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Lepton spectra in the earth's atmosphere

Paolo Lipari
Dipartimento di Fisica, Università di Roma, Piazzale Aldo Moro 2, Rome, Italy

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In this work we present an analytic calculation of the $\mu^\pm$ and neutrino fluxes in the earth's atmosphere. We discuss in detail the ratios $\nu_\mu/\bar{\nu}_\mu$, $\nu_e/\bar{\nu}_e$ and μ^+/μ^- as a function of particle energy and zenith angle. Precise knowledge of the neutrino/antineutrino ratios is necessary for the correct interpretation of neutrino oscillation experiments that use atmospheric neutrinos because of the different oscillation properties of neutrinos and antineutrinos in matter due to the Mikheyev–Smirnov–Wolfenstein mechanism. We also discuss the relation between muon and neutrino fluxes and the possibility to determine experimentally the non-oscillation neutrino fluxes from precise measurements of the muon flux.

1. Introduction

In this work we compute with analytic methods the spectra of μ^+, μ^-, and of neutrinos of different types (ν_μ, $\bar{\nu}_\mu$, ν_e, $\bar{\nu}_e$) produced by cosmic rays in the earth's atmosphere. We will make some important approximations: (i) the primary cosmic rays have a power law spectrum $\propto E^{-\alpha}$; (ii) the composition of cosmic rays is constant with energy; (iii) the interaction of heavy nuclei can be treated with the superposition model; (iv) the hadronic interactions lengths are constant; (v) we have exact Feynman scaling of hadronic cross sections.

Calculations along these lines have been discussed several times before (for a review see the textbook of Gaisser [1]), only some details in our calculation are original, in particular the improved treatment of the kaon–kaon regeneration, and the discussion of muon polarization.

The inclusive lepton spectra can also be calculated with Monte Carlo methods. In a Monte Carlo calculation it is possible to take into account the details of the energy spectrum and composition of the primary cosmic ray fluxes, and to include the energy dependence of the hadronic interactions lengths and differential cross sections. The analytic technique is nevertheless useful, because it gives accurately the relations among the fluxes of $\mu^\pm$, $\nu_\mu(\bar{\nu}_\mu)$, $\nu_e(\bar{\nu}_e)$, and allows one to study how they depend on the assumptions used in the calculation. These questions are of great current interest, because of the interest in the atmospheric neutrino fluxes stimulated by the observation of an anomaly in the e/μ ratio of the contained events in underground water Cherenkov detectors [6,7].

The aims of this work are twofold. The first is to discuss in detail the ratios $\nu_\mu/\bar{\nu}_\mu$ and $\nu_e/\bar{\nu}_e$ as a function of energy and zenith angle. Volkova [2] in her widely used calculation of neutrino fluxes discusses only in passing the question of neutrino–antineutrino ratios, and her tables of numerical results refer to the sums $(\nu_\mu + \bar{\nu}_\mu)$ and $(\nu_e + \bar{\nu}_e)$. However, it is important to consider separately the neutrino and antineutrino fluxes, as on one hand the cross sections of neutrinos and antineutrinos are different, and therefore the detectability of the two fluxes is not the same; still more important is the fact that the oscillation properties of ν's and $\bar{\nu}$'s in matter are different because of the Mikheyev–Smirnov–Wolfenstein effect [20,21], and the two fluxes have to be treated separately in their propagation through the earth. The neutrino–neutrino ratio is discussed in detail also in the work of Butkevich et al. [5].

Correspondence to: P. Lipari, Dipartimento di Fisica, Università di Roma, Piazzale Aldo Moro 2, Rome, Italy.

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A second goal of this work is to discuss the relation between the neutrino and muon fluxes. The fluxes of $\mu^{\pm}$ can be measured relatively easily with great accuracy. On the other hand the neutrino fluxes can be used to study neutrino oscillation with long baselines ($\sim 10^4$ km). Obviously a precise knowledge of the neutrino fluxes in the absence of oscillations is an important tool in the interpretation of these experiments. The uncertainties in a calculation of the neutrino fluxes arise essentially from two sources: (1) the uncertainty in the primary cosmic ray flux, (2) uncertainties in the modeling of hadronic interactions. Neutrinos and muons are produced in the weak decays of the ground state mesons (pions and kaons) produced in the hadronic cascades, additional neutrinos are then produced in the subsequent decay of the muons themselves. Therefore muons and neutrinos originate from the same mesons, and the main sources of uncertainties, primary flux and hadronic cross sections, cancel in a comparison. In this comparison we have to deconvolve the effects of the different decay kinematics and the effects of muon energy loss and decay. The physics of muon decay and energy loss is very well known, and we can hope to determine the properties of the neutrino fluxes from precise direct measurements of the muon fluxes. To develop this program it is very useful to perform as far as possible the calculations with analytic methods, in order to relate the different fluxes among each other in a transparent way. To reformulate our second aim in different words, we are asking the following questions: (i) how well are the neutrino fluxes determined from the current knowledge of the μ^+/μ^- fluxes? (ii) Can new more precise measurements improve the situation?

In this work we are not discussing prompt muons and neutrinos produced in charm decay. This source of leptons could become significant for energies of the order of ~ 100 TeV.

This work is organized as follows: In the next section, following standard methods we solve the diffusion equation for stable hadronic particles (nucleons and ground state mesons). In section 3 we will compute the neutrino fluxes coming directly from the decay of these meson fluxes. In section 4 we discuss the muon fluxes, taking into account energy loss and decay. In section 5 we discuss muon polarization. In section 6 we discuss the neutrinos produced in muon decay. In section 7 we present some numerical results. In section 8 we discuss in more detail the ratios μ^+/μ^-, $\nu_\mu/\bar{\nu}_\mu$ and $\nu_e/\bar{\nu}_e$. In section 9 we make a final discussion of the relation between the muon and neutrino fluxes.

2. Hadron fluxes

The evolution with depth of $\phi_j(E, t)$, the flux of hadron of type j, is described in general (neglecting energy loss) by the equation

$$\frac{\partial \phi_j}{\partial t} = -\frac{\phi_j}{\lambda_j} - \frac{\phi_j}{\lambda_{\rm dec}^{(j)}} + \sum_k S(k \to j), \tag{1}$$

which contains two sink terms (for interaction and decay), and several source terms corresponding to production in the interactions of hadrons of type k. With t we indicate the slant depth along a given direction of zenith angle θ, $\lambda_j(E)$ is the interaction length in air, and $\lambda_{\rm dec}^{(j)}(E)$ is the decay length (in g cm^{-2}) of particle j:

$$\lambda_{\rm dec}^{(j)}(E, t, \theta) = c\beta\tau_j \frac{E}{m_j} \rho(t, \theta), \tag{2}$$

m_j and τ_j are the mass and lifetime of the particle, $\rho(t, \theta)$ is the air density at the point considered. We are considering the showers induced by cosmic-ray particles as unidimensional. Under this approximation the zenith angle plays simply the role of a parameter. The source term $S(k \to j)$ can be written explicitly as:

$$S(k \to j;\, E,\, t) = \int_E^\infty \mathrm{d}E_k \frac{\phi_k(E_k,\, t)}{\lambda_k(E_k)} \frac{\mathrm{d}n_{k \to j}(E_j;\, E_k)}{\mathrm{d}E_j}. \tag{3}$$

It is well known [1] that under three assumptions, (i) the hadron flux $\phi_k(E, t)$ can be factorized in the form $\phi_k(E, t) = E^{-\alpha}\phi_k(t)$, (ii) the interaction length λ_k is independent of energy, (iii) the differential cross section is Feynman scaling, the source term takes the simple form

$$S(k \to j) = E^{-\alpha} \frac{\phi_k(t)}{\lambda_k} Z_{kj}, \tag{4}$$

where

$$Z_{kj} = \int_0^1 \mathrm{d}x\, x^{\alpha - 1} \frac{\mathrm{d}n_{k \to j}}{\mathrm{d}x}. \tag{5}$$

We will consider 7 types of stable hadrons (p, n, $\pi^\pm$, $K^\pm$ and K_{L}). The π^0 decay rapidly into two photons, and because the hadronic cross section of the photon is small, they decouple immediately from the evolution of the hadronic component. The K_{s} with a decay length $c\tau = 2.675$ cm can be taken into account as a contribution to the pion fluxes.

In principle we should solve a set of 7 coupled partial differential equations, of form (1). The problem is significantly simplified if we make the following approximations:

$$S(\pi^\pm \to p,\, n) = S(K \to p,\, n,\, \pi^\pm) = 0. \tag{6}$$

The contribution of the mesons to the nucleon fluxes is certainly small, the additional assumption $S(K \to \pi) = 0$ is obviously incorrect. However, because the kaon fluxes are a factor ~ 10 smaller that the pion fluxes, the approximation is reasonable. Under approximation (6) the nucleon fluxes develop independently from the meson fluxes, and the evolution of the pion fluxes is decoupled from the kaon fluxes. We can then solve the problem of the hadron fluxes in three separate steps: we can compute first the nucleon fluxes, then we can use this solution to compute the $\pi^\pm$ fluxes, and finally compute the kaon fluxes taking into account the nucleon and pion sources.

We will assume as an initial condition a primary flux composed of protons and neutrons with an exact power law spectrum of differential index α. The initial conditions of the set of differential equations is therefore:

$$\phi_p(E,\, t = 0) = p_0 E^{-\alpha}, \tag{7}$$

$$\phi_n(E,\, t = 0) = n_0 E^{-\alpha} = p_0 \delta_0 E^{-\alpha}. \tag{8}$$

In our numerical results we will use: $\alpha = 2.7$, $p_0(1 - \delta_0) = 1.7$ (cm^2 s sr GeV$^{1.7}$)$^{-1}$, $\delta_0 = 0.14$.

2.1. Nucleon fluxes

The proton and neutron components will have the form $p(t)E^{-\alpha}$ and $n(t)E^{-\alpha}$. The functions $p(t)$ and $n(t)$ satisfy the coupled differential equations:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{p}{\lambda_p} + \frac{p}{\lambda_p} Z_{pp} + \frac{n}{\lambda_n} Z_{np}, \tag{9}$$

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -\frac{n}{\lambda_n} + \frac{p}{\lambda_p} Z_{pn} + \frac{n}{\lambda_n} Z_{nn}. \tag{10}$$

Using the approximation $\lambda_p = \lambda_n$ and the consequence of isospin symmetry $Z_{pp} = Z_{nn}$ and $Z_{pn} = Z_{np}$ the solution is immediately found

$$(p \pm n) = (p_0 \pm n_0) \exp\left(\frac{-t}{\Lambda_{1,2}}\right), \tag{11}$$

with

$$\Lambda_{1,2} = \frac{\lambda_p}{1 - Z_{pp} \mp Z_{pn}}. \tag{12}$$

It is important to note that the ratio n/p grows with depth reaching asymptotically ($t \to \infty$) the value of 1. The nucleon fluxes are independent from zenith angle.

2.2. $\pi^\pm$ fluxes

The equations that describe the evolution of the pion fluxes are:

$$\frac{\partial \pi^\pm}{\partial t} = -\frac{\pi^\pm}{\lambda_\pi} - \frac{\pi^\pm}{\lambda_{\mathrm{dec}}^\pi} + S(p \to \pi^\pm) + S(n \to \pi^\pm) + S(\pi^+ \to \pi^\pm) + S(\pi^- \to \pi^\pm). \tag{13}$$

Because the decay length decreases, at sufficiently low energy the pion interaction terms are negligible, and therefore also the pion regeneration terms can be neglected. The equations then decouple and we find the solution:

$$\pi_{\mathrm{L}}^\pm(E, t) = \left(\frac{p(t)}{\lambda_p} Z_{p\pi^\pm} + \frac{n(t)}{\lambda_n} Z_{n\pi^\pm}\right) \lambda_{\mathrm{dec}}^\pi(E, t), \tag{14}$$

where the subscript L indicates that the solution is only correct for asymptotically low pion energies.

At very high energy pion decay becomes very rare, and neglecting the decay term, we can again solve the coupled equations. The source and sink terms are both exponential, we can use the result

$$\int_0^x \mathrm{d}y\, \mathrm{e}^{-y/\lambda}\, \mathrm{e}^{-(x-y)/\Lambda} = \frac{\Lambda\lambda}{\Lambda - \lambda}\left(\mathrm{e}^{-x/\Lambda} - \mathrm{e}^{-x/\lambda}\right) \tag{15}$$

and the isospin relations $Z_{\pi^+\pi^+} = Z_{\pi^-\pi^-}$ and $Z_{\pi^+\pi^-} = Z_{\pi^-\pi^+}$ to obtain the solution

$$(\pi^+ \pm \pi^-)_{\mathrm{H}} = \frac{(p_0 \pm n_0)}{\lambda_p} (Z_{p\pi^+} \pm Z_{p\pi^-}) \frac{\Lambda_{\pi 1,2} \Lambda_{1,2}}{\Lambda_{\pi 1,2} - \Lambda_{1,2}} \left[\exp\left(\frac{-t}{\Lambda_{\pi 1,2}}\right) - \exp\left(\frac{-t}{\Lambda_{1,2}}\right)\right] E^{-\alpha}, \tag{16}$$

where the subscript H indicates that this solution is only correct for asymptotically high energies, $\Lambda_{1,2}$ are defined in (12) and:

$$\Lambda_{\pi 1,2} = \frac{\lambda_\pi}{1 - Z_{\pi^+\pi^+} \mp Z_{\pi^+\pi^-}}. \tag{17}$$

It is important to note that the $\pi^\pm$ fluxes have a slightly different development with the depth t. The π^- flux that receives a larger contribution from the neutron component develops somewhat deeper in the atmosphere.

2.3. Kaon fluxes

At low energy the only source of kaons are the proton and neutron fluxes, the pions and the kaons themselves interact too rarely. The kaon fluxes then are

$$K_{\mathrm{L}}^{j}(E,t)=\left(\frac{p(t)}{\lambda_p}Z_{pK^j}+\frac{n(t)}{\lambda_n}Z_{nK^j}\right)\lambda_{\mathrm{dec}}^{K^j}E^{-\alpha}. \tag{18}$$

At very large energies, when kaon decay can be treated as a perturbation, we can again find an explicit solution. When considering $K \to K$ regeneration we have to take into account the complication that we have three different particles: K^+, K^- and K_{L}. In matrix form we have to solve the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{K}=\frac{1}{\lambda_K}\boldsymbol{K}+\frac{1}{\lambda_K}\mathbf{Z}_K\cdot\boldsymbol{K}+\boldsymbol{S}, \tag{19}$$

or more explicitly:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}K^+\\K^-\\K_{\mathrm{L}}\end{bmatrix}=\frac{1}{\lambda_{\mathrm{K}}}\begin{bmatrix}K^+\\K^-\\K_{\mathrm{L}}\end{bmatrix}+\frac{1}{\lambda_{\mathrm{K}}}\begin{bmatrix}Z_{++} & Z_{-+} & Z_{0+}\\Z_{+-} & Z_{--} & Z_{0-}\\Z_{-0} & Z_{-0} & Z_{00}\end{bmatrix}\begin{bmatrix}K^+\\K^-\\K_{\mathrm{L}}\end{bmatrix}+\begin{bmatrix}S_+\\S_-\\S_{\mathrm{L}}\end{bmatrix}. \tag{20}$$

The source terms have the form

$$S_j=S(p\to K^j)+S(n\to K^j)+S(\pi^+\to K^j)+S(\pi^-\to K^j)$$

$$=\frac{p}{\lambda_p}Z_{pK^j}+\frac{n}{\lambda_n}Z_{nK^j}+\frac{\pi^+}{\lambda_\pi}Z_{\pi^+K^j}+\frac{\pi^-}{\lambda_\pi}Z_{\pi^-K^j} \tag{21}$$

$$=\sum_{\sigma=1}^{4}c(j,\sigma)\exp\left(-\frac{t}{\Lambda_\sigma}\right). \tag{22}$$

In the last equation we have used the explicit solutions for the pion and nucleon fluxes, and when the index σ runs from 1 to 4, then Λ_σ runs through Λ_1, Λ_2, $\Lambda_{\pi 1}$, $\Lambda_{\pi 2}$. The coefficients $c(j,\sigma)$ then are:

$$c(j,1)=\frac{1}{\lambda_p}\frac{p_0+n_0}{2}\left(Z_{pK^j}+Z_{nK^j}\right)-\frac{1}{\lambda_\pi}\frac{A_\pi}{2}\left(Z_{\pi^+K^j}+Z_{\pi^-K^j}\right), \tag{23}$$

$$c(j,2)=\frac{1}{\lambda_p}\frac{p_0-n_0}{2}\left(Z_{pK^j}-Z_{nK^j}\right)-\frac{1}{\lambda_\pi}\frac{B_\pi}{2}\left(Z_{\pi^+K^j}-Z_{\pi^-K^j}\right), \tag{24}$$

$$c(j,3)=\frac{1}{\lambda_\pi}\frac{A_\pi}{2}\left(Z_{\pi^+K^j}+Z_{\pi^-K^j}\right), \tag{25}$$

$$c(j,4)=\frac{1}{\lambda_\pi}\frac{B_\pi}{2}\left(Z_{\pi^+K^j}-Z_{\pi^-K^j}\right). \tag{26}$$

The constants A_π and B_π are (compare with eq. (16))

$$A_\pi(B_\pi)=\frac{(p_0\pm n_0)}{\lambda_p}\left(Z_{p\pi^+}\pm Z_{p\pi^-}\right)\frac{\Lambda_{\pi 1,2}\Lambda_{1,2}}{\Lambda_{\pi 1,2}-\Lambda_{1,2}}. \tag{27}$$

The solution of eq. (19) will depend on the exact form of the kaon regeneration matrix $\mathbf{Z}_K$. In general $\mathbf{Z}_K$ will be symmetric with real eigenvalues (Z_{K1}, Z_{K2}, Z_{K3}), and will have an orthogonal diagonalization matrix $\mathbf{U}$. We can define three effective lengths:

$$\Lambda_{K,j} = \frac{\lambda_K}{1 - Z_{K,j}}. \tag{28}$$

The general solution for the evolution of the three kaon fluxes is

$$K_{\mathrm{H}}^{j}(t) = \sum_{a=1}^{3} U_{aj} \sum_{\sigma=1}^{4} \frac{\Lambda_{Ka}\Lambda_{\sigma}}{\Lambda_{Ka} - \Lambda_{\sigma}} \left(\sum_{k=1}^{3} U_{ak} c(k, \sigma) \right) \left[\exp\left(\frac{-t}{\Lambda_{K,a}} \right) - \exp\left(\frac{-t}{\Lambda_{\sigma}} \right) \right]. \tag{29}$$

In the numerical work we have chosen the simple form

$$\mathbf{Z}_K = \begin{bmatrix} Z_1 & 0 & Z_2 \\ 0 & Z_1 & Z_2 \\ Z_1 & Z_2 & Z_1 \end{bmatrix} \tag{30}$$

that neglects the strangeness changing contributions: $(K^{\pm} \to K^{\mp})$ and assumes the validity of the relation $(K^{\pm} \to K_{\mathrm{L}}) = (K_{\mathrm{L}} \to K^{\pm})$ that can be obtained with simple considerations on the valence quark content of the different mesons. The eigenvalues of the matrix (30) are Z_1 and $(Z_1 \pm \sqrt{2} Z_2)$. The corresponding normalized eigenvectors are $(K^+ - K^-)/\sqrt{2}$ and $(K^+ + K^- \pm \sqrt{2} K_{\mathrm{L}})/2$. These linear combinations of the physical states have the simplest development in depth, and play the same role as the combinations $(\pi^+ \pm \pi^-)$ for the pion fluxes. The diagonalization matrix in this case is

$$\mathbf{U} = \frac{1}{2} \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \end{bmatrix}. \tag{31}$$

2.4. General expression for the meson fluxes

In general we have found asymptotic solutions for the meson fluxes; at very low energy, when the probability of hadronic interaction is negligible, the meson fluxes have the form

$$M_{\mathrm{L}}(E, t, \theta) = S_{M,\mathrm{L}}(t)\, \lambda_{\mathrm{dec}}^{M}(E, t, \theta)\, E^{-\alpha} = S_{M,\mathrm{L}}(t)\, \rho(t, \theta) \frac{c\tau_m}{m_M} E^{-(\alpha-1)}, \tag{32}$$

where the function $S_{M,\mathrm{L}}(t)$ is the source of meson M at low energy (explicit expressions are given in eqs. (14) and (18)). At very high energy when the decay probability is negligible, the meson fluxes have the form

$$M_{\mathrm{H}}(E, t) = M_{\mathrm{H}}(t)\, E^{-\alpha}. \tag{33}$$

$M_{\mathrm{H}}(t)$ is the depth dependence of the high energy meson flux. Explicit expressions for the functions $M_{\mathrm{H}}(t)$ are given in formula (16) and (29). In general we use the interpolation function

$$M(E, t, \theta) = \frac{S_{M,\mathrm{L}}(t)\, \lambda_{\mathrm{dec}}^{M}(E, t, \theta)}{1 + S_{M,\mathrm{L}}(t)\, \lambda_{\mathrm{dec}}^{M}(E, t, \theta)/M_{\mathrm{H}}(t)} E^{-\alpha}. \tag{34}$$

3. Neutrinos from meson decay

The meson fluxes are the sources of neutrino and muon fluxes via their weak decays. For neutrinos we do not have to consider energy loss or decay and their fluxes obey the simple differential equation

$$\frac{\partial \nu_j(E,t,\theta)}{\partial t} = \sum_M S(M \to \nu_j;\, E,t,\theta), \tag{35}$$

where the subscript j specifies the neutrino (or antineutrino) type considered, and the sum is over all meson types. The equation that describes the flux of $\mu^\pm$ (formula (54)) contains additional terms that describe decay and energy loss. The source of neutrinos of type j from the decay of meson M (also the source of $\mu^\pm$ can be obtained with obvious replacements) is

$$S(M \to \nu_j;\, E,t,\theta) = B(M \to \nu_j)\int_E^\infty \mathrm{d}E_0 \frac{M(E_0,t)}{\lambda_{\mathrm{dec}}^M(E_0,t,\theta)} F_{M\to\nu_j}(E, E_0), \tag{36}$$

where $B(M \to \nu)$ is the branching ratio for decay into a state with the neutrino ν_j (more generally we should sum over all decay channels, but in practice we will only have a single channel to consider), and $F_{M\to\nu_j}(E, E_0)$ is the neutrino spectrum after the decay of meson M. A discussion of the inclusive lepton spectra in meson decay is contained in appendix A. In general it is possible to show that in the ultrarelativistic limit the inclusive spectrum of particle b in the decay $a \to b + X$ takes the simple scaling form

$$F_{a\to b}(E_b;\, E_a) = \frac{1}{E_a} F_{a\to b}\left(\frac{E_b}{E_a}\right). \tag{37}$$

It is useful to define in analogy with eq. (5) the decay Z factors:

$$Z(a \to b, \alpha) = \int_0^1 \mathrm{d}x\, x^{\alpha-1} F_{a\to b}(x). \tag{38}$$

The physical meaning of these quantities is that if we have a beam of particle of type a with a power law energy spectrum: $f_a(E) \propto E^{-\alpha}$, after complete decay of a the resulting spectrum of particle b is $f_b = Z(a \to b;\, \alpha) f_a$. Numerical values of the relevant decay Z factor are given in table 1.

If the meson flux is factorizable in the form $M(t)\, E^{-\beta}$ the integral over the energy in (36) can be performed analytically:

$$S(M \to \nu_j;\, E,t,\theta) = B(M \to \nu_j) \frac{m_M}{c\tau_m} \frac{M(t)}{\rho(t,\theta)} Z(M \to \nu_j, \beta+1)\, E^{-(\beta+1)}. \tag{39}$$

The neutrino flux from direct meson decay at slant depth t_f is obtained integrating the sources along the line of sight:

$$\begin{aligned}\phi(M \to \nu_j) &= \int_0^{t_f} \mathrm{d}t\, S(M \to \nu_j) \\ &= Br(M \to \nu_j)\, Z(M \to \nu_j, \beta+1) \frac{m_M}{c\tau_M}\left(\int_0^{t_f} \mathrm{d}t \frac{M(t)}{\rho(t,\theta)}\right) E^{-(\beta+1)}.\end{aligned} \tag{40}$$

The integral over the slant depth can be performed analytically in the limit of high energy and low energy.

Table 1
Decay constants $Z(a \to b;\ \alpha)$.

Decay	$\alpha = 2$	$\alpha = 2.7$	$\alpha = 3.7$
$\pi^+ \to \mu^+$	0.787	0.676	0.552
$\pi^+ \to \nu_\mu$	0.213	0.087	0.027
$K^+ \to \mu^+$	0.523	0.388	0.283
$K^+ \to \nu_\mu$	0.477	0.342	0.238
$K^+ \to \nu_e$	0.266	0.135	0.062
$K_L \to \nu_e$	0.264	0.134	0.061
$\mu_0^- \to \nu_\mu$	0.350	0.213	0.123
$\mu_R^- \to \nu_\mu$	0.300	0.166	0.085
$\mu_L^- \to \nu_\mu$	0.400	0.260	0.161
$\mu_0^- \to \bar{\nu}_e$	0.300	0.166	0.085
$\mu_R^- \to \bar{\nu}_e$	0.400	0.242	0.134
$\mu_L^- \to \bar{\nu}_e$	0.200	0.090	0.036
$\pi \to \mu \to \nu_\mu$	0.265	0.133	0.059
$\pi \to \mu \to \nu_e$	0.257	0.129	0.058
$K \to \mu \to \nu_\mu$	0.159	0.065	0.024
$K \to \mu \to \nu_e$	0.205	0.092	0.037

3.1. Low energy limit

At low energy the meson flux has the form (32). We can then introduce the quantity

$$D_L(M,\ t_f) = \int_0^{t_f} \mathrm{d}t\ S_{M,L}(t) = \left(\frac{p_0 + n_0}{2}\right)(Z_{pM} + Z_{nM})\frac{\Lambda_1}{\lambda_p}[1 - \mathrm{e}^{-t_f/\Lambda_1}]$$
$$+ \left(\frac{p_0 - n_0}{2}\right)(Z_{pM} - Z_{nM})\frac{\Lambda_2}{\lambda_p}[1 - \mathrm{e}^{-t_f/\Lambda_2}]. \tag{41}$$

The flux of neutrinos due to the decay of meson M is then

$$\phi_L(M \to \nu_j;\ E,\ t) = B(M \to \nu_j)\, D_L(M,\ t)\, Z(M \to \nu_j,\ \alpha)\, E^{-\alpha}. \tag{42}$$

Note that all dependence on the zenith angle and on the density $\rho(t)$ has canceled. The dependence on the atmospheric depth is contained in $D_L(M,\ t)$ in the two terms in square parenthesis. The fluxes develop rapidly and then remain stable. We take the convention that if we drop the dependence on the depth of observation we are assuming a sufficiently large t_f to have reached the asymptotic value. In

Table 2
$\phi(X \to l,\ E)/\phi_p(E)$ at low energy. (μ decay and energy loss are not included.)

	π^+	π^-	K^+	K^-	K_L	Total
μ^+	4.22×10^{-2}	–	3.01×10^{-3}	–	2.11×10^{-4}	4.54×10^{-2}
μ^-	–	3.38×10^{-2}	–	9.83×10^{-4}	2.11×10^{-4}	3.50×10^{-2}
ν_μ	5.44×10^{-3}	–	2.66×10^{-3}	–	1.33×10^{-4}	8.23×10^{-3}
$\bar{\nu}_\mu$	–	4.36×10^{-3}	–	8.66×10^{-4}	1.33×10^{-4}	5.36×10^{-3}
ν_e	–	–	7.98×10^{-5}	–	1.91×10^{-4}	2.70×10^{-4}
$\bar{\nu}_e$	–	–	–	2.60×10^{-5}	1.91×10^{-4}	2.17×10^{-4}

summary, at low energy, when meson interactions can be neglected the neutrino fluxes from meson decay are simply proportional to the primary flux. Numerical values of the proportionality constants, are given in table 2 (the values of the parameters used are fully described in section 7).

3.2. High energy limit

In the high energy limits the meson fluxes have the form (33), where again the functions $M_H(t)$ are linear combinations of exponentials. To perform the integral (40), we need to specify the exact form of the density $\rho(t, \theta)$. It is a good approximations (in the following we will discuss its limitations) to consider the density of form (see appendix B)

$$\rho(t, \theta) = \frac{t \cos\theta}{h_0}, \tag{43}$$

with $h_0 = 6.334$ km. This expression is valid in the stratosphere $h \geqslant 11$ km, and for zenith angles sufficiently small so that the curvature of the earth can be neglected. In this approximation (taking the limit $t_f \to \infty$) we define

$$D_H(M, t_f) = \int_0^{t_f} dt \frac{M_H(t)}{t}. \tag{44}$$

For large t_f we have explicitly:

$$D_H(\pi^{\pm}) = \tfrac{1}{2}A_\pi \ln\left(\frac{\Lambda_{\pi 1}}{\Lambda_1}\right) \pm \tfrac{1}{2}B_\pi \ln\left(\frac{\Lambda_{\pi 2}}{\Lambda_2}\right), \tag{45}$$

$$D_H(K^j) = \sum_{a=1}^{3} U_{aj} \sum_{\sigma=1}^{4} \frac{\Lambda_{Ka}\Lambda_\sigma}{\Lambda_{Ka} - \Lambda_\sigma} \ln\left(\frac{\Lambda_{Ka}}{\Lambda_\sigma}\right)\left(\sum_{k=1}^{3} U_{ak} c(k, \sigma)\right). \tag{46}$$

If we perform the integral in the depth only up to a finite value t_f we have to make the replacement

$$\ln\left(\frac{\Lambda}{\lambda}\right) \to \ln\left(\frac{\Lambda}{\lambda}\right)\left\{1 - \frac{1}{\ln(\Lambda/\lambda)}\left[E_1\left(\frac{t_f}{\Lambda}\right) - E_1\left(\frac{t_f}{\lambda}\right)\right]\right\}, \tag{47}$$

where the function $E_1(x)$ is the exponential integral:

$$E_1(x) = \int_x^\infty dy \, \frac{e^{-y}}{y}. \tag{48}$$

The spectrum at high energy takes the form

$$\phi(M \to \nu) = Br(M \to \nu) \, Z(M \to \nu, \alpha + 1) \, D_H(M) \frac{\varepsilon_M}{\cos\theta} E^{-(\alpha+1)}, \tag{49}$$

with

$$\varepsilon_M = \frac{m_M h_0}{c\tau_M}. \tag{50}$$

In summary at high energy, when meson decay is rare (and when the zenith angle is not too large), the neutrino fluxes from meson decay is $\propto \phi_p(E)/(E \cos\theta)$. Values of the proportionality constants are given in table 3.

Table 3
$[\phi(X \to l,\ E,\ \theta = 0°)E]/\phi_p(E)$ at high energy (TeV).

	π^+	π^-	K^+	K^-	K_L	Total
μ^+	4.46×10^0	–	2.38×10^0	–	3.51×10^{-2}	6.87×10^0
μ^-	–	3.56×10^0	–	8.80×10^{-1}	3.51×10^{-2}	4.48×10^0
ν_μ	2.19×10^{-1}	–	2.00×10^0	–	1.73×10^{-2}	2.23×10^0
$\bar{\nu}_\mu$	–	1.75×10^{-1}	–	7.40×10^{-1}	1.73×10^{-2}	9.32×10^{-1}
ν_e	–	–	3.93×10^{-2}	–	2.48×10^{-2}	6.41×10^{-2}
$\bar{\nu}_e$	–	–	–	1.45×10^{-2}	2.48×10^{-2}	3.93×10^{-2}

3.3. Very large angles

The expression (49) has the well known angular dependence $\propto 1/\cos\theta$. More generally, for large zenith angles we have to go back to eq. (40). The $1/\cos\theta$ angular dependence is replaced by

$$\frac{1}{\cos\theta} \to F_M(\theta) = \frac{1}{D_{\mathrm{H}}(M,\ t_{\mathrm{f}})\, h_0} \int_0^{t_{\mathrm{f}}} \frac{M_{\mathrm{H}}(t)}{\rho(t,\ \theta)}. \tag{51}$$

In principle different mesons will have a slightly different angular enhancement. In practice the development of the different meson fluxes is so similar, that to a good approximation we have a universal curve.

For a qualitative understanding let us consider the particles that arrive horizontally at a detector at sea level (zenith angle $\theta = 90°$). The points along an horizontal line of sight at sea level that have a slant depth of {10, 100, 200} g cm^{-2}, are at a distance from the detection point of {750, 624, 581} km; at these three points the zenith angle of the line sight θ^* is {83.3, 84.4, 84.8} degrees. The high energy enhancement can be obtained with a precision of 5% or so with the simple minded replacement

$$\frac{1}{\cos\theta} \to \frac{1}{\cos\theta^*\ (\theta,\ h = 30\ \mathrm{km})}. \tag{52}$$

The column density in the horizontal direction from seal level is 36 280 g cm^{-2}. The large column density and long flight path in the horizontal direction causes a strong suppression of the low energy muon fluxes, due to energy loss and decay.

3.4. Form of interpolation

The low energy and high energy solutions can be tied together using the interpolation form

$$\phi(M \to \nu_j;\ E,\ t,\ \theta) = Br(M \to \nu_j)\, D_{\mathrm{L}}(M,\ t)\, Z(M \to \nu_j,\ \alpha)$$
$$\times \left(1 + \frac{E}{\varepsilon_M F_M(\theta)} \frac{D_{\mathrm{L}}(M,\ t)\, Z(M \to \nu_j,\ \alpha)}{D_{\mathrm{H}}(M,\ t)\, Z(M \to \nu_j,\ \alpha + 1)}\right)^{-1} E^{-\alpha}. \tag{53}$$

4. Muon fluxes

For the correct calculation of the muon fluxes, we need to take into account energy loss and decay probability. If the energy loss is treated as a continuous deterministic process, it is straightforward to take into account these effects. Under this approximation the muon fluxes obey the differential equation [3]

$$\frac{\partial \mu^{\pm}(E, t, \theta)}{\partial t} = -\frac{\mu^{\pm}(E, t, \theta)}{\lambda^{\mu}_{\rm dec}(E, t, \theta)} + \frac{\partial[\mathrm{d}E/\mathrm{d}t(E)]\,\mu^{\pm}(E, t, \theta)}{\partial E} + \sum_M S(M \to \mu^{\pm}; E, t, \theta). \quad (54)$$

The first term on the right-hand side takes into account decay, and the second describes energy loss. The source terms are given in eq. (36). A simple approximation of the exact formula [22] for the energy loss is

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = a + bE = a\left(1 + \frac{E}{\varepsilon}\right). \quad (55)$$

If energy loss is treated as a continuous process, the energy of a muon is a well defined function of the position along its path. A muon created at slant depth t_0 with energy E_0 and zenith angle θ will reach the depth $t_{\rm f}$ with an energy $E(t_{\rm f} - t_0;\, E_0)$ with a probability

$$p_{\rm surv}(E, t, t_0, \theta) = \exp\left(-\frac{m_\mu}{c\tau_\mu}\int_{t_0}^{t_{\rm f}} \frac{\mathrm{d}t}{E(t - t_0, E_0)\,\rho(t, \theta)}\right). \quad (56)$$

In the approximation (55) we have

$$E(t_{\rm f} - t_0, E_0) = (E_0 + \varepsilon)\,\mathrm{e}^{-b(t_{\rm f} - t_0)} - \varepsilon \simeq E_0 - a(t_{\rm f} - t_0). \quad (57)$$

In appendix C we will discuss some analytical solutions for the integral in the argument of the exponential.

The flux of muons reaching an observation point of slant depth $t_{\rm f}$ with zenith angle θ can be calculated from the source of $S_\mu(E_0, t)$ as

$$\phi_\mu(E_{\rm f}, t_{\rm f}) = \int_0^{t_{\rm f}} \mathrm{d}t_0\, S_\mu(E_0, t_0)\, p_{\rm surv}(E_0, t_0, t_{\rm f}) \left(\frac{\mathrm{d}E_{\rm f}(E_0, t_{\rm f} - t_0)}{\mathrm{d}E_0}\right)^{-1}, \quad (58)$$

where the integral is over all positions of creation of the muons, $E_{\rm f}$ is the detected energy, E_0 is the initial energy, and the last factor is a Jacobian term that in the approximation (55) become simply $\mathrm{e}^{b(t_{\rm f} - t_0)}$.

5. Muon polarization

The muons produced in cosmic rays are polarized, μ^+ have on average negative helicity and μ^- positive helicity (the helicity is the expectation value of the angular momentum in the direction of motion in units of $\frac{1}{2}\hbar$). The polarization of muons is important because the neutrino spectra produced in μ decay depend on the polarization [18,19].

In the following we will discuss the decay $\pi^- \to \mu^- \bar{\nu}_\mu$, the charge conjugate reaction is obtained reversing all spins, the two body $K^{\pm}$ decays can be obtained replacing m_π with m_K. In the rest frame of the π^- the μ^- is produced completely polarized with positive helicity, this is a simple consequence of angular momentum conservation and of the facts that the π has spin zero and the $\bar{\nu}$ is right-handed. In the laboratory frame where the initial π has velocity β_π, a muon produced at c.m. angle θ^* with respect to the direction of the boost has helicity

$$P(\beta_\pi, \theta^*) = \frac{1}{\beta_\mu}\left(\frac{(1 - r_\pi) + (1 + r_\pi)\cos\theta^*\beta_\pi}{(1 + r_\pi) + (1 - r_\pi)\cos\theta^*\beta_\pi}\right), \quad (59)$$

where $r_\pi = (m_\mu/m_\pi)^2$ and β_μ is the velocity of the muon in the lab frame and is implicitly a function of β_π and of the c.m. angle of emission θ^*. In the limit $\beta_\pi = 0$, the velocity of the muon is: $\beta_\mu = (1 - r_\pi)/(1 - r_\pi)$ and we have $P = +1$ independently from the angle of emission of the muon. We can express the angle θ^* as a function of the ratio of the laboratory energy of the μ and of the parent pion using the relation

$$x = \frac{E_\mu}{E_\pi} = \tfrac{1}{2}\left[(1 + r_\pi) + \beta_\pi (1 - r_\pi) \cos\theta^*\right] \tag{60}$$

and write the polarization as a function of β_π and x. In the ultrarelativistic limit ($\beta_\pi \to 1$, $\beta_\mu \to 1$) we obtain

$$P(x) = \frac{1 + r_\pi}{1 - r_\pi} - \frac{2 r_\pi}{(1 - r_\pi) x}. \tag{61}$$

The slowest (fastest) μ^- produced in the decay of an ultrarelativistic π^-, with fractional energy $x = r_\pi$, ($x = 1$), has helicity $P = -1$ ($P = +1$). The slow pions are produced backward in the c.m. frame, and the Lorentz boost to the laboratory frame reversing their direction reverses also their helicity.

It is easy to compute the average polarization of the muons of a given energy produced at a given point (defined by (θ, t), zenith angle and slanted depth) by the decay of pions:

$$\langle P \rangle = \frac{1}{S_\mu^{(\pi)}(E, \theta, t)} \frac{1}{(1 - r_\pi)} \int_E^{E/r_\pi} \frac{\mathrm{d}E_\pi}{E_\pi} \frac{\pi(E_\pi, \theta, t)}{\lambda_{\mathrm{dec}}(E_\pi, \theta, t)} P\left(\frac{E}{E_\pi}\right), \tag{62}$$

where $P(x)$ is defined by eq. (61), and $S_\mu^{(\pi)}(E, \theta, t)$ is the source of muons (due to π decay) of energy E at the point considered:

$$S_\mu^{(\pi)}(E, \theta, t) = \frac{1}{(1 - r_\pi)} \int_E^{E/r_\pi} \frac{\mathrm{d}E_\pi}{E_\pi} \frac{\pi(E_\pi, \theta, t)}{\lambda_{\mathrm{dec}}(E_\pi, \theta, t)}. \tag{63}$$

If the pion spectrum at the point (θ, t) has the energy dependence: $\propto E_\pi^{-\alpha+1}$ (and therefore $S_\mu^{(\pi)} \propto E_\mu^{-\alpha}$), then the integrals can be easily performed and the average polarization of the muons expressed as a function of α is

$$\langle P(\alpha) \rangle = \frac{1 + r_\pi}{1 - r_\pi} - \frac{2 r_\pi}{(1 - r_\pi)} \frac{\alpha}{(\alpha - 1)} \frac{\left(1 - r_\pi^{\alpha-1}\right)}{\left(1 - r_\pi^{\alpha}\right)}. \tag{64}$$

Numerical values of this average polarization are given in table 4.

Two important points are: (i) The polarization of the muons coming from kaon decay is much larger that the polarization of muons that have a pion parent. (ii) The polarization increases with the steepening of the parent spectrum. Both these effects are the consequence of a simple kinematical effect. A μ^- of energy E will have helicity $+1$ if the parent has exactly the same energy (and therefore the

Table 4
Average muon polarization for a power law meson source: $S_\mu^{(M)} \propto E_\mu^{-\alpha}$.

	$\alpha = 2$	$\alpha = 2.7$	$\alpha = 3.7$
$\langle P_\mu \rangle_{\pi^- \to \mu^-}$	0.271	0.329	0.406
$\langle P_\mu \rangle_{K^- \to \mu^-}$	0.912	0.944	0.964

muon was emitted exactly forward in the parent rest frame); and helicity -1 if the parent has the maximum allowed energy (the muon was emitted backward). However, there are fewer mesons at higher energy (this is the reason of the polarization in the first place); the polarization will therefore be stronger for steeper spectra, when the suppression of the higher energies is larger, and also stronger in the case of kaon decay because the maximum energy of the parent of a muon of energy E is $E(m_M/m_\mu)^2$ and is larger for kaons.

Until now we have considered the polarization of the muons produced at the point (θ, t), however, we need to calculate the polarization of the muons at the point, integrating over all source points and taking into account muon decay and energy loss. In a Monte Carlo calculation this can be easily achieved generating the muons in the correct state of spin and transporting them to the detection point. In an analytic calculation the best method is to consider separately the evolution of the left-handed and right-handed muons, therefore calculating 4 types of muon fluxes (μ_R^+, μ_L^+, μ_R^-, μ_L^-). This can be obtained considering that a μ^- produced in a π^- decay with fractional energy x has a probability of being right-handed (left-handed) P_R (P_L): $P_{R,L} = \frac{1}{2}(1 \pm P)$ ($P(x)$ is defined in eq. (61)) or:

$$P_R(x) = \frac{1}{1-r_\pi}\left(1 - \frac{r_\pi}{x}\right), \tag{65}$$

$$P_L(x) = \frac{1}{1-r_\pi}\left(\frac{r_\pi}{x} - r_\pi\right). \tag{66}$$

We can then assume that a π^- will produce spectra of right-handed and left-handed μ^- with spectra

$$F_{\pi^- \to \mu^-_{R,L}}(x) = \frac{1}{(1-r_\pi)} P_{R,L}(x)\, \theta(x - r_\pi). \tag{67}$$

The corresponding decay Z factors are

$$Z_{\pi^- \to \mu_R^-} = \frac{1}{(1-r_\pi)^2}\left(\frac{1-r_\pi^\alpha}{\alpha} - r_\pi \frac{1-r_\pi^{\alpha-1}}{\alpha-1}\right), \tag{68}$$

$$Z_{\pi^- \to \mu_L^-} = \frac{1}{(1-r_\pi)^2}\left(\frac{1-r_\pi^\alpha}{\alpha} - \frac{1-r_\pi^{\alpha-1}}{\alpha-1}\right). \tag{69}$$

It is easy to check (in shortened notation) that

$$Z_R + Z_L = \frac{1-r_\pi^\alpha}{\alpha(1-r_\pi)}, \tag{70}$$

which is the usual result for $Z_{\pi \to \mu}$ (see appendix A), and

$$\frac{Z_R - Z_L}{Z_R + Z_L} = \langle P(\alpha) \rangle, \tag{71}$$

with $\langle P(\alpha) \rangle$ defined in eq. (64).

One may think that with the method described above some information about the spin status of the muons is lost, however this is not so. It is true that we lose track of the quantum spin status of each produced muon, but this has no observable consequence. The spin status of muons of energy E at the

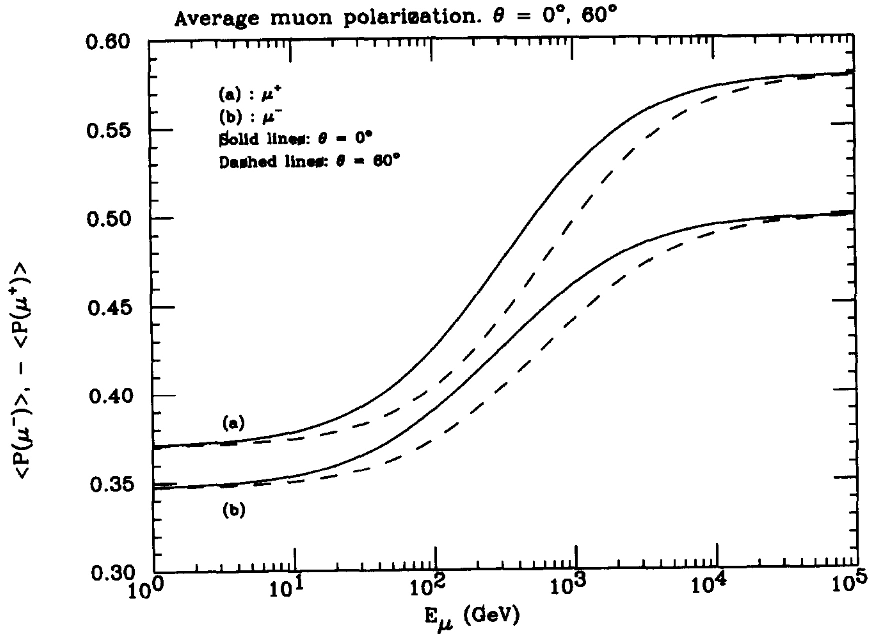

Fig. 1. Polarization of the muon fluxes at sea level. The 4 curves refer to μ^+ and μ^-, at two values of the zenith angle: $\theta = 0°$ and $\theta = 60°$.

point (θ, t) is fully defined by a (2×2) density matrix $\hat{\rho}$. Choosing a basis of orthonormal states $\{|1\rangle, |2\rangle\}$, the elements of the density matrix are

$$\hat{\rho}_{ij} = \sum_n \omega_n \langle \psi_n | i \rangle \langle j | \psi_n \rangle, \tag{72}$$

where ω_n is the statistical weight of the pure state $|\psi_n\rangle$. The expectation value of any spin operator Q can be obtained as: $\langle Q \rangle = \mathrm{Tr}[Q\hat{\rho}]$ where Tr indicates the operation of trace. It is easy to show that in the basis $\{|\mathrm{R}\rangle, |\mathrm{L}\rangle\}$ the density matrix takes the form

$$\hat{\rho} = \begin{bmatrix} p_\mathrm{R} & 0 \\ 0 & 1 - p_\mathrm{R} \end{bmatrix} \tag{73}$$

(this result is also obvious because of azimuthal symmetry). The entire information of the state of spin of the muons (for a given set (E, θ, t)) is contained in a single number p_R, which in our treatment can be exactly calculated as $p_\mathrm{R} = \mu_\mathrm{R}/(\mu_\mathrm{R} + \mu_\mathrm{L})$.

A graph of the muon polarization at sea level as a function of energy for two values of the zenith angle is given in fig. 1. The polarization has a dependence on energy and zenith angle that reflects the steepening of the meson source (because of the meson decay probability) with increasing energy. Because of the different contribution of charged kaons, the $\mu^\pm$ fluxes have different absolute values of the polarization (μ^- (μ^+) have positive (negative) helicity). Negative muons at low energy have a polarization of $P_- \simeq +0.35$ at low energy that grows to $P_- \simeq +0.50$. The positive muons receive a larger contribution from positive kaons, at low energies the polarization is $P_+ \simeq -0.38$ and goes to $P_+ \simeq -0.58$ at very large energies. It is interesting to observe that in principle a precise measurement of muon polarization would determine the relative contribution of kaons and pions to the muon flux.

6. Neutrinos from $\mu^{\pm}$ decay

The neutrinos coming from $\mu^{\pm}$ decay can be easily calculated from equation

$$\frac{\partial \nu_j}{\partial t}(E, t) = \sum_{s=\mathrm{L,R}} \int_E^{\infty} \mathrm{d}E_\mu \, \frac{\mu_s^{\pm}(E_\mu, t)}{\lambda_{\mathrm{dec}}^{\mu}(E_\mu, t)} \, \frac{1}{E_\mu} F_{(\mu_s^{\pm} \to \nu_j)}\left(\frac{E}{E_\mu}\right), \tag{74}$$

where we indicate with $\mu_s^{\pm}(E, t; \theta)$ the flux of muons of a given sign and helicity, the dependence on the zenith angle θ is considered as a parameter. The spectra for the decay $(\mu_{\mathrm{R,L}}^{\pm} \to \nu_j)$ are given in appendix A.2. The neutrino fluxes at the point t_{obs} are obtained integrating over all possible source points:

$$\nu_j(E, t_{\mathrm{obs}}) = \int_0^{t_{\mathrm{obs}}} \mathrm{d}t \, \frac{\partial \nu_j}{\partial t}(E, t). \tag{75}$$

The neutrino fluxes produced by muon decay can therefore be obtained with a straightforward double integral from the calculated $\mu^{\pm}$ fluxes. It is convenient to exchange the order of the integrals over slant depth and muon energy as follows:

$$\nu_j(E, t_{\mathrm{obs}}) = \sum_{s=\mathrm{L,R}} \int_E^{\infty} \mathrm{d}E_\mu \, \frac{1}{E_\mu} F_{(\mu_s^{\pm} \to \nu_j)}\left(\frac{E}{E_\mu}\right) \left[\int_0^{t_{\mathrm{obs}}} \mathrm{d}t \, \frac{\mu_s^{\pm}(E_\mu, t)}{\lambda_{\mathrm{dec}}^{\mu}(E_\mu, t)}\right]. \tag{76}$$

7. Numerical results

The calculation of the spectra of $(\mu^+, \mu^-, \nu_\mu, \bar{\nu}_\mu, \nu_e, \bar{\nu}_e)$ along the lines described above has the following ingredients:

(i) Define a primary flux: $\phi_p = p_0 E^{-\alpha}$, $\phi_n = p_0 \delta_0 E^{-\alpha}$; this requires the choice of three parameters: p_0, δ_0 and α.

(ii) Choose values for the hadronic Z factors, and for the interaction lengths λ_p, λ_π, λ_K.

(iii) Define a model of the atmosphere. Obviously one could at this point choose not only the earth's atmosphere but, for example, the density distribution of another astrophysical object. If the composition of this medium is different from the one of the earth's atmosphere we need to modify accordingly the choice of the hadronic Z factors and interactions lengths of point (ii).

The calculation of the lepton spectra (for each given value of the zenith angle θ) can be outlined as follows:

1. Compute the meson fluxes $M(E, t)$ (with $M = \pi^{\pm}, K^{\pm}, K_{\mathrm{L}}$) making use of eq. (34).
2. Compute the neutrino fluxes produced by direct meson decay, using eq. (40).
3. Compute the source of $\mu_{\mathrm{L,R}}^{\pm}$ due to the decay of the $\pi^{\pm}$ and $K^{\pm}$ fluxes by using the polarized muon spectra (67) in eq. (36).
4. Compute the $\mu_{\mathrm{L,R}}^{\pm}(E, t)$ at a sufficient number of points in energy and depth using eq. (58).
5. Compute the neutrino from muon decay according to eq. (76).

It is a straightforward program that requires only some numerical integrations. The entire calculation for a given zenith angle can be performed in a quite limited amount (few minutes on a VAX) of CPU time. The most time consuming part of the calculation is the propagation of the muon flux (eq. (56)), some shortcuts are explored in appendix C, but also a detailed account of the muons energy loss (neglecting fluctuations) is possible.

As discussed in the introduction, this method of calculating the muon and neutrino fluxes has been used by several authors before, see for example refs. [8–12] (see ref. [1] for a review and a complete

Table 5
Z factors for hadronic interactions ($\alpha = 2.7$).

	p	n	π^+	π^-	K^+	K^-	K_L
p	0.2630	0.0350	0.0460	0.0330	0.0090	0.0028	0.0059
n	0.0350	0.2630	0.0330	0.0460	0.0065	0.0028	0.0014
π^+	0	0	0.2430	0.0280	0.0067	0.0067	0.0067
π^-	0	0	0.0280	0.2430	0.0067	0.0067	0.0067
K^+	0	0	0	0	0.2110	0	0.0070
K^-	0	0	0	0	0	0.2110	0.0070
K_L	0	0	0	0	0.0070	0.0070	0.2110

bibliography). A clear description of these techniques has been made for example by Dar [13,14]. We are improving on this work in a few details: (i) The proton and neutron components are treated separately with different cross sections. (ii) We are considering the interactions of mesons produced in cosmic rays showers (ignored in ref. [13]). (iii) In the expressions of Dar it is implicitly assumed that the interaction lengths of pions and kaons are equal to the interaction length of nucleons (and in fact the meson interaction lengths are never introduced by him). (iv) We are including the effects of the muon polarization. (v) We have not tried to parametrize in simple form the muon energy loss and decay. This part of the calculation is under complete theoretical control, and the calculation is in principle straightforward. The shortcuts proposed in ref. [13] have valuable heuristic merits, and can be useful for good approximation estimates.

We have checked that we can reproduce exactly the results of Dar, if we assume that the differential cross sections of neutrons are equal to the ones of protons, if we neglect meson reinteractions, ignore

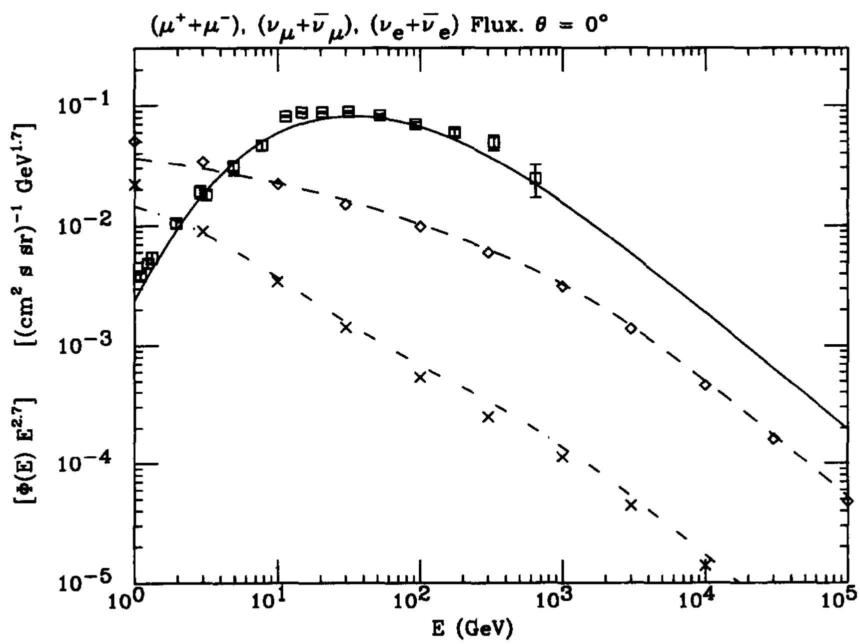

Fig. 2. Comparison of the vertical muon flux (solid line) with the data from Allkofer et al. [15]. Comparison of the vertical $(\nu_\mu + \bar{\nu}_\mu)$ (dashed line) and $(\nu_e + \bar{\nu}_e)$ (dot–dash line) fluxes with the calculation of Volkova [2].

Table 6
Flux of $\mu^+ + \mu^-$ at sea level. The energy is in GeV, the flux in units $(\mathrm{cm}^2\ \mathrm{s\ sr\ GeV})^{-1}$. The two numbers in each column are the mantissa and the decimal exponent of the flux.

E	$\cos\theta$							
	1	0.6	0.4	0.3	0.2	0.1	0.05	0.0
1.00×10^0	2.24, −3	4.04, −4	1.19, −4	3.62, −5	9.00, −6	1.08, −6	2.37, −7	5.15, −8
3.16×10^0	8.16, −4	2.80, −4	1.06, −4	4.42, −5	1.33, −5	1.88, −6	4.23, −7	6.45, −8
1.00×10^1	1.10, −4	6.78, −5	3.95, −5	2.37, −5	1.05, −5	2.28, −6	6.16, −7	8.88, −8
3.16×10^1	6.80, −6	6.09, −6	5.07, −6	4.11, −6	2.80, −6	1.17, −6	4.75, −7	9.54, −8
1.00×10^2	2.48, −7	2.81, −7	2.91, −7	2.84, −7	2.59, −7	1.87, −7	1.21, −7	4.55, −8
3.16×10^2	6.06, −9	8.12, −9	9.78, −9	1.08, −8	1.18, −8	1.18, −8	1.04, −8	6.78, −9
1.00×10^3	1.14, −10	1.69, −10	2.24, −10	2.67, −10	3.28, −10	4.02, −10	4.19, −10	3.69, −10
3.16×10^3	1.87, −12	2.91, −12	4.06, −12	5.06, −12	6.65, −12	9.14, −12	1.04, −11	1.05, −11
1.00×10^4	2.81, −14	4.50, −14	6.47, −14	8.26, −14	1.13, −13	1.64, −13	1.95, −13	2.07, −13
3.16×10^4	4.06, −16	6.58, −16	9.60, −16	1.24, −15	1.72, −15	2.60, −15	3.16, −15	3.42, −15
1.00×10^5	5.78, −18	9.41, −18	1.38, −17	1.79, −17	2.51, −17	3.85, −17	4.72, −17	5.16, −17
3.16×10^5	8.18, −20	1.33, −19	1.96, −19	2.55, −19	3.59, −19	5.53, −19	6.81, −19	7.47, −19
1.00×10^6	1.16, −21	1.89, −21	2.78, −21	3.61, −21	5.08, −21	7.86, −21	9.68, −21	1.06, −20

muon polarization and put appropriate values of the interactions length and proton Z factors (according to Liland [12]).

A crucial element of the calculation is obviously the choice of the input parameters: the primary flux, the hadronic Z factors and the hadronic interactions lengths. A detailed critical discussion of the best possible choice and of the systematic uncertainty connected with this choice is beyond the scope of this work. As an example we have used for the primary flux: $\delta_0 = 0.14$, $p_0(1+\delta_0) = 1.7$, $\alpha = 2.7$. For the interaction length we have used the values suggested by Gaisser [1] (in units g cm^{-2}: $\lambda_p = 86$, $\lambda_\pi = 116$, $\lambda_K = 138$). We have also used the hadronic Z factors tabulated in ref. [1], and for those not tabulated the values given in table 5.

We have calculated the lepton fluxes for 8 values of the zenith angle (the same used by Volkova [2]: $\cos\theta = 0$, 0.05, 0.1, 0.2, 0.3, 0.4, 0.6, 1). The fluxes of $(\mu^+ + \mu^-)$, $(\nu_\mu + \bar{\nu}_\mu)$ and $(\nu_e + \bar{\nu}_e)$ are given in tables 6, 7 and 8. In figs. 2 and 3 we compare the vertical and horizontal fluxes with data for the $\mu^\pm$ flux)

Table 7
Flux of $\nu_\mu + \bar{\nu}_\mu$ at sea level. The energy is in GeV, the flux in units $(\mathrm{cm}^2\ \mathrm{s\ sr\ GeV})^{-1}$. The two numbers in each column are the mantissa and the decimal exponent of the flux.

E	$\cos\theta$							
	1	0.6	0.4	0.3	0.2	0.1	0.05	0.0
1.00×10^0	3.64, −2	4.11, −2	4.43, −2	4.63, −2	4.85, −2	5.12, −2	5.26, −2	5.30, −2
3.16×10^0	1.31, −3	1.52, −3	1.70, −3	1.82, −3	1.96, −3	2.11, −3	2.17, −3	2.19, −3
1.00×10^1	4.46, −5	5.20, −5	5.90, −5	6.43, −5	7.18, −5	8.22, −5	8.73, −5	8.98, −5
3.16×10^1	1.42, −6	1.69, −6	1.92, −6	2.10, −6	2.37, −6	2.81, −6	3.10, −6	3.30, −6
1.00×10^2	4.08, −8	5.00, −8	5.84, −8	6.47, −8	7.38, −8	8.79, −8	9.77, −8	1.07, −7
3.16×10^2	1.09, −9	1.37, −9	1.63, −9	1.83, −9	2.12, −9	2.56, −9	2.84, −9	3.07, −9
1.00×10^3	2.59, −11	3.48, −11	4.27, −11	4.87, −11	5.72, −11	6.98, −11	7.71, −11	8.17, −11
3.16×10^3	4.93, −13	7.28, −13	9.63, −13	1.15, −12	1.42, −12	1.81, −12	2.01, −12	2.12, −12
1.00×10^4	7.90, −15	1.24, −14	1.74, −14	2.18, −14	2.87, −14	3.98, −14	4.61, −14	4.91, −14
3.16×10^4	1.17, −16	1.88, −16	2.72, −16	3.49, −16	4.79, −16	7.07, −16	8.48, −16	9.18, −16
1.00×10^5	1.67, −18	2.72, −18	3.97, −18	5.15, −18	7.18, −18	1.09, −17	1.33, −17	1.46, −17
3.16×10^5	2.37, −20	3.87, −20	5.67, −20	7.38, −20	1.03, −19	1.59, −19	1.96, −19	2.14, −19

Table 8
Flux of $\nu_e + \bar{\nu}_e$ at sea level. The energy is in GeV, the flux in units (cm^2 s sr GeV)$^{-1}$. The two numbers in each column are the mantissa and the decimal exponent of the flux.

E	$\cos\theta$							
	1	0.6	0.4	0.3	0.2	0.1	0.05	0.0
1.00×10^0	1.46, −2	1.90, −2	2.22, −2	2.41, −2	2.63, −2	2.89, −2	3.02, −2	3.05, −2
3.16×10^0	3.76, −4	5.64, −4	7.28, −4	8.41, −4	9.77, −4	1.13, −3	1.18, −3	1.20, −3
1.00×10^1	7.42, −6	1.24, −5	1.78, −5	2.23, −5	2.91, −5	3.87, −5	4.37, −5	4.62, −5
3.16×10^1	1.34, −7	2.21, −7	3.32, −7	4.39, −7	6.25, −7	9.80, −7	1.24, −6	1.43, −6
1.00×10^2	2.71, −9	4.00, −9	5.69, −9	7.40, −9	1.07, −8	1.81, −8	2.50, −8	3.26, −8
3.16×10^2	5.77, −11	8.16, −11	1.08, −10	1.33, −10	1.80, −10	2.89, −10	3.99, −10	5.49, −10
1.00×10^3	1.10, −12	1.62, −12	2.17, −12	2.65, −12	3.43, −12	4.96, −12	6.31, −12	8.09, −12
3.16×10^3	1.80, −14	2.81, −14	3.93, −14	4.91, −14	6.50, −14	9.26, −14	1.12, −13	1.30, −13
1.00×10^4	2.70, −16	4.34, −16	6.25, −16	7.99, −16	1.09, −15	1.61, −15	1.95, −15	2.17, −15
3.16×10^4	3.90, −18	6.32, −18	9.23, −18	1.19, −17	1.66, −17	2.52, −17	3.08, −17	3.39, −17
1.00×10^5	5.54, −20	9.02, −20	1.32, −19	1.72, −19	2.41, −19	3.70, −19	4.55, −19	5.00, −19
3.16×10^5	7.85, −22	1.28, −21	1.88, −21	2.45, −21	3.44, −21	5.30, −21	6.53, −21	7.17, −21

and with the Volkova calculation. The agreement is in general good even if some small differences can be seen. The largest disagreement is at the lowest energies. Our results are in better agreement with the calculation of Mitsui et al. [4]. In the high energy region the disagreement can be traced to different choices of hadronic Z factors, in particular for K_L production.

In figs. 4 and 5 we show the depth dependence of the $(\mu^+ + \mu^-)$ fluxes for the vertical and horizontal.

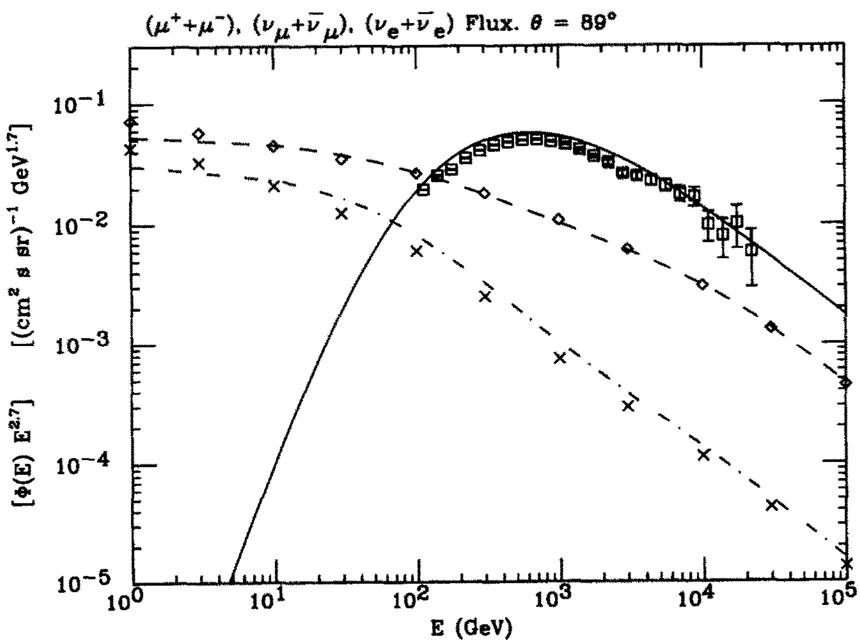

Fig. 3. Comparison of the nearly horizontal ($\theta = 89°$) muon flux at sea level (solid line(, with the data from Matsuno et al. [17]. Comparison of the horizontal ($\theta = 90°$) $(\nu_\mu + \bar{\nu}_\mu)$ (dashed line) and $(\nu_e + \bar{\nu}_e)$ (dot–dash line) fluxes, with the calculation of Volkova [2].

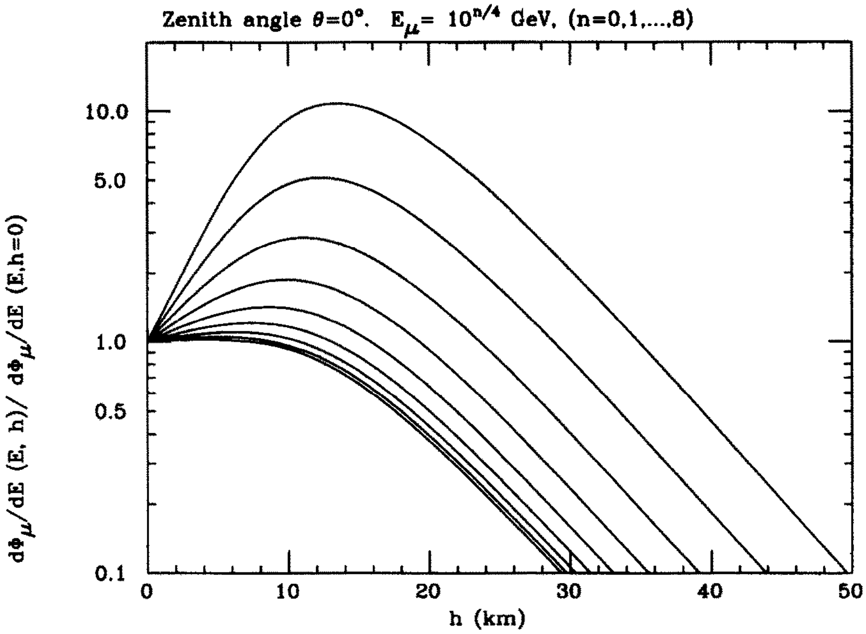

Fig. 4. Flux of vertical muons (zenith angle $\theta = 0°$) as a function of height above sea level. The ratio $\phi(h)/\phi(0)$ is plotted as a function of the height h for 8 different values of the muon energy: ($E_\mu = 10^{n/4}$, with $n = 0, 1, \ldots, 8$). The lowest energy corresponds to the curve that grows most with increasing height.

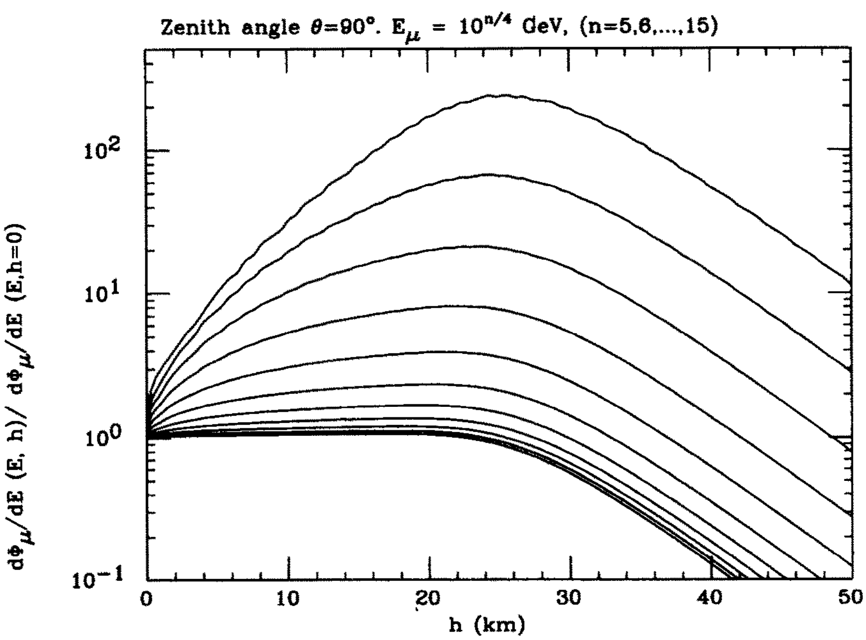

Fig. 5. Flux of horizontal muons (zenith angle $\theta = 90°$) as a function of height above sea level. The ratio $\phi(h)/\phi(0)$ is plotted as a function of the height h for 8 different values of the muon energy: ($E_\mu = 10^{n/4}$, with $n = 5, 6, \ldots, 15$). The lowest energy corresponds to the curve that grows most with increasing height.

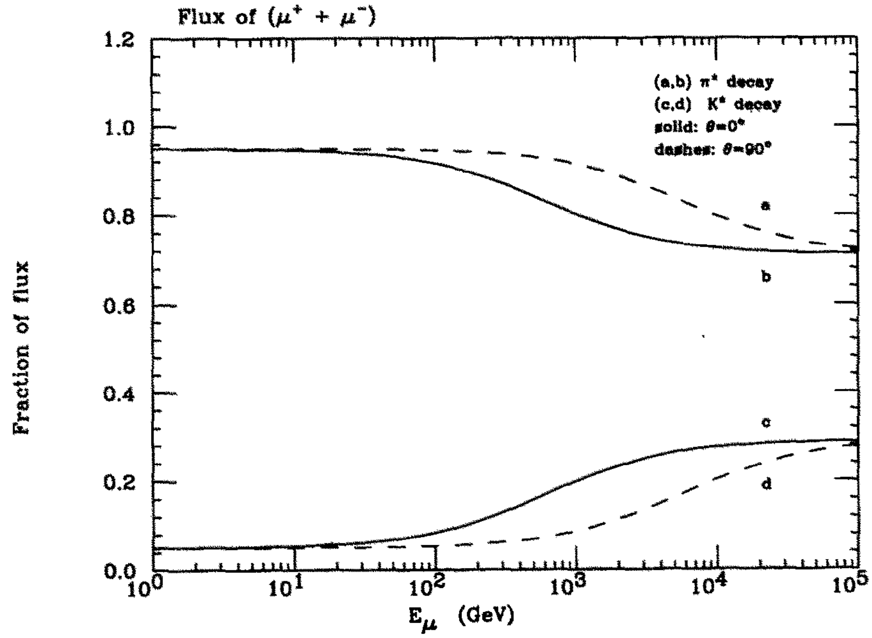

Fig. 6. Fractional contributions of $\pi^{\pm}$ and $K^{\pm}$ decay to the flux of $\mu^{\pm}$ at sea level. The π and K contributions are plotted as a function of muon energy, for zenith angles $\theta = 0°$ (solid lines) and $\theta = 90°$ (dashed lines).

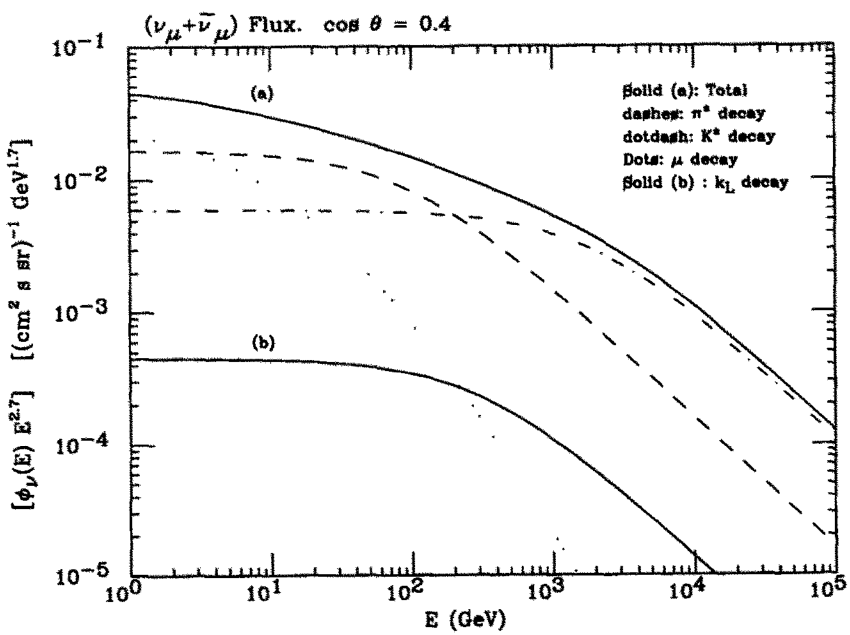

Fig. 7. Flux of $(\nu_\mu + \bar{\nu}_\mu)$ as a function of energy for cos $\theta = 0.4$. The separate contributions of $\pi^{\pm}$, $K^{\pm}$, $\mu^{\mp}$ and K_L decay are also shown (the K_L contribution is negligible.

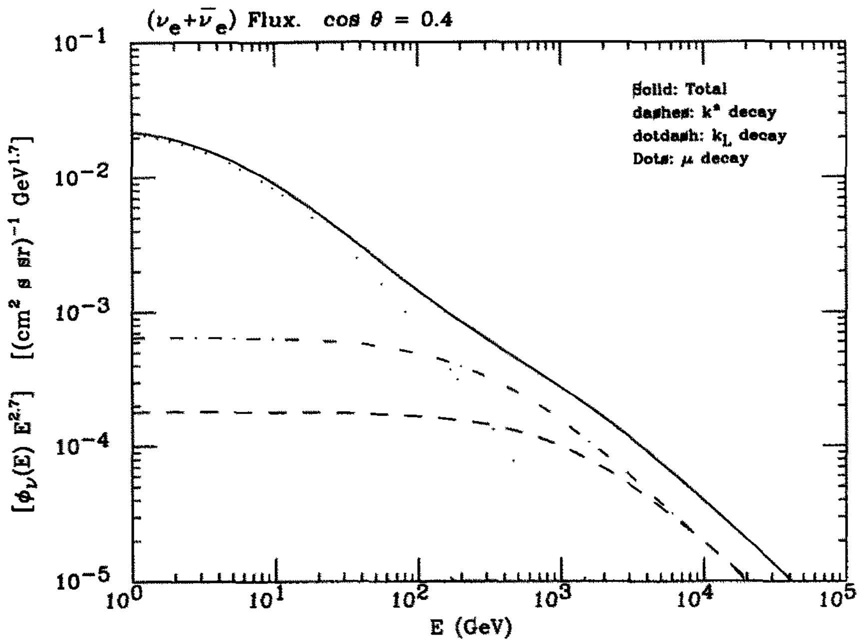

Fig. 8. Flux of ($\nu_e = \bar{\nu}_e$) as a function of energy for cos $\theta = 0.4$. The separate contributions of $K^\pm$, K_L and $\mu^\pm$ decay are also shown.

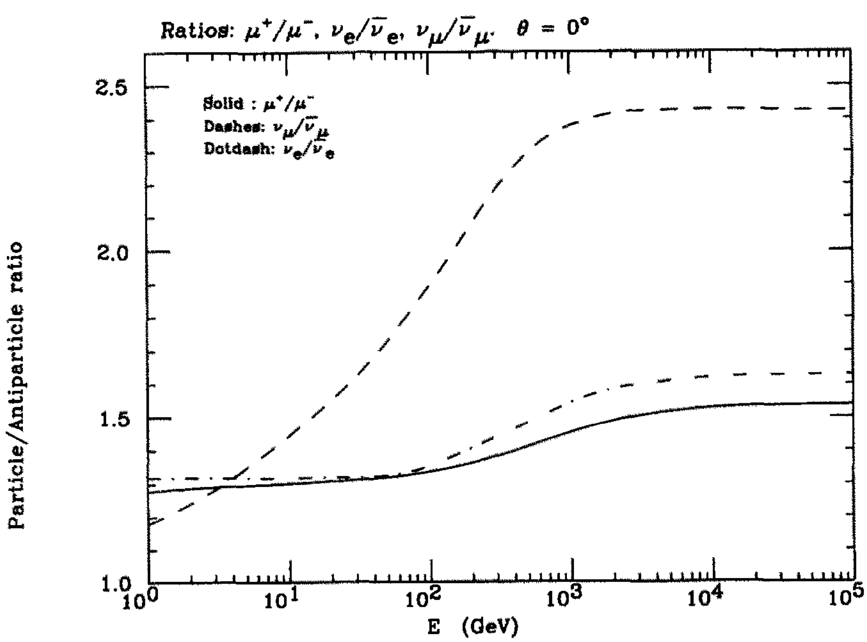

Fig. 9. Ratios μ^+/μ^-, $\nu_\mu/\bar{\nu}_\mu$, $\nu_e/\bar{\nu}_e$ plotted as a function of energy for zenith angle $\theta = 0°$.

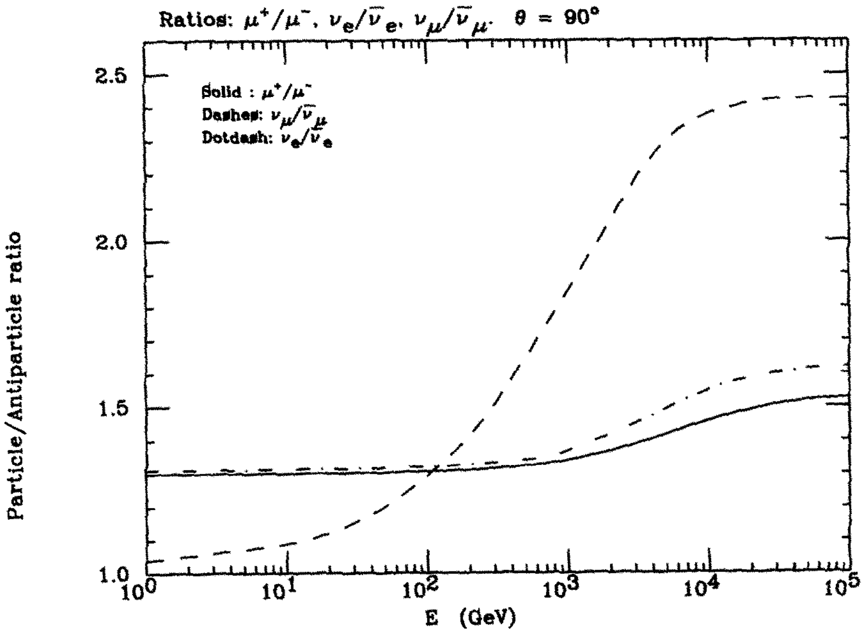

Fig. 10. Ratios μ^+/μ^-, $\nu_\mu/\bar{\nu}_\mu$, ν_e/ν ay$_e$ plotted as a function of energy for zenith angle $\theta = 90°$.

In fig. 6 we show that the contributions of $\pi^\pm$ and $K^\pm$ to the fluxes of $\mu^\pm$, K decay accounts for approximately 5% of the muon flux at low energy, the charged kaon contribution raises to $\sim 30\%$ at high energy when meson decays are rare because of the shorter lifetime of the kaons, and for the smaller Lorentz factor due to their large mass. For neutrinos the source situation is more complicated. For ν_μ ($\bar{\nu}_\mu$) we have to consider the three sources $\pi^\pm$, $K^\pm$ and $\mu^\mp$, and for ν_e ($\bar{\nu}_e$) the three sources $K^\pm$, K_L and $\mu^\pm$. A plot of the different contributions for $\cos\theta = 0.4$ is shown in figs. 7 and 8.

A discussion of the particle/antiparticle ratios is contained in the next section.

8. Particle–antiparticle ratios

In figs. 9 and 10 we show the calculated ratios μ^+/μ^-, $\nu_\mu/\bar{\nu}_\mu$ and $\nu_e/\bar{\nu}_e$ as a function of energy for vertical and horizontal fluxes. The ratios are also given in tables 9, 10 and 11. These lepton–antilepton asymmetries reflect the asymmetries in the meson fluxes π^+/π^- and K^+/K^-. All the asymmetries are non-trivial functions of E, θ and, weakly, of the depth of observation t.

The pion asymmetry is due to the fact that in the primary cosmic rays there is an excess of protons over neutrons, and that $Z_{p\pi^+} > Z_{p\pi^-}$, reflecting the valence quark content of protons and pions. The asymmetry for kaons is significantly stronger reflecting the fact that $K^- = [s\bar{u}]$ does not receive any contribution from the valence quark content of the incident nucleon. In nucleon interactions positive kaons are produced both in $K\bar{K}$ events and in ΛK^+ events, and are more numerous and faster than K^-.

8.1. Ratio μ^+/μ^-

Neglecting energy loss and decay we can compute the ratio μ^+/μ^- using the expressions

$$\mu_L^\pm = \left[D_L(\pi^\pm)\, Z(\pi \to \mu;\, \alpha) + D_L(K^\pm)\, Z(K \to \mu;\, \alpha)\, Br(K \to \mu\nu)\right] E^{-\alpha}. \tag{77}$$

Table 9
Ratio μ^+/μ^-.

E_μ (GeV)	$\cos\theta$							
	1	0.6	0.4	0.3	0.2	0.1	0.05	0.0
1.00×10^0	1.28	1.28	1.29	1.29	1.29	1.29	1.29	1.30
3.16×10^0	1.29	1.29	1.29	1.29	1.29	1.29	1.30	1.30
1.00×10^1	1.30	1.30	1.30	1.30	1.30	1.29	1.30	1.30
3.16×10^1	1.31	1.31	1.30	1.30	1.30	1.30	1.30	1.30
1.00×10^2	1.34	1.32	1.32	1.31	1.31	1.31	1.30	1.31
3.16×10^2	1.39	1.36	1.35	1.34	1.33	1.32	1.32	1.32
1.00×10^3	1.45	1.42	1.40	1.39	1.37	1.35	1.34	1.34
3.16×10^3	1.50	1.49	1.47	1.45	1.43	1.41	1.39	1.39
1.00×10^4	1.53	1.52	1.51	1.51	1.49	1.47	1.46	1.46
3.16×10^4	1.53	1.53	1.53	1.53	1.52	1.51	1.51	1.51
1.00×10^5	1.54	1.54	1.54	1.54	1.53	1.53	1.53	1.53

Table 10
Ratio $\nu_\mu/\bar{\nu}_\mu$.

E_ν (GeV)	$\cos\theta$							
	1	0.6	0.4	0.3	0.2	0.1	0.05	0.0
1.00×10^0	1.18	1.13	1.10	1.08	1.07	1.05	1.04	1.04
3.16×10^0	1.29	1.21	1.16	1.13	1.10	1.07	1.06	1.06
1.00×10^1	1.44	1.35	1.28	1.23	1.18	1.12	1.10	1.09
3.16×10^1	1.63	1.53	1.45	1.40	1.33	1.23	1.19	1.16
1.00×10^2	1.90	1.76	1.67	1.60	1.52	1.42	1.35	1.29
3.16×10^2	2.20	2.07	1.96	1.89	1.79	1.66	1.59	1.52
1.00×10^3	2.38	2.32	2.26	2.20	2.11	1.98	1.91	1.85
3.16×10^3	2.42	2.41	2.40	2.38	2.34	2.27	2.23	2.19
1.00×10^4	2.43	2.43	2.42	2.42	2.42	2.40	2.39	2.38
3.16×10^4	2.42	2.43	2.43	2.43	2.43	2.42	2.42	2.42
1.00×10^5	2.42	2.42	2.42	2.42	2.43	2.43	2.43	2.43

Table 11
Ratio $\nu_e/\bar{\nu}_e$.

E_ν (GeV)	$\cos\theta$							
	1	0.6	0.4	0.3	0.2	0.1	0.05	0.0
1.00×10^0	1.32	1.32	1.32	1.31	1.31	1.31	1.31	1.31
3.16×10^0	1.32	1.32	1.32	1.32	1.31	1.31	1.31	1.31
1.00×10^1	1.32	1.32	1.32	1.32	1.32	1.32	1.31	1.31
3.16×10^1	1.32	1.32	1.32	1.32	1.32	1.32	1.32	1.32
1.00×10^2	1.35	1.33	1.33	1.32	1.32	1.32	1.32	1.32
3.16×10^2	1.44	1.40	1.37	1.36	1.34	1.33	1.33	1.33
1.00×10^3	1.54	1.51	1.47	1.45	1.42	1.38	1.37	1.37
3.16×10^3	1.60	1.58	1.56	1.54	1.52	1.48	1.46	1.45
1.00×10^4	1.62	1.61	1.61	1.60	1.59	1.57	1.55	1.55
3.16×10^4	1.63	1.62	1.62	1.62	1.61	1.61	1.60	1.60
1.00×10^5	1.63	1.63	1.63	1.63	1.62	1.62	1.62	1.62

This gives a ratio $\mu^+/\mu^- \simeq 1.31$. This estimate is, however, too large by $\sim 3\%$ as can be tested with a careful numerical calculation. This is because the positive pions are created slightly higher in the atmosphere than negative pions, and therefore the positive muons have a larger probability to decay, and at a fixed energy their number is more depleted than the number of negative muons. This effect introduces a weak but in principle detectable dependence of the μ^+/μ^- ratio in the region 1–20 GeV.

The μ^+/μ^- becomes larger with increasing energy. This is because the role of kaons is enhanced with respect to pions, when both meson types have a small decay probability, by a factor $(m_K/\tau_K)/(m_\pi/\tau_\pi)$. At very high energy we have in fact

$$\mu_{\rm H}^{\pm} = \left[D_{\rm H}(\pi^{\pm})\, Z(\pi \to \mu;\, \alpha+1) \frac{m_\pi}{c\tau_\pi} \right.$$
$$\left. + D_{\rm H}(K^{\pm})\, Z(K \to \mu;\, \alpha+1)\, Br(K \to \mu\nu) \frac{m_K}{c\tau_K} \right] h_0 \frac{E^{-(\alpha+1)}}{\cos\theta}. \tag{78}$$

Kaons that where the source of $\sim 5\%$ of the muons at low energy become the source of $\sim 25\%$ of the muons, and the muon ratio increases to $\mu^+/\mu^- = 1.54$.

The data on the μ^+/μ^- ratio [16,17] is in reasonable agreement with this calculation.

8.2. Ratio ν_μ/ν_μ

In the case of muon neutrinos we have to consider three sources: $\mu^{\mp}$, $\pi^{\pm}$ and $K^{\pm}$ decay. At low energy

$$\nu_\mu = \left\{ D_{\rm L}(\pi^+)\, Z(\pi \to \nu;\, \alpha) + D_{\rm L}(\pi^-)\, Z(\pi^- \to \mu^- \to \nu_\mu;\, \alpha) \right.$$
$$\left. + \left[D_{\rm L}(K^+)\, Z(K \to \nu;\, \alpha) + D_{\rm L}(K^-)\, Z(K^- \to \mu^- \to \nu_\mu;\, \alpha) \right] Br(K \to \mu\nu) \right\} E^{-\alpha} \tag{79}$$

(the $\bar{\nu}_e$ flux is obtained with obvious substitutions). The flux of ν_μ receives contributions both from the positive mesons (via a direct decay) and from negative mesons (via chain decay), and the ratio is close to unity ($\nu_\mu/\bar{\nu}_\mu \simeq 1.05$. When the energy increases the muon decay term becomes less and less important and the ratio $\nu_\mu/\bar{\nu}_\mu$ grows. At asymptotic energies the flux becomes

$$\nu_\mu = \left[D_{\rm H}(\pi^{\pm})\, Z(\pi \to \nu;\, \alpha+1) \frac{m_\pi}{c\tau_\pi} \right.$$
$$\left. + D_{\rm H}(K^{\pm})\, Z(K \to \nu_\mu;\, \alpha+1)\, Br(K \to \mu\nu) \frac{m_K}{c\tau_K} \right] h_0 \frac{E^{-(\alpha+1)}}{\cos\theta}. \tag{80}$$

The asymptotic value of the ratio becomes $\nu_\mu/\bar{\nu}_\mu = 2.52$. The reason why this ratio is so different from the muon ratio, is that the contribution of pions to the neutrino flux is very suppressed for kinematical reasons. In the decay $\pi \to \mu\nu$ the muon mass is close to the pion mass so that the muon takes away most (0.785) of the parent energy; this effect enhances (depresses) the muon (neutrino) spectrum from π decay. Therefore $K^{\pm}$ decay is much more important for the neutrino flux, and accounts for $\sim 85\%$ of the very high energy neutrino flux.

8.3. Ratio $\nu_e/\bar{\nu}_e$

Also in the case of electron neutrinos, we have to consider three sources ($\mu^{\pm}$, $K^{\pm}$ and $K_{\rm L}$ decay). The three sources have different importance in different regions of energy and zenith angles. The asymptotic values are easily calculated. At low energy,

$$\begin{aligned}\nu_e = \{&D_{\rm L}(\pi^+)Z(\pi^+ \to \mu^+ \to \nu_e;\, \alpha)\\ &+D_{\rm L}(K^+)\, Z(K^+ \to \mu^+ \to \nu_e;\, \alpha)\, Br(K^+ \to \mu^+\nu_\mu)\\ &+D_{\rm L}(K^+)\, Z(K^+ \to \nu_e;\, \alpha)\, Br(K^+ \to \pi^0 e^+ \nu_e)\\ &+D_{\rm L}(K_{\rm L})[Z(K_{\rm L} \to \nu_e;\, \alpha)\, Br(K_{\rm L} \to \pi^- e^+ \nu_e)]\}E^{-\alpha},\end{aligned} \tag{81}$$

where we have the contribution from muon decay and from the direct decay of $K^\pm$ and $K_{\rm L}$. The dominant contribution is the chain decay $\pi \to \mu \to \nu_e$, and we have $(\nu_e/\bar{\nu}_e)_{\rm L} \simeq 1.30$ which is very close to the low energy muon ratio. At very high energy we can neglect the contribution due to muon decay:

$$\begin{aligned}\nu_e = \Bigg[&D_{\rm H}(K^-)\, Z(K^+ \to \nu_e;\, \alpha+1)\, Br(K^+ \to \pi^0 e^+ \nu_e) \frac{m_{K^+}}{c\tau_{K^+}}\\ &+D_{\rm H}(K_{\rm L})\, Z(K_{\rm L} \to \nu_e;\, \alpha+1)\, Br(K_{\rm L} \to \pi^- e^+ \nu_e) \frac{m_{K_{\rm L}}}{c\tau_{K_{\rm L}}}\Bigg] h_0 \frac{E^{-(\alpha+1)}}{\cos\theta}.\end{aligned} \tag{82}$$

The charged kaon contribution grows in relative importance with energy because its lifetime is ~ 4.2 times shorter than the $K_{\rm L}$ lifetime, and the electron neutrino ratio grows to an asymptotic value $\nu_e/\bar{\nu}_e = 1.62$.

9. Discussion and conclusions

In this work we have calculated the fluxes of $\mu^\pm$, ν_μ, $\bar{\nu}_\mu$, ν_e and $\bar{\nu}_e$, and the $\mu^\pm$ polarization under the assumptions of exact scaling of hadronic interactions and of a power law primary flux ($\phi_p = p_0 E^{-\alpha}$, $\phi_n = p_0 \delta_0 E^{-\alpha}$). This approach has two obvious limits with respect to a Monte Carlo calculation: it is not possible to investigate the effects of deviations of the primary spectrum from the power law form, and of the mildly non-scaling properties of the hadronic cross sections. In fact our calculation cannot be very reliable at energies below a few GeV, because the primary flux begins to flatten (and a time dependence is introduced by solar modulation), and because the energy is too low and the scaling region is not yet fully reached. However, the analytic calculation even with these obvious limitations remains an important tool in the study and understanding of the muon and neutrino fluxes in cosmic rays above an energy of a few GeV. The uncertainty in the primary cosmic ray flux is such that it is today still difficult to go beyond a power law fit (the important feature of the "knee" will produce some structure in the muon and neutrino spectra at energies of approximately 100 TeV). There is also still considerable uncertainty about the properties of particle production in the fragmentation region of high energy hadronic interactions, especially of the properties of strange mesons productions (the Z factors for K production in nucleon–air interactions have probably still an uncertainty of the order of 30%). Still less is known about scaling violations; in any case most theoretical models [1] predict small scaling violation effects in the fragmentation region. The analytic calculation has the advantage that the assumptions made about the primary flux and the properties of hadronic interactions (Z factors and interactions lengths) are directly controlled, the calculation can be easily repeated with modified assumptions, and thus an uncertainty range established.

The most important numerical results of this work are contained in tables 10 and 11, which contain the energy and zenith angle dependence of the ratios $\nu_\mu/\bar{\nu}_\mu$ and $\nu_e/\bar{\nu}_e$ calculated with the set of assumptions discussed in section 7. In studies of the upward muon fluxes it is important to consider correctly the non-trivial dependence on energy and zenith angle of the $\nu/\bar{\nu}$ ratio.

The polarization of μ^+ and μ^- contains useful information about the nature of the parent particles ($\pi^\pm$ or $K^\pm$).

Some final comments about the relation between the muon and the neutrino fluxes. The cosmic ray μ^+ fluxes are produced mostly in the decay of π mesons, the K decay contribution grows with energy becoming (at energies of a few TeV) of the order of $\sim 30\%$. Because of the different kinematics, $K^\pm$ decays account for more than 80% of the $\nu_\mu + \bar{\nu}_\mu$ spectra at high energies. The inclusive cross sections for kaon production are less well determined than the corresponding ones for $\pi^\pm$ production, and therefore the uncertainties in the calculations of the neutrino fluxes are higher than for the muon fluxes. Because of this large difference in the relative importance of the kaon contribution, the $\nu_\mu(\bar{\nu}_\mu)$ fluxes are not very constrained by the present measurements of the muon fluxes, which have still uncertainties of the order of $\sim 10\%$. On the other hand it is clear that a precision measurement of the μ^+ and μ^- fluxes, at the level of the $\sim 1\%$ precision, would allow one to determine the K contribution, which can be extracted from the data because of a different energy and angular dependence. In this way one could pinpoint the muon–neutrino fluxes with an uncertainty of $\sim 3\%$.

Electron neutrinos and antineutrinos have an important contribution due to K_L decay. This decay gives small contributions (in a fractional sense) to the muon and muon-neutrino fluxes (approximately 0.5% and 1% at large energies). The properties of K_L production therefore cannot be determined by measurements of the muon fluxes, however, it should be possible with some solid theoretical considerations to relate K_L production to the $K^\pm$ production.

A large fraction of the $\nu_e(\bar{\nu}_e)$ fluxes is due to $\mu^\mp$ decay (this is also true at low energy for the $\bar{\nu}_\mu(\nu_\mu)$ fluxes); a good determination of the height dependence of the muon fluxes, using detectors located on high mountains ($h \leqslant 5000$ m) or on airplanes ($h \simeq 10^4$ m) could also be of significant help in reducing the uncertainty of this neutrino source.

In conclusion the calculation of the neutrino fluxes suffers a significant uncertainty due to lack of knowledge of the properties of strange meson production in the fragmentation region of hadron–nucleus interactions. An experimental program to determine with high precision the $\mu^\pm$ flux could significantly reduce this uncertainty. In this way the limits on the parameters of neutrino oscillations obtainable from the detection of the atmospheric neutrino fluxes could become more stringent.

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Appendix A. Particle decays

A.1. Meson decay

In the ultrarelativistic limit the spectrum of particle b in the decay $a \to b + X$ takes the simple scaling form

$$F_{a\to b}(E_b;\, E_a) = \frac{1}{E_a} F_{a\to b}\left(\frac{E_b}{E_a}\right). \tag{83}$$

in the case of the 2 body leptonic decays: $\pi^+ \to \mu^+ \nu_\mu$, $K^+ \to \mu^+ \nu_\mu$ and charge conjugates we have

$$F_{\pi^+ \to \mu^+}(x) = \frac{1}{1 - r_\pi} \theta(x - r_\pi), \tag{84}$$

$$F_{\pi^+ \to \nu_\mu}(x) = \frac{1}{1 - r_\pi} [1 - \theta(x - 1 + r_\pi)], \tag{85}$$

where $r_\pi = (m_\mu / m_\pi)^2$, and similar expressions for $K^\pm$ decays. The decay Z factors defined as

$$Z(a \to b, \alpha) = \int_0^1 \mathrm{d}x \, x^{\alpha - 1} F_{a \to b}(x) \tag{86}$$

are

$$Z(\pi^+ \to \mu^+; \alpha) = \frac{1 - r_\pi^\alpha}{\alpha(1 - r_\pi)}, \tag{87}$$

$$Z(\pi^+ \to \nu_\mu; \alpha) = \frac{(1 - r_\pi)^{\alpha - 1}}{\alpha}. \tag{88}$$

For the neutrino spectrum in three body semileptonic kaon decay ($K^+ \to \pi^0 e^+ \nu_e$, $K_\mathrm{L} \to \pi^\pm e^\mp \nu_e$), in the rest frame of the kaon we have approximated the spectrum with the expression

$$G(x) = \frac{12 x^2 (1 - \epsilon^2 - x)^2}{g(\epsilon)(1 - x)}, \tag{89}$$

where $x = 2E_\nu^* / m_K$ (with kinematical limits $0 \leqslant x \leqslant 1 - \epsilon^2$), $\epsilon = m_\pi / m_K$, and

$$g(\epsilon) = 1 - 8\epsilon^2 - 24\epsilon^4 \ln \epsilon + 8\epsilon^6 - \epsilon^8. \tag{90}$$

In a frame where the kaon is ultrarelativistic the neutrino spectrum becomes ($y = E_\mu / E_K$):

$$F_{K \to \nu}(y) = \frac{1}{g(\epsilon)} \left[\frac{(12 - 24\epsilon^2)(1 - \epsilon^2)^2}{2} - 4(1 - \epsilon^2)^3 - 12\epsilon^4 (1 - \epsilon^2) + 12\epsilon^4 y \right.$$
$$\left. - \frac{(12 - 24\epsilon^2)}{2} y^2 + 4y^3 + 12\epsilon^4 \ln\left(\frac{1 - y}{\epsilon^2} \right) \right]. \tag{91}$$

The Z factors cannot be calculated analytically because of the term involving the logarithm but they are readily obtained with a numerical integration.

For the decay ($K_\mathrm{L} \to \pi^\pm \mu^\mp \nu_\mu$) the expressions for the spectrum of the neutrino and the muons are more complicated because of the non-negligible effect of the muon mass. The inclusive spectra can be obtained from the matrix element $(p_k \cdot p_{\nu_\mu})(p_\pi \cdot p_\mu)$. The effect of this decay channel of the lepton fluxes is small, however; it accounts for $\sim 0.5\%$ of the $\mu^\pm$ flux, and $\sim 1\%$ of the $\nu_\mu(\bar{\nu}_\mu)$ flux (see tables 2 and 3 and fig. 7). The lengthy expressions of the inclusive spectra are not written here.

A.2. Muon decay

When considering muon decay, we have to take into consideration the polarization of the muons. The $\mu^\pm$ polarization is important because the spectra of the neutrinos produced in muon decay depend on this polarization. We will consider the decay $\mu^- \to \nu_\mu e^- \bar{\nu}_e$. The charge conjugate reaction can be obtained with a CP transformation reversing all spins (or equivalently with the replacement $\cos\theta \to -\cos\theta$). Neglecting the electron mass and radiative corrections, the spectra of the ν_μ and of the electron are

$$G_{\nu_\mu}(x, \cos\theta) = x^3(3-2x) + \cos\theta\, x^2(1-2x), \tag{92}$$

the spectrum of the $\bar{\nu}_e$ is

$$G_{\bar{\nu}_e}(x, \cos\theta) = 6x^2(1-x) + \cos\theta\, 6x^2(1-x), \tag{93}$$

where $x = 2E_\nu/m_\mu$ (with limits $0 \leqslant x \leqslant 1$), $\cos\theta$ is the angle of the emitted particle with respect to the spin of the muon. The normalization is chosen so that

$$\int_0^1 \mathrm{d}x \int_{-1}^{+1} \mathrm{d}\cos\theta\; G(x, \cos\theta) = 1. \tag{94}$$

Integrating over the momentum, the ν_μ spectrum has the angular dependence $\propto (1 - 1/3\cos\theta)$, and has its maximum in the direction opposite to the muon spin. The $\bar{\nu}_e$ angular distribution has the form $\propto (1 + \cos\theta)$ with the maximum in the direction parallel to the muon spin. This different behaviour can be easily explained from angular momentum conservation in the decay of a spin $\frac{1}{2}$ particle into three particles, two left-handed and one right-handed.

In an ultrarelativistic frame where the μ^- has energy E_μ and helicity P (P is the expectation value of the angular momentum in the direction of motion in units of $\frac{1}{2}\hbar$), the neutrino spectra are ($y = E_\nu/E_\mu$)

$$F_{\mu^- \to \nu_\mu}(y) = \left(\tfrac{5}{3} - 3y^2 + \tfrac{4}{3}y^3\right) + P\left(\tfrac{1}{3} - 3y^2 + \tfrac{8}{3}y^3\right), \tag{95}$$

$$F_{\mu^- \to \bar{\nu}_e}(y) = (2 - 6y^2 + 4y^3) + P(-2 + 12y - 18y^2 + 8y^3). \tag{96}$$

The spectra for μ^+ decay are obtained with the replacement $P \to -P$.

The decay Z factors are

$$Z_{\mu^- \to \nu_\mu}(\alpha) = \left(\frac{5}{3\alpha} - \frac{3}{(\alpha+2)} - \frac{4}{3(\alpha+3)}\right) + P\left(\frac{1}{3\alpha} - \frac{3}{(\alpha+2)} + \frac{8}{3(\alpha+3)}\right), \tag{97}$$

$$Z_{\mu^- \to \bar{\nu}_e}(\alpha) = \left(\frac{2}{\alpha} - \frac{6}{(\alpha+2)} + \frac{4}{(\alpha+3)}\right) + P\left(-\frac{2}{\alpha} + \frac{12}{(\alpha+1)} - \frac{18}{(\alpha+2)} + \frac{8}{(\alpha+3)}\right). \tag{98}$$

For $\alpha = 2$ the Z factor is the average fractional energy carried by the neutrino in the final state:

$$\langle y \rangle_{\nu_\mu} = \tfrac{7}{20} - \tfrac{1}{20}P, \tag{99}$$

$$\langle y \rangle_{\bar{\nu}_e} = \tfrac{3}{10} + \tfrac{1}{10}P. \tag{100}$$

A.3. Neutrino spectra in the chain decay $\pi \to \mu \to \nu$

The neutrino spectra $g(z)$ after the chain decay ($\pi \to \mu \to \nu_\mu$) can be simply written in the form

$$g(z) = \begin{cases} G(z/r) - G(z), & \text{for } z \leqslant r_\pi, \\ G(1) - G(z), & \text{for } z \geqslant r_\pi, \end{cases} \tag{101}$$

where $r_\pi = (m_\mu/m_\pi)^2$. The spectrum can be decomposed as the sum: $g(z) = g^0(z) = g^{\rm pol}(z)$, where $g^0(z)$ is the spectrum calculated neglecting the polarization effects. The explicit expressions are:

$$G^0_{\pi^+\to\mu^+\to\bar\nu_\mu}(y) = \frac{1}{1-r_\pi}\left[\tfrac{5}{3}\ln y - \tfrac{3}{2}y^2 + \tfrac{4}{9}y^3\right], \tag{102}$$

$$G^0_{\pi^+\to\mu^+\to\nu_e}(y) = \frac{1}{1-r_\pi}\left[2\ln y - 3y^2 + \tfrac{4}{3}y^3\right], \tag{103}$$

$$G^{\rm pol}_{\pi^+\to\mu^+\to\bar\nu_\mu}(y) = \frac{1}{(1-r_\pi)^2}\left\{\frac{1+r_\pi}{3}\ln y - \frac{2r_\pi}{3z}y - \tfrac{3}{2}(1+r_\pi)y^2 + \left[\tfrac{8}{9}(1+r_\pi) + \frac{2r_\pi}{z}\right]y^3 - \frac{4r_\pi}{3z}y^4\right\}, \tag{104}$$

$$G^{\rm pol}_{\pi^+\to\mu^+\to\nu_e}(y) = \frac{1}{(1-r_\pi)^2}\left\{-2(1+r_\pi)\ln y + \left[12(1+r_\pi) + \frac{4r_\pi}{z}\right]y - \left[9(1+r_\pi) + \frac{12r_\pi}{z}\right]y^2 + \left[\tfrac{8}{3}(1+r_\pi) + \frac{12r_\pi}{z}\right]y^3 - \frac{4r_\pi}{z}y^4\right\}. \tag{105}$$

The Z factors for chain decay are:

$$Z_{\pi^+\to\mu^+\to\bar\nu_\mu}(\alpha) = \left(\frac{1-r_\pi^\alpha}{\alpha(1-r_\pi)}\right)\left[\left(\frac{5}{3\alpha} - \frac{3}{(\alpha+2)} - \frac{4}{3}\frac{1}{(\alpha+3)}\right) + \langle P(\alpha)\rangle\left(\frac{1}{3\alpha} - \frac{3}{(\alpha+2)} + \frac{8}{3}\frac{1}{(\alpha+3)}\right)\right], \tag{106}$$

$$Z_{\pi^+\to\mu^+\to\nu_e}(\alpha) = \left(\frac{1-r_\pi^\alpha}{\alpha(1-r_\pi)}\right)\left[\left(\frac{2}{\alpha} - \frac{6}{(\alpha+2)} + \frac{4}{(\alpha+3)}\right) + \langle P(\alpha)\rangle\left(-\frac{2}{\alpha} + \frac{12}{(\alpha+1)} - \frac{18}{(\alpha+2)} + \frac{8}{(\alpha+3)}\right)\right], \tag{107}$$

where $\langle P(\alpha)\rangle$ is given in eq. (64), and can be interpreted as the average polarization of the muons produced after the complete decay of a pion spectrum $\propto E_\pi^{-\alpha}$.

Appendix B. Model of the atmosphere

In the calculation of cosmic ray fluxes in the atmosphere one needs a model of $\rho(h)$, the atmospheric density as a function of the height above sea level, or equivalently of $X(h)$, the vertical column density as a function of height. We have used the fit of Maeda [10] to the average US standard atmosphere. For completeness we give here the parameters of this fit. We can observe that under 3 assumptions: (i) the composition of air does not change with height, (ii) the equation of state of air is well approximated by the equation for a perfect gas, (iii) the atmosphere is in hydrostatic equilibrium: $g\rho(h) = -\mathrm{d}p(h)/\mathrm{d}h$ (g is the gravity acceleration, p the pressure); the density profile of the atmosphere is completely defined by the temperature profile $T(h)$, and the total column density at sea level X_0. The fit of Maeda corresponds to choosing a constant temperature T_s in the stratosphere $h \geqslant h_t$, and a linear dependence in the troposphere ($h \leqslant h_t$) with a lapse rate Γ (the sea level temperature is then $T_0 = T_s - \Gamma h_t$). The 4 parameters $\{X_0, T_s, h_t, \Gamma\}$ and the average mass of air moleculae $\langle m \rangle$, fully define the atmosphere. We have used $X_0 = 1030$ g cm^{-2}, $T_s = -56.5$°C, $\Gamma = 6.5$°C km^{-1}, $h_t = 11$ km, and an average mass $\langle m \rangle = 4.811 \times 10^{-23}$ g (or 28.07 atomic mass units). The functions $\rho(h)$ and $X(h)$ take the form

$$\rho(h) = \begin{cases} \rho_0 \exp(-h/h_0), & \text{for } h \geqslant h_t, \\ B(h_b - h)^{\alpha}, & \text{for } h \leqslant h_t, \end{cases} \tag{108}$$

$$X(h) = \begin{cases} \rho_0 h_0 \exp(-h/h_0), & \text{for } h \geqslant h_t, \\ A(h_b - h)^{(\alpha+1)}, & \text{for } h \leqslant h_t. \end{cases} \tag{109}$$

The density ρ can also be expressed directly as a function of the column density as

$$\rho(X) = \begin{cases} X/h_0, & \text{for } X \leqslant x_t, \\ CX^{\beta}, & \text{for } X \geqslant X_t. \end{cases} \tag{110}$$

The different constants appearing in these formulae are combinations of the 4 quantities $\{X_0, T_s, h_t, \Gamma\}$ (g is the gravitational acceleration at sea level, K_B is the Boltzmann constant):

$$h_0 = \frac{T_s K_B}{g\langle m \rangle}, \tag{111}$$

$$h_a = \frac{T_0 K_B}{g\langle m \rangle}, \qquad h_b = \frac{T_0}{\Gamma}, \qquad \alpha = \frac{h_b}{h_a} - 1, \qquad \beta = \frac{\alpha}{\alpha+1}, \tag{112}$$

$$\rho_0 = \frac{X_0}{h_b} \exp(h_t/h_0)\,(\alpha+1)\left(1 - \frac{h_t}{h_0}\right)^{\alpha}, \qquad A = \frac{X_0}{h_b h_a^{\alpha}}\left(\frac{h_0 - h_t}{h_b - h_t}\right)^{\alpha}, \tag{113}$$

$$B = A(\alpha+1), \qquad C = \frac{B}{A^{\alpha/(\alpha+1)}}, \qquad X_t = \rho_0 h_0 \exp(-h_t/h_0). \tag{114}$$

Measuring the distances in km, the density in g cm^{-3} and the column density in g cm^{-2}, the constants take values: $h_0 = 6.344$, $X_t = 230.2$, $\rho_0 = 2.054 \times 10^{-3}$, $h_b = 44.33$, $A = 2.303 \times 10^{-6}$, $B = 1.210 \times 10^{-10}$, $C = 4.439 \times 10^{-6}$, $\alpha = 4.253$, $\beta = 0.8096$.

In fig. 11 we show a graph of $\rho(h)$, the density as a function of the height above sea level. In fig. 12 we show the relation between the slant depth t, and the height above sea level h.

Appendix C. Muon decay probability

A muon produced at depth t_i with initial energy E_0, because of its finite lifetime will arrive at depth t_f with a probability less than unity. Because of energy loss (that we will treat as a continuous process) the muon will arrive at the point t_f with a smaller energy E_f, and will be able to reach only points such that $(t_f - t_i) < R(E_0)$, where $R(E)$ is the muon range. The probability that a muon of initial energy E_0, created at a the point t_i will reach the point t_f without decaying is given in eq. (56).

The survival probability depends on 4 variables, the initial (or final) energy of the muon, the creation point, the final point, and the zenith angle. For $\theta \leqslant 60°$ the zenith angle dependence is simply: $\ln p_{\text{surv}} \propto (\cos\theta)^{-1}$. This complex dependence makes the numerical calculation of the survival probability rather time consuming, even though it involves a single straightforward integral (56) per value. It is

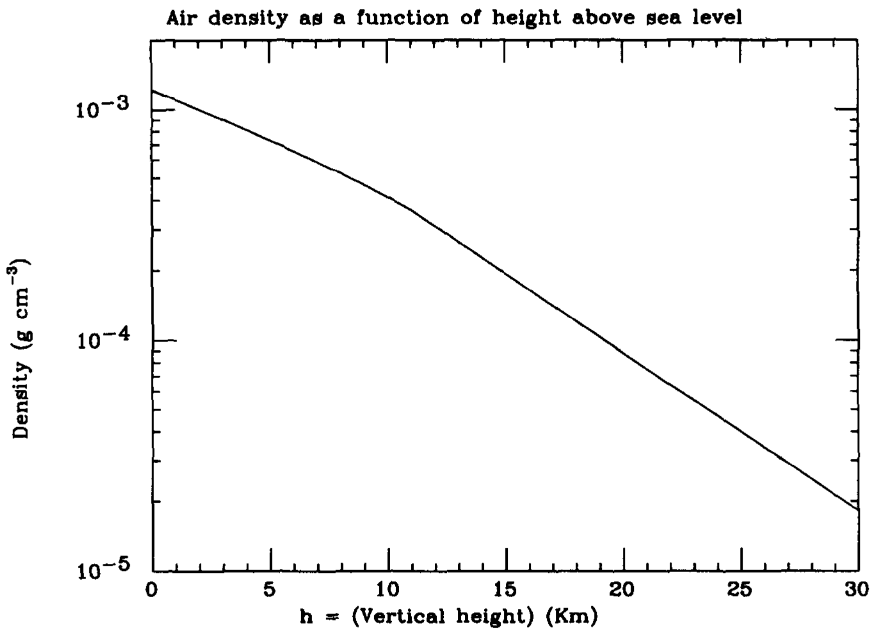

Fig. 11. Atmospheric density as a function of height above sea level. Note the discontinuity in the derivative at the troposphere $h_t = 11$ km.

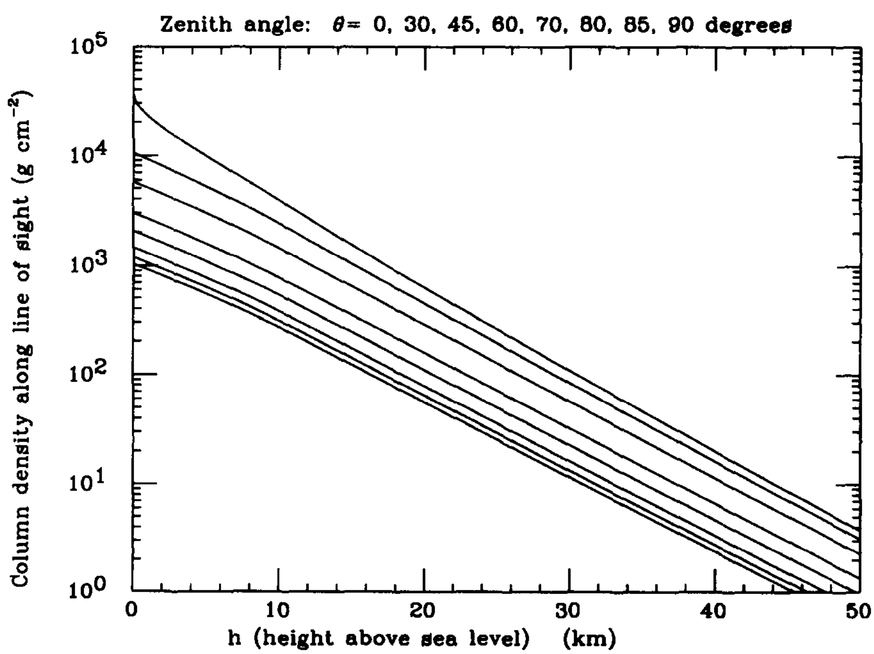

Fig. 12. Relation between the height above sea level and the slant depth, for different values of the zenith angle (at sea level): $\theta = 0, 30, 45, 60, 70, 80, 85, 90$ degrees.

therefore useful to know that there is an exact analytic solution under some reasonable approximations. If the energy loss is approximated as a constant ionization loss, $\mathrm{d}E/\mathrm{d}t = a$, and if the density distribution of the atmosphere is described by an exponential or polytropic form (eq. (109), see appendix A), and the zenith angle is not too large ($\theta \leqslant 60°$), then eq. (56) can be integrated analytically.

If both t_i and t_f are in the stratosphere, where the density has an exponential dependence on height, then

$$p_{\text{surv}}(E_0, t_i, t_f, \theta) = \exp\left[-\frac{m_\mu}{c\tau_\mu} \frac{1}{\cos\theta} \frac{h_0}{E_0 + at_i} \ln\left(\frac{t_f}{t_i} \frac{E_0}{E_0 - a(t_f - t_i)} \right) \right], \tag{115}$$

where h_0 (see q. (111)) is the scale height of the atmosphere, and m_μ and τ_μ are the muon mass and lifetime.

In the troposphere where the atmospheric density has a polytropic form, we have

$$p_{\text{surv}}(E_0, t_i, t_f, \theta) = \exp\left\{ -\frac{1}{\cos\theta} \frac{m_\mu}{c\tau_\mu} \frac{1}{Ca} \left(\frac{a}{E_0 + at_i} \right)^\beta \left[L\left(\frac{at_f}{E_0 + at_i} \right) - L\left(\frac{at_i}{E_0 + at_i} \right) \right] \right\}. \tag{116}$$

The constants β and C are related to the atmospheric density and are defined in eqs. (112), (114). The function $L(z)$ (defined for argument $0 \leqslant z < 1$), can be calculated numerically from its definition:

$$L(z) = \int_0^z \frac{\mathrm{d}z}{z^\beta (1-z)}. \tag{117}$$

When $z \to 1$ the function diverges as $L(z) \to c - \ln(1-z)$, (where c is a constant). When $z \to 0$, $L(z)$ vanishes. A useful expansion is

$$L(z) = \sum_{n=0}^{\infty} \frac{z^{-\beta+n+1}}{1+n-\beta}. \tag{118}$$

Equations (115), (116) can be easily expressed as a function of E_f instead of E_0 with the replacement $E_0 - E_f + a(t_f - t_i)$.

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