Mean-Variance-Optimization in the Era of Financial Digitalization

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July 23, 2021

Consider a portfolio X, consisting of the following five crypto assets: $\{$*Bitcoin*, *Ethereum*, *Cardano*, *Chainlink*, *Binance Coin*$\}$. We want to find the optimal portfolio X^* using different optimization algorithms.

1 Markowitz Framework

Let the expected return and the individual risk be represented by the mean μ_i and the standard deviation σ_i, for the asset S_i. Our goal is to find the optimal collection of weights w_i for each asset in the portfolio. We can express these inputs in matrix form:

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_5 \end{bmatrix}; \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1,5} \\ \sigma_{2,1} & \cdots & \sigma_{2,5} \\ \vdots & \ddots & \vdots \\ \sigma_{5,1} & \cdots & \sigma_5^2 \end{bmatrix}; \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_5 \end{bmatrix}$$

Where $\sigma_i^2 = \mathrm{Var}[S_i]$ represents the variance[1] of asset i and $\sigma_{i,j} = \mathrm{Cov}[S_i, S_j]$ for $i \neq j$ is the covariance[2] between asset i and j. $\boldsymbol{\Sigma}$ is also often called variance-covariance matrix. Therefore, we can write the expected return and the variance of the portfolio X as follows[3]:

$$E[X] = \boldsymbol{\mu}^\top \boldsymbol{w}$$
$$\mathrm{Var}[X] = \boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$$

The vector of weights $\boldsymbol{w}$ is constrained, so the addition of all weights for each asset will add up to one: $\boldsymbol{w}^\top \boldsymbol{\iota} = 1$, where $\boldsymbol{\iota} = (1, \dots, 1)$.

[1] $\mathrm{Var}[S_i] = E[(S_i - \mu_i)^2] = E[S_i^2] - E[S_i]^2$

[2] $\mathrm{Cov}[S_i, S_j] = E[(S_i - \mu_i)(S_j - \mu_j)] = E[S_i S_j] - E[S_i]E[S_j]$

[3] $E[X] = \mu_1 w_1 + \cdots + \mu_5 w_5$
$\mathrm{Var}[X] = \sum_i^5 \sum_j^5 \sigma_{i,j} w_i w_j$

We can formulate the Markowitz optimization problem in four different ways, with each variation focusing on a different objective:

- Minimum Variance: Solves for the portfolio with the minimum variance out of all possible portfolios.

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w} \\ \text{s.t.} \quad & \boldsymbol{w}^\top \boldsymbol{\iota} = 1 \end{aligned} \tag{1}$$

- Maximum Return: Solves for the portfolio with the maximum return out of all possible portfolios.

$$\begin{aligned} \max_{\boldsymbol{w}} \quad & \boldsymbol{\mu}^\top \boldsymbol{w} \\ \text{s.t.} \quad & \boldsymbol{w}^\top \boldsymbol{\iota} = 1 \end{aligned} \tag{2}$$

- Risk-Aversion: Explicit model for the trade-off between risk and return. The objective is to maximize the return, penalized by the variance of the portfolio. The penalization is determined by the risk-aversion parameter α, where a small value for α leads to a riskier portfolio and vice versa.

$$\begin{aligned} \max_{\boldsymbol{w}} \quad & \boldsymbol{\mu}^\top \boldsymbol{w} - 0.5\alpha(\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}) \\ \text{s.t.} \quad & \boldsymbol{w}^\top \boldsymbol{\iota} = 1 \end{aligned} \tag{3}$$

- Maximum Sharpe-Ratio: Solves for the portfolio with the maximum sharpe ratio out of all possible portfolios. The sharpe ratio measures the excess return for a given measure of risk and can be defined by:

$$SharpeRatio = \frac{\boldsymbol{\mu}^\top \boldsymbol{w} - r_f}{\sqrt{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}}}$$

  Where r_f represents the risk free rate. Thus, the problem can be formulated like the following:

$$\begin{aligned} \max_{\boldsymbol{w}} \quad & \frac{\boldsymbol{\mu}^\top \boldsymbol{w} - r_f}{\sqrt{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}}} \\ \text{s.t.} \quad & \boldsymbol{w}^\top \boldsymbol{\iota} = 1 \end{aligned} \tag{4}$$

Since we assume that $\boldsymbol{\Sigma}$ is a positive semidefinite matrix, we can solve the problem using the Lagrange multiplier method, solving the linear equations from the First-Order-Conditions (FOC). [1]

2 Portfolio Optimization

We will solve for the optimal portfolio X^* using the minimum variance, risk aversion and maximum sharpe ratio algorithms mentioned in chapter 1. The results will then be compared in chapter 3.

2.1 Minimum Variance Optimization

According to equation (1), we can formulate the Lagrangian as follows:

$$\max_{\boldsymbol{w}} \; L(w, \lambda) = \boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w} + \lambda(\boldsymbol{w}^\top \boldsymbol{\iota} - 1)$$

We proceed by taking the partial derivatives of the Lagrangian:

$$\frac{\partial L}{\partial w} = 2\boldsymbol{\Sigma}\boldsymbol{w} + \lambda\boldsymbol{\iota} = 0$$
$$\frac{\partial L}{\partial \lambda} = \boldsymbol{w}^\top \boldsymbol{\iota} - 1 = 0$$

Now we can solve the equation system above with respect to w and receive the following vector of optimal weights for the minimum variance portfolio:

$$\boldsymbol{w}^*_{MV} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}{\boldsymbol{\iota}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}} \tag{5}$$

2.2 Risk-Aversion Optimization

According to equation (3), we can formulate the Lagrangian as follows:

$$\max_{\boldsymbol{w}} \; L(w, \lambda) = \boldsymbol{\mu}^\top \boldsymbol{w} - 0.5\alpha(\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}) + \lambda(\boldsymbol{w}^\top \boldsymbol{\iota} - 1)$$

We proceed by taking the partial derivatives of the Lagrangian:

$$\frac{\partial L}{\partial w} = \boldsymbol{\mu}^\top - \alpha\boldsymbol{\Sigma}\boldsymbol{w} + \lambda\boldsymbol{\iota} = 0$$
$$\frac{\partial L}{\partial \lambda} = \boldsymbol{w}^\top \boldsymbol{\iota} - 1 = 0$$

We can then again solve the equation system with respect to w and receive the vector of weights for the optimal portfolio:

$$\boldsymbol{w}^*_{RA}(\alpha) = \frac{1}{\alpha}\left[\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}\left(\frac{\alpha - \boldsymbol{\iota}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\boldsymbol{\iota}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}\right)\right] \tag{6}$$

2.3 Maximum Sharpe-Ratio Optimization

According to equation (4), we can formulate the Lagrangian as follows:

$$\max_{\boldsymbol{w}} \ L(w, \lambda) = \frac{\boldsymbol{\mu}^\top \boldsymbol{w} - r_f}{\sqrt{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}}} + \lambda(\boldsymbol{w}^\top \boldsymbol{\iota} - 1)$$

Because of the complex objective function of the above problem, we simplify under the assumption that $\boldsymbol{\mu}^\top \boldsymbol{w} - r_f > 0$ and reduce the problem into a convex risk minimization form. We also assume that the risk free asset has zero variance and is uncorrelated with the other assets. Solving the optimization problem gives the optimal weights:

$$\boldsymbol{w}_S^* = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\boldsymbol{\iota}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}} \tag{7}$$

With the given solution looking similar to that of the minimum variance optimization problem (5). [2]

2.4 Estimation of μ and σ

In order to run the optimization and find the optimal weights w_i, we need to estimate the expected return μ_i and the individual risk σ_i. We will use the following estimation functions:

$$\begin{aligned}
\hat{\mu}_i &= \frac{1}{n}\sum_{t=1}^{n} r_{it} \\
\hat{\sigma}_i^2 &= \frac{1}{n-1}\sum_{t=1}^{n}(r_{it} - \hat{\mu}_i)^2 \\
\hat{\sigma}_{i,j} &= \frac{1}{n-1}\sum_{t=1}^{n}(r_{it} - \hat{\mu}_i)(r_{jt} - \hat{\mu}_j)
\end{aligned}$$

With r_{it} being the simple returns[4] of asset S_i at time t. We also assume that the returns are i.i.d with $r_{it} \sim N(\mu_i, \sigma_i^2)$. The covariance can also be computed using the correlation coefficient $\rho_{i,j}$ for $i \neq j$. The estimation function for $\rho_{i,j}$ is given by:

$$\hat{\rho}_{i,j} = \frac{\hat{\sigma}_{i,j}}{\hat{\sigma}_i \hat{\sigma}_j} \quad \longrightarrow \quad \hat{\sigma}_{i,j} = \hat{\rho}_{i,j}\hat{\sigma}_i\hat{\sigma}_j$$

Therefore, the estimated variance-covariance matrix $\widehat{\boldsymbol{\Sigma}}$ consists of the estimated parameters $\hat{\sigma}_i^2$ and $\hat{\sigma}_{i,j}$. [2]

[4] $r_{it} = \frac{p_{it} - p_{i,t-1}}{p_{i,t-1}} = \frac{p_{it}}{p_{i,t-1}} - 1$ with p_{it} being the price of asset S_i at time t.

3 Results

We start by providing an overview of the descriptive statistics for each individual asset in Table 1. Next, the optimal weights for each optimization problem from chapter 2 are computed. Finally, the efficient frontier is plotted. The programming language `Python` is used for all associated computations and visualizations.

3.1 Descriptive Statistics

The data consists of the daily simple returns of the above mentioned cryptocurrencies. The daily returns correspond to the period of July 6, 2016 to July 6, 2021. As we can see from Table 1, mean and median are close to zero, but the minimum and maximum values show extreme cases of high volatility, implying very irrational movements for each asset.

	N	Mean	Median	Min	Max	Volatility	$Q_{0.25}$	$Q_{0.75}$
ETH	1826	0.4559	0.0972	-42.3472	33.6621	5.6645	-2.0723	2.7832
BTC	1826	0.2999	0.2157	-37.1695	25.2472	4.0989	-1.3168	1.9322
ADA	1374	0.5908	0.1007	-39.5672	136.6809	-8.3563	-2.9799	3.3058
LINK	1385	0.6499	0.0000	-45.9130	61.7069	-8.0457	-3.6122	4.3335
BNB	1442	0.8474	0.1150	-41.9046	96.4374	8.0023	-2.4466	3.3558

Table 1: Descriptive statistics of simple returns for the period of July 6, 2016 to July 6, 2021 for each individual asset.

3.2 Optimization

For estimation of the variance-covariance matrix we need the same number of observations for each asset, hence we drop all missing values from the data which results in an dataframe with $N = 1374$ for each asset. Estimating the expected returns and the variance-covariance matrix yields[5]:

$$\hat{\boldsymbol{\mu}} = \begin{bmatrix} 0.2892 \\ 0.2389 \\ 0.5908 \\ 0.5794 \\ 0.6082 \end{bmatrix}; \widehat{\boldsymbol{\Sigma}} = \begin{bmatrix} 27.5731 & 16.2100 & 24.9813 & 24.0029 & 20.4680 \\ 16.2100 & 17.6783 & 17.6541 & 15.3843 & 16.7486 \\ 24.9813 & 17.6541 & 69.8279 & 27.2128 & 23.8954 \\ 24.0029 & 15.3843 & 27.2128 & 61.7816 & 23.2200 \\ 20.4680 & 16.7486 & 23.8954 & 23.2200 & 46.2452 \end{bmatrix}$$

[5] $\{S_1, S_2, \ldots, S_5\} \longrightarrow \{\textit{Ethereum}, \textit{Bitcoin}, \textit{Cardano}, \textit{Chainlink}, \textit{Binance Coin}\}$

Running each optimization algorithm gives the following optimal weights:

$$\boldsymbol{w}_{MV}^* = \begin{bmatrix} 0.0901 \\ 0.8753 \\ 0.0000 \\ 0.0271 \\ 0.0073 \end{bmatrix}; \boldsymbol{w}_{RA}^* = \begin{bmatrix} 0.0195 \\ 0.7491 \\ 0.0292 \\ 0.0847 \\ 0.1172 \end{bmatrix}; \boldsymbol{w}_{S}^* = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.2140 \\ 0.2549 \\ 0.5310 \end{bmatrix}$$

With a risk-aversion parameter of $\alpha = 0.1$ implying an investor with a low risk aversion. We also use a risk free rate of $r_f = 0$ representing the all-time low levels of interest rates in the developed economies around the world. It can be seen that minimum-variance and risk-aversion optimization yield similar results with Bitcoin constituting roughly 87% and 75% of the portfolio respectively. Interestingly, the maximum sharpe-ratio portfolio suggests to neglect both Bitcoin and Ethereum altogether, with the Binance Coin representing the largest share of all assets included with 53%, followed by Chainlink (25%).

It should be noted that using the basic sample estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ usually leads to portfolios that may perform poorly and have counter-intuitive asset allocation weights. This has also been referred to as the "Markowitz-Optimization-Enigma". Therefore, more complex estimation methods have to be used in order to yield better portfolios when using the optimization algorithms mentioned in chapter 1. [3]

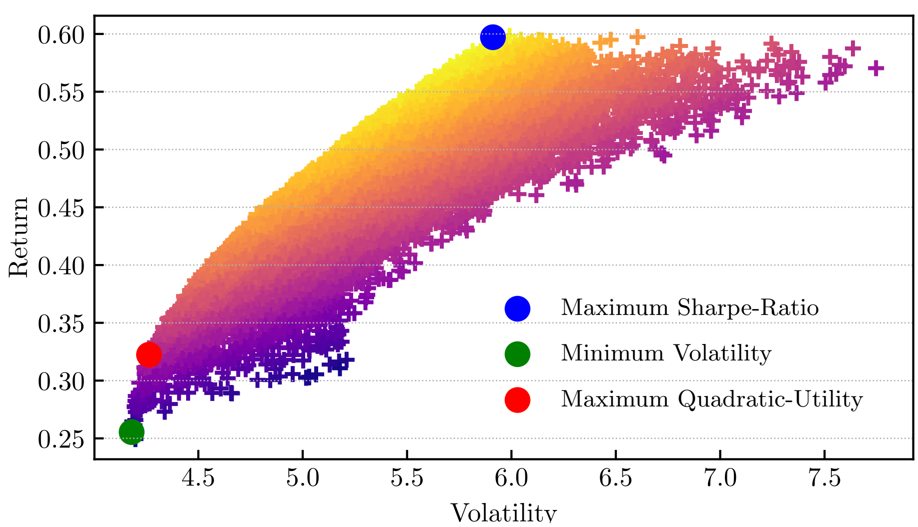

Figure 1: Simulation of 10.000 random portfolios, the individual optimal portfolios are shown on the efficient frontier ($r_f = 0$ and $\alpha = 0.1$).

References

[1] G. M. M. Takane, "Automated cryptocurrency portfolios: Portfolio optimization, an empirical study," Master's thesis, Humboldt-Universität zu Berlin, 2020.

[2] A. Geyer, "Analyse und entscheidung im finanzmanagement," 2009.

[3] Z. C. Tze Leung Lai, Haipeng Xing, "Mean-variance portfolio optimization when means and covariances are unknown," tech. rep., Stanford University, 2009.