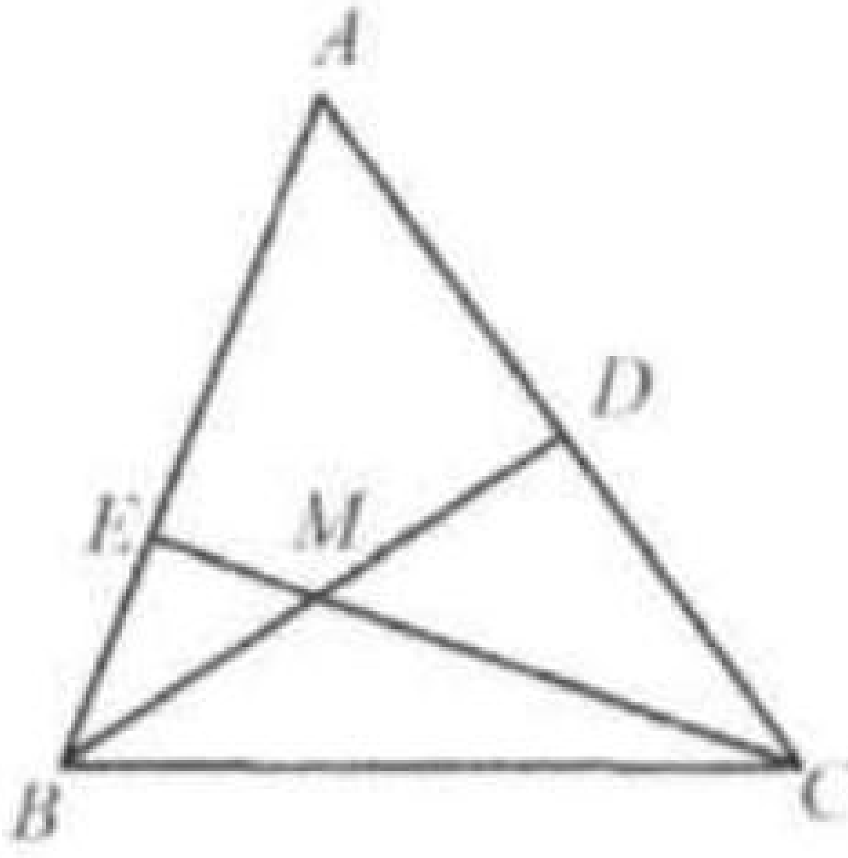
Example 12

In $\triangle ABC, D, E$ are points on ACB, and AB, respectively. CE and CD meet at $M. \angle DBC = \angle A. BM = MD$. Prove that $\frac{BC^2}{AC^2} = \frac{BE}{AE}$.

Solution: Method 1:



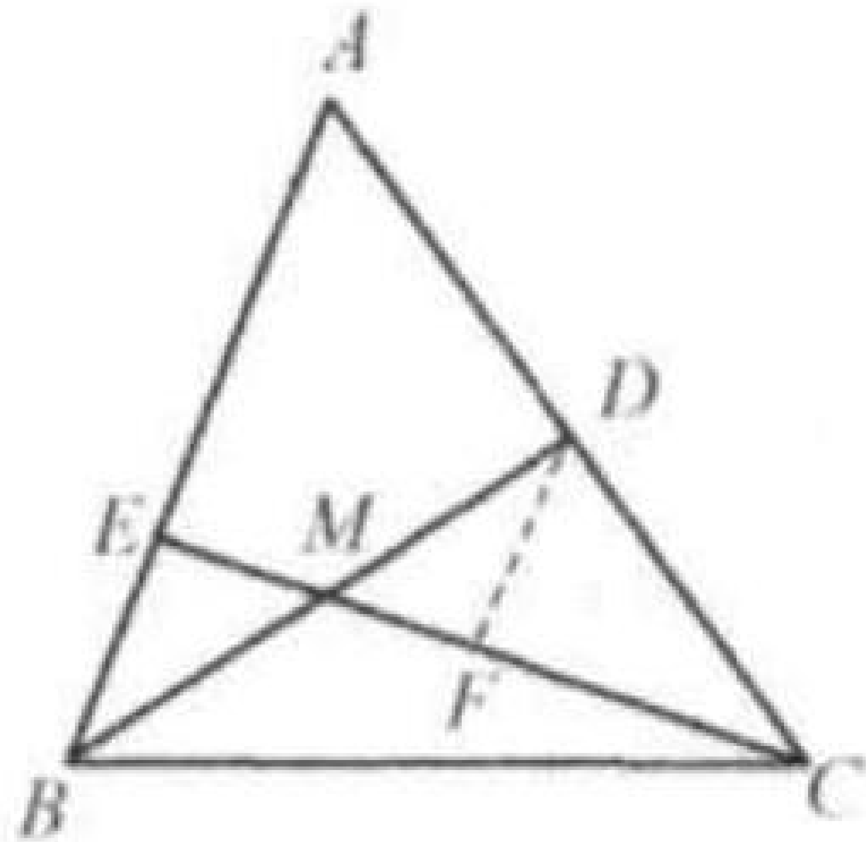
Since $\angle DBC = \angle A, \angle ACB = \angle BCD, \triangle ABC \sim \triangle BDC$.

$$\frac{BC}{CD} = \frac{AC}{BC} \quad \Rightarrow \quad \frac{BC^2}{AC^2} = \frac{CD}{AC}$$

Draw $DF//BE$ to meet CE at $F. \triangle ACE \sim \triangle DCF$.

$\frac{CD}{AC} = \frac{DF}{AE}$

We know that $\triangle BEM \sim \triangle DF$.



So we have $\frac{DF}{MD} = \frac{BE}{BM}$ or $\frac{DF}{MD} = \frac{BE}{MD}$.

So $DF = BE$.

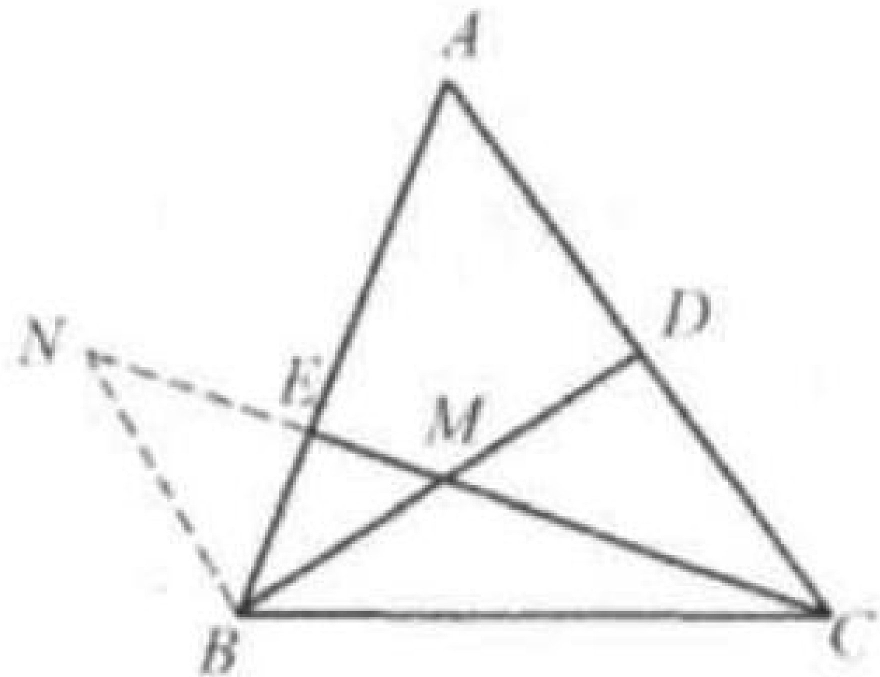
Thus $\frac{BC^2}{AC^2} = \frac{CD}{AC} = \frac{DF}{AE} = \frac{BE}{AE}$.

Method 2:

Since $\angle DBC = \angle A, \angle ACB = \angle BCD, \triangle ABC \sim \triangle BDC$.

$$\frac{BC}{CD} = \frac{AC}{BC} \quad \Rightarrow \quad \frac{BC^2}{AC^2} = \frac{CD}{AC}$$

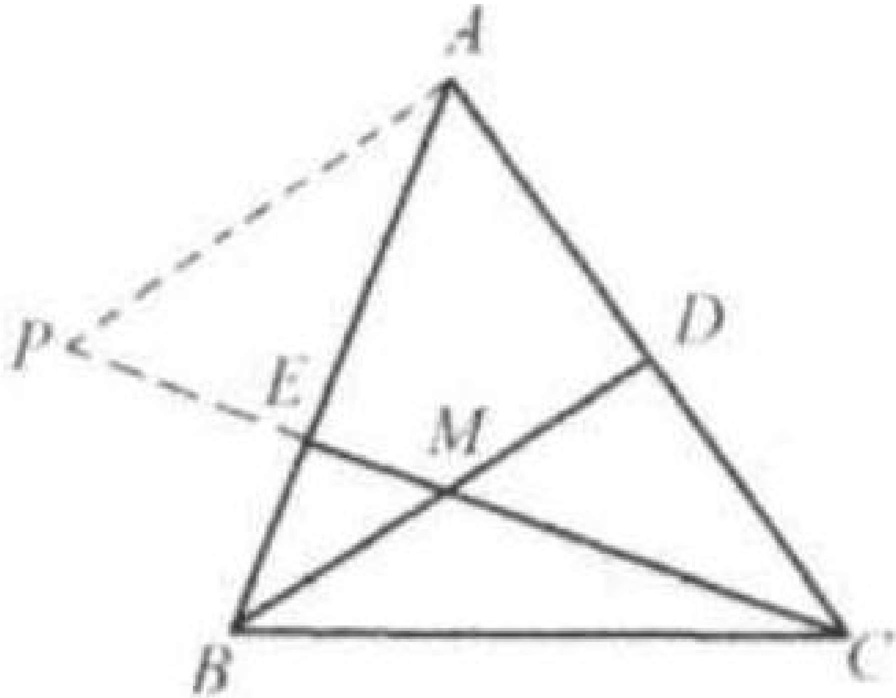
Draw $BN//AC$ to meet the extension of CE at N. Since



$\angle NMB = \angle CMD, BM = MD, \angle NBM = \angle CDM, \triangle NMB \cong \triangle CMD.$
Then $BN = CD.$
Since $\angle N = \angle ECA, \angle NEB = \angle CEA, \triangle NEB \sim \triangle CEA.$
So $\frac{BN}{AC} = \frac{BE}{AE} \quad \Rightarrow \quad \frac{CD}{AC} = \frac{BE}{AE}.$
Therefore, $\frac{BC^2}{AC^2} = \frac{BE}{AE}.$
Method 3:
Since $\angle DBC = \angle A, \angle ACB = \angle BCD, \triangle ABC \sim \triangle BDC.$

$$\frac{BC}{CD} = \frac{AC}{BC} \quad \Rightarrow \quad \frac{BC^2}{AC^2} = \frac{CD}{AC}$$

Draw $AP//BD$ to meet the extension of CE at P.
We know that $\triangle AEP \sim \triangle BEM(AA) \cdot \frac{BM}{AP} = \frac{BE}{AE}.$



We know that $\triangle ACP \sim \triangle DCM(AA) \cdot \frac{CD}{AC} = \frac{DM}{AP}$.
Since $BM = MD, \frac{CD}{AC} = \frac{BM}{AP} = \frac{BE}{AE}$.
Therefore, $\frac{BC^2}{AC^2} = \frac{BE}{AE}$.