IMC 2020 Online

Day 1, July 26, 2020

Problem 1. Let n be a positive integer. Compute the number of words w (finite sequences of letters) that satisfy all the following three properties:

(1) w consists of n letters, all of them are from the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$;

(2) w contains an even number of letters $\mathtt{a}$;

(3) w contains an even number of letters $\mathtt{b}$.

(For example, for $n = 2$ there are 6 such words: aa, bb, cc, dd, cd and dc.)

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Solution 1. Let $N = \{1, 2, \ldots, n\}$. Consider a word w that satisfies the conditions and let $A, B, C, D \subset N$ be the sets of positions of letters a, b, c and d in w, respectively. By the definition of the words we have $A \sqcup B \sqcup C \sqcup D = N$. The sets A and B are constrained to have even sizes.

In order to construct all suitable words w, choose the set $S = A \cup B$ first; by the conditions, $|S| = |A| + |B|$ must be even. It is well-known that an n-element set (with $n \geq 1$) has 2^{n-1} even subsets, so there are 2^{n-1} possibilities for S.

If $S = \emptyset$ then we can choose $C \subset N$ arbitrarily, and then the set $D = S \setminus C$ is determined D uniquely. Since N has 2^n subsets, we have 2^n options for set C and therefore 2^n suitable words w with $S = \emptyset$.

Otherwise, if $k = |S| > 0$, we have to choose an arbitrary subset C of $N \setminus S$ and an even subset A of S; then $D = (N \setminus S) \setminus C$ and $B = S \setminus A$ are determined and $|B| = |S| - |A|$ will automatically be even. We have 2^{n-k} choices for C and 2^{k-1} independent choices for A; so for each nonempty even S we have $2^{n-k} \cdot 2^{k-1} = 2^{n-1}$ suitable words.

The number of nonempty even sets S is $2^{n-1} - 1$, so in total, the number of words satisfying the conditions is

$$1 \cdot 2^n + (2^{n-1} - 1) \cdot 2^{n-1} = 4^{n-1} + 2^{n-1}.$$

Solution 2. Let a_n denote the number of words of length n over $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ such that a and b appear even number of times. Further, we define the following sequences for the number of words of length n, all over $\mathcal{A}$.

- b_n - the number of words with an odd number of a's and even number of b's
- c_n - the number of words with even number of a's and an odd number of b's
- d_n - the number of words with an odd number of a's and an odd number of b's

We will call them A-words, B-words, C-words and D-words, respectively.
It is clear that $a_1 = 2$ and that

$$a_n + b_n + c_n + d_n = 4^n.$$

First, we find a recurrence relation for a_n. If an A-word of length n begins with c or d, it can be followed by any A-word of length $n-1$, contributing with $2a_{n-1}$. If an A-word of length n begins with a, it can be followed by any word of length $n-1$ that contains an odd number

of $\mathtt{a}$'s and even number of $\mathtt{b}$'s, thus contributing with b_{n-1}. If an A-word of length n begins with $\mathtt{b}$, it can be followed by any word of length $n-1$ that contains even number of $\mathtt{a}$'s and an odd number of $\mathtt{b}$'s, thus contributing with c_{n-1}. Therefore we have the following recurrence relation:

$$a_n = 2a_{n-1} + b_{n-1} + c_{n-1}. \tag{1}$$

Next, we find a recurrence relation for b_n.
If a B-word of length n begins with $\mathtt{c}$ or $\mathtt{d}$, it can be followed by any B-word of length $n-1$, contributing with $2b_{n-1}$. If a B-word of length n begins with $\mathtt{a}$, it can be followed by any word of length $n-1$ that contains even number of $\mathtt{a}$'s and even number of $\mathtt{b}$'s, contributing with a_{n-1}. If a B-word of length n begins with $\mathtt{b}$, it can be followed by any word of length $n-1$ that contains an odd number of $\mathtt{a}$'s and an odd number of $\mathtt{b}$'s, contributing with $d_{n-1} = 4^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}$. Therefore we have the following recurrence relation:

$$b_n = b_{n-1} + 4^{n-1} - c_{n-1}. \tag{2}$$

Now observe that $b_k = c_k$ for all k, since simultaneously replacing $\mathtt{a}$'s to $\mathtt{b}$'s and vice versa we get a C-word from a B-word. Therefore (2) yields $b_n = 4^{n-1}$. Now (1) yields

$$a_n = 2 \cdot a_{n-1} + 2 \cdot 4^{n-2}.$$

Solving the last recurrence relation (for example, diving by 2^n we get $x_n := a_n 2^{-n}$ satisfies $x_n - x_{n-1} = 2^{n-3}$, and it remains to sum up consecutive powers of 2) we get

$$a_n = 2^{n-1} + 4^{n-1}.$$

Solution 3. Consider the sum

$$\frac{(a+b+c+d)^n + (-a-b+c+d)^n + (-a+b+c+d)^n + (a-b+c+d)^n}{4}. \tag{$*$}$$

Expanding the parentheses as

$$(a+b+c+d)^n = (a+b+c+d)(a+b+c+d)\dots(a+b+c+d),$$

we get a sum of products $x_1 \dots x_n$, $x_i \in \{a, b, c, d\}$, naturally corresponding to the words of length n over the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$. Consider the other terms in the numerator similarly.

If a word $x_1 \dots x_n$ contains A, B, C, D letters $\mathtt{a}$,$\mathtt{b}$,$\mathtt{c}$ and $\mathtt{d}$ respectively, we get $a^A b^B c^C d^D$ with the coefficient

$$\frac{1 + (-1)^{A+B} + (-1)^A + (-1)^B}{4} = \frac{(1 + (-1)^A)(1 + (-1)^B)}{4} = \begin{cases} 1, & \text{if } A \text{ and } B \text{ are even} \\ 0, & \text{otherwise.} \end{cases}$$

Hence, by substituting $a = b = c = d = 1$ in $(*)$ we get the answer $(4^n + 2^{n+1})/4 = 4^{n-1} + 2^{n-1}$.

Problem 2. Let A and B be $n \times n$ real matrices such that

$$\mathrm{rk}(AB - BA + I) = 1$$

where I is the $n \times n$ identity matrix.

Prove that

$$\mathrm{trace(ABAB)} - \mathrm{trace(A^2B^2)} = \frac{1}{2}\mathrm{n(n-1)}.$$

($\mathrm{rk}(M)$ denotes the rank of matrix M, i.e., the maximum number of linearly independent columns in M. $\mathrm{trace(M)}$ denotes the trace of M, that is the sum of diagonal elements in M.)

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Solution. Let $X = AB - BA$. The first important observation is that

$$\mathrm{trace(X^2)} = \mathrm{trace(ABAB - ABBA - BAAB + BABA)} = \mathrm{2trace(ABAB)} - \mathrm{2trace(A^2B^2)}$$

using that the trace is cyclic. So we need to prove that $\mathrm{trace(X^2)} = \mathrm{n(n-1)}$.

By assumption, $X + I$ has rank one, so we can write $X + I = v^t w$ for two vectors v, w. So

$$X^2 = (v^t w - I)^2 = I - 2v^t w + v^t w v^t w = I + (wv^t - 2)v^t w.$$

Now by definition of X we have $\mathrm{trace(X)} = 0$ and hence $wv^t = \mathrm{trace(wv^t)} = \mathrm{trace(v^tw)} = \mathrm{n}$ so that indeed

$$\mathrm{trace(X^2)} = \mathrm{n + (n-2)n = n(n-1)}.$$

An alternative way to use the rank one condition is via eigenvalues: Since $X + I$ has rank one, it has eigenvalue 0 with multiplicity $n-1$. So X has eigenvalue -1 with multiplicity $n-1$. Since $\mathrm{trace(X)} = 0$ the remaining eigenvalue of X must be $n-1$. Hence

$$\mathrm{trace(X^2)} = \mathrm{(n-1)^2 + (n-1) \cdot 1^2 = n(n-1)}.$$

Problem 3. Let $d \geq 2$ be an integer. Prove that there exists a constant $C(d)$ such that the following holds: For any convex polytope $K \subset \mathbb{R}^d$, which is symmetric about the origin, and any $\varepsilon \in (0,1)$, there exists a convex polytope $L \subset \mathbb{R}^d$ with at most $C(d)\varepsilon^{1-d}$ vertices such that

$$(1-\varepsilon)K \subseteq L \subseteq K.$$

(For a real α, a set $T \subset \mathbb{R}^d$ with nonempty interior is a *convex polytope with at most α vertices*, if T is a convex hull of a set $X \subset \mathbb{R}^d$ of at most α points, i.e., $T = \{\sum_{x\in X} t_x x \mid t_x \geq 0, \sum_{x\in X} t_x = 1\}$. For a real λ, put $\lambda K = \{\lambda x \mid x \in K\}$. A set $T \subset \mathbb{R}^d$ is *symmetric about the origin* if $(-1)T = T$.)

Fedor Petrov, St. Petersburg State University

Solution [in elementary terms] Let $\{p_1, \ldots, p_m\}$ be an inclusion-maximal collection of points on the boundary ∂K of K such that the homothetic copies $K_i := p_i + \frac{\varepsilon}{2}K$ have disjoint interiors. We claim that the convex hull $L := \operatorname{conv}\{p_1, \ldots, p_m\}$ satisfies all the conditions.

First, note that by convexity of K we have $aK + bK = (a+b)K$ for $a, b > 0$. It follows that $K_i \subset (1 + \frac{\varepsilon}{2})K$. On the other hand, if $k \in K$, $a > 0$ and and $ak \in K_i$, then

$$p_i \in ak - \frac{\varepsilon}{2}K = ak + \frac{\varepsilon}{2}K \subset (a + \frac{\varepsilon}{2})K,$$

and since p_i is a boundary point of K, we get $a + \frac{\varepsilon}{2} \geqslant 1$, $a \geqslant 1 - \frac{\varepsilon}{2}$. It means that all K_i lie between $(1 - \frac{\varepsilon}{2})K$ and $(1 + \frac{\varepsilon}{2})K$. Since their interiors are disjoint, by the volume counting we obtain

$$m\left(\frac{\varepsilon}{2}\right)^d \leqslant \left(1 + \frac{\varepsilon}{2}\right)^d - \left(1 - \frac{\varepsilon}{2}\right)^d \leqslant (3/2)^d \varepsilon$$

(since $F(\varepsilon) = (1 + \frac{\varepsilon}{2})^d - (1 - \frac{\varepsilon}{2})^d$ is a polynomial in ε without constant term with non-negative coefficients which sum up to $(3/2)^d - (1/2)^d$), therefore $m \leqslant 3^d \varepsilon^{1-d}$.

It is clear that $L \subseteq K$, so it remains to prove that $(1-\varepsilon)K \subseteq L$. Assume the contrary: there exists a point $p \in (1-\varepsilon)K \setminus L$. Separate p from L by a hyperplane: Choose a linear functional ℓ such that $\ell(p) > \max_{x\in L} \ell(x) = \max_i \ell(p_i)$. Choose $x \in K$ such that $\ell(x) =: a$ is maximal possible. Note that by our construction $x + \frac{\varepsilon}{2}K$ has a common point with some K_i: there exists a point $z \in (x + \frac{\varepsilon}{2}K) \cap (p_i + \frac{\varepsilon}{2}K)$. We have

$$\ell(p_i) + \frac{\varepsilon}{2}a \geqslant \ell(z) \geqslant \ell(x) - \frac{\varepsilon}{2}a,$$

and therefore $\ell(p_i) \geqslant a(1-\varepsilon)$. Since $p \in (1-\varepsilon)K$, we obtain $\ell(p) \leqslant a(1-\varepsilon)$. A contradiction.

Solution [in the language of Banach spaces] Equip $\mathbb{R}^d$ with the norm $\|\cdot\|$, whose unit ball is K, call this Banach space V. Choose an inclusion maximal set $X \subset \partial K$ whose pairwise distances are $\geq \varepsilon$. Put $L = \operatorname{conv} X$.

The inclusion $L \subseteq K$ follows from the convexity of K. If the inclusion $(1-\varepsilon)K \subseteq L$ fails then the Hahn–Banach theorem provides a unit linear functional $\lambda \in V^*$ such that $\max\{\lambda(L)\} = \max\{\lambda X\} \leq 1 - \varepsilon$. Then the point $x \in K$, where the maximum $\max\{\lambda(K)\} = 1$ is attained (thanks to the finite dimension and compactness) is in ∂K and, as λ witnesses, at distance $\geq \varepsilon$ from all other points of L and X, contradicting the inclusion-maximality of X.

The upper bound for the cardinality $|X|$ is obtained by noting that the $\varepsilon/2$ balls centered at the points of X are pairwise disjoint and lie in the difference of balls $(1+\varepsilon/2)K \setminus (1-\varepsilon/2)K$, whose volume is $\big((1+\varepsilon/2)^d - (1-\varepsilon/2)^d\big)\operatorname{vol}K$, the volume of each of the small balls being $\varepsilon^d/2^d \operatorname{vol}K$. Hence

$$|X| \leq \frac{(2+\varepsilon)^d - (2-\varepsilon)^d}{\varepsilon^d} = O(\varepsilon^{1-d}).$$

Problem 4. A polynomial p with real coefficients satisfies the equation $p(x+1)-p(x)=x^{100}$ for all $x\in\mathbb{R}$. Prove that $p(1-t)\geqslant p(t)$ for $0\leqslant t\leqslant 1/2$.

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Solution 1. Denote $h(z)=p(1-\bar{z})-p(z)$ for complex z. For $t\in\mathbb{R}$ we have $h(it)=p(1+it)-p(it)=t^{100}$, $h(1/2+it)=0$.

If $p(z)=c_nz^n+\ldots+c_0, c_n\neq 0$, we have

$$h(a+it)=p((1-a)+it)-p(a+it)=(1-2a)\left(nc_ni^{n-1}t^{n-1}+Q(t,a)\right)$$

for some polynomial Q having degree at most $n-2$ with respect to the variable t. Substituting $a=0$ we get $n=101$, $c_n=1/101$.

Next, for large $|t|$ we see that $\Re(h(a+it))>0$ for $0\leqslant a<1/2$.

Therefore by Maximum Principle for the harmonic function $\Re h$ and the rectangle $[0,1/2]\times[-N,N]$ for large enough N we conclude that $\Re h$ is non-negative in this rectangle, in particular on $[0,1/2]$, as we need.

Solution 2. Let $p(x)=\sum_{j=0}^m a_jx^j$. Then

$$p(x+1)-p(x)=\sum_{j=0}^m a_j\left((x+1)^j-x^j\right)=a_1+a_2(2x+1)+\cdots+a_m\left(mx^{m-1}+\binom{m}{2}x^{m-2}+\cdots+1\right).$$

This implies that $m=101$, $ma_m=1$ so $a_{101}=\frac{1}{101}$, $(m-1)a_{m-1}+a_m\binom{m}{2}=0$ so $a_{100}=-\frac{1}{2}$ etc. For $j\geqslant 1$ a_j is uniquely defined, a_0 may be chosen arbitrarily.

The equality $p_{2n}(\frac{1}{2})=0$ holds because $0=p_{2n}(\frac{1}{2})+p_{2n}(1-\frac{1}{2})=2p_{2n}(\frac{1}{2})$. Let $n\geqslant 1$ be an integer and let p_n be a polynomial such that $p_n(x+1)-p_n(x)=x^n$ for all x and $p_n(0)=0=p_n(1)$. The above considerations prove the uniqueness of p_n. We have $p_1(x)=\frac{1}{2}x^2-\frac{1}{2}x$. Also $p_n'(x+1)-p_n'(x)=nx^{n-1}=n\left(p_{n-1}(x+1)-p_{n-1}(x)\right)$. Therefore $p_n'(x)=np_{n-1}(x)+c_{n-1}$ for a properly chosen constant c_{n-1}. We shall prove that

(1) $p_{2n-1}(x)-p_{2n-1}(1-x)=0,\ p_{2n}(x)+p_{2n}(1-x)=0,\ c_{2n}=0,\ p_{2n}''(x)=2n(2n-1)p_{2n-2}(x)$

for $n=1,2,\ldots$ and for all x. Simple computation shows that $p_1(x)-p_1(1-x)=0$. We have $(p_2(x)+p_2(1-x))'=2p_1(x)+c_1-(2p_1(1-x)+c_1)=0$ so the map $x\mapsto p_2(x)+p_2(1-x)$ is constant thus $p_2(x)+p_2(1-x)=p_2(0)+p_2(1-0)=0$. If the first two equalities hold for some n then $(p_{2n+1}(x)-p_{2n+1}(1-x))'=(2n+1)p_{2n}(x)+c_{2n}+(p_{2n}(1-x)+c_{2n})=2c_{2n}$ so there exists $b\in R$ such that $p_{2n+1}(x)-p_{2n+1}(1-x)=2c_{2n}x+b$ for all x. $p_{2n+1}(0)-p_{2n+1}(1-0)=0$ and $p_{2n+1}(1)-p_{2n+1}(1-1)=0$ so $2c_{2n}=0=b$. This proves that $p_{2n+1}(x)-p_{2n+1}(1-x)=0$ for all x. In a similar way we shall prove the second equality: $(p_{2n+2}(x)+p_{2n+2}(1-x))'=(2n+2)p_{2n+1}'(x)+c_{2n+1}-(2n+2)\left(p_{2n+1}(1-x)+c_{2n+1}\right)=0$ so the map $x\mapsto p_{2n+2}(x)+p_{2n+2}(1-x)$ is constant hence $p_{2n+2}(x)+p_{2n+2}(1-x)=p_{2n+2}(0)+p_{2n+2}(1-0)=0$ for all x. Now $p_{2n+2}''(x)=((2n+2)p_{2n+1}(x)+c_{2n+1})'=(2n+2)p_{2n+1}'(x)=(2n+2)((2n+1)p_{2n}(x)+c_{2n})=(2n+2)(2n+1)p_{2n}(x)$. Since $p_2'(x)=2p_1(x)+c_1=x^2-x+c_1$ we obtain $p_2''(x)=2x-1<0$ for $x<\frac{1}{2}$.The function p_2 is strictly concave on $[0,\frac{1}{2}]$ and $p_2(0)=0=p_2(\frac{1}{2})$. Therefore $p_2(x)>0$ for $x\in(0,\frac{1}{2})$. This together with the equality $p_4(x)=12p_2(x)$ implies that p_4 is strictly convex on $[0,\frac{1}{2}]$ so in view of $p_4(0)=0=p_4(\frac{1}{2})$ we conclude that $p_4(x)<0$ for $x\in(0,\frac{1}{2})$. Easy induction shows that for $x\in(0,\frac{1}{2})$ one has $p_{2n}(x)>0$ for an odd n and $p_{2n}(x)<0$ for an even n. If $t\in(0,\frac{1}{2})$ then by (1) we get $p_{100}(1-t)-p_{100}(t)=-2p_{100}(t)>0$ as required.

IMC 2020 Online

Day 2, July 27, 2020

Problem 5. Find all twice continuously differentiable functions $f:\mathbb{R}\to(0,+\infty)$ satisfying

$$f''(x)f(x)\ge 2(f'(x))^2$$

for all $x\in\mathbb{R}$.

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Solution. We shall show that only positive constant functions satisfy the condition.

Let $g(x)=\dfrac{1}{f(x)}$. Notice that

$$g''=\left(\frac{1}{f}\right)''=\left(\frac{-f'}{f^2}\right)'=\left(\frac{2(f')^2-f''f}{f^3}\right)'\le 0,$$

so the positive function $g(x)$ is concave. We show that g must be constant.

Take two arbitrary real numbers $a<b$. By the concavity of g, for all $u<a$ and $v>b$ we have

$$\frac{g(a)-g(u)}{a-u}\ge\frac{g(b)-g(a)}{b-a}\ge\frac{g(v)-g(b)}{v-b}.$$

Combining this with $g(u),g(v)>0$ we get

$$\frac{g(a)}{a-u}>\frac{g(b)-g(a)}{b-a}>\frac{-g(b)}{v-b}$$

Now by taking limits $u\to-\infty$ and $v\to\infty$ we obtain

$$0\ge\frac{g(b)-g(a)}{b-a}\ge 0,$$

so $g(a)=g(b)$. This holds for any pair (a,b), so $g(x)$ is constant and $f(x)=1/g(x)$ also is constant.

If f is constant then $f'=f''=0$, so the condition is satisfied.

Remark. Instead of the function $1/f(x)$, the same idea works with $\arctan f(x)$:

$$(\arctan f(x))''=\frac{f''(1+f^2)-2(f')^2}{(1+f^2)^2}=\frac{f''(1+f^2)-2(f')^2(1+f^2)}{(1+f^2)^2}=\frac{f''-2(f')^2}{1+f^2}\ge 0.$$

As can be seen, $\arctan f(x)$ is a bounded convex function, therefore it must be constant.

Problem 6. Find all prime numbers p for which there exists a unique $a\in\{1,2,\dots,p\}$ such that a^3-3a+1 is divisible by p.

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Solution 1. We show that $p=3$ the only prime that satsfies the condition.

Let $f(x) = x^3 - 3x + 1$. As preparation, let's compute the roots of $f(x)$. By Cardano's formula, it can be seen that the roots are

$$2\mathrm{Re}\sqrt[3]{\frac{-1}{2} + \sqrt{\left(\frac{-1}{2}\right)^2 - \left(\frac{-3}{3}\right)^3}} = 2\mathrm{Re}\sqrt[3]{\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}} = \left\{2\cos\frac{2\pi}{9}, 2\cos\frac{4\pi}{9}, 2\cos\frac{8\pi}{9}\right\}$$

where all three values of the complex cubic root were taken.

Notice that, by the trigonometric identity $2\cos 2t = (2\cos t)^2 - 2$, the map $\varphi(x) = x^2 - 2$ cyclically permutes the three roots. We will use this map to find another root of f, when it is considered over $\mathbb{F}_p$.

Suppose that $f(a) = 0$ for some $a \in \mathbb{F}_p$ and consider

$$g(x) = \frac{f(x)}{x-a} = \frac{f(x) - f(a)}{x-a} = x^2 + ax + (a^2 - 3).$$

We claim that $b = a^2 - 2$ is a root of $g(x)$. Indeed,

$$g(b) = (a^2 - 2)^2 + a(a^2 - 2) + (a^2 - 3) = (a+1) \cdot f(a) = 0.$$

By Vieta's formulas, the other root of $g(x)$ is $c = -a - b = -a^2 - a + 2$.

If f has a single root then the three roots must coincide, so

$$a = a^2 - 2 = -a^2 - a + 2.$$

Here the quadratic equation $a = a^2 - 2$, or equivalently $(a+1)(a-2) = 0$, has two solutions, $a = -1$ and $a = 2$. By $f(-1) = f(2) = 3$, in both cases we have $0 = f(a) = 3$, so the only choice is $p = 3$.

Finally, for $p = 3$ we have $f(1) = -1$, $f(2) = 3$ and $f(3) = 19$, from these values only $f(2)$ is divisible by 3, so $p = 3$ satisfies the condition.

Solution 2 (outline) Define $f(x)$ and $g(x)$ like in Solution 1. The discriminant of $g(x)$ is

$$\Delta_g = a^2 - 4(a^2 - 3) = 12 - 3a^2.$$

We show that Δ_g has a square root in $\mathbb{F}_p$.

Take two integers k, m (to be determinated later) and consider

$$\Delta_g = \Delta_g + (ka + m)f(a) = ka^4 + ma^3 - (3k+1)a^2 + (k - 3m)a + (m + 12).$$

Our goal is to choose k, m in such a way that the last expression is a complete square. Either by direct calculations or guessing, we can find that $k = m = 4$ works:

$$\Delta_g = \Delta_g + (4a+4)f(a) = 4a^4 + 4a^3 - 15a^2 - 8a + 16 = (2a^2 + a - 4)^2.$$

If $p \neq 2$ then we can conclude that $f(x)$ has either no or three roots, therefore p is suitable if and only is $f(x)$ is a complete cube: $x^3 - 3x + 1 = (x-a)^3$. From Vieta's formulas $a^3 = 1$, so $a \neq 0$ and $3a = 0$, which is possible if $p = 3$.

For $p = 3$ we have $f(x) = (x+1)^3$, so $p = 3$ is suitable.

The case $p = 2$ must be checked separately because the quadratic formula contains a division by 2. $f(1) = -1$ and $f(2) = 3$, so $p = 2$ is not suitable.

Solution 3 (outline) Assume $p > 3$; the cases $p = 2$ and $p = 3$ will be checked separately.

Let $f(x) = x^3 - 3x + 1$ and suppose that $a \in \mathbb{F}_p$ is a root of $f(x)$, and let $b, c \in \mathbb{F}_{p^2}$ be the other two roots. The discriminant Δ_f of $f(x)$ can be expressed by the elementary symmetric polynomials of a, b, c; it can be calculated that

$$\Delta_f = (b-c)^2(a-b)^2(a-c)^2 = 81 = 9^2,$$

so

$$(b-c)(a-b)(a-c) = \pm 9 \in \mathbb{F}_p.$$

Notice that $\Delta_f \neq 0$, so the three roots are distinct.

Either $b, c \in \mathbb{F}_p$ or b, c are conjugate elements in $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$, we have $(a-b)(a-c) \in \mathbb{F}_p$, so $b - c = \frac{(b-c)(a-b)(a-c)}{(a-b)(a-c)} \in \mathbb{F}_p$. From Vieta's formulas we have $b + c \in \mathbb{F}_p$ as well; since $p \neq 2$, it follows that $b, c \in \mathbb{F}_p$. Now $f(x)$ has three distinct roots in $\mathbb{F}_p$, so p cannot be suitable.

$p = 2$ does not satisfies the condition because both $f(1) = -1$ and $f(2) = 3$ are odd. $p = 3$ is suitable, because $f(2) = 3$ is divisible by 3 while $f(1) = -1$ and $f(3) = 19$ are not.

Problem 7. Let G be a group and $n \geq 2$ be an integer. Let H_1 and H_2 be two subgroups of G that satisfy

$$[G : H_1] = [G : H_2] = n \quad \text{and} \quad [G : (H_1 \cap H_2)] = n(n-1).$$

Prove that H_1 and H_2 are conjugate in G.

(Here $[G : H]$ denotes the *index* of the subgroup H, i.e. the number of distinct left cosets xH of H in G. The subgroups H_1 and H_2 are *conjugate* if there exists an element $g \in G$ such that $g^{-1}H_1g = H_2$.)

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Solution 1. Denote $K = H_1 \cap H_2$. Since

$$n(n-1) = [G : K] = [G : H_1][H_1 : K] = n[H_1 : K],$$

we obtain that $[H_1 : K] = n - 1$. Thus, the subgroup H_1 is partitioned into $n-1$ left cosets of K, say $H_1 = \bigsqcup_{i=1}^{n-1} h_iK$. Therefore, the set $H_1H_2 = \{ab \colon a \in H_1,\ b \in H_2\}$ is partitioned as

$$H_1H_2 = \left(\bigsqcup_{i=1}^{n-1} h_iK\right) H_2 = \bigsqcup_{i=1}^{n-1} h_iKH_2 = \bigsqcup_{i=1}^{n-1} h_iH_2.$$

The last equality holds because $K \subseteq H_2$, so $KH_2 = H_2$. The last expression is a disjoint union since

$$h_iH_2 \cap h_jH_2 \neq \varnothing \iff h_i^{-1}h_j \in H_2 \iff h_i^{-1}h_j \in K \iff h_i = h_j.$$

Thus, H_1H_2 is a disjoint union of $n-1$ left cosets with respect to H_2; hence $L = G \setminus (H_1H_2)$ is the remaining such left coset. Similarly, L is a right coset with respect to H_1. Therefore, for each $g \in L$ we have $L = gH_2 = H_1g$, which yields $H_2 = g^{-1}H_1g$.

Solution 2. Put $G/H_1 = X$ and $G/H_2 = Y$, those are n-element sets acted on by G from the left. Let G act on $X \times Y$ from the left coordinate-wise, consider this product as a table, with rows being copies of X and columns being copies of Y.

The stabilizer of a point (x, y) in $X \times Y$ is $H_1 \cap H_2$. By the orbit-stabilizer theorem, we obtain that the orbit Z of (x, y) has size $[G : H_1 \cap H_2] = n(n-1)$.

If Z contains a whole column then there is a subgroup G_1 of G that stabilizes x and acts transitively on Y. If we conjugate G_1 to a group G_1', then its action remains transitive on Y,

so by conjugation we obtain columns of the table. Since G acts transitively on X, we cover all the columns. It follows that $Z = X \times Y$, so

$$n(n-1) = |Z| = |X \times Y| = n^2,$$

which is a contradiction.

Hence every column of $X \times Y$ has an element not from Z. The same holds for the rows of $X \times Y$. There are n elements not from Z in total and they induce a bijection between X and Y which allows us to identify $X = Y$.

After this identification, every element (x, x) from the diagonal of $X \times X$ (i.e. from $(X \times X) \setminus Z$) is moved to a diagonal element by any $g \in G$, because $gx = gx$. In this formula the action of g in the left hand side and the action of g in the right hand side are the actions of g on X and Y respectively.

Therefore our bijection between X and Y is an isomorphism of sets with a left action of G. Since H_1 and H_2 are stabilizers of the points in the same transitive action of G, we conclude that they are conjugate.

Remark. The situation in the problem is possible for every $n \geq 2$: let $G = S_n$ and let H_1 an H_2 be the stabilizer subgroups of two elements.

Problem 8. Compute

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} (-1)^k \binom{n}{k} \log k.$$

(Here log denotes the natural logarithm.)

Fedor Petrov, St. Petersburg State University

Solution 1. Answer: 1.

The idea is that if $f(k) = \int g^k$, then

$$\sum (-1)^k \binom{n}{k} f(k) = \int (1-g)^n.$$

To relate this to logarithm, we may use the Frullani integrals

$$\int_0^\infty \frac{e^{-x} - e^{-kx}}{x} dx = \lim_{c\to+0} \int_c^\infty \frac{e^{-x}}{x} dx - \int_c^\infty \frac{e^{-kx}}{x} dx = \lim_{c\to+0} \int_c^\infty \frac{e^{-x}}{x} dx - \int_{kc}^\infty \frac{e^{-x}}{x} dx =$$

$$\lim_{c\to+0} \int_c^{kc} \frac{e^{-x}}{x} dx = \log k + \lim_{c\to+0} \int_c^{kc} \frac{e^{-x}-1}{x} dx = \log k.$$

This gives the integral representation of our sum:

$$A := \sum_{k=1}^{n} (-1)^k \binom{n}{k} \log k = \int_0^\infty \frac{-e^{-x} + 1 - (1-e^{-x})^n}{x} dx.$$

Now the problem is reduced to a rather standard integral asymptotics.

We have $(1-e^{-x})^n \geqslant 1 - ne^{-x}$ by Bernoulli inequality, thus $0 \leqslant -e^{-x} + 1 - (1-e^{-x})^n \leqslant ne^{-x}$, and we get

$$0 \leqslant \int_M^\infty \frac{-e^{-x} + 1 - (1-e^{-x})^n}{x} dx \leqslant n \int_M^\infty \frac{e^{-x}}{x} dx \leqslant nM^{-1} \int_M^\infty e^{-x} dx = nM^{-1}e^{-M}.$$

So choosing M such that $Me^M = n$ (such M exists and goes to ∞ with n) we get

$$A = O(1) + \int_0^M \frac{-e^{-x} + 1 - (1-e^{-x})^n}{x} dx.$$

Note that for $0 \leqslant x \leqslant M$ we have $e^{-x} \geqslant e^{-M} = M/n$, and $(1-e^{-x})^{n-1} \leqslant e^{-e^{-x}(n-1)} \leqslant e^{-M(n-1)/n}$ tends to 0 uniformly in x. Therefore

$$\int_0^M \frac{(1-e^{-x})(1-(1-e^{-x})^{n-1})}{x} dx = (1+o(1)) \int_0^M \frac{1-e^{-x}}{x} dx.$$

Finally

$$\int_0^M \frac{1-e^{-x}}{x} dx = \int_0^1 \frac{1-e^{-x}}{x} dx + \int_1^M \frac{-e^{-x}}{x} dx + \int_1^M \frac{dx}{x} =$$
$$\log M + O(1) = \log(M + \log M) + O(1) = \log\log n + O(1),$$

and we get $A = (1+o(1)) \log\log n$.

Solution 2. We start with a known identity (a finite difference of $1/x$).

Expand the rational function

$$f(x) = \frac{m!}{x(x+1)\dots(x+m)}$$

as the linear combination of simple fractions $f(x) = \sum_{j=0}^{m} c_j/(x+j)$. To find c_j we use

$$c_j = ((x+j)f(x))\,|_{x=-j} = (-1)^j\binom{m}{j}.$$

So we get

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{1}{x+k} = \frac{m!}{x(x+1)\dots(x+m)}. \tag{1}$$

Another known identity we use is

$$\sum_{k=j+1}^{n}(-1)^k\binom{n}{j} = \sum_{k=j+1}^{n}(-1)^k\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right) = (-1)^{j+1}\binom{n-1}{j}. \tag{2}$$

Finally we write $\log k = \int_1^k \frac{dx}{x} = \sum_{j=1}^{k-1} I_j$, where $I_j = \int_0^1 \frac{dx}{x+j}$.

Now we have

$$S := \sum_{k=1}^{n}(-1)^k\binom{n}{k}\log k = \sum_{k=1}^{n}(-1)^k\binom{n}{k}\sum_{j=1}^{k-1}I_j = \sum_{j=1}^{n-1}I_j\sum_{k=j+1}^{n}(-1)^k\binom{n}{k} \underset{(2)}{=} \sum_{j=1}^{n-1}I_j(-1)^{j+1}\binom{n-1}{j} =$$

$$\int_0^1\sum_{j=1}^{n-1}(-1)^{j+1}\binom{n-1}{j}\frac{dx}{x+j} = \int_0^1\left(\frac{1}{x}-\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\frac{dx}{x+j}\right)dx \underset{(1)}{=}$$

$$\int_0^1\left(\frac{1}{x}-\frac{(n-1)!}{x(x+1)\dots(x+(n-1))}\right)dx = \int_0^1\frac{dx}{x}\left(1-\frac{1}{(1+x)(1+x/2)\dots(1+x/(n-1))}\right).$$

So S is again expressed as an integral, for which it is not hard to get an asymptotics.

Since $e^t \geqslant 1+t$ for all real t (by convexity or any other reason), we have $e^{y^2-y} \geqslant 1+y^2-y = \frac{1+y^3}{1+y} \geqslant \frac{1}{1+y}$ and $\frac{1}{1+y} \geqslant \frac{1}{e^y} = e^{-y}$ for $y>0$. Therefore

$$e^{y^2-y} \geqslant \frac{1}{1+y} \geqslant e^{-y},\ y>0.$$

Using this double inequality we get

$$e^{x^2\left(1+\frac{1}{2^2}+\dots+\frac{1}{(n-1)^2}\right)-x\left(1+\frac{1}{2}+\dots+\frac{1}{n-1}\right)} \geqslant \frac{1}{(1+x)(1+x/2)\dots(1+x/(n-1))} \geqslant e^{-x\left(1+\frac{1}{2}+\dots+\frac{1}{n-1}\right)}.$$

Since $x^2(1+1/2^2+\dots) \leqslant 2x^2 \leqslant 2x$, we conclude that

$$\frac{1}{(1+x)(1+x/2)\dots(1+x/(n-1))} = e^{-C_n x}, \text{where } -2+\sum_{j=1}^{n-1}\frac{1}{j} \leqslant C_n \leqslant \sum_{j=1}^{n-1}\frac{1}{j},$$

i.e., $C_n = \log n + O(1)$. Thus

$$S = \int_0^1\frac{dx}{x}(1-e^{-C_n x}) = \int_0^{C_n}\frac{dt}{t}(1-e^{-t}) = \int_1^{C_n}\frac{dt}{t}+\int_0^1(1-e^{-t})\frac{dt}{t}+\int_1^{C_n}e^{-t}\frac{dt}{t}$$

$$= \log C_n + O(1) = \log\log n + O(1).$$

IMC 2021 Online

First Day, August 3, 2021

Solutions

Problem 1. Let A be a real $n \times n$ matrix such that $A^3 = 0$.

(a) Prove that there is a unique real $n \times n$ matrix X that satisfies the equation

$$X + AX + XA^2 = A.$$

(b) Express X in terms of A.

(proposed by Bekhzod Kurbonboev, Institute of Mathematics, Tashkent)

Hint: (a) Multiply the equation by some power of A from left and another power of A from right.

(b) Substitute repeatedly $X = A - AX - XA^2$.

Solution 1. First suppose that some matrix X satisfies the equation. We can obtain new equations if we multiply the given equation by some power of A from left and another power of A from right. For example,

$$A^2(X + AX + XA^2)A^2 = A^2XA^2 + A^3 \cdot XA^2 + A^2XA \cdot A^3 = A^2XA^2.$$

The right-hand side is $A^2 \cdot A \cdot A^2 = A^3 \cdot A^2 = 0$, so

$$\begin{aligned}
A^2XA^2 &= A^2(X + AX + XA^2)A^2 = A^5 = 0. \qquad \text{Similarly,} \\
A^2X &= A^2(X + AX + XA^2) = A^3 = 0 \\
AXA &= A(X + AX + XA^2)A = A^3 = 0 \\
XA^2 &= (X + AX + XA^2)A^2 = A^3 = 0 \\
AX &= A(X + AX + XA^2)A = A^2. \qquad \text{Finally} \\
X &= A - AX - XA^2 = A - A^2.
\end{aligned}$$

Hence, no matrix other than $A - A^2$ can satisfy the equation.

Note that the argument above does not prove that the matrix $X = A - A^2$ satisfies the equation, because the steps cannot be done in reverse order. That must be verified separately. Indeed,

$$X + AX + XA^2 = (A - A^2) + A(A - A^2) + (A - A^2)A^2 = A - A^4 = A.$$

Hence, $X = A - A^2$ is the unique solution of the equation.

Remark. By multiplying the equation by A^n from left and by A^k from right we can get 9 different equations:

$$\begin{array}{ccc}
X + AX + XA^2 = A & XA + AXA = A^2 & XA^2 + AXA^2 = 0 \\
AX + A^2X + AXA^2 = A^2 & AXA + A^2XA = 0 & AXA^2 + A^2XA^2 = 0 \\
A^2X + A^2XA^2 = 0 & A^2XA = 0 & A^2XA^2 = 0
\end{array}$$

These formulas provide a system of linear equations for the nine matrices X, AX, A^2X, XA, AXA, A^2XA, XA^2, AXA^2 and A^2XA^2.

Solution 2. We use a different approach to express X in terms of A. If some matrix X satisfies the equation then

$$X = A - AX - XA^2.$$

Let us substitute this identity in the right-hand side repeatedly until X cancels out everywhere. Notice that by the condition $A^3 = 0$ we have $A^3 = A^4 = A^5 = A^3X = XA^4 = AXA^4 = A^3XA^2 = 0$, so

$$\begin{aligned}
X &= A - AX - XA^2 \\
&= A - A(A - AX - XA^2) - (A - AX - XA^2)A^2 \\
&= A - (A^2 - A^2X - AXA^2) - (A^3 - AXA^2 - XA^4) \\
&= A - A^2 + A^2X + 2AXA^2 \\
&= A - A^2 + A^2(A - AX - XA^2) + 2A(A - AX - XA^2)A^2 \\
&= A - A^2 + (A^3 - A^3X - A^2XA^2) + 2(A^4 - A^2XA^2 - AXA^4) \\
&= A - A^2 - 3A^2XA^2 \\
&= A - A^2 - 3A^2(A - AX - XA^2)A^2 \\
&= A - A^2 - 3(A^5 - A^3XA^2 - A^2XA^4) \\
&= A - A^2.
\end{aligned}$$

To complete the solution, we have to verify that $X = A - A^2$ is indeed a solution. This step is the same as in Solution 1.

Solution 3. Let $B = I - A + A^2$ so that B is the inverse of $I + A$. Multiplying by B from the left, the equation is equivalent to

$$X + BXA^2 = BA. \tag{1}$$

Now assume X satisfies the equation. Multiplying by A^2 from the right and using $A^3 = 0$ we get $XA^2 = 0$. Hence the equation simplifies to $X = BA = A - A^2$.

On the other hand, $X = BA$ obviously satisfies (1).

Problem 2. Let n and k be fixed positive integers, and let a be an arbitrary non-negative integer. Choose a random k-element subset X of $\{1, 2, \dots, k + a\}$ uniformly (i.e., all k-element subsets are chosen with the same probability) and, independently of X, choose a random n-element subset Y of $\{1, \dots, k + n + a\}$ uniformly.

Prove that the probability

$$\mathsf{P}\Big(\min(Y) > \max(X)\Big)$$

does not depend on a.

(proposed by Fedor Petrov, St. Petersburg State University)

Hint: The sets X and Y with $\min(Y) > \max(X)$ are uniquely determined by $X \cup Y$.

Solution 1. The number of choices for (X, Y) is $\binom{k+a}{k} \cdot \binom{n+k+a}{n}$.

The number of such choices with $\min(Y) > \max(X)$ is equal to $\binom{n+k+a}{n+k}$ since this is the number of choices for the $n+k$-element set $X \cup Y$ and this union together with the condition $\min(Y) > \max(X)$ determines X and Y uniquely (note in particular that no elements of X will be larger than $k + a$). Hence the probability is

$$\frac{\binom{n+k+a}{n+k}}{\binom{k+a}{k} \cdot \binom{n+k+a}{n}} = \frac{1}{\binom{n+k}{k}}$$

where the identity can be seen by expanding the binomial coefficients on both sides into factorials and canceling.

Since the right hand side is independent of a, the claim follows.

Solution 2. Let f be the increasing bijection from $\{1, 2, \ldots, k+a\}$ to $\{1, \ldots, k+a+n\} \setminus Y$. Note that $\min(Y) > \max(X)$ if and only if $\min(Y) > \max(f(X))$.

Note that

$$\{Z_n := Y, Z_k := f(X), Z_a := f(\{1, 2, \ldots, k+a\} \setminus X)\}$$

is a random partition of

$$\{1, \ldots, n+k+a\} = Z_n \sqcup Z_k \sqcup Z_a$$

into an n-subset, k-subset, and a-subset.

If an a-subset Z_a is fixed, the conditional probability that $\min(Z_k) > \max(Z_n)$ always equals $1/\binom{n+k}{k}$. Therefore the total probability also equals $1/\binom{n+k}{k}$.

Problem 3. We say that a positive real number d is *good* if there exists an infinite sequence $a_1, a_2, a_3, \ldots \in (0, d)$ such that for each n, the points $a_1, \ldots, a_n$ partition the interval $[0, d]$ into segments of length at most $1/n$ each. Find

$$\sup\Big\{d \;\Big|\; d \text{ is good}\Big\}.$$

(proposed by Josef Tkadlec)

Hint: To get an upper bound, use that some of the gaps after n steps are still intact some steps later.

Solution. Let $d^\star = \sup\{d \mid d \text{ is good}\}$. We will show that $d^\star = \ln(2) \doteq 0.693$.

1. $d^\star \le \ln 2$:

   Assume that some d is good and let $a_1, a_2, \ldots$ be the witness sequence.

   Fix an integer n. By assumption, the prefix $a_1, \ldots, a_n$ of the sequence splits the interval $[0, d]$ into $n+1$ parts, each of length at most $1/n$.

   Let $0 \le \ell_1 \le \ell_2 \le \cdots \le \ell_{n+1}$ be the lengths of these parts. Now for each $k = 1, \ldots, n$ after placing the next k terms $a_{n+1}, \ldots, a_{n+k}$, at least $n+1-k$ of these initial parts remain intact. Hence $\ell_{n+1-k} \le \frac{1}{n+k}$. Hence
   $$d = \ell_1 + \cdots + \ell_{n+1} \le \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}. \tag{2}$$
   As $n \to \infty$, the RHS tends to $\ln(2)$ showing that $d \le \ln(2)$.

   Hence $d^\star \le \ln 2$ as desired.

2. $d^\star \ge \ln 2$:

   Observe that
   $$\ln 2 = \ln 2n - \ln n = \sum_{i=1}^{n} \ln(n+i) - \ln(n+i-1) = \sum_{i=1}^{n} \ln\left(1 + \frac{1}{n+i-1}\right).$$
   Interpreting the summands as lengths, we think of the sum as the lengths of a partition of the segment $[0, \ln 2]$ in n parts. Moreover, the maximal length of the parts is $\ln(1 + 1/n) < 1/n$.

   Changing n to $n+1$ in the sum keeps the values of the sum, removes the summand $\ln(1+1/n)$, and adds two summands
   $$\ln\left(1 + \frac{1}{2n}\right) + \ln\left(1 + \frac{1}{2n+1}\right) = \ln\left(1 + \frac{1}{n}\right).$$

This transformation may be realized by adding one partition point in the segment of length $\ln(1+1/n)$.

In total, we obtain a scheme to add partition points one by one, all the time keeping the assumption that once we have $n-1$ partition points and n partition segments, all the partition segments are smaller than $1/n$.

The first terms of the constructed sequence will be $a_1 = \ln\frac{3}{2}, a_2 = \ln\frac{5}{4}, a_3 = \ln\frac{7}{4}, a_4 = \ln\frac{9}{8}, \ldots$.

Remark. This remark describes in fact the same solution from a different view and some ideas behind it. It could be erased after marking is finished. Estimate (2) is quite natural. To prove that RHS tends to $\ln 2$ we use some integral estimates by

$$\int_n^{2n+1} \frac{1}{x} dx = \ln(2n+1) - \ln n.$$

Here we can observe that

$$\int_n^{2n} \frac{1}{x} dx = \ln 2$$

is independent of n. This can help us with the construction since the above equality means

$$I_1 = \int_n^{n+1} \frac{1}{x} dx = \int_{2n}^{2n+1} \frac{1}{x} dx + \int_{2n+1}^{2n+2} \frac{1}{x} dx = I_2 + I_3,$$

so, interval of length I_1 can be splitted into two intervals of lengths I_2 and I_3. In fact, after placing the point a_n in the construction for $d = \ln 2$, the lengths of the $n+1$ intervals are

$$\int_{n+1}^{n+2} \frac{1}{x}, \int_{n+2}^{n+3} \frac{1}{x}, \ldots, \int_{2n+1}^{2n+2} \frac{1}{x}$$

with total length

$$d = \int_{n+1}^{2n+2} \frac{1}{x} = \ln 2.$$

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that for every $\varepsilon > 0$, there exists a function $g : \mathbb{R} \to (0, \infty)$ such that for every pair (x, y) of real numbers,

$$\text{if} \quad |x-y| < \min\{g(x), g(y)\}, \quad \text{then} \quad |f(x) - f(y)| < \varepsilon.$$

Prove that f is the pointwise limit of a sequence of continuous $\mathbb{R} \to \mathbb{R}$ functions, i.e., there is a sequence $h_1, h_2, \ldots$ of continuous $\mathbb{R} \to \mathbb{R}$ functions such that $\lim_{n\to\infty} h_n(x) = f(x)$ for every $x \in \mathbb{R}$.

(proposed by Camille Mau, Nanyang Technological University, Singapore)

Hint: Start from a segment in place of $\mathbb{R}$ and use its compactness. Or recall the cool things called "the Lebesgue characterization theorem" and "the Baire characterization theorem".

Solution 1. Since g depends also on ε, let us use the notation $g(x, \varepsilon)$. Considering only $\varepsilon = 1/n$ for positive integer n will suffice to reach our conclusions, hence we may use $\min\{g(x, 1/m) \mid m \le n\}$ in place of $g(x, 1/n)$ and thus assume $g(x, \varepsilon)$ decreasing in ε.

For any $x \in \mathbb{R}$, choose $\delta_n(x) = \min\{1/n, g(x, 1/n)\}$. Of the $\delta_n(x)$-neighborhoods of all x select (using local compactness of the reals) an inclusion-minimal locally finite covering $\{U_i\}$. From its inclusion-minimality it follows that we may enumerate U_i with $i \in \mathbb{Z}$ so that $U_i \cap U_j \ne \varnothing$ only when $|i-j| \le 1$ and the enumeration goes from left to right on the real line. For an assumed n, let x_i be the center of U_i and $\delta_i = \delta_n(x_i)$, so that $U_i = (x_i - \delta_i, x_i + \delta_i)$ and $\delta_i < 1/n$ for all i.

Now define a continuous $f_n : \mathbb{R} \to \mathbb{R}$ so that it equals $f(x_i)$ in $U_i \setminus (U_{i-1} \cup U_{i+1})$, and so that f_n changes continuously between $f(x_{i-1})$ and $f(x_i)$ in the intersection $U_{i-1} \cap U_i$.

Now we show that $f_n \to f$ pointwise. Fix a point x and $\varepsilon = 1/m > 0$, and choose

$$n > \max\{1/g(x, \varepsilon), m\}.$$

Examine the construction of f_n for any such n. Observe that $g(x, \varepsilon) > 1/n > \delta_i$ and $1/n < 1/m$. There are two cases:

- x belongs to the unique U_i. Then using the monotonicity of $g(x,\varepsilon)$ in ε we have

$$|x_i - x| < \delta_i \le \min\left\{g\left(x_i, \frac{1}{n}\right), g(x,\varepsilon)\right\} \le \min\{g(x_i,\varepsilon), g(x,\varepsilon)\}.$$

  Hence

$$|f(x) - f_n(x)| = |f(x) - f(x_i)| < \varepsilon.$$

- x belongs to $U_{i-1} \cap U_i$. Similar to the previous case,

$$|f(x) - f(x_{i-1})|, |f(x) - f(x_i)| < \varepsilon.$$

  Since $f_n(x)$ is between $f_n(x_{i-1}) = f(x_{i-1})$ and $f_n(x_i) = f(x_i)$ by construction, we have

$$|f(x) - f_n(x)| < \varepsilon.$$

We have that $|f(x) - f_n(x)| < \varepsilon$ holds for sufficiently large n, which proves the pointwise convergence.

Solution 2. This solution uses the Baire characterization theorem: *A function $f : \mathbb{R} \to \mathbb{R}$ is a pointwise limit of continuous functions if and only if its restriction to every non-empty closed subset of $\mathbb{R}$ has a point of continuity.*

Assume the contrary in view of the above theorem: $A \subseteq \mathbb{R}$ is a non-empty closed set and f has no point of continuity in A. Let's think that f is defined only on A.

Then for all $x \in A$ there exist rationals $p < q$ for which $\limsup_x f > q$, $\liminf_x f < p$. Apply the Baire category theorem: *If a complete metric space A is a countable union of sets then some of the sets is dense in a positive radius metric ball of A.* It follows that there exist p and q, which serve for a subset $B \subset A$ which is dense on a certain ball (in the induced metric of the real line) $A_1 \subset A$. It yields that both sets $Q = f^{-1}(q,\infty)$ and $P = f^{-1}(-\infty, p)$ are dense in A_1.

Choose $\varepsilon = (q-p)/10$ and find k for which the set $S = \{x : g(x) > 1/k\}$ is also dense on a certain ball $A_2 \subset A_1$. Partition S into subsets where $f(x) > (p+q)/2$ and $f(x) \leqslant (p+q)/2$, one of them is again dense somewhere in A_3, say the latter.

Now take any point $y \in A_3 \cap Q$ and a very close (within distance $\min(1/k, g(y))$) to y point x with $g(x) > 1/k$ but $f(x) \leqslant (p+q)/2$. This pair x, y contradicts the property of f from the problem statement.

Solution 3. This solution uses the Lebesgue characterization theorem: *If $f : \mathbb{R} \to \mathbb{R}$ is a function and, for all real c, the sublevel and superlevel sets $\{x \mid f(x) \geqslant c\}$, $\{x \mid f(x) \leqslant c\}$ are countable intersections of open sets then f is a pointwise limit of continuous functions.*

Now the solution follows from the formula with a countable intersection of the unions of intervals:

$$\{x \mid f(x) \geqslant c\} = \bigcap_{n,k=1}^{\infty} \bigcup_{\substack{y\in\mathbb{R}\\ f(y)\geqslant c}} \left(y - \min\left\{\frac{1}{k}, g\left(y, \frac{1}{n}\right)\right\}, y + \min\left\{\frac{1}{k}, g\left(y, \frac{1}{n}\right)\right\}\right) \qquad (*)$$

and the similar formula for $\{x : f(x) \leqslant c\}$. It remains to prove $(*)$.

The left hand side is obviously contained in the right hand side, just put $y = x$.

To prove the opposite inclusion assume the contrary, that $f(x) < c$, but x is contained in the right hand side. Choose a positive integer n such that $f(x) < c - 1/n$ and k such that $g(x, 1/n) > 1/k$. Then, since x belongs to the right hand side, we see that there exists y such that $f(y) \geqslant c$ and

$$|x - y| < \min\left\{g\left(y, \frac{1}{n}\right), \frac{1}{k}\right\} \leqslant \min\left\{g\left(y, \frac{1}{n}\right), g\left(x, \frac{1}{n}\right)\right\},$$

which yields $f(x) \geqslant f(y) - 1/n \geqslant c - 1/n$, a contradiction.

IMC 2021 Online

Second Day, August 4, 2021

Solutions

Problem 5. Let A be a real $n \times n$ matrix and suppose that for every positive integer m there exists a real symmetric matrix B such that

$$2021B = A^m + B^2.$$

Prove that $|\det A| \le 1$.

(proposed by Rafael Filipe dos Santos, Instituto Militar de Engenharia, Rio de Janeiro)

Hint: The determinant is the product of the eigenvalues.

Solution. Let B_m be the corresponding matrix B depending on m:

$$2021B_m = A^m + B_m^2.$$

For $m = 1$, we obtain $A = 2021B_1 - B_1^2$. Since B_1 is real and symmetric, so is A. Thus A is diagonalizable and all eigenvalues of A are real.

Now fix a positive integer m and let λ be any real eigenvalue of A. Considering the diagonal form of both A and B_m, we know that there exists a real eigenvalue μ of B_m such that

$$2021\mu = \lambda^m + \mu^2 \Rightarrow \mu^2 - 2021\mu + \lambda^m = 0.$$

The last equation is a second degree equation with a real root. Therefore, the discriminant is non-negative:

$$2021^2 - 4\lambda^m \ge 0 \Rightarrow \lambda^m \le \frac{2021^2}{4}.$$

If $|\lambda| > 1$, letting m even sufficiently large we reach a contradiction. Thus $|\lambda| \le 1$.

Finally, since $\det A$ is the product of the eigenvalues of A and each of them has absolute value less then or equal to 1, we get $|\det A| \le 1$ as desired.

Solution. Different solution can be found in paper s2002

Problem 6. For a prime number p, let $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ be the group of invertible 2×2 matrices of residues modulo p, and let S_p be the symmetric group (the group of all permutations) on p elements. Show that there is no injective group homomorphism $\varphi : \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z}) \to S_p$.

(proposed by Thiago Landim, Sorbonne University, Paris)

Hint: First find what the monomorphism must do with elements of order p.

Solution. For $p = 2$, just note that $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ has more than $2 = |S_2|$ elements.

From now on, let p be an odd prime and suppose that there exists such a homomorphism.

The matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has order p and commutes with the matrix

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

of order 2, hence AB has order $2p$. But there is no permutation in S_p of order $2p$ since only p-cycles have order divisible by p, and their order is exactly p.

Problem 7. Let $D \subseteq \mathbb{C}$ be an open set containing the closed unit disk $\{z \ : \ |z| \le 1\}$. Let $f : D \to \mathbb{C}$ be a holomorphic function, and let $p(z)$ be a monic polynomial. Prove that

$$\left|f(0)\right| \le \max_{|z|=1} \left|f(z)p(z)\right|.$$

(proposed by Lars Hörmander)

Hint: Apply the maximum principle or the Cauchy formula to a suitable function $f(z)q(z)$.

Solution.

Let $q(z) = z^n \cdot \overline{p\left(1/\overline{z}\right)}$, or more explicitly, if

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0,$$

let

$$q(z) = 1 + \overline{a_{n-1}}z + \cdots + \overline{a_0}z^n.$$

Note that for $|z| = 1$ we have $1/\overline{z} = z$ and hence $|q(z)| = |p(z)|$. Hence by the maximum principle or the Cauchy formula for the product of f and q, it follows that

$$|f(0)| = |f(0)q(0)| \le \max_{|z|=1} |f(z)q(z)| = \max_{|z|=1} |f(z)p(z)|.$$

Problem 8. Let n be a positive integer. At most how many distinct unit vectors can be selected in $\mathbb{R}^n$ such that from any three of them, at least two are orthogonal?

(proposed by Alexander Polyanskii, Moscow Institute of Physics and Technology; based on results of Paul Erdős and Moshe Rosenfeld)

Hint: Play with the Gram matrix of these vectors.

Solution 1. $2n$ is the maximal number.

An example of $2n$ vectors in the set is given by a basis and its opposite vectors. In the rest of the text we prove that it is impossible to have $2n + 1$ vectors in the set.

Consider the Gram matrix A with entries $a_{ij} = e_i \cdot e_j$. Its rank is at most n, its eigenvalues are real and non-negative. Put $B = A - I_{2n+1}$, this is the same matrix, but with zeros on the diagonal. The eigenvalues of B are real, greater or equal to -1, and the multiplicity of -1 is at least $n + 1$.

The matrix $C = B^3$ has the following diagonal entries

$$c_{ii} = \sum_{i \ne j \ne k \ne i} a_{ij}a_{jk}a_{ki}.$$

The problem statement implies that in every summand of this expression at least one factor is zero. Hence $\operatorname{tr} C = 0$. Let $x_1, \ldots, x_m$ be the positive eigenvalues of B, their number is $m \le n$ as noted above. From $\operatorname{tr} B = \operatorname{tr} C$ we deduce (taking into account that the eigenvalues between -1 and 0 satisfy $\lambda^3 \ge \lambda$):

$$x_1 + \cdots + x_m \ge x_1^3 + \cdots + x_m^3$$

Applying $\operatorname{tr} C = 0$ once again and noting that C has eigenvalue -1 of multiplicity at least $n+1$, we obtain

$$x_1^3 + \cdots + x_m^3 \geq n + 1.$$

It also follows that

$$\left(x_1 + \cdots + x_m\right)^3 \geq \left(x_1^3 + \cdots + x_m^3\right)(n+1)^2.$$

By Hölder's inequality, we obtain

$$\left(x_1^3 + \cdots + x_m^3\right) m^2 \geq \left(x_1 + \cdots + x_m\right)^3,$$

which is a contradiction with $m \leq n$.

Solution 2. Let P_i denote the projection onto i-th vector, $i = 1, \ldots, N$. Then our relation reads as $\operatorname{tr}\left(P_i P_j P_k\right) = 0$ for distinct i, j, k. Consider the operator $Q = \sum_{i=1}^{N} P_i$, it is non-negative definite, let $t_1, \ldots, t_n$ be its eigenvalues, $\sum t_i = \operatorname{tr} Q = N$. We get

$$\sum t_i^3 = \operatorname{tr} Q^3 = N + 6 \sum_{i<j} \operatorname{tr} P_i P_j = N + 3(\operatorname{tr} Q^2 - N) = 3 \sum t_i^2 - 2N$$

(we used the obvious identities like $\operatorname{tr} P_i P_j P_i = \operatorname{tr} P_i^2 P_j = \operatorname{tr} P_i P_j$). But $(t_i - 2)^2(t_i + 1) = t_i^3 - 3t_i^2 + 4 \geqslant 0$, thus $-2N = \sum t_i^3 - 3t_i^2 \geqslant -4n$ and $N \leqslant 2n$.

IMC 2022

First Day, August 3, 2022

Solutions

Problem 1. Let $f : [0, 1] \to (0, \infty)$ be an integrable function such that $f(x) \cdot f(1 - x) = 1$ for all $x \in [0, 1]$. Prove that

$$\int_0^1 f(x)\ \mathrm{d}x \geq 1.$$

(proposed by Mike Daas, Universiteit Leiden)

Hint: Apply the AM–GM inequality.

Solution 1. By the AM–GM inequlity we have

$$f(x) + f(1 - x) \geq 2\sqrt{f(x)f(1 - x)} = 2.$$

By integrating in the interval $[0, \frac{1}{2}]$ we get

$$\int_0^1 f(x)\mathrm{d}x = \int_0^{\frac{1}{2}} f(x)\mathrm{d}x + \int_0^{\frac{1}{2}} f(1 - x)\mathrm{d}x = \int_0^{\frac{1}{2}} \big(f(x) + f(1 - x)\big)\mathrm{d}x \geq \int_0^{\frac{1}{2}} 2\mathrm{d}x = 1.$$

Solution 2. From the condition, we have

$$\int_0^1 f(x)dx = \int_0^1 f(1 - x)dx = \int_0^1 \frac{1}{f(x)}dx$$

and hence, using the positivity of f, the claim follows since

$$\left(\int_0^1 f(x)dx\right)^2 = \int_0^1 f(x)dx \cdot \int_0^1 \frac{1}{f(x)}dx \geq \left(\int_0^1 1dx\right)^2 \geq 1$$

by the Cauchy-Schwarz inequality.

Problem 2. Let n be a positive integer. Find all $n \times n$ real matrices A with only real eigenvalues satisfying

$$A + A^k = A^T$$

for some integer $k \geq n$.

(A^T denotes the transpose of A.)

(proposed by Camille Mau, Nanyang Technological University)

Hint: Consider the eigenvalues of A.

Solution 1. Taking the transpose of the matrix equation and substituting we have

$$A^T + (A^T)^k = A \implies A + A^k + (A + A^k)^k = A \implies A^k(I + (I + A^{k-1})^k) = 0.$$

Hence $p(x) = x^k(1 + (1 + x^{k-1})^k)$ is an annihilating polynomial for A. It follows that all eigenvalues of A must occur as roots of p (possibly with different multiplicities). Note that for all $x \in \mathbb{R}$ (this can be seen by considering even/odd cases on k),

$$(1 + x^{k-1})^k \geq 0,$$

and we conclude that the only eigenvalue of A is 0 with multiplicity n.

Thus A is nilpotent, and since A is $n \times n$, $A^n = 0$. It follows $A^k = 0$, and $A = A^T$. Hence A can only be the zero matrix: A is real symmetric and so is orthogonally diagonalizable, and all its eigenvalues are 0.

Remark. It's fairly easy to prove that eigenvalues must occur as roots of any annihilating polynomial. If λ is an eigenvalue and v an associated eigenvector, then $f(A)v = f(\lambda)v$. If f annihilates A, then $f(\lambda)v = 0$, and since $v \neq 0$, $f(\lambda) = 0$.

Solution 2. If λ is an eigenvalue of A, then $\lambda + \lambda^k$ is an eigenvalue of $A^T = A + A^k$, thus of A too. Now, if k is odd, then taking λ with maximal absolute value we get a contradiction unless all eigenvalues are 0. If k is even, the same contradiction is obtained by comparing the traces of A^T and $A + A^k$.

Hence all eigenvalues are zero and A is nilpotent. The hypothesis that $k \geq n$ ensures $A = A^T$. A nilpotent self-adjoint operator is diagonalizable and is necessarily zero.

Problem 3. Let p be a prime number. A flea is staying at point 0 of the real line. At each minute, the flea has three possibilities: to stay at its position, or to move by 1 to the left or to the right. After $p-1$ minutes, it wants to be at 0 again. Denote by $f(p)$ the number of its strategies to do this (for example, $f(3) = 3$: it may either stay at 0 for the entire time, or go to the left and then to the right, or go to the right and then to the left). Find $f(p)$ modulo p.

(proposed by Fedor Petrov, St. Petersburg)

Hint: Find a recurrence for $f(p)$ or use generating functions.

Solution 1. The answer is $f(p) \equiv 0 \mod 3$ for $p = 3$, $f(p) \equiv 1 \mod 3$ for $p = 3k + 1$, and $f(p) \equiv -1 \mod 3$ for $p = 3k - 1$.

The case $p = 3$ is already considered, let further $p \neq 3$. For a residue i modulo p denote by $a_i(k)$ the number of Flea strategies for which she is at position i modulo p after k minutes. Then $f(p) = a_0(p-1)$. The natural recurrence is $a_i(k+1) = a_{i-1}(k)+a_i(k)+a_{i+1}(k)$, where the indices are taken modulo p. The idea is that modulo p we have $a_0(p) \equiv 3$ and $a_i(p) \equiv 0$. Indeed, for all strategies for p minutes for which not all p actions are the same, we may cyclically shift the actions, and so we partition such strategies onto groups by p strategies which result with the same i. Remaining three strategies correspond to $i = 0$. Thus, if we denote $x_i = a_i(p-1)$, we get a system of equations $x_{-1}+x_0+x_1 = 3$, $x_{i-1}+x_i+x_{i+1} = 0$ for all $i = 1, \ldots, p-1$. It is not hard to solve this system (using the 3-periodicity, for example). For $p = 3k + 1$ we get $(x_0, x_1, \ldots, x_{p-1}) = (1, 1, -2, 1, 1, -2, \ldots, 1)$, and $(x_0, x_1, \ldots, x_{p-1}) = (-1, 2, -1, -1, 2, \ldots, 2)$ for $p = 3k + 2$.

Solution 2. Note that $f(p)$ is the constant term of the Laurent polynomial $(x + 1 + 1/x)^{p-1}$ (the moves to right, to left and staying are in natural correspondence with x, $1/x$ and 1.) Thus, working with power series over $\mathbb{F}_p$ we get (using the notation $[x^k]P(x)$ for the coefficient of x^k in P)

$$\begin{aligned} f(p) = [x^{p-1}](1+x+x^2)^{p-1} &= [x^{p-1}](1-x^3)^{p-1}(1-x)^{1-p} = [x^{p-1}](1-x^3)^p(1-x)^{-p}(1-x^3)^{-1}(1-x) \\ &= [x^{p-1}](1-x^{3p})(1-x^p)^{-1}(1-x^3)^{-1}(1-x) = [x^{p-1}](1-x^3)^{-1}(1-x), \end{aligned}$$

and expanding $(1 - x^3)^{-1} = \sum x^{3k}$ we get the answer.

Problem 4. Let $n > 3$ be an integer. Let Ω be the set of all triples of distinct elements of $\{1, 2, \ldots, n\}$. Let m denote the minimal number of colours which suffice to colour Ω so that whenever $1 \le a < b < c < d \le n$, the triples $\{a, b, c\}$ and $\{b, c, d\}$ have different colours. Prove that

$$\frac{1}{100} \log\log n \leqslant m \leqslant 100 \log\log n.$$

(proposed by Danila Cherkashin, St. Petersburg)

Hint: Define two graphs, one on Ω and another graph on pairs (2-element sets).

Solution. For $k = 1, 2, \ldots, n$ denote by Ω_k the set of all $\binom{n}{k}$ k-subsets of $[n]$. For each $k = 1, 2, \ldots, n-1$ define a directed graph G_k whose vertices are elements of Ω_k, and edges correspond to elements of Ω_{k+1} as follows: if $1 \leqslant a_1 < a_2 < \ldots < a_{k+1} \leqslant n$, then the edge of G_k corresponding to $(a_1, \ldots, a_{k+1})$ goes from $(a_1, \ldots, a_k)$ to $(a_2, \ldots, a_{k+1})$.

For a directed graph $G = (V, E)$ we call a subset $E_1 \subset E$ *admissible*, if E_1 does not contain a directed path $a - b - c$ of length 2. Define *b-index* $b(G)$ of the G as the minimal number of admissible sets which cover E. As usual, a subset $V_1 \subset V$ is called *independent*, if there are no edges with both endpoints in V_1; a *chromatic number* of G is defined as the minimal number of independent sets which cover V.

A straightforward but crucial observation is the following

Lemma. For all $k = 2, 3, \ldots, n$ a subset $A_k \subset \Omega_k$ is independent in G_k if and only if it is admissible as a set of edges of G_{k-1}.

Corollary. $\chi(G_k) = b(G_{k-1})$ for all $k = 2, 3, \ldots, n$.

Now the bounds for numbers $\chi(G_k)$ follow by induction using the following general

Lemma. For a directed graph $G = (V, E)$ we have

$$\log_2 \chi(G) \leqslant b(G) \leqslant 2\lceil \log_2 \chi(G) \rceil.$$

Proof. 1) Denote $b(G) = m$ and prove that $\log_2 \chi(G) \leqslant m$. For this we take a covering of E by m admissible subsets $E_1, \ldots, E_m$ and define a color $c(v)$ of a vertex $v \in V$ as the following subset of $[m]$: $c(v) := \{i \in [m] : \exists vw \in E_i\}$. Note that for any edge $vw \in E$ there exists i such that $vw \in E_i$ which yields $i \in c(v)$ and $i \notin c(w)$, therefore $c(v) \neq c(w)$. So, each color class is an independent set and we get $\chi(G) \leqslant 2^m$ as needed.

2) Denote $\chi(G) = k$ and prove that $b(G) \leqslant 2\lceil \log_2 k \rceil$. Take a proper coloring $\tau \colon V \to \{0, 1, \ldots, k-1\}$ (that means that $\tau(u) \neq \tau(v)$ for all edges $vu \in E$). For an integer $x \in \{0, 1, \ldots, k-1\}$ take a binary representation $x = \sum_{i=0}^{r-1} \varepsilon_i(x) 2^i$, $\varepsilon_i(x) \in \{0, 1\}$, where $r = \lceil \log_2 k \rceil$. Consider the following $2r$ subsets of E, two subsets $E_{i,+}$ and $E_{i,-}$ for each $i \in \{0, 1, \ldots, k-1\}$:

$$E_{i,+} = \{vu \in E \colon \varepsilon_i(\tau(v)) = 0, \varepsilon_i(\tau(u)) = 1\},$$
$$E_{i,-} = \{vu \in E \colon \varepsilon_i(\tau(v)) = 1, \varepsilon_i(\tau(u)) = 0\}.$$

Each of them is admissible, and they cover E, thus $b(G) \leqslant 2r$.

Note that $\chi(G_1) = n$, thus $b(G_1) \geqslant \log_2 n$. Actually we have $b(G_1) = \lceil \log_2 n \rceil$: indeed, if we define $\tau(v) = v - 1$ for all $v \in [n] = \Omega_1$, then the above sets $E_{i,+}$ cover all edges of G_1.

The Lemma above now yields for our number $m = \chi(G_3) = b(G_2)$ the following bounds, which are better than required:

$$b(G_2) \geqslant \log_2 \chi(G_2) = \log_2 b(G_1) = \log_2 \lceil \log_2 n \rceil$$
$$b(G_2) \leqslant 2\lceil \log_2 \chi(G_2) \rceil = 2\lceil \log_2 b(G_1) \rceil = 2\lceil \log_2 \lceil \log_2 n \rceil \rceil.$$

Remark. Actually the upper bound in the Lemma may be improved to $(1 + o(1)) \log_2 \chi(G)$ that yields $m = (1 + o(1)) \log_2 \log_2 n$.

IMC 2022

Second Day, August 4, 2022

Solutions

Problem 5. We colour all the sides and diagonals of a regular polygon P with 43 vertices either red or blue in such a way that every vertex is an endpoint of 20 red segments and 22 blue segments. A triangle formed by vertices of P is called monochromatic if all of its sides have the same colour. Suppose that there are 2022 blue monochromatic triangles. How many red monochromatic triangles are there?

(proposed by Mike Daas, Universiteit Leiden)

Hint: Call two connecting edges a *cherry*. Double-count cherries.

Solution. 1 Define a *cherry* to be a set of two distinct edges from K_{43} that have a vertex in common. We observe that a monochromatic triangle always contains three monochromatic cherries, and that a polychromatic triangle always contains one monochromatic cherry and two polychromatic cherries. Therefore we study the quantity $2M - P$, where M is the number of monochromatic cherries and P is the number of polychromatic cherries. By observing that every cherry is part of a unique triangle, we can split this quantity up into all the distinct triangles in K_{43}. By construction the contribution of a polychromatic triangle will vanish, whereas a monochromatic triangle will contribute 6. We conclude that

$$2M - P = 6 \cdot \{\text{number of monochromatic triangles}\}.$$

Consider any vertex v. Let M_v be the number of monochromatic cherries with central vertex v and P_v the number such polychromatic cherries. It then follows that

$$M_v = \frac{20 \cdot 19}{2} + \frac{22 \cdot 21}{2} = 421 \quad \text{and} \quad P_v = 20 \cdot 22 = 440.$$

In other words, for any vertex v it holds that $2M_v - P_v = 402$. Adding up all these contributions, we find that

$$2M - P = 43 \cdot 402.$$

We conclude that there are $43 \cdot 402/6 = 43 \cdot 67 = 2881$ monochromatic triangles in total. Since 2022 of these were blue, 859 must be red.

Problem 6. Let $p > 2$ be a prime number. Prove that there is a permutation $(x_1, x_2, ..., x_{p-1})$ of the numbers $(1, 2, ..., p-1)$ such that

$$x_1x_2 + x_2x_3 + ... + x_{p-2}x_{p-1} \equiv 2 \pmod{p}.$$

(proposed by Giorgi Arabidze, Tbilisi Free University, Georgia)

Hint:

Solution 1. We show such a permutation.

Let $x_i \equiv i^{-1} \pmod{p}$ for $i = 1, 2, \cdots, p-1$. Then

$$\sum_{i=1}^{p-2} x_i x_{i+1} \equiv \sum_{i=1}^{p-2} \frac{1}{i} \cdot \frac{1}{i+1} \equiv \sum_{i=1}^{p-2} \left(\frac{1}{i} - \frac{1}{i+1}\right) \equiv 1 - \frac{1}{p-1} \equiv \frac{p-2}{p-1} \equiv 2 \pmod{p}$$

Solution 2. We begin by noting that the identity permutation yields the value

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (p-2)(p-1) = 2 \cdot \binom{p}{3} \equiv 0 \pmod{p}$$

as soon as $p > 3$. The idea now is to perturb that permutation to obtain the desired value 2.

One thing we can do is to replace $(i, i+1, i+2, i+3)$ by $(i, i+2, i+1, i+3)$. Indeed, this will decrease the sum by 3. So if $p \equiv 2 \pmod 3$, we can just take the permutation $(1, 3, 2, 4, 6, 5, 7, \ldots, p-4, p-2, p-3, p-1)$ i.e. exchanging $3k-1$ and $3k$ whenever $k = 1, 2, \ldots, \frac{p-2}{3}$. This means we decrease the sum $\frac{p-2}{3}$ times by 3, leading to a remaining sum of $-(p-2) \equiv 2 \pmod{p}$.

If $p \equiv 1 \pmod 3$, this strategy does not work immediately. Instead, we can change $(1, 2, 3, 4, 5)$ to $(1, 4, 3, 2, 5)$ resulting in a decrement of the sum by 8. If we then exchange $3k$ and $3k+1$ for $k = 2, 3, \ldots, \frac{p-7}{3}$ as before, we get another $\frac{p-10}{3}$ times a decrement by 3, leading to a remaining sum of $-8 - \frac{p-10}{3} \cdot 3 \equiv 2 \pmod{p}$.

Of course this only works if $p \geq 13$. It thus remains to consider the cases $p = 3$ and $p = 7$ by hand. For $p = 3$, we just take $(1, 2)$ and for $p = 7$ we can take $(1, 4, 5, 2, 3, 6)$.

Problem 7. Let $A_1, A_2, \dots, A_k$ be $n \times n$ idempotent complex matrices such that

$$A_i A_j = -A_j A_i \quad \text{for all } i \neq j.$$

Prove that at least one of the given matrices has rank $\leq \frac{n}{k}$.

(A matrix A is called idempotent if $A^2 = A$.)

(proposed by Danila Belousov, Novosibirsk)

Hint: Consider the trace and the rank of A.

Solution 1.

Lemma. For any idempotent matrix B

$$\operatorname{tr}(B) = \operatorname{rank}(B)$$

Proof. Observe that an idempotent matrix satisfies the equation $\lambda(1-\lambda) = 0$. Hence the minimal polynomial is a product of linear factors and the matrix is diagonalizable. Therefore, the rank of the matrix equals the number of non-zero eigenvalues. Since the matrix has eigenvalues 0 or 1, this provides that the trace is equal to the number of unity eigenvalues, or non-zero eigenvalues.

It can be shown that $\sum\limits_{i=1}^{k} A_i$ is also an idempotent. Indeed,

$$\left(\sum_{i=1}^{k} A_i\right)^2 = \sum_{i=1}^{k} A_i^2 + \sum_{i \neq j} (A_i A_j + A_j A_i) = \sum_{i=1}^{k} A_i$$

Applying the lemma one can obtain

$$\sum_{i=1}^{k} rank(A_i) = \sum_{i=1}^{k} tr(A_i) = tr\left(\sum_{i=1}^{k} A_i\right) = rank\left(\sum_{i=1}^{k} A_i\right) \leqslant n$$

The required inequality follows.

Solution 2. We first prove that for idempotents A, B with $AB = -BA$ we already must have $AB = BA = 0$. Indeed, it is clear that $ABx = BAx = 0$ for $x \in \ker(A)$ so it suffices to prove the same for $x \in \operatorname{im}(A)$, i.e. when $Ax = x$. But then writing $Bx = y$ we have $Ay = -y$ i.e. $y = -Ay = -A^2 y = Ay = -y$ and hence $y = 0$ so that again $ABx = BAx = 0$.

Henceforth, we can assume the stronger condition $A_i A_j = 0$ for all $i \neq j$. We next claim that all the image spaces V_i of A_i are linearly independent. This will imply the claim, since then the sum of their dimensions can be at most n, and so one of them has to be $\leq \frac{n}{k}$. Now, for the sake of contradiction, suppose that $\sum_i v_i = 0$ with $v_i \in V_i$ and w.l.o.g. $v_1 \neq 0$. But then

$$0 = A_1(v_1 + \dots + v_k) = v_1 + A_1 v_2 + \dots + A_1 v_k = v_1 + A_1 A_2 v_2 + \dots + A_1 A_k v_k = v_1$$

since $A_1 A_i = 0$ for all i.

Remark. Here is a different argument for $AB = BA = 0$, without eigenvectors: multiplying by A and using its idempotence and the super-commutativity , we have

$$-BA = AB = A^2 B = AAB = -ABA = BAA = BA^2 = BA$$

thus $BA = 0$.

Problem 8. Let $n, k \geq 3$ be integers, and let S be a circle. Let n blue points and k red points be chosen uniformly and independently at random on the circle S. Denote by F the intersection of the convex hull of the red points and the convex hull of the blue points. Let m be the number of vertices of the convex polygon F (in particular, $m = 0$ when F is empty). Find the expected value of m.

(proposed by Fedor Petrov, St. Petersburg)

Hint:

Solution 1. We prove that

$$E(m) = \frac{2kn}{n+k-1} - 2\frac{k!n!}{(k+n-1)!}.$$

Let $A_1, \ldots, A_n$ be blue points. Fix $i \in \{1, \ldots, n\}$. Enumerate our $n+k$ points starting from a blue point A_i counterclockwise as $A_i, X_{1,i}, X_{2,i}, \ldots, X_{(n+k-1),i}$. Denote the minimal index j for which the point $X_{j,i}$ is blue as $m(i)$. So, $A_i X_{m(i),i}$ is a side of the convex hull of blue points. Denote by b_i the following random variable:

$$b_i = \begin{cases} 1, & \text{if the chord } A_i X_{m(i),i} \text{ contains a side of } F \\ 0, & \text{otherwise.} \end{cases}$$

Define analogously k random variables $r_1, \ldots, r_k$ for the red points. Clearly,

$$m = b_1 + \ldots + b_n + r_1 + \ldots + r_k. \qquad (\heartsuit)$$

We proceed with computing the expectation of each b_i and r_j. Note that $b_i = 0$ if and only if all red points lie on the side of the line $A_i X_{m(i),i}$. This happens either if $m(i) = 1$, *i.e., the point $X_{i,1}$ is blue* (which happens with probability $\frac{n-1}{k+n-1}$), or if $i = k+1$, *points $X_{1,i}, \ldots, X_{k,i}$ are red, and points $X_{k+1,i}, \ldots, X_{k+n-1,i}$ are blue* (which happens with probability $1/\binom{k+n-1}{k}$, since all subsets of size k of $\{1, 2, \ldots, n+k-1\}$ have equal probabilities to correspond to the indices of red points between $X_{1,i}, \ldots, X_{n+k-1,i}$). Thus the expectation of b_i equals $1 - \frac{n-1}{k+n-1} - 1/\binom{k+n-1}{k} = \frac{k}{n+k-1} - \frac{k!(n-1)!}{(k+n-1)!}$. Analogously, the expectation of r_j equals $\frac{n}{n+k-1} - \frac{n!(k-1)!}{(k+n-1)!}$. It remains to use $(\heartsuit)$ and linearity of expectation.

Solution 2. Let $C_1, \ldots, C_{n+k}$ be the colours of the points, scanned counterclockwise from a fixed point on the circle. We consider the sequence as cyclic (so C_{n+k} is also adjacent to C_1). There are two cases: Either (i) all red points appear contiguously, followed by all blue points contiguously, or (ii) the red and blue points alternate at least twice. It can be seen that in the second case, m is exactly equal to the number of colour changes in the C_i sequence: For example, if C_i is red and C_{i+1} is blue, then the intersection of the red chord from C_i to the next red point with the blue chord from C_{i+1} to the previous blue point is a vertex of F, and every vertex is of this form. Case (i) is exceptional, as we have two colour changes, but $m = 0$, so it is 2 less than the number of changes in that case.

Now observe that the distribution of C_i is purely combinatorial: Each of the $\binom{n+k}{n,k}$ distributions of colours is equally likely (for example, because we can generate the distribution by first choosing all $n+k$ points on the circle, and then assigning colours uniformly). In particular the probability that $C_i C_{i+1}$ is a colour change is exactly $\frac{2nk}{(n+k)(n+k-1)}$, and by lineraity of expectation, the total expected number of color changes (including $i = n+k$) is $n+k$ times this, i.e. $\frac{2nk}{n+k-1}$.

To get the expected value of m, we must subtract from the above 2 times the probability of case (i). Exactly $n+k$ of the $\binom{n+k}{n,k}$ distributions belong to case (i), so we must subtract $2(n+k)\binom{n+k}{n,k}^{-1} = 2\frac{n!k!}{(n+k-1)!}$, as claimed.

Solution 3. Let $A_1, \ldots, A_n$ be the blue points and $B_1, \ldots, B_k$ be the red points. For every pair of blue points A_i, A_j, $1 \leqslant i < j \leqslant n$, we evaluate the probability p that A_iA_j contains a side of F (it obviously does not depend on the choice of i and j). By q denote the analogous probability for the red points. Then by linearity of expectation we have $\mathbb{E}m = \binom{n}{2}p + \binom{k}{2}q$.

We proceed with finding p. Without loss of generality $i = 1$, $j = 2$. Let the length of the circle be 1, and the length of arc A_1A_2 (counterclockwise from A_1 to A_2) be x. Then x is uniformly distributed on $[0, 1]$. Then A_1A_2 contains a side of F if

(i) all blue points are on the same side of A_1A_2, but

(ii) the red points are not on the same side of A_1A_2.

The probability of (i) is $x^{n-2} + (1-x)^{n-2}$. The probability of (ii) is $1 - (x^k + (1-x)^{n-k})$.

Thus, using Beta function value $B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = B(a, b) = \frac{(a-1)!(b-1)!}{(a+b-1)!}$ for positive integers a, b

$$\begin{aligned}
p &= \int_0^1 (x^{n-2} + (1-x)^{n-2})(1 - (x^k + (1-x)^{n-k}))dx = \frac{2}{n-1} - \frac{2}{n+k-1} - 2B(n-1, k+1) \\
&= \frac{2}{n-1} - \frac{2}{n+k-1} - 2\frac{(n-2)!k!}{(n+k-1)!}.
\end{aligned}$$

Next,

$$\binom{n}{2}p = n - \frac{n(n-1)}{n+k-1} - \frac{n!k!}{(n+k-1)!} = \frac{nk}{n+k-1} - \frac{n!k!}{(n+k-1)!},$$

and by symmetry $\binom{k}{2}q$ takes the same value (that is in agreement with the observation that red and blue sides of F alternate).

IMC 2023

First Day, August 2, 2023

Solutions

Problem 1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that have a continuous second derivative and for which the equality $f(7x+1) = 49f(x)$ holds for all $x \in \mathbb{R}$.

(proposed by Alex Avdiushenko, Neapolis University Paphos, Cyprus)

Hint:

- The fixed point of $7x+1$ is $-1/6$.
- Differentiating twice cancels out the coefficient 49.

Solution. Differentiating the equation twice, we get

$$f''(7x+1) = f''(x) \quad \text{or} \quad f''(x) = f''\left(\frac{x-1}{7}\right). \tag{1}$$

Take an arbitrary $x \in \mathbb{R}$, and construct a sequence by the recurrence

$$x_0 = x, \quad x_{k+1} = \frac{x_k - 1}{7}.$$

By (1), the values of f'' at all points of this sequence are equal. The limit of this sequence is $-\frac{1}{6}$, since $\left|x_{k+1} + \frac{1}{6}\right| = \frac{1}{7}\left|x_k + \frac{1}{6}\right|$.

Due to the continuity of f'', the values of f'' at all points of this sequence are equal to $f''\left(-\frac{1}{6}\right)$, which means that $f''(x)$ is a constant.

Then f is an at most quadratic polynomial, $f(x) = ax^2 + bx + c$. Substituting this expression into the original equation, we get a system of equations, from which we find $a = 36c$, $b = 12c$, and hence

$$f(x) = c(6x+1)^2.$$

Problem 2. Let A, B and C be $n \times n$ matrices with complex entries satisfying

$$A^2 = B^2 = C^2 \quad \text{and} \quad B^3 = ABC + 2I.$$

Prove that $A^6 = I$.

(proposed by Mike Daas, Universiteit Leiden)

Hint: Factorize $B^3 - ABC$.

Solution. Note that $B^3 = A^2B$, from which it follows that

$$A^2B - ABC = 2I \implies A(AB - BC) = 2I.$$

Similarly, using that $B^3 = BC^2$, we find that

$$BC^2 - ABC = 2I \implies (BC - AB)C = 2I.$$

It follows that A is a left-inverse of $(AB - BC)/2$, whereas $-C$ is a right inverse. Hence $A = -C$ and as such, it must hold that $ABA = 2I - B^3$. It follows that ABA must commute with B, and so it follows that $(AB)^2 = (BA)^2$. Now we compute that

$$(AB - BA)(AB + BA) = (AB)^2 + AB^2A - BA^2B - (BA)^2 = (AB)^2 + A^4 - B^4 - (AB)^2 = 0.$$

However, we noted before that the matrix $AB - BC = AB + BA$ must be invertible. As such, it must follow that $AB = BA$. We conclude that $ABA = A^2B = B^3$ and so it readily follows that $B^3 = I$. Finally, $A^6 = B^6 = (B^3)^2 = I^2 = I$, completing the proof.

Problem 3. Find all polynomials P in two variables with real coefficients satisfying the identity

$$P(x, y)P(z, t) = P(xz - yt, xt + yz).$$

(proposed by Giorgi Arabidze, Free University of Tbilisi, Georgia)

Hint: The polynomials $(x+iy)^n$ and $(x-iy)^m$ are trivial complex solutions. Suppose that $P(x, y) = (x+iy)^n(x-iy)^m Q(x, y)$, where $Q(x, y)$ is divisible neither by $x+iy$ nor $x = iy$ and consider $Q(x, y)$.

Solution. First we find all polynomials $P(x, y)$ with complex coefficients which satisfies the condition of the problem statement. The identically zero polynomial clearly satisfies the condition. Let consider other polynomials.

Let $i^2 = -1$ and $P(x, y) = (x + iy)^n(x - iy)^m Q(x, y)$, where n and m are non-negative integers and $Q(x, y)$ is a polynomial with complex coefficients such that it is not divisible neither by $x + iy$ nor by $x - iy$. By the problem statement we have $Q(x, y)Q(z, t) = Q(xz - yt, xt + yz)$. Note that $z = t = 0$ gives $Q(x, y)Q(0, 0) = Q(0, 0)$. If $Q(0, 0) \neq 0$, then $Q(x, y) = 1$ for all x and y. Thus $P(x, y) = (x + iy)^n(x - iy)^m$. Now consider the case when $Q(0, 0) = 0$.

Let $x = iy$ and $z = -it$. We have $Q(iy, y)Q(-it, t) = Q(0, 0) = 0$ for all y and t. Since $Q(x, y)$ is not divisible by $x - iy$, $Q(iy, y)$ is not identically zero and since $Q(x, y)$ is not divisible by $x + iy$, $Q(-it, t)$ is not identically zero. Thus there exist y and t such that $Q(iy, y) \neq 0$ and $Q(-it, t) \neq 0$ which is impossible because $Q(iy, y)Q(-it, t) = 0$ for all y and t.

Finally, $P(x, y)$ polynomials with complex coefficients which satisfies the condition of the problem statement are $P(x, y) = 0$ and $P(x, y) = (x + iy)^n(x - iy)^m$. It is clear that if $n \neq m$, then $P(x, y) = (x + iy)^n(x - iy)^m$ cannot be polynomial with real coefficients. So we need to require $n = m$, and for this case $P(x, y) = (x + iy)^n(x - iy)^n = (x^2 + y^2)^n$.

So, the answer of the problem is $P(x, y) = 0$ and $P(x, y) = (x^2+y^2)^n$ where n is any non-negative integer.

Problem 4. Let p be a prime number and let k be a positive integer. Suppose that the numbers $a_i = i^k + i$ for $i = 0, 1, \ldots, p-1$ form a complete residue system modulo p. What is the set of possible remainders of a_2 upon division by p?

(proposed by Tigran Hakobyan, Yerevan State University, Armenia)

Hint: Consider $\prod_{i=0}^{p-1}(i^k + i)$.

Solution. First observe that $p = 2$ does not satisfy the condtion, so p must be an odd prime.

Lemma. If $p > 2$ is a prime and $\mathbb{F}_p$ is the field containing p elements, then for any integer $1 \le n < p$ one has the following equality in the field $\mathbb{F}_p$

$$\prod_{\alpha \in \mathbb{F}_p^*} (1 + \alpha^n) = \begin{cases} 0, & \text{if } \dfrac{p-1}{\gcd(p-1, n)} \text{ is even} \\ 2^n, & \text{otherwise} \end{cases}$$

Proof. We may safely assume that $n | p-1$ since it can be easily proved that the set of n-th powers of the elements of $\mathbb{F}_p^*$ coincides with the set of $\gcd(p-1, n)$-th powers of the same elements. Assume that $t_1, t_2, ..., t_n$ are the roots of the polynomial $t^n + 1 \in \mathbb{F}_p[x]$ in some extension of the field $\mathbb{F}_p$. It follows that

$$\prod_{\alpha \in \mathbb{F}_p^*} (1 + \alpha^n) = \prod_{\alpha \in \mathbb{F}_p^*} \prod_{i=1}^{n} (\alpha - t_i) = \prod_{i=1}^{n} \prod_{\alpha \in \mathbb{F}_p^*} (\alpha - t_i) = \prod_{i=1}^{n} \prod_{\alpha \in \mathbb{F}_p^*} (t_i - \alpha) = \prod_{i=1}^{n} \Phi(t_i),$$

where we define $\Phi(t) = \prod_{\alpha \in \mathbb{F}_p^*} (t - \alpha) = t^{p-1} - 1$. Therefore

$$\prod_{\alpha \in \mathbb{F}_p^*} (1 + \alpha^n) = \prod_{i=1}^{n} (t_i^{p-1} - 1) = \prod_{i=1}^{n} ((t_i^n)^{\frac{p-1}{n}} - 1) = \prod_{i=1}^{n} ((-1)^{\frac{p-1}{n}} - 1) = \begin{cases} 0, & \text{if } \frac{p-1}{n} \text{ is even} \\ 2^n, & \text{otherwise} \end{cases}$$

Let us now get back to our problem. Suppose the numbers $i^k + i, 0 \le i \le p-1$ form a complete residue system modulo p. It follows that

$$\prod_{\alpha \in \mathbb{F}_p^*} (\alpha^k + \alpha) = \prod_{\alpha \in \mathbb{F}_p^*} \alpha$$

so that $\prod_{\alpha \in \mathbb{F}_p^*} (\alpha^{k-1} + 1) = 1$ in $\mathbb{F}_p$. According to the Lemma, this means that $2^{k-1} = 1$ in $\mathbb{F}_p$, or equivalently, that $2^{k-1} \equiv 1 (\text{mod } p)$. Therefore $a_2 = 2^k + 2 \equiv 4 (\text{mod } p)$ so that the remainder of a_2 upon division by p is either 4 when $p > 3$ or is 1, when $p = 3$.

Problem 5. Fix positive integers n and k such that $2 \le k \le n$ and a set M consisting of n fruits. A *permutation* is a sequence $x = (x_1, x_2, \ldots, x_n)$ such that $\{x_1, \ldots, x_n\} = M$. Ivan *prefers* some (at least one) of these permutations. He realized that for every preferred permutation x, there exist k indices $i_1 < i_2 < \ldots < i_k$ with the following property: for every $1 \le j < k$, if he swaps x_{i_j} and $x_{i_{j+1}}$, he obtains another preferred permutation.

Prove that he prefers at least $k!$ permutations.

(proposed by Ivan Mitrofanov, École Normale Superieur Paris)

Hint: For every permutation z of M, choose a preferred permutation x such that $\sum\limits_{m \in M} x^{-1}(m) z^{-1}(m)$ is maximal.

Solution. Let S be the set of all $n!$ permutations of M, and let P be the set of preferred permutations. For every permutation $x \in S$ and $m \in M$, let $x^{-1}(m)$ denote the unique number $i \in \{1, 2, \ldots, n\}$ with $x_i = m$.

For every $x \in P$, define

$$A(x) = \left\{ z \in S \ : \ \forall y \in P \ \ \sum_{m \in M} x^{-1}(m) z^{-1}(m) \ge \sum_{m \in M} y^{-1}(m) z^{-1}(m) \right\}.$$

For every permutation $z \in S$, we can choose a permutation $x \in P$ for which $\sum\limits_{m \in M} x^{-1}(m) z^{-1}(m)$ is maximal, and then we have $z \in A(x)$; hence, all $z \in S$ is contained in at least one set $A(x)$.

So, it suffices to prove that $\left|A(x)\right| \le \dfrac{n!}{k!}$ for every preferred permutation x. Fix $x \in P$, and consider an arbitrary $z \in A(x)$. Let the indices $i_1 < \ldots < i_k$ be as in the statement of the problem, and let $m_j = x_{i_j}$ for $j = 1, 2, \ldots, k$.

For $s = 1, 2, \ldots, k-1$ consider the permutation y obtained from x by swapping m_s and m_{s+1}. Since $y \in P$, the definition of $A(x)$ provides

$$i_s z^{-1}(m_s) + i_{s+1} z^{-1}(m_{s+1}) \ge i_{s+1} z^{-1}(m_s) + i_s z^{-1}(m_{s+1}),$$

$$z^{-1}(m_{s+1}) \ge z^{-1}(m_s).$$

Therefore, the elements $m_1, m_2, \ldots, m_k$ appear in z in this order. There are exactly $n!/k!$ permutations with this property, so $\left|A(x)\right| \le \dfrac{n!}{k!}$.

IMC 2023

Second Day, August 3, 2023

Solutions

Problem 6. Ivan writes the matrix $\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$ on the board. Then he performs the following operation on the matrix several times:

- he chooses a row or a column of the matrix, and
- he multiplies or divides the chosen row or column entry-wise by the other row or column, respectively.

Can Ivan end up with the matrix $\begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$ after finitely many steps?

(proposed by Alex Avdiushenko, Neapolis University Paphos, Cyprus)

Hint: Construct an invariant quantity that does not change during Ivan's prcedure.

Solution. We show that starting from $A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$, Ivan cannot reach the matrix $B = \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$.

Notice first that the allowed operations preserve the positivity of entries; all matrices Ivan can reach have only positive entries.

For every matrix $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ with positive entries, let $L(X) = \begin{pmatrix} \log_2 x_{11} & \log_2 x_{12} \\ \log_2 x_{21} & \log_2 x_{22} \end{pmatrix}$. By taking logarithms of the entries, the steps in Ivan game will be replaced by adding or subtracting a row or column to the other row. Such standard row and column operations preserve the determinant.

Hence, if the matrices in the game are $A = X_0, X_1, X_2, \ldots$, then we have $\det L(A) = \det L(X_1) = \det L(X_2) = \ldots$, and it suffices to verify that $\det L(A) \neq \det L(B)$.

Indeed,

$$\det L(A) = \log_2 2 \cdot \log_2 4 - \log_2 4 \cdot \log_2 3 = \log_2(4/3) > 0$$

and similarly $\det L(B) < 0$, so $\det L(A) \neq \det L(B)$.

Problem 7. Let V be the set of all continuous functions $f\colon [0,1] \to \mathbb{R}$, differentiable on $(0,1)$, with the property that $f(0) = 0$ and $f(1) = 1$. Determine all $\alpha \in \mathbb{R}$ such that for every $f \in V$, there exists some $\xi \in (0,1)$ such that

$$f(\xi) + \alpha = f'(\xi).$$

(proposed by Mike Daas, Leiden University)

Hint: Find a function $h \in V$ such that $h' - h$ is constant, then apply Rolle's theorem to $f - h$.
Alternatively, you can apply Cauchys's mean value theorem with some auxiliary functions.

Solution 1. First consider the function

$$h(x) = \frac{e^x - 1}{e - 1}, \quad \text{which has the property that} \quad h'(x) = \frac{e^x}{e-1}.$$

Note that $h \in V$ and that $h'(x) - h(x) = 1/(e-1)$ is constant. As such, $\alpha = 1/(e-1)$ is the only possible value that could possibly satisfy the condition from the problem. For $f \in V$ arbitrary, let

$$g(x) = f(x)e^{-x} + h(-x), \quad \text{with} \quad g(0) = 0 \quad \text{and also} \quad g(1) = e^{-1} + \frac{e^{-1} - 1}{e - 1} = 0.$$

We compute that

$$g'(x) = f'(x)e^{-x} - f(x)e^{-x} - h'(-x).$$

Now apply Rolle's Theorem to g on the interval $[0,1]$; it yields some $\xi \in (0,1)$ with the property that

$$g'(\xi) = 0 \implies f'(\xi)e^{-\xi} - f(\xi)e^{-\xi} - \frac{e^{-\xi}}{e-1} = 0 \implies f'(\xi) = f(\xi) + \frac{1}{e-1},$$

showing that $\alpha = 1/(e-1)$ indeed satisfies the condition from the problem.

Solution 2. Notice that the expression $f'(x) - f(x)$ appears in the derivative of the function $F(x) = f(x) \cdot e^{-x}$: $F'(x) = \big(f'(x) - f(x)\big)e^{-x}$.

Apply Cauchy's mean value theorem to $F(x)$ and the function $G(x) = -e^{-x}$. By the theorem, there is some $\xi \in (0,1)$ such that

$$\frac{F'(\xi)}{G'(\xi)} = \frac{F(1) - F(0)}{G(1) - G(0)}$$

$$f'(\xi) - f(\xi) = \frac{e^{-1} - 0}{-e^{-1} + 1} = \frac{1}{e-1}.$$

This proves the required property for $a = \dfrac{1}{e-1}$.

Now we show that no other α is possible. Choose f and F in such a way that $\dfrac{F'(x)}{G'(x)} = f'(x) - f(x) = \dfrac{1}{e-1}$ is constant. That means

$$F'(x) = \frac{G'(x)}{e-1} = \frac{e^{-x}}{e-1},$$

$$F(x) = \frac{1 - e^{-x}}{e-1},$$

$$f(x) = F(x) \cdot e^x = \frac{e^x - 1}{e-1}.$$

With this choice we have $f(0) = 0$ and $f(1) = 1$, so $f' \in V$, and $f'(x) - f(x) \equiv \dfrac{1}{e-1}$ for all x, so for this function the only possible value for α is $\dfrac{1}{e-1}$.

Problem 8. Let T be a tree with n vertices; that is, a connected simple graph on n vertices that contains no cycle. For every pair u, v of vertices, let $d(u,v)$ denote the distance between u and v, that is, the number of edges in the shortest path in T that connects u with v.

Consider the sums

$$W(T) = \sum_{\substack{\{u,v\}\subseteq V(T)\\ u\neq v}} d(u,v) \quad \text{and} \quad H(T) = \sum_{\substack{\{u,v\}\subseteq V(T)\\ u\neq v}} \frac{1}{d(u,v)}.$$

Prove that

$$W(T)\cdot H(T) \geq \frac{(n-1)^3(n+2)}{4}.$$

(proposed by Slobodan Filipovski, University of Primorska, Koper)

Hint: There are $n-1$ pairs u, v with $d(u,v)=1$; in all other cases $d(u,v)\geq 2$.

Solution. Let $k=\binom{n}{2}$ and let $x_1 \leq x_2 \leq \ldots \leq x_k$ be the distances between the pairs of vertices in the tree T. Thus

$$W(T)\cdot H(T) = (x_1+x_2+\ldots+x_k)\cdot\left(\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_k}\right).$$

Since the tree has exactly $n-1$ edges, there are exactly $n-1$ pairs of vertices at distance one, that is, $x_1 = x_2 = \ldots = x_{n-1} = 1$. Thus

$$W(T)\cdot H(T) = (n-1+x_n+x_{n+1}+\ldots+x_k)\cdot\left(n-1+\frac{1}{x_n}+\frac{1}{x_{n+1}}+\ldots+\frac{1}{x_k}\right) =$$

$$= (n-1)^2 + (n-1)\left(\left(x_n+\frac{1}{x_n}\right)+\ldots+\left(x_k+\frac{1}{x_k}\right)\right)+$$

$$+(x_n+\ldots+x_k)\left(\frac{1}{x_n}+\ldots+\frac{1}{x_k}\right).$$

From Cauchy inequality we have

$$(x_n+\ldots+x_k)\left(\frac{1}{x_n}+\ldots+\frac{1}{x_k}\right) \geq (1+1+\ldots+1)^2 = (k-n+1)^2 = \frac{(n-1)^2(n-2)^2}{4}.$$

The equality holds if and only if $x_n = x_{n+1} = \ldots = x_k$.
Now we minimize the expression $\left(x_n+\frac{1}{x_n}\right)+\ldots+\left(x_k+\frac{1}{x_k}\right)$, where $x_i \in [2, n-1]$.
It is clear that the minimal value is achieved for $x_n = x_{n+1} = \ldots = x_k = 2$. Therefore we get

$$W(T)\cdot H(T) \geq (n-1)^2 + (n-1)\left(\left(2+\frac{1}{2}\right)(k-n+1)\right) + \frac{(n-1)^2(n-2)^2}{4} = \frac{(n-1)^3(n+2)}{4}.$$

The equality holds for $x_1 = \ldots = x_{n-1} = 1$ and $x_n = x_{n+1} = \ldots = x_k = 2$, that is, the smallest value is achieved for the tree where $n-1$ pairs are at distance one, and the remaining $k-(n-1) = \frac{(n-1)(n-2)}{2}$ pairs are at distance two. The unique tree which satisfies these conditions is the star graph S_n. In this case it holds

$$W(S_n)\cdot H(S_n) = (n-1)^2\cdot\frac{(n-1)(n+2)}{4} = \frac{(n-1)^3(n+2)}{4}.$$

Problem 9. We say that a real number V is *good* if there exist two closed convex subsets X, Y of the unit cube in $\mathbb{R}^3$, with volume V each, such that for each of the three coordinate planes (that is, the planes spanned by any two of the three coordinate axes), the projections of X and Y onto that plane are disjoint.

Find $\sup\{V \mid V \text{ is good}\}$.

(proposed by Josef Tkadlec and Arseniy Akopyan)

Hint: The two bodies can be replaced by a pair symmetric to the midpoint of the cube.

Solution. We prove that $\sup\{V \mid V \text{ is good}\} = 1/4$.

We will use the unit cube $U = [-1/2, 1/2]^3$.

For $\varepsilon \to 0$, the axis-parallel boxes $X = [-1/2, -\varepsilon] \times [-1/2, -\varepsilon] \times [-1/2, 1/2]$ and $Y = [\varepsilon, 1/2] \times [\varepsilon, 1/2] \times [-1/2, 1/2]$ show that $\sup\{V\} \geq 1/4$.

To prove the other bound, consider two admissible convex bodies X, Y. For any point $P = [x, y, z] \in U$ with $xyz \neq 0$, let $\overline{P} = \{[\pm x, \pm y, \pm z]\}$ be the set consisting of 8 points (the original P and its 7 "symmetric" points). If for each such P we have $|\overline{P} \cap (X \cup Y)| \leq 4$, then the conclusion follows by integrating. Suppose otherwise and let P be a point with $|\overline{P} \cap (X \cup Y)| \geq 5$. Below we will complete the proof by arguing that:

(1) we can replace one of the two bodies (the "thick" one) with the reflection of the other body about the origin, and

(2) for such symmetric pairs of bodies we in fact have $|\overline{P} \cap (X \cup Y)| \leq 4$, for all P.

To prove Claim (1), we say that a convex body is *thick* if each of its three projections contains the origin. We claim that one of the two bodies X, Y is thick. This is a short casework on the 8 points of $\overline{P}$. Since $|\overline{P} \cap (X \cup Y)| \geq 5$, by pigeonhole principle, we find a pair of points in $\overline{P} \cap (X \cup Y)$ symmetric about the origin. If both points belong to one body (say to X), then by convexity of X the origin belongs to X, thus X is thick. Otherwise, label $\overline{P}$ as $ABCDA'B'C'D'$. Wlog $A \in X$, $C' \in Y$ is the pair of points in $\overline{P}$ symmetric about the origin. Wlog at least 3 points of $\overline{P}$ belong to X. Since X, Y have disjoint projections, we have $C, B', D' \notin X$, so wlog $B, D \in X$. Then Y can contain no other point of $\overline{P}$ (apart from C'), so X must contain at least 4 points of $\overline{P}$ and thus $A' \in X$. But then each projection of X contains the origin, so X is indeed thick.

Note that if X is thick then none of the three projections of Y contains the origin. Consider the reflection $Y' = -Y$ of Y about the origin. Then (Y, Y') is an admissible pair with the same volume as (X, Y): the two bodies Y and Y' clearly have equal volumes V and they have disjoint projections (by convexity, since the projections of Y miss the origin). This proves Claim (1).

Claim (2) follows from a similar small casework on the 8-tuple $\overline{P}$: For contradiction, suppose $|\overline{P} \cap Y'| = |\overline{P} \cap Y| \geq 3$. Wlog $A \in Y'$. Then $C' \in Y$, so $C, B', D' \notin Y'$, so wlog $B, D \in Y'$. Then $B', D' \in Y$, a contradiction with (Y, Y') being admissible.

Remark. There are more examples with $V \to 1/4$, e.g. X a union of two triangular pyramids with base ACD' – one with apex D, one with apex at the origin (and Y symmetric with X about the origin).

Remark. The word "convex" matters. E.g., in a $3 \times 3 \times 3$ cube, one can set X to be a $2 \times 2 \times 2$ sub-cube, and Y to be the (non-convex) 3D L-shape consisting of 7 unit cubes. This shows that without convexity we have $V \geq 7/27 > 1/4$.

Problem 10. For every positive integer n, let $f(n), g(n)$ be the minimal positive integers such that

$$1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}=\frac{f(n)}{g(n)}.$$

Determine whether there exists a positive integer n for which $g(n)>n^{0.999\,n}$.

(proposed by Fedor Petrov, St. Petersburg State University)

Solution. We show that there does exist such a number n.

Let $\varepsilon=10^{-10}$. Call a prime p *special*, if for certain $k\in\{1,2,\ldots,p-1\}$ there exist at least $\varepsilon\cdot k$ positive integers $j\le k$ for which p divides $f(j)$.

Lemma. There exist only finitely many special primes.

Proof. Let p be a special prime number, and p divides $f(j)$ for at least $\varepsilon\cdot k$ values of $j\in\{1,2,\ldots,k\}$. Note that if p divides $f(j)$ and $f(j+r)$, then p divides

$$(j+r)!\left(\frac{f(j+r)}{g(j+r)}-\frac{f(j)}{g(j)}\right)=1+(j+r)+(j+r)(j+r-1)+\ldots+(j+r)\ldots(j+2)$$

that is a polynomial of degree $r-1$ with respect to j. Thus, for fixed j it equals to 0 modulo p for at most $r-1$ values of j. Look at our $\ge\varepsilon\cdot k$ values of $j\in\{1,2,\ldots,k\}$ and consider the gaps between consecutive j's. The number of such gaps which are greater than $2/\varepsilon$ does not exceed $\varepsilon\cdot k/2$ (since the total sum of gaps is less than k). Therefore, at least $\varepsilon\cdot k/2-1$ gaps are at most $2/\varepsilon$. But the number of such small gaps is bounded from above by a constant (not depending on k) by the above observation. Therefore, k is bounded, and, since p divides $f(1)f(2)\ldots f(k)$, p is bounded too.

Now we want to bound the product $g(1)g(2)\ldots g(n)$ (for a large integer n) from below. Let $p\le n$ be a non-special prime. Our nearest goal is to prove that

$$\nu_p\big(g(1)g(2)\ldots g(n)\big)\ge(1-\varepsilon)\nu_p(1!\cdot 2!\cdot\ldots\cdot n!) \qquad (1)$$

Partition the numbers $p,p+1,\ldots,n$ onto the intervals of length p (except possibly the last interval which may be shorter): $\{p,p+1,\ldots,2p-1\}$, $\ldots$, $\{p\lfloor n/p\rfloor,\ldots,n\}$. Note that in every interval $\Delta=[a\cdot p,a\cdot p+k]$, all factorials $x!$ with $x\in\Delta$ have the same p-adic valuation, denote it $T=\nu_p\big((ap)!\big)$. We claim that at least $(1-\varepsilon)(k+1)$ valuations of $g(x)$, $x\in\Delta$, are equal to the same number T. Indeed, if $j=0$ or $1\le j\le k$ and $f(j)$ is not divisible by p, then

$$\frac{1}{(ap)!}+\frac{1}{(ap+1)!}+\ldots+\frac{1}{(ap+j)!}=\frac{1}{(ap)!}\cdot\frac{A}{B}$$

where $A\equiv f(j)\pmod p$, $B\equiv g(j)\pmod p$, so, this sum has the same p-adic valuation as $1/(ap)!$, which is strictly less than that of the sum $\sum_{i=0}^{ap-1}1/i!$, that yields $\nu_p\big(g(ap+j)\big)=\nu_p\big((ap)!\big)$. Using this for every segment Δ, we get (1).

Now, using (1) for all non-special primes, we get

$$A\cdot g(1)g(2)\ldots g(n)\ge(1!\cdot 2!\cdot\ldots\cdot n!)^{1-\varepsilon},$$

where $A=\prod_{p,k}p^{\nu_p(g(k))}$, p runs over non-special primes, k from 1 to n. Since $\nu_p(g(k))\le\nu_p(k!)=\sum_{i=1}^{\infty}\lfloor k/p^i\rfloor\le k$, we get

$$A\le(\prod_p p)^{1+2+\ldots+n}\le C^{n^2}$$

for some constant C. But if we had $g(n)\le n^{0.999n}\le e^n n!^{0.999}$ for all n, then

$$\log\big(A\cdot g(1)g(2)\ldots g(n)\big)\le O(n^2)+0.999\log(1!\cdot 2!\cdot\ldots\cdot n!)<(1-\varepsilon)\log(1!\cdot 2!\cdot\ldots\cdot n!)$$

for large n, a contradiction.

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IMC 2024

Second Day, August 8, 2024

Solutions

Problem 6. Prove that for any function $f\colon \mathbb{Q} \to \mathbb{Z}$, there exist $a, b, c \in \mathbb{Q}$ such that $a < b < c$, $f(b) \geq f(a)$, and $f(b) \geq f(c)$.

(proposed by Mehdi Golafshan & Markus A. Whiteland, University of Liège, Liège)

Solution 1. We can replace $f(x)$ by the function $g(x) = f(1-x)$, so without loss of generality we can assume $f(0) \leq f(1)$.

If $f(1) \geq f(2)$ then we can choose $(a, b, c) = (0, 1, 2)$. Otherwise we have $f(0) \leq f(1) < f(2)$.

If there is some $x \in (1, 2)$ such that $f(x) \geq f(2)$ then we can chose $(a, b, c) = (1, x, 2)$; similarly, if there is some $x \in (1, 2)$ with $f(x) \leq f(1)$ then choose $(a, b, c) = (0, 1, x)$. Hence, in the remaining cases we have $f(1) \leq f(x) \leq f(2)$ for all $x \in (1, 2)$.

Now f is bonded on the interval $[1, 2]$, so it has only finitely many values on this interval. Since there are infintely many rational numbers in $[0, 1]$, there is a value y that is attained infinitely many times. The we can choose $1 \leq a < b < c \leq 2$ such that $f(a) = f(b) = f(c) = y$.

Solution 2. Assume towards a contradiction that there is a function f which does not satisfy the claim: for all rationals a, b, c with $a < b < c$ we have $f(b) < f(a)$ or $f(b) < f(c)$.

Let x and y be arbitrary rationals with $x < y$. Let $I(x, y) = [x, y] \cap \mathbb{Q}$. We first observe that $\inf f(I(x, y)) = -\infty$. Indeed, if the infimum was finite, then, as the set $f(I(x, y))$ is bounded ($\sup f(I(x, y)) = \max\{f(x), f(y)\}$) and thus finite, there are three points having the same value under f, which leads to a contradiction regarding our assumption on f.

So, going back to the question at hand, let x, b, y be arbitrary rationals with $x < b < y$. Applying the above observation to the set $I(x, b)$, there exists a point $a \in I(x, b)$ such that $f(a) < f(b)$. Similarly, there exists a point $c \in I(b, y)$ such that $f(c) < f(b)$. Hence we have the points a, b, c with $a < b < c$ and $f(b) > \max\{f(a), f(c)\}$, which contradicts our assumption on f.

Problem 7. Let n be a positive integer. Suppose that A and B are invertible $n \times n$ matrices with complex entries such that $A + B = I$ (where I is the identity matrix) and

$$(A^2 + B^2)(A^4 + B^4) = A^5 + B^5.$$

Find all possible values of $\det(AB)$ for the given n.

(proposed by Sergey Bondarev, Sergey Chernov, Belarusian State University, Minsk)

Hint: Find a polynomial $p(x)$ such that $p(AB) = 0$.

Solution 1. Notice first that $AB = A(I - A) = A - A^2 = (I - A)A = BA$, so A and B commute.
Let $C = AB = BA$; then

$$\begin{aligned} A^2 + B^2 &= (A + B)^2 - 2AB = I - 2C, \\ A^4 + B^4 &= (A + B)^4 - 4AB(A + B)^2 + 2A^2B^2 = I - 4C + 2C^2, \\ A^5 + B^5 &= (A + B)^5 - 5AB(A + B)^3 + 5A^2B^2(A + B) = I - 5C + 5C^2, \end{aligned}$$

so

$$\begin{aligned} 0 &= (A^5 + B^5) - (A^2 + B^2)(A^4 + B^4) = (I - 5C + 5C^2) - (I - 2C)(I - 4C + 2C^2) \\ &= 4C^3 - 5C^2 + C = 4C(C - I)(C - \tfrac{1}{4}I); \end{aligned}$$

since C is invertible, we have

$$(C - I)(C - \tfrac{1}{4}I) = 0.$$

Hence, the polynomial $p(x) = (x - 1)(x - \frac{1}{4})$ annihilates the matrix $C = AB$ and therefore all eigenvalues of C are roots of $p(x)$, so the possible eigenvalues are 1 and $\frac{1}{4}$. The determinant is the product of the n eigenvalues, so

$$\det(AB) = \det C \in \left\{1, \tfrac{1}{4}, \tfrac{1}{4^2}, \dots, \tfrac{1}{4^n}\right\}.$$

Now show that these values are indeed possible.
If

$$A = \operatorname{diag}\Big(\underbrace{\tfrac{1}{2}, \dots, \tfrac{1}{2}}_{k}, \underbrace{e^{i\pi/3}, \dots, e^{i\pi/3}}_{n-k}\Big) \quad \text{and} \quad B = \operatorname{diag}\Big(\underbrace{\tfrac{1}{2}, \dots, \tfrac{1}{2}}_{k}, \underbrace{e^{-i\pi/3}, \dots, e^{-i\pi/3}}_{n-k}\Big),$$

then $A + B = I$, $AB = \operatorname{diag}\Big(\underbrace{\tfrac{1}{4}, \dots, \tfrac{1}{4}}_{k}, \underbrace{1, \dots, 1}_{n-k}\Big)$ and $\det(AB) = \frac{1}{4^k}$.

Problem 8. Define the sequence $x_1, x_2, \ldots$ by the initial terms $x_1 = 2$, $x_2 = 4$, and the recurrence relation

$$x_{n+2} = 3x_{n+1} - 2x_n + \frac{2^n}{x_n} \quad \text{for } n \geq 1.$$

Prove that $\lim\limits_{n\to\infty} \dfrac{x_n}{2^n}$ exists and satisfies

$$\frac{1+\sqrt{3}}{2} \leq \lim_{n\to\infty} \frac{x_n}{2^n} \leq \frac{3}{2}.$$

(proposed by Karen Keryan, Yerevan State University & American University of Armenia, Armenia)

Hint: Prove that $2x_n \leq x_{n+1} \leq 2x_n + n$.

Solution. Let's prove by induction that $x_{n+1} \geq 2x_n$. It holds for $n = 1$. Assume it holds for n. Then by the induction hypothesis we have that $x_n \geq 2x_{n-1} \geq \ldots \geq 2^{n-1}x_1 > 0$ and

$$x_{n+2} = 2x_{n+1} + (x_{n+1} - 2x_n) + \frac{2^n}{x_n} > 2x_{n+1}.$$

Similarly we prove that $x_{n+1} \leq 2x_n + n$. Again it holds for $n = 1$. Assume that the inequality holds for n. Then using that $x_n \geq 2^n$ and the induction hypothesis we obtain

$$x_{n+2} \leq 3x_{n+1} - 2x_n + 1 \leq 2x_{n+1} + (2x_n + n) - 2x_n + 1 = 2x_{n+1} + n + 1.$$

Using the previous inequalities we obtain that the sequence $y_n = \dfrac{x_n}{2^n}$ is increasing and $y_{n+1} \leq y_n + \frac{n}{2^n} \leq \ldots \leq y_1 + \sum_{k=1}^{n} \frac{k}{2^k} < \infty$, thus $\lim\limits_{n\to\infty} y_n = \dfrac{x_n}{2^n} = c$ exists.

The recurrence relation has the following form for y_n:

$$4y_{n+2} - 2y_{n+1} = 4y_{n+1} - 2y_n + \frac{1}{2^n \cdot y_n}.$$

By summing up the above equality for $n = 1, \ldots, m$ we obtain

$$4y_{m+2} - 2y_{m+1} = 4y_2 - 2y_1 + \sum_{n=1}^{m} \frac{1}{2^n \cdot y_n} = 2 + \sum_{n=1}^{m} \frac{1}{2^n \cdot y_n}. \tag{1}$$

Now using the facts that $y_1 = 1$, y_n increases and $\lim_{n\to\infty} y_n = c$ we obtain $1 \leq y_n \leq c$. Hence

$$\frac{1}{c} \leq \sum_{n=1}^{\infty} \frac{1}{2^n \cdot y_n} \leq 1.$$

Thus we get from (1)

$$2c = \lim_{m\to\infty} (4y_{m+2} - 2y_{m+1}) = 2 + \sum_{n=1}^{\infty} \frac{1}{2^n \cdot y_n} \in \left[2 + \frac{1}{c}, 3\right].$$

So we have $2c^2 \geq 2c + 1$ and $2c \leq 3$. Recall that $c \geq 1$. Therefore $1 + \sqrt{3} \leq 2c \leq 3$, which finishes the proof.

Problem 9. A matrix $A = (a_{ij})$ is called *nice*, if it has the following properties:

(i) the set of all entries of A is $\{1, 2, \ldots, 2t\}$ for some integer t;

(ii) the entries are non-decreasing in every row and in every column: $a_{i,j} \leq a_{i,j+1}$ and $a_{i,j} \leq a_{i+1,j}$;

(iii) equal entries can appear only in the same row or the same column: if $a_{i,j} = a_{k,\ell}$, then either $i = k$ or $j = \ell$;

(iv) for each $s = 1, 2, \ldots, 2t - 1$, there exist $i \neq k$ and $j \neq \ell$ such that $a_{i,j} = s$ and $a_{k,\ell} = s + 1$.

Prove that for any positive integers m and n, the number of nice $m \times n$ matrices is even.

For example, the only two nice 2×3 matrices are $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & 4 \end{pmatrix}$.

(proposed by Fedor Petrov, St Petersburg State University)

Solution. Define a *standard Young tableaux* of shape $m \times n$ as an $m \times n$ matrix with the set of entries $\{1, 2, \ldots, mn\}$, increasing in every row and in every column as in (ii).

Call two standard Young tableaux Y_1, Y_2 *friends*, if they differ by a switch of two consecutive numbers $x, x+1$ (the places of x and $x+1$ must be not neighbouring, for such a switch preserving the monotonicity in rows and columns).

For a nice $m \times n$ matrix A we construct a standard Young tableaux Y_A of shape $m \times n$ as follows: if A has n_i entries equal to i ($i = 1, 2, \ldots, 2t$), we replace them by the numbers from $n_1 + \ldots + n_{i-1} + 1$ to $n_1 + \ldots + n_i$ preserving monotonicity.

Note that our Y_A has exactly $2t - 1$ friends, where $2t$ is the number of distinct entries in A, and moreover, every standard Young tableaux with odd number of friends corresponds to a unique nice matrix. It remains to apply the handshaking lemma (i.e., the sum of the degrees equals twice the number of edges in this graph).

Problem 10. We say that a square-free positive integer n is *almost prime* if

$$n \mid x^{d_1} + x^{d_2} + ... + x^{d_k} - kx$$

for all integers x, where $1 = d_1 < d_2 < ... < d_k = n$ are all the positive divisors of n. Suppose that r is a Fermat prime (i.e. it is a prime of the form $2^{2^m} + 1$ for an integer $m \geq 0$), p is a prime divisor of an almost prime integer n, and $p \equiv 1 \pmod r$. Show that, with the above notation, $d_i \equiv 1 \pmod r$ for all $1 \leq i \leq k$.

(An integer n is called *square-free* if it is not divisible by d^2 for any integer $d > 1$.)

(proposed by Tigran Hakobyan, Yerevan State University, Vanadzor, Armenia)

Solution. We first prove the following claims.

Lemma 1. If n is almost prime then $\gcd(n, \varphi(n)) = 1$.

Proof. Assume to the contrary that $\gcd(n, \varphi(n)) > 1$ so that there are primes p and q dividing n such that $p \equiv 1 (\text{mod } q)$. For $0 \leq i \leq p-2$ let h_i be the number of positive divisors of n congruent to i modulo $p-1$ and similarly for $0 \leq j \leq q-1$ let ν_j denote the number of positive divisors of n congruent to j modulo q. Observe that the polynomial $F_n(x) = x^{d_1} + x^{d_2} + ... + x^{d_k} - kx$ defines the zero function on $\mathbb{F}_p$ due to the condition of the problem. On the other hand, $F_n(x) = (h_1 - k)x + \sum_{i \neq 1} h_i x^i$ in $\mathbb{F}_p[x]$, so that $p | h_i$ for all $0 \leq i \leq p-2, i \neq 1$. It follows that $2^{\omega(n)-1} = \nu_0 = h_0 + h_q + h_{2q} + ... \equiv 0 (\text{mod } p)$ which is a contradiction (here $\omega(n)$ means the number of distinct prime divisors of n). Therefore our assumption was wrong and the lemma is proved. $\square$

Lemma 2. Let q be a prime number and let h be a positive integer coprime to $q-1$. If l is the order of h modulo $q-1$, then there exists $a \in \mathbb{F}_q$ such that $a^{h^l} = a$ and

$$a - a^h + a^{h^2} - ... + (-1)^{l-1} a^{h^{l-1}} \neq 0$$

Proof. Observe that $a^{h^l} = a$ for any $a \in \mathbb{F}_q$ since $q - 1 | h^l - 1$. On the other hand, the numbers $h^0, h^1, ..., h^{l-1}$ leave different remainders upon division by $q-1$ and therefore the polynomial

$$f(x) = x - x^h + x^{h^2} - ... + (-1)^{l-1} x^{h^{l-1}}$$

defines a function on $\mathbb{F}_q$, which is not identically zero. Hence the existence of an element with the required properties is proved. $\square$

Lemma 3. If n is almost prime then for any primes p and q dividing n, the order of p modulo $q-1$ is an odd number.

Proof. Observe that due to Lemma 1 the order l of p modulo $q-1$ is well defined and assume to the contrary that l is an even number. According to Lemma 2 there exists $a \in \mathbb{F}_q$ such that $a^{p^l} = a$ and $f(a) \neq 0$, where $f(x) = x - x^p + x^{p^2} - ... + (-1)^{l-1} x^{p^{l-1}}$. Let us consider the sequence $(a_i)_{i=0}^{l} \subset \mathbb{F}_q$ defined by $a_0 = a$ and $a_{i+1} = -a_i^p$ for $0 \leq i \leq l-1$. Notice that since l is even by the assumption, we have $a_l = a_0^{p^l} = a_0$. It follows that

$$\sum_{i=0}^{l-1} \sum_{d|n} a_i^d = \sum_{i=0}^{l-1} \left(\sum_{d|\frac{n}{p}} a_i^d + \sum_{d|\frac{n}{p}} a_i^{pd} \right) = \sum_{i=0}^{l-1} \sum_{d|\frac{n}{p}} \left(a_{i+1}^d + a_i^{pd} \right) = 0,$$

since d is always odd being a divisor of n (Recall that $\gcd(n, \varphi(n)) = 1$ due to Lemma 1, so that n is odd, except the trivial case $n = 2$), and $a_{i+1} = -a_i^p$ for all $0 \leq i \leq l-1$. On the other hand, according to the condition of the problem, $\sum_{d|n} a_i^d = ka_i$ in $\mathbb{F}_q$ for all i, which shows that

$$kf(a) = k \sum_{i=0}^{l-1} a_i = \sum_{i=0}^{l-1} ka_i = \sum_{i=0}^{l-1} \sum_{d|n} a_i^d = 0$$

in $\mathbb{F}_q$ which is impossible, since $f(a) \neq 0$ by construction and $k = 2^{\omega(n)-1}$ is coprime to q. The attained contradiction shows that our assumption was wrong and concludes the proof of the lemma. $\square$

Let us get back to the problem. Suppose that $p|n$ is prime and $r = 2^{2^m}+1$ is a Fermat's prime such that $p \equiv 1 (\mathrm{mod}\ r)$. If q is any prime divisor of n, then by Lemma 3 we have that $q^l \equiv 1 (\mathrm{mod}\ p-1)$ for some odd l, so that $q^l \equiv 1 (\mathrm{mod}\ r)$ and therefore $q = q^{\mathrm{gcd}(l,r-1)} \equiv 1 (\mathrm{mod}\ r)$. Hence $d \equiv 1 (\mathrm{mod}\ r)$ for any divisor d of n. $\square$