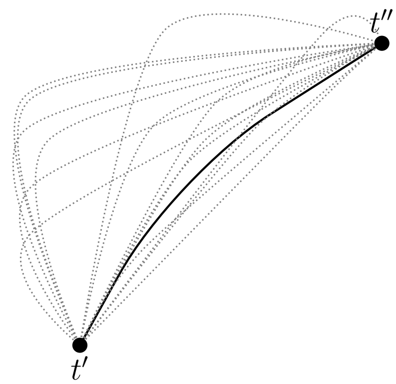
Thus we take $N \longrightarrow \infty$ such that $q_j - q_{j-1}$ being proportional to $\frac{1}{N}$ goes against 0. While we do not discuss convergence and well-definedness at this point, the reader is invited to start looking critically at these notions from now on! Formally this gives us the following form of U which indeed has many problems with convergence of integrals

$$U(q,q',t''-t') = \int \dots \int_{t'}^{t''} \exp\left[i\int_{t'}^{t''}(p(t)\dot{q}(t) - h(p(t),q(t)))\ dt\right]\prod_i \frac{dp(t)dq(t)}{2\pi}$$

Where the first fraction defines our measure on the space of paths from $[t', t'']$ called the **Liouville-measure** and the term in the exponential is called the **Action functional**.

Note that this is only a one-dimensional "toy model" which already boasts some pretty severe convergence problems. Since we ultimately aim to define an infinite-dimensional generalization, the rest of the course will deal with the "taming" of these problems.

In the following we give a short introduction to **Feynman Path Integrals**, visually respresented in the following picture:



Feynman's approach to Quantum Field Theory was the idea of a "sum over all possible histories" of a particle. In *chapter 2* 2.5 we already mentioned scalar field theories where $\psi \in \mathcal{C}^\infty(M)$ is a scalar field. The physical phenomenon corresponding to the "sum over histories" is described by a superposition of all possible scalar theories, weighted by $e^{iS(\psi)/\hbar}$ for $S(\psi)$ the action of ψ.

This in turn has an interpretation coming from statistical mechanics: Here, ψ would represent the state of a statistical system, $S(\psi)$ would be the systems energy. Given a temperature T, the system could be in any state $e^{-S(\psi)/T}$. Thus we obtain the following correspondence:

$$T \longleftrightarrow i\hbar$$

This gives us a sufficient motivation to repeat the assessment of dynamics for statistical systems:

In statistical mechanics, observables are maps $O : \mathcal{C}^\infty(M, \mathbb{R}) \longrightarrow \mathbb{C}$ and correlation functions between observables are defined as

$$\langle O_1, ..., O_n \rangle = \int_\psi e^{-S(\psi)/T} O_1(\psi)...O_n(\psi) D\psi$$

$$\Longrightarrow \qquad \langle O_1, ..., O_n \rangle = \int_{\psi \in \mathcal{C}^\infty(M)} e^{iS(\psi)/\hbar} O_1(\psi)...O_n(\psi) D\psi$$

The main problem in this situation is that $\mathcal{C}^\infty(M)$ is an infinite-dimensional vector space.

***Remark* 3.3**

(1) One approach to resolve this is *perturbative QFT* where we work in the limit $\hbar \longrightarrow 0$ such that we can take formal power series in $\hbar$. This provides us with a formal procedure to go

from a quantum description to a classical one. Note that this endows $\hbar$ with the role of a formal parameter.

(2) In addition to a perturbative treatment, we can drop the Lorentzian signature and stick to Riemannian ones since the mathematical background is better understood.

3.2 The Effective Action

In this section, we try to "tame" the previously exposed integrals over infinite-dimensional vector spaces. We were stuck with expressions of the form $\langle O_1, ..., O_n \rangle$ which indeed could not be calculated with the tools we previously used. One approach to expressions of this form are *"Wilson low-energy theories"* which will make up the first part of this subchapter.

The main idea of *"Wilson low-energy theories"* is that observables can only measure phenomena with an energy below a fixed constant energy Λ. Thus let us fix a Riemannian manifold M, $I \subset [0, \infty)$ and denote by $\mathcal{C}^\infty(M)_I \subset \mathcal{C}^\infty(M)$ the set of functions f that are sums of eigenfunctions of the *Laplacian* 2.50 with eigenvalues in I.

Lemma 3.4 If I is bounded, $\mathcal{C}^\infty(M)_I$ is a finite-dimensional vector space.

Proof. The proof is left as an exercise to the reader. □

Now let $\Lambda \in I$ and define $J = [0, \Lambda)$. We thus define

$$\mathcal{C}^\infty(M)_J := \mathcal{C}^\infty(M)_{\leq \Lambda} \subseteq \mathcal{C}^\infty(M)$$

and call it the **space of fields with energy at most** Λ. Observables in this framework, i.e. restricted to such fields, are functionals

$$O \colon \mathcal{C}^\infty(M)_{\leq \Lambda} \longrightarrow \mathbb{R}[\![\hbar]\!]$$

where $\mathbb{R}[\![\hbar]\!]$ denotes formal power series in $\hbar$. We can extend each such operator O to $\mathcal{C}^\infty(M)$ by composing it with the evident projection to $\mathcal{C}^\infty(M)_{\leq \Lambda}$. We define $Obs_{\leq \Lambda}$ to be the thus arising observables.

Now we investigate $\langle O_1, ..., O_n \rangle$ where $O_i \in Obs_{\leq \Lambda}$. We define quantities of this form as

$$\langle O_1, ..., O_n \rangle := \int_{\varphi \in \mathcal{C}^\infty(M)_{\leq \Lambda}} e^{S_{eff}[\Lambda](\varphi)/\hbar} \, O_1 ... O_n \; D\varphi$$

Here, $D\varphi$ is the Lebesgue measure on $\mathcal{C}^\infty(M)_{\leq \Lambda}$, $S_{eff}[\Lambda]$ is a function on $\mathcal{C}^\infty(M)_{\leq \Lambda}$ and thus a formal power series in $\hbar$. We call such an action a **low energy effective action**.

***Remark* 3.5** The quadratic part of $S_{eff}[\Lambda]$ is negative-definite.

Definition 3.6 (Scalar Quantum Field Theory) A **scalar (perturbative) quantum field theory** is a collection of effective actions

$$S_{eff}[\Lambda]\colon \mathcal{C}^\infty(M)_{\leq\Lambda} \longrightarrow \mathbb{R}[\![\hbar]\!]$$

for all $\Lambda \in [0,\infty)$ such that

1. $S_{eff}[\Lambda]$ is a formal power series in both the field $\varphi \in \mathcal{C}^\infty(M)_{\leq\Lambda}$ and the variable $\hbar$.
2. When setting $\hbar = 1$, $S_{eff}[\Lambda]$ must be of the form
$$S_{eff}[\Lambda] = -\frac{1}{2}\int \varphi(D+m^2)\varphi + \mathcal{O}(\varphi^3)$$
where D is the positive-definite Laplacian.
3. If $\Lambda' \leq \Lambda$, $S_{eff}[\Lambda']$ is determined by $S_{eff}[\Lambda]$ by the renormalisation group equation.
4. The effective actions $S_{eff}[\Lambda]$, when translative in length scale terms, satisfy the asymptotic locality axiom.

Note that the notions mentioned in points 3. and 4. have not been discussed yet. To fill in these gaps will be the function of this subchapter.

We will subsequently fix the notions mentioned in the previous definition. From point 2. we know that $S_{eff}[\Lambda]$ must be of the following form:

$$S_{eff}[\Lambda](\varphi) = -\frac{1}{2}\langle\varphi,(D+m^2)\varphi\rangle + I[\Lambda](\varphi)$$

Here, $\langle\cdot,\cdot\rangle$ denotes the L^2-inner product on $\mathcal{C}^\infty(M)$ defined via

$$\langle\varphi,\psi\rangle := \int_M \varphi(x)\psi(x)dx$$

D is the Laplacian and m is a positive real parameter. Further, $I[\Lambda]$, the **effective interaction**, is understood as a formal power series in $\hbar$:

$$I[\Lambda](\varphi) = I_0[\Lambda](\varphi) + \hbar I_1[\Lambda](\varphi) + ...$$

Here, $I_0[\Lambda](\varphi)$ is at least cubic in φ and all the I_i are formal power series in φ.

3.3 Renormalization group equation

Now our goal is to "translate" a system $S_{eff}[\Lambda], Obs_{\leq\Lambda}, \mathcal{C}^\infty(M)_{\leq\Lambda}$ into one with threshold $\Lambda' \leq \Lambda$. There exists an evident inclusion $Obs_{\leq\Lambda'} \hookrightarrow Obs_{\leq\Lambda}$. The correlation functions should not change if we compute them in $Obs_{\leq\Lambda'}$ rather than in $Obs_{\leq\Lambda}$:

$$\int_{\varphi\in\mathcal{C}^\infty(M)_{\leq\Lambda'}} e^{S_{eff}[\Lambda'](\varphi)/\hbar}\, O_1(\varphi)...O_n(\varphi)\ D\varphi^{\Lambda'} = \int_{\varphi\in\mathcal{C}^\infty(M)_{\leq\Lambda}} e^{S_{eff}[\Lambda](\varphi)/\hbar}\, O_1(\varphi)...O_n(\varphi)\ D\varphi^{\Lambda} \quad (2)$$

Further note that we can split

$$\mathcal{C}^\infty(M)_{\leq\Lambda} = \mathcal{C}^\infty(M)_{\leq\Lambda'} \oplus \mathcal{C}^\infty(M)_{(\Lambda',\Lambda]}$$

Using this splitting to separate (2) allows us to formulate:

$$\begin{aligned} RHS &= \int_{\varphi_L\in\mathcal{C}^\infty(M)_{\leq\Lambda'}} \int_{\varphi_H\in\mathcal{C}^\infty(M)_{(\Lambda',\Lambda]}} e^{S_{eff}[\Lambda](\varphi_L+\varphi_H)/\hbar}\, O_1(\varphi_L)...O_n(\varphi_L)\ D\varphi^{\Lambda'}\ D\varphi^{\Lambda\Lambda'} \\ &= \int_{\varphi_L\in\mathcal{C}^\infty(M)_{\leq\Lambda'}} \left[\int_{\varphi_H\in\mathcal{C}^\infty(M)_{(\Lambda',\Lambda]}} e^{S_{eff}[\Lambda](\varphi_L+\varphi_H)/\hbar}\ D\varphi^{\Lambda\Lambda'}\right] O_1(\varphi_L)...O_n(\varphi_L)\ D\varphi^{\Lambda'} \end{aligned}$$

This in turn allows us to identify

$$e^{S_{eff}[\Lambda'](\varphi_L)/\hbar} = \int_{\varphi_H\in\mathcal{C}^\infty(M)_{(\Lambda',\Lambda]}} e^{S_{eff}[\Lambda](\varphi_L+\varphi_H)/\hbar}\ D\varphi^{\Lambda\Lambda'}$$

which ultimately leads to the **Renormalisation group equation (RGE)**

$$S_{eff}[\Lambda'](\varphi_L) = \hbar\log\left[\int_{\varphi_H\in\mathcal{C}^\infty(M)_{(\Lambda',\Lambda]}} e^{S_{eff}[\Lambda](\varphi_L+\varphi_H)/\hbar}\ D\varphi^{\Lambda\Lambda'}\right] \quad \text{(RGE)}$$

Having a rather well-defined system of such effective actions we may wonder, what their relation to the original classical action S is. This correspond to a "limit" of the form $S_{eff}[\infty]$ that is $\Lambda \longrightarrow \infty$. Note that the space of fields we used in the integral of (RGE) is ill-defined for $\Lambda = \infty$, thus this discussion is not trivially dispatched.

One approach is to describe (RGE) in terms of effective interactions. Note that $\mathcal{C}^\infty(M)_{\Lambda'}$ and $\mathcal{C}^\infty(M)_{\Lambda}$ are D-orthogonal, thus

$$\begin{aligned} \langle\varphi_L+\varphi_H, D(\varphi_L+\varphi_H)\rangle &= \langle\varphi_L, D\varphi_L\rangle + \langle\varphi_H, D\varphi_H\rangle \\ \langle\varphi_L+\varphi_H, m^2(\varphi_L+\varphi_H)\rangle &= \langle\varphi_L, m^2\varphi_L\rangle + \langle\varphi_H, m^2\varphi_H\rangle \end{aligned}$$

Now if we define

$$F(\varphi) = -\frac{1}{2}\left\langle\varphi, (D+m^2)\varphi\right\rangle, \qquad s.t. \qquad F(\varphi_L+\varphi_H) = F(\varphi_L) + F(\varphi_H)$$

we can rewrite our effective action as

$$S_{eff}[\Lambda](\varphi) = F(\varphi) + I[\Lambda](\varphi), \qquad s.t. \qquad S_{eff}[\Lambda](\varphi_L + \varphi_H) = F(\varphi_L) + F(\varphi_H) + I[\Lambda](\varphi_L + \varphi_H)$$

Combining this with (RGE) brings us to

$$\begin{aligned} F(\varphi_L) + I[\Lambda'](\varphi_L) &= \hbar \log \left[\int_{\varphi_H \in \mathcal{C}^\infty(M)_{(\Lambda',\Lambda]}} \exp\left(\frac{F(\varphi_H)}{\hbar} + \frac{F(\varphi_L)}{\hbar} + \frac{I[\Lambda](\varphi_H + \varphi_L)}{\hbar} \right) \, D\varphi^{\Lambda'\Lambda} \right] \\ &= \hbar \log \left[e^{\frac{F(\varphi_L)}{\hbar}} \int_{\varphi_H \in \mathcal{C}^\infty(M)_{(\Lambda',\Lambda]}} \exp\left(\frac{F(\varphi_H)}{\hbar} + \frac{I[\Lambda](\varphi_H + \varphi_L)}{\hbar} \right) \, D\varphi^{\Lambda'\Lambda} \right] \\ &= F(\varphi_L) + \hbar \log \left[\int_{\varphi_H \in \mathcal{C}^\infty(M)_{(\Lambda',\Lambda]}} \exp\left(\frac{F(\varphi_H)}{\hbar} + \frac{I[\Lambda](\varphi_H + \varphi_L)}{\hbar} \right) \, D\varphi^{\Lambda'\Lambda} \right] \end{aligned}$$

Thus we arrive at the **Interaction form of the RGE**:

$$I[\Lambda'](a) = \hbar \log \left[\int_{\varphi_H \in \mathcal{C}^\infty(M)_{(\Lambda',\Lambda]}} \exp\left(\frac{F(\varphi_H)}{\hbar} + \frac{I[\Lambda](\varphi_H + a)}{\hbar} \right) \, D\varphi^{\Lambda'\Lambda} \right] \tag{RGE I}$$

This form has two major advantages over (RGE):

1. We are not integrating over $a \in \mathcal{C}^\infty(M)_{\leq \Lambda'}$, the equation can be extended to any $\varphi_L \in \mathcal{C}^\infty(M)$.
2. It is invertible. It remains valid when choosing $\Lambda' > \Lambda$.

While we have indeed explored the meaning and framework of *point 3.* of 3.6, we still have to settle the notion of **locality**. On a physical level, locality means that interactions between fundamental particles occur at a point of the manifold.

Definition 3.7 (Local action functional) A functional $S\colon \mathcal{C}^\infty(M) \longrightarrow \mathbb{R}[\![\hbar]\!]$ is a **local action functional** if it can be written as a sum

$$S(\varphi) = \sum_k S_k(\varphi), \quad \text{where} \quad S_k(\varphi) = \int_M (D_1\varphi)...(D_k\varphi) \ \mathrm{Vol}_M$$

for some differential operators D_i on M. This allows us to write S as

$$S(\varphi) = \int_M \mathcal{L}(\varphi)(x) \ dx$$

where $\mathcal{L}(\varphi)$ is a **Lagrangian** and depends only on the Taylor expansion of φ at x.

In our current formulation this condition on S does not really make sense yet which is why a definition using $S_{eff}[\Lambda]$ would be favourable. A tentative definition could be the following:

Definition 3.8 A collection of low-energy effective actions $S_{eff}[\Lambda]$ satisfying the renormalisation group equation RGE is **asymptotically local** if there exists an asymptotic expansion for large Λ

$$S_{eff}[\Lambda](\varphi) \cong \sum_i f_i(\Lambda)\ \Theta_i(\varphi)$$

where the $\Theta_i(\varphi)$ are local action functionals.

While tentative, this definition certainly is not a good idea: Supposing $S_{eff}[\Lambda]$ is close to a local action functional, then using the RGE one obtains that $S_{eff}[\Lambda']$ is in fact entirely non local for $\Lambda' < \Lambda$. The solution to this problem is to consider length scales instead of energy scales.

The theory based on length scales takes as fundamental objects the propagator of a differential operator. In order to rewrite the RGE in terms of such propagators, we will need to introduce *Feynmann graphs* as a way to compute integrals.

3.4 Feynmann graphs

The RGE contains an integral of the form

$$\int_{x\in U} \exp(\Phi(x)/\hbar + I(x+a)/\hbar)$$

where Φ is a quadratic negative definite form on a vector space U. From now on we will assume as a convention that the measure on U is the *Lebesgue measure* normalized such that

$$\int_{x\in U} \exp(\Phi(x)/\hbar) = 1$$

Note that the measure does in fact depend on $\hbar$. Now this type of integral can be expressed as a sum over *Feynman graphs* where we understand the integral as an asymptotic series in $\hbar$. To introduce *Feynman graphs*, we start by givin two separate but equivalent definitions of *stable graphs*:

Definition 3.9 (Stable Graphs - Graph Version) A **stable graph** is a graph γ possibly with external edges (edges that only connect to one vertex/node) such that

1. to each vertex (node) $v \in V(\gamma)$ there is an associated number $g(\gamma) \in \mathbb{Z}_{\geq 0}$ called the genus of the vertex.
2. each vertex of genus 0 is at least trivalent (that is of degree 3).
3. every vertex of genus 1 is at least 1-valent (that is of degree 1).

Using the genus of vertices in a graph we can define the genus of a stable graph in a canonical way by taking into account the topology of the graph:

Definition 3.10 (Genus of a Stable Graph) Let γ be a stable graph. Its **genus** $g(\gamma)$ is defined by

$$g(\gamma) \; = \; b_1(\gamma) + \sum_{v \in V(\gamma)} g(v)$$

where $b_1(\gamma)$ is the first *Betti number* of γ defined by $b_1(\gamma) = |E| + |C| - |V|$ where E is the set of edges, C the set of connected components of γ and V the set of vertices (nodes).

As promised we give the following alternative yet equivalent definition of a stable graph:

Definition 3.11 (Stable Graph - Formal Version) Let $H(\gamma)$ and $V(\gamma)$ be two finite sets (half edges or vertices) and let

$$\begin{aligned} &\sigma\colon H(\gamma) \longrightarrow H(\gamma) \qquad \text{an involution} \\ &\pi\colon H(\gamma) \longrightarrow V(\gamma) \\ &g\colon V(\gamma) \longrightarrow \mathbb{Z}_{\geq 0} \qquad \text{(the genus map)} \end{aligned}$$

then a **graph** is the topological space

$$V(\gamma) \sqcup (H(\gamma) \times [0, 0.5])/\sim$$

where $(h, 0) \sim \pi(h)$ and $(h, 0.5) \sim (\sigma(h), 0.5)$. A graph is **stable** if every vertex $v \in V(\gamma)$ is such that if

$$\begin{aligned} g(v) = 0 \quad &\text{then} \quad \#\pi^{-1}(v) \geq 3 \\ g(v) = 1 \quad &\text{then} \quad \#\pi^{-1}(v) \geq 1 \end{aligned}$$

Definition 3.12 An **automorphism** F of a graph γ is a pair of maps

$$\begin{aligned} H(F)\colon H(\gamma) \longrightarrow H(\gamma) \\ V(F)\colon V(\gamma) \longrightarrow V(\gamma) \end{aligned}$$

such that $H(F)$ commutes with σ and such that the following diagram commutes:

$$\begin{tikzcd} H(\gamma) \arrow[r, "H(F)"] \arrow[d, "\pi"'] & H(\gamma) \arrow[d, "\pi"] \\ V(\gamma) \arrow[r, "V(F)"'] & V(\gamma) \end{tikzcd}$$

The automorphisms form a finite group, the proof is left as an exercise.

We denote by $T(\gamma)$ the set of fixed points of the involution σ and by $E(\gamma)$ the set of two element orbits of σ. As such, $T(\gamma)$ stands for the tails or external edges and $E(\gamma)$ for the internal edges of the graph.

Now we turn towards the data associated to *Feynmann graphs* using the previously presented tools. Let us fix a finited graded vector space U over a field $\Bbbk$. Let further $\mathcal{O}(U)$ be the (completed) symmetric algebra on the dual vector space U^*. This algebra coincides with $\mathcal{O}(U)$ being the ring of formal power series in variables in U. Given a basis $\mathcal{B}$ of U there exists a canonical isomorphism

$$\mathcal{O}(U) \longleftrightarrow \Bbbk[\mathcal{B}]$$

Definition 3.13 Let $f \in \mathcal{O}(U)$ be homogenous of degree k. Then we can define an S_k-invariant linear map

$$\begin{aligned} D^k f \colon U^{\otimes k} &\longrightarrow \Bbbk \\ u_1 \otimes ... \otimes u_k &\longmapsto \left(\frac{\partial}{\partial u_1} ... \frac{\partial}{\partial u_k} f \right)(0) \end{aligned}$$

Note that $D^k f \in \mathrm{Hom}(U^{\otimes k}, \Bbbk)$ and it maps

$$\mathcal{O}(U) \longrightarrow \mathcal{O}(U)[\![\hbar]\!]$$

thus to elements of the form $I = \sum_{i,k} \hbar^i I_{i,k}$ where $I_{i,k}$ is homogenous of degree k in U.

Our next goal is to assign some algebraic data to our graphs:

Definition 3.14 Let us denote by $\mathcal{O}^+(U)[\![\hbar]\!] \subset \mathcal{O}(U)[\![\hbar]\!]$ the subset of functionals at least cubic modulo $\hbar$. Now fix an ordering of the set of tails of γ

$$T(\gamma) \overset{\psi}{\longleftrightarrow} \{1, ..., n\}$$

Also fix an element $P \in \mathrm{Sym}^2(U) \subset U^{\otimes 2}$ and some $a_1, ..., a_n \in U$. Then we have

$$\begin{aligned} 2E(\gamma) + T(\gamma) &= H(\gamma) \\ U^{\otimes 2E(\gamma)} \otimes U^{\otimes T(\gamma)} &\cong U^{\otimes H(\gamma)} \\ P \otimes ... \otimes P \otimes a_1 \otimes ... \otimes a_n &=: \mathbb{P} \end{aligned}$$

Now fix $I \in \mathcal{O}(U)[\![\hbar]\!]$ and pick a vertex v with arbitrary valency k and genus i. We associate to it

$$D^k I_{i,k} \in \mathrm{Hom}(U^{\otimes k}, \Bbbk)$$

Taking the tensor product of these elements yields

$$\mathbb{I} = \bigotimes_v D^k I_{i,k}$$
$$\mathbb{I} \in \operatorname{Hom}(U^{\otimes H(\sigma)}, \Bbbk)$$

Now define

$$\begin{aligned} w_{\gamma,\psi}(P,I)(a_1,...,a_n) &:= \mathbb{I}(\mathbb{P}) \in \Bbbk \\ w_\gamma(P,I)(a) &:= w_{r,\psi}(P,I)(a,...,a) \end{aligned}$$

which assigns, for chosen ψ, a number to our graph.

Exercise 3.15 Show the following statements:

1. $w_\gamma(P,I) \in \mathcal{O}(U)$
2. $w_\gamma(P,I)$ is homogenous of degree n and has the property

$$\frac{\partial}{\partial a_1} \cdots \frac{\partial}{\partial a_n} w_\gamma(P,I) = \sum_\psi w_{\gamma,\psi}(P,I)(a_1,...,a_n)$$

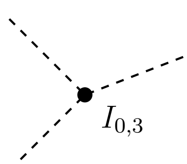
3. If $v_{i,k}$ is the graph with one vertex of genus i and valency k and no internal edges, then $w_{v_{i,k}} = k!\ I_{i,k}$

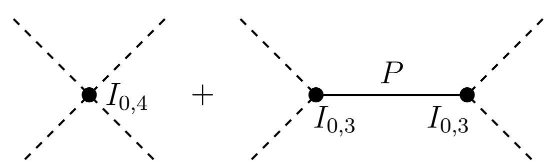
Using $w_\gamma(P,I) \in \mathcal{O}(U)$ we define

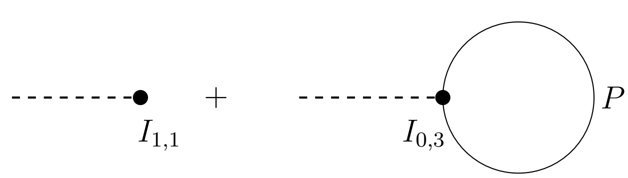
$$W(P,I) = \sum_\gamma \frac{1}{|\operatorname{Aut}(\gamma)|} \hbar^{g(\gamma)} w_\gamma(P,I) \ \in \ \mathcal{O}^+(U)[\![\hbar]\!]$$

where we sum over all connected stable graphs. This is indeed a power series in $u \in U$ and in $\hbar$. We present some insightful examples:

Example 3.16

$W_{0,3}(P,I) =$ 

$W_{0,4}(P,I) =$ 

$W_{1,1}(P,I) =$ 

Lemma 3.17 Setting P to 0 we obtain $W(0, I) = I$.

Proof. As a *hint*, note that only graphs with no edges can contribute. □

Lemma 3.18 For $a_1, ..., a_k \in U$ we get

$$\left(\frac{\partial}{\partial a_1} ... \frac{\partial}{\partial a_n} W(P, I)\right)(0) = \sum_{\gamma,\psi} \frac{\hbar^{g(\gamma)}}{|\operatorname{Aut}(\sigma, \psi)|} w_{\gamma,\psi}(P, I)(a_1, ..., a_k)$$

Proof. The proof is left as an exercise to the reader. □

Now we define some helpful notation to formulate our next result: Let $P \in \operatorname{Sym}^2(U)$. We define $\partial_P \colon \mathcal{O}(U) \longrightarrow \mathcal{O}(U)$ using the decomposition $P = \sum_i P_i' \otimes P_i''$ as

$$\partial_P := \frac{1}{2} \sum_i \frac{\partial}{\partial P_i'} \frac{\partial}{\partial P_i''}$$

This leads us to the following lemma:

Lemma 3.19

$$W(P, I)(a) = \hbar \ \log\{\exp(\hbar \partial_P) \ \exp(I/\hbar)\}(a) \quad \in \mathcal{O}^+(U)[\![\hbar]\!]$$

Proof. The case of $P = 0$ is treated in 3.17. Thus assume $P \neq 0$ such that

$$\exp(\hbar^{-1} W(P, I)) = \exp(\hbar \partial_P) \exp(I/\hbar) \tag{$\diamond$}$$

Now let $\varepsilon > 0$ small such that we can neglect terms of order ε^2 and let $P' \in \operatorname{Sym}^2(U)$. Thus we arrive at

$$(\diamond) \;\Leftrightarrow\; \exp(\hbar^{-1} W(P + \varepsilon P', I)) = (1 + \hbar\varepsilon \partial_{P'}) \exp(\hbar^{-1} W(P, I)) \tag{$\diamond\diamond$}$$

Thus we get $(1 + \hbar\varepsilon\partial_{P'}) \approx \exp(\hbar\varepsilon\partial_{P'})$. Now if we assume in $(\diamond)$ that $P \longrightarrow P + \varepsilon P'$ we get:
So we want to prove $(\diamond\diamond)$

$$\frac{d}{d\varepsilon} \left(\exp(\hbar^{-1} W(P + \varepsilon P', I))\right) = \hbar \partial_{P'} \exp(W(P, I)/\hbar)$$

which is equivalent to $(\diamond\diamond)$. Thus we have already done it and can integrate it. The arising constant can be computed using $\varepsilon = 0$.

$$\left(\frac{\partial}{\partial a_1} ... \frac{\partial}{\partial a_n} W(P, I)\right)(0) = \sum_{\gamma,\psi} \frac{\hbar^{g(\gamma)}}{|\operatorname{Aut}(\sigma, \psi)|} w_{\gamma,\psi}(P, I)(a_1, ..., a_k)$$

$$\frac{\partial}{\partial a_1} ... \frac{\partial}{\partial a_n} \exp(\hbar W(P, I))(0) = \sum_{\gamma,\psi} \frac{\hbar^{g(\gamma)}}{|\operatorname{Aut}(\sigma, \psi)|} w_{\gamma,\psi}(P, I)(a_1, ..., a_k)$$

This formula will be used for $P+\varepsilon P'$. Consider

$$w_{\gamma,\psi}(P+\varepsilon P', I)(a_1, ..., a_k)$$

and recall $\varepsilon^2 \approx 0$. Now note that

$$P+\varepsilon P' = P \otimes ... \otimes \varepsilon P' \otimes a_1 ... a_k + P \otimes ... \otimes \varepsilon P \otimes a_1 ... a_k$$

where P' is at edge e. We thus inspect the derivative

$$\begin{aligned} \frac{d}{d\varepsilon}\left(\frac{\partial}{\partial a_1}...\frac{\partial}{\partial a_k}\exp(\hbar W(P+\varepsilon P', I))\right)(0) &= \sum_{\gamma,e,\psi}\frac{\hbar^{g(\gamma)}}{|\operatorname{Aut}(\sigma,e,\psi)|}w_{\gamma,e,\psi}(P,I)(a_1,...,a_k) \\ &= \frac{1}{2}\sum_{\gamma,\psi}\frac{\hbar^{g(\gamma)}}{|\operatorname{Aut}(\sigma,\psi)|}w_{\gamma,\psi}(P,I)(a_1,...,a_k,U',U'') \end{aligned}$$

where we used $P' = \sum u' \otimes u''$. Now this brings us to

$$= \frac{1}{2}\frac{\partial}{\partial a_1}...\frac{\partial}{\partial a_k}\frac{\partial}{\partial u'}\frac{\partial}{\partial u''}\exp(\hbar^{-1}W(P,I))(0)$$

$\square$

Let us now connect Feynman diagrams to integrals. Let U now have the base field $\mathbb{R}$ and let Φ be a non-degenerate quadratic form on it. Further let $P = \sum_i e_i \otimes e_i$ where $\{e_i\}$ forms an orthonormal basis for $-\Phi$.

Proposition 3.20

$$W(P,I)(a) = \hbar \ \log \int_{x\in U} \exp\left(\frac{1}{2\hbar}\Phi(x,x) + \frac{1}{\hbar}I(x+a)\right) \quad \forall a \in U$$

Proof. Note that this is but a sketch of the proof, you can fill in the details as an exercise. First note

$$\int_{x\in U} \exp\left(\frac{1}{2\hbar}\Phi(x,x)\right) f(x+a) = (\exp(\hbar\partial_P)f)(a) \quad \forall f \in \mathcal{O}(U)$$

where we identified the exponential of $I(x+a)$ with $f(x+a)$. Note that the integral of only the first exponential in above equation amounts to 1. Thus for $f=1$ the equation holds. Thus let us fix $l \in U^*$ thus $lf \in \mathcal{O}(U)$. Note that

$$\begin{aligned} \Phi\colon U &\longrightarrow U^* \quad \text{isomorphism} \\ a &\longmapsto \Phi(a,\cdot) := a^* \end{aligned}$$

Using the dual idea we take for $l \in U^*$ the corresponding $l^* \in U$ such that $l = \Phi(l^*, \cdot)$. The following three statements are given as an exercise:

1. $[\partial_P, l] = -\frac{\partial}{\partial l^*}$
2. $e^{\hbar\partial_P}(lf) - le^{\hbar\partial_P}(f) = \hbar[\partial_P, l]e^{\hbar\partial_P}(f)$
3. $\frac{\partial}{\partial l^*}e^{\Phi(x,x)/2\hbar} = \hbar^{-1}l(x)e^{\Phi(x,x)/\hbar}$

Now using these statements we have:

$$\begin{aligned}\int_{x\in U}\exp\left(\frac{1}{2\hbar}\Phi(x,x)\right)l(x)f(x+a) &= \hbar\int_{x\in U}\left(\frac{\partial}{\partial l^*}e^{\Phi(x,x)/2\hbar}\right)f(x+a)\\ &= -\hbar\int_{x\in U}e^{\Phi(x,x)/2\hbar}\,l(x)\frac{\partial}{\partial l_a^*}f(x+a)\\ &= -\hbar\frac{\partial}{\partial l_a^*}\int_{x\in U}e^{\Phi(x,x)/2\hbar}f(x+a)\end{aligned}$$

This, together with the three previous results, can be used to prove the main claim. □

If U should not be finite-dimensional we can still define $W(P,I)$ and ∂_P. We also still have

$$W(P,I) = \hbar\log\{\exp(\hbar\partial_P)\exp(I/\hbar)\}$$

This concludes our discussion of Feynman graphs which will now be used to make a transition from energy scales to length scales. We start by introducing some common concepts used in the arising theory:

Definition 3.21 (Integral Kernels) Given $F\colon \mathcal{C}^\infty(M)\longrightarrow \mathcal{C}^\infty(M)$ an **integral kernel** of F is a map $K\colon M\times M\longrightarrow \mathbb{R}$ such that

$$F(f)(y) = \int_{x\in M} f(y)K(x,y)$$

A certain type of integral kernels will be of particular importance:

Definition 3.22 (Propagators) Given a scalar field theory, we define its **propagator** to be the integral kernel of the operator $(D+m^2)^{-1}$ where $m\in\mathbb{R}_+$ and D is the *Laplacian* 2.50.

Now we fix a basis $\{e_i\}$ of $\mathcal{C}^\infty(M)$ of orthonormal eigenvectors of D with eigenvalues $\lambda_i\in\mathbb{R}_{\geq 0}$. Then

$$P = \sum_i \frac{1}{\lambda_i^2+m^2}e_i\otimes e_i$$

In the spirit of cutting our action into many effective actions, we define a cut-off operator by considering the following quantity for $U \subseteq \mathbb{R}_{\geq 0}$

$$P_U := \sum_{i|\lambda_i \in U} \frac{1}{\lambda_i^2 + m^2} e_i \otimes e_i$$

Note that P_U is smooth as long as U is a bounded subset of $\mathbb{R}_{\geq 0}$. Now we consider

$$W(P, I)(a) = \hbar \log \int_{\varphi \in \mathcal{C}^\infty(M)} \exp\left(-\frac{1}{2\hbar} \langle \varphi, (D + m^2)\varphi \rangle + \frac{1}{\hbar} I(\varphi + a) \right)$$

Clearly this is not well-defined, thus we utilize the bounded subsets to define

$$W(P_U, I)(a) = \hbar \log \int_{\varphi \in \mathcal{C}^\infty(M)_U} \exp\left(-\frac{1}{2\hbar} \langle \varphi, (D + m^2)\varphi \rangle + \frac{1}{\hbar} I(\varphi + a) \right)$$

which is indeed well-defined. This leads us to rewrite

$$I[\Lambda'](a) = \hbar \log \int_{\varphi_H \in \mathcal{C}^\infty(M)_{[\Lambda', \Lambda)}} \exp(F(\varphi_H) + I[\Lambda](\varphi_H + a)/\hbar)$$

into the **rephrased RGE**:

$$I[\Lambda'] = W\left(P_{[\Lambda', \Lambda)}, I[\Lambda]\right) \tag{RGE}$$

Now Feynman proved the following:

$$P(x, y) = \int_{\tau=0}^{\infty} e^{-\tau m^2} \kappa_\tau(x, y)\ d\tau$$

where $\kappa_\tau(x, y)$ is the **heat kernel** i.e. the integral kernel of $e^{\tau D}$ where D is the Laplacian. We will try to work towards understanding this result.

To this end let $l \in \mathbb{R}_{\geq 0}$ and κ_l^0 the heat kernel of lD. For all $\varphi \in \mathcal{C}^\infty(M)$ this gives us

$$\left(e^{-lD}\varphi\right)(x) = \int_{y \in M} \kappa_l^0(x, y)\varphi(y) dy$$

Using the basis above we get

$$\kappa_l^0 = \sum_i e^{-l\lambda_i} e_i \otimes e_i$$

Now define

$$\kappa_l := e^{-lm^2}\kappa_l^0 \quad \text{which is the kernel of the operator} \quad e^{-l(D+m^2)}$$

All of this construction lets us denote the propagator (integral kernel of $(D + m^2)^{-1}$) as

$$P = \int_{l=0}^{\infty} \kappa_l\ dl$$

Before this, we had the following form in the energy scale:

$$P = \sum_i \frac{1}{x_i^2 + m^2} e_i \otimes e_i$$

where the e_i denoted the eigenfunctions to eigenvalues λ_i. We then defined a cut-off via

$$P_U = \sum_{i|\lambda_i \in U} \sum_i \frac{1}{x_i^2 + m^2} e_i \otimes e_i$$

In an analogy we define the cut-off propagator as

$$P(\varepsilon, L) = \int_\varepsilon^L \kappa_l dl$$

for $\varepsilon, L \in (0, \infty)$ and $\varepsilon < L$. Using coordinates, one can write this propagator as

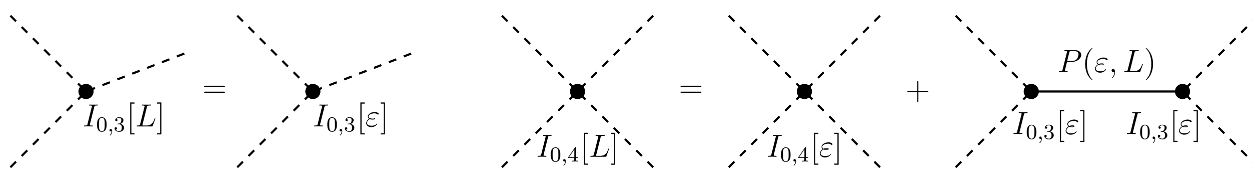
$$P(\varepsilon, L) = \frac{e^{-\varepsilon\lambda_i} - e^{-l\lambda_i}}{\lambda_i^2 + m^2} e_i \otimes e_i$$

As an **exercise** you can prove this claim.

Now we know how to express the (RGE) in terms of length sclaes:

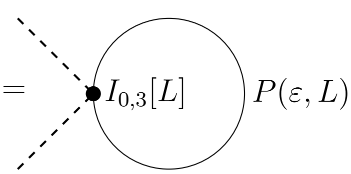
$$I[L] = W(P(\varepsilon, L), I(\varepsilon)) \tag{L-RGE}$$

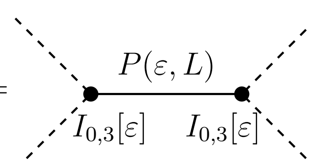
We can indeed give a pictorial explanation for some examples of the (L-RGE) in form of Feynman diagrams:



The following ideas on taking limits of the above construction for $\varepsilon \longrightarrow 0$ will be presented as a sketch rather than a more strict formulation. Note that this is due to time restrictions and the respectable overhead a strict treatment demands.

Sketch 3.23 We consider $W(P(\varepsilon, L), I(\varepsilon))$ and $I(\varphi) = \frac{1}{3!}\int_M \varphi^3$. In particular we consider two graphs:

$\gamma_1 =$ 

$\gamma_2 =$ 

Now note that

$$w_{\gamma_1}(P(\varepsilon,L),I(\varepsilon))(a) = \int_{l\in[\varepsilon,L]}\int_{x\in M} a^2(x)\ \kappa_l(x,x)\ d\,\mathrm{Vol}_M\, dl$$

Now if we take the limit $\varepsilon \longrightarrow 0$ we obtain

$$\kappa_l(x,x) \simeq l^{-(\dim M)/2} + h.o.t.$$

The problem is, that the integral over this term does not converge and thus

$$\lim_{e\longrightarrow 0} w_{\gamma_1}(P(\varepsilon,L),I(\varepsilon))(a)$$

does not exist. Now for γ_2 we note that

$$\begin{aligned} w_{\gamma_2}(P(\varepsilon,L),I(\varepsilon))(a) &= \int_{l\in[\varepsilon,L]}\int_{x,y\in M} a^2(x)\ \kappa_l(x,y)\ a^2(y)\ dxdydl \\ &= \left\langle a^2, \int_\varepsilon^L e^{lD}a^2\ dl\right\rangle \end{aligned}$$

Now the limit of $\lim_{\varepsilon\longrightarrow 0} w_{\gamma_2}$ does indeed exist!

Now this rough sketch can indeed be linked to a more general result:

Fact 3.24

- **graphs with loops do not have a limit $\varepsilon \longrightarrow 0$**
- **graphs without loops (tree-level graphs) do have a limit $\varepsilon \longrightarrow 0$**

We will see in the subsequent material that and how we need to react to the infinities that arise when working with graphs that do contain loops.

3.4.1 Intepretation of Feynman diagrams

The heat kernel can be interpreted as a form of transition propability such that we arrive at the following propability for a transition from x to y:

$$P(x,y) = \int_{l=0}^{\infty} e^{-lm^2}\int_{f\in X}\exp\left(-\int_0^l ||df||^2\right)$$

where $X = \{f\colon [0,l] \longrightarrow M | f(0) = x, f(l) = y\}$ is the space of paths from x to y. Remember that we defined the energy of such a path as

$$E(f) := -\int_0^l \langle df, df\rangle$$

Now we can collect this as

$$P(x, y) = \int_0^\infty \kappa_l(x, y)\ dl$$

To make this integral well-defined, we also need the *Wiener* measure, which we won't go into detail about:

$$\kappa_f(x, y) = \int_X D_{Wiener} f$$

To consider a particular interaction and think about why it physically corresponds to Feynman graphs, we take

$$S(\varphi) = \int_M -\frac{1}{2}\varphi D\varphi + \frac{1}{3!}\varphi^3$$

note that this consideration is not necessarily well-defined and does contain quite some problems. Thus this discussion is more of a sketch. For n particles we consider the expectation value

$$\mathbb{E}(x_1, x_2, ..., x_n) = \int_{\varphi \in \mathcal{C}^\infty(M)} e^{S(\varphi)/\hbar} \varphi(x_1)...\varphi(x_n)$$

Now we associate to n particles n external edges:

$$\mathbb{E}(x_1, ..., x_n) = \sum_\gamma \frac{1}{|\operatorname{Aut}(\gamma)|} \hbar^{-\sigma(\gamma)} \int_{g \in Met\gamma} \int_x e^{-E(\gamma)}$$

where $Met\gamma$ stands for the metric on the space of curves γ. Also consider the inverse assignment

$$f : \gamma \longrightarrow M, \qquad E(\gamma) \longmapsto (x_1, ..., x_n)$$

Definition 3.25 (Local action functional) A functional $I \in \mathcal{O}(\mathcal{C}^\infty(M))$ that can be written as $I = \sum_k I_k$ where I_k is homogenous of degree k in the variable $a \in \mathcal{C}^\infty(M)$, that is

$$I_k(\lambda a) = \lambda^k I_k(a)$$

and such that the I_k can be written as

$$I_k(a) = \sum_{j=1}^{S} \int_M D_{1,j}(a)...D_{k,j}(a)$$

where $D_{i,j}$ are differential operators on M is called a **local action functional**.

We will subsequently denote by $\mathcal{O}_{loc}(\mathcal{C}^\infty(M))$ the **subspace of local action functionals**. This definition extends to the formal power series in $\hbar$ that is $I \in \mathcal{O}(\mathcal{C}^\infty(M))[\![\hbar]\!]$ as follows:

$$I = \sum_{i,k} \hbar^i I_{i,k}$$

where the $I_{i,k}$ need to be in $\mathcal{O}_{loc}(\mathcal{C}^\infty(M))$. We further define

$$\mathcal{O}^+_{loc}(\mathcal{C}^\infty(M))[\![\hbar]\!] \subset \mathcal{O}_{loc}(\mathcal{C}^\infty(M))[\![\hbar]\!]$$

the subspae of local functionals which are at least cubic modulo $\hbar$. Finally we are prepared to define scalar QFTs:

Definition 3.26 (Perturbative scalar quantum field theories) A **perturbative scalar quantum field theory** is given by a set of effective interactions

$$I[L] \in \mathcal{O}^+_{loc}(\mathcal{C}^\infty(M))[\![\hbar]\!] \quad \forall L \in [0,\infty]$$

(plus a kinetic action $-\frac{1}{2}\langle\varphi,(D+m^2)\varphi\rangle$ for $\varphi \in \mathcal{C}^\infty(M)$) such that

1. $I[L] = W(P(\varepsilon, L), I[\varepsilon]) \quad \forall \varepsilon, L \in (0,\infty)$
2. Let
$$I[L] = \sum_{i,k} \hbar^i I_{i,k}[L]$$
For each i,k we require a small L asymptotic expansion
$$I_{i,k}[L] \simeq \sum_{r\in\mathbb{Z}} g_r(L)\Phi_r$$
where $g_r(L) \in \mathcal{C}^\infty((0,\infty)_L)$ and $\Phi_r \in \mathcal{O}_{loc}(\mathcal{C}^\infty(M))$. Small L asymptotic expansion means that there exists a non-decreasing sequence
$$d_R \in \mathbb{Z}, \quad d_R \longrightarrow \infty, \quad R \longrightarrow \infty$$
such that for all R
$$\lim_{L\longrightarrow 0} L^{-d_R}\left(I_{i,k}[L](a) - \sum_{r=0}^{R} g_r(L)\Phi_r(a)\right) = 0$$
for all $a \in \mathcal{C}^\infty(M)$.

We will denote by $\mathcal{Z}^{(a)}$ the set of perturbative quantum field theories and by $\mathcal{Z}^{(n)}$ the set of such theories defined modulo $\hbar^{n+1}$. Further we denote

$$\mathcal{Z}^{(\infty)} = \lim_{\longleftarrow} \mathcal{Z}^{(n)}$$

Theorem 3.27 $\mathcal{Z}^{(n+1)}$ is a principal bundle over $\mathcal{Z}^{(n)}$ with structure group the abelian group $\mathcal{O}_{loc}(\mathcal{C}^\infty(M))$ (local action functionals on M). In particular, $\mathcal{Z}^{(0)}$ is canonically isomorphic to the space $\mathcal{O}^+_{loc}(\mathcal{C}^\infty(M))$

Theorem 3.28 If we fix a *normalisation scheme* (which we will define and explain later), we can find a section for each torsor $\mathcal{Z}^{(n+1)} \longrightarrow \mathcal{Z}^{(n)}$ and consequently a bijection between the set of perturbative quantum fields theories and the set of local action functionals $I \in \mathcal{O}^+_{loc}(\mathcal{C}^\infty(M))[\![\hbar]\!]$.

While we might not find the time to discuss the proofs of the two above theorems, their statements are the most important points. The first one provides convenient and interesting structure, mainly telling us that once we restrict to the classical setting, we recover a classical action functional, hence providing a canonical transition from quantum to classical. The second theorem provides a direct link between local action functionals and perturbative QFTs, thus enabling the transition shown in the first.

We now discuss a *strategy* to work with both of these statements: First fix $I \in \mathcal{O}^+_{loc}(\mathcal{C}^\infty(M))[\![\hbar]\!]$. We want to build the effective action $I[L]$ satisfying the properties of its definition. So first of all we want to construct counterterms $I^{CT}(\varepsilon)$ such that

$$\lim_{\varepsilon \longrightarrow 0} W(P(\varepsilon, L), I - I^{CT}(\varepsilon))$$

exists and in particular defines $I[L]$. Conversely given $\{I[L]\}$ we construct I as a renormalized limit, subtracting the suitable counterterms. To this end, we define some new helpful notions:

Definition 3.29 (Periods) Into the classical sequence $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ we want to insert $\mathbb{Q} \subset \mathbb{P} \subset \mathbb{R}$ where $\mathbb{P}$ forms a ring contained in transcendental numbers. A **period** is a complex number whose real and imaginary part are values of absolutely converging integrals of rational functions with rational coefficients over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Example 3.30 Two numbers included in $\mathbb{P}$ and clearly not included in $\mathbb{Q}$ are:

$$\sqrt{2} = \int_{2x^2 \leq 1} dx$$
$$\pi = \iint_{x^2+y^2 \leq 1} dxdy = 2\int_{-1}^{+1} \sqrt{1-x^2}dx$$

Now let $t \in (0, \infty)$ be a real parameter. Formally we might write a "period depending smoothly on t" as

$$\alpha(t) = \int_{\gamma(t)} \omega(t)$$

where we allow $\gamma(t)$ and $\omega(t)$ to depend smoothly on t. But of course we further need to require that $\alpha(t)$ is a period, thus $\in \mathbb{P}$, for every rational number $t \in \mathbb{Q} \cap (0, \infty)$. Collecting these requirements leads us to a definition from *algebraic geometry*:

Definition 3.31 (Periods in Algebraic Geometry) We call **rational periods** those functions in $\mathcal{C}^\infty((0, \infty))$ which are of this form and denote by $P_{\mathbb{Q}}((0, \infty))$ the set of rational periods.

Definition 3.32 We define $\mathcal{P}((0,\infty))$ to be the real vector space spanned by the space of rational periods:

$$\mathcal{P}((0,\infty)) = P_{\mathbb{Q}}((0,\infty)) \otimes \mathbb{R} \subset \mathcal{C}^\infty((0,\infty))$$

By an abuse of notation, we call elements of $\mathcal{P}((0,\infty))$ **periods**.

Now we still investigate $w_\gamma(P(\varepsilon, L), I(\varepsilon))(a)$ in the limit $\varepsilon \longrightarrow 0$. Now we interpret w_γ as a function of ε, L, a, thus in

$$\mathcal{O}(\mathcal{C}^\infty(M), \mathcal{C}^\infty((0,\infty)_\varepsilon)) \otimes \mathcal{C}^\infty((0,\infty)_L)$$

In particular, if we fix ε, we have

$$w_\gamma \in \mathcal{O}(\mathcal{C}^\infty(M), \mathcal{C}^\infty((0,\infty)_L))$$

This leads us to the following theorem:

Theorem 3.33 Let $I \in \mathcal{O}_{loc}(\mathcal{C}^\infty(M))[\![\hbar]\!]$ be a local action functional and let γ be a connected stable graph. Then there exists a small-ε asymptotic expansion

$$w_\gamma(P(\varepsilon, L), I[\varepsilon]) \simeq \sum_{i=0}^{\infty} g_i(\varepsilon)\psi_i$$

where $g_i \in \mathcal{P}((0,\infty)_\varepsilon)$ are periods and $\psi_i \in \mathcal{O}(\mathcal{C}^\infty(M), \mathcal{C}^\infty((0,\infty)_L))$ such that

1. $$\lim_{\varepsilon \longrightarrow 0} L^{-d_R} \left(w_\gamma - \sum_{r=0}^{R} g_r(\varepsilon)\psi_r \right) = 0 \quad \forall a \in \mathcal{C}^\infty(M)$$

2. the $g_i(\varepsilon)$ have finite order poles at 0, i.e. for $\forall i\ \exists k$ such that

$$\lim_{\varepsilon \longrightarrow 0} \varepsilon^k g_i(\varepsilon) = 0$$

3. the ψ_i have a small-L asymptotic expansion of the form

$$\psi_i \simeq \sum_{j=0}^{\infty} f_{i,j}(L)\psi_{i,j}$$

where $\psi_{i,j} \in \mathcal{O}_{loc}(\mathcal{C}^\infty(M))$ and $f_{i,j} \in \mathcal{C}^\infty((0,\infty)_L)$.

Sadly we won't have the time to prove the above theorem since it is rather lengthy and technical. Thus we go on with the following definition:

Definition 3.34 Define $\mathcal{P}((0,\infty))_{\geq 0} \subseteq \mathcal{P}((0,\infty))$ to be the subspace of functions of ε that are periods and which admit a limit as $\varepsilon \longrightarrow 0$.

Using this definition we can finally expand upon the previously mentioned *renormalisation schemes*:

Definition 3.35 Renormalisation Schemes A choice of a subspace

$$\mathcal{P}((0,\infty))_{<0} \subset \mathcal{P}((0,\infty))$$

complementary to $\mathcal{P}((0,\infty))_{\geq 0}$ is called a **renormalisation scheme**. Hence a renormalisation scheme provides a direct sum decomposition

$$\mathcal{P}((0,\infty))_{\geq 0}((0,\infty)) = \mathcal{P}((0,\infty))_{\geq 0}((0,\infty))_{\geq 0} \oplus \mathcal{P}((0,\infty))_{<0}$$

Introducing further notation using the decomposition at hand we define

Definition 3.36 (Singular Parts) Given a period $f \in \mathcal{P}((0,\infty))$ define its **singular part** to be the projection $\mathrm{sing}(f)$ of f onto $\mathcal{P}((0,\infty))_{<0}$.

From now on, we fix a renormalisation scheme. From the previous theorem, take

$$w_\gamma(P(\varepsilon,L), I[\varepsilon]) \simeq \sum_{i=0}^{\infty} g_i(\varepsilon)\psi_i$$

Now there exists an N such that $\forall n > N$ the $g_n(\varepsilon)$ admits a limit $\varepsilon \longrightarrow 0$. As an **exercise** you can prove this claim. Now define

$$\psi_N := \sum_{i=0}^{N} g_i(\varepsilon)\psi_i$$

and note that

$$\mathrm{sing}_\varepsilon(w_\gamma(P(\varepsilon,L), I[\varepsilon])) := \mathrm{sing}(\psi_N(\varepsilon)) = \sum_{i=0}^{N} \mathrm{sing}_\varepsilon(g_i(\varepsilon))\psi_i$$

We can collect these results and definitions into a theorem:

Theorem 3.37 Let $I \in \mathcal{O}_{loc}(\mathcal{C}^\infty(M))[\![\hbar]\!]$ be a local action functional and let γ be a connected stable graph. Then

$$\mathrm{sing}_\varepsilon(w_\gamma(P(\varepsilon,L), I[\varepsilon])) = \sum_i f_i(\varepsilon)\psi_i$$

where $f_i(\varepsilon) \in \mathcal{P}((0,\infty))_{\geq 0}$ are singular periods (periods equivalent to their singular part) and the ψ_i have a small-L asymptotic expansions of the form

$$\varphi_i \simeq \sum_{i=0}^{\infty} f_{i,j}\psi_{i,j}$$

where $\psi_{i,j} \in \mathcal{O}_{loc}(\mathcal{C}^\infty(M))$ and $f_{i,j} \in \mathcal{C}^\infty((0,\infty)_L)$. Furthermore the limit

$$\lim_{\varepsilon \longrightarrow 0} (w_\gamma(P(\varepsilon, L), I[\varepsilon]) - \mathrm{sing}_\varepsilon(w_\gamma(P(\varepsilon, L), I[\varepsilon])))$$

exists in $\mathcal{O}_{loc}(\mathcal{C}^\infty(M), \mathcal{C}^\infty((0,\infty)_L))$.

Theorem 3.38 There exists a unique series of local counterterms

$$I^{CT}_{i,k}(\varepsilon) \in \mathcal{O}_{loc}(\mathcal{C}^\infty(M)) \otimes \mathcal{P}((0,\infty))_{<0}$$

for any i, k with $I^{CT}_{i,k}$ homogenous of degree k as a function of $a \in \mathcal{C}^\infty(M)$ and such that for all $L \in (0,\infty)$ the limit

$$\lim_{\varepsilon \longrightarrow 0} W(P(\varepsilon, L), I - \sum_{i,k} \hbar^i I^{CT}_{i,k}(\varepsilon))$$

exists.

4 Gauge Theories

5 Outlook

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