Baden-Wuerttemberg
Cooperative State University Stuttgart
Course: STG-TINF18A
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Analysis II for computer scientists

Problem sheet 5

Problem 21 (**T**hings **t**o **t**hink)
Correct or incorrect? If you think the assertion is correct then justify your answer. Provide a counterexample otherwise.

(i) If x is a stationary point of the Lagrangian corresponding to an equality constrained optimization problem **(ECOP)**, then it yields a minimum or maximum for problem (ECOP).

(ii) Only local extreme points of the Lagrangian can be solutions to the corresponding ECOP. **Hint:** Have a look at the example of maximizing $f(x) = x^2$ subject to $x^2 = 1$.

(iii) Is the vector-valued function $f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (|x - 1|,\ yx^2)$ totally differentiable?

(iv) What is the Jacobian of $f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (x,\ x^2 + 2y)$?

(v) Determine two zeros of the Jacobian of

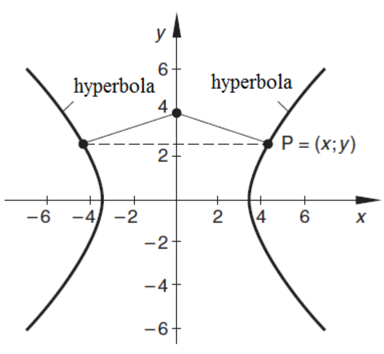
$$f : [0, 2\pi] \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} 5\cos(t) - \cos(5t) \\ 5\sin(t) - \sin(5t) \end{pmatrix}.$$

Problem 22 (Extreme Points)
Solve the following constrained optimization problems.

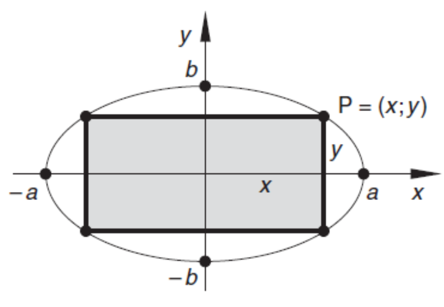
(a) Which point $P = (x, y)$ of the hyperbola $x^2 - y^2 = 12$ has the minimum euclidian distance to $Q = (0, 4)$ (compare Fig.(a)).

(b) For an ellipse given in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $(a, b > 0)$, what are the physical dimensions of a rectangle of maximum surface inside this ellipse (compare Fig.(b))?

(c) Maximize $f(x, y) = x^2 y$ subject to $g(x, y) = x^2 + y^2 = 3$.



(a) illustration of hyperbola problem



(b) illustration of ellipse problem

Problem 23 (Application examples - Constrained optimization)

(a) Assume you are designing a portfolio where you can invest your money in three different stocks, say

$$\alpha_j \in [0,1] : \text{investment part in stock j}$$
$$1 = \alpha_1 + \alpha_2 + \alpha_3.$$

The risk of your portfolio depends on the splitting and it shall be given by the function

$$r(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{2}\alpha_1^2 + \alpha_2^2 + \frac{1}{2}\alpha_3^2.$$

Which splitting serves to create a portfolio of minimum risk? Determine your solution using the Lagrange method!

(b) In theoretical computer science the following fact is known: If characters from an alphabet $\{a_j \mid j = 1, \dots n\}$ are sent via a communication channel with a relative frequency of $p_i > 0$ $(i = 1, \dots, n)$ for each character, then the mean information gain, when the next character is published is given by **Shannon's entropy**

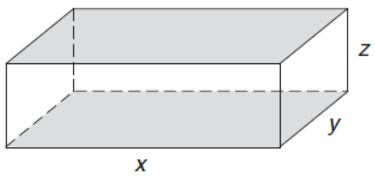
$$I(p_1, \dots, p_n) = -\sum_{i=1}^{n} p_i \ln(p_i).$$

If we can choose the relative frequencies arbitrarily, but always such that $\sum_{j=1}^{n} p_j = 1$, which frequencies maximize the mean information gain?

(c) A random experiment has four possible outcomes, which appear with probabilities $p_1, \dots,$ p_4. Since one of the outcomes must appear, we have $\sum_{j=1}^{4} p_j = 1$. Which probabilities maximize the function

$$Z(p_1, p_2, p_3, p_4) = p_1 \cdot p_2 \cdot p_3 \cdot p_4?$$

(d) A cuboid pool of volume $V = 108\ m^3$ shall be constructed such that the surfaces (ground and side walls) are as small as possible. How shall we choose the physical dimensions (x, y, z) of the pool?



Problem 24 (Sufficient conditions)
Using the modified Lagrangian from Proposition III.3.5, find all local extreme points of

$$f(x_1, x_2, x_3) = x_1 \cdot x_2 + x_2 \cdot x_3 + x_1 \cdot x_3$$

subject to $g(x, y) = x_1 + x_2 + x_3 = 3$ and check for a maximum or minimum.

Problem 25 (Differential operators)
For the following scalar fields f_i and vector fields F_i calculate $grad(f_i)$, $\Delta(f_i)$, $div(F_i)$ and $curl(F_i)$.

$$f_1(x,y,z) = x^2y + y^2 + z^3 \quad f_2(x,y,z) = \sin(x) + y \cdot \cos(z) \quad f_3(x,y,z) = e^{xyz}$$
$$F_1(x,y,z) = (x^2z^2,\ \ y^3,\ \ \tfrac{1}{z}) \quad F_2(x,y,z) = (x,\ \ y^2,\ \ z^3) \quad F_3(x,y,z) = \tfrac{(x,\ y,\ z)}{x^2+y^2+z^2}$$

The problems are due on 15.10.2019.