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Mathematical Foundations of Quantum Field Theory

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0 Introduction

The goal of Quantum Field Theory is to merge the principles of Quantum Mechanics and Classical Mechanics into one fundamental field theory. However there is currently not **one** accepted understanding of QFT but rather a huge collection of different models, approaches and techniques, partly general in nature and partly specialised to certain applications. Unlike many other physical fields, QFT has been a huge influence in the development of contemporary mathematics.

At its core there are two pillars supporting QFT. First of all, there are powerful calculation methods, second, there is an axiomatic understanding of QFT which is completely invariant under the specific theory at hand. In this lecture we will mainly focus on the axiomatic approach to the field.

Rooting in classical Lagrangian mechanics, the axiomatic frameworks (see Wightman, Segal, Haag-Kastler) propose that a QFT is an assignment that associates to $(d - 1)$ -manifolds Hilbert spaces (or other structures) and of d -manifolds to (unitary) operators. This assignment does define a functor between certain categories which sums up certain axiomatic approaches and provides a first link to more modern mathematics.

Most field theories which are of interest for physics (in fact most *known* field theories) are so-called "gauge theories" meaning that they display a fundamental redundancy in the description of physical quantities. To treat such occurrences, the methods of cohomology have been rediscovered for physical applications. This is largely due to Batalin and Vilkovisky who provided an enrichment to the axiomatic picture.

The main question still remains: What exactly **is** a QFT? In this lecture we will explore many ideas introduced by Wilson, Kadanoff, Polchinski and others and which have been extended and framed in recent years by Costello, Giulliani, Cattaneo, Mnev, Reshtekhin, Fredenhagen, Rejzner and others.

1 Geometric Preliminaries

This chapter is aimed at giving an introduction to the geometric setting of Quantum Field Theory. While trying to be as self-contained as possible, we will have to assume that the reader is fluent in basic Differential Geometry. While there are many excellent textbooks on Differential Geometry, the textbook by Lee [1] offers a highly self-contained take on the topic. Note that in the following, the notion of fibre- and vector-bundles will be especially crucial. For an insightful treatment of the matter and a more compact introduction to Differential Geometry, see Will Merry's website [2].

The main sources for this section will be **Renormalisation and Effective Field Theory** by K. Costello [3] and **Quantum Fields and Strings: A Course for Mathematicians** by various authors [4].

We will further make use of excerpts from **Introduction to Supergeometry** by A.S. Cattaneo and F. Schaetz [5], chapter 2.1 of **Algebraic Geometry** by R. Hartshorne [6], chapters 5.2, 5.3 and 6.1 of **Modern differential geometry for physicists** by C.J. Isham [7] and chapter 2 of A. Hatcher's **Algebraic Topology** [8]. Note that most of these texts are available online, respective links are provided in the References.

1.1 The Graded Setting

In the following section we will generalize the notion of Differential Geometry to a graded setting. To this end we start with the preliminary definitions of a graded structure and discuss some interesting examples.

DEFINITION 1.1 (Graded Vector Space)

A **graded vector space** V^\bullet is a collection $\{V^k\}_{k \in \mathbb{Z}}$ of vector spaces. The *realisation* of V^\bullet is the direct sum

$$V^\bullet = \bigoplus_{k \in \mathbb{Z}} V^k. \quad (1.1)$$

To extend the notion of graded vector spaces to a category, we need morphisms between them:

DEFINITION 1.2 (Graded Morphisms)

Let V^\bullet, W^\bullet be graded vector spaces. A **graded morphism** $\varphi : V^\bullet \rightarrow W^\bullet$ is a collection of linear maps $\varphi^k : V^k \rightarrow W^k \ \forall k \in \mathbb{Z}$. We extend this notion to that of a graded morphism of **degree** r by requiring that $\varphi^k : V^k \rightarrow W^{k+r} \ \forall k \in \mathbb{Z}$. In the case $V^\bullet = W^\bullet$ we call φ a **graded endomorphism** (of degree r , respectively). The category **gVect** of graded vector spaces has objects graded vector spaces and morphisms graded morphisms.

Given V^\bullet we call $v \in V^k$ an homogeneous element of degree $|v| := k$.

EXAMPLE 1.3

- The most common examples of graded vector spaces emerge from the tensor product. The k -fold tensor product of a vector space V denoted by $V^{\otimes k}$, the k -fold exterior algebra of V denoted by $\bigwedge^k V$ and the space of differential k -forms on a smooth manifold M denoted by $\Omega^k(M)$ each generate a graded vector space, graded by polynomial degree, denoted by $T^\bullet(V)$, $\bigwedge^\bullet V$ and $\Omega^\bullet(M)$ respectively.
- For the space of differential forms on a smooth manifold M given by $\Omega^\bullet(M)$ we can define a graded morphism of degree 1 given by the *de Rham differential* $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$.
- The *contraction* of a differential form by a vector field defines another graded morphism on the space of differential forms $\Omega^\bullet(M)$. For $X \in \mathfrak{X}(M)$ we define

$$\iota_X : \Omega^k(M) \longrightarrow \Omega^{k-1}(M), \quad (1.2)$$

$$\omega \longmapsto \iota_X \omega = \omega(X, \cdot, \dots, \cdot). \quad (1.3)$$

The contraction is a graded morphism of degree -1 .

- Given $X \in \mathfrak{X}(M)$ the *Lie Derivative* \mathcal{L}_X defines, by Cartan's magic formula, a graded morphism of degree 0 by

$$\mathcal{L}_X : \Omega^k(M) \longrightarrow \Omega^k(M), \quad (1.4)$$

$$\omega \longmapsto \mathcal{L}_X \omega := \iota_X \circ d\omega + d \circ \iota_X \omega \quad (1.5)$$

- Given two graded vector spaces V^\bullet, W^\bullet we obtain another graded vector space through the set of graded morphisms

$$\mathrm{Hom}^\bullet(V^\bullet, W^\bullet) := \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}^k(V^\bullet, W^\bullet), \quad (1.6)$$

where Hom^k denotes degree- k maps.

- Given a graded vector space V^\bullet we define the dual graded vector space $(V^*)^\bullet$ via the collection $(V^*)_i = (V_{-i})^*$.¹

DEFINITION 1.4 (Shifted Vector Spaces) Let V^\bullet be a graded vector space. We **shift** V^\bullet by $k \in \mathbb{Z}$ by defining the **shifted vector bundle**

$$V[k] := \bigoplus_{i \in \mathbb{Z}} V[k]^i = \bigoplus_{i \in \mathbb{Z}} V^{k+i} \quad (1.7)$$

¹Note that in the case where $V^\bullet = T^\bullet(W)$ where W is a classical vector space, $(V^*)_i$ becomes trivially zero for positive indices.

Given a vector space V^\bullet such that $V^i = \{0\}$ for all $i \neq p$, we call V^\bullet **concentrated in p** . In particular for any vector space V , the shifted vector space $V[k]$ is concentrated in degree $-k$. For a dual space V^* we obtain

$$V^*[k] = \bigoplus_{i \in \mathbb{Z}} V^*[k]_i = \bigoplus_{i \in \mathbb{Z}} (V^*)_{k+i} = \bigoplus_{i \in \mathbb{Z}} (V^{k+i})^*$$

Due to their importance for the forthcoming discussion, we classify certain special cases of graded endomorphisms.

DEFINITION 1.5 Let φ be a graded endomorphism such that $\varphi \circ \varphi = 0$. We call φ a **boundary operator**, if φ is of degree -1 . If φ is of degree $+1$, we call it a **coboundary operator**. A graded vector space V^\bullet together with a coboundary operator is called a **complex**, or **cochain complex**. If it is equipped with a boundary operator, we call it a **cocomplex**, or **chain complex**.

A very important example for a cochain complex is $\Omega^\bullet(M)$ equipped with the *de Rham differential* used for *de Rham cohomology*. To define further structure on graded vector spaces, we turn to algebraic relations defined on its realisation.

DEFINITION 1.6 (Graded Algebra)

A **graded algebra** is a graded vector space A^\bullet together with a collection of bilinear maps called the *product* on A^\bullet

$$A^k \times A^l \rightarrow A^{k+l}, \quad (a, b) \mapsto ab \quad \forall k, l \in \mathbb{Z} \quad (1.8)$$

- A^\bullet is **associative**, if

$$(ab)c = a(bc) \quad \forall a, b, c \in A^\bullet \quad (1.9)$$

- A^\bullet is **graded commutative**, if

$$ab = (-1)^{kl} ba \quad \forall a \in A^k, b \in A^l \quad (1.10)$$

- A^\bullet is **graded skew-commutative**, if

$$ab = (-1)^{kl+1} ba \quad \forall a \in A^k, b \in A^l \quad (1.11)$$

EXAMPLE 1.7

- The tensor algebra $T^\bullet(V)$ of a graded vector space V^\bullet is a graded algebra with its product given by the tensor product \otimes . Note that this makes $T^\bullet(V)$ an associative graded algebra.
- The two spaces $(\bigwedge^\bullet V, \wedge)$, with V an ordinary vector space, and $(\Omega^\bullet(M), \wedge)$, with M a smooth manifold, are associative commutative graded algebras as can be readily deduced from the definition of the wedge-product.

- Given a graded vector space V^\bullet we can take the graded vector space of endomorphisms $\text{End}^\bullet(V^\bullet)$ and elevate it to an associative graded algebra. To this end, we define the product of two endomorphisms $\varphi \in \text{End}^r(V^\bullet)$ and $\psi \in \text{End}^s(V^\bullet)$ as the graded endomorphism given by the collection of linear maps

$$\varphi\psi : \varphi^{k+s}\psi^k : V^k \rightarrow V^{k+r+s} \quad \forall k \in \mathbb{Z} \quad (1.12)$$

resulting in a new graded endomorphism of degree $(r + s)$. It is left as an easy exercise to prove that this does indeed define an associative graded algebra on $\text{End}^\bullet(V^\bullet)$.

- The *graded symmetric algebra* $S^\bullet(V)$ over a graded vector space V^\bullet is the quotient of the tensor algebra $T^\bullet(V)$ by the ideal generated by all elements of the form

$$v \otimes w - (-1)^{|v||w|} w \otimes v \quad \forall v, w \in V^\bullet \quad (1.13)$$

This graded algebra is graded commutative with respect to the tensor product, the proof is left as an exercise.

- The *algebra of polynomial functions* on V^\bullet is the graded symmetric algebra $S^\bullet((V^*)^\bullet)$ over $(V^*)^\bullet$. This notion will be particularly useful when talking about graded manifolds later in the course.

Extending the notion of vector subspaces and closed subspaces of algebras, so-called subalgebras, we go on to define the respective notions in the graded context:

DEFINITION 1.8 A **graded subspace** $W^\bullet \subset V^\bullet$ of a graded vector space V^\bullet is a collection of subspaces $\{W^k\}_{k \in \mathbb{Z}}$ such that $W^k \subset V^k \quad \forall k \in \mathbb{Z}$.

A **graded subalgebra** is a graded subspace that is closed under the product.

We go on to further classify certain graded morphisms by restricting to those satisfying an adapted *Leibniz Rule*. The idea is to generalize the notion of a derivation on an algebra, which is required to satisfy the classical *Leibniz Rule*, to general graded algebras. This will provide us with an excellent example of a graded subalgebra:

DEFINITION 1.9 (Graded Derivations)

Let V^\bullet be a graded algebra. A **graded derivation** of degree n is a graded endomorphism \mathfrak{D} of degree n that satisfies the **graded Leibniz rule**

$$\mathfrak{D}(ab) = \mathfrak{D}(a) b + (-1)^{rn} a \mathfrak{D}(b) \quad \forall a \in V^r, b \in V^s \quad (1.14)$$

The following examples inspired the above general definition:

EXAMPLE 1.10

- The *de Rham differential* d is a graded derivation of degree $+1$
- The *contraction* ι_X is a graded derivation of degree -1
- The *Lie Derivative* \mathcal{L}_X is a graded derivation of degree 0

On a graded algebra A^\bullet the set of graded derivations $\text{Der}^\bullet(A^\bullet)$ is a subspace of $\text{End}^\bullet(A^\bullet)$ in the sense that $\text{Der}^k(A^\bullet) \subset \text{End}^k(A^\bullet)$.

Remark 1.11 Derivations *do not* form a subalgebra of endomorphisms, although they do form a graded Lie subalgebra. This is analogous to the fact that if X and Y are vector fields on a manifold, XY or YX (their compositions as derivations on the algebra of functions) are *not* vector fields, but $[X, Y] = XY - YX$ *is*.

Combining the different classifications for graded endomorphisms, we go on to define yet another utterly important subspecies:

DEFINITION 1.12 A **differential** is a coboundary operator that is also a graded derivation. A **codifferential** is a boundary operator that is a graded derivation.

The *de Rham differential* is a differential in the sense of the above definition. We consider a graded *Lie Algebra*. The definition is easily anticipated from that of a usual Lie algebra

DEFINITION 1.13 (Graded Lie Algebra)

For simplicity, let us assume that the base field has characteristic zero. A **graded Lie algebra** is a graded algebra \mathfrak{g}^\bullet whose product, $[-, -]: \mathfrak{g}^\bullet \times \mathfrak{g}^\bullet \rightarrow \mathfrak{g}^\bullet$ is graded skew-commutative and satisfies the graded Jacobi-Identity, i.e.

$$(-1)^{|a||c|}[a, [b, c]] + (-1)^{|a||b|}[b, [a, c]] + (-1)^{|b||c|}[c, [a, b]] = 0. \quad (1.15)$$

As usual we provide some insightful examples to support the understanding of the above definition:

EXAMPLE 1.14

- Take $(A^\bullet, [\cdot, \cdot])$ such that A^\bullet is a graded associative algebra. We define for $a \in A^k, b \in A^l$:

$$[a, b] := ab - (-1)^{kl} ba \quad (1.16)$$

This constitutes a graded Lie algebra as can be checked through tedious calculation.

- The set $(\text{End}^\bullet(V^\bullet), [\cdot, \cdot])$ with the bracket defined as in the previous example is a graded Lie algebra seeing $\text{End}^\bullet(V^\bullet)$ as a graded associative algebra with composition as its product.

Remark 1.15 We note that if both $a, b \in A^\bullet$ are odd, the bracket as defined in the above

examples turns out to be the anticommutator

$$[a, b] = ab + ba \quad (1.17)$$

Thus, if a is odd, we obtain $[a, a] = 2a^2 \neq 0$.

DEFINITION 1.16 A **graded Lie subalgebra** is a graded subalgebra of a graded Lie algebra.

An example for a graded Lie subalgebra would be the space of Derivations as a subset of the space of endomorphisms, namely $\text{Der}^\bullet(A^\bullet) \subset \text{End}^\bullet(A^\bullet)$.

Remark 1.17 In all of our previous discussion, we used \mathbb{Z} as the set that defines our graded setting. Now if we would like to generalize the graded setting ideas to more general spaces, we need some extra work:

Take a set G . We define a G -graded vector space $\{V^k\}_{k \in G}$. But in order to define further notions like that of graded morphisms etc. we previously used the specific structure of \mathbb{Z} and thus need to generalize the required properties:

- We require a composition law $G \times G \rightarrow G$ to define graded morphisms.
- If we want to define graded associated algebras, G has to be a group (or a monoid) such that the order of composition does not matter.
- G has to be abelian (or a commutative monoid), if we want to define graded commutative algebras.

So far we did everything on \mathbb{Z} . Taking $\mathbb{Z}/2\mathbb{Z}$ as the underlying set, we obtain the notion of *super-linear algebra*. To prove the above claims is an insightful and interesting exercise! We will only need \mathbb{Z} - and $\mathbb{Z}/2\mathbb{Z}$ -gradings.

After this small detour, we continue to define further structure in our graded setting. Next, we go back to \mathbb{Z} -grading and discuss the so-called *Cartan Calculus* using d and ι .

PROPOSITION 1.18 The span over \mathbb{R} of the set

$$d, \iota_X, \mathcal{L}_X : X \in \mathfrak{X}(M) \quad (1.18)$$

is a graded Lie subalgebra of $\text{Der}^\bullet(\Omega^\bullet(M))$. Moreover it satisfies the following algebraic properties with regard to the usual bracket as defined before:

$$\begin{array}{lll} (1) [d, d] = 0 & (2) [d, \iota_X] = \mathcal{L}_X & (3) [d, \mathcal{L}_X] = 0 \\ (4) [\iota_X, \iota_Y] = 0 & (5) [\iota_X, \mathcal{L}_Y] = \iota_{[X, Y]} & (6) [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X, Y]} \end{array}$$

Proof. We only need to prove the claims for functions and exact 1-forms as all of the above are local operators. First we note that $[d, d] = 0$ simply due to $d \circ d = 0$. For the second identity we note that for functions

$$\mathcal{L}_X(f) = X(f) = \iota_X df = [\iota_X, d]f \quad (1.19)$$

Now for exact 1-forms we obtain

$$[d, \iota_X]df = d \circ \iota_X df = d \circ \mathcal{L}_X f = \mathcal{L}_X df \quad (1.20)$$

Now the third claim is again easily obtained using the Cartan-Magic-Formula and the graded Jacobi-Identity:

$$[d, \mathcal{L}_X] = [d, [d, \iota_X]] = [[d, d], \iota_X] - [d, [d, \iota_X]] = -[d, \mathcal{L}_X] \quad (1.21)$$

Thus it trivially becomes 0. For the fourth identity we note $[\iota_X, \iota_Y] = \iota_X \iota_Y + \iota_Y \iota_X$. Now if we define $I_{XY} := [\iota_X, \iota_Y]$, we can apply to the wedge product of two forms to obtain

$$I_{XY}(\alpha \wedge \beta) = (I_{XY}\alpha) \wedge \beta + \alpha \wedge (I_{XY}\beta) \quad (1.22)$$

Thus I_{XY} is defined by its action on 1-forms, for which we see $I_{XY}(\alpha) = 0$ thus $I_{XY} = 0$. For the fifth identity we simply note

$$[\iota_X, \mathcal{L}_Y](f) = \iota_X \iota_Y(f), \quad [\iota_X, \mathcal{L}_Y](df) = \iota_{[X, Y]} df \quad (1.23)$$

Now the only one left to prove is the sixth identity. First, for functions, we get

$$[\mathcal{L}_X, \mathcal{L}_Y](f) = [X, Y](f) \quad (1.24)$$

Now for 1-forms we obtain, using the Cartan-Magic-Formula and the Jacobi-Identity:

$$\mathcal{L}_{[X, Y]} = [d, \iota_{[X, Y]}] = [\mathcal{L}_X, \mathcal{L}_Y] \quad (1.25)$$

□

We go on to find an interesting expression for the de Rham differential d . To this end, we note that for a closed form ω (thus $d\omega = 0$) we have

$$\mathcal{L}_X \omega = d \circ \iota_X \omega \quad (1.26)$$

Now we use some induction and the above observation to prove the following statement:

LEMMA 1.19 Given $k + 1 \geq 2$ vector fields $X_0, \dots, X_k \in \mathfrak{X}(M)$, the following identity holds:

$$[\iota_{X_k} \dots \iota_{X_1}, \mathcal{L}_{X_0}] = \sum_{i=1}^k (-1)^{i+1} \iota_{X_k} \dots \widehat{\iota_{X_i}} \dots \iota_{X_1} \iota_{[X_i, X_0]} \quad (1.27)$$

Proof. We use induction on k . Since we have already shown the formula for $k = 1$, let the statement hold for some $(k - 1)$. Now for k we get

$$[\iota_{X_k} \dots \iota_{X_1}, \mathcal{L}_{X_0}] = \iota_{X_k} \dots \iota_{X_2} [\iota_{X_1}, \mathcal{L}_{X_0}] + [\iota_{X_k} \dots \iota_{X_2}, \mathcal{L}_{X_0}] \iota_{X_1} \quad (1.28)$$

Using our assumption for $k - 1$, we obtain the required form which proves the statement. \square

Now we use the previous lemma, to constitute a way of calculating the de Rham differential of a form.

PROPOSITION 1.20 Let again $X_0, \dots, X_k \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$. Then

$$d\omega(X_0, \dots, X_k) = \sum_{i=0}^k (-1)^i X_i(\omega(X_0, \dots, \widehat{X_i}, \dots, X_k)) \quad (1.29)$$

$$+ \sum_{0 \leq i, j \leq k} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k) \quad (1.30)$$

Proof. We again use induction on k . For $k = 0$ we already know the formula holds. Now given for some $k - 1$, we want to show that it also holds for k . To this end, note that

$$(d\iota_{X_0}\omega)(X_1, \dots, X_k) = \sum_{i=1}^k (-1)^{i+1} X_i(\iota_{X_0}\omega(X_1, \dots, \widehat{X_i}, \dots, X_k)) \quad (1.31)$$

$$+ \sum_{1 \leq i, j \leq k} (-1)^{i+j} \iota_{X_0}\omega([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k) \quad (1.32)$$

$$= \sum_{i=1}^k (-1)^{i+1} X_i(\omega(X_0, X_1, \dots, \widehat{X_i}, \dots, X_k)) \quad (1.33)$$

$$+ \sum_{1 \leq i, j \leq k} (-1)^{i+j} \omega([X_i, X_j], X_0, X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k) \quad (1.34)$$

Now we further note

$$d \circ \iota_{X_0} \omega(X_1, \dots, X_k) = \iota_{X_k} \dots \iota_{X_1} \circ d \circ \iota_{X_0} \omega \quad (1.35)$$

$$= -d\omega(X_k, X_0) + \sum_{i=1}^k (-1)^{i+1} \iota_{X_k} \dots \widehat{\iota_{X_i}} \dots \iota_{X_1} \iota_{[X_i, X_0]} - X_0(\omega(X_1, \dots, X_k)) \quad (1.36)$$

Putting those two together proves the Proposition. \square

1.2 Sheaves

DEFINITION 1.21 (Presheaf) Let X be a topological space. A **presheaf** \mathcal{F} of rings on X consists of the following data:

- (1) For every open subset $U \subset X$ there exists a ring $\mathcal{F}(U)$
- (2) For every inclusion $V \subset U$ of open subsets of X , there exists a morphism of rings

$$\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V) \quad (1.37)$$

such that the following properties hold

- a) $\mathcal{F}(\emptyset) = 0$
- b) ρ_{UU} is the identity morphism
- c) If $W \subset V \subset U$ then

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV} \quad (1.38)$$

DEFINITION 1.22 (Sheaf) A presheaf \mathcal{F} on a topological space X is a **sheaf**, if it satisfies the following additional properties:

- d) If U is an open subset of X and $\{V_i\}$ an open cover of U , $s \in \mathcal{F}(U)$ such that $\rho_{UV_i}(s) = 0 \quad \forall i$ then $s = 0$.
- e) If U is an open subset of X and $\{V_i\}$ an open cover of U . Let $s_i \in \mathcal{F}(V_i)$ such that

$$\rho_{V_i \cap V_j}(s_i) = \rho_{V_j \cap V_i}(s_j) \quad \forall i, j \quad (1.39)$$

then there exists an $s \in \mathcal{F}(U)$ such that $\rho_{UV_i}(s) = s_i \quad \forall i$.

Alternatively, one can discard property d) and ask in e) that there exists a unique such $s \in \mathcal{F}(U)$.

We again provide some interesting examples of sheaves and presheaves:

EXAMPLE 1.23

- Let X be a topological space. For each open set $U \subseteq X$ let $C(U)$ be the ring of continuous functions with codomain \mathbb{R} . The maps $\rho_{UV} : C(U) \rightarrow C(V)$ for $V \subset U$ are now given by the restrictions of these functions in the usual sense. This makes C a sheaf of rings on X .
- Let $f : X \rightarrow Y$ be a continuous map between topological spaces X, Y . Define

$$\mathcal{F}(U) := \{s : U \subset Y \rightarrow X, f \circ s = \text{id}_U\} \quad (1.40)$$

This constitutes a sheaf on Y . The maps s in $\mathcal{F}(U)$ are called **sections** of the sheaf.

- Let $X = \mathbb{R}$ and define $\mathcal{F}(U)$ to be the ring of bounded functions of an open subset $U \subseteq \mathbb{R}$. Now this is not a sheaf: If $U = \mathbb{R}$, we choose $V_i := \{x \in \mathbb{R} \text{ s.t. } |x| < i\}$. Now let $\text{id}: x \mapsto x$ denote the identity map. We note that $\text{id} \in \mathcal{F}(V_i) \forall i \in \mathbb{N}$. Thus there cannot exist $s \in \mathcal{F}(\mathbb{R})$ s.t. $s|_{V_i} = \text{id}|_{V_i}$ since $\text{id}_{\mathbb{R}} \notin \mathcal{F}(\mathbb{R})$ as it is not bounded. Thus we found a presheaf that is not a full sheaf.

The next natural step is to again identify the data needed for morphisms in the category of presheaves:

DEFINITION 1.24 (Morphisms of Sheaves) Let \mathcal{F} and \mathcal{G} be two presheaves (or sheaves) on a topological space X . A **morphism of (pre-) sheaves** consist of a morphisms of rings $\varphi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for all open sets $U \in X$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \rho_{UV} \downarrow & & \downarrow \rho'_{UV} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array} \quad (1.41)$$

Which sums up to saying $\rho' \circ \varphi = \varphi \circ \rho$.

To further expand upon the agricultural language of our setting, we go on to discuss the notion of germs:

DEFINITION 1.25 (Germs) Let X be a topological space and $x \in X$. A **germ** on X at x is an equivalence class of functions $f : X \rightarrow Y$ for a set Y where we define $f \sim g$ if there exists a neighbourhood $U \subseteq X$ of x such that $f|_U = g|_U$.

Now it is very reasonable to ask how a sheaf behaves at an arbitrary point of the base topological space X . To this end, we can look at its behaviour in a small neighbourhood of a chosen point and then implement a fitting limit to come arbitrarily close. To this end, we use the direct limit known from category theory.

DEFINITION 1.26 (Stalks) Let \mathcal{F} be a presheaf on X , $p \in X$. The **stalk** \mathcal{F}_p of \mathcal{F} at p is the direct limit of the rings $\mathcal{F}(U)$ for all open sets U containing p through the restriction maps ρ .

EXERCISE 1.27 As an interesting exercise you can show that the space of continuous functions at a point $p \in X$ is the stalk at p of the sheaf of continuous functions on X .

DEFINITION 1.28 (Locally Ringed Spaces) A ring R is a **local ring** if it has a unique maximal left or right ideal.

DEFINITION 1.29 (Ringed Space) A **ringed space** (X, \mathcal{O}_X) is a topological space X together

with a sheaf of rings O_X on X . O_X is called the **structure sheaf** of X .

DEFINITION 1.30 (Locally Ringed Space) A **locally ringed space** is a ringed space such that all the stalk of O_X are local rings.

In the following we will mostly use notion of locally ringed spaces. Before continuing, we again provide an example for the above constructs:

EXAMPLE 1.31

- Let X be a topological space. X is a locally ringed space if it is equipped with the sheaf of real valued continuous functions on X . The same holds for X a manifold equipped with the sheaf of smooth functions. As an exercise you can prove that X is indeed locally ringed. Hint: The unique maximal ideal consists of the germs for which the value at $x \in X$ is 0.
- A smooth manifold is a locally ringed space locally isomorphic to $(\mathbb{R}^n, \mathcal{C}^\infty(M, \mathbb{R}))$.

1.3 Graded Manifolds

DEFINITION 1.32 (Graded Manifold) A **graded manifold** M is a locally ringed space (M, O_M) where M is a manifold and O_M is a sheaf such that, for $U \subseteq \mathbb{R}^n$ an arbitrary open subset, $O_M|_U \cong (U, \mathcal{C}^\infty(U) \otimes S^\bullet((W_U^*)^\bullet))$. W_U is a graded vector space such that $(W_U)^0 = \{0\}$ (here, the index 0 indicates the grade). Recall that $S^\bullet((W_U^*)^\bullet)$ denotes the graded algebra of graded polynomial functions on W_U .

The sections of the sheaf O_M are called **graded functions** on the graded manifold M . Note that the graded functions form a graded algebra.

For later use, consider $S^\bullet((W_U^*)^\bullet)$ as a generalisation of formal power series (say in \hbar) and $\mathcal{C}^\infty(U)$ as smooth functions depending on n variables.

Remark 1.33 The local isomorphism required above should preserve the grading of

$$\bigoplus_{k \geq 0} \mathcal{C}^\infty(U) \otimes S^k W_U^* \quad (1.42)$$

Note further that $|f \otimes v| = |v|$.

On a graded manifold we do have local coordinates. But aside from the usual coordinates defined using open coverings of M , we also have coordinates that do justice to the graded setting:

We denote the coordinates by $(x_i)_{i=1, \dots, m}^k$ with $x^k: U \longrightarrow S^\bullet((W_U^*)^k)$. Now coordinates of degree 0 are coordinates on the open sets of an open cover U_i . Meanwhile coordinates with $|x_i| \neq 0$ are

coordinates on $S^\bullet(W^*)$.

Now we once again turn to the respective morphisms of graded manifolds. Looking at the definition of a graded manifold, we first need to define the morphisms of (locally) ringed spaces:

DEFINITION 1.34 (Morphisms of Ringed Spaces)

A **morphism between ringed spaces** (X, O_X) and (Y, O_Y) is a pair (f, ψ) where $f : X \rightarrow Y$ is continuous and $\psi : O_Y \rightarrow f_*O_X$ is a local (see below) morphism of sheaves (on Y). f_*O_X denotes the direct image of the structure sheaf of X defined for opens $U \subseteq Y$ as

$$(f_*O_X)(U) := O_X(f^{-1}(U)) \quad (1.43)$$

Unpacking the above definition we note: We have $f : X \rightarrow Y$ continuous and a family of morphisms of rings $\psi_U : O_Y(V) \rightarrow O_X(f^{-1}(V))$ for all $V \subseteq Y$ open. Now this makes the following diagram commute for some $V_1 \subseteq V_2$:

$$\begin{array}{ccc} O_Y(V_2) & \xrightarrow{\psi_{V_2}} & O_Y(f^{-1}(V_2)) \\ \rho_{V_2 V_1}^Y \downarrow & & \downarrow \rho_{f^{-1}(V_2) f^{-1}(V_1)}^X \\ O_Y(V_1) & \xrightarrow{\psi_{V_1}} & O_Y(f^{-1}(V_1)) \end{array}$$

An algebraic version of the diagram can be formed as such: Let $s \in O_Y(V_2)$ be a section. Then $\psi(s) \in O_Y(f^{-1}(V_2))$ and $\rho^X(\psi(s)) \in O_X(f^{-1}(V_1))$. Using ρ^Y instead we finally obtain:

$$\rho^X \circ \psi = \psi \circ \rho^Y \quad (1.44)$$

Now in the case of a locally ringed space, we have an additional condition that must uphold. Namely the ring homomorphism induced by ψ between the stalks of Y and X has to be a local homomorphism i.e. it has to map the maximal ideal to the maximal ideal.

DEFINITION 1.35 (The Algebra of Smooth Functions)

The **algebra of smooth functions** $\mathcal{C}^\infty(M)$ of a graded manifold (M, O_M) is the algebra of global sections of O_M . It automatically inherits a \mathbb{Z} grading and thus forms a graded algebra.

Note that the graded symmetric algebra is the generalisation of both the symmetric algebra and the exterior algebra. We use the previous definitions to pull the concept of a vector bundle to the graded setting:

DEFINITION 1.36 (Graded Vector Bundles)

A **graded vector bundle** over a manifold M is a collection of ordinary vector bundles $(E_i)_{i \in \mathbb{Z}}$ over M .

We try to bring the matter closer to the reader by giving one example:

EXAMPLE 1.37 Given a manifold M and a vector bundle $E \rightarrow M$, the sheaf of sections

$$U \mapsto \Gamma(U, S^\bullet(E^*|_U)) \quad (1.45)$$

is a graded manifold denoted by E . As a rather complicated exercise, you can prove the above statement.

EXAMPLE 1.38

- Let V be a vector bundle. We can construct a graded manifold as in Example 1.37 by taking $E := V[1]$. Here, $V[1]$ is the graded vector bundle with V at grade 1 and no other content.
- In the same manner we can take $T[1]M$ to get the graded manifold $\Omega^\bullet(M)$.

Remark 1.39 (Serre–Swan Theorem) Under certain hypotheses one could prove that any graded manifold is isomorphic to a graded manifold associated to a graded vector-bundle. This result is a version of the Serre–Swan Theorem.

This theorem tells us that all graded manifolds can be considered to be of the form seen in the aforementioned example regarding the sheaf of sections. We go on to define *graded vector fields* to fit some analysis into our graded setting.

DEFINITION 1.40 (Graded Vector Fields) A **graded vector field** on a graded manifold M is a graded linear map

$$X : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)[k] \quad (1.46)$$

satisfying the **graded Leibniz rule**

$$X(fg) = X(f)g + (-1)^{k|f|} f X(g) \quad \forall f, g \in \mathcal{C}^\infty(M) \quad (1.47)$$

It is easy to show that the graded vector fields form a graded vector space.

Now we use the coordinates x^i , we previously defined on a graded manifold, to locally express graded vector fields as

$$X := X^i \frac{\partial}{\partial x^i} \quad (1.48)$$

EXAMPLE 1.41 The *graded Euler vector field* is a graded vector field existing for any graded manifold M defined as

$$E(f) := |f| f \quad (1.49)$$

for f any homogenous function. As an exercise you can show that E is a graded derivation of degree 0.

In local coordinates, we can express the Euler vector field as

$$E = \sum_i |x^i| x^i \frac{\partial}{\partial x^i} \quad (1.50)$$

Remark 1.42 The **graded commutator** equips the graded vector space of graded vector fields with the structure of a graded Lie algebra:

$$[X, Y] := X \circ Y - (-1)^{kl} Y \circ X \quad (1.51)$$

It is an easy exercise to prove this statement.

Now we further investigate the graded Euler vector field and its remarkable properties.

PROPOSITION 1.43 For any graded vector field X we have

$$[E, X] = \deg(X) X \quad (1.52)$$

Proof.

$$[E, X](f) = E(X(f)) - X(E(f)) = |X(f)|X(f) - X(|f|f) \quad (1.53)$$

$$= (|X(f)| - |f|)X(f) = \deg(X) X(f) \quad (1.54)$$

□

To bring further notions of our usual geometric background to the graded setting, we turn towards the notion of homology as a powerful tool to classify manifolds using their inert topological properties:

DEFINITION 1.44 (Cohomological Vector Fields)

A **cohomological vector field** is a graded vector field of degree +1 which commutes with itself, thus

$$[Q, Q] = 0 \quad \Rightarrow \quad 2Q \cdot Q = 0 \quad \Rightarrow \quad Q^2 = 0 \quad (1.55)$$

Note that thus every cohomological vector field corresponds to a differential on the graded algebra of smooth functions $\mathcal{C}^\infty(M)$.

Again, we provide some interesting examples to further illuminate the definitions:

EXAMPLE 1.45

- Consider $T[1]M$. The algebra of smooth functions on it is the algebra of differential forms on M , namely $\Omega^\bullet(M)$ using the de Rham differential. Note that M is just a normal manifold in this example. This shows that the de Rham differential is a cohomological vector field on

$T[1]M$ seen as the graded manifold $(U, \mathcal{C}^\infty(U) \otimes S(T[1]M^*))$.

- Let \mathfrak{g} be a finite dimensional Lie algebra. We look at the graded manifold with body a point and $W = \mathfrak{g}[1]$. This graded manifold carries a cohomological vector field Q corresponding to the Chivally–Eilenberg differential on $\bigwedge^\bullet \mathfrak{g}^* = \mathcal{C}^\infty(\mathfrak{g}[1])$. We use $(e_i)_{i=1}^n$ as a basis of \mathfrak{g} and denote by f_{ij}^k the induced structure constants

$$[e_i, e_j] = \sum_{k=1}^n f_{ij}^k e_k \quad (1.56)$$

Now we can define Q to be

$$Q := \frac{1}{2} \sum_{i,j=1}^n x^i x^j f_{ij}^k \frac{\partial}{\partial x^k} \quad (1.57)$$

Where we denoted by $(x^i)_{i=1}^n$ the coordinates on $\mathfrak{g}[1]$ dual to the basis $(e_i)_{i=1}^n$. As an interesting exercise, prove that Q is indeed cohomological, i.e. of degree 1 and $Q^2 = 0$.

DEFINITION 1.46 (Differential Graded Manifolds)

A graded manifold together with a cohomological vector field is called a **differential graded manifold** or short **dg-manifold**. The morphisms of dg-manifolds are morphisms of graded manifolds with respect to which the respective cohomological vector fields are related.

We turn our heads towards integration in the graded setting. To this end, we investigate differential forms. The main problem will be the problematic transformation behaviour of differential forms that will dramatically complicate the notion of integration. As a result, we will turn towards the symplectic setting which eliminates many of the upcoming difficulties.

Locally, we reconstruct the algebra of differential forms on M by adding a new set of coordinates $(dx^i)_{i=1}^n$ with total degree (grading plus degree of differential form) $(|x^i| + 1)$. Note that generally $(dx^i)^2 \neq 0$. The rough outline of the procedure looks as follows:

1. Differential forms are reconstructed as a graded manifold starting from $T[1]M$.
2. The de Rham differential has already been shown to be a cohomological vector field on $\mathcal{C}^\infty(T[1]M)$.

Now we interpret $T[1]M$ as a dg-manifold where Q is the de Rham differential. In coordinates, we have

$$Q = \sum_{i=1}^n dx^i \frac{\partial}{\partial x^i} \quad (1.58)$$

DEFINITION 1.47 (Graded de Rham Complex)

The **graded de Rham complex** (cochain complex) $(\Omega^\bullet(M), d)$ is defined as $\mathcal{C}^\infty(T[1]M)$ equipped with the differential (coboundary operator) Q as defined above. Thus the elements of the space of

this complex are differential forms on M .

Now our next goal is to extend the Cartan Calculus to this setting. The main point is thus to define the respective operators. First note that $\iota_{\frac{\partial}{\partial x^i}}$ has degree $(|x^i| - 1)$ and $\iota_{\frac{\partial}{\partial x^i}} dx^j = \delta_i^j$. Note further that

$$\mathcal{L}_X \omega := \iota_X \circ d\omega + (-1)^{|X|} d \circ \iota_X \omega \quad (1.59)$$

Now it would be convenient to pinpoint the degree of a differential form. To no surprise, the graded Euler vector field can be used to do just that:

$$\mathcal{L}_E x^i = E(x^i) = |x^i| x^i \quad (1.60)$$

$$\mathcal{L}_E dx^i = d \circ \iota_E(dx^i) = d \circ \mathcal{L}_E(x^i) = |x^i| dx^i \quad (1.61)$$

Thus we can unambiguously define the degree of a differential form:

DEFINITION 1.48 We define the degree $|\omega|$ of a differential form ω through the formula

$$\mathcal{L}_E \omega = |\omega| \omega \quad (1.62)$$

With this definition at hand, we are ready to introduce the symplectic setting.

DEFINITION 1.49 (Graded Symplectic Forms)

A **graded symplectic form** of degree k is a two-form ω which has the following properties:

1. ω is homogeneous of degree k
2. ω is closed wrt. the de Rham operator
3. ω is non-degenerate, i.e. the induced musical morphism

$$\omega^i : TM \rightarrow T^*[k]M, \quad (1.63)$$

$$X \mapsto \omega(X, \cdot) \quad (1.64)$$

is an isomorphism.

DEFINITION 1.50 (Graded Symplectic Manifolds)

A **graded symplectic manifold of degree k** is a pair (M, ω) where M is a graded manifold and ω is a graded symplectic form on M of degree k .

This definition implies some interesting properties of graded symplectic manifolds:

LEMMA 1.51 Given (M, ω) , let $\text{gr}(\omega) = k \neq 0$. This implies that ω is exact.

Proof. We easily show that

$$k \cdot \omega = \mathcal{L}_E \omega = d \circ \iota_E \omega \quad \Rightarrow \quad \omega = d \left(\frac{1}{k} \iota_E \omega \right) \quad (1.65)$$

□

We extend the notion of a symplectic form to vector fields:

DEFINITION 1.52

- Let ω be a graded symplectic form on M . A vector field X is a **symplectic vector field** if

$$\mathcal{L}_X \omega = 0 \quad (1.66)$$

- A vector field X is an **hamiltonian vector field** if the contraction of X and ω is an exact one-form, i.e. $\exists H \in \mathcal{C}^\infty(M)$ s.t.

$$\iota_X \omega = dH \quad (1.67)$$

LEMMA 1.53 Let ω be a graded symplectic form on M with $\text{gr}(\omega) = k$ and let X be a symplectic vector field of degree l . If $k + l \neq 0$, then X is hamiltonian.

Proof. First, we note that

$$[E, X] = lX, \quad \mathcal{L}_X \omega = d \circ \iota_X \omega = 0, \quad \mathcal{L}_E \omega = k\omega \quad (1.68)$$

Now we can define $H := \iota_E \iota_X \omega$ to obtain

$$dH = d \circ \iota_E \iota_X \omega = \mathcal{L}_E \iota_X \omega - \iota_E \circ d \iota_X \omega \quad (1.69)$$

$$= \mathcal{L}_E \iota_X \omega = \iota_{[E, X]} \omega + \iota_X \mathcal{L}_E \omega = \iota_X \omega + \iota_X (k\omega) \quad (1.70)$$

$$= \iota_X \omega + k \iota_X \omega = (l + k) \iota_X \omega \quad (1.71)$$

and thus

$$\iota_X \omega = \frac{dH}{k + l} \quad (1.72)$$

which proves the claim. □

DEFINITION 1.54 (Differential Graded Symplectic Manifolds)

A graded symplectic manifold of degree k with a symplectic cohomological vector field is called a **differential graded symplectic manifold of degree k** and is denoted by (M, ω, Q) .

Using the previous lemma we see that, given $\text{gr}(\omega) \neq -1$, the cohomological vector field Q is necessarily hamiltonian. As an interesting analogy, we can define a structure similar to that of the well-known *Poisson-bracket* using the symplectic form of a differential graded symplectic manifold.

To this end, define

$$\{\cdot, \cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M), \quad (1.73)$$

$$(f, g) \mapsto \{f, g\} := (-1)^{|f|+1} X_f(g) \quad (1.74)$$

where we denote by X_f the unique hamiltonian graded vector field corresponding to f , thus satisfying

$$\iota_{X_f} \omega = df \quad (1.75)$$

As an exercise you can show that the thus defined bracket satisfies relations strikingly similar to those of the Poisson-bracket.

Using the notion of a hamiltonian graded vector field, we take a differential graded symplectic manifold (M, ω, Q) and use the innert cohomological vector field Q to define the unique function S that satisfies (modulo sign)

$$\iota_Q \omega = dS \quad (1.76)$$

Now since $Q = \{S, \cdot\}$ we see

$$Q(f) = \{S, f\} = (-1)^{|S|+1} X_S(f) = X_S(f) \quad (1.77)$$

implying that $[Q, Q](f) = \{\{S, S\}, f\} = 0$ making $\{S, S\}$ a constant. Now since $\{S, S\}$ has degree $k+2$, ω has degree k and S has degree $k+1$ we see that

$$\{S, S\} = 0 \quad (\text{CME})$$

since constants have degree 0. This equation is called the **Classical Master Equation** and will become increasingly important towards the end of the course.

1.4 Principal Bundles

Remark 1.55 In the following subsection, we consider everything to be smooth which makes a lot of subtle details easier.

We begin by giving the barebone definition of a Fibre bundle. Good additional material on this topic can be found in [1] and [2].

DEFINITION 1.56 (Fibre Bundles and Sections)

Let M be a manifold. A **bundle** over M is a triple $E \xrightarrow{\pi} M$ where E is a smooth manifold and π a smooth surjective map $\pi : E \rightarrow M$.

A **fibre bundle** is a bundle for which all fibres, that is preimages of the form $\pi^{-1}(p)$ for $p \in M$, are diffeomorphic to one another.

Now a **section** (or cross section) of a bundle (E, π, M) is a smooth map $s : M \rightarrow E$ satisfying the

section property $\pi \circ s = id_M$.

Now recall that a **G -space** X is a topological space X equipped with a Lie Group G and a smooth right (or left) action $\sigma : G \times X \rightarrow X$. We now use the notion of orbit spaces to form special fibre bundles using the quotient spaces under a given action:

DEFINITION 1.57 A bundle (E, π, M) is a **G -bundle** if E is a right G -space and if (E, π, M) is isomorphic to the bundle $(E, \rho, E/G)$ where E/G denotes the quotient space of the respective action and ρ is the standard projection.

We introduce further classification of G -bundles by restricting to certain types of actions:

DEFINITION 1.58 (Principal G -bundles) A G -bundle (E, π, M) is a **principal G -bundle** if G acts freely on E and transitively on fibres.

Remember that an action σ is free, if for any $g \in G$ and $x \in X$ the equation $\sigma_g(x) = x$ implies $g = e_G$. Now a free action results in each orbit, and thus each fibre, being homomorphic to G itself and hence (E, π, M) is indeed a fibre bundle as implied before. We give some interesting examples to illuminate the definitions:

EXAMPLE 1.59

- The **product bundle** $(M \times G, \pi_1, M)$ is a principal G -bundle under the action $(p, g_0)g = (p, g_0g)$.
- Let again G be a Lie group and H be a closed Lie subgroup of G . Then H acts freely on G via the right action and we denote the orbit space by G/H . Now the triple $(G, \pi, G/H)$ where π is the standard projection to the orbit space is a principal H -bundle.
- Let M be an m -dimensional manifold. A linear frame b_p at $p \in M$ is an ordered set (b_1, \dots, b_m) of vectors that forms a basis of the tangent space T_pM . Now we define the set $B(M)$ to be the set of all frames at all points of M and the projection $\pi : B(M) \rightarrow M$ that maps $b_p \mapsto p$. Using the vector space structure of the tangent spaces, we can define a natural free action on $B(M)$ using the Lie group $GL(m, \mathbb{R})$ via

$$(b_1, \dots, b_m)g := \left(\sum_{j_1}^m b_{j_1} g_{j_1 1}, \dots, \sum_{j_m}^m b_{j_m} g_{j_m 1} \right) \quad (1.78)$$

Now as an exercise $B(M)$ can be given the structure of a smooth manifold. This enables us to show that $(B(M), \pi, M)$ is a $GL(m, \mathbb{R})$ -principal bundle, known as the frame bundle.

Having defined a new set of objects, we again define the morphisms between them to enable comparison and transition.

DEFINITION 1.60 (Principal Bundle Morphisms)

Let (P, π_1, M) be a principal G -bundle and (Q, π_2, N) be a principal H -bundle. A **principal bundle map** is a triple (u, φ, Λ) such that $u : P \rightarrow Q$ and $\varphi : M \rightarrow N$ are smooth maps and $\Lambda : G \rightarrow H$ is a group homomorphism and such that

$$\pi_2 \circ u = \varphi \circ \pi_1, \quad (1.79)$$

$$u(pg) = u(p)\Lambda(g) \quad \forall p \in P, g \in G. \quad (1.80)$$

We thus arrive at an important conclusion regarding bundles over the same base manifold.

THEOREM 1.61 Let (u, id_M, id_G) be a principal bundle map between the principal G -bundles (P, π_1, M) and (Q, π_2, M) . Then u is an isomorphism.

Proof. This proof is left as an easy but important exercise left to the reader. \square

We further classify special cases of principal G -bundles using the aforementioned concepts:

DEFINITION 1.62 A principal G -bundle (P, π, M) is **trivial** if there is a principal bundle map to the product bundle $(M \times G, \pi_1, M)$.

An important property to classify trivial bundles is the existence of smooth global sections (which imply global frames):

THEOREM 1.63 A principal G -bundle (P, π, M) is trivial iff there exists a smooth global section.

Proof. Let $\sigma : M \rightarrow P$ be a section such that $\pi \circ \sigma(x) = x$ for all $x \in M$. Now for all $p \in P$ there exists a $\chi_\sigma(p) \in G$ such that

$$p = \sigma(\pi(p))\chi_\sigma(p) \quad (1.81)$$

Now $\chi_\sigma : P \rightarrow G$ is uniquely defined since G acts freely on P . Furthermore we have

$$\chi_\sigma(pg) = \chi_\sigma(p)g \quad (1.82)$$

and thus

$$\sigma(\pi(pg))\chi_\sigma(pg) = pg \quad (1.83)$$

$$\Rightarrow \sigma(\pi(p))\chi_\sigma(p)g = \sigma(\pi(p))\chi_\sigma(pg) \quad (1.84)$$

Thus we can define a map

$$u_\sigma : P \longrightarrow M \times G, \quad (1.85)$$

$$p \longmapsto u_\sigma(p) := (\pi(p), \chi_\sigma(p)) \quad (1.86)$$

Now note that $\pi \circ u_\sigma = \pi$ and further $u_\sigma(pg) = u_\sigma(p)g$. Thus u_σ is a principal bundle map and

conversely we can define $h : M \times G \longrightarrow P$ as a principal bundle map via

$$\sigma_u : M \longrightarrow P, \quad (1.87)$$

$$x \longmapsto h(x, e_G) \quad (1.88)$$

This proves the theorem. \square

We take a step back and investigate a natural product defined on G -spaces. This will enable us to define associated bundles, a very important naturally occurring form of principal bundles.

DEFINITION 1.64 Let X and Y be a pair of G -spaces. Then their **G -product** $X \times_G Y$ is the space of orbits of the G -action on the cartesian product $X \times Y$, namely $X \times_G Y = (X \times Y)/G$ where

$$(x, y) \sim (x', y') \quad \text{if} \quad \exists g \in G \text{ s.t. } x' = xg, \quad y' = yg \quad (1.89)$$

DEFINITION 1.65 (Associated Bundles) Let $\xi = (P, \pi, M)$ be a principal G -bundle and let F be a left G -space. Define

$$P_F := P \times_G F \quad (1.90)$$

$$(p, v)g := (pg, g^{-1}v) \quad (1.91)$$

and further $\pi_F : P_F \rightarrow M$ via $\pi_F([p, v]) = \pi(p)$. Then $\xi[F] := (P_F, \pi_F, M)$ is a fibre bundle on M with fibre F . We call this bundle the **associated bundle** to the principal bundle ξ .

To show that the above is indeed a well-defined concept, we need to prove that the introduced map π_F does not care about the choice of the representative. Also we need to show that all fibres of the given bundle are indeed homeomorphic to one another:

Proof. This proof can be split in two steps:

- (1) Let $(p_1, v_1) \sim (p_2, v_2)$ thus there exists $g \in G$ such that $(p_2, v_2) = (p_1g, g^{-1}v_1)$. Now our goal is to show that $\pi_F(p_1, v_1) = \pi_F(p_2, v_2)$ and $\pi(p_1g) = \pi(p_1)$. But since $\pi_F(p_1, v_1) = \pi(p_1)$, we easily see that

$$\pi_F(p_2, v_2) = \pi(p_2) = \pi(p_1g) = \pi(p_1) = \pi_F(p_1, v_1) \quad (1.92)$$

- (2) For each $x \in M$ the fibre $\pi_F^{-1}(\{x\})$ is homeomorphic to F . As an exercise, you can prove this claim.

\square

Now we use associated bundles, to build a bridge to a very important type of bundle that is used heavily throughout physics and will play a major role in our course:

THEOREM 1.66 (Associated Bundle Theorem)

Let V be any real, finite-dimensional vector space and let $GL(V, \mathbb{R})$ be the group of automorphisms of V . Then the associated bundle $\xi[V]$ is a **vector bundle** (note that here, ξ is a principal $GL(V, \mathbb{R})$ -bundle). Conversely, every vector bundle is bundle-isomorphic to an associated bundle of this type. Note that this is not unique since every such principal bundle qualifies for this transition.

Proof. A proof of this claim can be found in [7]. □

The next important construction is that of a connection on a bundle and the induces notion of curvature both of which will play a major role when we talk about Yang-Mills-Theory later in the course.

DEFINITION 1.67 (Vertical Subspace)

Let $P \xrightarrow{\pi} M$ be a principal G -bundle. The **vertical subspace** of $T_p P$ is defined and denoted as

$$V_p P := \{\tau \in T_p P \mid \pi_* \tau = 0\} \quad (1.93)$$

Where we denote by π_* the pushforward $TP \rightarrow TM$.

Now defining the vertical subspace leaves us with the converse concept, a kind of horizontal space. This will be achieved by defining connections on a bundle:

DEFINITION 1.68 (Connections)

A **connection** on a principal bundle $P \rightarrow M$ is choice of subspaces of TP denoted by HP defined for each $p \in P$ such that $H_p P \subset T_p P$ and such that

1. $T_p P \simeq H_p P \oplus V_p P \quad \forall p \in P$
2. $(\delta_g)_*(H_p P) = H_{pg} P \quad \forall p \in P, \forall g \in G$

where we denoted by $\delta_g(p) = pg$ the action of G .

The main point is that, given a connection on a principal bundle, we can take any $\tau \in T_p P$ and uniquely decompose it as $\tau = \text{hor}(\tau) + \text{vert}(\tau)$. The same is possible for $X \in \mathfrak{X}(P)$ where $X = \text{hor}(X) + \text{ver}(X)$ such that $\text{hor}(X)_p \in H_p P$ and $\text{ver}(X)_p \in V_p P$ for all $p \in P$. Note that these two vector fields are smooth.

Now in order to avoid working with spaces, we aim to give an equivalent notion using Lie algebras and some geometry. First recall that, given a Lie group G , we can identify the Lie algebra $\mathfrak{g} \simeq T_e G$. Now for $A \in \mathfrak{g}$ and a principal G -bundle P , we can define a vector field on TP for $f : P \rightarrow \mathbb{R}$ via

$$X_p^A(f) := \left. \frac{d}{dt} \right|_{t=0} f(p \exp(tA)) \quad (1.94)$$

where $t \in \mathbb{R}$. We denote by \exp the well-known exponential map

$$\exp : \mathfrak{g} \longrightarrow G \quad (1.95)$$

$$x \longmapsto \gamma(1) \quad (1.96)$$

where $\gamma : \mathbb{R} \longrightarrow G$ is the unique 1-parameter subgroup with tangent vector x at the identity. The exponential map is particularly important because it allows us to prove the following theorem:

THEOREM 1.69 Let P be a manifold on which G has a right action. Then the map

$$\psi : \mathfrak{g} \longrightarrow \mathfrak{X}(P) \quad (1.97)$$

$$A \longmapsto X^A \quad (1.98)$$

is a homomorphism of \mathfrak{g} into the infinite dimensional Lie algebra of all vector fields on P , i.e.

$$[X^A, X^B] = X^{[A, B]} \quad \forall A, B \in \mathfrak{g} \quad (1.99)$$

Proof. The proof is quite complicated and can be found on page 196 of [7]. \square

Another result which further explains, why we care about these results in the context of vertical and horizontal vector fields is presented in the following proposition:

PROPOSITION 1.70 For a principal bundle P , the map $\psi_p : A \rightarrow X_p^A$ is an isomorphism of \mathfrak{g} onto $V_p P$. This can be seen by using $D\pi(X)(f) = 0$ due to $\pi(pg) = \pi(p)$ for any $g \in G$. Note that $\dim V_p P = \dim G = \dim \mathfrak{g}$

Hence we can denote the inverse

$$\psi_p^{-1} : V_p P \longrightarrow \mathfrak{g} \quad (1.100)$$

mapping the vertical space at p to the Lie algebra. Using the isomorphism at hand, we can define a uniquely determined one-form on a principal G -bundle which will turn out to be equivalent to a unique connection.

DEFINITION 1.71 (Connection Form)

A **connection form** ω is a \mathfrak{g} -valued one-form on a principal G -bundle $P \rightarrow M$ such that

1. $\omega_p(X^A) = A \quad \forall p \in P, \forall A \in \mathfrak{g}$
2. $\delta_g(p) = pg \quad (\delta_g)_* \omega = Ad_{g^{-1}} \omega \quad \forall g \in G$

where we denote by Ad the adjoint map defined as

$$Ad : G \longrightarrow Aut(G) \quad (1.101)$$

$$g \longmapsto \psi_g[h \in G \longmapsto ghg^{-1}] \quad (1.102)$$

Note that we can also interpret Ad as a map to the automorphisms of \mathfrak{g} since

$$Ad_g = (d\psi_g)_e : T_e G \longrightarrow T_e G \quad (1.103)$$

$$\Rightarrow Ad : G \longrightarrow Aut(\mathfrak{g}) \quad (1.104)$$

Now take a smooth map $\varphi : M \rightarrow N$ and look at the corresponding pullback map

$$\varphi^* : \Omega^k(N) \longrightarrow \Omega^k(M) \quad (1.105)$$

$$\alpha \longrightarrow \varphi^* \alpha \quad (1.106)$$

Now if we take $\varphi = \delta_g : P \rightarrow P$ that maps $p \mapsto pg$, we obtain a pullback map $\Omega^1(P) \rightarrow \Omega^1(P)$. Using a connection form ω we see that $\delta_g^* \omega = Ad_{g^{-1}} \omega$ amounts to the claim

$$\omega_{pg}(D\delta_g(\tau)) = Ad_{g^{-1}}(\omega_p(\tau)) \quad (1.107)$$

This leads us to the following theorem:

THEOREM 1.72 A connection form on a principal bundle P uniquely defines a connection. Vice versa a connection on P uniquely defines a connection form. We can thus use the two interchangeably.

Proof. With some details left out, we give the main sketch of this proof and encourage the reader to fill in the details:

Let ω be a connection form on P . Define $H_p P := \ker \omega_p$. As an exercise one can prove that this definition of $H_p P$ indeed satisfies the properties of a connection. Basically one needs to use that ω_p maps elements of the horizontal space to zero by definition while for vertical vectors we obtain nonzero elements of the Lie algebra. Thus we obtain a way to identify vertical vectors. One then verifies, using the second property of connection forms, the second property of 1.68.

Starting with a connection, we can define

$$\omega_p(\tau) := \psi_p^{-1}(\text{ver}(\tau)) \quad (1.108)$$

which clearly defines a one-form. As an exercise you can check that the thus defined form satisfies the properties of a connection form. For the first property, note that $\omega_p(X^A) = \psi_p^{-1}(\psi_p(A)) = A$. For the second, we remember $H_{pg} P = (\delta_g)_* H_p P$. \square

We will now turn towards the notion of curvature: Given a connection on a principal G -bundle P we have the following splittings for bundles and their sections at hand:

- $TP = VP \oplus HP$
- $T^*P = V^*P \oplus H^*P$

- $\bigwedge^2 T^*P = (\bigwedge^2 V^*P) \oplus (V^*P \wedge H^*P) \oplus (\bigwedge^2 H^*P)$
- $\Omega^1(P) = \Omega_{vert}^1(P) \oplus \Omega_{hor}^1(P)$
- $\Omega^2(P) = \Omega_{vert}^2(P) \oplus \Omega_{mix}^2(P) \oplus \Omega_{hor}^2(P)$

Note that thus a connection form ω lies in $\Omega_{vert}^1(P) \otimes \mathfrak{g}$. Now we use the exterior derivative (de Rham differential) to state

$$d\omega \in \Omega^2(P) \otimes \mathfrak{g} = (\Omega_{vert}^2(P) \oplus \Omega_{mix}^2(P) \oplus \Omega_{hor}^2(P)) \otimes \mathfrak{g} \quad (1.109)$$

and thus

$$d\omega = d\omega_{vert} + d\omega_{mix} + d\omega_{hor} \quad (1.110)$$

which leads us to the following proposition:

PROPOSITION 1.73

- $d\omega_{vert}(X, Y) = [\omega(X), \omega(Y)]$
- $d\omega_{mix} = 0$

Proof. The proof uses the previously presented properties of the action of forms on vector fields and the connection form. It is left as an exercise to the reader. \square

We can now define the notion of curvature on a principal bundle using the differential of a connection form:

DEFINITION 1.74 (Curvature)

Let P be a principal G -bundle and ω a connection form on P . The **curvature** of ω is the unique 2-form

$$F_\omega := d\omega_{hor} \in \Omega_{hor}^2(P) \otimes \mathfrak{g} \quad (1.111)$$

Keep in mind that this notion will be particularly useful in Yang-Mills-Theory. Now one simple case of curvature is trivial curvature:

DEFINITION 1.75 A connection with zero curvature is called **flat**.

DEFINITION 1.76 (Torsors)

A **Torsor** for a group G is a non-empty set X on which G acts freely and transitively.

Above we used an arbitrary set X . If we take G to be a Lie group and X a manifold, we need the action to be smooth to define a torsor of G on X . Note that the definition of a torsor can vary depending on the context and is thus given in a rather general form above.

Unpacking the definition, we first note that we have an action

$$X \times G \longrightarrow X \quad s.t. \quad (1.112)$$

$$x \cdot e = x \quad \forall x \in X \quad (1.113)$$

$$x(gh) = (xg)h \quad \forall g, h \in G \quad (1.114)$$

and such that the following map is an isomorphism of X

$$X \times G \longrightarrow X \times X \quad (1.115)$$

$$(x, g) \longrightarrow (x, xg) \quad (1.116)$$

Thus X and G are isomorphic as sets or manifolds depending on the context. We present some insightful examples of a torsor:

EXAMPLE 1.77

- A group G is a torsor itself using group multiplication as action on itself.
- Take an affine space, thus a set A together with a vector space \vec{A} with a free and transitive action of \vec{A} on A . An affine space A underlying a vector space \vec{A} is a torsor for \vec{A} acting as the additive group of translations.
- A principal bundle $P \xrightarrow{\pi} \{x\}$ over a single point as base space trivially defines a torsor.

resource: Wiki principal homogeneous space.

1.5 (Co-)chain Complexes

In this subsection, we will completely change topic and turn towards the notion of *chain complexes* and *cochain complexes* and thus *homology* and *cohomology*. We start by giving the definition of a (co-)chain complex:

DEFINITION 1.78 A **Chain complex** is a sequence of homomorphisms of abelian groups denoted as

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \quad (1.117)$$

such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{N}$. In other words: $\text{im } \partial_{n+1} \subseteq \ker \partial_n$

DEFINITION 1.79 We call elements of C_n that lie in $\ker \partial_n$ **cycles** and denote the set of cycles by \mathcal{Z}_n . We call elements of C_n that lie in the image of ∂_{n+1} **boundaries** and denote their set by \mathcal{B}_n .

We now use the cycles and boundaries to define homology groups as quotient groups:

DEFINITION 1.80 (Homology groups)

Let $(C_\bullet, \partial_\bullet)$ be a chain complex. We define its **n-th homology group** to be the quotient group

$$\mathcal{H}_n := \ker \partial_n / \operatorname{im} \partial_{n+1} = \mathcal{Z}_n / \mathcal{B}_n \quad (1.118)$$

The equivalence classes that form \mathcal{H}_n (thus cosets of \mathcal{Z}_n) are called **homology classes**.

Like for any introduced category, we are interested in the morphisms between them:

DEFINITION 1.81 (Chain maps)

A **chain map** between two chain complexes $(C_{1\bullet}, \partial_{1\bullet})$ and $(C_{2\bullet}, \partial_{2\bullet})$ is a sequence of homomorphisms $f_n : C_{1n} \rightarrow C_{2n}$ such that

$$\partial_{2n} \circ f_n = f_{n-1} \circ \partial_{1n} \quad (1.119)$$

We can pictorially denote a chain map in the following commuting diagram which of course only represents an excerpt of the full commuting diagram spanned using the chain map:

$$\begin{array}{ccccccc} \dots & \longrightarrow & C_{1\ n+1} & \xrightarrow{\partial_{1n+1}} & C_{1\ n} & \xrightarrow{\partial_{1n}} & C_{1\ n-1} \longrightarrow \dots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \dots & \longrightarrow & C_{2\ n+1} & \xrightarrow{\partial_{2n+1}} & C_{2\ n} & \xrightarrow{\partial_{2n}} & C_{2\ n-1} \longrightarrow \dots \end{array}$$

We use the above diagram to arrive at an interesting intuitive lemma connecting the homology groups of two chain complexes:

LEMMA 1.82 Let $f_\bullet : (C_{1\bullet}, \partial_{1\bullet}) \rightarrow (C_{2\bullet}, \partial_{2\bullet})$ be a chain map. Then f_\bullet induces an homomorphism of homology groups

$$\mathcal{H}_n(f_\bullet) : \mathcal{H}_n(C_{1\bullet}) \rightarrow \mathcal{H}_n(C_{2\bullet}) \quad (1.120)$$

for each $n \in \mathbb{N}$, defined by

$$[\mathcal{Z}_n] \rightarrow [f_n(\mathcal{Z}_n)] \quad (1.121)$$

Proof. Our main goal is to show that f_\bullet maps cycles to cycles and boundaries to boundaries. This is equivalent to showing that it is a homomorphism of homology groups.

We begin by fixing $n \in \mathbb{N}$. Note that $f_n(\mathcal{Z}_n(C_{1\bullet})) \subseteq \mathcal{Z}_n(C_{2\bullet})$ and let $z_n \in \mathcal{Z}_n(C_{1\bullet})$ which means $\partial_{1n} z_n = 0$. Now we calculate

$$\partial_{2n}(f_n(z_n)) = f_{n-1}(\partial_{1n}(z_n)) = 0 \quad (1.122)$$

which shows that f_\bullet maps cycles to cycles. For boundaries, we note $f_n(\mathcal{B}_n(C_{1\bullet})) \subseteq \mathcal{B}_n(C_{2\bullet})$. Now

choose $b \in \mathcal{B}_n(C_{1\bullet})$ i.e. $b = \partial_{1 \ n+1}(a)$ for some $a \in C_{1 \ n+1}$. Now we calculate

$$f_n(b) = f_n(\partial_{1 \ n+1}(a)) = \partial_{2 \ n+1}(f_{n+1}(a)) \quad (1.123)$$

$$\Rightarrow f_n(b) \in \text{im } \partial_{2 \ n+1} \quad (1.124)$$

Thus f_n is constant on equivalence classes. Let now $[z_n^1]$ be an equivalence class and let x^1, x^2 be two representatives. Thus

$$x^1, x^2 \in \mathcal{Z}_n \quad \text{and} \quad x^1 - x^2 = b \in \mathcal{B}_n \quad (1.125)$$

Which proves that the equivalence class of $f_n(x^1)$ is the same as that of $f_n(x^2)$. \square

We go on by classifying chain maps which will prove useful later in the course.

DEFINITION 1.83 A chain map $f_\bullet : C_{1\bullet} \longrightarrow C_{2\bullet}$ is called a **quasi-isomorphism** if $\mathcal{H}_n(f_\bullet)$ is an isomorphism for all $n \in \mathbb{N}$.

Note that the name "quasi-isomorphism" makes sense since the isomorphisms only identify the respective abelian groups and not the entire complex. While it gives us a tool to identify their *homology groups* which will, in most cases, carry most of the important information, it does not tell us anything about the respective homomorphism sequences on the chain complexes. For some examples of chain complexes and their (quasi-)isomorphisms, which require far more work than the scope of this lecture allows, see [8].

DEFINITION 1.84 A sequence of homomorphisms

$$\dots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \dots \quad (1.126)$$

is said to be an **exact sequence** if $\ker \alpha_n = \text{im } \alpha_{n+1} \quad \forall n \in \mathbb{N}$. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0 \quad (1.127)$$

is called a **short exact sequence**. Note that for a short exact sequence $\ker(j) = \text{im}(i)$ as well as

$$\text{im}(j) = \ker(0) \quad \Rightarrow j \text{ is surjective} \quad (1.128)$$

$$\text{im}(0) = \ker(i) \quad \Rightarrow i \text{ is injective} \quad (1.129)$$

We can use short exact sequences for many important proofs and definitions. Thus we provide some extra classification properties:

DEFINITION 1.85 A short exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0 \quad (1.130)$$

is said to **split** if it satisfies one of the following equivalent properties:

1. There is a homomorphism

$$p : B \longrightarrow A \quad (1.131)$$

such that $p \circ i = 1_A : A \longrightarrow A$

2. There is a homomorphism

$$s : C \longrightarrow B \quad (1.132)$$

such that $j \circ s = 1_C : C \longrightarrow C$

3. There exists an isomorphism

$$\psi : B \longrightarrow A \oplus C \quad (1.133)$$

making the following diagram commute:

$$\begin{array}{ccccccc}
 & & & B & & & \\
 & & i \nearrow & \uparrow \psi & \searrow j & & \\
 0 & \longrightarrow & A & & C & \longrightarrow & 0 \\
 & & i' \searrow & \downarrow & \nearrow j' & & \\
 & & & A \oplus C & & &
 \end{array}$$

where $i'(a) = (a, 0)$ and $j'(a, c) = c$.

LEMMA 1.86 The three properties above are equivalent.

Proof.

$1 \rightarrow 3$ We have $p : B \longrightarrow A$. Now define $\psi(b) := (p(b), j(b))$ which can be proven to be an isomorphism and makes the above diagram commute.

$2 \rightarrow 3$ We use s, i and j : Define

$$\psi^{-1}(a, c) := i(a) + s(c) \quad (1.134)$$

which is an isomorphism and makes the above diagram commute. Note that $j(i(a)) + j(s(c)) = c$.

$3 \rightarrow 1/2$ Here we use ψ, i, j . Define $p(b) := pr_1 \circ \psi(b)$ and $s(c) := \psi^{-1} \circ (j')^{-1}(c)$. Now note that $p \circ i = id_A$ due to

$$p(i(a)) = pr_1(\psi(i(a))) = pr_1(i'(a)) = a \quad (1.135)$$

We further have $j \circ s = id_C$ due to

$$j \circ s(c) = j(\psi^{-1}(0, c)) = j'(0, c) = c \quad (1.136)$$

$2 \rightarrow 1$ The last thing we need to show uses s, i, j . To this end define $p(b) := i^{-1}(b - s \circ j(b))$.

□

We now move to the important concept of cochain complexes, which form the dual concept to chain complexes. We will thus encounter *cohomology groups* and the like:

DEFINITION 1.87 A **cochain complex** is a sequence of homomorphisms of abelian groups

$$\dots \longleftarrow D^{n+1} \xleftarrow{\delta^n} D^n \xleftarrow{\delta^{n-1}} D^{n-1} \longleftarrow \dots \quad (1.137)$$

such that $\delta^n \circ \delta^{n-1} = 0$ for each $n \in \mathbb{N}$, namely $\text{im}(\delta^{n-1}) \subseteq \ker(\delta^n)$

DEFINITION 1.88 Elements of D^n are called **cocycles** if they lie in $\ker(\delta^n)$ and **coboundaries** if they lie in $\text{im}(\delta^{n-1})$.

As promised this allows us to define the dual concept to homology groups:

DEFINITION 1.89 (Cohomology groups)

Let $(D^\bullet, \delta^\bullet)$ be a cochain complex. We define the **n-th cohomology group** to be the quotient group

$$\mathcal{H}^n := \ker(\delta^n) / \text{im}(\delta^{n-1}) \quad (1.138)$$

Elements of \mathcal{H}^n (thus cosets of $\ker(\delta^n)$) are called **cohomology classes**.

Now we aim to prove that cochain complexes are dual to chain complexes. To this end, let G be a group and $(C_\bullet, \partial_\bullet)$ be a chain complex. Now define its dual

$$D^{n+1} := (C_n)^* = \text{Hom}(C_n, G) \quad (1.139)$$

as the group of homomorphisms $C_n \rightarrow G$. Now remember that a **dual homomorphism** is defined as follows: Take a group homomorphism $\alpha : A \rightarrow B$. One defines the dual homomorphism (sometimes called "pullback") via

$$\alpha^* : \text{Hom}(A, B) \rightarrow \text{Hom}(A, G), \quad (1.140)$$

$$\varphi \mapsto \alpha^*(\varphi) = \varphi \circ \alpha \quad (1.141)$$

We apply the same concept to our chain complex $\partial_n : C_n \rightarrow C_{n-1}$ which lets us define

$$\delta^n := \partial_n^* : \text{Hom}(C_{n-1}, G) \rightarrow \text{Hom}(C_n, G) \quad (1.142)$$

These maps satisfy $\delta^n \circ \delta^{n-1} = 0$ due to the corresponding equation for ∂_n . There exists a theorem, found in [8], stating that the cohomology groups of C_\bullet under G denoted by $\mathcal{H}^n(C; G)$ are fully determined by G and the homology groups $\mathcal{H}_n(C_\bullet)$.

Remark 1.90 Cochain maps and all of the other notions defined for chain complexes carry over to cochains. For example the analogous equation for cochain maps is

$$f^n \circ \delta_1^{n-1} = \delta_2^{n-1} \circ f^{n-1} \quad (1.143)$$

for maps $f^n : D_1^n \rightarrow D_2^n$. As an exercise, you can transport all of these results to the cochains.

We conclude this subchapter by bringing the notion of substructures to (co-)chain complexes.

DEFINITION 1.91 A **subcomplex** C'_\bullet of a chain complex $(C_\bullet, \partial_\bullet)$ is a sequence of subgroups C'_n of C_n for all $n \in \mathbb{N}$ such that $\partial_n(C'_n) \subseteq C'_{n-1}$ for all $n \in \mathbb{N}$.

Remark 1.92 A subcomplex is a chain complex itself. The inclusion map acts as a chain complex map.

2 An Introduction to Classical Field Theories

In this section, we will explore classical field theories in order to give a solid foundation to their concepts, especially locality. While field theories come in many forms like thermodynamics, electrodynamics, general relativity, the standard model of particle physics and even string theory, we will mainly focus on classical Lagrangian field theories.

In Lagrangian mechanics, as taught in undergrad physics, one uses the notion of the action functional, being a measure for the "excitednes" of a system, to find physically favoured and thus realized trajectories. A common action in this context would be

$$S(q) = \int_{\mathbb{R}^+} \left(\frac{m}{2} \dot{q}^i(t) \dot{q}^i(t) - V(q(t)) \right) dt \quad (2.1)$$

For $q \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^n)$ and $V: \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^n) \rightarrow \mathbb{R}$. Now the condition of vanishing variation $\delta S \equiv 0$ imposes two conditions

$$\Rightarrow \quad \frac{m}{2} \ddot{q}^i(t) + \nabla^i V = 0 \quad \text{Euler-Lagrange-equations} \quad (2.2)$$

$$\Rightarrow \quad \delta q^i \Big|_{t=0} = 0 \quad \text{"Boundary conditions"} \quad (2.3)$$

Now the **boundary form** $\frac{m}{2} \delta q^i(0) \dot{q}^i = \alpha$ can be thought of as a 1-form on $T^*\mathbb{R}^n$ such that the boundary term, using $p^i(0) = \frac{m}{2} \dot{q}^i$, can be written as $\omega := \delta \alpha = \delta q^i(0) \delta p^i(0)$ which is a symplectic form.

This marks the starting point of a procedure called **canonical quantisation** of classical mechanics. As a Hilbert space we use $\mathcal{H} := L^2(\mathbb{R}^n)$. Our main goal will be to generalize this procedure.

2.1 Spaces of fields and Locality

We will usually work on a fibre bundle $\pi: F \rightarrow M$ over some smooth manifold M . For simplicity, we will assume that M is compact and without boundary, furthermore we assume that M is oriented and connected.

DEFINITION 2.1 (Sections and local sections) A **section** of $\pi: F \rightarrow M$ is a smooth map $\varphi: M \rightarrow F$ such that $\pi \circ \varphi = \text{id}_M$. We call φ a **local section** if it is only defined on an open subset $U \subseteq M$ such that $\pi \circ \varphi = \text{id}_U$. We further denote the space of sections by

$$\Gamma(M, F) := \{ \varphi \in \mathcal{C}^\infty(M, F) \mid \pi \circ \varphi = \text{id}_M \} \equiv \mathcal{F} \quad (2.4)$$

We often refer to \mathcal{F} as the **space of fields**. Note that if we work with a vector bundle, \mathcal{F} inherits a linear structure.

We are mainly interested in **locality**. Thus we work with equivalence classes of such sections that coincide in a neighbourhood of a point up to some arbitrary k -th derivative:

DEFINITION 2.2 For $p \in M$ denote by $\Gamma(p)$ the space of local sections whose domain contains p .

Using the thus defined local spaces allows us to define the utterly important notion of *Jets of sections*:

DEFINITION 2.3 (k -Jets of sections) Let $\pi: F \longrightarrow M$ be a fibre bundle and k any integer. We say that two local sections of π at $p \in M$ **have the same k -th jet at p** if their partial derivatives agree at p up to k -th order in some chart around p . We denote by $J_p^k F$ the set of such equivalence classes and use

$$j_p^k \varphi: = [(\varphi, p)]_k \quad (2.5)$$

to denote such equivalence classes.

Remark 2.4 As a rather abstract but interesting exercise, you can show that the above definition does not depend on the choice of coordinate charts. *Hint:* Introduce multiindices I such that if U^α is a chart for F , we look at

$$\left. \frac{\partial^{|I|}}{\partial x^I} (u^\alpha \circ \varphi) \right|_p \quad (2.6)$$

As we will see later, this will also introduce coordinates on the objects $J_p^k F$

DEFINITION 2.5 (Jet bundles) Given a fibre bundle $\pi: F \longrightarrow M$ and an integer k we denote

$$J^k F: = \{j_p^k \varphi | p \in M, \varphi \in \Gamma(p)\} \quad (2.7)$$

and $J^0 F \equiv F$ together with the maps

$$\pi_k: J^k F \longrightarrow M, \quad \mathbf{k\text{-th Jet bundles}} \quad (2.8)$$

$$j_p^k \varphi \longmapsto p \quad (2.9)$$

$$\pi_k^l: J^k F \longrightarrow J^l F, \quad 1 \leq l \leq k \quad (2.10)$$

$$j_p^k \varphi \longmapsto j_p^l \varphi \quad (2.11)$$

such that $\pi_k = \pi \circ \pi_k^0$, $\pi_k^l = \pi_m^l \circ \pi_k^m$ for $0 \leq l \leq m \leq k$. Moreover if φ is a section of our fibre bundle local in some $U \subseteq M$, we define the **Jet Prolongations**

$$j^k: \mathcal{F}_p U \longrightarrow J^k F, \quad (2.12)$$

$$(\varphi, p) \longmapsto j^k(\varphi)(p) := j_p^k(\varphi) \quad (2.13)$$

such that the following diagram commutes for $i \geq k \geq l$:

$$\begin{array}{ccccccc}
\mathcal{F}_p M & \xrightarrow{j^k} & J^k F & \xrightarrow{\pi_k^l} & J^l F & \xrightarrow{\pi_l} & M \\
& & \nwarrow \pi_i^k & & \nearrow \pi_i^l & & \\
& & J^i F & & & &
\end{array}$$

j^i (from $\mathcal{F}_p M$ to $J^i F$) and π_i (from $J^i F$ to M)

The next proposition will go without proof (see Saunders, Ch6).

PROPOSITION 2.6 There exists a sequence of smooth fibre bundles

$$\dots \longrightarrow J^k F \xrightarrow{\pi_k^{k-1}} J^{k-1} F \longrightarrow \dots \longrightarrow F \longrightarrow M \quad (2.14)$$

for every k . Furthermore that maps π_k^{k-1} are *surjective* with *surjective tangent map* (submersions).

Now if (U, u) is an adapted chart (U) of the fibre bundle F such that $(x^i, u^\alpha) \equiv u$ and thus

$$\Rightarrow U^k = \{j_p^k \varphi : p \in M, \varphi(p) \in U \subset F\} \quad (2.15)$$

$$u^k = (x^i, u^\alpha, u_I^\alpha) \quad (2.16)$$

$$u_I^\alpha(j_p^k \varphi) = \left. \frac{\partial^{|I|}(u^\alpha \circ \varphi)}{\partial x^I} \right|_p \quad (2.17)$$

Where I is a multiindex with $|I| = k$ and $U^0 \cong U$. Now to make precise Jets agreeing "for all k " in the above proposition, we need to introduce some extra structures:

DEFINITION 2.7 (Inverse/Projective systems) Define $Sys_X := (\{X_i\}, f_{ij} | i, j \in I \subset \mathbb{N})$ with

- $\{x_i\}$ a collection of spaces (generalizes to objects in a category)
- $f_{ij} : x_i \longrightarrow x_j$ for all i, j s.t. $i \leq j$ and

$$f_{ik} = f_{ij} \circ f_{jk} \quad (2.18)$$

We call this construct an **inverse system** or **projective system**.

Now we denote by $\varprojlim X_i$ the subset of elements $x \in \prod_i X_i$ such that $x_i = f_{ij}(x_j) \quad \forall i \leq j$. We call \varprojlim the **projective/inverse** limit of the inverse system.

DEFINITION 2.8 The sequence $\{J^k F\}_{n \in \mathbb{N}}$ together with $\pi_l^k : J^l F \longrightarrow J^k F$ defines an inverse system (of fibre bundles). We thus use the inverse limit to define $J^\infty F := \varprojlim J^k F$.

To be precise: $J^\infty F$ is the space of equivalence classes of sections $\varphi : M \longrightarrow F$ such that two sections s_1 and s_2 are equivalent, if their partial derivatives agree at all orders. We denote these equivalence classes by $j^\infty(\varphi)$. As an exercise you can show that the germs of functions surject over

$J^\infty \mathcal{R}$ where $\mathcal{R} = M \times \mathbb{R} \longrightarrow R$. You can also find a counterexample for the opposite statement.

Now the interesting question for our physical analysis will be if $J^\infty F$ can be given a smooth manifold structure. It will turn out to be very convenient to consider smooth functions on $J^\infty F$ first:

DEFINITION 2.9 Let $\pi : F \longrightarrow M$ be a fibre bundle and let $J^k F$ denote the space of k -jets. Since we are dealing with regular finite dimensional manifolds, we can consider $\mathcal{C}^\infty(J^k F, P)$ for some manifold P . For every $l \geq k$ there are connecting maps

$$\tilde{\pi}_k^l : \mathcal{C}^\infty(J^k(E), P) \longrightarrow \mathcal{C}^\infty(J^l(E), P) \quad (2.19)$$

which can be constructed using precomposition, thus $(\pi_l^k)^* f := \tilde{\pi}_k^l(f)$. Thus we obtain a so-called **direct system**

$$\mathcal{C}^\infty(F) \longrightarrow \mathcal{C}^\infty(J^1 F) \longrightarrow \mathcal{C}^\infty(J^2 F) \longrightarrow \dots \quad (2.20)$$

with functions $f_{ij} : x_i \longrightarrow x_j$ for $i \leq j$. We further define the **direct limit**

$$\mathcal{C}^\infty(J^\infty F) := \lim_{\longrightarrow} \mathcal{C}^\infty(J^k F) = \Pi_k \mathcal{C}^\infty(J^k F) / \sim \quad (2.21)$$

where $g_i \sim g_l$ iff $\exists k \geq i, k \geq l$ s.t. $\tilde{\pi}_i^k g_i = \tilde{\pi}_l^k g_l$.

Note that by construction $f \in \mathcal{C}^\infty(J^\infty F)$ is fully represented by functions $\hat{f}_k \in \mathcal{C}^\infty(J^k F)$ which only depend on a finite number of derivatives. This will be the essence of the notion of locality which we will unfold in the following pages. Thus if we consider $f \in \mathcal{C}^\infty(J^\infty F)$ represented by \hat{f} on some k -jet then on each coordinate neighbourhood $(\pi^\infty)^{-1}(U)$ and each point $\sigma = j^\infty(\varphi)(p) \in (\pi^\infty)^{-1}(U)$ we have

$$f(\sigma) = \hat{f}(x^i, u^\alpha, u_{i_1}^\alpha, u_{i_1 i_2}^\alpha, \dots, u_{i_1 \dots i_k}^\alpha) \quad (2.22)$$

From now on, we will, in a slight abuse of notation, not distinguish between functions on $J^\infty F$ and their representatives.

2.2 Fréchet Manifolds

Since field theory inherently works with infinite dimensional manifolds, we have a real need for a respective mathematical theory. We saw that if we look at such infinite dimensional manifolds in a local setting, they are "tame" in that we can consider many of their concepts using finite dimensional structures. This will bring us to the notion of a *Fréchet manifold*. While one can approach this concept from a very categorical direction, we will give a less abstract introduction. Note that this part will in no way be exhaustive and not even be enough to understand everything we will use it for later on. Its goal is rather to give a feeling for the formal background which is completely omitted in many introductions to the field due to its depth and level of abstraction.

DEFINITION 2.10 (Seminorm) Let V be a vector space. A **seminorm** is a map $|\cdot| : V \longrightarrow \mathbb{R}$ that satisfies

1. $|v| \geq 0 \quad \forall v \in V$
2. $|v + w| \leq |v| + |w| \quad \forall v, w \in V$
3. $|a \cdot v| = |a| \cdot |v| \quad \forall v \in V, a \in \mathbb{R}$

A **family of seminorms** (for our purposes) is a set $\{|\cdot|_i\}_{i \in I}$ with $|\cdot|_i$ a seminorm for each $i \in I$.

DEFINITION 2.11 (Locally convex topological vector spaces) A **locally convex topological vector space** is a vector space V together with a family of seminorms Γ . We denote a locally convex topological vector space with such a family of seminorms by (V, Γ) .

Using families of seminorms, we can assign a unique topology to the vector space they live on:

DEFINITION 2.12 A family of seminorms Γ on a vector space V defines a **unique topology** \mathcal{T}_Γ compatible with the vector space structure. The neighbourhood base of \mathcal{T}_Γ is given by the family

$$B_\Gamma := \{U_\varepsilon^J \mid \varepsilon > 0, J \subset I \text{ finite}\} \quad (2.23)$$

with $U_\varepsilon^J := \{v \in V \mid |v|_i < \varepsilon \quad \forall i \in J\}$

Using the above notions, we arrive at a proposition that sheds further light onto such spaces.

PROPOSITION 2.13 Let (V, Γ) be a locally convex topological vector space. Then the following statements hold:

1. \mathcal{T}_Γ is the finest topology in which all included seminorms are continuous.
2. (V, Γ) is Hausdorff iff

$$v = 0 \quad \Leftrightarrow \quad (|v|_i = 0 \quad \forall i \in I) \quad (2.24)$$

3. If (V, Γ) is Hausdorff then it is metrisable iff the family Γ is countable (i.e. there exists a metric $d : V \times V \longrightarrow \mathbb{R}_{\geq 0}$ s.t. the topology induced by d is \mathcal{T}_Γ).
4. Convergence of sequences is controlled by the seminorms, i.e.:

$$(v_n)_{n \in \mathbb{N}} : v_n \longrightarrow v \quad \Leftrightarrow \quad |v_n - v|_i \longrightarrow 0 \quad \forall i \in I \quad (2.25)$$

5. V is complete with respect to \mathcal{T}_Γ iff every Cauchy sequence converges, i.e. iff every $(v_n)_{n \in \mathbb{N}}$ with $\lim_{n, m \rightarrow \infty} |v_n - v_m|_i = 0$ converges $\forall i \in I$.

DEFINITION 2.14 (Fréchet spaces) A **Fréchet space** is a sequentially complete, Hausdorff, metrisable, locally convex vector space.

We give some popular examples to illuminate the above definition:

EXAMPLE 2.15

- Every Banach space is a Fréchet space.
- $\mathbb{R}^\infty = \prod_{n \in \mathbb{N}} \mathbb{R}^n$ with either the cartesian topology or the corresponding family of seminorms $\{p_n(x_1, \dots, x_n) = |x_1| + \dots + |x_n|\}$ together with the metric $d(x, y) := \sum_i \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}$ is a Fréchet space.
- The space of smooth sections on a vector bundle $V \longrightarrow M$ where (M, g) is a Riemannian manifold is Fréchet with $\|f\|_n = \sum_{i=0}^n \sup_x |\nabla^i f(x)|$ for $n \in \mathbb{N}$ with ∇ a covariant derivative and ∇^i denoting its i -th iteration.

Our next step is to generalize the notion of a derivative (differential) to infinite dimensional spaces. This will in turn enable us to define the idea of smoothness in this infinite dimensional setting.

DEFINITION 2.16 (Gâteaux-differential) Let V, W be locally convex topological vector spaces, $U \subset V$ open and $F : V \longrightarrow W$ then $dF(u)(v)$, the **Gâteaux-differential** of F at $u \in U$ along $v \in V$, is defined as

$$dF(u)(v) := \lim_{\tau \rightarrow 0} \frac{F(u + \tau v) - F(u)}{\tau} = \frac{d}{d\tau} F(u + \tau v) \Big|_{\tau=0} \quad (2.26)$$

If the limit exists $\forall v \in V$, F is **Gâteaux differentiable** at $u \in U$.

Using this idea of a differential or derivative, we can define what it means for a homeomorphism to be smooth. Thus being able to use Fréchet spaces instead of \mathbb{R}^n as our target space, we can define a corresponding manifold structure:

DEFINITION 2.17 (Fréchet manifolds) A Hausdorff topological space M is a **Fréchet manifold** if it is equipped with an atlas of homeomorphisms to open sets U of a Fréchet space V such that the transition functions are smooth in the sense that the Gâteaux-derivations $D^{k+1}f : U \times \dots \times U \longrightarrow V$ with

$$Df(u)v := \lim_{t \rightarrow 0} \frac{f(u + tv) - f(u)}{t} \quad (2.27)$$

are continuous for all $k \in \mathbb{N}$.

Applying the above concept to 2.7 leads us to the following idea:

LEMMA 2.18 Inverse limits of normed spaces are Fréchet manifolds.

Proof. Consider the system $f_n^m : V_n \longrightarrow V_m$ with $(V_m, |\cdot|_m)$ a normed vector space $\forall m \in \mathbb{N}$. The inverse limit of the system, denoted by V , is endowed with the linear maps $f_\infty^n : V \longrightarrow V_n$. The norms $|\cdot|_n$ induce a family of seminorms on V via $\|\cdot\|_n := |\cdot|_n \circ f_\infty^n$, such that

1. $\|v\|_n = |f_\infty^n(v)|_n \geq 0 \quad \forall v \in V$
2. $\|v + w\|_n = \dots \leq |v|_n + |w|_n$
3. $\|av\|_n = |f_\infty^n(a \cdot v)|_n = |a \cdot f_\infty^n(v)|_n = |a| \cdot \|v\|_n$

As can be seen in *Schäfer A.2.3*, the space is also metrisable. Since

$$\{v = 0 \Leftrightarrow f_\infty^n(v) = 0 \quad \forall n \in \mathbb{N} \Leftrightarrow |f_\infty^n(v)|_n = 0 \quad \forall n \in \mathbb{N} \Leftrightarrow |v|_n = 0 \quad \forall n \in \mathbb{N}\} \quad (2.28)$$

which proves that our space is indeed Hausdorff which proves the lemma. \square

Now let $F \longrightarrow M$ be a fibre bundle with coordinate charts (U_a, u_a) . Consider the induced cover $U^\infty = \{(\pi_\infty^0)^{-1}(U_a)\}$ and the induced maps $u_a^\infty : (\pi_\infty^0)^{-1}(U_a) \longrightarrow \mathbb{R}^\infty$ of F respecting the ∞ -jet bundle. This defines an atlas as can be seen in *Saunders 7.2.4..* We thus arrive at one of the corner stones of our infinite-dimensional analysis:

PROPOSITION 2.19 The infinite jet bundle $J^\infty F$ together with an atlas given by $(J^\infty F, \{u_a^\infty : (\pi_\infty^0)^{-1}(U_a) \longrightarrow \mathbb{R}^\infty\}_{a \in A})$ is a Fréchet manifold.

Furthermore, we can state the following:

PROPOSITION 2.20 Let $\pi : F \longrightarrow M$ be a smooth fibre bundle. For every $k \in \mathbb{N} \cup \{\infty\}$,

- $\pi_\infty^k : J^\infty F \longrightarrow J^k F$ is a smooth fibre bundle
- $\pi_\infty : J^\infty F \longrightarrow M$ is a smooth fibre bundle
- $j_\varphi^\infty : M \longrightarrow J^\infty F$ is smooth $\forall \varphi \in \mathcal{F}$

2.3 The Variational Bicomplex

Tangent Bundle: There are several alternative ways to progress further with differential geometry on Jet bundles:

- The tangent bundle $T_x J^\infty F$ at a point $x \in J^\infty F$ can be seen as the limit of the system $\{(T_{\pi_\infty^k(x)} J^k F, T\pi_k)\}$ such that the bundle $T(J^\infty F) = \bigcup_{x \in J^\infty F} T_x(J^\infty F)$ is modeled on the $T(J^k F)$ and the projection

$$pr_{J^\infty F} : T(J^\infty F) \longrightarrow J^\infty F \quad (2.29)$$

is represented by $\{pr_\infty^k = pr_{J^k F}\}$ where $pr_{J^k F} : T(J^k F) \longrightarrow J^k F$.

- Observe that if φ_t is a smooth 1-parameter family of sections, we can define $\dot{\varphi}_0 : M \longrightarrow TF$. But φ is a section such that $d\pi(\dot{\varphi}_0) = \frac{d}{dt}\big|_{t=0}(\pi(\varphi_t(x))) = 0_m$ and thus $\dot{\varphi}_0 \in \ker(d\pi)$. Hence we can associate a tangent bundle to $J^\infty F$ in the context of Fréchet manifolds via

$$T(J^\infty F) := \mathcal{C}^\infty(\mathbb{R}, J^\infty F) / \sim \quad (2.30)$$

where $c \sim c'$ iff $c(0) = c'(0)$ and further

$$D(\varphi \circ c)(0, 1) = D(\varphi \circ c')(0, 1) \quad (2.31)$$

with φ any chart around $c(0) \in J^\infty F$ and D denoting the Gâteaux-derivative.

These two notions can be shown to be equivalent. Now in order to unambiguously deal with vector fields and their relations to derivations of the respective algebra of functions, we still need to work a bit more. The following definitions can be carried over from the finite-dimensional setting:

DEFINITION 2.21 (Complex of differential forms) Let $F \rightarrow M$ be a smooth fibre bundle. We define the **complex of differential forms** $\Omega^\bullet(J^\infty F)$ to be the direct limit of the sequence

$$\Omega^\bullet(F) = \Omega^\bullet(J^0 F) \rightarrow \Omega^\bullet(J^1 F) \rightarrow \dots \quad (2.32)$$

together with the morphism $d : \Omega^\bullet(J^\infty F) \rightarrow \Omega^\bullet(J^\infty F)$ given by the collection $\{d^k : \Omega^\bullet(J^k F) \rightarrow \Omega^\bullet(J^{k+1} F)\}$ such that $d \circ d = 0$.

We can now endow $\Omega^\bullet(J^\infty F)$ with the morphism

$$\wedge : \Omega^\bullet(J^\infty F) \otimes \Omega^\bullet(J^\infty F) \rightarrow \Omega^\bullet(J^\infty F) \quad (2.33)$$

given by the collection of morphisms

$$\{\wedge_{k,l} : \Omega^\bullet(J^k F) \otimes \Omega^\bullet(J^l F) \rightarrow \Omega^\bullet(J^{\max(k,l)} F)\}_{k,l} \quad (2.34)$$

This in turn makes the morphism d into a graded derivation of the wedge product. In the following we use the derivation to split the complex of differential forms into two parts. To this end, we first define certain derived differentials using d :

DEFINITION 2.22 (Horizontal differential) Consider a local chart on $J^k F$ given by (x^i, u_I^α) for I running over all multiindices of length lesser than k . The concrete form of the maps is given by

$$x^i(j^k(\varphi, p)) := x^i(p) \quad (2.35)$$

$$u_I^\alpha(j^k(\varphi, p)) := \left. \frac{\partial^{|I|}(u^\alpha \circ \varphi)}{\partial x^I} \right|_p \quad (2.36)$$

Both are clearly smooth and uniquely determine the equivalence class $j^k(\varphi, p)$. We now define the **horizontal differential** for any $f \in \mathcal{C}^\infty(J^\infty F)$ to be

$$d_H f := \frac{\partial f}{\partial x^i} dx^i + \frac{i_1! \dots i_m!}{k!} \frac{\partial f}{\partial u_I^\alpha} u_{I,i}^\alpha dx^i \quad (2.37)$$

where $k = |I|$, i_l denotes the number of occurrences of l in I , $m = \dim(M)$ and $u_{I,i}^\alpha = \partial_{x^i} u_I^\alpha$.

Now for convenience, denote

$$\partial_\alpha^I := \frac{i_1! \dots i_m!}{k!} \frac{\partial}{\partial u_I^\alpha} \quad (2.38)$$

$$D_i := \frac{\partial}{\partial x^i} + u_{I,i}^\alpha \cdot \partial_\alpha^I \quad (2.39)$$

We can thus denote $d_H f = D_i f dx^i$ and further define the **vertical differential**

$$d_V f = (d - d_H) f = \partial_\alpha^I f (d - d_H) u_I^\alpha = \partial_\alpha^I f d_V u_I^\alpha \quad (2.40)$$

We can also prove

$$(D_i f)(j^\infty \varphi) = \frac{\partial}{\partial x^i} (f(j^\infty \varphi)) \quad (2.41)$$

$$d_H x^i = dx^i, \quad d_H(u_I^\alpha) = u_{I,i}^\alpha dx^i \quad (2.42)$$

We follow our successful splitting of the operator d with a collection of useful definitions that will in turn lead us to a splitting of the complex $(\Omega^\bullet(J^\infty F), d)$:

DEFINITION 2.23 (Vertical vector fields) Let $\pi_\infty : J^\infty F \rightarrow M$ be the fibre bundle of infinite jets. We define the subbundle of **vertical vector fields** to be

$$V(J^\infty F) := \ker(T\pi_\infty), \text{ i.e. for } \chi \in T(J^\infty F) \quad (2.43)$$

$$V_\chi(J^\infty F) := \{x_\chi \in T_x(J^\infty F) \mid T\pi_k(T\pi_\infty^k x_\chi) = 0 \ \forall k\} \quad (2.44)$$

Using vertical vector fields, we can now define certain types of differential form that are classified by their behaviour when acting on vertical vector fields.

DEFINITION 2.24

- The set of **horizontal** (p, s) -forms is defined as

$$\Omega_H^{(p,s)} := \{\omega \in \Omega^p(J^\infty F) : \omega_\chi(x_1, \dots, x_{p-s+1}, \cdot, \dots, \cdot) = 0 \mid x_i \in V_\chi(J^\infty F)\} \quad (2.45)$$

- A **contact form** is a differential 1-form θ on $J^\infty F$ that is annihilated by all jets of the form $j^\infty \varphi : M \rightarrow J^\infty F$ via pullback, i.e. $(j^\infty \varphi)^* \theta = 0$. Their set is denoted by \mathcal{C} .
- The set of **vertical** (p, r) -forms are defined as

$$\Omega_V^{(p,r)} := \{\omega \in \Omega^p(J^\infty F) : \theta_1 \wedge \theta_2 \wedge \dots \wedge \theta_r \wedge \tilde{\omega} \mid \theta_i \text{ a contact form}\} \quad (2.46)$$

Thus in particular $\Omega_V^{(s,p)}$ trivially vanishes for $s > p$ and $\Omega_V^{(0,p)} = \Omega^p$. In the same manner, $\Omega_H^{(r,p)}$ trivially vanishes for $r > p$ and $\Omega_H^{(0,p)} = \Omega^p$. For contact forms, we can attain a helpful extra property as well as a basis forming the space of contact forms:

THEOREM 2.25 Contact forms \mathcal{C} form a differential ideal. Its basis is given by $\theta_I^\alpha = du_I^\alpha - u_{I,i}^\alpha dx^i$.

Proof. Let θ be a contact form, thus $(j^\infty \varphi)^* \theta = 0$ for all $\varphi \in \mathcal{F}$. Then for all $\alpha \in \Omega^\bullet(J^\infty F)$ we have

$$(j^\infty \varphi)^*(\theta \wedge \alpha) = (j^\infty \varphi)^* \theta \wedge (j^\infty \varphi)^* \alpha = 0 \quad (2.47)$$

$$\Rightarrow (j^\infty \varphi)^* d(\theta \wedge \alpha) = d(j^\infty \varphi)^*(\theta \wedge \alpha) = 0 \quad (2.48)$$

Turning to the basis, we see that

$$(j^\infty \varphi)^* d_V u_I^\alpha = (j^\infty \varphi)^*(du_I^\alpha - u_{I,i}^\alpha dx^i) \quad (2.49)$$

$$= \frac{\partial(u_I^\alpha \circ \varphi)}{\partial x^i} dx^i - (u_{I,i}^\alpha \circ \varphi) dx^i \quad (2.50)$$

Thus we have a basis of the given form, since locally any 1-form θ splits to $\theta = d_H f + d_V f$. \square

Thus we obtain $\Omega_V^{s,p} = \mathcal{C}^s \cap \Omega^p$. We can now use the ideal of contact forms to investigate local sections of the bundle π_∞ :

LEMMA 2.26 Let $F \longrightarrow M$ be a smooth fibre bundle. A local section ξ of $\pi_\infty : J^\infty F \longrightarrow M$ is holonomic, i.e. a ∞ -prolongation of a section of F , if and only if $\xi^* \mathcal{C} = 0$.

Proof. Note that

$$\xi^* d_V u_I^\alpha = 0 \quad \text{iff} \quad \frac{\partial}{\partial x^i} (u_I^\alpha \circ \xi) = u_{I,i}^\alpha \circ \xi \quad (2.51)$$

for all indices I, i, α such that ξ can be constructed inductively from $u_i \circ \xi$. Thus define $\varphi := \pi_\infty^0 \circ \xi$ we see that $\xi = j^\infty \varphi$. \square

Using the thus defined structures, we can approach the splitting of the complex $(\Omega^\bullet(J^\infty F), d)$ by defining forms of bi-degree:

DEFINITION 2.27 (Forms of bi-degree) Let $F \longrightarrow M$ and $J^\infty F$ be as above. Define the space of **forms of bi-degree** (r, s) to be the intersection

$$\Omega^{r,s}(J^\infty F) := \Omega_V^{p,r}(J^\infty F) \cap \Omega_H^{s,s}(J^\infty F) \quad (2.52)$$

where $p := r + s$. A form $\omega \in \Omega^\bullet(J^\infty F)$ is of bi-degree (r, s) iff it is of the form

$$\omega_{\alpha_1 \dots \alpha_r; i_1 \dots i_s}^{I_1 \dots I_r} \theta_{I_1}^{\alpha_1} \wedge \dots \wedge \theta_{I_r}^{\alpha_r} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s} \quad (2.53)$$

where all the functions $\omega_{(\cdot)}^{(\cdot)}$ are smooth in $\mathcal{C}^\infty(J^\infty F)$.

We are now in a position to "split" $(\Omega^\bullet(J^\infty F), d)$. First we define

$$\Omega^p(J^\infty F) := \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty F) \quad (2.54)$$

Since $d(dx^i) = 0$ we also have $d_H(dx^i) = 0$ and $d_V(dx^i) = 0$. Now we can also use $\theta_I^\alpha = du_I^\alpha - u_{I,i}^\alpha dx^i$ such that

$$d\theta_I^\alpha = -du_{I,i}^\alpha \wedge dx^i = -\theta_{I,j}^\alpha \wedge dx^j \quad (2.55)$$

and thus $d_V\theta_I^\alpha = 0$ and $d_H\theta_I^\alpha = -\theta_{I,j}^\alpha \wedge dx^j$. This brings us to the following theorem, which is the defining theorem for the *variational bicomplex*:

THEOREM 2.28 (Variational bicomplex) The complex of differential forms on the infinite jet bundle $(\Omega^\bullet(J^\infty F), d)$ splits into the bicomplex $(\Omega^{\bullet,\bullet}(J^\infty F), d_H, d_V)$, i.e. $d_H^2 = d_V^2 = 0$ and further $d_H \circ d_V = -d_V \circ d_H$. We call this complex the **variational bicomplex**.

2.4 Local Lagrangian Field Theory

So far, we have been working on the infinite jet bundle. Our goal is to work on \mathcal{F} in a *local* way. Thus we need to find some local calculus on $\mathcal{F} \times M$.

To this end, denote by $VF := \ker(d\pi)$ the vertical tangent bundle. Recall that if φ_t is a smooth curve in $\mathcal{F} = \Gamma(M, F)$, $\dot{\varphi}|_{t=0}$ is vertical and

$$0 = d\pi(\dot{\varphi}|_{t=0}) = \frac{d}{dt}(\pi(\varphi_t))|_{t=0} \quad (2.56)$$

Thus we can think of $\dot{\varphi}_0 : M \longrightarrow VF$ as a section covering $\varphi_0 = pr_F \circ \dot{\varphi}_0$. We can express this in the following diagram:

$$\begin{array}{ccc} VF & \xrightarrow{pr_F} & F \\ \dot{\varphi}_0 \uparrow & \nearrow \varphi_0 & \downarrow \pi \\ M & \xrightarrow{id} & M \end{array}$$

For $\varphi \in \Gamma(M, F) = \mathcal{F}$ consider sections $\Gamma(M, VF)_\varphi = \Gamma(M, \varphi^*VF)$. An element of $\Gamma(M, VF)_\varphi$ is a map that associates to every $x \in M$ a vector $v_{\varphi(x)} \in V_{\varphi(x)}F$. We can think of $v_\varphi \in \Gamma(M, \varphi^*VF)$.

DEFINITION 2.29 We define the tangent bundles

$$T\mathcal{F} := \Gamma(M, VF), \quad T(\mathcal{F} \times M) := \Gamma(M, VF) \times TM, \quad (2.57)$$

$$pr_{\mathcal{F}} : T\mathcal{F} \longrightarrow \mathcal{F} \quad pr_{\mathcal{F}} \times pr_M : T(\mathcal{F} \times M) \longrightarrow \mathcal{F} \times M \quad (2.58)$$

An element of $T_\varphi \mathcal{F}$ is often called an **infinitesimal variation**.

Now let $E \longrightarrow M$ and $F \longrightarrow M$ be two fibre bundles with projections π_E and π_F respectively. Note that the following diagram commutes and thus defines the *product bundle*:

$$E \times F := \{(e, f) \in E \times F \mid \pi_E(e) = \pi_F(f)\} \quad (2.59)$$

such that the following diagram commutes:

$$\begin{array}{ccc} E \times_M F & \xrightarrow{pr_F} & F \\ pr_E \downarrow & & \downarrow \pi_F \\ E & \xrightarrow{\pi_E} & M \end{array}$$

In general, this allows us to write $\Gamma(M, E \times_M F) \cong \mathcal{E} \times \mathcal{F}$ where we identify $\mathcal{E} = \Gamma(M, E)$ and $\mathcal{F} = \Gamma(M, F)$. We can also write

$$T(\mathcal{E} \times \mathcal{F}) \cong \Gamma(M, V(E \times_M F)) \cong \Gamma(M, VE \times_M VF) \cong T\mathcal{E} \times T\mathcal{F} \cong (T\mathcal{E} \times \mathcal{F}) \times_{\mathcal{E} \times \mathcal{F}} (\mathcal{E} \times T\mathcal{F}) \quad (2.60)$$

As an exercise, you can prove the above chain of isomorphisms. Since we can further write $T(\mathcal{F} \times M) \cong (T\mathcal{F} \times M) \times_{\mathcal{F} \times M} (\mathcal{F} \times TM)$ we can for the above statements take a pair $(\varphi, x) \in \mathcal{F} \times M$ and denote

$$T_{(\varphi, x)}(\mathcal{F}_x M) \cong T_\varphi \mathcal{F} \times T_x M \quad (2.61)$$

We denote by $T_\varphi \mathcal{F}$ the **vertical tangent space** and by $T_x M$ the **horizontal tangent space**.

Taking the ∞ -prolongation j^∞ and its tangent map we obtain the following commuting diagram:

$$\begin{array}{ccc} T(\mathcal{F} \times M) & \xrightarrow{Tj^\infty} & TJ^\infty F \\ pr_{\mathcal{F}} \times pr_M \downarrow & & \downarrow pr_{J^\infty F} \\ \mathcal{F} \times M & \xrightarrow{j^\infty} & J^\infty F \end{array}$$

This leads us to the following interesting piece of information:

PROPOSITION 2.30 The tangent map

$$T_{(\varphi, x)} j^k : T_\varphi \mathcal{F} \times T_x M \longrightarrow T_{j_x^k \varphi} J^k F \quad (2.62)$$

is given by

$$(T_{(\varphi, x)} j^k)(\xi_\varphi, v_x) := \sum_{|I|=0}^k \dot{u}_I^\alpha(j_x^k \xi_\varphi) \frac{\partial}{\partial u_I^\alpha} + v_x^i \left(\frac{\partial}{\partial x^i} + \sum_{|I|=0}^k u_{I,i}^\alpha(j_x^{k+1} \varphi) \frac{\partial}{\partial u_I^\alpha} \right) \quad (2.63)$$

Proof. While the proof is left as an exercise to the reader, we give the following hint: To compute tangent maps in coordinates, compute the time derivative of a path $t \mapsto (\varphi_t, x(t))$. For example

$$\left. \frac{d}{dt} (u_I^\alpha(j^k(\varphi_t, x(t)))) \right|_{t=0} = \left. \frac{d}{dt} \left(\frac{\partial^{|\mathbf{I}|}}{\partial x^{\mathbf{I}}} \varphi^\alpha(t, x(t)) \right) \right|_{t=0} \quad (2.64)$$

Now set $\xi_\varphi = \dot{\varphi}_0$ and $v_x = \dot{x}(0)$. □

Remark 2.31 The above proposition induces the following maps:

$$\tau_k: J^k(VF) \longrightarrow TJ^k(F), \quad \sigma_k: J^{k+1}F \times TM \longrightarrow TJ^kF, \quad (2.65)$$

$$j_x^k \dot{\varphi}(0) \longmapsto \frac{d}{dt} (j_x^k \varphi_t)_{t=0} \quad (j_x^{k+1} \varphi, \dot{x}(0)) \longmapsto \frac{d}{dt} (j_{x(t)}^k \varphi)_{t=0} \quad (2.66)$$

Now using our results for TJ^kF we obtain a splitting for $TJ^\infty F$:

THEOREM 2.32 There exists a splitting of $TJ^\infty F$ that takes the form of

$$\begin{array}{ccc} (T\mathcal{F} \times M) \times_{\mathcal{F} \times M} (\mathcal{F} \times TM) & \xrightarrow{\cong} & T(\mathcal{F} \times M) \\ \downarrow j_{T\mathcal{F}}^\infty \times (j_{\mathcal{F}}^\infty \times \text{id}) & & \downarrow Tj^\infty \\ J^\infty(VF) \times_{J^\infty F} (J^\infty F \times_M TM) & \xrightarrow{\cong} & TJ^\infty F \end{array}$$

One can read this diagram as the splitting into

$$J^\infty(VF) \longrightarrow TJ^\infty F \quad \text{vertical tangent bundle} \quad (2.67)$$

$$J^\infty F \times_M TM \longrightarrow TJ^\infty F \quad \text{horizontal tangent bundle} \quad (2.68)$$

COROLLARY 2.33 The vector space of vector fields on $J^\infty F$ decomposes via $\mathfrak{X}(J^\infty F) = \mathfrak{X}_{vert} \oplus \mathfrak{X}_{hor}$ where

$$\mathfrak{X}_{vert} \cong \Gamma(J^\infty F, J^\infty(VF)) \quad (2.69)$$

$$\mathfrak{X}_{hor} \cong \text{Hom}(J^\infty F, TM) \quad (2.70)$$

Thus we can split any $V \in \mathfrak{X}(J^\infty F)$ into

$$V = V^i \frac{\partial}{\partial x^i} + \sum_{|\mathbf{I}|=0}^{\infty} V_I^\alpha \frac{\partial}{\partial u_I^\alpha} \quad (2.71)$$

where both V^i and V_I^α are in $\mathcal{C}^\infty(J^\infty F)$.

Now we can finally unambiguously answer the question "*What is Locality?*". Essentially, locality is

the requirement that we work with $(j^\infty)^*\Omega^{\bullet,\bullet}(J^\infty F)$ where $(j^\infty)^*: TJ^\infty F \longrightarrow T(\mathcal{F} \times M)$.

Remark 2.34 Note that we always assume that our base manifold M is orientable. There exist generalisations for non-orientable manifolds (see Deligne/Freed).

DEFINITION 2.35 (Local differential forms) A differential form on $\mathcal{F}_x M$ is called a **local differential form**, iff it is the pullback of a form on $J^\infty F$ by the ∞ -jet prolongation $j^\infty: \mathcal{F}_x M \longrightarrow J^\infty F$. We define the **bicomplex of local forms** as

$$(\Omega_{loc}^{\bullet,\bullet}(\mathcal{F} \times M) := ((j^\infty)^*\Omega^{\bullet,\bullet}(J^\infty F), d, \delta) \quad (2.72)$$

such that

$$\delta((j^\infty)^*\alpha) := (j^\infty)^*d_V\alpha, \quad d((j^\infty)^*\alpha) := (j^\infty)^*d_H\alpha \quad (2.73)$$

We can also extend this *local* setting to vector fields:

DEFINITION 2.36 (Local vector fields) A **local vector field** χ on $\mathcal{F} \times M$ is a section of the tangent bundle $T(\mathcal{F} \times M)$ covered by a section $\chi^\infty: J^\infty F \longrightarrow TJ^\infty F$ and supported on $j^\infty(\mathcal{F} \times M)$. In other words, the following diagram commutes:

$$\begin{array}{ccc} T(\mathcal{F} \times M) & \xrightarrow{Tj_F^\infty} & TJ^\infty F \\ \chi \curvearrowright \downarrow pr & & \downarrow pr \curvearrowright \chi^\infty \\ \mathcal{F} \times M & \xrightarrow{j_F^\infty} & J^\infty F \end{array}$$

Practically speaking, a vector field is **local**, if for any φ there exists an integer k such that the value of $\chi_\varphi \in \Gamma(M, \varphi^*VF)$ at $x \in M$ depends only on the k -th jet of φ at x . We denote the set of local vector fields by $\mathfrak{X}_{loc}(\mathcal{F} \times M)$. A local vector field in $\mathfrak{X}(\mathcal{F})$ is called **evolutionary**.

Notice that by definition, local functions $\mathcal{C}_{loc}^\infty(\mathcal{F} \times M)$ are represented by a pair (f, f^∞) meaning that they factor through some infinite jet as:

$$\begin{array}{ccc} f: \mathcal{F} \times M & \longrightarrow & \mathbb{R} \\ j^\infty \downarrow & \nearrow f^\infty & \\ & J^\infty F & \end{array}$$

We also port the idea of a derivation to our newly defined setting by defining them on $\mathcal{C}_{loc}^\infty(\mathcal{F} \times M)$:

DEFINITION 2.37 (Local derivations) A derivation D of $\mathcal{C}_{loc}^\infty(\mathcal{F} \times M)$ is called a **local derivation**, iff there exists a vector field $\chi^\infty: J^\infty F \longrightarrow TJ^\infty F$ such that

$$Df = (\chi^\infty f^\infty) \circ j^\infty \quad (2.74)$$

for every (f, f^∞) . We denote a local derivation by (D, χ^∞) .

A proposition coming directly from usual differential geometry is the correspondence between local vector fields and local derivations:

PROPOSITION 2.38 Local vector fields are in 1:1 correspondence with local derivations of $\mathcal{C}_{loc}^\infty(\mathcal{F} \times M)$. Moreover they form a Lie subalgebra of $\text{Der}(\mathcal{C}_{loc}^\infty)$.

Proof. The proof is left to the reader as it is fairly similar to the case in classical differential geometry. \square

We go on with a short remark on local coordinates: If we identify

$$(j^\infty)^* x^i \equiv x^i, \quad (j^\infty)^* u_I^\alpha \equiv u_I^\alpha \quad (2.75)$$

we can interpret $dx^i \in \Omega_{loc}^{0,1}(\mathcal{F} \times M)$ and $\delta u_I^\alpha \in \Omega_{loc}^{1,0}(\mathcal{F} \times M)$ as a basis for $\Omega_{loc}^{\bullet,\bullet}(\mathcal{F} \times M)$. This allows us to write a form $\omega \in \Omega_{loc}^{p,q}(\mathcal{F} \times M)$ as

$$\omega = \omega_{\alpha_1 \dots \alpha_p i_1 \dots i_q}^{I_1 \dots I_p} \delta u_{I_1}^{\alpha_1} \wedge \dots \wedge \delta u_{I_p}^{\alpha_p} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_q} \quad (2.76)$$

for $\omega_{\alpha_1 \dots \alpha_p i_1 \dots i_q}^{I_1 \dots I_p} \in \mathcal{C}_{loc}^\infty(\mathcal{F} \times M)$, $df = D_i f dx^i$ and $\delta f = \partial_\alpha f \delta u_I^\alpha$. Thus if we take $X \in \mathfrak{X}_{loc}(\mathcal{F})$ and $\omega \in \Omega_{loc}^{\bullet,\bullet}(\mathcal{F} \times M)$, we obtain a variational *Cartan Calculus*

$$\mathcal{L}_X \omega = \iota_X \circ \delta \omega + \delta \circ \iota_X \omega \quad (2.77)$$

This can be generalized to $\mathfrak{X}_{loc}(\mathcal{F} \times M)$.

2.5 Classical Field Theory

DEFINITION 2.39 (Classical Lagrangian Field Theory) A **Classical Lagrangian Field Theory** is a triple (M, F, L) where $F \xrightarrow{\pi} M$ is a smooth fibre bundle and $L \in \Omega_{loc}^{0,top}(\mathcal{F} \times M)$ called a **Lagrangian (density)**. We call the integral

$$S = \int_M L \quad (2.78)$$

the **action functional**.

This means

$$L = (j^\infty)^* L^\infty \quad \text{for} \quad L^\infty \in \Omega_{loc}^{0,top}(J^\infty F) \quad (2.79)$$

and thus

$$L^k(x^i, u^\alpha, u_{i_1}^\alpha, \dots, u_{i_1 \dots i_n}^\alpha) dx^1 \dots dx^n \quad (2.80)$$

If we evaluate L at φ , we obtain

$$L(\varphi) = L\left(x^i, \varphi^\alpha, \frac{\partial \varphi^\alpha}{\partial x^i}, \dots, \frac{\partial^k \varphi^\alpha}{\partial x^{i_1} \dots \partial x^{i_n}}\right) \quad (2.81)$$

DEFINITION 2.40 (Source forms) Let $\alpha \in \Omega_{loc}^{1,top}(\mathcal{F} \times M)$. α is called a **source form**, iff it only depends on the dx^i and δu^α , not the δu_I^α .

Thus a source form, if evaluated on a vectorfield, depends only on the projection of that vectorfield to on F . This definition is a working version and does not really explain itself. The following theorem sheds some light onto the situation:

THEOREM 2.41 Let (M, F, L) be a Lagrangian field theory. Then there exists a unique source form $EL \in \Omega_{loc}^{1,top}(\mathcal{F} \times M)$ and two unique "boundary" forms

$$\alpha \in \Omega_{loc}^{1,top-1}(\mathcal{F} \times M), \quad \omega \in \Omega_{loc}^{2,top-1}(\mathcal{F} \times M) \quad (2.82)$$

such that

- $\delta L = EL - d\alpha$
- $\omega = \delta\alpha \Rightarrow \delta\omega = 0$
- $d\omega = -\delta EL$

This is the statement that by means of "integration by parts" we can write δL as

$$EL = E_\alpha \delta u^\alpha \wedge dx^1 \wedge \dots \wedge dx^n \quad (2.83)$$

with $E_\alpha: J^k F \rightarrow \mathbb{R}$ which defines the **Euler-Lagrange-equations** as

$$E_\alpha\left(x^i, \varphi^\beta, \frac{\partial \varphi^\beta}{\partial x^i}, \dots, \frac{\partial^k \varphi^\beta}{\partial x^{i_1} \dots \partial x^{i_n}}\right) = 0 \quad (2.84)$$

To prove the above statement, we would need to reformulate integration by parts in "cohomological terms", thus in terms of operators on local forms, which does not fit into the scope of this course. Doing this, allows us to define the "Euler Operator" \mathbb{E} such that $\mathbb{E}(L) \equiv EL$.

With the following statements, we introduce some notions necessary to make the concept *evolutionary vector fields* more rigid and well-defined:

DEFINITION 2.42 (Strictly vertical/horizontal vector fields) A vector field $v \in \mathfrak{X}(J^\infty F)$ is called **strictly vertical** or **strictly horizontal**, iff $[v, d_H] = 0$ or $[v, d_V] = 0$ respectively.

We use this definition to formulate the following proposition:

PROPOSITION 2.43 There are two commuting Cartan calculi: For ξ, ξ' strictly vertical and X, X' strictly horizontal we have:

$$[\iota_\xi, d_V] = \mathcal{L}_\xi, \quad [\mathcal{L}_\xi, \iota_{\xi'}] = \iota_{[\xi, \xi']}, \quad [\mathcal{L}_\xi, \mathcal{L}_{\xi'}] = \mathcal{L}_{[\xi, \xi']} \quad (2.85)$$

$$[d_V, d_V] = [\iota_\xi, \iota_{\xi'}] = [\mathcal{L}_\xi, d_V] = 0 \quad (2.86)$$

The analogous equations for X, X' also hold.

Proof. Follows directly from representing them on finite jets together with the bicomplex structure and the definition of strictly vertical/horizontal vector fields 2.42. \square

LEMMA 2.44 A vector field $v \in \mathfrak{X}(J^\infty F)$ is strictly horizontal, iff $v = v^i D_i$ for $v^i \in \mathcal{C}^\infty(M)$.

Proof. First note that $[\iota_v, d_V] = 0$ iff it annihilates functions and 1-forms generated by $\{d_H x^i, d_V u_I^\alpha\}$.

$$\Rightarrow [\iota_v, d_V]f = \iota_v \frac{\partial f}{\partial u_I^\alpha} d_V u_I^\alpha = \frac{\partial f}{\partial u_I^\alpha} (v_I^\alpha - u_{I,i}^\alpha v^i) = 0 \quad (2.87)$$

$$\Leftrightarrow v_I^\alpha = u_{I,i}^\alpha v^i \quad (2.88)$$

$$\Leftrightarrow v = v^i \frac{\partial}{\partial x^i} + u_{I,i}^\alpha v^i \frac{\partial}{\partial u_I^\alpha} = v^i D_i \quad (2.89)$$

Thus we can conclude by checking what happens on 1-forms. \square

LEMMA 2.45 $v \in \mathfrak{X}(J^\infty F)$ is strictly vertical iff it is of the form $v = (D_I v^\alpha) \frac{\partial}{\partial u_I^\alpha}$ for some $v^\alpha \in \mathcal{C}^\infty(J^\infty F)$.

Proof. The proof is left as an exercise to the reader. *Hint:* The proof can, as done in the previous proof, be concluded by checking the statement on the generators $\{x^i, u_I^\alpha\}$ and $\{dx^i, d_V u_I^\alpha\}$. \square

DEFINITION 2.46 A vector field $\xi: \mathcal{F} \rightarrow T\mathcal{F}$ is said to **project** to a vector field v on $J^\infty F$, iff the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F} \times M & \xrightarrow{\xi \times \text{id}} & T\mathcal{F} \times M \\ \downarrow j^\infty & & \downarrow Tj^\infty \\ J^\infty F & \xrightarrow{v} & TJ^\infty F \end{array}$$

THEOREM 2.47 A vector field v on $J^\infty F$ is strictly vertical, iff there exists a local vector field $\xi \in \Gamma(\mathcal{F}, T\mathcal{F})$ that projects to it. Moreover ξ is unique and called an **evolutionary** vector field.

DEFINITION 2.48 (Local symmetries) An evolutionary vector field $\xi \in \mathfrak{X}_{loc}(\mathcal{F} \times M)$ is called a **local symmetry** of the Lagrangian field theory (M, F, L) iff $\exists A_\xi \in \Omega_{loc}^{0, \text{top}-1}(\mathcal{F} \times M)$ such that

$$\mathcal{L}_\xi L = \iota_\xi \delta L = dA_\xi \quad (2.90)$$

We denote a **symmetry** by the pair (ξ, A_ξ) .

Showing the power of the presented formalism, we can now prove *Noether's first theorem*:

THEOREM 2.49 (Noether's first) Let (M, F, L) be a Lagrangian field theory such that $\delta L = EL - d\alpha$ and let (ξ, A_ξ) be a local symmetry of the theory. Then

$$j := A_\xi - \iota_\xi \alpha \in \Omega_{loc}^{0, top-1}(\mathcal{F} \times M) \quad (2.91)$$

is **on shell**, i.e.

$$dj = 0 \mod (EL) \quad (2.92)$$

Here, "on shell" means "on those configurations φ s.t. $EL_\varphi = 0$ that satisfy the Euler-Lagrange equations".

Proof. To prove this, we can simply calculate:

$$dj = d(A_\xi - \iota_\xi \alpha) = \mathcal{L}_\xi L - d \circ \iota_\xi \alpha = \mathcal{L}_\xi L + \iota_\xi \circ d\alpha \quad (2.93)$$

$$= \iota_\xi(\delta L + d\alpha) = \iota_\xi EL = 0 \mod (EL) \quad (2.94)$$

□

Some terminology: j is usually called a **Noether current**, thus 2.49 is a conservation statement. One can introduce *Noether Charges* by integration of j over codimension 1 submanifolds (hyper-surfaces).

If we allow dependency of currents on a vector bundle $W \rightarrow M$ together with a linear map

$$\Gamma(M, W) \rightarrow \mathfrak{X}_W(\mathcal{F} \times M) \times \Omega_{loc}^{0, top-1}(\mathcal{F} \times M) \quad (2.95)$$

$$w \mapsto (\xi_w, A_{\xi_w}) \quad (2.96)$$

such that

$$\mathcal{L}_{\xi_w} L = dA_{\xi_w} \Rightarrow j_w = A_{\xi_w} - \iota_{\xi_w} \alpha \quad (2.97)$$

We move on to some examples of classical Lagrangian field theories:

EXAMPLE 2.50 (Classical Mechanics) Let $F = \mathbb{R} \times M \rightarrow \mathbb{R}$ such that $\mathcal{F} = \Gamma(\mathbb{R}, F)$ and take

$$L = \left(\frac{m}{2} \dot{q}_i \dot{q}^i - V(q) \right) dt \quad (2.98)$$

as a Lagrangian density. Note that L is defined on $J^1(M \times \mathbb{R}) \simeq TM \times \mathbb{R}$. We find the variation

of L to be

$$\delta L = (m \delta \dot{q} \cdot \dot{q} - \delta q \nabla V) \wedge dt \quad (2.99)$$

$$= \left(\frac{d}{dt} (m \delta q \cdot \dot{q}) - \delta q (m \ddot{q} + \nabla V) \right) \wedge dt \quad (2.100)$$

$$= d(m \delta q \cdot \dot{q}) - (m \ddot{q} + \nabla V) \delta q \wedge dt \quad (2.101)$$

Thus we have $\alpha = m \dot{q}^i \delta q_i$ and $E_q = m \ddot{q} + \nabla V$. Now if we consider a time translation $t \mapsto t + \varepsilon$ we find the induced evolutionary vector field $\xi = -\dot{q}^i \frac{\partial}{\partial q^i} - \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} \dots$

Exercise: Show that j_ξ is an energy, i.e. $j = m \dot{q} \cdot \dot{q} + V(q)$.

Proof. First note $\iota_\xi \alpha = -m \dot{q}^i \dot{q}_i$ and thus let $A_\xi = V(q)$. We obtain $j = V(q) + m \dot{q}^i \dot{q}_i$. \square

2.6 Scalar Field Theory and Yang-Mills-Theory

Consider a smooth, oriented, compact manifold M equipped with a (pseudo-)Riemannian metric g . This metric and the orientation define a volume form on M given by $\text{Vol}_g := \sqrt{|\det(g)|} dx^1 \wedge \dots \wedge dx^n$. This so-called Riemannian volume form has this form independently of the choice of local coordinates.

DEFINITION 2.51 (Hodge Duality) The **Hodge-Duality map** (Hodge-*) is the $\mathcal{C}^\infty(M)$ -linear map

$$*: \Omega^k(M) \longrightarrow \Omega^{n-k}(M) \quad (2.102)$$

uniquely determined by the equation

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle_g \text{Vol}_g \quad (2.103)$$

The Hodge star acts on a local basis as

$$* (dx^1 \wedge \dots \wedge dx^n) = \frac{\sqrt{|\det(g)|}}{(n-k)!} g^{i_1 j_1} \dots g^{i_k j_k} \varepsilon_{j_1 \dots j_k} dx^{j_k+1} \dots dx^{j_n} \quad (2.104)$$

Using the Hodge star we can define the

DEFINITION 2.52 (Codifferential) The **Codifferential** is the operator

$$\delta_g := \Omega^k(M) \longrightarrow \Omega^{k-1}(M) \quad (2.105)$$

defined as

$$\delta_g := (-1)^{n(k-1)+1} \text{sgn}(g) * d * \quad (2.106)$$

Using the codifferential, one can define the **Laplace-deRham operator**

$$\Delta_g := (d + \delta_g)^2 = d\delta_g + \delta_g d \quad (2.107)$$

Using the above definition, one can directly show some interesting properties of the codifferential:

LEMMA 2.53 The following properties hold true:

- $* * \alpha = (-1)^{k(n-k)} \text{sgn}(g) \alpha \quad \forall \alpha \in \Omega^k(M)$
- $*^{-1}: \eta \longmapsto (-1)^{k(n-k)} \text{sgn}(g) * \eta$
- $(\delta_g)^2 = 0$

DEFINITION 2.54 (Scalar Field Theory) Let (M, g) as above and $\mathcal{F} = \mathcal{C}^\infty(M)$ seen as the sections of a trivial \mathbb{R} -bundle. Then (a.k.a. non-interacting) **free Scalar field theory** on \mathcal{F} is defined using the Lagrangian

$$L = \frac{1}{2} d\varphi \wedge *d\varphi \equiv g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \sqrt{|\det(g)|} d^n x \quad (2.108)$$

EXERCISE 2.55 Show that $\delta L = [\Delta_g \varphi] \delta \varphi \wedge \text{Vol}_g + d(\delta \varphi \wedge *d\varphi)$.

Proof.

$$\delta L = \delta \left(\frac{1}{2} d\varphi \wedge *d\varphi \right) = \frac{1}{2} [\delta d\varphi \wedge *d\varphi + d\varphi \wedge \delta *d\varphi] \quad (2.109)$$

$$= \frac{1}{2} [-d\delta\varphi \wedge *d\varphi - d\varphi \wedge *d\delta\varphi] = -\frac{1}{2} [d(\delta\varphi \wedge *d\varphi) - \delta\varphi \wedge d *d\varphi + d\varphi \wedge *d\delta\varphi] \quad (2.110)$$

$$= -\frac{1}{2} [d(\delta\varphi \wedge *d\varphi) - \delta\varphi \wedge *\delta_g d\varphi + d\varphi \wedge *d\delta\varphi] \quad (2.111)$$

$$= -\frac{1}{2} [d(\delta\varphi \wedge *d\varphi) - \delta\varphi \wedge *\delta_g d\varphi + d\delta\varphi \wedge *d\varphi] \quad (2.112)$$

$$= -\frac{1}{2} [d(\delta\varphi \wedge *d\varphi) - \delta\varphi \wedge *\delta_g d\varphi + d(\delta\varphi \wedge *d\varphi) - \delta\varphi \wedge d *d\varphi] \quad (2.113)$$

$$= -\frac{1}{2} [d(\delta\varphi \wedge *d\varphi) - \delta\varphi \wedge *\delta_g d\varphi + d(\delta\varphi \wedge *d\varphi) - \delta\varphi \wedge *\delta_g d\varphi] \quad (2.114)$$

$$= -d(\delta\varphi \wedge *d\varphi) + \delta\varphi \wedge *\delta_g d\varphi = -d(\delta\varphi \wedge *d\varphi) + \delta\varphi \wedge (\Delta_g \varphi \cdot \text{Vol}_g) \quad (2.115)$$

The last equation holds since $\delta_g \varphi = 0$ and thus $\delta_g d\varphi = (\delta_g d + d\delta_g)\varphi$. □

The equations of motion for scalar field theory are

$$\Delta_g \varphi = 0 \quad (2.116)$$

Remark 2.56 One can add "potential (a.k.a. interaction) terms" to the Lagrangian. A common

example would be $V(\varphi) = \frac{m^2}{2}\varphi^2 + \frac{\beta}{4}\varphi^4$. Note further that our theory *requires* a metric g on M and is thus a "Non-topological field theory".

DEFINITION 2.57 (Pure, d-dimensional Yang-Mills-Theory) Let (M, g) be as above and $P \rightarrow M$ a principal G -bundle where G is a compact Lie group ($U(n)$). Denote by A a connection 1-form and let F_A denote its curvature. In a local chart we can write it as

$$F_A = dA + \frac{1}{2}[A, A] \quad (2.117)$$

Assume for simplicity that $P \rightarrow M$ is a trivial bundle and $A \in \Omega^1(P, \mathfrak{g})$ descends to a \mathfrak{g} -valued 1-form on M . Then a **Pure Yang-Mills-Theory** is the Lagrangian field theory given by $\mathcal{F} = \mathcal{A}$ (space of principal connections on the principal bundle) and the Lagrangian (density)

$$L_{YM} = \frac{1}{2} \text{Tr}[F_A \wedge *F_A] \quad (2.118)$$

EXERCISE 2.58 As an exercise, you can show that the variation of L_{YM} is given by

$$\delta L_{YM} = \text{Tr}[d_A * F_A \wedge \delta A + d(\delta A * F_A)] \quad (2.119)$$

Remember that $d_A \varphi = d\varphi + A\varphi$ for $\varphi \in \mathcal{C}^\infty(M, R)$ and R a representation of \mathfrak{g} . For any $\beta \in \Omega^1(M, \mathfrak{g})$ one has $d_A \beta = d\beta + [A, \beta]$ where the latter term denotes the adjoint action/representation.

DEFINITION 2.59 (Yang-Mills-Theory with scalar "matter") Working with the same setting as usual Yang-Mills-Theory, one takes the Lagrangian (density)

$$L_{YM}^{scalar} = \text{Tr}[F_A \wedge *F_A] + \langle d_A \varphi, *d_A \varphi \rangle_R \quad (2.120)$$

for $\varphi \in \mathcal{C}^\infty(M, R)$ where R is a representation that comes with an inner product $\langle \cdot, \cdot \rangle_R$. One can then add

$$+ \text{mass terms } \frac{m^2}{2} \langle \varphi, \varphi \rangle_R + \beta \varphi^4 \quad (2.121)$$

to form the "Higgs-mechanism" Lagrangian that appears in the *Standard Modell of Particle Physics*. Here, φ plays the role of the "Higgs" field.

Due to its convenience in upcoming applications in Yang-Mills-Theory, we reformulate some parts of our setting. This induces the *First order formulation of YM-theory*. First note that

$$L_{YM} = \frac{1}{2} \text{Tr}[F_A \wedge *F_A] \quad (2.122)$$

The trick is to introduce a field $B := *F_A$ allowing for the following description:

$$\mathcal{F}^{1st} := \mathcal{A}_P \times \Omega^{n-2}(M, y) \quad (2.123)$$

$$L_{YM}^{1st} := \text{Tr}[B \wedge F_A - \frac{1}{2}B \wedge *B] \quad (2.124)$$

This results in the following variation

$$\delta L_{YM}^{1st} = (F_A - *B)\delta B + d_A B \delta A + d(B \wedge \delta A) \quad (2.125)$$

Further note that $F_A = *B$ together with $d_A B = 0$ is equivalent to $d_A *F_A = 0$. First and second order formulations are **classically equivalent**. This concludes our introduction to classical field theory.

3 Quantisation and Renormalisation Group Flow

This section will treat *scalar perturbative quantum field theory*. We will start by giving a general introduction to the matter and then talk about the *renormalisation group equation*. After introducing *Feynman graphs*, we will finally arrive at the first full definition of a *QFT* and discuss some immediate results. Finally we will discuss the *renormalisation of QFT*.

3.1 Introduction

The main idea of quantum mechanics is to define a deterministic evolution of a system as a superposition of possible evolutions where one weights them by how likely they are. Our ultimate goal is to describe quantum mechanics and classical mechanics in terms of the sets of objects, states and observables. The main idea is to take a state ω and an observable A such that the state associates a probability distribution to the observable on the real line in the form of $\omega_A(\lambda)$. To illuminate the concepts and systems we aim to unify, we present a short overview of both:

In classical mechanics, states are nothing but normalized measures on the *phase space* (a symplectic manifold). Meanwhile observables are functions on the phase space given by (p, q) such that the symplectic form $\Omega = dp \wedge dq$. The probability distribution is then given by

$$\omega_A(\lambda) := \int_M \theta(\lambda - f_A(p, q)) d\mu_\Omega \quad (3.1)$$

where θ denotes the step function, f_A an observable and $d\mu_\Omega$ a measure corresponding to the state. The dynamics on the other hand are encoded in the *Poisson brackets* in that

$$\frac{df}{dt} = \{H, f\}, \quad (3.2)$$

defined using the symplectic form ω , for some function H which turns out to be the *Hamiltonian* of the system.

In quantum mechanics, observables are linear operators on some complex *Hilbert space* \mathcal{H} . States are positive trace-class operators with unit trace defined as $\text{Tr } A = \sum_n \langle A e_n, e_n \rangle$. The probability distribution for a state is defined by

$$\omega_A(\lambda) := \text{Tr}(M P_A(\lambda)) \quad (3.3)$$

where M is a trace-class (state) and $P_A(\lambda)$ is a projection (observable). Note that we are looking at the field from a mathematical perspective, thus the reader might encounter unfamiliar technical details in this and upcoming discussions.

The commutator of two states A, B is defined as

$$\{A, B\} = \frac{i}{\hbar}(AB - BA) \quad (3.4)$$

such that the time evolution, given the Hamiltonian operator H , is defined as

$$\frac{dA}{dt} = \{H, A\} \quad (3.5)$$

With these classifications of the two fields in mind, we arrive at a first sketch of a transition between them:

DEFINITION 3.1 (Quantisation Scheme) A map passing from classical mechanics to quantum mechanics is called a **quantisation scheme**.

To illuminate this rather vague definition, we discuss a "toy example":

EXAMPLE 3.2 (Weyl Quantisation) This quantisation treats a mechanical system with one degree of freedom. The classical system has \mathbb{R}^2 as its phase space with coordinates (p, q) . The dynamics are given by the Poisson brackets with the Hamiltonian function. Now we want to transition to a quantum description:

We take the Hilbert-space \mathcal{H} defined as

$$\mathcal{H} = L^2(\mathbb{R}) := \left\{ \psi(q) \in C(\mathbb{R}) \mid \int_{-\infty}^{\infty} |\psi(q)|^2 dq < \infty \right\} \quad (3.6)$$

Observables are now operators on \mathcal{H} . We define the two operators

$$(P\psi)(q) = \frac{\hbar}{i} \frac{d}{dq} \psi(q), \quad (Q\psi)(q) = q\psi(q) \quad (3.7)$$

Now we can define products

$$(A\psi)(q) = \int_{-\infty}^{\infty} A(q, q') \psi(q') dq' \quad (3.8)$$

where $A(q, q')$ denotes the integral kernel of A defined as

$$A(q, q') = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} f\left(p, \frac{q+q'}{2}\right) e^{ip(q-q')/\hbar} dp \quad (3.9)$$

where the f are functions that represent A in classical mechanics. As an exercise you can show that with these definitions you obtain the previously defined operators P, Q if you set $f = p$ and $f = q$ respectively.

Next we need to define a product. Since generally $A_{fg} \neq A_f A_g$ we investigate

$$A_f A_g(q, q') = \int_{-\infty}^{\infty} A_f(q, q'') A_g(q'', q') dq'' \quad (3.10)$$

and then define

$$f *_h g = fg + \frac{i\hbar}{2} \{f, g\} + \mathcal{O}(\hbar^2) \quad \Rightarrow \quad \frac{i}{\hbar} (f *_h g - g *_h f) = \{f, g\} + \mathcal{O}(\hbar^2) \quad (3.11)$$

We now encode the dynamics via the time derivative using the Hamiltonian operator

$$\frac{dA(t)}{dt} = \{H, A(t)\}_h, \quad A(0) = A \quad (3.12)$$

where we define $A(t)$ as

$$A(t) = U^{-1}(t) A U(t), \quad U(t) = e^{-iHt/\hbar} \quad (3.13)$$

As an exercise you can prove this in a formal setting. Note that we do not yet have a concrete way to define the exponential of an operator such as H . Thus, setting $\hbar = 1$ for simplicity, we investigate the integral kernel of $U = e^{-iH(t''-t')}$ in terms of $h(p, q)$. To this end let $\Delta \ll 1$ such that

$$U_\Delta := e^{-iH\Delta} \cong 1 - iH\Delta \quad (3.14)$$

Thus setting $t'' - t' = N\Delta$ we obtain $e^{-iH(t''-t')} = (U_\Delta)^N$. Now we calculate

$$U_\Delta(q, q') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{ip(q-q')} \left(1 - ih \left(p, \frac{q+q'}{2} \right) \Delta \right) \quad (3.15)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{ip(q-q') - ih \left(p, \frac{q+q'}{2} \right) \Delta} + \mathcal{O}(\Delta^2) \quad (3.16)$$

Now setting $q = q_N, q' = q_0$ and q_j the $N - 1$ intermediate variables, we arrive at the following form:

$$U(q, q', t'' - t') = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{dp_1 dq_1}{2\pi} \dots \frac{dp_{N-1} dq_{N-1}}{2\pi} \frac{dp_N}{2\pi} \quad (3.17)$$

$$\times e^{ip_N(q_N - q_{N-1}) + \dots + ip_1(q_1 - q_0)} \cdot e^{-ih \left(p_N, \frac{q_N + q_{N-1}}{2} \right) \dots - ih \left(p_1, \frac{q_1 + q_0}{2} \right)} \quad (3.18)$$

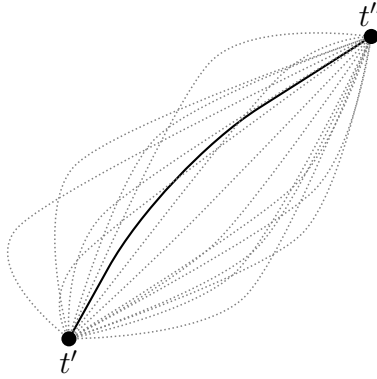
Thus we take $N \rightarrow \infty$ such that $q_j - q_{j-1}$ being proportional to $\frac{1}{N}$ goes against 0. While we do not discuss convergence and well-definedness at this point, the reader is invited to start looking critically at these notions from now on! Formally this gives us the following form of U which indeed has many problems with convergence of integrals

$$U(q, q', t'' - t') = \int \dots \int \exp \left[i \int_{t'}^{t''} (p(t) \dot{q}(t) - h(p(t), q(t))) dt \right] \prod_i \frac{dp_i(t) dq^i(t)}{2\pi} \quad (3.19)$$

Where the fraction defines our measure on the space of paths from $[t', t'']$ called the **Liouville-measure** and the term in the exponential is called the **Action functional**.

Note that this is only a one-dimensional "toy model" which already boasts some pretty severe convergence problems. Since we ultimately aim to define an infinite-dimensional generalisation, the rest of the course will deal with the "taming" of these problems.

In the following we give a short introduction to **Feynman Path Integrals**, visually represented in the following picture:



Feynman's approach (historically first introduced by Dirac many years earlier) to Quantum Field Theory was the idea of a "sum over all possible histories" of a particle. In *chapter 2* 2.5 we already mentioned scalar field theories where $\psi \in \mathcal{C}^\infty(M)$ is a scalar field. The physical phenomenon corresponding to the "sum over histories" is described by a superposition of all possible trajectories ψ , weighted by $e^{iS(\psi)/\hbar}$ for $S(\psi)$ the action of ψ .

This in turn has an interpretation (in Euclidean signature) coming from statistical mechanics: Here, ψ would represent the state of a statistical system, $S(\psi)$ would be the systems energy. Given a temperature T , the system could be in any state $e^{-S(\psi)/T}$. Thus we obtain the following correspondence:

$$T \longleftrightarrow i\hbar$$

This gives us a sufficient motivation to repeat the assessment of dynamics for statistical systems:

In statistical mechanics, observables are maps $O : \mathcal{C}^\infty(M, \mathbb{R}) \longrightarrow \mathbb{C}$ and correlation functions between observables are defined as

$$\langle O_1, \dots, O_n \rangle = \int_{\psi} e^{-S(\psi)/T} O_1(\psi) \dots O_n(\psi) D\psi \quad (3.20)$$

$$\implies \langle O_1, \dots, O_n \rangle = \int_{\psi \in \mathcal{C}^\infty(M)} e^{iS(\psi)/\hbar} O_1(\psi) \dots O_n(\psi) D\psi \quad (3.21)$$

The main problem in this situation is that $\mathcal{C}^\infty(M)$ is an infinite-dimensional vector space.

Remark 3.3 One approach to resolve this is *perturbative QFT* where we work in the limit $\hbar \rightarrow 0$ such that we can take formal power series in \hbar . This provides us with a formal procedure to go from a quantum description to a classical one. Note that this endows \hbar with the role of a formal parameter. In addition to a perturbative treatment, we can drop the Lorentzian signature and stick to Riemannian ones since the mathematical background is better understood.

3.2 The Effective Action

In this section, we try to "tame" the previously exposed integrals over infinite-dimensional vector spaces. We were stuck with expressions of the form $\langle O_1, \dots, O_n \rangle$ which indeed could not be calculated with the tools we previously used. One approach to expressions of this form are "*Wilson low-energy theories*" which will make up the first part of this subchapter.

The main idea of "*Wilson low-energy theories*" is that observables can only measure phenomena with an energy below a fixed constant energy Λ . Thus let us fix a Riemannian manifold M , $I \subset [0, \infty)$ and denote by $\mathcal{C}^\infty(M)_I \subset \mathcal{C}^\infty(M)$ the set of functions f that are sums of eigenfunctions of the *Laplacian* 2.52 with eigenvalues in I .

LEMMA 3.4 If I is bounded, $\mathcal{C}^\infty(M)_I$ is a finite-dimensional vector space.

Proof. The proof is left as an exercise to the reader. □

Now let $\Lambda \in I$ and define $J = [0, \Lambda)$. We thus define

$$\mathcal{C}^\infty(M)_J := \mathcal{C}^\infty(M)_{\leq \Lambda} \subseteq \mathcal{C}^\infty(M) \quad (3.22)$$

and call it the **space of fields with energy at most Λ** . Observables in this framework, i.e. restricted to such fields, are functionals

$$O: \mathcal{C}^\infty(M)_{\leq \Lambda} \longrightarrow \mathbb{R}[[\hbar]] \quad (3.23)$$

where $\mathbb{R}[[\hbar]]$ denotes formal power series in \hbar . We can extend each such operator O to $\mathcal{C}^\infty(M)$ by composing it with the evident projection from $\mathcal{C}^\infty(M)_{\leq \Lambda}$. We define $Obs_{\leq \Lambda}$ to be the thus arising observables.

Now we investigate $\langle O_1, \dots, O_n \rangle$ where $O_i \in Obs_{\leq \Lambda}$. We define quantities of this form as

$$\langle O_1, \dots, O_n \rangle := \int_{\varphi \in \mathcal{C}^\infty(M)_{\leq \Lambda}} e^{S_{eff}[\Lambda](\varphi)/\hbar} O_1 \dots O_n D\varphi \quad (3.24)$$

Here, $D\varphi$ is the Lebesgue measure on $\mathcal{C}^\infty(M)_{\leq \Lambda}$, $S_{eff}[\Lambda]$ is a function on $\mathcal{C}^\infty(M)_{\leq \Lambda}$ and thus a formal power series in \hbar . We call such an action a **low energy effective action**.

Remark 3.5 The quadratic part of $S_{eff}[\Lambda]$ is negative-definite.

DEFINITION 3.6 (Scalar Quantum Field Theory) A **scalar (perturbative) quantum field theory** is a collection of effective actions

$$S_{eff}[\Lambda]: \mathcal{C}^\infty(M)_{\leq \Lambda} \longrightarrow \mathbb{R}[[\hbar]] \quad (3.25)$$

for all $\Lambda \in [0, \infty)$ such that

1. $S_{eff}[\Lambda]$ is a formal power series in both the field $\varphi \in \mathcal{C}^\infty(M)_{\leq \Lambda}$ and the variable \hbar .
2. When setting $\hbar = 1$, $S_{eff}[\Lambda]$ must be of the form

$$S_{eff}[\Lambda] = -\frac{1}{2} \int \varphi(D + m^2)\varphi + \mathcal{O}(\varphi^3) \quad (3.26)$$

where D is the positive-definite Laplacian.

3. If $\Lambda' \leq \Lambda$, $S_{eff}[\Lambda']$ is determined by $S_{eff}[\Lambda]$ by the renormalisation group equation.
4. The effective actions $S_{eff}[\Lambda]$, when translatable in length scale terms, satisfy the asymptotic locality axiom.

Note that the notions mentioned in points 3. and 4. have not been discussed yet. To fill in these gaps will be the function of this subchapter.

We will subsequently fix the notions mentioned in the previous definition. From point 2. we know that $S_{eff}[\Lambda]$ must be of the following form:

$$S_{eff}[\Lambda](\varphi) = -\frac{1}{2} \langle \varphi, (D + m^2)\varphi \rangle + I[\Lambda](\varphi) \quad (3.27)$$

Here, $\langle \cdot, \cdot \rangle$ denotes the L^2 -inner product on $\mathcal{C}^\infty(M)$ defined via

$$\langle \varphi, \psi \rangle := \int_M \varphi(x)\psi(x)dx \quad (3.28)$$

is the Laplacian and m is a positive real parameter. Further, $I[\Lambda]$, the **effective interaction**, is understood as a formal power series in \hbar :

$$I[\Lambda](\varphi) = I_0[\Lambda](\varphi) + \hbar I_1[\Lambda](\varphi) + \dots \quad (3.29)$$

Here, $I_0[\Lambda](\varphi)$ is at least cubic in φ and all the I_i are formal power series in φ .

3.3 Renormalisation group equation

Now our goal is to "translate" a system $S_{eff}[\Lambda], Obs_{\leq \Lambda}, \mathcal{C}^\infty(M)_{\leq \Lambda}$ into one with threshold $\Lambda' \leq \Lambda$. There exists an evident inclusion $Obs_{\leq \Lambda'} \hookrightarrow Obs_{\leq \Lambda}$. The correlation functions should not change if we compute them in $Obs_{\leq \Lambda'}$ rather than in $Obs_{\leq \Lambda}$ after including:

$$\int_{\varphi \in \mathcal{C}^\infty(M)_{\leq \Lambda'}} e^{S_{eff}[\Lambda'](\varphi)/\hbar} O_1(\varphi) \dots O_n(\varphi) D\varphi^{\Lambda'} = \int_{\varphi \in \mathcal{C}^\infty(M)_{\leq \Lambda}} e^{S_{eff}[\Lambda](\varphi)/\hbar} O_1(\varphi) \dots O_n(\varphi) D\varphi^\Lambda \quad (3.30)$$

Further note that we can split

$$\mathcal{C}^\infty(M)_{\leq \Lambda} = \mathcal{C}^\infty(M)_{\leq \Lambda'} \oplus \mathcal{C}^\infty(M)_{(\Lambda', \Lambda]} \quad (3.31)$$

Using this splitting to separate (3.30) allows us to formulate:

$$RHS = \int_{\varphi_L \in \mathcal{C}^\infty(M)_{\leq \Lambda'}} \int_{\varphi_H \in \mathcal{C}^\infty(M)_{(\Lambda', \Lambda]}} e^{S_{eff}[\Lambda](\varphi_L + \varphi_H)/\hbar} O_1(\varphi_L) \dots O_n(\varphi_L) D\varphi^{\Lambda'} D\varphi^{\Lambda\Lambda'} \quad (3.32)$$

$$= \int_{\varphi_L \in \mathcal{C}^\infty(M)_{\leq \Lambda'}} \left[\int_{\varphi_H \in \mathcal{C}^\infty(M)_{(\Lambda', \Lambda]}} e^{S_{eff}[\Lambda](\varphi_L + \varphi_H)/\hbar} D\varphi^{\Lambda\Lambda'} \right] O_1(\varphi_L) \dots O_n(\varphi_L) D\varphi^{\Lambda'} \quad (3.33)$$

$$(3.34)$$

This in turn allows us to identify

$$e^{S_{eff}[\Lambda'](\varphi_L)/\hbar} = \int_{\varphi_H \in \mathcal{C}^\infty(M)_{(\Lambda', \Lambda]}} e^{S_{eff}[\Lambda](\varphi_L + \varphi_H)/\hbar} D\varphi^{\Lambda\Lambda'} \quad (3.35)$$

which ultimately leads to the **Renormalisation group equation (RGE)**

$$S_{eff}[\Lambda'](\varphi_L) = \hbar \log \left[\int_{\varphi_H \in \mathcal{C}^\infty(M)_{(\Lambda', \Lambda]}} e^{S_{eff}[\Lambda](\varphi_L + \varphi_H)/\hbar} D\varphi^{\Lambda\Lambda'} \right] \quad (\text{RGE})$$

Having a rather well-defined system of such effective actions we may wonder, what their relation to the original classical action S is. This correspond to a "limit" of the form $S_{eff}[\infty]$ that is $\Lambda \rightarrow \infty$. Note that the space of fields we used in the integral of (RGE) is ill-defined for $\Lambda = \infty$, thus this discussion is not trivially dispatched.

One approach is to describe (RGE) in terms of effective interactions. Note that $\mathcal{C}^\infty(M)_{\Lambda'}$ and

$\mathcal{C}^\infty(M)_\Lambda$ are D -orthogonal, thus

$$\langle \varphi_L + \varphi_H, D(\varphi_L + \varphi_H) \rangle = \langle \varphi_L, D\varphi_L \rangle + \langle \varphi_H, D\varphi_H \rangle \quad (3.36)$$

$$\langle \varphi_L + \varphi_H, m^2(\varphi_L + \varphi_H) \rangle = \langle \varphi_L, m^2\varphi_L \rangle + \langle \varphi_H, m^2\varphi_H \rangle \quad (3.37)$$

Now if we define

$$F(\varphi) = -\frac{1}{2} \langle \varphi, (D + m^2)\varphi \rangle, \quad s.t. \quad F(\varphi_L + \varphi_H) = F(\varphi_L) + F(\varphi_H) \quad (3.38)$$

we can rewrite our effective action as

$$S_{eff}[\Lambda](\varphi) = F(\varphi) + I[\Lambda](\varphi), \quad s.t. \quad S_{eff}[\Lambda](\varphi_L + \varphi_H) = F(\varphi_L) + F(\varphi_H) + I[\Lambda](\varphi_L + \varphi_H) \quad (3.39)$$

Combining this with (RGE) brings us to

$$F(\varphi_L) + I[\Lambda'](\varphi_L) = \hbar \log \left[\int_{\varphi_H \in \mathcal{C}^\infty(M)_{(\Lambda', \Lambda)}} \exp \left(\frac{F(\varphi_H)}{\hbar} + \frac{F(\varphi_L)}{\hbar} + \frac{I[\Lambda](\varphi_H + \varphi_L)}{\hbar} \right) D\varphi^{\Lambda'} \right] \quad (3.40)$$

$$= \hbar \log \left[e^{\frac{F(\varphi_L)}{\hbar}} \int_{\varphi_H \in \mathcal{C}^\infty(M)_{(\Lambda', \Lambda)}} \exp \left(\frac{F(\varphi_H)}{\hbar} + \frac{I[\Lambda](\varphi_H + \varphi_L)}{\hbar} \right) D\varphi^{\Lambda'} \right] \quad (3.41)$$

$$= F(\varphi_L) + \hbar \log \left[\int_{\varphi_H \in \mathcal{C}^\infty(M)_{(\Lambda', \Lambda)}} \exp \left(\frac{F(\varphi_H)}{\hbar} + \frac{I[\Lambda](\varphi_H + \varphi_L)}{\hbar} \right) D\varphi^{\Lambda'} \right] \quad (3.42)$$

Thus we arrive at the **Interaction form of the RGE**:

$$I[\Lambda'](a) = \hbar \log \left[\int_{\varphi_H \in \mathcal{C}^\infty(M)_{(\Lambda', \Lambda)}} \exp \left(\frac{F(\varphi_H)}{\hbar} + \frac{I[\Lambda](\varphi_H + a)}{\hbar} \right) D\varphi^{\Lambda'} \right] \quad (\text{RGE I})$$

This form has two major advantages over (RGE):

1. We are not integrating over $a \in \mathcal{C}^\infty(M)_{\leq \Lambda'}$, the equation can be extended to any $\varphi_L \in \mathcal{C}^\infty(M)$.
2. It is invertible. It remains valid when choosing $\Lambda' > \Lambda$.

While we have indeed explored the meaning and framework of *point 3.* of 3.6, we still have to settle the notion of **locality**. On a physical level, locality means that interactions between fundamental particles occur at a point of the manifold.

DEFINITION 3.7 (Local action functional) A functional $S: \mathcal{C}^\infty(M) \longrightarrow \mathbb{R}[[\hbar]]$ is a **local action**

functional if it can be written as a sum

$$S(\varphi) = \sum_k S_k(\varphi), \quad \text{where} \quad S_k(\varphi) = \int_M (D_1\varphi)\dots(D_k\varphi) \text{ Vol}_M \quad (3.43)$$

for some first-order differential operators D_i on M . This allows us to write S as

$$S(\varphi) = \int_M \mathcal{L}(\varphi)(x) \text{ Vol}_M \quad (3.44)$$

where $\mathcal{L}(\varphi)$ is a **Lagrangian** and depends only on the Taylor expansion of φ at x .

In our current formulation this condition on S does not really make sense yet which is why a definition using $S_{eff}[\Lambda]$ would be favourable. A tentative definition could be the following:

DEFINITION 3.8 A collection of low-energy effective actions $S_{eff}[\Lambda]$ satisfying the renormalisation group equation RGE is **asymptotically local** if there exists an asymptotic expansion for large Λ

$$S_{eff}[\Lambda](\varphi) \cong \sum_i f_i(\Lambda) \Theta_i(\varphi) \quad (3.45)$$

where the $\Theta_i(\varphi)$ are local action functionals and $f_i \in \mathcal{C}^\infty(\mathbb{R})$.

While tentative, this definition certainly is not a good idea: Supposing $S_{eff}[\Lambda]$ is close to a local action functional, then using the RGE one obtains that $S_{eff}[\Lambda']$ is in fact entirely non local for $\Lambda' < \Lambda$. The solution to this problem is to consider length scales instead of energy scales.

The theory based on length scales takes as fundamental objects the propagator of a differential operator. In order to rewrite the RGE in terms of such propagators, we will need to introduce *Feynman graphs* as a way to compute integrals.

3.4 Feynman graphs

The RGE contains an integral of the form

$$\int_{x \in U} \exp(\Phi(x)/\hbar + I(x+a)/\hbar) \quad (3.46)$$

where Φ is a quadratic negative definite form on a vector space U . From now on we will assume as a convention that the measure on U is the *Lebesgue measure* normalized such that

$$\int_{x \in U} \exp(\Phi(x)/\hbar) = 1 \quad (3.47)$$

Note that the measure does in fact depend on \hbar . Now this type of integral can be expressed as a sum over *Feynman graphs* where we understand the integral as an asymptotic series in \hbar . To introduce *Feynman graphs*, we start by giving two separate but equivalent definitions of *stable graphs*:

DEFINITION 3.9 (Stable Graphs – Graph Version) A **stable graph** is a graph γ possibly with external edges (edges that only connect to one vertex/node) such that

1. to each vertex (node) $v \in V(\gamma)$ there is an associated number $g(v) \in \mathbb{Z}_{\geq 0}$ called the genus of the vertex.
2. each vertex of genus 0 is at least trivalent (that is of degree 3).
3. every vertex of genus 1 is at least 1-valent (that is of degree 1).

Using the genus of vertices in a graph we can define the genus of a stable graph in a canonical way by taking into account the topology of the graph:

DEFINITION 3.10 (Genus of a Stable Graph) Let γ be a stable graph. Its **genus** $g(\gamma)$ is defined by

$$g(\gamma) = b_1(\gamma) + \sum_{v \in V(\gamma)} g(v) \quad (3.48)$$

where $b_1(\gamma)$ is the first *Betti number* of γ defined by $b_1(\gamma) = |E| + |C| - |V|$ where E is the set of *internal* edges, C the set of connected components of γ and V the set of vertices (nodes).

As promised we give the following alternative yet equivalent definition of a stable graph:

DEFINITION 3.11 (Stable Graph - Formal Version) Let $H(\gamma)$ and $V(\gamma)$ be two finite sets (half-edges and vertices) and let

$$\sigma: H(\gamma) \longrightarrow H(\gamma) \quad \text{an involution} \quad (3.49)$$

$$\pi: H(\gamma) \longrightarrow V(\gamma) \quad (3.50)$$

$$g: V(\gamma) \longrightarrow \mathbb{Z}_{\geq 0} \quad (\text{the genus map}) \quad (3.51)$$

then a **graph** is the topological space

$$V(\gamma) \sqcup (H(\gamma) \times [0, 1/2]) / \sim \quad (3.52)$$

where $(h, 0) \sim \pi(h)$ and $(h, 0.5) \sim (\sigma(h), 0.5)$. A graph is **stable** if every vertex $v \in V(\gamma)$ is such that if

$$g(v) = 0 \quad \text{then} \quad \#\pi^{-1}(v) \geq 3 \quad (3.53)$$

$$g(v) = 1 \quad \text{then} \quad \#\pi^{-1}(v) \geq 1 \quad (3.54)$$

DEFINITION 3.12 An **automorphism** F of a graph γ is a pair of maps

$$H(F): H(\gamma) \longrightarrow H(\gamma) \quad (3.55)$$

$$V(F): V(\gamma) \longrightarrow V(\gamma) \quad (3.56)$$

such that $H(F)$ commutes with σ and such that the following diagram commutes:

$$\begin{array}{ccc} H(\gamma) & \xrightarrow{H(F)} & H(\gamma) \\ \pi \downarrow & & \downarrow \pi \\ V(\gamma) & \xrightarrow{V(F)} & V(\gamma) \end{array}$$

The automorphisms form a finite group, the proof is left as an exercise.

We denote by $T(\gamma)$ the set of fixed points of the involution σ and by $E(\gamma)$ the set of two element orbits of σ . As such, $T(\gamma)$ stands for the tails or external edges and $E(\gamma)$ for the internal edges of the graph.

Now we turn towards the data associated to *Feynman graphs* using the previously presented tools. Let us fix a finite graded vector space U over a field \mathbb{k} . Let further $\mathcal{O}(U)$ be the (completed) symmetric algebra on the dual vector space U^* . This algebra coincides with $\mathcal{O}(U)$ being the ring of formal power series in variables in U . Thus given a basis \mathcal{B} of U there exists a canonical isomorphism

$$\mathcal{O}(U) \longleftrightarrow \mathbb{k}[[\mathcal{B}]] \quad (3.57)$$

DEFINITION 3.13 Let $f \in \mathcal{O}(U)$ be homogenous of degree k . Then for a local action S it defines an S_k -invariant linear map

$$D^k f: U^{\otimes k} \longrightarrow \mathbb{k} \quad (3.58)$$

$$u_1 \otimes \dots \otimes u_k \longmapsto \left(\frac{\partial}{\partial u_1} \dots \frac{\partial}{\partial u_k} f \right) (0) \quad (3.59)$$

Our next goal is to assign some algebraic data to our graphs:

DEFINITION 3.14 Let us denote by $\mathcal{O}^+(U)[[\hbar]] \subset \mathcal{O}(U)[[\hbar]]$ the subset of functionals at least cubic modulo \hbar . An element $I \in \mathcal{O}(U)[[\hbar]]$ can be decomposed as $I = \sum_{i,k} \hbar^i I_{i,k}$ where $I_{i,k}$ is homogenous of degree k in U . Now fix an ordering of the set of tails of γ

$$T(\gamma) \xleftrightarrow{\psi} \{1, \dots, n\} \quad (3.60)$$

Also fix an element $P \in \text{Sym}^2(U) \subset U^{\otimes 2}$ and some $a_1, \dots, a_n \in U$. Then we have

$$2E(\gamma) + T(\gamma) = H(\gamma) \quad (3.61)$$

$$U^{\otimes 2E(\gamma)} \otimes U^{\otimes T(\gamma)} \cong U^{\otimes H(\gamma)} \quad (3.62)$$

$$P \otimes \dots \otimes P \otimes a_1 \otimes \dots \otimes a_n =: \mathbb{P} \in U^{\otimes H(\gamma)} \quad (3.63)$$

Now fix $I \in \mathcal{O}(U)[[\hbar]]$ and pick a vertex v with arbitrary valency k and genus i . We associate to it

$$D^k I_{i,k} \in \text{Hom}(U^{\otimes k}, \mathbb{k}) \quad (3.64)$$

Taking the tensor product of these elements yields

$$\mathbb{I} = \bigotimes_{v \in \gamma} D^k I_{i,k} \quad (3.65)$$

$$\mathbb{I} \in \text{Hom}(U^{\otimes H(\gamma)}, \mathbb{k}) = (U^{\otimes H(\gamma)})^* \subset \mathcal{O}(U) \quad (3.66)$$

Now define the **weight** of a graph γ with fixed ordering ψ , elements P and a_1, \dots, a_n and interaction I as

$$w_{\gamma, \psi}(P, I)(a_1, \dots, a_n) := \mathbb{I}(\mathbb{P}) \in \mathbb{k} \quad (3.67)$$

$$w_{\gamma}(P, I)(a) := w_{\gamma, \psi}(P, I)(a, \dots, a) \quad (3.68)$$

which assigns, for chosen ψ , a number to our graph.

EXERCISE 3.15 Show the following statements:

1. $w_{\gamma}(P, I)(a_1, \dots, a_n) \in \mathcal{O}(U)$
2. $w_{\gamma}(P, I)(a_1, \dots, a_n)$ is homogenous of degree n and has the property

$$\frac{\partial}{\partial a_1} \dots \frac{\partial}{\partial a_n} w_{\gamma}(P, I)(a_1, \dots, a_n) = \sum_{\psi \in \pi(T(\gamma))} w_{\gamma, \psi}(P, I)(a_1, \dots, a_n) \quad (3.69)$$

3. If $v_{i,k}$ is the graph with one vertex of genus i and valency k and no internal edges, then $w_{v_{i,k}}(1, I)(a) = k! I_{i,k}$

Using $w_{\gamma}(P, I) \in \mathcal{O}(U)$ we define

$$W(P, I) = \sum_{\gamma} \frac{1}{|\text{Aut}(\gamma)|} \hbar^{g(\gamma)} w_{\gamma}(P, I) \in \mathcal{O}^+(U)[[\hbar]] \quad (\text{Weight})$$

where we sum over all connected stable graphs. This is a power series in $u \in U$ and in \hbar . We further define $W_{i,j}(P, I)$ to be the sum of all terms of $W(P, I)$ where $g(\gamma) = i$ and $|T(\gamma)| = j$.

EXAMPLE 3.16

$$\begin{aligned}
W_{0,3}(P, I) &= \text{Diagram: a vertex with three dashed lines (tails) meeting at it, labeled } I_{0,3} \\
W_{0,4}(P, I) &= \text{Diagram: a vertex with four dashed lines (tails) meeting at it, labeled } I_{0,4} \quad + \quad \text{Diagram: two vertices connected by a solid line labeled } P, \text{ each vertex has two dashed lines (tails), labeled } I_{0,3} \\
W_{1,1}(P, I) &= \text{Diagram: a vertex with one dashed line (tail) and one solid line (loop) attached, labeled } I_{1,1} \quad + \quad \text{Diagram: a vertex with one dashed line (tail) and one solid line (loop) attached, labeled } I_{0,3} \text{ and } P
\end{aligned}$$

LEMMA 3.17 Setting P to 0 we obtain $W(0, I) = I$.

Proof. As a *hint*, note that only graphs with no (internal) edges, i.e. with only tails, can contribute. \square

LEMMA 3.18 For $a_1, \dots, a_k \in U$ we get

$$\left(\frac{\partial}{\partial a_1} \dots \frac{\partial}{\partial a_n} W(P, I) \right) (0) = \sum_{\gamma, \psi} \frac{\hbar^{g(\gamma)}}{|\text{Aut}(\gamma, \psi)|} w_{\gamma, \psi}(P, I)(a_1, \dots, a_k) \quad (3.70)$$

Proof. The proof is left as an exercise to the reader. \square

Now we define some helpful notation to formulate our next result: Let $P \in \text{Sym}^2(U)$. We define $\partial_P: \mathcal{O}(U) \rightarrow \mathcal{O}(U)$ using the decomposition $P = \sum_i P'_i \otimes P''_i$ as

$$\partial_P := \frac{1}{2} \sum_i \frac{\partial}{\partial P'_i} \frac{\partial}{\partial P''_i} \quad (3.71)$$

This leads us to the following lemma:

LEMMA 3.19

$$W(P, I)(a) = \hbar \log \{ \exp(\hbar \partial_P) \exp(I/\hbar) \}(a) \in \mathcal{O}^+(U)[[\hbar]] \quad (3.72)$$

Proof. The case of $P = 0$ is treated in 3.17. Thus assume $P \neq 0$ such that

$$\exp(\hbar^{-1} W(P, I)) = \exp(\hbar \partial_P) \exp(I/\hbar) \quad \forall P \quad (\diamond)$$

Now let $\varepsilon > 0$ small such that we can neglect terms of order ε^2 and let $P' \in \text{Sym}^2(U)$. Thus we arrive at

$$(\diamond) \Leftrightarrow \exp(\hbar^{-1} W(P + \varepsilon P', I)) = (1 + \hbar \varepsilon \partial_{P'}) \exp(\hbar^{-1} W(P, I)) \quad \forall P, P' \quad (\diamond\diamond)$$

Thus we get $(1 + \hbar \varepsilon \partial_{P'}) \approx \exp(\hbar \varepsilon \partial_{P'})$. Now if we assume in (\diamond) that $P \rightarrow P + \varepsilon P'$ we get:

So we want to prove $(\diamond\diamond)$

$$\frac{d}{d\varepsilon} (\exp(\hbar^{-1}W(P + \varepsilon P', I))) = \hbar \partial_{P'} \exp(W(P, I)/\hbar) \quad (3.73)$$

which is equivalent to $(\diamond\diamond)$. Thus we have already done it and can integrate it. The arising constant can be computed using $\varepsilon = 0$.

$$\left(\frac{\partial}{\partial a_1} \dots \frac{\partial}{\partial a_n} W(P, I) \right) (0) = \sum_{\gamma, \psi} \frac{\hbar^{g(\gamma)}}{|\text{Aut}(\sigma, \psi)|} w_{\gamma, \psi}(P, I)(a_1, \dots, a_k) \quad (3.74)$$

$$\frac{\partial}{\partial a_1} \dots \frac{\partial}{\partial a_n} \exp(\hbar W(P, I))(0) = \sum_{\gamma, \psi} \frac{\hbar^{g(\gamma)}}{|\text{Aut}(\sigma, \psi)|} w_{\gamma, \psi}(P, I)(a_1, \dots, a_k) \quad (3.75)$$

This formula will be used for $P + \varepsilon P'$. Consider

$$w_{\gamma, \psi}(P + \varepsilon P', I)(a_1, \dots, a_k) \quad (3.76)$$

and recall $\varepsilon^2 \approx 0$. Now note that

$$P + \varepsilon P' = P \otimes \dots \otimes \varepsilon P' \otimes a_1 \dots a_k + P \otimes \dots \otimes \varepsilon P \otimes a_1 \dots a_k \quad (3.77)$$

where P' is at edge e . We thus inspect the derivative

$$\frac{d}{d\varepsilon} \left(\frac{\partial}{\partial a_1} \dots \frac{\partial}{\partial a_k} \exp(\hbar W(P + \varepsilon P', I)) \right) (0) = \sum_{\gamma, e, \psi} \frac{\hbar^{g(\gamma)}}{|\text{Aut}(\sigma, e, \psi)|} w_{\gamma, e, \psi}(P, I)(a_1, \dots, a_k) \quad (3.78)$$

$$= \frac{1}{2} \sum_{\gamma, \psi} \frac{\hbar^{g(\gamma)}}{|\text{Aut}(\sigma, \psi)|} w_{\gamma, \psi}(P, I)(a_1, \dots, a_k, U', U'') \quad (3.79)$$

where we used $P' = \sum u' \otimes u''$. Now this brings us to

$$= \frac{1}{2} \frac{\partial}{\partial a_1} \dots \frac{\partial}{\partial a_k} \frac{\partial}{\partial u'} \frac{\partial}{\partial u''} \exp(\hbar^{-1}W(P, I))(0) \quad (3.80)$$

□

Let us now connect Feynman diagrams to integrals. Let U now have the base field \mathbb{R} and let Φ be a non-degenerate quadratic form on it. Further let $P = \sum_i e_i \otimes e_i$ where $\{e_i\}$ forms an orthonormal basis for $-\Phi$.

PROPOSITION 3.20

$$W(P, I)(a) = \hbar \log \int_{x \in U} \exp \left(\frac{1}{2\hbar} \Phi(x, x) + \frac{1}{\hbar} I(x + a) \right) \quad \forall a \in U \quad (3.81)$$

Proof. Note that this is but a sketch of the proof, you can fill in the details as an exercise. First note

$$\int_{x \in U} \exp\left(\frac{1}{2\hbar}\Phi(x, x)\right) f(x + a) = (\exp(\hbar\partial_P)f)(a) \quad \forall f \in \mathcal{O}(U) \quad (3.82)$$

where we identified the exponential of $I(x + a)$ with $f(x + a)$. Note that the integral of only the first exponential in above equation amounts to 1. Thus for $f = 1$ the equation holds. Note that

$$\Phi: U \longrightarrow U^* \quad \text{isomorphism} \quad (3.83)$$

$$a \longmapsto \Phi(a, \cdot) := a^* \quad (3.84)$$

Using the dual idea we take for $l \in U^*$ the corresponding $l^* \in U$ such that $l = \Phi(l^*, \cdot)$. The following three statements are given as an exercise:

1. $[\partial_P, l] = -\frac{\partial}{\partial l^*}$
2. $e^{\hbar\partial_P}(lf) - le^{\hbar\partial_P}(f) = \hbar[\partial_P, l]e^{\hbar\partial_P}(f)$
3. $\frac{\partial}{\partial l^*}e^{\Phi(x, x)/2\hbar} = \hbar^{-1}l(x)e^{\Phi(x, x)/\hbar}$

Now using these statements we have:

$$\int_{x \in U} \exp\left(\frac{1}{2\hbar}\Phi(x, x)\right) l(x)f(x + a) = \hbar \int_{x \in U} \left(\frac{\partial}{\partial l^*}e^{\Phi(x, x)/2\hbar}\right) f(x + a) \quad (3.85)$$

$$= -\hbar \int_{x \in U} e^{\Phi(x, x)/2\hbar} l(x) \frac{\partial}{\partial l_a^*} f(x + a) \quad (3.86)$$

$$= -\hbar \frac{\partial}{\partial l_a^*} \int_{x \in U} e^{\Phi(x, x)/2\hbar} f(x + a) \quad (3.87)$$

This, together with the three previous results, can be used to prove the main claim. \square

Should U not be finite-dimensional, we can still define $W(P, I)$ and ∂_P . We also still have

$$W(P, I) = \hbar \log\{\exp(\hbar\partial_P) \exp(I/\hbar)\} \quad (3.88)$$

This concludes our discussion of Feynman graphs which will now be used to make a transition from energy scales to length scales.

3.5 Towards a definition of perturbative QFTs

We start by introducing some common concepts used in the arising theory:

DEFINITION 3.21 (Integral Kernels) Given $F: \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ an **integral kernel** of F is a map $K: M \times M \longrightarrow \mathbb{R}$ such that

$$F(f)(y) = \int_{x \in M} f(x)K(x, y) \quad (3.89)$$

A certain type of integral kernels will be of particular importance:

DEFINITION 3.22 (Propagators) The **propagator** of a scalar field theory is the integral kernel of the operator $(D + m^2)^{-1}$ where $0 < m \in \mathbb{R}$ and D denotes the *Laplacian* 2.52.

Given a basis $\{e_i\}$ for $\mathcal{C}^\infty(M)$ formed by orthonormal eigenvectors of D with eigenvalues $\lambda_i \in \mathbb{R}_{\geq 0}$ the propagator takes the following form:

$$P = \sum_i \frac{1}{\lambda_i^2 + m^2} e_i \otimes e_i \quad (3.90)$$

Define a cut-off operator by considering the following quantity for any subset $U \subseteq \mathbb{R}_{\geq 0}$:

$$P_U := \sum_{i|\lambda_i \in U} \frac{1}{\lambda_i^2 + m^2} e_i \otimes e_i \quad (3.91)$$

P_U is smooth as long as U is a bounded subset of $\mathbb{R}_{\geq 0}$. Using these bounded subsets, we denote subsets of $\mathcal{C}^\infty(M)$ whose Laplacian eigenvalues lie in U by $\mathcal{C}^\infty(M)_U$. Thus consider

$$W(P, I)(a) = \hbar \log \int_{\varphi \in \mathcal{C}^\infty(M)} \exp \left(-\frac{1}{2\hbar} \langle \varphi, (D + m^2) \varphi \rangle + \frac{1}{\hbar} I(\varphi + a) \right). \quad (3.92)$$

Due to the integration over $\mathcal{C}^\infty(M)$ this is not well-defined. We utilise the bounded subsets to define

$$W(P_U, I)(a) = \hbar \log \int_{\varphi \in \mathcal{C}^\infty(M)_U} \exp \left(-\frac{1}{2\hbar} \langle \varphi, (D + m^2) \varphi \rangle + \frac{1}{\hbar} I(\varphi + a) \right). \quad (3.93)$$

This expression is well-defined since $\mathcal{C}^\infty(M)_U$ is of finite dimension for U bounded. Hence take

$$I[\Lambda'](a) = \hbar \log \int_{\varphi_H \in \mathcal{C}^\infty(M)_{[\Lambda', \Lambda]}} \exp(F(\varphi_H)/\hbar + I[\Lambda](\varphi_H + a)/\hbar) \quad (3.94)$$

and let $U = [\Lambda', \Lambda]$. This yields a **rephrased RGE**:

$$I[\Lambda'] = W(P_{[\Lambda', \Lambda]}, I[\Lambda]) \quad (\text{RGE})$$

This result shows that the weight of a Feynman graph for an interaction term at energy Λ allows the calculation of an interaction at energy Λ' by rule of (RGE I). This new formulation of the same process allows for a transition to length scales using the previously defined propagators of a theory.

For any operator $P: \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$ define the **heat kernel** $\kappa_l(x, y)$ for $\varphi \in \mathcal{C}^\infty(M)$ via

$$(e^{-lP} \varphi)(x) = \int_M \kappa_l(x, y) \varphi(y) dy \quad (3.95)$$

Given an operator like P , one often calls e^{-lP} the *heat operator*; the heat kernel of P is its kernel. Feynman proved the following result relating the heat kernel of the Laplacian and the propagator:

$$P(x, y) = \int_{l=0}^{\infty} e^{-lm^2} \kappa_l(x, y) dl \quad (3.96)$$

This equation allows for a transition from an energy scale, thus energy eigenvalues of D , to length scales. Understanding this result and its implications will be the goal of the next pages.

Let $l \in \mathbb{R}_{\geq 0}$ and κ_l^0 the heat kernel of D . Thus for all $\varphi \in \mathcal{C}^\infty(M)$

$$(e^{-lD}\varphi)(x) = \int_{y \in M} \kappa_l^0(x, y) \varphi(y) dy. \quad (3.97)$$

Using the energy eigenvector basis $\{e_i\}$ of D we write

$$\kappa_l^0 = \sum_i e^{-l\lambda_i} e_i \otimes e_i. \quad (3.98)$$

Now define

$$\kappa_l := e^{-lm^2} \kappa_l^0 = \sum_i e^{-l(\lambda_i + m^2)} e_i \otimes e_i \quad (3.99)$$

which is the integral kernel of $e^{-l(D+m^2)}$ which in turn is the heat operator of $(D + m^2)$. The propagator of a theory is the integral kernel of $(D + m^2)^{-1}$. Thus, by an abuse of notation, we can write it in a "length-scale form" as

$$P = \int_{l=0}^{\infty} \kappa_l dl. \quad (3.100)$$

In the energy form, we had the following representation:

$$P = \sum_i \frac{1}{\lambda_i^2 + m^2} e_i \otimes e_i \quad (3.101)$$

For a subset $U \subseteq \mathbb{R}$ we defined a cut-off propagator via

$$P_U = \sum_{i|\lambda_i \in U} \frac{1}{\lambda_i^2 + m^2} e_i \otimes e_i. \quad (3.102)$$

In a similar fashion, define the cut-off propagator in length scales as

$$P(\varepsilon, L) = \int_{\varepsilon}^L \kappa_l dl \quad (3.103)$$

where $\varepsilon, L \in (0, \infty)$ and $\varepsilon < L$. This corresponds to a propagator with an infrared cutoff at L and an ultraviolet cutoff at ε . Note that length scales behave inverse to energy scales; small energies correspond to large length scales and vice versa. Using the basis of energy eigenfunctions $\{e_i\}$, one

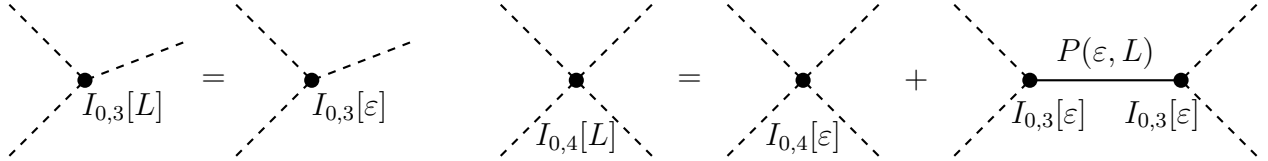
can write this propagator as

$$P(\varepsilon, L) = \sum_i \frac{e^{-\varepsilon\lambda_i} - e^{-L\lambda_i}}{\lambda_i^2 + m^2} e_i \otimes e_i \quad (3.104)$$

Thus for fixed ε and L the coefficients decay for growing λ_i . Since we can express the interaction term in length scales, we can rewrite (RGE) to:

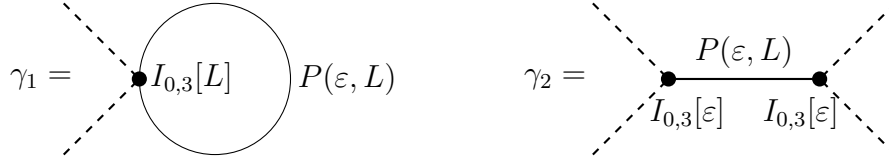
$$I[L] = W(P(\varepsilon, L), I(\varepsilon)) \quad (\text{L-RGE})$$

Again, a pictorial explanation for some examples of the (L-RGE) in form of Feynman diagrams greatly helps understanding this relation:



Our initial problem was the limit for high energies. So far we have just rephrased the issue. Note that the high-energy limit now corresponds to the limit $\varepsilon \rightarrow 0$. The following ideas are just a sketch of the formal procedure which has to be cut short due to time constraints and depth of the material.

SKETCH 3.23 We consider $W(P(\varepsilon, L), I(\varepsilon))$ and $I(\varphi) = \frac{1}{3!} \int_M \varphi^3$. In particular we consider two graphs:



For γ_1 we obtain:

$$w_{\gamma_1}(P(\varepsilon, L), I(\varepsilon))(a) = \int_{l \in [\varepsilon, L]} \int_{x \in M} a^2(x) \kappa_l(x, x) d\text{Vol}_M dl. \quad (3.105)$$

In the limit $l \rightarrow 0$ we obtain

$$\kappa_l(x, x) \simeq l^{-(\dim M)/2} + h.o.t. \quad (3.106)$$

However the integral over this term does not converge and thus

$$\lim_{\varepsilon \rightarrow 0} w_{\gamma_1}(P(\varepsilon, L), I(\varepsilon))(a) \quad (3.107)$$

does not exist. For γ_2 we note that

$$w_{\gamma_2}(P(\varepsilon, L), I(\varepsilon))(a) = \int_{l \in [\varepsilon, L]} \int_{x, y \in M} a^2(x) \kappa_l(x, y) a^2(y) dx dy dl \quad (3.108)$$

$$= \left\langle a^2, \int_{\varepsilon}^L e^{-lD} a^2 dl \right\rangle \quad (3.109)$$

In this case the limit $\lim_{\varepsilon \rightarrow 0} w_{\gamma_2}$ does exist!

This fairly rough sketch is linked to a much deeper result whose proof we omit due to time constraints:

FACT 3.24 Only graphs without loops (tree-level graphs) have a limit $\varepsilon \rightarrow 0$.

We will see in the subsequent material that and how we need to react to the infinities that arise when working with graphs that do contain loops.

3.5.1 Interpretation of Feynman diagrams

The heat kernel can be interpreted as a form of transition propability. Namely the kernel $\kappa_t(x, y)$ can be interpreted as the propability density that a particle in Brownian motion on M starts at x and reaches y in times t . We arrive at the following propability for a transition from x to y :

$$P(x, y) = \int_{l=0}^{\infty} e^{-lm^2} \int_{f \in X} \exp \left(- \int_0^l \|df\|^2 \right) \quad (3.110)$$

where $X_l = \{f: [0, l] \rightarrow M | f(0) = x, f(l) = y\}$ is the space of paths from x to y . The energy of such a path is defined as

$$E(f) := \int_0^l \langle df, df \rangle. \quad (3.111)$$

Now we can collect this as

$$P(x, y) = \int_0^{\infty} \kappa_l(x, y) dl. \quad (3.112)$$

To make this integral well-defined, we also need the *Wiener* measure D_{Wiener} , which we won't go into detail about:

$$\kappa_l(x, y) = \int_{X_l} D_{Wiener} f. \quad (3.113)$$

Using the previously defined energy we can loosely write

$$\kappa_l(x, y) = \int_{X_l} e^{-E(f)}. \quad (3.114)$$

Now consider the following action:

$$S(\varphi) = \int_M \left(-\frac{1}{2} \varphi D \varphi + \frac{1}{3!} \varphi^3 \right). \quad (3.115)$$

Note that this discussion is more of a sketch since the actual treatment has quite some problems already. For n particles we consider the expectation value

$$\mathbb{E}(x_1, \dots, x_n) = \int_{\varphi \in \mathcal{C}^\infty(M)} e^{S(\varphi)/\hbar} \varphi(x_1) \dots \varphi(x_n) \quad (3.116)$$

Now we associate to n particles n external edges:

$$\mathbb{E}(x_1, \dots, x_n) = \sum_{\gamma} \frac{1}{|\text{Aut}(\gamma)|} \hbar^{-g(\gamma)} \int_{g \in \text{Met}_{\gamma}} \int_{f: \gamma \rightarrow M} e^{-E(f)} \quad (3.117)$$

where Met_{γ} stands for the space of metrics on the space of curves γ and f are those maps $\gamma \rightarrow M$ that take the endpoints of the n external edges of γ to x_1, \dots, x_n . Also consider the inverse assignment

$$f: \gamma \rightarrow M, \quad E(\gamma) \mapsto (x_1, \dots, x_n) \quad (3.118)$$

DEFINITION 3.25 (Local action functional) A functional $I \in \mathcal{O}(\mathcal{C}^\infty(M))$ that can be written as $I = \sum_k I_k$ where I_k is homogenous of degree k in the variable $a \in \mathcal{C}^\infty(M)$, that is

$$I_k(\lambda a) = \lambda^k I_k(a) \quad (3.119)$$

and such that each I_k can, for some integer s , be written as

$$I_k(a) = \sum_{j=1}^s \int_M D_{1,j}(a) \dots D_{k,j}(a) \quad (3.120)$$

where $D_{i,j}$ are differential operators on M is called a **local action functional**.

We will subsequently denote by $\mathcal{O}_{loc}(\mathcal{C}^\infty(M))$ the **subspace of local action functionals**. This definition extends to local formal power series in \hbar that is $I \in \mathcal{O}_{loc}(\mathcal{C}^\infty(M))[[\hbar]]$ as follows:

$$I = \sum_{i,k} \hbar^i I_{i,k} \quad (3.121)$$

where the $I_{i,k}$ need to be in $\mathcal{O}_{loc}(\mathcal{C}^\infty(M))$ for all i, k . We further define

$$\mathcal{O}_{loc}^+(\mathcal{C}^\infty(M))[[\hbar]] \subset \mathcal{O}_{loc}(\mathcal{C}^\infty(M))[[\hbar]] \quad (3.122)$$

the subspace of local functionals which are at least cubic modulo \hbar . Finally we are prepared to

define scalar QFTs:

DEFINITION 3.26 (Perturbative Scalar Quantum Field Theories) A **perturbative scalar quantum field theory** is given by a set of effective interactions

$$I[L] \in \mathcal{O}_{loc}^+(\mathcal{C}^\infty(M))[[\hbar]] \quad \forall L \in [0, \infty] \quad (3.123)$$

(plus a kinetic action $-\frac{1}{2}\langle\varphi, (D + m^2)\varphi\rangle$ for $\varphi \in \mathcal{C}^\infty(M)$) such that

$$1. \quad I[L] = W(P(\varepsilon, L), I[\varepsilon]) \quad \forall \varepsilon, L \in (0, \infty)$$

2. Let

$$I[L] = \sum_{i,k} \hbar^i I_{i,k}[L] \quad (3.124)$$

For each i, k we require a small- L asymptotic expansion

$$I_{i,k}[L] \simeq \sum_{r \in \mathbb{Z}} g_r(L) \Phi_r \quad (3.125)$$

where $g_r(L) \in \mathcal{C}^\infty((0, \infty)_L)$ and $\Phi_r \in \mathcal{O}_{loc}(\mathcal{C}^\infty(M))$. Small- L asymptotic expansion means that there exists a non-decreasing sequence

$$d_R \in \mathbb{Z}, \quad d_R \longrightarrow \infty, \quad R \longrightarrow \infty \quad (3.126)$$

such that for all R

$$\lim_{L \rightarrow 0} L^{-d_R} \left(I_{i,k}[L](a) - \sum_{r=0}^R g_r(L) \Phi_r(a) \right) = 0 \quad (3.127)$$

for all $a \in \mathcal{C}^\infty(M)$.

The small- L asymptotic expansion will later allows us to find divergences in the limit $\varepsilon \longrightarrow 0$ and introduce counterterms in the form of a sum.

3.6 The structure of scalar perturbative QFTs

We now have a definition of quantum field theory and we might ask how we can connect it to a classical definition of quantum field theory without any effective interactions. Denote by $\mathcal{Z}^{(n)}$ the **set of perturbative quantum field theories** modulo \hbar^{n+1} , that is ignoring terms in \hbar of order higher than n . Further we denote

$$\mathcal{Z}^{(\infty)} = \varprojlim \mathcal{Z}^{(n)} \quad (3.128)$$

THEOREM 3.27 $\mathcal{Z}^{(n+1)}$ is a principal bundle over $\mathcal{Z}^{(n)}$ with structure group the abelian group $\mathcal{O}_{loc}(\mathcal{C}^\infty(M))$ (local action functionals on M). In particular, $\mathcal{Z}^{(0)}$ is canonically isomorphic to the space $\mathcal{O}_{loc}^+(\mathcal{C}^\infty(M))$

THEOREM 3.28 If we fix a **renormalisation scheme** (which we will define and explain later), we can find a section for each torsor $\mathcal{Z}^{(n+1)} \rightarrow \mathcal{Z}^{(n)}$ and consequently a bijection between the set of perturbative quantum fields theories $\mathcal{Z}^{(\infty)}$ and the set of local action functionals $I \in \mathcal{O}_{loc}^+(\mathcal{C}^\infty(M))[[\hbar]]$.

While we might not find the time to discuss the proofs of the two above theorems, their statements are the most important points. The first one provides convenient and interesting structure, mainly telling us that once we restrict to the classical setting, we recover a classical action functional, hence providing a canonical transition from quantum to classical. The second theorem provides a direct link between local action functionals and perturbative QFTs, thus enabling the transition shown in the first.

We now discuss a *strategy* to work with both of these statements: First fix $I \in \mathcal{O}_{loc}^+(\mathcal{C}^\infty(M))[[\hbar]]$. We want to build the effective interaction $I[L]$ satisfying the properties of its definition. So first of all we want to construct counterterms $I^{CT}(\varepsilon)$ such that

$$\lim_{\varepsilon \rightarrow 0} W(P(\varepsilon, L), I - I^{CT}(\varepsilon)) \quad (3.129)$$

exists and in particular defines $I[L]$. Conversely given $\{I[L]\}$ we construct I as a renormalised limit, subtracting the suitable counterterms. To this end, we define some new helpful notions:

DEFINITION 3.29 (Periods) Into the classical sequence $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{A} \subset \mathbb{R}$ we want to insert $\mathbb{A} \subset \mathbb{P} \subset \mathbb{R}$ where \mathbb{P} forms a ring contained in transcendental and algebraic numbers. A **period** is a complex number whose real and imaginary part are values of absolutely converging integrals of rational functions with rational coefficients over domains in \mathbb{R}^n given by polynomial inequalities with rational coefficients.

EXAMPLE 3.30 Two numbers included in \mathbb{P} and clearly not included in \mathbb{Q} are:

$$\sqrt{2} = \int_{2x^2 \leq 1} dx \quad (3.130)$$

$$\pi = \iint_{x^2+y^2 \leq 1} dx dy = 2 \int_{-1}^{+1} \sqrt{1-x^2} dx \quad (3.131)$$

Now let $t \in (0, \infty)$ be a real parameter. Formally we might write a "period depending smoothly on t " as

$$\alpha(t) = \int_{\gamma(t)} \omega(t) \quad (3.132)$$

where we allow $\gamma(t)$ and $\omega(t)$ to depend smoothly on t . But of course we further need to require that $\alpha(t)$ is a period, thus $\in \mathbb{P}$, for every rational number $t \in \mathbb{Q} \cap (0, \infty)$. Collecting these requirements leads us to a definition from *algebraic geometry*:

DEFINITION 3.31 (Periods in Algebraic Geometry) We call **rational periods** those functions in $\mathcal{C}^\infty((0, \infty))$ which are of this form and denote by $P_{\mathbb{Q}}((0, \infty))$ the set of rational periods.

DEFINITION 3.32 We define $\mathcal{P}((0, \infty))$ to be the real vector space spanned by the space of rational periods:

$$\mathcal{P}((0, \infty)) = P_{\mathbb{Q}}((0, \infty)) \otimes \mathbb{R} \subset \mathcal{C}^\infty((0, \infty)) \quad (3.133)$$

By an abuse of notation, we call elements of $\mathcal{P}((0, \infty))$ **periods**.

Now we still investigate $w_\gamma(P(\varepsilon, L), I(\varepsilon))(a)$ in the limit $\varepsilon \rightarrow 0$. We interpret w_γ as a function of ε, L, a , thus in

$$\mathcal{O}(\mathcal{C}^\infty(M), \mathcal{C}^\infty((0, \infty)_\varepsilon) \otimes \mathcal{C}^\infty((0, \infty)_L)) \quad (3.134)$$

In particular, if we fix ε , we have

$$w_\gamma \in \mathcal{O}(\mathcal{C}^\infty(M), \mathcal{C}^\infty((0, \infty)_L)) \quad (3.135)$$

This leads us to the following theorem which allows us to extract the singular terms in the finite-dimensional integrals w_γ in order to extract singular points in our Feynman graphs:

THEOREM 3.33 Let $I \in \mathcal{O}_{loc}(\mathcal{C}^\infty(M))[[\hbar]]$ be a local action functional and let γ be a connected stable graph. Then there exists a small- ε asymptotic expansion

$$w_\gamma(P(\varepsilon, L), I[\varepsilon]) \simeq \sum_{i=0}^{\infty} g_i(\varepsilon) \psi_i \quad (3.136)$$

where $g_i \in \mathcal{P}((0, \infty)_\varepsilon)$ are periods and $\psi_i \in \mathcal{O}(\mathcal{C}^\infty(M), \mathcal{C}^\infty((0, \infty)_L))$ such that

1.

$$\lim_{\varepsilon \rightarrow 0} L^{-d_R} \left(w_\gamma - \sum_{r=0}^R g_r(\varepsilon) \psi_r \right) = 0 \quad \forall a \in \mathcal{C}^\infty(M), \quad (3.137)$$

2. the $g_i(\varepsilon)$ have finite order poles at 0, i.e. for $\forall i \exists k$ such that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^k g_i(\varepsilon) = 0, \quad (3.138)$$

3. the ψ_i have a small- L asymptotic expansion of the form

$$\psi_i \simeq \sum_{j=0}^{\infty} f_{i,j}(L) \psi_{i,j} \quad (3.139)$$

where $\psi_{i,j} \in \mathcal{O}_{loc}(\mathcal{C}^\infty(M))$ and $f_{i,j} \in \mathcal{C}^\infty((0, \infty)_L)$.

Sadly we won't have the time to prove the above theorem since it is rather lengthy and technical.

Thus we go on with the following definition:

DEFINITION 3.34 Define $\mathcal{P}((0, \infty))_{\geq 0} \subseteq \mathcal{P}((0, \infty))$ to be the subspace of functions of ε that are periods and which admit a limit as $\varepsilon \rightarrow 0$.

Using this definition we can finally expand upon the previously mentioned *renormalisation schemes*:

DEFINITION 3.35 (Renormalisation Schemes) A choice of a subspace

$$\mathcal{P}((0, \infty))_{<0} \subset \mathcal{P}((0, \infty)) \quad (3.140)$$

complementary to $\mathcal{P}((0, \infty))_{\geq 0}$ is called a **renormalisation scheme**. Hence a renormalisation scheme provides a direct sum decomposition

$$\mathcal{P}((0, \infty)) = \mathcal{P}((0, \infty))_{\geq 0} \oplus \mathcal{P}((0, \infty))_{<0} \quad (3.141)$$

Note that this decomposition is not unique, hence fixing a normalisation scheme is a **real choice**, since $\mathcal{P}((0, \infty))$ has the structure of a vector space.

DEFINITION 3.36 (Singular Parts) Given a period $f \in \mathcal{P}((0, \infty))$ define its **singular part** to be the projection $\text{sing}(f)$ of f onto $\mathcal{P}((0, \infty))_{<0}$.

From now on, we fix a renormalisation scheme.

From the previous theorem, take

$$w_\gamma(P(\varepsilon, L), I[\varepsilon]) \simeq \sum_{i=0}^{\infty} g_i(\varepsilon) \psi_i \quad (3.142)$$

Now there exists an N such that $\forall n > N$ the $g_n(\varepsilon)$ admits a limit $\varepsilon \rightarrow 0$. As an **exercise** you can prove this claim. Now define

$$\psi_N := \sum_{i=0}^N g_i(\varepsilon) \psi_i \quad (3.143)$$

and note that

$$\text{sing}_\varepsilon(w_\gamma(P(\varepsilon, L), I[\varepsilon])) := \text{sing}(\psi_N(\varepsilon)) = \sum_{i=0}^N \text{sing}_\varepsilon(g_i(\varepsilon)) \psi_i \quad (3.144)$$

We can collect these results and definitions into a theorem:

THEOREM 3.37 Let $I \in \mathcal{O}_{loc}(\mathcal{C}^\infty(M))[[\hbar]]$ be a local action functional and let γ be a connected stable graph. Then

$$\text{sing}_\varepsilon(w_\gamma(P(\varepsilon, L), I[\varepsilon])) = \sum_i f_i(\varepsilon) \psi_i \quad (3.145)$$

where $f_i(\varepsilon) \in \mathcal{P}((0, \infty))_{<0}$ are singular periods (periods equivalent to their singular part) and the ψ_i have small- L asymptotic expansions of the form

$$\psi_i \simeq \sum_{j=0}^{\infty} f_{i,j} \psi_{i,j} \quad (3.146)$$

where $\psi_{i,j} \in \mathcal{O}_{loc}(\mathcal{C}^\infty(M))$ and $f_{i,j} \in \mathcal{C}^\infty((0, \infty)_L)$. Furthermore the limit

$$\lim_{\varepsilon \rightarrow 0} (w_\gamma(P(\varepsilon, L), I[\varepsilon]) - \text{sing}_\varepsilon(w_\gamma(P(\varepsilon, L), I[\varepsilon]))) \quad (3.147)$$

exists in $\mathcal{O}_{loc}(\mathcal{C}^\infty(M), \mathcal{C}^\infty((0, \infty)_L))$.

THEOREM 3.38 There exists a unique series of local counterterms

$$I_{i,k}^{CT}(\varepsilon) \in \mathcal{O}_{loc}(\mathcal{C}^\infty(M)) \otimes \mathcal{P}((0, \infty))_{<0} \quad (3.148)$$

for any i, k with $I_{i,k}^{CT}$ homogenous of degree k as a function of $a \in \mathcal{C}^\infty(M)$ and such that for all $L \in (0, \infty)$ the limit

$$\lim_{\varepsilon \rightarrow 0} W \left(P(\varepsilon, L), I - \sum_{i,k} \hbar^i I_{i,k}^{CT}(\varepsilon) \right) \quad (3.149)$$

exists.

Proof. First note that

$$W(P, I) = \sum_{i,k} \hbar^i W_{i,k}(P, I) \quad (3.150)$$

$$W_{i,k}(P, I) = \sum_{\gamma \in \Gamma_{i,k}} w_\gamma(P(\varepsilon, L), I[\varepsilon]) \quad (3.151)$$

where $\Gamma_{i,k}$ is the set of all stable graphs of genus i with k external edges. Now for $i = 0$ the term

$$\lim_{\varepsilon \rightarrow 0} w_\gamma(P(\varepsilon, L), I[\varepsilon]) \quad (3.152)$$

converges. Now for $i \neq 0$ we denote

$$(i, j) < (k, l) \quad \Leftrightarrow \quad i < k \text{ or } (i = k \text{ and } j < l) \quad (3.153)$$

To investigate the base case we define

$$I_{1,1}^{CT}(\varepsilon, L) = \text{sing}_\varepsilon(W_{1,1}(P(\varepsilon, L), I[\varepsilon])) \quad (3.154)$$

To show that this is indeed a fitting counterterm, we calculate

$$W_{1,1}(P(\varepsilon, L), I[\varepsilon] - \hbar I_{1,1}^{CT}(\varepsilon, L)) = W_{1,1}(P(\varepsilon, L), I[\varepsilon]) - I_{1,1}^{CT}(\varepsilon, L) \quad (3.155)$$

$$\Rightarrow W_{1,1}(P(\varepsilon, L), \hbar I_{1,1}^{CT}(\varepsilon, L)) = I_{1,1}^{CT} \quad (3.156)$$

Thus taking the limit we obtain

$$\lim_{\varepsilon \rightarrow 0} W_{1,1}(P(\varepsilon, L), I[\varepsilon] - \hbar I_{1,1}^{CT}(\varepsilon, L)) \quad (3.157)$$

$$= \lim_{\varepsilon \rightarrow 0} [\text{reg}_\varepsilon(W_{1,1}(P(\varepsilon, L))) + \text{sing}_\varepsilon(W_{1,1}(P(\varepsilon, L))) - \text{sing}_\varepsilon(W_{1,1}(P(\varepsilon, L)))] \quad (3.158)$$

and thus the limit exists. Now we want to show that $I_{1,1}^{CT}$ is local. To this end note that

$$\frac{d}{dL} W_{1,1}(P(\varepsilon, L)) \quad (3.159)$$

Since K_L is smooth the singular point of the derivative is zero. Further the singular point of $W_{1,1}$ does not depend on L which implies

$$I_{1,1}^{CT}(\varepsilon, L) = I_{1,1}^{CT}(\varepsilon) \quad (3.160)$$

Now recall that

$$\text{sing}_\varepsilon w_\gamma = \sum_i f_i(\varepsilon) \psi_i \quad (3.161)$$

where the ψ_i have a small- L asymptotic expansion. Thus $\text{sing}_\varepsilon w_\gamma$ is local. Having established the base case, we now use induction to conclude the proof: Suppose we have constructed $I_{j,l}^{CT}(\varepsilon)$ for all $(j, l) < (i, k)$ such that they satisfy our theorem. For convenience we define

$$W_{<(i,k)}(P, I) = \sum_{(j,l) < (i,k)} \hbar^j W_{j,l}(P, I) \quad (3.162)$$

$$= \sum_{\gamma \in \Gamma_{<(i,k)}} \frac{\hbar^{g(\gamma)}}{|\text{Aut}(\gamma)|} w_\gamma(P, I) \quad (3.163)$$

Now we define the counterterms as

$$I_{i,k}^{CT}(\varepsilon, L) = \text{sing}_\varepsilon W_{i,k} \left(P(\varepsilon, L), I - \sum_{(j,l) < (i,k)} \hbar^j I_{j,l}^{CT}(\varepsilon) \right) \quad (3.164)$$

Thus we again investigate the limit of

$$W_{i,k} \left(P(\varepsilon, L), I - \sum_{(j,l) < (i,k)} \hbar^j I_{j,l}^{CT}(\varepsilon) - \hbar^i I_{i,k}^{CT}(\varepsilon, L) \right) \quad (3.165)$$

$$= W_{i,k} \left(P(\varepsilon, L), I - \sum_{(j,l) < (i,k)} \hbar^j I_{j,l}^{CT}(\varepsilon) \right) - I_{i,k}^{CT}(\varepsilon, L) \quad (3.166)$$

We see that the limit for $\varepsilon \rightarrow 0$ exists. Now all we need to show is that $I_{i,k}^{CT}(\varepsilon, L)$ is local which is done as before by showing that it is independent on L . \square

Now let $I \in \mathcal{O}^+(\mathcal{C}^\infty(M))[[\hbar]]$ be a local action functional. We define

$$I[L] := \lim_{\varepsilon \rightarrow 0} (W(P(\varepsilon, L), I - I^{CT}(\varepsilon))) =: W^R(P(0, L), I) \quad (3.167)$$

where $I \simeq I[\varepsilon \rightarrow 0]$ and the superscript R stands for *renormalised*.

EXERCISE 3.39 Prove that $I[L]$ satisfies the (L-RGE) and the small- L asymptotic expansion of the definition of a scalar PQFT.

Conversely, let $\{I[L]\}$ be a collection of effective interactions defining a theory. Now for

$$I = \sum_{i,k} \hbar^i I_{i,k} \quad (3.168)$$

we want to construct the terms $I_{i,k}$. For $I_{0,0}$ just take the limit $\lim_{\varepsilon \rightarrow 0} W(P(\varepsilon, L), I[L])$. Now let us suppose we have constructed $I_{r,s}$ for $(r, s) < (i, k)$ such that

$$W_{a,b}^R \left(P(0, L), \sum_{(r,s) < (i,k)} \hbar^r I_{r,s} \right) = I_{a,b}(L) \quad (3.169)$$

for all $(a, b) < (i, k)$. Now

$$I_{i,k} = I_{i,k}[L] - W_{i,k}^R \left(P(0, L), \sum_{(r,s) < (i,k)} \hbar^r I_{r,s} \right) \quad (3.170)$$

Thus recall

$$W^R(P(0, L), I) = \sum_{i,k} \hbar^i W_{i,k}^R(P(0, L), I) \quad (3.171)$$

EXERCISE 3.40 Show that this quantity does not depend on L (use (L-RGE)) and is local (use

the locality of $\{I[L]\}$.

Thus we have established a bijection between theories and local power series in \hbar

$$\mathcal{Z}^{(\infty)} \xleftrightarrow[RS]{} \mathcal{O}_{loc}^+(\mathcal{C}^\infty(M))[[\hbar]] \quad (3.172)$$

$$\mathcal{Z}^{(n)} \xleftrightarrow[RS]{} \mathcal{O}_{loc}^+(\mathcal{C}^\infty(M))[[\hbar]]/\hbar^{n+1} \quad (3.173)$$

3.7 Renormalisability

Problem: The space of theories is an infinite-dimensional manifold. Physically, to specify a particular theory we would need an infinite number of experiments.

The goal is to restrict to a finite subset of these theories, namely renormalisable theories. **Classically** a theory is renormalisable, if it has a finite number of counterterms. But we saw that the number of counterterms depends on the choice of renormalisation scheme.

PRINCIPLE 1 A theory $\{S^{eff}[\Lambda]\}$ is renormalisable if $S^{eff}[\Lambda]$ does not grow too fast when $\Lambda \rightarrow \infty$, measured in the "right" units appropriate to the energy scale.

If $M = \mathbb{R}^n$, it is not compact, but we can define a scalar PQFT in (more or less) the same way and get similar results. Define

$$R_l: \mathcal{C}^\infty(\mathbb{R}^n) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n), \quad \varphi(x) \mapsto l^{1-n/2} \varphi(l^{-1}x) \quad (3.174)$$

and additionally the map

$$RG_l(S^{eff}[\Lambda])(\varphi) := S^{eff}[l^{-2}\Lambda](R_l(\varphi)) \quad (3.175)$$

LEMMA 3.41 RG_l is a flow on the space of theories. If $\{S^{eff}[\Lambda]\}$ satisfies the (RGE), then so does $\{RG_l(S^{eff}[\Lambda])\}$. We call RG_l the **local renormalisation group flow**.

DEFINITION 3.42 A theory is **renormalisable** if $RG_l(S^{eff}[\Lambda])$ grows, in terms of Λ , at most logarithmically as $l \rightarrow 0$. Suppose that S is translation invariant. We say that S is of dimension m if

$$S(R_l(\varphi)) = l^m S(\varphi) \quad (3.176)$$

EXAMPLE 3.43

- $\int_{\mathbb{R}^4} \varphi D\varphi, \int_{\mathbb{R}^4} \varphi^4$ are of dim 0.
- $\int_{\mathbb{R}^4} \varphi^2$ is of dim 2.

THEOREM 3.44 Let $\mathfrak{R}^{(k)}(\mathbb{R}^n)$ be the space of renormalisable scalar field theories of \mathbb{R}^n invariant

under translation, defined modulo \hbar^{k+1} . Then

$$\mathfrak{R}^{(k+1)}(\mathbb{R}^n) \longrightarrow \mathfrak{R}^{(k)}(\mathbb{R}^n) \quad (3.177)$$

is a torsor for the vector space of local action functionals $S(\varphi)$ which are a sum of terms of non-negative dimensions. Furthermore $\mathfrak{R}^{(0)}(\mathbb{R}^n)$ is canonically isomorphic to the space of local action functionals of the form

$$S(\varphi) = \frac{1}{2} \int_{\mathbb{R}^n} \varphi D\varphi + I \quad (3.178)$$

where the I are cubic or higher terms of non-negative dimensions.

COROLLARY 3.45 On \mathbb{R}^4 renormalisable scalar field theories invariant under $SO(4) \times \mathbb{R}^4$ and $\varphi \mapsto -\varphi$ are in bijection with Lagrangians of the form

$$\mathcal{L}(\varphi) = a\varphi D\varphi + b\varphi^4 + c\varphi^2 \quad (3.179)$$

for $a, b, c \in \mathbb{R}[[\hbar]]$ and such that $a = -\frac{1}{2}$ modulo \hbar and $c = 0$ modulo \hbar .

4 Gauge Theories

4.1 Quantisation in the Presence of Local Symmetries

Recall that our chosen approach to the quantisation of (local) Lagrangian field theories deals with expressions of the form

$$\langle O_1 \dots O_n \rangle := \frac{1}{\mathcal{Z}} \int_{\mathcal{F}} e^{i/\hbar S(\varphi)} O_1(\varphi) \dots O_n(\varphi) D\varphi \quad (4.1)$$

where $S = \int_M L$, the Lagrangian L is a local $(0, \text{top})$ -form and $\mathcal{F} = \Gamma(M, F)$. Further the O_i are local action functionals on \mathcal{F} and we define

$$\mathcal{Z} := \int_{\mathcal{F}} e^{i/\hbar S(\varphi)} D\varphi \quad (4.2)$$

where $D\varphi$ is supposed to be "a measure" on \mathcal{F} which **does not exist in general**. Now for **finite dimensions**, given a measure μ on X we have the **Stationary Phase Formula**

$$\begin{aligned} \mathcal{Z} := \int_X \mu e^{i/\hbar S_0} &\underset{\hbar \rightarrow 0}{\sim} \left[\sum_{x_0 \in \text{Crit}(S_0)} e^{i/\hbar S(x_0)} |\det \mathcal{H}[S](x_0)|^{-1/2} \exp\left(i\frac{\pi}{4} \text{sgn}(S(x_0))\right) \right] \quad (\text{SPF}) \\ &\cdot (2\pi\hbar)^{\dim(X)/2} \sum_{\gamma \text{ graph}} \frac{\hbar^{-\chi(\gamma)}}{|\text{Aut}(\gamma)|} \Phi_\gamma \end{aligned}$$

Where γ are *Feynman graphs*, $\chi(\gamma)$ is the Euler characteristic of γ (thus $|V| - |E|$) and $\Phi(\gamma)$ is the weight of a graph. Note that here we have an *implicit assumption*, namely that $\mathcal{H}[S](x_0)$ is non-degenerate. In field theory, we use the stationary phase formula as a *definition* of the "path integral".

Now we can ask, if the possible degeneracy of $\mathcal{H}[S](x_0)$ really is an issue. The answer is: **Yes!** Consider $\mathcal{F} = \Gamma(M, F)$ with M compact without boundary. Let further $S = \int_M L$. Now assume we are given a collection $\{V_\alpha\}^M$ of local symmetries 2.48 such that

$$\mathcal{L}_{V_\alpha} S = V_\alpha^{(i)} \frac{\delta S}{\delta \varphi^{(i)}} = 0 \quad (4.3)$$

Namely we obtain the *Noether Identities*. Now consider

$$0 = \frac{\delta}{\delta \varphi^{(j)}} \left(V_\alpha^{(i)} \frac{\delta S}{\delta \varphi^{(i)}} \right) = \mathcal{H}[S]_{(ij)} V_\alpha^{(i)} + \frac{\delta S}{\delta \varphi^{(i)}} \frac{\delta V_\alpha^{(i)}}{\delta \varphi^{(j)}} \quad (4.4)$$

Thus we restrict to $x_0 \in \text{Crit}(S) = \{x \in \mathcal{F} \text{ s.t. } \delta S|_x = 0\}$ such that

$$\mathcal{H}[S]_{(ij)}(x_0)V_\alpha^{(i)} \stackrel{\neq}{=} 0 \quad \Leftrightarrow \quad V_\alpha \text{ is a local symmetry} \quad (4.5)$$

EXERCISE 4.1 Show that $\{V_\alpha \in \mathfrak{X}_{evol}(\mathcal{F})\}$ which are local symmetries of S forms a Lie subalgebra of $\mathfrak{X}_{evol}(\mathcal{F})$. We further denote it by $\tilde{\mathfrak{g}}$.

Note that not all symmetries are "interesting" and thus not all symmetries are created equal. Consider a bivector field $\mu \in \Gamma(\bigwedge^2(T\mathcal{F}))$ and look at the vector field

$$\omega := \mu(\delta S), \quad \mu = \mu^{ij} \frac{\delta}{\delta \varphi^{(i)}} \wedge \frac{\delta}{\delta \varphi^{(j)}}, \quad \omega = \mu^{ij} \frac{\delta S}{\delta \varphi^{(i)}} \frac{\delta S}{\delta \varphi^{(j)}}, \quad \mu^{ij} = -\mu^{ji} \quad (4.6)$$

Now we claim that ω is a local symmetry $\forall \mu$:

$$\textit{Proof. } \mathcal{L}_\omega S = \frac{\delta S}{\delta \varphi^{(i)}} \omega^i = \frac{\delta S}{\delta \varphi^{(i)}} \frac{\delta S}{\delta \varphi^{(j)}} \mu^{ij} \equiv 0 \quad \square$$

This leads us to the following definition:

DEFINITION 4.2 (Trivial Symmetries) A local symmetry of the type $\omega = \mu(\delta S)$ for $\mu \in \Gamma(\bigwedge^2(T\mathcal{F}))$ is called a **trivial symmetry**. Denote the space of such trivial symmetries by \mathfrak{t} .

LEMMA 4.3 \mathfrak{t} is an ideal in $\tilde{\mathfrak{g}}$, thus $[\tilde{\mathfrak{g}}, \mathfrak{t}] \subset \mathfrak{t}$.

Proof. The proof is left as an exercise to the reader. \square

Remark 4.4 Note that if ω is trivial, $\mathcal{L}_\omega S = 0$ does **not** impose nontrivial conditions, thus no Noether Identities. They are not associated to nontrivial conservation laws, currents, charges etc. Since they particularly vanish on $\text{Crit}(S)$ they are still considered symmetries.

When considering theories with symmetries, we effectively want to work with a kind of quotient

$$\mathfrak{g} := \tilde{\mathfrak{g}}/\mathfrak{t} \quad (4.7)$$

Notice that \mathfrak{g} is not necessarily a Lie subalgebra:

- $[\mathfrak{t}, \mathfrak{g}] \subset \mathfrak{t}$ thus \mathfrak{t} is an ideal in \mathfrak{g} .
- $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{t} + \mathfrak{g}$ thus \mathfrak{g} is closed "on-shell".

Thus \mathfrak{g} is a subalgebra at least on $\text{Crit}(S)$. There are two main approaches to this:

- Dealing with \mathfrak{g} 's that are closed everywhere (Lie subalgebras). This treatment is due to Beach, Rovet, Stara, Tyutin (**BRST**)
- More generally we search for a formalism that can treat Lie algebras \mathfrak{g} that only form Lie subalgebras on $\text{Crit}(S)$. This leads to the Batalin-Vilkovisky formalism (**BV**).

Most cases of physical interest are such that \mathfrak{g} is a Lie subalgebra but the conceptual generalisation of BV will allow for

- more general **gauge-fixing conditions**,
- a treatment of boundary data,
- observables in BV that are more general than those in $BRST$.

Thus for the rest of the script assume that \mathfrak{g} is a subalgebra.

4.2 Lie algebra actions & Lie algebra cohomology

DEFINITION 4.5 (Lie group/algebra actions) Let G be a Lie group with lie algebra \mathfrak{g} and let M be a smooth manifold.

- A **Lie group action** is a smooth map $R: G \times M \rightarrow M$ such that for all $g \in G$ the map $R_g: M \rightarrow M$ is a diffeomorphism. This is equivalent to $\tilde{R}: G \rightarrow \text{Diff}(M)$ being a group homomorphism.
- A **Lie algebra action** on M is a Lie algebra morphism

$$\rho: \mathfrak{g} \rightarrow \mathfrak{X}(M) \quad (4.8)$$

- If $\{t_a\}$ is a basis of \mathfrak{g} , the image $\{V_a = \rho(t_a)\}$ is a basis of **fundamental vector fields**.

EXAMPLE 4.6 Given \tilde{R} a Lie group action we obtain a Lie algebra action by

$$d\tilde{R}|_{id}: \mathfrak{g} \rightarrow \mathfrak{X}(M) \quad (4.9)$$

DEFINITION 4.7 (Tangent Distributions) Let M be a smooth manifold. A **smooth (tangent) distribution** Δ of rank k on M is a subset $\Delta \subset TM$ such that $\forall p \in M$ the set $\Delta_p \subset T_p M$ is a k -dimensional subspace and such that there exists a neighbourhood N_p over which there are k linearly independent smooth vector fields with $\Delta_y = \text{span}(X_1(y), \dots, X_k(y))$ for $y \in N_p$. A smooth (tangent) distribution is **involutive** if $\Gamma(\Delta)$ forms a Lie subalgebra of $\Gamma(TM)$.

EXERCISE 4.8 Show that a Lie algebra action on M induces an involutive distribution on TM .

Remark 4.9 Unsurprisingly, in infinite dimensions one needs to be very careful. Say that $\delta \subset TM$ is a (smooth) subbundle. In order to keep the notion of a "finite rank" k one can say that Δ_y is finitely generated as a $\mathcal{C}^\infty(M)$ -span. Notice that in finite dimensions **Frobenius' Theorem** states that an involutive smooth distribution is tangent to an integral submanifold $N \subset M$ s.t. $TN \simeq \Delta$. Unfortunately this does not generally hold true in infinite dimensions. It does when the manifolds are Banach, however for Fréchet manifolds it does not necessarily. **Warning:** The Inverse Function Theorem can also fail in Fréchet manifolds.

Now consider a Lie algebra action $\rho: \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ and recall that $\mathfrak{X}(M) \simeq \text{Der}(\mathcal{C}^\infty(M))$.

DEFINITION 4.10 The space of $\mathcal{C}^\infty(M)$ -valued k -forms on \mathfrak{g} is defined to be

$$C_{CE}^k(\mathfrak{g}; \mathcal{C}^\infty(M)) := \text{Hom} \left(\bigwedge^k \mathfrak{g}, \mathcal{C}^\infty(M) \right) \cong \bigwedge^k \mathfrak{g}^* \otimes \mathcal{C}^\infty(M). \quad (4.10)$$

This is called the **Chevalley–Eilenberg** complex for the Lie algebra action $\rho: \mathfrak{g} \longrightarrow \text{Der}(\mathcal{C}^\infty(M))$. Now define the map

$$d_{CE}: C_{CE}^k(\mathfrak{g}; \mathcal{C}^\infty(M)) \longrightarrow C_{CE}^{k+1}(\mathfrak{g}; \mathcal{C}^\infty(M)) \quad (4.11)$$

such that the following properties hold:

1. $\forall f \in \mathcal{C}^\infty(M)$ we have $df(X) := \rho(X)(f) \ \forall X \in \mathfrak{g}$.
2. $\forall \alpha \in \mathfrak{g}^*$ we have $d_{CE}\alpha(X, Y) := -\alpha([X, Y]) \ \forall X, Y \in \mathfrak{g}$.
3. We extend to $\bigwedge^\bullet \mathfrak{g}^*$ using the Leibniz rule

$$d_{CE}(\alpha \wedge \beta) = d_{CE}\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d_{CE}\beta \quad (4.12)$$

where $|\alpha| = k$ if $\alpha \in \bigwedge^k \mathfrak{g}^*$.

4. We extend to $\bigwedge^\bullet \mathfrak{g}^* \otimes \mathcal{C}^\infty(M)$ by

$$d_{CE}(\omega \otimes f) = d_{CE}\omega \otimes f + (-1)^{|\omega|} \omega \otimes d_{CE}f. \quad (4.13)$$

PROPOSITION 4.11 $d_{CE}^2 = 0$ and hence $(C_{CE}^\bullet(\mathfrak{g}; \mathcal{C}^\infty(M)))$ is a cochain complex.

Proof. Since by property 2. in the definition of d_{CE} it is just the dual of the Lie bracket, it satisfies the Jacobi-Identity and thus squares to zero:

$$d_{CE}^2\alpha(X, Y, Z) = -d_{CE}[\alpha([X, Y], Z) + \alpha([Z, X], Y) + \alpha([Y, Z], X)] \quad (4.14)$$

$$= \alpha([X, Y], Z) + \alpha([Z, X], Y) + \alpha([Y, Z], X) = 0 \quad (4.15)$$

□

If $\{t_a\}$ is a basis of \mathfrak{g} and $V_a = \rho(t_a)$, denote by $C^a \in \mathfrak{g}^*$ the coordinates in \mathfrak{g}^* and structure constants f_{ab}^c for the Lie algebra we obtain

$$d_{CE} = C^a V_a - \frac{1}{2} f_{ab}^d C^a C^b \frac{\partial}{\partial C^d} \quad (4.16)$$

Now it is immediate to prove the above proposition.

The cohomology of the thus arising cochain complex is

$$H^k(C_{CE}^\bullet; \mathcal{C}^\infty(M)) := \frac{\ker(d_{CE}|_{C^k})}{\ker(d_{CE}|_{C^{k-1}})} \quad (4.17)$$

Remark 4.12 The set $H^0(C_{CE}^\bullet; \mathcal{C}^\infty(M))$ is the set of $f \in \mathcal{C}^\infty(M)$ such that

$$d_{CE}f = 0 \quad \Leftrightarrow \quad d_{CE}f(X) = 0 \quad \forall X \in \mathfrak{g} \quad (4.18)$$

hence $\rho(X)(f) = 0$, thus $f \in H^0(C_{CE}^\bullet; \mathcal{C}^\infty(M))$ iff it is **invariant** under the Lie algebra action:

$$H^0(C_{CE}^\bullet; \mathcal{C}^\infty(M)) \simeq \mathcal{C}^\infty(M)^\mathfrak{g} \quad (4.19)$$

where $\mathcal{C}^\infty(M)^\mathfrak{g}$ are \mathfrak{g} -invariant smooth functions on M .

On $\bigwedge^\bullet \mathfrak{g}^*$ there are the following natural operations:

$$\forall \alpha \in \mathfrak{g}^* \quad \mathcal{E}(\alpha): \bigwedge^k \mathfrak{g}^* \longrightarrow \bigwedge^{k+1} \mathfrak{g}^*, \quad \omega \longmapsto \alpha \wedge \omega \quad (4.20)$$

$$\forall X \in \mathfrak{g} \quad \iota(X): \bigwedge^k \mathfrak{g}^* \longrightarrow \bigwedge^{k-1} \mathfrak{g}^*, \quad \omega \longmapsto \iota_X(\omega) \quad (4.21)$$

Now we extend ι_X by the graded Leibniz rule

$$\iota_X(\alpha \wedge \beta) = (\iota_X \alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge (\iota_X \beta) \quad (4.22)$$

Now if $\{t_a\}$ and $\{\alpha^a\}$ are basis for \mathfrak{g} and \mathfrak{g}^* respectively, then

$$d_{CE} = \mathcal{E}(\alpha^a) \rho(t_a) - \frac{1}{2} \mathcal{E}(\alpha^a) \mathcal{E}(\alpha^b) \iota_{[t_a, t_b]} \quad (4.23)$$

where we sum over repeated indices. Now define $C^a \equiv \mathcal{E}(\alpha^a)$ and $\frac{\partial}{\partial C^a} \equiv \iota_{t_a}$ to obtain

$$C^a C^b = \mathcal{E}(\alpha^a) \alpha^b = -\mathcal{E}(\alpha^b) \alpha^a = -C^b C^a \quad (4.24)$$

such that

$$d_{CE} = C^a V_a - \frac{1}{2} f_{ab}^d C^a C^b \frac{\partial}{\partial C^d} \quad (4.25)$$

Now we need to consider the collection $\{C^a\}$ as **odd coordinates**. This corresponds to the notion of **Ghost Variables** or **Ghost Fields** in physics. Thus we can formulate

$$\mathcal{C}^\infty(M \times \mathfrak{g}[1]) \simeq \bigwedge^\bullet \mathfrak{g}^* \otimes \mathcal{C}^\infty(M) \cong \text{Hom} \left(\bigwedge^\bullet \mathfrak{g}, \mathcal{C}^\infty(M) \right) \quad (4.26)$$

where $\mathfrak{g}[1]$ is a graded vector space, concentrated in degree -1 . This leads us to

$$C_{CE}^\bullet(\mathfrak{g}; \mathcal{C}^\infty(M)) \simeq \mathcal{C}^\infty(M \times \mathfrak{g}[1]) \quad (4.27)$$

which allows us to think of d_{CE} as a vector field of degree 1 which squares to zero. Thus we can also think of it as a **cohomological vector field** on $M_{min} := M \times \mathfrak{g}[1]$ such that

$$\frac{1}{2}[d_{CE}, d_{CE}]_{\mathfrak{X}(M_{min})} = d_{CE}^2 = 0 \quad (4.28)$$

4.3 BRST Formalism

This subsection is dedicated to a concise formulation of the **Becchi–Rouet–Stora–Tyutin** (short BRST) formalism. Let F be a smooth manifold endowed with a Lie algebra action

$$\rho: \mathfrak{g} \longrightarrow \mathfrak{X}(F) \simeq \text{Der}(\mathcal{C}^\infty(M)) \quad (4.29)$$

We associate to it the Chevalley-Eilenberg complex where $F_{min} = F \times \mathfrak{g}[1]$

$$C_{CE}^\bullet(\mathfrak{g}, \mathcal{C}^\infty(F)) \simeq \bigwedge^\bullet \mathfrak{g}^* \otimes \mathcal{C}^\infty(F) \simeq \mathcal{C}^\infty(F \times \mathfrak{g}[1]) = \mathcal{C}^\infty(F_{min}) \quad (4.30)$$

Now consider the product $F_{Aux} := \mathfrak{g}^* \times \mathfrak{g}^*[-1]$. We want to extend d_{CE} to a cohomological vector field on

$$F_{BRST} := F_{min} \times F_{Aux} = F \times \mathfrak{g}[1] \times \mathfrak{g}^* \times \mathfrak{g}^*[-1] \quad (4.31)$$

Luckily on F_{Aux} there is a natural cohomological vector field:

LEMMA 4.13 On F_{Aux} consider the "vector field" $d_{Aux}\bar{c}_a = \lambda_a$ such that $d_{Aux}\lambda_a = 0$ where we are given a basis $\{e^a\}$ of \mathfrak{g}^* , the \bar{c}_a are coordinates in $\mathfrak{g}^*[-1]$ and the λ_a are coordinates in \mathfrak{g}^* . Then $d_{Aux}^2 = 0$ and thus it is cohomological.

Note that our conventions make \bar{c}_a coordinates of degree -1 and λ of degree 0 .

EXERCISE 4.14 Show that $(\mathcal{C}^\infty(F_{Aux}), d_{Aux}) \simeq (\Omega^\bullet(\mathfrak{g}^*[-1]), d_{dR})$. In particular, the cohomology of $\mathcal{C}^\infty(F_{Aux})$ vanishes.

We do not want to change the cohomology of F_{min} which represents invariant functions. Thus we aim for a prescription of the form

$$F_{min} \longrightarrow F_{BRST} = F_{min} \times F_{Aux} \quad (4.32)$$

such that we only "add" fields that do not contribute to the cohomology class. This allows for the

following splitting:

$$\mathcal{C}^\infty(F_{BRST}) \simeq \mathcal{C}^\infty(F) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes S^\bullet(\mathfrak{g}) \otimes \bigwedge^\bullet \mathfrak{g} \simeq C_{CE}^\bullet(\mathfrak{g}, \mathcal{C}^\infty(F)) \otimes \Omega_{dR}^\bullet(\mathfrak{g}^*[-1]) \quad (4.33)$$

which in turn allows us to define

$$d_{BRST} := d_{CE} + d_{Aux} \quad (4.34)$$

Note that the auxiliary variables (fields) are often called **Nakanishi-Lautrup fields**. Now let $S \in \mathcal{C}^\infty(F)$ such that $d_{CE}S = 0$. Namely S is lifted to F_{min} since it is invariant under d_{CE} and thus in particular a \mathfrak{g} -invariant function. Thus use $H^0(C_{CE}^\bullet; \mathcal{C}^\infty(M)) \simeq \mathcal{C}^\infty(M)^\mathfrak{g}$.

Now think of $S \in \mathcal{C}^\infty(F_{BRST})$ such that $S \in H^0(C_{BRST}^\bullet, d_{BRST})$. Thus applying **gauge fixing** we get for a gauge field $\psi_{gf} \in \mathcal{C}^\infty(F_{Aux})^{(-1)}$ with $|\psi_{gf}| = 1$:

$$S_{gf} = S + d_{BRST}\psi_{gf} \quad \implies \quad [S_{gf}]_{BRST} = [S]_{BRST} \quad (4.35)$$

A valid question is, why this should be relevant. To understand this, recall that if $S \in C(F)^\mathfrak{g}$ this means that $\mathcal{H}[S](x_0)$ is degenerate. How about if we choose a different representative $S_{gf} = S + d_{BRST}\psi_{gf}$ such that $\mathcal{H}[S_{gf}](x_0)$ is non-degenerate? We will pick up this thought after a short digression:

Remark 4.15 (Integration on Super/Graded manifolds) Note that everything formulated for super manifolds also applies to \mathbb{Z} -graded manifolds by sending a degree to the parity. Let $E \rightarrow M$ be a rank m vector bundle over M where $\dim(M) = n$. Denote by $\Pi E =: \mathcal{M}$ the (super) vector bundle obtained by shifting the degree of the fibre vector spaces by 1 (which just reverses the parity in $\mathbb{Z}/2$ (super) grading).

DEFINITION 4.16 (Berezin line bundle) The (real) line bundle over M defined by

$$Ber(\mathcal{M}) := \bigwedge^{top} T^*M \otimes \bigwedge^m E \quad (4.36)$$

is called the **Berezin line bundle** over \mathcal{M} . A **Berezinian** is a smooth section of $Ber(\mathcal{M})$ denoted by $\mu \in \Gamma(M, Ber(\mathcal{M}))$.

DEFINITION 4.17 (Berezin Integral) Given a section $\mu \in \Gamma(M, Ber(\mathcal{M}))$ define the integration map

$$\int_{\mathcal{M}} \cdot \mu : \mathcal{C}_c^\infty(\mathcal{M}) \longrightarrow \mathbb{R} \quad (4.37)$$

defined as

$$\int_{\mathcal{M}} f \mu = \int_M \langle \mu, (f)_m \rangle \quad (4.38)$$

where $(f)_m$ is the component in $\Gamma_c(M, \bigwedge^m E)$ and $\langle \cdot, \cdot \rangle : \bigwedge^m E \otimes \bigwedge^m E^* \longrightarrow \mathbb{R}$ is the fibrewise

paring.

Now consider a vector space V and its parity-shifted version ΠV . Now on ΠV let there be a function given as $f = f_0 + f_1\theta$ with $f_0, f_1 \in \mathbb{R}$ and θ an *odd* coordinate such that

$$\int d\theta = 0 \quad \int d\theta\theta = \quad (4.39)$$

In m dimensions we have

$$\int d^m\theta := \int d\theta^m \int \dots \int d\theta \quad (4.40)$$

such that $\int d^m\theta \theta^1 \dots \theta^m = 1$ and zero otherwise. Now $V^0 = V_m^0 \oplus V_m^1$ such that we can decompose

$$f = f^0 + f_i^1 \theta^i + f_{ij}^2 \theta^i \theta^j + \dots \quad (4.41)$$

where $f^k \in \mathcal{C}^\infty(V^0)$. Now

$$\int_V f d^n x d^m \theta = \int_{V^0} f^{top} d^n x \quad (4.42)$$

Thus changing coordinates in the form $\tilde{\theta}^i = A_j^i \theta^j$ amounts to

$$\tilde{\theta}^1 \dots \tilde{\theta}^m = \det(A) \theta^1 \dots \theta^m \quad (4.43)$$

Now we enforce

$$\int d^m \tilde{\theta} \tilde{\theta}^1 \dots \tilde{\theta}^m = 1 \quad (4.44)$$

such that $d^m \tilde{\theta} = (\det(A))^{-1} d^m \theta$. In general by pure construction

$$\langle \mu, (f)_{top} \rangle \in \Gamma \left(M, \bigwedge^n T^* M \right) \quad (4.45)$$

is a top-form and can be integrated. One can generalize this entire construction by looking at

$$\widetilde{Ber}(\mathcal{M}) := Ber(\mathcal{M}) \otimes \bigwedge^\bullet E^* \simeq \bigwedge^n T^* M \otimes \bigwedge^m E \otimes \bigwedge^\bullet E^* \quad (4.46)$$

which lets us define

$$BER(\mathcal{M}) = \Gamma(\mathcal{M}, \widetilde{Ber}(\mathcal{M})) \quad (4.47)$$

by letting $\mu \in BER(\mathcal{M})$ be non-constant along the fibres of $\rho: \mathcal{M} \rightarrow M$. Alternatively one can define $BER(\mathcal{M}) = \Gamma(\mathcal{M}, \rho^* Ber(M))$ as a $\mathcal{C}^\infty(\mathcal{M})$ -module.

EXAMPLE 4.18 Consider the shifted tangent bundle $\mathcal{M} := \Pi TM$ and recall that $\mathcal{C}^\infty(\Pi TM) \simeq \Omega^\bullet(M)$. Now let us denote by $\tilde{f} \in \Omega^\bullet(M)$ the differential form associated to $f \in \mathcal{C}^\infty(\Pi TM)$.

Define $\mu_{\Pi TM}$ as the Berezinian

$$\int_{\mathcal{M}} \mu_{\Pi TM} f = \int_M \tilde{f} \quad (4.48)$$

where we work in local coordinates $\{x^i, \theta_i\}$ and $\mu_{\Pi TM} = dx^i D\theta_i$.

Now to define the generalized version of the change of variables, consider the super vector space $\mathbb{R}^{n|m} = \mathcal{R}$, recall that $\mathbb{R}^{n|m} = \mathbb{R}^n \oplus \Pi \mathbb{R}^m$ and take an endomorphism of \mathcal{R} . Now parametrise the endomorphism, namely look at $J \in \text{End}(\mathcal{R}) \otimes \mathcal{C}^\infty(S)$ where S denoted the parameter space.

$$J = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (4.49)$$

$$A \in (\text{End}(\mathbb{R}^n) \otimes \mathcal{C}^\infty(S))_{\text{even}} \quad (4.50)$$

$$D \in (\text{End}(\mathbb{R}^m) \otimes \mathcal{C}^\infty(S))_{\text{even}} \quad (4.51)$$

$$B \in (\text{Hom}(\mathbb{R}^m, \mathbb{R}^n) \otimes \mathcal{C}^\infty(S))_{\text{odd}} \quad (4.52)$$

$$C \in (\text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \otimes \mathcal{C}^\infty(S))_{\text{odd}} \quad (4.53)$$

DEFINITION 4.19 (Super determinant) The **super determinant** of J is the $\mathcal{C}^\infty(S)$ -function

$$\text{sdet}(J) = \det(D)^{-1} \cdot \det(A - BD^{-1}C) \quad (4.54)$$

LEMMA 4.20 The super determinant is

- Multiplicative: $\text{sdet}(JK) = \text{sdet}(J) \text{sdet}(K)$
- $\text{sdet}(1 + \varepsilon J) = 1 + \varepsilon \text{str}(J) + \mathcal{O}(\varepsilon^2)$ where $\text{str}(J) := \text{tr}(A) - \text{tr}(D)$

COROLLARY 4.21 $\text{sdet}(\exp(J)) = \exp(\text{str}(J))$

THEOREM 4.22 Let $\mathbb{R}_1^{n|m}$ and $\mathbb{R}_2^{n|m}$ be two copies of an $(n|m)$ super space with coordinates $\{x^i, \theta_a\}_k$ for $k = 1, 2$. Consider the integral w.r.t the "coordinate" Berezinian

$$\mu_1 = \prod_{i,a} dx^i d\theta_a, \quad \mu_2 = \prod_{j,b} dy^j d\psi_b \quad (4.55)$$

Now let $\varphi: \mathbb{R}_1^{n|m} \rightarrow \mathbb{R}_2^{n|m}$ be a smooth map of supermanifolds and let $f \in \mathcal{C}_c^\infty(\mathbb{R}_2^{n|m})$. Then

$$\int_{\mathbb{R}_2^{n|m}} \mu_2 f = \int_{\mathbb{R}_1^{n|m}} \mu_1 \text{sgn}(\det(J_f^{nn})) \cdot \text{sdet}(J_f) \varphi^* f \quad (4.56)$$

where

$$J_f = \begin{pmatrix} J_f^{(nn)} & J_f^{(mn)} \\ J_f^{(nm)} & J_f^{(mm)} \end{pmatrix} = \begin{pmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial \theta} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial \theta} \end{pmatrix}. \quad (4.57)$$

Here we think of J_f as a function in $\text{End}(\mathbb{R}^{n|m}) \otimes \mathcal{C}^\infty(\mathbb{R}^{n|m})$.

DEFINITION 4.23 Let $V \in \mathfrak{X}(\mathcal{M})$ for a super/graded manifold \mathcal{M} and let $\mu \in \text{BER}(\mathcal{M})$. The **divergence** of V is the $\mathcal{C}^\infty(\mathcal{M})$ function $\text{div}_\mu(V) \in \mathcal{C}^\infty(\mathcal{M})$ such that

$$\forall f \in \mathcal{C}_c^\infty(\mathcal{M}), \quad \int_{\mathcal{M}} \mu V(f) = - \int_{\mathcal{M}} \mu \text{div}_\mu(V) f \quad (4.58)$$

LEMMA 4.24 Let $\mu_0, \mu_1 \in \text{BER}(\mathcal{M})$ with

$$\mu_1 = \rho \mu_0 \quad \rho \in \mathcal{C}^\infty(\mathcal{M}) \quad (4.59)$$

then $V \in \mathfrak{X}(\mathcal{M})$, $\text{div}_{\mu_1}(V) = \text{div}_{\mu_0}(V) + V(\log(\rho))$

Proof.

$$\int \text{div}_{\mu_1}(V) f \mu_1 = - \int \mu_1 V(f) = - \int \rho \mu_0 V(f) \quad (4.60)$$

$$= - \int \mu_0 V(\rho f) \mu_0 + \int V(\rho) f \mu_0 \frac{\rho}{\rho} \quad (4.61)$$

$$= \int \text{div}_{\mu_0}(V)(\rho f) \mu_0 + V(\log(\rho)) f \mu_0 \rho \quad (4.62)$$

□

LEMMA 4.25 Let $\{x^i, \theta_a\}$ denote coordinates on the super manifold $\mathbb{R}^{n|m}$ such that

$$\mu_{\text{cord}} = d^n x^i \prod_a D\theta_a \quad (4.63)$$

Thus we write a vector field locally as

$$v = v^i \frac{\partial}{\partial x^i} + v^a \frac{\partial}{\partial \theta_a} \quad (4.64)$$

its divergence reads

$$\text{div}_{\mu_{\text{cord}}} v = \partial_{x^i} v^i - (-1)^{|v|} \partial_{\theta_a} v^a \quad (4.65)$$

LEMMA 4.26

$$\text{div}_{\mu_{\text{cord}}}(d_{\text{Aux}}) = 0 \quad (4.66)$$

$$\text{div}_{\mu_{\text{cord}}}(d_{\text{BRST}}) = \text{div}_{\mu_{\text{cord}}}(d_{\text{CE}}) = \sum_a c^a \left(\sum_b f_{ba}^b + \text{div}_F v_a \right) \quad (4.67)$$

where $v_a = \rho(t_a)$ for $\{t_a\}$ a basis of \mathfrak{g} .

Proof. First note that

$$\operatorname{div}_{\mu_{\text{coord}}}(d_{Aux}) = -\frac{\partial}{\partial \bar{c}_a} \lambda_a \equiv 0 \quad (4.68)$$

which leads us to

$$\operatorname{div}_{\mu_{\text{coord}}}(d_{CE}) = \frac{\partial}{\partial x^i} (c^a v_a^i) - \frac{\partial}{\partial c^d} \left(-\frac{1}{2} f_{ab}^d c^a c^b \right) = c^a \operatorname{div}_F(v_a) + f_{db}^d c^b \quad (4.69)$$

□

This lemma lets us formulate and prove the main theorem of BRST:

THEOREM 4.27 (BRST) Let $\rho: \mathfrak{g} \rightarrow \mathfrak{X}(F)$ be a Lie algebra action on F and let $\tilde{\mu}$ be a Berezinian on F_{BRST} such that $\operatorname{div}_{\tilde{\mu}}(d_{BRST}) = 0$. Define

$$\langle h \rangle_\psi := \frac{\int_{F_{BRST}} \exp(\frac{i}{\hbar} S_{gf}) h \tilde{\mu}}{\int_{F_{BRST}} \exp(\frac{i}{\hbar} S_{gf}) \tilde{\mu}} \quad (4.70)$$

such that

$$1. \quad S_{gf} = S + d_{BRST}\psi, \quad \psi \in \mathcal{C}^\infty(F_{BRST})^{(-1)} \quad (4.71)$$

$$2. \quad d_{BRST}S = 0, \quad d_{BRST}h = 0 \quad (4.72)$$

Then the following statements hold true:

a. $\langle h \rangle_\psi$ is locally constant in ψ

b. If $a = d_{BRST}b \Rightarrow \langle a \rangle_\psi = 0 \forall \psi$

Proof. Let ψ_t be a smooth family of "gauge fixing functions". Define

$$I_{\psi_t} = \int_{F_{BRST}} \exp(\frac{i}{\hbar} (S + d_{BRST}\psi_t)) h \tilde{\mu} \quad (4.73)$$

such that we can denote

$$\frac{d}{dt} I_{\psi_t} = \frac{i}{\hbar} \int_{F_{BRST}} d_{BRST} \left(\frac{d}{dt} \psi_t \right) \exp(\frac{i}{\hbar} (S_{gf}(t))) h \tilde{\mu} \quad (4.74)$$

$$= \frac{i}{\hbar} \int_{F_{BRST}} d_{BRST} \left(\frac{d}{dt} \psi_t \exp(\frac{i}{\hbar} S_{gf}(t)) h \right) \tilde{\mu} \quad (4.75)$$

$$= -\frac{i}{\hbar} \int_{F_{BRST}} (\dots) \operatorname{div}_{\tilde{\mu}}(d_{BRST}) \tilde{\mu} = 0 \quad (4.76)$$

Now for the second part note that

$$\int_{F_{BRST}} \exp(i/\hbar S) d_{BRST} b \tilde{\mu} = \int_{F_{BRST}} d_{BRST}(\dots) \tilde{\mu} = - \int_{F_{BRST}} \operatorname{div}_{\tilde{\mu}}(d_{BRST}) \tilde{\mu} = 0 \quad (4.77)$$

which proves our theorem. \square

Now given a Berezinian $\tilde{\mu}$ we always have $\operatorname{div}_{\tilde{\mu}}(d_{BRST}) = 0$. Thus BRST integrals descend to the BRST cohomology. Namely the value of the integrals do not change if you change the representative of a cohomology class.

$$\int_{F_{BRST}} \tilde{\mu} : H_{BRST}^{\bullet}(\mathcal{C}^{\infty}(F_{BRST})) \longrightarrow \mathbb{R} \quad (4.78)$$

If S has a degenerate Hessian, we make our transition to $S_{gf} = S + d_{BRST}\psi$. However the divergence-free condition is a strong restriction. In general we are interested in d_{BRST} -closed quantities like $\exp(i/\hbar S_{gf})h$ but not in d_{BRST} -exact quantities.

When \mathfrak{g} is unimodular then $\sum f_{ba}^b = 0$ and we can find a \mathfrak{g} -invariant measure on F , then $\operatorname{div}(v_a) \equiv 0$ and thus in particular $\operatorname{div}(d_{BRST}) = 0$.

4.3.1 Interpretation of "gauge fixing"

Consider, as before, a manifold F together with a Lie group/algebra action of G (or \mathfrak{g}). Further suppose that we are given an invariant function $S: F \longrightarrow \mathbb{R}$ where $S \in \mathcal{C}^{\infty}(F)^G$ and an invariant volume form $\mu_F \in \Omega^n(F)^G$ (if F is non compact, take μ_F compactly supported). The action of the Lie group traces out orbits along which S is constant. Let us look at the integral

$$I = \int_F \mu_F e^{i/\hbar S} \longrightarrow I = \operatorname{Vol}_G \int_{F/G} \tilde{\mu}_F e^{i/\hbar \tilde{S}} \quad (4.79)$$

where $\rho: F \longrightarrow F/G$ projects to equivalence classes (the space of leaves of the foliation defined by the group action), and $\tilde{S} \in \mathcal{C}^{\infty}(F/G)$ is such that $S = \rho^* \tilde{S}$, while $\tilde{\mu} \in \Omega^{n-m}(F/G)$ is such that for a basis $\{v_a\}$ where $v_a = \rho(t_a)$

$$v_m \dots v_1 \mu = \rho^* \mu \quad (4.80)$$

Now this point of view does not generalise very well because a quotient can be very badly behaved and finding \tilde{S} might not be easy. Alternatively we look for a "section" of the projection ρ , denoted by J , such that $\operatorname{im}(J)$ cuts the G -orbits. Practically one looks for a function

$$\varphi: F \longrightarrow \mathfrak{g} \quad (4.81)$$

such that

1. $0 \in \mathfrak{g}$ is a regular value for φ
2. $\Sigma := \varphi^{-1}(0)$ intersects the G -orbits $N \geq 1$ times, transversally (locally, for $U \subset F$ a

neighbourhood of an intersection point $T_p U \simeq \mathfrak{g} \oplus T_p \Sigma$ for any $p \in \Sigma$)

SKETCH 4.28 (Faddeev-Popov trick) This trick lets us write

$$I = \frac{\text{Vol}(G)}{N} \int_{\Sigma} \iota_{v_1} \dots \iota_{v_m} \mu e^{i/\hbar S} \Big|_{\Sigma} = \frac{\text{Vol}(G)}{N} \int_F \delta^{(m)}(\varphi) \iota_{v_1} \dots \iota_{v_m} \mu e^{i/\hbar S} \quad (4.82)$$

where $\delta^{(m)}(\varphi) := \delta(\varphi) \cdot \bigwedge_a d\varphi^a$ is a distributional m -form.

LEMMA 4.29 $\bigwedge d\varphi^a \iota_{v_1} \dots \iota_{v_m} \mu = \det(FP(x)) \cdot \mu$ where

$$FP(x) = d_x \varphi \circ d_{(1,x)} \rho: \mathfrak{g} \longrightarrow \mathfrak{g} \quad (4.83)$$

In components

$$[FP(x)]_a^b = \langle d_x \varphi^b(x), v_a(x) \rangle = v_a(\varphi^b)|_x \quad (4.84)$$

Proof. $FP(x)$ is non-degenerate iff the intersection of $\varphi^{-1}(\varphi(x))$ with a G -orbit is transversal. In fact we have $\langle d\varphi, v \rangle_x = 0$ iff $v \in \ker(d\varphi_x)$ which is equivalent to v being tangent to $\varphi^{-1}(\varphi(x))$. Now assume transversality, i.e. non-degeneracy, and consider $V_x = \text{span}\{v_a^{(x)}\} \subset T_x F$. Now if we write the orbit as

$$\mathcal{O}_x := \{xg \mid g \in G\} \quad (4.85)$$

we observe $V_x \in T_x \mathcal{O}$. Moreover if $\{\alpha_1, \dots, \alpha_{n-m}\}$ is a basis, we can define

$$\text{Ann}(V_x) := \{\alpha \in T_x^* F \mid \langle \alpha, v \rangle = 0 \ \forall v \in V\} \subset T_x^* F \quad (4.86)$$

Now a basis for $T_x^* F$ is $\{d\varphi^1(x), \dots, d\varphi^m(x), \alpha_1, \dots, \alpha_{n-m}\}$. Thus we define

$$\mu = \bigwedge_{a=1}^m d\varphi^a \wedge \alpha_1 \wedge \dots \wedge \alpha_{n-m}. \quad (4.87)$$

With a little care we obtain

$$\iota_{v_1} \dots \iota_{v_m} \mu = \left(\sum_{\sigma \in S_m} (-1)^S \prod_{a=1}^m \langle d\varphi^a(x), v_{S(a)}(x) \rangle \right) \alpha_1 \wedge \dots \wedge \alpha_{n-m} \quad (4.88)$$

$$= \det(FP(x)) \cdot \alpha_1 \wedge \dots \wedge \alpha_{n-m}. \quad (4.89)$$

Now conclude the proof by "wedging" with $\bigwedge d\varphi^a$ to arrive at the statement of the lemma. \square

We will need to keep this lemma in mind when going forward. Another helpful lemma is the following.

LEMMA 4.30 Denote by $DcD\bar{c}$ the coordinate Berezinians on $\mathfrak{g}[1] \oplus \mathfrak{g}^*[-1]$. Then

$$\det(FP(x)) = \left(\frac{\hbar}{i}\right)^m \int_{\mathfrak{g}[1] \oplus \mathfrak{g}^*[-1]} Dc D\bar{c} \exp(i/\hbar \langle \bar{c}, FP(x) \cdot c \rangle) \quad (4.90)$$

The splitting $\mathfrak{g}[1] \oplus \mathfrak{g}^*[-1]$ was the original interpretation of *ghosts/antighosts*. Moreover we now have an integral representation of the delta function in the form of

$$\delta(\varphi(x)) = \frac{1}{(2\pi\hbar)^m} \int_{\mathfrak{g}^*} d^m \lambda e^{i/\hbar \langle \lambda, \varphi(x) \rangle} \quad (4.91)$$

THEOREM 4.31 (Faddeev-Popov)

$$I = \int_F \exp\left(\frac{i}{\hbar} S\right) = \frac{\text{Vol}(G)}{N} \int_F \mu \delta(\varphi) \det(FP) \exp\left(\frac{i}{\hbar} S\right) \quad (4.92)$$

$$= \int_{F \times \mathfrak{g}[1] \times \mathfrak{g}^*[-1] \times \mathfrak{g}^*} \mu Dc D\bar{c} d^m \lambda \exp\left(\frac{i}{\hbar} (S + \langle \lambda, \varphi(x) \rangle + \langle \bar{c}, FP(x)c \rangle)\right) \quad (4.93)$$

We define $S_{FP} := (S + \langle \lambda, \varphi(x) \rangle + \langle \bar{c}, FP(x)c \rangle)$.

THEOREM 4.32 (FP within BRST) Choose $\psi = \langle \bar{c}, \varphi \rangle \in \mathcal{C}^\infty(F_{BRST})^{(-1)}$. Then

$$S_{gf} = S + d_{BRST}\psi = S_{FP} \quad (4.94)$$

Proof. Note that due to $d_{Aux}\bar{c} = \lambda$ we get

$$d_{BRST}\psi = \langle d_{BRST}\bar{c}, \varphi \rangle - \langle \bar{c}, d_{BRST}\varphi \rangle = \langle \lambda, \varphi \rangle - \langle \bar{c}, d_{BRST}\varphi \rangle \quad (4.95)$$

such that we can write

$$d_{BRST}\varphi = c^a v_a(\varphi) = -\langle d\varphi, v_a \rangle c^a \quad (4.96)$$

□

Remark 4.33 FP 4.31 is quite restrictive if you formulate it in a mathematically precise way. Nevertheless we *can* interpret it as a cohomological procedure which works with fewer assumptions. However this is not fully satisfying for the following reasons:

1. Cumbersome minimal/nonminimal extension is **needed** to introduce ψ .
2. BRST is inherently limited to Lie algebra actions. Thus $\widetilde{\mathfrak{g}} = \mathfrak{t} \oplus \mathfrak{g}$ where \mathfrak{g} is a subalgebra.
3. The space of invariant functions, in physical terminology called "**The Observables**" are quantities that, in principle, can be measured. In BRST they do **not** "know" anything about S : It is not required in the definition of $H^\bullet(C_{CE}^\bullet)$.

Solving all of those problems we will look at functions that are invariant on $\text{Crit}(S)$, not globally. To incorporate this idea from the start will lead us to the **BV formalism**.

4.4 BV Formalism

The main idea of the BV formalism is to look at the space of functions invariant on $\text{Crit}(S)$ and not necessarily global. This less restrictive setting will eliminate many of the problems the BRST formulation inherently attains from its rather restrictive formulation.

4.4.1 Symplectic Preliminaries

First let us recall some background on graded symplectic geometry:

If \mathcal{M} is a graded/super manifold, we define $\Omega^\bullet(\mathcal{M})$ as $\mathcal{C}^\infty(T[1]\mathcal{M})$. So if $\{x^i, \theta_a\}$ are coordinates on \mathcal{M} , we can introduce $\{dx^i, d\theta_a\}$ as coordinates on the shifted fibres. Generally in \mathbb{Z} -grading there are two relevant degrees:

- The de Rham "form degree" denoted by \deg_{dR}
- The internal degree (grade) that comes from the \mathbb{Z} -grading denoted by

$$\text{gr}(x^i) = \text{gr}(dx^i), \quad \text{gr}(\theta_a) = \text{gr}(d\theta_a) \quad (4.97)$$

This is useful to decide the signs in the explicit calculations by defining the **total degree**:

$$\text{td} = \deg_{dR} + \text{gr} \quad (4.98)$$

Now recall that a graded k -symplectic manifold is a \mathbb{Z} -graded manifold \mathcal{M} together with a closed nondegenerate 2-form ω of grade $\text{gr}(\omega) = k$ inducing a natural isomorphism

$$\omega^\#: T\mathcal{M} \xrightarrow{\sim} T^*\mathcal{M}[k] \quad (4.99)$$

Now in infinite dimensions, one speaks of *weak symplectic* forms, if $d\omega = 0$, such that $\omega^\#$ is *injective*. For special cases we can use the following result due to Schwarz

THEOREM 4.34 (Schwarz) Let (\mathcal{M}, ω) be a (-1) -symplectic graded manifold with body manifold M . Then

1. In the neighbourhood of any point of M , there are local coordinates on \mathcal{M} denoted by $\{x^i, \xi_i\}$ such that $\omega = \sum_i dx^i \wedge d\xi_i$.
2. There exists a global (non-canonical) symplectomorphism

$$\varphi: (\mathcal{M}, \omega) \longrightarrow (T^*[-1]\mathcal{M}, \omega_{can}), \quad \omega_{can} = \sum_i dx^i \wedge d\xi_i \quad (4.100)$$

Note that the first statement of this theorem corresponds to a generalized variant of the *Darboux Theorem* from usual symplectic geometry.

DEFINITION 4.35 (Lagrangian Submanifolds) A submanifold $\mathcal{L} \subset \mathcal{M}$ for k -symplectic \mathcal{M} is a **Lagrangian submanifold** in \mathcal{M} iff it is maximally isotropic, i.e. \mathcal{L} is isotropic ($\omega|_{\mathcal{L}} = 0$) and it is not properly contained in any other isotropic submanifold.

In finite dimensions one can prove that \mathcal{L} being Lagrangian is equivalent to it being isotropic and coisotropic. This means that \mathcal{L} is isotropic and its symplectic complement \mathcal{L}^ω is too. Thus we also have $\dim(\mathcal{L}) = \frac{1}{2} \dim(\mathcal{M})$.

DEFINITION 4.36 (Conormal Bundle) Let C be a submanifold of M . We can look at $N^*C \subset (T^*M)|_C$ as the bundle over C with fibre

$$N_x^*C := \{ \alpha \in T_x^*M \mid \langle \alpha, v \rangle = 0 \ \forall v \in T_x C \} \quad (4.101)$$

We call N^*C the **conormal bundle** of $C \subset M$.

EXAMPLE 4.37 (Conormal Lagrangian) Let $C \subset M$ be a smooth submanifold. The shifted conormal bundle

$$N^*[-1]C \subset T^*[-1]M \quad (4.102)$$

is a Lagrangian submanifold. The proof is left as an interesting exercise to the reader.

Another important theorem, which will be given without proof due to the lack of time in this course, is the following results from symplectic geometry:

THEOREM 4.38 (Weinstein Tubular nbhd Theorem in (-1) -symplectic context) Let $\mathcal{L} \subset \mathcal{M}$ be a Lagrangian submanifold in \mathcal{M} where \mathcal{M} is (-1) -symplectic. Then there exist

1. a tubular neighbourhood $U \subseteq \mathcal{M}$ of \mathcal{L} , i.e. a vector bundle $p: E \rightarrow \mathcal{L}$ together with a smooth map $J: E \rightarrow \mathcal{M}$ such that

$$J \circ 0_E = i, \quad i: \mathcal{L} \rightarrow \mathcal{M} \quad (4.103)$$

and there exists $V \subseteq E$ with $0_E(\mathcal{L}) \subset V$ such that $J|_V \xrightarrow{\sim} U$ is a diffeomorphism.

2. a tubular neighbourhood $U_0 \subset T^*[-1]\mathcal{L}$ (with projection p_0) of the zero section $\mathcal{L}_0 \equiv \mathcal{L}$.
3. a symplectomorphism $\varphi: U \xrightarrow{\sim} U_0$ such that $\varphi(\mathcal{L}) = \mathcal{L}_0$ and $p_0 \circ \varphi = \varphi|_{\mathcal{L}} \circ p$.

Thus in a neighbourhood of a Lagrangian submanifold, the ambient (-1) -symplectic manifold is locally symplectomorphic to $T^*[-1]\mathcal{L}$.

EXAMPLE 4.39 (BRST Graph Lagrangian) Consider F_{BRST} as we defined it in the previous

section ($F_{BRST} = F \times \mathfrak{g}[1] \times \mathfrak{g}^*[-1] \times \mathfrak{g}^*$) and let $\psi \in \mathcal{C}^\infty(F_{BRST})^{(-1)}$. Then

$$\Gamma_\psi := \text{graph}(d\psi) \subset T^*[-1]F_{BRST} \quad (4.104)$$

is a Lagrangian submanifold. Denote by $\{\varphi^a\}$ coordinates in F_{BRST} and by $\{\xi_a\}$ coordinates in the fibre. Then

$$\text{graph}(d\psi) = \left\{ (\varphi^a, \xi_a) \left| \xi_a = \frac{\partial}{\partial \varphi^a} \psi \right. \right\} \quad (4.105)$$

We also define

$$\omega|_{\Gamma_\psi} := \langle d\xi, d\varphi \rangle|_{\Gamma_\psi} = d\xi_a d\varphi^a|_{\Gamma_\psi} \quad (4.106)$$

$$= d\varphi^a d(\partial_a \psi) = d\varphi^a \partial_a \partial_b \psi^b = 0 \quad (4.107)$$

As an exercise check that $\dim(\text{graph}(d\psi)) = \frac{1}{2} \dim(T^*[-1]F_{BRST})$. For $\psi = 0$ we get Γ_0 equal the zero section, thus Γ_ψ is a kind of *deformation* of the zero section.

4.4.2 Classification of Lagrangians

Let \mathcal{M} be a graded (-1) -symplectic manifold. Then

1. for $\mathcal{L} \subset \mathcal{M}$ a Lagrangian submanifold, there exists a submanifold $C \subset M$ and a symplectomorphism $\varphi: \mathcal{M} \xrightarrow{\sim} T^*[-1]M$ such that $\varphi(\mathcal{L}) = \mathcal{L}_C \equiv \mathcal{N}^*[-1]C \subset T^*[-1]M$.
2. for $\mathcal{L} \subset \mathcal{M}$ a Lagrangian submanifold obtained from $\mathcal{L}_C = \mathcal{N}^*[-1]C$ where $C \subset M$ as a graph of $d\psi$ for some $\psi \in \mathcal{C}^\infty(\mathcal{L}_C)^{(-1)}$ we can use the tubular neighbourhood theorem to identify the neighbourhood of \mathcal{L}_C with $T^*[-1]\mathcal{L}_C$.

We say that $\mathcal{L} \sim \mathcal{L}'$ homotopic as Lagrangian submanifolds, iff there exists a family of Lagrangian submanifolds \mathcal{L}_t smoothly varying over such that $\mathcal{L}_0 = \mathcal{L}$ and $\mathcal{L}_1 = \mathcal{L}'$ for $t \in [0, 1]$.

DEFINITION 4.40 Let (\mathcal{M}, ω) be a $(n|n)$ -dimensional (-1) -symplectic graded manifold. A Berezinian μ on \mathcal{M} is called **compatible with** ω , if there exists an atlas of Darboux charts $\{x^i, \xi_i\}$ such that $\mu = d^n x D^n \xi$ in every chart. We also say that the triple $(\mathcal{M}, \omega, \mu)$ is compatible.

Now for $(\mathcal{M}, \omega, \mu)$ compatible we introduce with

$$\Delta_{BV}: \mathcal{C}^\infty(\mathcal{M}) \longrightarrow \mathcal{C}^\infty(\mathcal{M}) \quad (4.108)$$

the "Batalin-Vilkovisky" Laplacian defined locally in any Darboux chart as the degree $+1$ second-order operator

$$\Delta_{BV} = \sum_i \frac{\partial}{\partial x^i} \frac{\partial}{\partial \xi_i} \quad (4.109)$$

LEMMA 4.41 Δ_{BV} is well defined and $\Delta_{BV}^2 = 0$.

Proof. Compatibility of ω and μ implies that the transition functions are unimodular i.e.

$$\text{sdet} \left(\frac{\partial(x, \xi)}{\partial(y, \theta)} \right) = 1 \quad (4.110)$$

Now the fact that the BV-Laplacian squares to 0 can be seen by looking at

$$\Delta_{BV}^2 = \sum_{i,j} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \xi_j} \quad (4.111)$$

and then mapping $(j, i) \rightarrow (i, j)$. Using antisymmetry we obtain the statement. \square

DEFINITION 4.42 Let (\mathcal{M}, ω) be a (-1) -symplectic graded manifold with Berezinian μ . Define the μ -Laplacian

$$\Delta_\mu: \mathcal{C}^\infty(\mathcal{M}) \longrightarrow \mathcal{C}^\infty(\mathcal{M}), \quad \Delta_\mu f := \frac{1}{2} \text{div}_\mu(X_f) \quad (4.112)$$

where $\iota_{X_f} \omega = df$, i.e. X_f is the Hamiltonian vector field of f .

If $\{\cdot, \cdot\}$ denotes the Poisson-brackets, we get $X_f = \{f, \cdot\}$ such that $\{f, g\} = X_f(g)$. For a Darboux chart $\{x^i, \xi_i\}$ and assuming that $\mu = \rho \cdot \mu_{\text{coord}}$ we obtain the following result:

LEMMA 4.43 $\Delta_\mu = \Delta_{BV} + \frac{1}{2} \{f, \log(\rho)\}$.

Proof. First note that

$$\Delta_{BV} f = -\frac{1}{2} \text{div}_{\mu_{\text{coord}}}(X_f) \quad (4.113)$$

and further

$$\text{div}_{\mu_{\text{coord}}}(X_f) = \frac{\partial}{\partial x^i} X_f^{(x^i)} - \frac{\partial}{\partial \xi_i} X_f^{(\xi_i)} \quad (4.114)$$

Together this lets us write

$$\iota_{X_f} \omega = df \iff \iota_{X_f} dx^i d\xi_i = \frac{\partial f}{\partial x^i} dx^i - \frac{\partial f}{\partial \xi_i} d\xi_i \quad (4.115)$$

$$\iff X_f^{(x^i)} = \frac{\partial f}{\partial x^i} \quad \text{and} \quad X_f^{(\xi_i)} = \frac{\partial f}{\partial \xi_i} \quad (4.116)$$

This allows us to reformulate the divergence of the Hamiltonian vector field as

$$\text{div}_{\mu_{\text{coord}}}(X_f) = -\frac{\partial}{\partial x^i} \frac{\partial}{\partial \xi_i} f - \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial x^i} f \quad (4.117)$$

$$= -2 \sum_i \frac{\partial}{\partial x^i} \frac{\partial}{\partial \xi_i} f = -2 \Delta_{BV} f \quad (4.118)$$

Now using $\operatorname{div}_{\rho\mu_{\text{coord}}}(X) = \operatorname{div}_{\mu_{\text{coord}}}(X) + X(\log(\rho))$ we can conclude

$$\Delta_\mu f = \Delta_{BV} f + \frac{1}{2}\{f, \log(\rho)\} \quad (4.119)$$

which proves the statement. \square

Remark 4.44 Note that Δ_μ does not necessarily square to 0 as Δ_{BV} does. However it always will if μ is compatible with ω .

4.4.3 BV Integrals

Notice that so far we did not talk about physics at all. We merely introduced several notions about (-1) -symplectic manifolds. Having a well-defined integration will bring us quite a few steps closer to a physical theory using the given framework. To this end recall that $\operatorname{Ber}(\mathcal{M}) = \bigwedge^m T^*M \otimes \bigwedge^m E$. If $\mathcal{M} = T^*[-1]M$ we obtain $\operatorname{Ber}(\mathcal{M}) = (\bigwedge^n T^*M)^{\otimes 2}$. Similarly if \mathcal{N} is a graded manifold $\operatorname{Ber}(T^*[-1]\mathcal{N})|_{\mathcal{N}} \cong \operatorname{Ber}(\mathcal{N})^{\otimes 2}$.

There is a canonical map sending a Berezinian μ on $T^*[-1]\mathcal{N}$ to a Berezinian on \mathcal{N} . We denote the induced Berezinian by $\sqrt{\mu|_{\mathcal{N}}}$. Locally, denoting mixed-parity/degree coordinates on \mathcal{N} by $\{X^a\}$ and with $\{X^a, \Xi_a\}$ a Darboux chart for $T^*[-1]\mathcal{N}$ we can write

$$\mu = \rho\mu_{\text{coord}} = \rho(X, \Xi) \prod_a DX^a D\Xi_a \quad (4.120)$$

which is mapped to the following Berezinian on \mathcal{N} :

$$\sqrt{\mu|_{\mathcal{N}}} := \sqrt{\rho(X, \Xi)} DX \quad (4.121)$$

DEFINITION 4.45 (BV Integral) Let $(\mathcal{M}, \omega, \mu)$ be compatible. A **BV integral** is an integral of the form

$$I_{\mathcal{L}} = \int_{\mathcal{L} \subset \mathcal{M}} f \sqrt{\mu|_{\mathcal{L}}} \quad (4.122)$$

where \mathcal{L} is a Lagrangian submanifold of \mathcal{M} and $f \in \mathcal{C}^\infty(\mathcal{M})$ such that $\Delta_\mu f = 0$.

Now consider $T[1]M$ and $T^*[-1]M$ with coordinates $\{x^i, \xi_i\}$ and $\{x^i, \theta^i\}$ respectively. We can think of the deRham differential on M as a cohomological vector field

$$D: \mathcal{C}^\infty(T[1]M) \simeq \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M) \quad (4.123)$$

$$D^2 = 0, \quad D = \theta^i \frac{\partial}{\partial x^i}, \quad \theta^i = dx^i \quad (4.124)$$

Now if $\operatorname{Vol}_M = \rho(x) dx^1 \dots dx^n$ is a volume form on M we arrive at the following statement:

LEMMA 4.46 The following two statements hold true when we consider coordinate transformations of the form $\tilde{\theta} = \tilde{\theta}(x, \theta)$ or $\tilde{\xi} = \tilde{\xi}(x, \xi)$:

1. $\int d^n \tilde{\theta} \tilde{\rho}^{-1} = \int d^n \theta \rho^{-1}$
2. $\int d^n \tilde{\xi} \tilde{\rho} = \int d^n \xi \rho$

Proof. First note that

$$\tilde{\text{Vol}}_M = \text{Vol}_M \implies \tilde{\rho} = \left[\det \frac{\partial \tilde{x}}{\partial x} \right]^{-1} \rho \quad (4.125)$$

However

$$d^m \tilde{\theta} = \left[\det \frac{\partial \tilde{x}}{\partial x} \right] d^m \theta \quad (4.126)$$

$$d^m \tilde{\xi} = \left[\det \frac{\partial \tilde{x}}{\partial x} \right]^{-1} d^m \xi \quad (4.127)$$

which proves the statement. \square

Now we define an adapted version of the well-known *Fourier Transformation*:

DEFINITION 4.47 (Odd Fourier Transformation) Let $f \in \mathcal{C}^\infty(T^*[-1]M)$ with $f = f(x, \xi)$. The **odd Fourier Transformation** is the map

$$\mathcal{F}_\rho: \mathcal{C}^\infty(T^*[-1]M) \longrightarrow \mathcal{C}^\infty(T[1]M) \quad (4.128)$$

given by its action on functions f as above by

$$\tilde{f} := \mathcal{F}_\rho[f](x, \theta) := \int d^n \xi \rho e^{\theta^i \xi_i} f(x, \xi) \quad (4.129)$$

An interesting result regarding the odd Fourier Transformation is the following:

PROPOSITION 4.48 $\mathcal{F} \circ \Delta_\mu = d \circ \mathcal{F}$.

Proof. The proof is left as an exercise to the reader. \square

Otherwise one can show that for $g \in \mathcal{C}^\infty(T[1]M)$ one can define

$$\mathcal{F}^{-1}[g] = \int d^n \theta \rho^{-1} e^{\xi_i \theta^i} g(x, \theta) \quad (4.130)$$

such that

$$\mathcal{F}^{-1}[dg(x, \theta)] = \Delta(\mathcal{F}^{-1}[g]) \quad (4.131)$$

Now we can finally formulate the main result shouldering the BV formalism:

THEOREM 4.49 (Batalin-Vilkovisky-Schwarz Theorem) Let $(\mathcal{M}, \omega, \mu)$ be a compatible (-1) -symplectic graded manifold with Berezinian μ . The following statements hold true:

1. For any $g \in \mathcal{C}^\infty(\mathcal{M})$ and a Lagrangian submanifold $\mathcal{L} \subset \mathcal{M}$

$$\int_{\mathcal{L}} \Delta_\mu g \sqrt{\mu_{\mathcal{L}}} = 0 \quad (4.132)$$

2. Let \mathcal{L} and \mathcal{L}' be two Lagrangian submanifolds whose bodies are homologous (as cycles) in the body of \mathcal{M} and let $f \in \mathcal{C}^\infty(M)$ such that $\Delta_\mu f = 0$. Then

$$\int_{\mathcal{L}} f \sqrt{\mu_{\mathcal{L}}} = \int_{\mathcal{L}'} f \sqrt{\mu_{\mathcal{L}'}} \quad (4.133)$$

Proof. This proof is more of a sketch. You can fill in the details with the material at hand. Assume that $\mu = \rho \mu_{coord}$ and $\mathcal{L} = \mathcal{L}_C = N^*[-1]C$ for $C \subset M$ a submanifold of M . Then the BV integral can be seen as the map

$$f \mapsto \int_{\mathcal{L}_C} f \sqrt{\mu_{\mathcal{L}_C}} = \int_C \tilde{f} \Big|_C \quad (4.134)$$

where \tilde{f} is the form representative of a function f and the right hand side is thus seen as the integral of a differential form on C . Then in particular

$$\int_{\mathcal{L}_C} \Delta_\mu g \sqrt{\mu_{\mathcal{L}_C}} = \int_C d\tilde{g} = 0 \quad (4.135)$$

and further

$$\int_{\mathcal{L}'_C} f \sqrt{\mu_{\mathcal{L}'_C}} - \int_{\mathcal{L}_C} f \sqrt{\mu_{\mathcal{L}_C}} = \left(\int_{C'} - \int_C \right) \tilde{f} = \int_D d\tilde{f} = 0 \quad (4.136)$$

We obtain the last equation by seeing that

$$d\tilde{f} = d\mathcal{F}(f) = \mathcal{F}\Delta f = 0 \quad (4.137)$$

as required per definition. Now since $\mathcal{L} \sim \mathcal{L}'$ are homologous as cycles, $\mathcal{L} \equiv \mathcal{L}_C$ and $\mathcal{L}' = \mathcal{L}'_C$ such that there necessarily exists a submanifold $D \subset M$ with $\partial D = C' - C$.

Now for a general Lagrangian submanifold part 1) of the classification of Lagrangian submanifolds (see 4.4.2) tells us that we can reconstruct the calculation for general Lagrangian submanifolds to that of conormal type to prove the first statement of the theorem. To prove the second, we use the second part of the classification together with the following:

Let \mathcal{L}_t be a smooth family of Lagrangian submanifolds such that for $\varepsilon > 0$

$$\mathcal{L}_{t+\varepsilon} = \text{graph}(\varepsilon d\psi_t + \mathcal{O}(\varepsilon^2)) \quad (4.138)$$

for a function $\psi_t \in \mathcal{C}^\infty(\mathcal{L}_t)^{(-1)}$. Then for $f \in \mathcal{C}^\infty(\mathcal{M})$ such that $\Delta_\mu f = 0$ we have

$$\frac{d}{dt} \int_{\mathcal{L}_t} f \sqrt{\mu_{\mathcal{L}_t}} = \int_{\mathcal{L}_t} \Delta_\mu(f \cdot \psi_t) \sqrt{\mu_{\mathcal{L}_t}} \quad (4.139)$$

This can be separately proven by investigating

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[\int_{\mathcal{L}_{t+\varepsilon}} f \sqrt{\mu_{\mathcal{L}_{t+\varepsilon}}} - \int_{\mathcal{L}_t} f \sqrt{\mu_{\mathcal{L}_t}} \right] \quad (\diamond) \quad (4.140)$$

To this end note that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\mathcal{L}_t} (f(x, \varepsilon d\psi_t + \varepsilon^2) - f(x, d\psi_t)) \sqrt{\mu_{\mathcal{L}_t}} \quad (4.141)$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\mathcal{L}_t} (f(x, \varepsilon d\psi_t) - f(x, d\psi_t) + \mathcal{O}(\varepsilon^2)) \sqrt{\mu_{\mathcal{L}_t}} \quad (4.142)$$

$$= \int_{\mathcal{L}_t} f(x, d\psi_t) \sqrt{\mu_{\mathcal{L}_t}} \quad (4.143)$$

Now since

$$\Delta_\mu(f \cdot \psi_t) = \Delta_\mu f \cdot \psi_t + f \cdot \Delta_\mu \psi_t + \frac{\partial f}{\partial \xi} \frac{\partial \psi_t}{\partial x} \quad (4.144)$$

$$= \frac{\partial f}{\partial \xi} \frac{\partial \psi_t}{\partial x} =: f(x, d\psi_t) \quad (4.145)$$

we can rewrite (\diamond) as

$$(\diamond) = \int_{\mathcal{L}_t} \Delta_\mu(f \cdot \psi_t) \sqrt{\mu_{\mathcal{L}_t}} \quad (4.146)$$

Since the last equation vanishes by rule of the first statement of the proof, we only need to find a family \mathcal{L}_t connecting \mathcal{L} to \mathcal{L}_C but this is immediate. \square

The BV integral vanishes on Δ -exact functions because regular integral vanish on d -exact forms. The BV integral does not depend on a particular Lagrangian inside a family of Lagrangians like \mathcal{L}_t if the function is Δ -closed, since d -closed forms have the same property.

4.5 Classical and Quantum BV formalism

DEFINITION 4.50 (BV Theory) The following set of data is called a **classical BV theory**:

1. A \mathbb{Z} -graded manifold \mathcal{F} (the space of BV fields)
2. A (-1) -symplectic structure $\omega \in \Omega^2(\mathcal{F})^{(-1)}$ (the BV 2-form)
3. A function $S \in \mathcal{C}^\infty(\mathcal{F})^{(0)}$ satisfying the Classical Master Equation CME $\{S, S\}_\omega = 0$ (the

BV action or master action)

Remark 4.51 Often it is convenient to work with the cohomological vector field $Q := \{S, \cdot\}$ or alternatively $\iota_Q \omega = dS$. Observe that $\mathcal{L}_Q \omega = 0$, i.e. $\iota_Q \omega$ is closed, but the degree reasons that we can always find S given Q with $\mathcal{L}_Q \omega = 0$ such that $\iota_Q \omega = dS$.

An important example which includes a huge part of interesting theories is the following:

EXAMPLE 4.52 (BV-BRST) Recall the construction of BRST data (F_{min}, d_{CE}, S_0) and $(F_{BRST}, d_{BRST}, S_0)$. Now

1. define $\mathcal{F} := T^*[-1]F_{min} \xrightarrow{p} F_{min}$ with ω_{BV} the canonical symplectic form.
2. define $S_{BV} = p^*S_0 + \widehat{d}_{CE}$ and recall that \widehat{d}_{CE} is the function in $\mathcal{C}^\infty(T^*[-1]F_{min})$ corresponding to a vector field on F_{min} .
3. define $Q_{BV} = X_{p^*S_0} + d_{CE}^{cl}$ where d_{CE}^{cl} is the cotangent lift of d_{CE} to $T^*[-1]F_{min}$.

Now in a Darboux chart $\{\varphi^a, \varphi_a^\dagger\}$ we can write

$$\omega_{BV} = \sum_a d\varphi^a \wedge d\varphi_a^\dagger \quad (4.146)$$

$$S_{BV} = S_0 + \sum_a \Phi^a d_{CE}^a \quad (4.147)$$

where $d_{CE}^a = d_{CE}(\Phi^a)$. Then we arrive at Q_{BV} such that

$$\iota_{Q_{BV}} \omega_{BV} = dS_{BV} \quad (4.148)$$

NOTATION 4.53 For $T^*[-1]F_{min}$ the coordinates (x^i, c^a) introduce coordinates on the fibre $(x_i^\dagger, c_a^\dagger)$ such that

$$\begin{aligned} |x^i| = 0 &\implies |x_i^\dagger| = -1, \\ |c^a| = 1 &\implies |c_a^\dagger| = -2, \end{aligned}$$

In literature one often call the x^i "physical fields" (degree 0), the c^a "ghost fields" (Chevalley-Eilenberg generators), the x_i^\dagger "anti-fields" (degree -1) and the c_a^\dagger "antighosts" (degree -2).

Note that for V^\bullet a graded vector space, $(V^\bullet)^*(-k) = (V^\bullet)^{(k)}$.

Now while we already have a rough idea of physical and ghost fields, we need further investigation on the role of anti-fields and antighosts.

4.5.1 Constructing BV data

The main goal of this section is to construct the data necessary for a complete BV theory. This in turn will allow for a more insightful description of anti-fields and antighosts. Let F be a smooth finite-dimensional manifold and let x^i be coordinates on F together with a function $S \in \mathcal{C}^\infty(F)$ in degree 0.

Now the set $\{\varphi_i := \frac{\partial S}{\partial x^i}\}$ determines an ideal $I_S \subset \mathcal{C}^\infty(F)$ of functions vanishing on $\text{Crit}(S) := \{x \in F \mid dS(x) = 0\}$. Assume for now that there are no nontrivial symmetries of S . Now we **want** a complex $(\mathcal{K}^\bullet, d_{\mathcal{K}})$ such that its cohomology in degree 0 is the quotient $\mathcal{C}^\infty(F)/I_S \simeq \mathcal{C}^\infty(\text{Crit}(S))$.

The following is due to Koszul:

Let V^\bullet be a graded vector space such that $\dim(V) = \dim(F)$, e.g. $V \simeq T_x^*F$. For every coordinate x^i generate a new coordinate $x_i^\dagger \in \mathcal{C}^\infty(V[-1]) \simeq \bigwedge^\bullet V^*$. Then we can define an operator $d_{\mathcal{K}}$ on $\mathcal{C}^\infty(F) \otimes \mathcal{C}^\infty(V[-1]) =: \mathcal{K}^\bullet$ by setting

$$d_{\mathcal{K}} x_i^\dagger := \varphi^i \equiv \frac{\partial S}{\partial x^i} \quad (4.149)$$

Thus in particular $d_{\mathcal{K}}^2 = 0$. Further for $f \in \mathcal{K}^{(0)}$ we have $d_{\mathcal{K}} f = 0 \Leftrightarrow f \in \mathcal{C}^\infty(F)$ but $f = d_{\mathcal{K}} g \Leftrightarrow f \in I_S$ i.e. $fg = \sum_i g^i x_i^\dagger$.

PROPOSITION 4.54 The **Koszul complex** $(\mathcal{K}^\bullet, d_{\mathcal{K}})$ is a **resolution** of $\mathcal{C}^\infty(F)/I_S$.

Proof. We want a complex C^\bullet such that $H^\bullet(C^\bullet) \simeq \mathcal{C}^\infty(F)/I_S \simeq \mathcal{C}^\infty(\text{Crit}(S))$ and further $H^{-i}(C^\bullet) = 0$ for all $i > 0$. In our case $H^{-1} \neq 0$ iff $\exists R_\alpha^i$ such that $d_{\mathcal{K}}(R_\alpha^i x_i^\dagger) = R_\alpha^i \varphi^i = 0$ since there is nothing in degree -2 that can compensate the cocycle $R_\alpha^i x_i^\dagger$.

However $R_\alpha^i \varphi^i \equiv R_\alpha^i \frac{\partial S}{\partial x^i} = 0$ (**Noether identity**) is equivalent to there being a vector field $V_\alpha \in \mathfrak{X}(F)$ with $V_\alpha = R_\alpha^i \frac{\partial}{\partial x^i}$ and such that $V_\alpha(S) = 0$. Namely such a cocycle exists iff S admits nontrivial symmetries which we ruled out in the definition. This proves the claim. \square

Now the coordinates x_i^\dagger that were introduced in the BV construction are in fact *Koszul generators*

$$\mathcal{F} = T^*[-1]F \implies d_{\mathcal{K}} = \{S_{cl}, \cdot\} \equiv Q \quad (4.150)$$

where $Q = \sum_i \varphi_i \frac{\partial}{\partial x_i^\dagger}$. Now in the case where nontrivial symmetries exist, we have nontrivial cocycles $R_\alpha^i x_i^\dagger \in H^{-1}(\mathcal{K}^\bullet)$. Thus we introduce new variables in degree -2 .

Let d_1 be the number of nontrivial Noether identities/symmetries. Namely it is the dimension of $\mathfrak{g} \in \tilde{\mathfrak{g}} = \mathfrak{t} \oplus \mathfrak{g}$. Thus we introduce $c_\alpha^\dagger \in \mathcal{C}^\infty(\mathbb{R}^{d_1}[-2])$ and set $d_{\mathcal{K}} c_\alpha^\dagger = R_\alpha^i x_i^\dagger$ such that by pure

construction $d_{\mathcal{K}}^2 = 0$. Thus we define a new complex

$$\mathcal{KT}^{(1)} := \mathcal{C}^\infty(F) \otimes \mathcal{C}^\infty(V[-1]) \otimes \mathcal{C}^\infty(\mathbb{R}^{d_1}[-2]) \quad (4.151)$$

$$\simeq \mathcal{C}^\infty(T^*[-1]F \times \mathbb{R}^{d_1}[-2]) \quad (4.152)$$

Now if the Noether identities are all independent, i.e. if the symmetries are not reducible which means that $\rho: \mathfrak{g} \rightarrow \mathfrak{X}(F)$ is injective, we have

$$H^\circ(\mathcal{KT}^{(1)}) \simeq \mathcal{C}^\infty(F)/I_S, \quad H^{-i}(\mathcal{KT}^{(1)}) = 0 \quad \forall i > 0 \quad (4.153)$$

Now it could happen that there exist functions $r_\beta^\alpha \in \mathcal{C}^\infty(F)$ such that $r_\beta^\alpha R_\alpha^i = 0 \quad \forall i$, namely relations among the symmetries. Thus there would exist a nontrivial cocycle $r_\beta^\alpha c_\alpha^\dagger \in H^{(-2)}(\mathcal{KT}^{(1)})$. Now we could repeat the procedure, adding $c_\beta^{(2)\dagger} \in \mathcal{C}^\infty(\mathbb{R}^{d_2}[-3])$ and setting d_2 as the number of nontrivial relations such that $r_\beta^\alpha R_\alpha^i = 0$. This procedure can be inductively repeated leading us to

$$\mathcal{KT}^{(\infty)} := \mathcal{C}^\infty \left(T^*[-1]F \times \bigtimes_{k=1}^{\infty} \mathbb{R}^{d_k}[-k-1] \right) \quad (4.154)$$

$$H^\circ(\mathcal{KT}^{(\infty)}) \simeq \mathcal{C}^\infty(F)/I_S, \quad H^{-i}(\mathcal{KT}^{(\infty)}) = 0 \quad \forall i > 0 \quad (4.155)$$

This is called the **Koszul-Tate resolution** of $\mathcal{C}^\infty(F)/I_S$ and $d_{\mathcal{K}}$ is seen as the cohomological vector field on $T^*[-1]F \times \bigtimes_{k=1}^{\infty} \mathbb{R}^{d_k}[-k-1]$.

The following is due to Batalin and Vilkovisky:

We can adjoin coordinates $c^{(k)\alpha}$ for every $c_\alpha^{(k)\dagger}$ that we have from the \mathcal{KT} construction where $-|c^{(k)\alpha}| = |c_\alpha^{(k)\dagger}| + 1$. In other words one looks at

$$BV^\bullet \mathcal{C}^\infty \left(T^*[-1] \left(F \times \bigtimes_{k=1}^{\infty} \mathbb{R}^{d_k}[-k-1] \right) \right) \quad (4.156)$$

Now there exists a canonical symplectic form Ω_{BV} .

Our next step is to reconstruct the "BV package". We start by looking at

$$d_{\mathcal{K}} x_i^\dagger = \{S_{cl}, \cdot\} \equiv Q_0 \quad (4.157)$$

where we set $S_{cl} = S_0$. Now extend $S_0 \mapsto S_1 := S_0 + x_i^\dagger R_\alpha^i c^\alpha$ and set $Q_1 := \{S_1, \cdot\}$ such that $Q_1 c_\alpha^\dagger = d_{\mathcal{K}} c_\alpha^\dagger \equiv R_\alpha^i x_i^\dagger$. Now we can calculate

$$Q_1 x^i = \{S_1, x^i\} = \pm \frac{\delta}{\delta x_k^\dagger} (S_0 + x_j^\dagger R_\alpha^j c^\alpha) \frac{\delta}{\delta x^k} x^i \quad (4.158)$$

$$= R_\alpha^i c^\alpha = c^\alpha V_\alpha(x^i) \quad (4.159)$$

EXERCISE 4.55 Show that $\{S_0, S_1\} = 0$ iff $\{R_\alpha^i\}$ represents a local symmetry. Also show that $\{S_1, S_1\} = 0$.

In order to really reproduce the Chevalley–Eilenberg cohomology, we need to further extend by

$$S_2 = S_1 + \frac{1}{2} c_\alpha^{(1)\dagger} + f_{\beta\gamma}^\alpha c^\beta c^\gamma \quad (4.160)$$

such that

$$[V_\beta, V_\alpha] = f^\gamma \beta \alpha V_\gamma, \quad f_{\beta\alpha}^\gamma = f_{\beta\alpha}^\gamma(x) \in \mathcal{C}^\infty(F) \quad (4.161)$$

No generally both R_α^i and the $f_{\beta\gamma}^\alpha$ are functions on F . So it is **not** guaranteed that $\{S_2, S_2\} = 0$ and thus $Q_2 := \{S_2, \cdot\}$ is not necessarily a differential. The following theorem provides the necessary information to fix such problems:

THEOREM 4.56 (Batalin-Vilkovisky) S_2 can be extended by terms of higher degree in $c_\alpha^{(k)\dagger}$ and $c^{(k)\alpha}$ to a function $S_\infty \in BV^\bullet$ such that $\{S_\infty, S_\infty\} = 0$ satisfies the *Classical Master Equation* (CME), i.e. $Q_\infty := \{S_\infty, \cdot\}$ is a differential and $(T^*[-1](F \times \bigtimes_{k=1}^\infty \mathbb{R}^{d_k}[-k-1]), \Omega_{BV}, S_\infty)$ is a classical BV theory.

Proof. This proof is a sketch but captures the main points. The result stems from cohomological perturbation theory where one observes that if for S_n , $\{S_n, S_n\} = 0$ one also has $d_K \{S_n, S_n\} = 0$. Then since the Koszul complex has vanishing cohomology, there exists f_{n+1} such that $\{S_n, S_n\} = d_K f_{n+1}$. \square

EXERCISE 4.57 Show that if $\{V_\alpha\}$ comes from an injective Lie algebra action, then one obtains $(T^*[-1](F \times \mathfrak{g}[-1]))$ with the BV-BRST data.

Now the BV complex as constructed above achieves two main things:

1. It localises to $\text{Crit}(S)$ by means of the Koszul-Tate construction.
2. It looks at invariants over $\text{Crit}(S)^\mathfrak{g}$. More generally than a Lie algebra, \mathfrak{g} is a subset of symmetries which will be involutive on $\text{Crit}(S)$.

Consider a complement \mathfrak{g} of trivial symmetries \mathfrak{t} inside all symmetries $\widetilde{\mathfrak{g}}$ (for a function $S_{cl} \equiv S_0: F \rightarrow \mathbb{R}$). We know that \mathfrak{g} is a subalgebra "on shell", i.e. on $\text{Crit}(S)$. Thus we construct $\mathcal{KT}^{(\infty)}$ such that $H^0(\mathcal{KT}^{(\infty)}) \simeq \mathcal{C}^\infty(\text{Crit}(S))$ and use it as a \mathfrak{g} -module which means looking at the Chevalley-Eilenberg complex for the \mathfrak{g} action on $\mathcal{C}^\infty(\text{Crit}(S))$. This leads us to the following result:

THEOREM 4.58 $(BV^\bullet, Q_\infty) \simeq C_{CE}^\bullet(\mathfrak{g}, H^\bullet(\mathcal{KT}^{(\infty)}))$

Proof. The proof is omitted since it is very technical and goes beyond the timescope of this course. \square

COROLLARY 4.59 $H^\bullet(BV^\bullet) \simeq \mathcal{C}^\infty(\text{Crit}(S))^\mathfrak{g} \simeq \mathcal{C}^\infty(\text{Crit}(S)/\mathfrak{g})$

Now this is all the data we need for a physical theory. We have a critical locust, symmetries and a set of evolution equations. Summing up, our procedure goes as follows:

We have F, S_0 and a set of nontrivial symmetries \mathfrak{g} . Now if \mathfrak{g} is a Lie algebra, we can look at pure BRST or more generally at $(T^*[-1]F_{BRST}, Q_{BV})$. If it isn't, we *need* to work with $(T^*[-1]F_{BRST}, Q_{BV})$. From here on we will take the tuple $(\mathcal{F}_{BV}, \Omega_{BV}, S_{BV}, Q_{BV})$ (where technically Q is induced) and construct a classical BV QFT. Note that in general, one could need an infinite number of ghosts, "ghosts for ghosts", anti-fields and antighosts. In standard situations where \mathfrak{g} is a lie subalgebra, the data of the BV construction goes under the name of L_∞ -algebra (we won't have time to discuss this). From now on, we can consider BV data as $(\mathcal{F} \simeq T^*[-1]M, \Omega, S, Q)$ where $\iota_Q \Omega = dS$ such that $(\mathcal{C}^\infty(\mathcal{F}), Q) = BV^\bullet$ is the BV complex.

4.6 Quantum Master Equations and Can. Transformations

We again consider the (-1) -symplectic manifold (\mathcal{F}, Ω) and formal power series $\mathcal{C}^\infty(\mathcal{F})[[\hbar]]$ therein.

DEFINITION 4.60 An element $S \in \mathcal{C}^\infty(\mathcal{F})[[\hbar]]$ is said to satisfy the **Quantum Master Equation** iff

$$\frac{1}{2}\{S, S\} - i\hbar\Delta_\mu S = 0 \quad (QME)$$

where μ is a Berezinian on \mathcal{F} .

Formally the above requirement is equivalent to $\Delta_\mu e^{i/\hbar S} = 0$.

EXERCISE 4.61 Prove this fact. Also explain why it is "formally" equivalent.

This follows from the following fact about BV data: The BV Laplacian is **not** a derivation of the product, i.e.

$$\Delta_\mu(f \cdot g) = \Delta_\mu f \cdot g + (-1)^{|f|} f \cdot \Delta_\mu g + (-1)^{|f|} \{f, g\} \quad (4.162)$$

$$\implies \Delta_\mu e^{i/\hbar S} = (-i\hbar)^{-2} \left(\frac{1}{2}\{S, S\} - i\hbar\Delta_\mu S \right) e^{i/\hbar S} \quad (4.163)$$

Note that we say that $(\mathcal{C}^\infty(\mathcal{F}), \Delta_\mu, \{\cdot, \cdot\})$ is a BV algebra if Δ_μ is a derivation of the bracket. Sometimes it is convenient to think of power series in $(-i\hbar)$ instead of just \hbar . Thus

$$S = S^{(0)} + (-i\hbar)S^{(1)} + \dots \quad (4.164)$$

and (QME) is equivalent to the following set of equations

$$\begin{aligned} \{S^{(0)}, S^{(0)}\} &= 0 & (CME) \\ \{S^{(1)}, S^{(1)}\} + \Delta_\mu S^{(0)} &= 0 \\ \{S^{(2)}, S^{(2)}\} + \frac{1}{2}\{S^{(1)}, S^{(1)}\} + \Delta_\mu S^{(1)} &= 0 \\ \dots \end{aligned}$$

Now starting from a solution of the Classical Master Equation (CME) for classical BV data, we can extend it to a power series in \hbar that satisfies (QME)

$$\Delta_\mu S^{(0)} = -\{S^{(0)}, S^{(1)}\} \quad (4.165)$$

the class of $\Delta_\mu S^{(0)}$ in the $Q := \{S^{(0)}, \cdot\}$ cohomology to be trivial (i.e. $\Delta_\mu S^{(0)} = Q$ -exact). Note that Q is a differential by assumption. We can find an appropriate $S^{(1)}$ such that

$$\frac{1}{2}\{S^{(1)}, S^{(1)}\} + \Delta_\mu S^{(1)} = -\{S^{(0)}, S^{(0)}\} \quad (4.166)$$

Now the question is if there is a Q -exact $S^{(1)}$ such that the above equation holds. There are indeed obstructions to this but in many cases this has a chance to work.

DEFINITION 4.62 If S and S' are two solutions of the (QME) they are said to be equivalent if there exists a **BV canonical transformation**, i.e. a family $S_t \in \mathcal{C}^\infty(\mathcal{F})^{(0)}[[\hbar]]$, $R_t \in \mathcal{C}^\infty(\mathcal{F})^{(-1)}[[\hbar]]$ such that $S_0 = S$, $S_1 = S'$ and

$$\frac{d}{dt}S_t = \{S_t, R_t\} - i\hbar\Delta_\mu R_t \quad (\blacksquare)$$

If we denote $\{S_t, \cdot\} - i\hbar\Delta_\mu =: \delta_t$, a Quantum BV operator for S_t , then we can express (\blacksquare) as

$$\dot{S}_t = \delta_t R_t \quad (4.167)$$

Remark 4.63 The Koszul-Tate condition eliminates cohomology in negative degrees. The problem of extending $CME \rightarrow QME$ however is related to cohomology in positive degrees. Now in field theory, thus when working with ∞ -dimensional manifolds, the requirement that BV^\bullet has vanishing negative cohomology is a quite strong one which can hinder the above transition. A strictly better requirement is that $\dim(H^{-i}) < +\infty$.

Let us look at (QME) for the family S_t :

$$\frac{1}{2}\{S_t, S_t\} - i\hbar\Delta_\mu S_t = 0 \quad (QME_t)$$

and thus in particular

$$\delta_t(\dot{S}_t) = 0 \quad (4.168)$$

LEMMA 4.64 The Quantum BV operator

$$\delta_S := \{S, \cdot\} - i\hbar\Delta_\mu = -i\hbar e^{-i/\hbar S} \Delta_\mu (e^{i/\hbar S} \cdot) \quad (4.169)$$

squares to zero iff S satisfies the (QME)

Thus we see from (QME_t) that $\delta_t^2 = 0$ which means that δ_t is a differential and further that \dot{S}_t is δ_t -closed. Additionally \dot{S}_t is δ_t -exact due to (■) which implies that

$$\frac{d}{dt}[(QME_t)] = 0 \quad (4.170)$$

if (QME_t) is satisfied. Hence it is satisfied for all t if S_t, R_t parametrise a canonical transformation.

PROPOSITION 4.65 $\frac{d}{dt}e^{i/\hbar S_t} = \Delta_\mu(-i\hbar e^{i/\hbar S_t} R_t)$ and for $S \sim S'$ we have $e^{i/\hbar S} - e^{i/\hbar S'} = \Delta_\mu\left(-i\hbar \int_0^t dt e^{i/\hbar S_t} R_t\right)$

DEFINITION 4.66 Define $\sigma := S_t + dt R_t$ such that $\sigma \in \Omega^\circ([0, 1]) \otimes \mathcal{C}^\infty(\mathcal{F})[[\hbar]]$. Then we have

$$\left(dt \wedge \frac{d}{dt} - i\hbar\Delta_\mu\right) e^{i/\hbar \sigma} = 0 \quad (\text{extQME})$$

This equation is sometimes referred to as the "extended QME" (Mnëv) or as "homotopies of solutions of the QME" (Costello).

This brings us to the following central definition:

DEFINITION 4.67 (Quantum BV Formalism) A finite-dimensional **Quantum BV theory** is specified by the following set of data:

1. A \mathbb{Z} -graded manifold \mathcal{F}
2. A (-1) -symplectic form Ω on \mathcal{F}
3. A Berezinian $\mu \in BER(\mathcal{F})$, compatible with Ω
4. A function $S \in \mathcal{C}^\infty(\mathcal{F})[[\hbar]]$ satisfying the (QME) (which is equivalent to $\delta_S := \{S, \cdot\}$ being a differential)

Remark 4.68 Note that the Hamiltonian vector field X_S for the **full** BV quantum master function S does **not** necessarily square to zero. However δ_S does. However we know that $\delta_S \bmod \hbar = X_{S(0)} =: Q_0$ will square to zero which is nothing but the (CME). Thus we always have a classical BV theory.

Given a Quantum BV theory we define the partition function to be

$$\mathcal{Z} := \int_{\mathcal{L} \subset \mathcal{F}} e^{i/\hbar S} \sqrt{\mu|_{\mathcal{L}}} \quad (4.171)$$

where \mathcal{L} is a Lagrangian submanifold of \mathcal{F} . Now since $\Delta_\mu e^{i/\hbar S} = 0$ the value of \mathcal{Z} does not depend on a particular choice of \mathcal{L} inside the homotopy class $[\mathcal{L}]$.

Now if $\mathcal{F} = T^*[-1]F_{min}$ then S is some lift of a classical action S_{cl} on F_{min} . The zero section $\mathcal{L} \equiv F_{min} \subset \mathcal{F}$ is a valid Lagrangian submanifold but $\mathcal{H}(S)$ is degenerate. Thus we deform $\mathcal{L}_0 \sim \mathcal{L}$ such that $\mathcal{H}(S)$ is no longer degenerate and compute the integral here. This sums up the concept of gauge-fixing.

Remark 4.69 Consider $\mathcal{O} \in \mathcal{C}^\infty(\mathcal{F})[[\hbar]]$. \mathcal{O} is an observable iff $\delta_S \mathcal{O} = 0$. Then we can compute its expectation value to be

$$\langle \mathcal{O} \rangle := \frac{1}{2} \int_{\mathcal{L} \subset \mathcal{F}} \mathcal{O} e^{i/\hbar S} \sqrt{\mu|_{\mathcal{L}}} \quad (4.172)$$

However a nontrivial fact is that this notion does not behave well under product because δ_S is not a derivation of the product of functions. The arising problems and the above definition are subject to current research.

If we again take the BV-BRST data from above, thus $\mathcal{F} := T^*[-1]F_{BRST}$ and $S_{BV} = p^*S_{cl} + d_{BRST}$ together with a function $\psi \in \mathcal{C}^\infty(F_{BRST})^{(-1)}$ we define

$$\mathcal{L}_\psi := \text{graph}(d\psi) \quad (4.173)$$

to obtain

$$S_{BV}|_{\mathcal{L}_\psi} = S_{BV}(\Phi^a, \Phi_a^\dagger = \frac{\partial}{\partial \Phi^a} \psi) = S_0 + \sum_a d_{BRST}^a \frac{\partial}{\partial \Phi^a} \psi \quad (4.174)$$

$$= S_0 + d_{BRST} \psi = S_{gf} \quad (4.175)$$

$$\int_{\mathcal{L}_\psi \subset T^*[-1]F_{BRST}} e^{i/\hbar S_{BV}} \sqrt{\mu|_{\mathcal{L}_\psi}} = \int_{F_{BRST}} e^{i/\hbar S_{gf}} D\Phi \quad (4.176)$$

Remark 4.70 To look at a ψ -dependent gauge-fixing in BRST we need the nonminimal extension F_{Aux} which is the same as adding $T^*[-1]F_{Aux}$ to $T^*[-1]F_{min}$. In principle in BV we can directly look at Lagrangian submanifolds in $T^*[-1]F_{min}$. Thus we have no further need for this nonminimal

extension.

4.7 BV Fibre Integrals

Let (\mathcal{F}', ω') and $(\mathcal{F}'', \omega'')$ two (-1) -symplectic manifold and $\mathcal{F} := \mathcal{F}' \times \mathcal{F}''$ with $\omega := \omega' \otimes 1 + 1 \otimes \omega'' \in \Omega^\bullet(\mathcal{F}') \otimes \Omega^\bullet(\mathcal{F}'')$.

DEFINITION 4.71 (BV Fibre Integral) IF $\mathcal{L} \subset \mathcal{F}''$ is a Lagrangian submanifold and if μ'' is a Berezinian on \mathcal{F}'' we have a map

$$P_*^{(\mathcal{L})} := \int_{\mathcal{L} \subset \mathcal{F}''} \cdot \sqrt{\mu''|_{\mathcal{L}}}: \mathcal{C}^\infty(\mathcal{F}) \longrightarrow \mathcal{C}^\infty(\mathcal{F}') \quad (4.177)$$

which is called a **BV fibre integral** or a **BV pushforward**.

THEOREM 4.72 (Stokes for BV fibre integrals)

1. $P_*^{(\mathcal{L})}$ is a chain map intertwining the BV Laplacians Δ_μ on \mathcal{F} and $\Delta_{\mu'}$ on \mathcal{F}' , i.e.

$$\Delta_{\mu'} P_*^{(\mathcal{L})} = P_*^{(\mathcal{L})} \Delta_\mu \quad (4.178)$$

where $\mu = \mu' \cdot \mu''$.

2. If $\mathcal{L} \sim \mathcal{L}'$ are Lagrangian submanifolds in \mathcal{F}'' and $\rho \in \mathcal{C}^\infty(\mathcal{F})$ such that $\Delta_\mu \rho = 0$ then

$$P_*^{(\mathcal{L})} \rho - P_*^{(\mathcal{L}')} \rho \quad (4.179)$$

is $\Delta_{\mu'}$ -exact in \mathcal{F}' .

DEFINITION 4.73 (Effective BV Actions) Let $S \in \mathcal{C}^\infty(\mathcal{F})[[\hbar]]$ be a solution of the (QME) on \mathcal{F} . Then we call $S' \in \mathcal{C}^\infty(\mathcal{F}')[[\hbar]]$ the **effective BV action** for S induced on \mathcal{F}' (given \mathcal{L}) iff

$$e^{i/\hbar S'} = P_*^{(\mathcal{L})} (e^{i/\hbar S}) \quad (4.180)$$

COROLLARY 4.74

1. If S is a solution of the (QME) on \mathcal{F} then S' is a solution of the (QME) on \mathcal{F}' .
2. If S is a solution of the (QME) on \mathcal{F} and $\mathcal{L} \sim \mathcal{L}'$ in \mathcal{F}'' then the corresponding effective actions are equivalent (i.e. related by a canonical transformation) (w.r.t. $\Delta_{\mu'}$).
3. IF $S \sim S'$ are two solutions of the (QME) (equivalent by means of a canonical transformation), then their respective effective actions w.r.t. the same Lagrangian submanifold \mathcal{L} are also equivalent.

In short the BV fibre integral defines a map

$$\mathrm{Sol}(QME(\mathcal{F}))/ \sim \xrightarrow{P_*^{[\mathcal{L}]}} \mathrm{Sol}(QME(\mathcal{F}'))/ \sim \quad (4.181)$$

where $[\mathcal{L}]$ is a homotopy class of Lagrangians in \mathcal{F}'' . We are going to use this fibre integral to interpret the renormalisation group flow/Wilson's effective action picture. Summing up we want a "tower" of theories

$$(F, S) \longrightarrow \dots \longrightarrow (F_\Lambda, S_\Lambda) \xrightarrow{P_*^{\Lambda\Lambda'}} (F_{\Lambda'}, S_{\Lambda'}) \longrightarrow \dots \longrightarrow (F_0, S_0) \quad (4.182)$$

where on the left we have a local theory, in the middle we have finite-energy theories and on the right we have the zero modes. This will lead us to

$$P_*^{\Lambda\Lambda'} e^{i/\hbar S_\Lambda(\varphi)} = \int_{[\Lambda, \Lambda']} D\tilde{\varphi} e^{i/\hbar S(\varphi + \tilde{\varphi})} = e^{i/\hbar S_{\Lambda'}} \quad (4.183)$$

Remark 4.75 Note that $P_*^{[\mathcal{L}]}$ is better thought of as an operator on Berezinians or even better on 1/2-densities

$$\mathcal{C}^\infty(\mathcal{M}) \xrightarrow{\sqrt{\mu}} \mathrm{Den}^{1/2}(\mathcal{M}) = \Gamma(\mathcal{M}, \mathrm{BER}(\mathcal{M})^{\otimes 1/2}) \quad (4.184)$$

4.8 Field Theory

Our goal in this chapter is to construct an ∞ -dimensional generalisation of the BV construction. To this end consider the data of a fibre bundle $F \longrightarrow M$ where $F = \bigtimes_{i=-\infty}^{\infty} F_i$ such that $F_i \longrightarrow M$ is a vector bundle for all $i \neq 0$ and $F_0 \longrightarrow M$ is just a fibre bundle.

Now the space of BV fields is given by the sections $\mathcal{F} = \Gamma(M, F)$ in the sense of classical field theory. Thus we know how to make sense of $J^\infty F$, $\mathcal{C}^\infty(J^\infty F)$, $\mathcal{C}_{loc}^\infty(\mathcal{F})$ and so on.

Remark 4.76 An important caveat in this setting is the definition of **duals**. We want to define $T^*[-1]\mathcal{E}$ where $\mathcal{E} = \Gamma(M, E)$. For simplicity we assume E to be a vector bundle. The symbol $*$ in E^* is used to denote the vector bundle $E^* := E^\vee \otimes \mathrm{Dens}(M) \longrightarrow M$ where E^\vee is the fibrewise dual. Thus define

$$\mathcal{E}^* := \Gamma(M, E^*), \quad T^*[-1]\mathcal{E} := \mathcal{E} \times \mathcal{E}^*[-1] \quad (4.185)$$

Alternatively one can use distributional sections and then consider the "strong dual" of the space of sections \mathcal{E} . This ansatz is chosen in *Perturbative Algebraic QFT*. With a little work one can extend the above ansatz to any fibre bundle $E \longrightarrow M$ by dualising TE instead.

The above remark lets us conclude the following: **Anti-fields** are "coordinates" in the cotangent fibre of $T^*[-1]\mathcal{E}$ where $\mathcal{E} = \Gamma(M, E)$. Thus $\varphi^\dagger \in \Gamma(M, E^\vee \otimes \mathrm{Dens}(M)) = \Gamma(M, E^*)$. Notice that

$\mathcal{E}^* = \Gamma(M, E^*)$. Now there exists a nondegenerate (weak) pairing

$$T_\varphi \mathcal{E} \otimes T_\varphi^* \mathcal{E} \longrightarrow \mathbb{C}/\mathbb{R} \quad (4.186)$$

$$(v_\varphi, \alpha_\varphi) \longmapsto \int_M \langle v_\varphi, \alpha_\varphi \rangle \quad (4.187)$$

where $\langle \cdot, \cdot \rangle$ is a $Dens(M)$ -valued fibrewise pairing. This can be written as the local 2-form

$$\Omega = \int_M \langle \delta\varphi, \delta\varphi^\dagger \rangle \quad (4.188)$$

Then given a local Lagrangian field theory (M, F, L) with $F \longrightarrow M$ a fibre bundle and $L \in \Omega_{loc}^{0, top}(\mathcal{F} \times M)$ together with a set of local symmetries $\{V_a\}$ for now assumed to originate from a Lie algebra action

$$\rho: \mathfrak{g} \longrightarrow \mathfrak{X}_{loc}(\mathcal{F}) \quad \text{for some Lie group } G \quad (4.189)$$

$$\text{im}(\rho) \simeq \Gamma(M, \mathfrak{g}) \quad M \times \mathfrak{g} \longrightarrow M \quad (4.190)$$

Note that this is a simplification, the generalisation works analogously to the previously discussed cases! In this setting the space of BV fields is the space of smooth section of $T^*[-1](F \times \mathfrak{g}[1])$, i.e. $\psi \in \Gamma(M, F)$, $\psi^\dagger \in \Gamma(M, T^*F \otimes Dens(M))$, $c \in \Gamma(M, \mathfrak{g}[1])$, $c^\dagger \in \Gamma(M, \mathfrak{g}^*[-2] \otimes Dens(M))$. Then Ghost fields c are functions on M with values in the Lie algebra. For an even basis $\{t_a\}$ we have

$$c = c^a(x)t_a \quad |c^a| = 1 \quad |t_a| = 0 \quad |c| = 1 \quad (4.191)$$

Now we go on and define the rest of the BV data

$$\Omega_{BV} = \int_M \langle \delta\varphi, \delta\varphi^\dagger \rangle = \int_M \langle \delta\psi, \delta\psi^\dagger \rangle + \langle \delta c^a, \delta c_a^\dagger \rangle \quad (4.192)$$

The BV action functional, if $S = \int_M L$ is given by

$$S_{BV} = p^*S + \int_M \langle \varphi^\dagger, Q_{CE}\varphi \rangle \quad (4.193)$$

where $p: \mathcal{F}_{BV} \longrightarrow \mathcal{F} \times \Gamma(M, \mathfrak{g}[1])$ for $\mathcal{F}_{BV} = T^*[-1](\mathcal{F} \times \Gamma(M, \mathfrak{g}[1]))$. Further Q_{CE} is a local (evolutionary) vector field on \mathcal{F}_{CE} such that

$$Q_{CE}(\psi) = c^a V_a(\psi) \quad Q_{CE}(c) = \frac{1}{2}[c, c] \quad (4.194)$$

EXAMPLE 4.77 (Yang–Mills theory) Let (M, g) be a 4-dimensional Riemannian manifold and let \mathfrak{g} be a semisimple Lie algebra with an invariant (a.k.a. cyclic) inner product $\langle \cdot, \cdot \rangle$. Here "invariant" means

$$\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle \quad \forall X, Y, Z \in \mathfrak{g} \quad (4.195)$$

Further assume that we have a trivial principal bundle $P \rightarrow M$ such that the connections are identified with \mathfrak{g} -valued 1-forms $\mathcal{A} \simeq \Omega^1(M, \mathfrak{g}) \simeq \Omega^1(M) \otimes \mathfrak{g}$ thus $A \in \mathcal{A}$ can be written as $A = A_i^a dx^i t_a$.

The "gauge group" $G = \mathcal{C}^\infty(M, G)$ acts on \mathcal{A} by $(g, A) \mapsto A^g := g^{-1}Ag + g^{-1}dg$ and infinitesimally we have the Lie algebra action

$$\rho: (X, A) \mapsto A + d_A X = A + dX + [A, X] \quad (4.196)$$

where $X \in \text{Lie}(G) \simeq \mathcal{C}^\infty(M, \mathfrak{g})$ and $d_A X \in \Omega^1(M, \mathfrak{g})$. Now the Yang-Mills action functional is

$$S_{YM_2} = \int \frac{1}{2} \langle F_A, *F_A \rangle_{\mathfrak{g}} \quad (4.197)$$

Note that $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ extends to a pairing on $\Omega^\bullet(M) \otimes \mathfrak{g}$ by denoting

$$\langle \omega_1 \otimes E_1, \omega_2 \otimes E_2 \rangle = \int \omega_1 \wedge \omega_2 \langle E_1, E_2 \rangle_{\mathfrak{g}} \quad (4.198)$$

LEMMA 4.78 S_{YM_2} is invariant under the Lie algebra action ρ .

Proof. Consider the vector field $d_A X \frac{\delta}{\delta A}$ representing $D_X(A) = d_A X$ on the algebra of functions. Now recall that $F_A = dA + \frac{1}{2}[A, A]$ such that

$$D_X(F_A) = d(d_A X) + [A, d_A X] = d[A, X] + [A, dX][A, [A, X]] \quad (4.199)$$

$$= [dA, X] + \frac{1}{2}[[A, A], X] = [F_A, X] \quad (4.200)$$

But then we can compute

$$\int D_X (\langle F_A, *F_A \rangle) = \int \langle [F_A, X], *F_A \rangle + \langle F_A, [*F_A, X] \rangle = 0 \quad (4.201)$$

Thus the pairing is indeed invariant. \square

Now we "promote" X to a ghost field C and look at $T^*[-1](\times \text{Lie}(G)[1])$ with fields $((A, A^\dagger), (C, C^\dagger))$. Then

$$\Omega_{BV} = \int_M \langle \delta A, \delta A^\dagger \rangle + \langle \delta C, \delta C^\dagger \rangle \quad (4.202)$$

where we tacitly identified $\mathfrak{g} \rightarrow \mathfrak{g}^*$ in the first pairing. Thus we arrive at the following BV action:

$$S_{BV} = \int \frac{1}{2} \langle F_A, *F_A \rangle + \langle A^\dagger, d_A C \rangle + \frac{1}{2} \langle C^\dagger, [C, C] \rangle \quad (4.203)$$

Assuming $\partial M = \emptyset$ we can compute $\iota_{Q_{BV}}\Omega_{BV} = \delta S_{BV}$ as well as

$$Q_{BV}A = d_A C \quad (4.204)$$

$$Q_{BV}C = \frac{1}{2}[C, C] \quad (4.205)$$

$$Q_{BV}A^\dagger = d_A * F_A + [C, A^\dagger] \quad (4.206)$$

$$Q_{BV}C^\dagger = -d_A A^\dagger - [C, C^\dagger] \quad (4.207)$$

Proof.

$$\iota_Q \Omega = \int \langle Q_A, \delta A^\dagger \rangle + \langle Q_{A^\dagger}, \delta A \rangle + \langle Q_C, \delta C^\dagger \rangle + \langle Q_{C^\dagger}, \delta C \rangle \quad (4.208)$$

$$\delta S_{BV} = \int \langle d_A * F_A, \delta A \rangle + \langle \delta A^\dagger, d_A C \rangle - \langle A^\dagger, [\delta A, C] \rangle \quad (4.209)$$

$$+ \langle A^\dagger, d_A \delta C \rangle + \frac{1}{2} \langle \delta C^\dagger, [C, C] \rangle - \langle C^\dagger, [C, \delta C] \rangle \quad (4.210)$$

$$= \int \langle d_A * F_A, \delta A \rangle + \langle d_A C, \delta A \rangle + \langle [A^\dagger, C], \delta A \rangle \quad (4.211)$$

$$- \langle d_A A^\dagger, \delta C \rangle + \frac{1}{2} \langle [C, C], \delta C^\dagger \rangle - \langle [C^\dagger, C], \delta C \rangle \quad (4.212)$$

$$(4.213)$$

□

Remark 4.79 Note that for signs we use the total degree:

$$td(A) = f.d. + g.h. = 1, \quad td(C) = f.d. + g.h. = 1 \quad (4.214)$$

$$\implies [A, C] = [C, A], \quad [C, C] \neq 0 \quad (4.215)$$

Further note that $td(\delta) = 1$ and $td(d) = 1$ such that $td(dA) = 2 = td(\delta A)$.

For clarity:

$$\mathcal{F}_{YM} = \Omega^1(M) \otimes \mathfrak{g} \times \Omega^0(M) \otimes \mathfrak{g} \times \Omega^3(M) \otimes \mathfrak{g}[-1] \times \Omega^{top}(M) \otimes \mathfrak{g}[-2] \quad (4.216)$$

where $T^*(\Omega^0(M) \otimes \mathfrak{g})$ such that $\langle \varphi, \varphi^\dagger \rangle \in Dens(M)$. Now we can turn to a first order formulation of Yang-Mills theory:

$$\mathcal{F}_{YM_1} = T^*[-1] (\Omega^1(M) \otimes \mathfrak{g} \times \Omega^2(M) \otimes \mathfrak{g} \times \Omega^0(M) \otimes \mathfrak{g}[-1]) \quad (4.217)$$

$$\Omega_{YM_1} = \int \langle \delta A, \delta A^\dagger \rangle + \langle \delta B, \delta B^\dagger \rangle + \langle \delta C, \delta C^\dagger \rangle \quad (4.218)$$

Further we get the following action:

$$S_{YM_1} = \int \langle B, F_A \rangle + \frac{1}{2} \langle B, *B \rangle + \langle A^\dagger, d_A C \rangle + \langle B^\dagger, [C, B] \rangle + \left\langle C^\dagger, \frac{1}{2} [C, C] \right\rangle \quad (4.219)$$

And thus $F_A = *B$ and $d_A B = 0$ which yields $d_A * F_A = 0$. Now the Lie algebra action on $\Omega^2(M) \otimes \mathfrak{g}$ is given by $B \mapsto [X, B]$. Thus BV is promoted to $QB = [C, B]$.

Remark 4.80 As a graded vector space \mathcal{F}_{YM_1} is

EXAMPLE 4.81 (Chern Simons theory) Let M be a 3-dimensional oriented manifold with a trivial principal G -bundle $P \rightarrow M$ over it. Then note $\mathfrak{g} = \text{Lie}(G)$ such that $\mathcal{A} \simeq \Omega^1(M) \otimes \mathfrak{g}$. Now the action is given by

$$S_{CS}^0 = \int \frac{1}{2} \langle A, dA \rangle + \frac{1}{6} \langle A, [A, A] \rangle \quad (4.220)$$

where $F_{CS} = \simeq \Omega^1(M) \otimes \mathfrak{g}$. Now we get

$$\delta L_{CS} = \frac{1}{2} \langle \delta A, dA \rangle + \frac{1}{2} \langle A, d\delta A \rangle + \frac{1}{6} \langle \delta A, [A, A] \rangle + \frac{1}{3} \langle A, [A, \delta A] \rangle \quad (4.221)$$

$$= \langle \delta A, F_A \rangle - d \left(\frac{1}{2} \langle A, \delta A \rangle \right) \quad (4.222)$$

The latter term gives us a symplectic form on (Σ, \mathfrak{g}) while the former gives the equation of motion for CS theory. When $F_A = 0$ we have **flat connections**.

EXERCISE 4.82 The usual Lie algebra action is $A \mapsto A + d_A X$. Show that S_{CS} is invariant under this action. Further compute the boundary term: $L_V(L_{CS}) = d(-)$.

Now we can define

$$S_{CS} = S_{CS}^0 + \int_M \langle A^\dagger, d_A C \rangle + \left\langle C^\dagger, \frac{1}{2} [C, C] \right\rangle \quad (4.223)$$

which, by defining $\mathbb{A} := (C, A, A^\dagger) \in \Omega^\bullet(M) \otimes \mathfrak{g}[1 - \bullet]$, can be rewritten as

$$S_{CS} = \int \frac{1}{2} \langle \mathbb{A}, d\mathbb{A} \rangle^{top} + \frac{1}{6} \langle \mathbb{A}, [\mathbb{A}, \mathbb{A}] \rangle^{top} \quad (4.224)$$

Now the question is, how we can extend out previous quantum considerations into the BV setting for field theory. To this end, consider a gauge-fixing Lagrangian submanifold as the image of a "gauge-fixing operator" [3].

DEFINITION 4.83 A **gauge-fixing operator** on a free BV theory S^0 is a quadratic functional given by

$$Q^{GF}: \mathcal{F} \rightarrow \mathcal{F} \quad (4.225)$$

such that the following conditions holds:

1. $|Q^{GF}| = -1$, $[Q^{GF}, Q^{GF}] = 0$ and it is self adjoint with respect to the pairing $\langle \cdot, \cdot \rangle$.
2. The commutator $[Q, Q^{GF}] =: D$ is a generalised Laplacian.

Remark 4.84 Scalar field theory can be formulated within the BV framework. Just take

$$\mathcal{F} = T^*[-1]\mathcal{C}^\infty(M) \simeq \mathcal{C}^\infty(M) \oplus \mathcal{C}^\infty(M)[-1] \otimes \text{Dens}(M) \quad (4.226)$$

with the following action:

$$S_{BV} = S_0 = \frac{1}{2} \int_M d\varphi \wedge *_g d\varphi \quad (4.227)$$

$$Q_{BV} \equiv d_K \implies Q_{BV}\varphi^\dagger = d *_g d\varphi \equiv \Delta_g \varphi \quad (4.228)$$

$$Q_{BV}: \mathcal{F}^{(0)} \longrightarrow \mathcal{F}^{(1)} \quad (4.229)$$

Thus the gauge-fixing operator is the identity seen as a map $Q^{GF}: \mathcal{F}^{(1)} \longrightarrow \mathcal{F}^{(0)}$ such that $[Q, Q^{GF}] = \Delta_g$. The Lagrangian submanifold $\mathcal{L}_0 \cong \mathcal{C}^\infty(M) \longleftrightarrow \mathcal{F}$ is obtained as $\text{im}(Q^{GF})$. Integrating on a BV path integral on this Lagrangian submanifold is the same as quantising scalar fields in the sense that we have previously seen.

4.8.1 Gauge fixing in CS and YM_1

THEOREM 4.85 (Hodge decomposition) Let (M, g) be a smooth Riemannian manifold. Then there exists an orhtogonal decomposition

$$\Omega^k(M) = d\Omega^{k-1}(M) \oplus \mathcal{H}^k(M) \oplus d^*\Omega^{k+1}(M) \quad (4.230)$$

where $\mathcal{H}^k(M) = \{\alpha \in \Omega^k(M) | \Delta_g \alpha = 0\}$. If M is compact, $\mathcal{H}^k(M)$ is the k -th de Rham cohomology group of M , namely $\mathcal{H}^k(M) = H_{dR}^k(M, d)$.

Remark 4.86 If M is such that $H_{dR}^k(M, d) = 0$ for any $k > 0$ then

$$\omega \text{ closed} \iff \omega \text{ exact} \quad (4.231)$$

$$\omega \text{ coclosed} \iff \omega \text{ coexact} \quad (4.232)$$

Further we have

$$d^* := *d*: \Omega^{k+1}(M) \longrightarrow \Omega^k(M) \quad (4.233)$$

$$\Delta_g \equiv (d + d^*)^2 = dd^* + d^*d \quad (4.234)$$

PROPOSITION 4.87 Assume $H_{dR}^k(M, d) = 0$ for any $k > 0$ and M compact. Then $\ker(d^*) \subset \Omega^0(M) \otimes \mathfrak{g}$ is a Lagrangian submanifold.

Proof. This is rather a sketch of the proof: For a submanifold to be Lagrangian, it needs to be isotropic with an isotropic complement. Thus if $W = V \oplus V'$ we have V and V' isotropic. Now $\mathcal{L} = \ker(d^*)$ is isotropic, i.e. Ω_{BV} vanishes on \mathcal{L} . Thus for $\omega_2 = d^*\alpha_2$ we obtain

$$\langle \omega_1, \omega_2 \rangle|_{\ker(d^*)} = \langle \omega_1, d^*\alpha_2 \rangle|_{\ker(d^*)} = \pm \langle d^*\omega_1, \alpha_2 \rangle|_{\ker(d^*)} = 0 \quad (4.235)$$

The same argument holds for d -exact and d -closed forms. Thus there exists a decomposition $\Omega^k(M) = d\text{-exact} \oplus d^*\text{-exact}$ into isotropic subspaces. \square

Remark 4.88 When looking at manifolds with nontrivial cohomology, this becomes very complicated since $\Omega^\bullet(M) = d\Omega \oplus d^*\Omega \oplus \mathcal{H}$ where the first two spaces harbour the path integral and the third the "effective fields".

A good gauge-fixing operator for CS -theory is precisely d^* since $d^{*2} = 0$, $|d^*| = -1$, $d^*: \Omega^k \rightarrow \Omega^{k-1}$ and $[Q_{BV}, d^*] = \Delta_g$ because $Q \equiv d$. Note that $Q_0 A = dC + [A, C]$. This sloppiness comes from the fact that Costello [3] defines a "free" BV operator as given by the quadratic part of the BV action:

$$S_{CS} = \int \frac{1}{2} \langle A, dA \rangle + \langle A^\dagger, dC \rangle + \frac{1}{6} \langle A, [A, A] \rangle + \langle A^\dagger, [A, C] \rangle \quad (4.236)$$

Remark 4.89 Chern Simons theory in principle does **not** require a specification of any metric in order to be defined and work. Thus it classifies as a so-called "Topological Theory" since it is independent of a metric. When working with gauge-fixing however, we need to fix a metric, even in this theory. Thus one needs to ask in how far the gauge-fixing procedure is independent of the choice of a metric. In his 1989 paper Witten has shown that one can remove "most" of the dependence on g . However it remains an open question if it is completely independent. Current belief is that it should be, although a full proof has yet to be developed. In YM_1 theory one can use the same gauge fixing using the same Riemannian structure (M, g) .

NOTATION 4.90 One typically calls $\ker(d^*)$ the **Lorenz gauge** (namely thinking of $\ker(d^*)$ as an isotropic subspace of the fields) when one chooses $d^*A = 0$, thus $A \in \ker(d^*)$. This is typically written as $*d*A \rightarrow \partial_\mu A^\mu = 0$, a form well-known from electrodynamics.

4.8.2 Quantum BV field theory

Classical BV field theories make perfect sense with minimal adaptations from the finite-dimensional case. This however turns out to be quite more complex for a quantum formulation. First remember that

$$\Delta_\mu f = -\frac{1}{2} \operatorname{div}_\mu(X_f) \quad (4.237)$$

However this *requires* a Berezinian which in infinite dimensions is generally not defined! Now there exist several types of "regularisation attempts" to define Δ_μ . Some try to regularise the heat kernel [3] while other probe triangulations/decompositions of manifolds into cells (a kind of finite-

dimensional filter) [?, ?]. Again others start from local functionals and deform the product among functionals. As a "byproduct" they obtain " Δ ". This ansatz is called "Deformation Quantisation" and can be found in [?]. In this subchapter we will only consider the first ansatz. Note that this is a very active field of research and while some approaches are more widespread, there is none that is considered the "right" approach.

Consider $\mathcal{E} = \Gamma(M, E)$ where $E \rightarrow M$ is a vector bundle, graded and endowed with a (-1) -symplectic pairing $\langle \cdot, \cdot \rangle$ such that $\Omega_{BV}(V, W) = \langle V, W \rangle$.

DEFINITION 4.91 Given any element $K \in \mathcal{E} \otimes \mathcal{E}$ we define a convolution

$$K * : \mathcal{E} \rightarrow \mathcal{E} \quad (4.238)$$

such that $\forall \psi \in \mathcal{E}$ we have

$$K * \psi := (-1)^{|K|} (1 \otimes \langle \cdot, \cdot \rangle) (K \otimes \psi) \quad (4.239)$$

Thus if $K = K_1 \otimes K_2$ we have $K * \psi = (-1)^{|K|} K_1 \langle K_2, \psi \rangle$

Now we denote $\tilde{Q} = Q \otimes 1 + 1 \otimes Q$ on $\mathcal{E}^{\otimes 2}$. This leads us to

LEMMA 4.92 For K as before we have $(\tilde{Q}K) * \psi \equiv [Q, K *] \psi$.

Proof.

$$((Q \otimes 1 + 1 \otimes Q)K) * \psi \quad (4.240)$$

$$= (QK_1 \otimes K_2 + (-1)^{|K_1|} K_1 \otimes QK_2) * \psi \quad (4.241)$$

$$= (-1)^{|K|} QK_1 \langle K_2, \psi \rangle + (-1)^{|K|} (-1)^{|K_1|} K_1 \langle QK_2, \psi \rangle \quad (4.242)$$

$$= Q \cdot (K * \psi) - (-1)^{|K|} K * Q\psi = \dots \quad (4.243)$$

$$= (-1)^{|K|} QK_1 \langle K_2, \psi \rangle + (-1)^{|K|} (-1)^{|K_1|} K_1 \langle QK_2, \psi \rangle \quad (4.244)$$

□

Now assume that we are given a gauge-fixing operator Q^{GF} such that $D = [Q, Q^{GF}]$ is our generalised Laplacian. This is sufficient to make sense of the heat kernel of e^{-lD} . Let $K_l \in \mathcal{E} \otimes \mathcal{E}$ be defined as $K_L * \psi = \exp(-lD)\psi$.

DEFINITION 4.93 The (gauge-fixed) propagator is defined as

$$P(\varepsilon, L) := \int_{\varepsilon}^L (Q^{GF} \otimes 1) K_l \, dl \quad (4.245)$$

such that $P(\varepsilon, L) * \psi = Q^{GF} \int_{\varepsilon}^L e^{-lD} \psi \, dl$. Thus it lands in (Q^{GF}) .

Now recall that given an element $K \in \mathcal{E}^{\otimes 2}$ we can define a second-order operator as

$$\partial_K := \frac{\partial}{\partial K_1} \otimes \frac{\partial}{\partial K_2}, \quad \text{where } K = K_1 \otimes K_2 \quad (4.246)$$

DEFINITION 4.94 The scale- l BV Laplacian is defined as

$$\Delta_L := -\partial_{K_l} : \mathcal{C}^\infty(\mathcal{E}) \longrightarrow \mathcal{C}^\infty(\mathcal{E}) \quad (4.247)$$

DEFINITION 4.95 A functional $S \in \mathcal{C}^\infty(\mathcal{E})[[\hbar]]$ satisfies the scale- L (QME) iff

$$\Delta_L \exp(1/\hbar S) = 0 \quad \Longleftrightarrow \quad \frac{1}{2}\{S, S\}_L + \hbar \Delta_L S = 0 \quad ([QME]_L)$$

where

$$\{f, g\} := (-1)^{|f|} \Delta_L(fg) - (-1)^{|f|} \Delta_L f \cdot g - f \Delta_L g \quad (4.248)$$

This means that we have a scale- L BV algebra.

Note that, in order to align with the respective parts in Costello [3], we did not choose a complex phase for the appearing exponentials. They are purely real.

It is often convenient to split S into a *free* (quadratic) part S_0 and an *interaction* (cubic) part I such that $S = S_0 + I$. Now choosing $Q_0 := \{S_0, \cdot\}$ we have a *classic free BV operator*. Thus we can reformulate the definition of the (QME) for interaction terms. I satisfies ($[QME]_L$) iff

$$(Q_0 + \Delta_L) \exp(1/\hbar I) = 0 \quad \Longleftrightarrow \quad Q_0 I + \frac{1}{2}\{I, I\}_L + \hbar \Delta_L I = 0 \quad (4.249)$$

DEFINITION 4.96 (Pre-Theories) A **Pre-Theory** is a collection of effective interactions $\{F[L]\}$ such that the following conditions hold:

1. Each $I \in \mathcal{C}^\infty(\mathcal{E})[[\hbar]]$ is of degree 0 and at least cubic modulo \hbar .
2. The *RGE* $I[L] = W(P(\varepsilon, L), I[\varepsilon])$ is satisfied, where we defined $W(P(\varepsilon, L), I[\varepsilon]) = \hbar \log(\exp(\hbar \partial_{P(\varepsilon, L)}) e^{I/\hbar})$ (formally analogous to the scalar field theory case).
3. Each $I_{i,k}[L]$ has a small- L asymptotic expansion in terms of local functionals, where (i, k) indicate the homogeneous part of degree k of the \hbar^i coefficient.

Further, a **Theory** is a Pre-Theory such that $I[L]$ satisfies ($[QME]_L$) for all L .

PROPOSITION 4.97 A functional $I \in \mathcal{C}^\infty(\mathcal{E})[[\hbar]]$ satisfies $[QME]_\varepsilon$ iff it satisfies $[QME]_L$. Note that this is the same as the BV theorem for fibre integrals.

Proof. Note that $P(\varepsilon, L)$ is the kernel of $\int_{\varepsilon}^L Q^{GF} e^{-lD} dl$. Now this lets us denote

$$P * \psi = \int_{\varepsilon}^L (Q^{GF} \otimes 1) K_l dl * \psi \int_{\varepsilon}^L Q^{GF} e^{-lD} \psi \quad (4.250)$$

Thus we can compute

$$\left[Q, \int_{\varepsilon}^L Q^{GF} e^{-lD} dl \right] = \int_{\varepsilon}^L [Q, Q^{GF}] e^{-lD} dl \quad (4.251)$$

$$= \int_{\varepsilon}^L D e^{-lD} dl = e^{-\varepsilon D} - e^{-lD} \quad (4.252)$$

$$\implies (K_{\varepsilon} - K_L) * \psi = (e^{-\varepsilon D} - e^{-lD}) \psi \quad (4.253)$$

Using the last lemma we obtain

$$(\tilde{Q}P(\varepsilon, L)) * \psi = [Q, P*]\psi \quad (4.254)$$

$$\implies \tilde{Q}P = K_{\varepsilon} - K_L \quad (4.255)$$

Now passing to second-order operators we get

$$[Q, \partial_{P(\varepsilon, L)}] = \partial_{K_{\varepsilon}} - \partial_{K_L} = \Delta_L - \Delta_{\varepsilon} \quad (4.256)$$

Thus $\partial_{P(\varepsilon, L)}$ can be seen as a chain homotopy between Δ_L and Δ_{ε} . Thus

$$(Q + \hbar \Delta_L) \exp(W(P(\varepsilon, L)I)/\hbar) \quad (4.257)$$

$$= (Q + \hbar \Delta_L) (\exp(\hbar \partial_P) \exp(I/\hbar)) \quad (4.258)$$

$$= \exp(\hbar \partial_P) (Q + \hbar \Delta_{\varepsilon}) \exp(I/\hbar) \quad (4.259)$$

□

This last result of the course poses as an entry point for even more sophisticated discussions in *Costello* [3] and other sources.

5 Outlook

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