Baden-Wuerttemberg
Cooperative State University Stuttgart
Course: STG-TINF18A
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Analysis II for computer scientists

Problem sheet 4

Problem 16 (**T**hings **t**o **t**hink)
Correct or incorrect? If you think the assertion is correct then justify your answer. Provide a counterexample otherwise.

(i) What is a positive definite matrix? State 3 examples in different dimensions.

(ii) A negative definite matrix has a negative determinant.

(iii) When does the Hessian of a multi-dimensional function exist? What are its entries?

(iv) The Hessian is always indefinite and symmetric.

(v) Assume that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is in $\mathcal{C}^2(\mathbb{R}^2)$ and has a stationary point at x_0. If all entries of the Hessian are positive, then f has a local minimum in x_0.

Problem 17 (Extreme Points)
Calculate the stationary points, local extreme points and saddle points of the following functions.

(a) $f(x,y) = 3xy - x^3 - y^3$,

(b) $f(x,y) = xy + \dfrac{1}{x} + \dfrac{1}{y}$, where $x \neq 0$ and $y \neq 0$,

(c) $f(x,y) = 3x^2 - 2x \cdot \sqrt{y} - 8x + y + 8$.

Problem 18 (Application examples - Extreme points)

(a) Assume you are designing a portfolio where you can invest your money in three different stocks, say

$$\begin{aligned} \alpha_1 \in [0,1]: &\quad \text{investment part in stock 1} \\ \alpha_2 \in [0,1]: &\quad \text{investment part in stock 2} \\ \alpha_3 \in [0,1]: &\quad \text{investment part in stock 3} \\ 1 = \alpha_1 + \alpha_2 + \alpha_3. & \end{aligned}$$

The risk of your portfolio depends of course on the splitting and it shall be given by the function

$$r(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{2}\alpha_1^2 + \alpha_2^2 + \frac{1}{2}\alpha_3^2.$$

Which splitting serves to create a portfolio of minimum risk?

(b) Assume you are a manufacturer of two products, for which you have storage and transport costs. The arising costs, when a volume order of (x,y) is addressed is given by the cost function

$$C(x,y) = \frac{a}{x} + \frac{b}{y} + cx + dy + e, \quad x,y > 0$$

where $a,b,c,d,e > 0$ are process-dependent constants. For which volume order do we have minimum costs?

Problem 19 (Positive Definiteness)
Assume that A and B are symmetric positive definite matrices.

(a) Prove that, $A \cdot B$ has only positive eigenvalues.
Hint: Consider the eigenvalue equation $(AB)x = \lambda x$ (where $x \neq \mathbb{0}$ is an eigenvector for λ) and multiply by $x^\top B$ from the left.

(b) Prove that $A \cdot B$ is not necessarily positive definite, even when both are symmetric and positive definite and consequently the eigenvalues of AB are all strictly positive (by (a)!). Use the (counter-)example matrices

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

(c) Is $A \cdot B$ always symmetric? Does this help you to understand the paradox sounding statement in (b)?

Problem 20 (Linear and quadratic approximation)

(a) Determine the Jacobian and the Hessian matrices for the function

$$f(x, y) = x \cos(x + y) + (y - 1)^2 e^{-x^2}.$$

Then provide a linear and a quadratic approximation for the function around the point $x_0 = (0, 0)$ according to Theorem 1.3. Compare the approximations in $x_1 = (0.1,\ 0.1)$ with the exact value.

(b) The oscillation period of a pendulum of length l (in m) is given by

$$T(l, g) = 2\pi \sqrt{\frac{l}{g}},$$

where g is the gravity factor. Determine a linear approximation around the point $(l, g) = (1, 9.81)$, that can be used to calculate the oscillation period at places where the gravitational constant differs a little bit from 9.81 or when the length is not exactly 1 m.

The problems are due on 08.10.2019.