Bitcoin Pricing with Stochastic Time Shift Model

Group Members: Liang Wang, Jingyan Wang, Kevin Tao

August 16, 2020

1 Introduction

On class, we derived many very useful results in martingale theorem. Almost all of those theorems, directly or indirectly, provides useful guidance for modern asset pricing methods. One asset that particularly out of the interests of the modern society is the Bitcoin currency. Here we explore some martingale theorem results and apply them to price the Bitcoin derivative prices.

1.1 Martingale and Asset Pricing

First consider an asset price S_t evolving over time t. Its derivative price $C_T = C(S_T)$ at time T is a random variable that can take over many different trajectories. The asset price out of our interests is $E[C_T]$ which could be found by taking the average of all the potential trajectories. The measure under which that we take the expectation is thus crucial. Followed by the assumption of "no-arbitrage", the first fundamental asset pricing theorem **[6]** provides us the existence of such a measure Q with density $f_T(S)$ under which S_T deflated by the risk free interests r is a martingale.

$$C_t = exp(-r(T-t))E^Q[C(S_T)|F_t] = exp(-rT)\int_0^\infty C(S_T)f_T(S_T)dS_T$$

1.2 Brownian Motion Martingale

Beside the martingale measure Q under which assets price S_t is a martingale, we also need some assumptions to describe the price moving dynamic. Commonly, a "multiplier $\frac{dS_t}{S_t}$" that matching the idea of asset return at time t is assumed. If we require the "multiplier" satisfies the Lindeberg conditions, then the central limit theorem will gives us the distribution of of the asset price.

Extending the definition for discrete processes, a continuous process $\{\frac{dS_t}{S_t} = X_t\}_{t\in T}$, with T being a continuous time interval, is said to be $\{F_t\}_{t\in T}$ adapted if X_t is F_t-measurable for all t. Similarly, a $\{F_t\}_{t\in T}$ adapted sequence $\{X_t\}_{t\in T}$ is a continuous time martingale if $EX_t < \infty$ for all t and:

$$E[X_t|F_s] = X_s \text{ a.s, for all } 0 \le s \le t \text{ and } s,t \in T$$

A $\{F_t\}_{t\in T}$ adapted Brownian Motion with drift parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma \in (0,\infty)$

is a continuous time process $\{W_t\}_{t \in T}$ that satisfies:

$$\begin{aligned} W_0 &= 0 \\ \text{Stationary increments: } W_t - W_s &= W_{t-s} \text{ in distribution for } t > s, t, s \in T \\ \text{Independent increments: } W_t - W_s & \text{ is independent to } F_s \text{ for } t > s, t, s \in T \\ \text{Normality: } X_t &\sim N(\mu t, \sigma^2 t), t \in T \end{aligned} \tag{1}$$

A Brownian Motion with drift of constant r under measure Q is the famous Black Scholes Model that founded the modern asset pricing methods.

1.3 Variance Gamma (VG) Process

However, real world financial market is highly volatile that might break our initial Lindeberg Condition assumption thus prevent us from using pure Brownian Motion to describe the dynamic. To incorporate the instantaneous price jumps of different level, poisson type of jump has been proposed to address the concerns. The variance gamma process **[3]** is one of those models that admits jump activities with the following form:

$$V_t = \mu G(t; 1, v) + \theta G(t, 1, v) + \sigma W(G(t; 1, v)), \text{ where } W(t) \text{ is a standard Brownian Motion} \tag{2}$$

It thus can be deemed as a generalization of Brownian motion (with drift μ, and scale σ), with the time parameter of the Brownian motion assumed be $G(t; 1, v)$. The derivative pricing under VG process often exploits its characteristic function with:

$$\phi_{Vt}(u) = E[exp(iuV_t)] = exp[i\mu tu][\frac{2\alpha - 2i\theta u + \sigma^2 u}{2\alpha}]^{-\alpha t} \tag{3}$$

Similar to Brownian Motion, the VG process satisfies the Levy properties and thus entitle independent and stationary increments.

2 Stochastic Time Shifted variance Gamma (STVG) Model

Even with jump being incorporated into assumption, the dynamic of certain financial assets, the Bitcoin currency for example, still lacks of decent description of contagion and leverage effects by having constant volatility. As for that, the STVG model**[1]** was proposed to incorporate long term stochastic volatility.

2.1 Dynamic Setup

We first define the stock pricing moving with a "mutiplier" over time $S_t = S_0 e^{V_{\mathscr{T}t}} / E[e^{V_{\mathscr{T}t}}]$ under the pricing measure. Then the moving dynamic is:

$$X_t = V_{\mathscr{T}t} = \mu_{\mathscr{T}t} + \theta G_{\mathscr{T}t} + \sigma W_{G_{\mathscr{T}t}} \tag{4}$$

where G_t is a gamma process with rate and shape both equal to $\alpha > 0$, μ is the drift term, θ is the Brownian motion location parameters and σ is the volatility term of Brownian Motion. Further, the idea of non-Gaussian Ornstein-Uhlenbeck processes **[4]** that are purely discontinuous was borrowed to describe stochastic time change:

$$\mathscr{T}_t = \int_0^t A_s ds$$

$$dA_t = \lambda(m - A_t)dt + dH_t$$

$$\Rightarrow A_t = A_0 exp(-\lambda t) + m[1 - exp(-\lambda t)] + \int_0^t exp[-\lambda(t-s)]dH_s$$

where $\lambda > 0$ is the reverting speed, m is the long term mean, $A_0 > 0$ is the initial state. (a,b) are shape and scale parameter of a gamma process. The characteristic function of such process is:

$$\phi_{\mathscr{T}|A_0}(u) = [\frac{\lambda b}{\lambda b - iu}]^{at} exp[i(mt + \frac{(A_0 - m)(1 - exp(-\lambda t))}{1 - exp(-\lambda t)})u] + \frac{a}{\lambda}[Li_2(\frac{iu}{iu - \lambda b}) - Li_2(\frac{iexp(-\lambda t)u}{iu - \lambda b})] \tag{5}$$

where Li_2 is Spence's function.

$$\phi_{X_t}(u) = \phi_{\mathscr{T}|A_0}[\mu u + i\alpha log(\frac{2\alpha - 2i\theta u + \sigma^2 u}{2\alpha})] \tag{6}$$

2.2 Verification of Martingale Process

Since we note that $E[e^{X_t}/E[e^{X_t}]] = 1$, assuming it defines the local martingale measure, we show that the price process $S_t = S_0 e^{rt} e^{X_t}/E[e^{X_t}]$ is a Q-martingale:

$$E^Q[S_t] = E[\frac{S_0 exp(rt) exp(X_t)}{\phi_{x_t}(-i)}|F_s]$$

$$= S_0 \frac{exp(rs + X_s)}{exp(X_s)} E[exp[r(t-s)] \frac{exp[X_t - X_s]}{\phi_{x_t - x_s}(-i)}|F_s]$$

$$= S_0 \frac{exp(rs + X_s)}{\phi_{X s}(-i)} = S_s \ , \forall s \le t$$

I.e, S_t is a Q measure martingale w.r.t $\{F_t\}_{t\in[0,T]}$ with $F_t = \sigma[(V_s)_{s\in[0,t]}, (A_s)_{s\in[0,T]}]$.

2.3 Comments on STVG process

Similar to variance gamma model, the STVG model can be considered as incorporating the randomness happening during different trading times. However, instead of treating it as a simple gamma process, it extended the process to be stochastic,(CIR) in particular, and thus is better suited in describing the complex trading phenomenon during different time period. Mathematically, the stochastic change in time could enlarge our analysis on stochastic volatility by taking stochastic jump into consideration.

3 Application to Bitcoin Option

Cryptocurrency, due to its characteristic of anonymity and free of country authority, has been a very popular financial asset for investors recently. However, Bitcoin, for example, sufferers from the high volatility problem to be used as the real currency. On July 24th 2017, US Commodity Futures Trading Commission (CFTC) granted permission of trade in Bitcoin derivative. European option in particular has been deemed as a very valuable financial asset that can potentialy hedge the high volatility possessed by Bitcoin currency and entitles a potential electronic currency system.

3.1 European Call Price Derivation

Here we derive the closed form expression for European Call Option price under STVG model.

$$\begin{aligned} \pi_0 &= E^Q[exp(-rT)(S_T-K)^+] \\ &= S_0E^Q[exp(X_T)1_{\{S_T>K\}}] - KE[exp(-rT)1_{\{S_T>K\}} \\ &= S_0E[\frac{exp(X_T)1_{\{S_T>K\}}}{\phi_{X_T}(-i)}] - KE[exp(-rT)1_{\{S_T>K\}}] \end{aligned} \tag{7}$$

$$where\ E^Q[exp(iuX_T)] = E[\frac{exp[(iu+1)X_T]}{\phi_{X_T}(-i)}] = \frac{\phi_{X_T}(u-i)}{\phi_{X_T}(-i)}$$

By Gil-Pelaez's theroem, we have for a univariate random variable X, if x is a continuity point,

$$F_X(x) = \frac{1}{2} - 1/\pi\int_0^\infty \frac{Im[exp(-itx)\phi_X(t)]}{t}dt$$

$$\Rightarrow \pi_0 = S_0C_1 - Kexp(-rT)C_2 \tag{8}$$

where

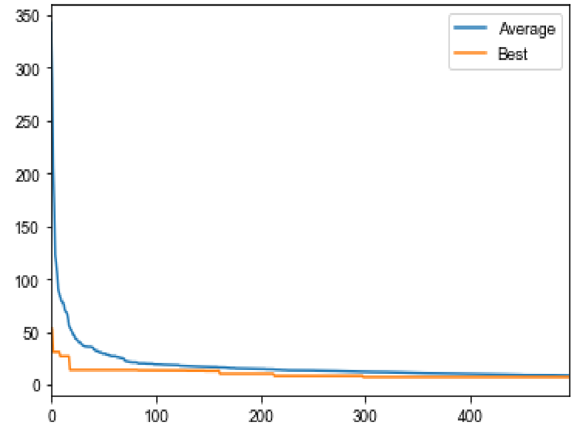
$$C_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{\phi(u-i)exp[-iulog\frac{K\phi_{X_T}(-i)}{S_0}]}{\phi_{X_T}(-i)iu}]du$$

$$C_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{\phi(u-i)exp[-iulog\frac{K\phi_{X_T}(-i)}{S_0}]}{iu}]du$$

3.2 Calibration

The calibration mainly used Eq 6 and Eq 7. We used the Bitcoin option data on 29 June 2018[**2**]. For simplicity, we used r=0. This is also reasonable since for such a volatile asset, the risk free rate is not a good assumption here to provide arbitrage opportunity. Because our parameterization of the model has 9 parameters, it is expensive to evaluate its gradients and inevitably suffers from the local minimum problem. We thus adopted evolutionary algorithm, particularly the genetic algorithm, which has no gradient evaluation and is robust to the change of starting point. For the optimization algorithm, we used 30 population size and 10% mutation rate to address the local minimum problem.

Figure 1: Genetic Algorithm Convergence Result



As we can see the iteration stabilized after roughly 300 iteration with the 30 samples' minimum matching the best minimum. The computation is done through Boston University Shared Computing Clustering [BUSCC] after parallelization. The optimal parameters turn out to be $\alpha = 10, \theta = -0.2, \mu = 10, \sigma = 0.47, \lambda = 4.58, b = 9.4, m = 0.55, a = 7.7, A_0 = 2.3$

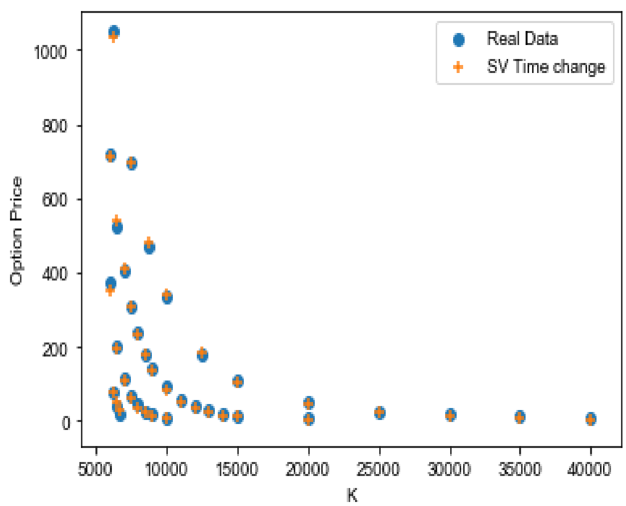


Figure 2: Stochastic Time Shift VG Model

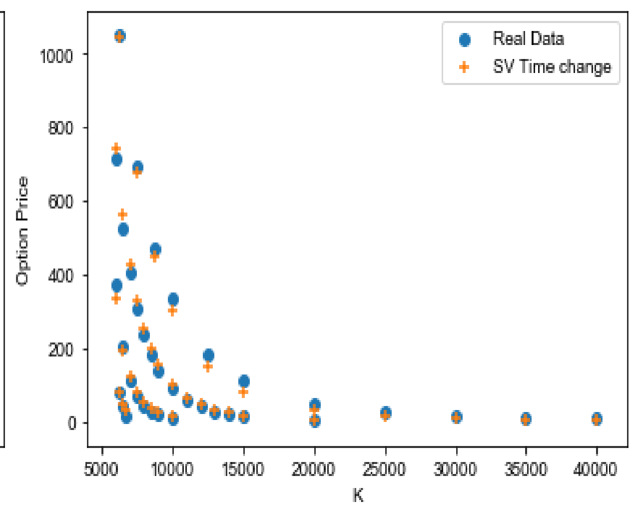


Figure 3: Variance Gamma Model

As we can see that comparing to the Variance Gamma Model (with RMSE=17.28), the stochastic time shift VG model with RMSE=7.12 has better fit to the bitcoin option data. In particular, The VG model seems to miss the long term structure of the bitcoin volatility.

3.3 Connection to American Option

After deriving the price of European call option price for Bitcoin under our model framework, we are particularly interested in the American option price application. One probability out of many financial practitioners is the tail probability of the financial asset. However, deriving such an explicit formula for American option is challenging without the computation methods. One way to derivative its upper bound is by using the martingale inequality theorem we proved on class. Notice that call option price $C = (S - K)^+$ is a non-negative convex function w.r.t S, and thus combined with 2.2, we know the tail probability of American call price is bounded above by Kolmogorov Doob inequality;

$$\lambda P(\max_{1 \leq k \leq T} C_k > \lambda) \leq E[C_T]$$

where $E[C_T]$ is the explicit European formula we derived in 3.1. For example, one particular interesting example is to find λ such that $\frac{E[C_T]}{\lambda} = 0.05$, which corresponds to the $VaR_{5\%}$

4 Conclusion

In conclusion, we studied and employed martingale theorem in asset pricing through exploring the the Stochastic Time Shifted model's explainability in bitcoin option pricing. We found that the Stochastic Time Shifted model improves the performance of Variance Gamma Model by capturing the long term volaitlity embeded in the Bitcoin option. The pricing of Bitcoin option under the STVG dynamic thus would be a better alternative model than variance gamma model for investor to conduct hedging on their highly volatile assets such as Bitcoin currency. Moreover, the martingale inequality could be used to provide an upper bound for the value at risk analysis.

References

[1] A Stochastic-Volatility Model for Pricing Power Variants of Exchange Options, Weixuan Xia, The Journal of Derivatives Summer 2019

[2] Advanced model calibration on bitcoin options, Dilip B. Madan · Sofie Reyners · Wim Schoutens, Digital Finance 2019

[3] Levy Processes in Finance: Pricing Financial Derivatives. Wim Schouten, John Wiley Sons, Ltd, 2003

[4] Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. Journal of the Royal Statistical Society, 63(2), Barndorff-Nielsen, O.E. Shephard, N. 2001

[5] Computational Methods in Finance, Book by Ali Hirsa, 2012

[6] Martingales and arbitrage in multiperiod securities markets, J. MICHAEL HARRISON AND DAVID M. KREPS, Journal of Economic Theory Volume 20, Issue 3, June 1979, Pages 381-408