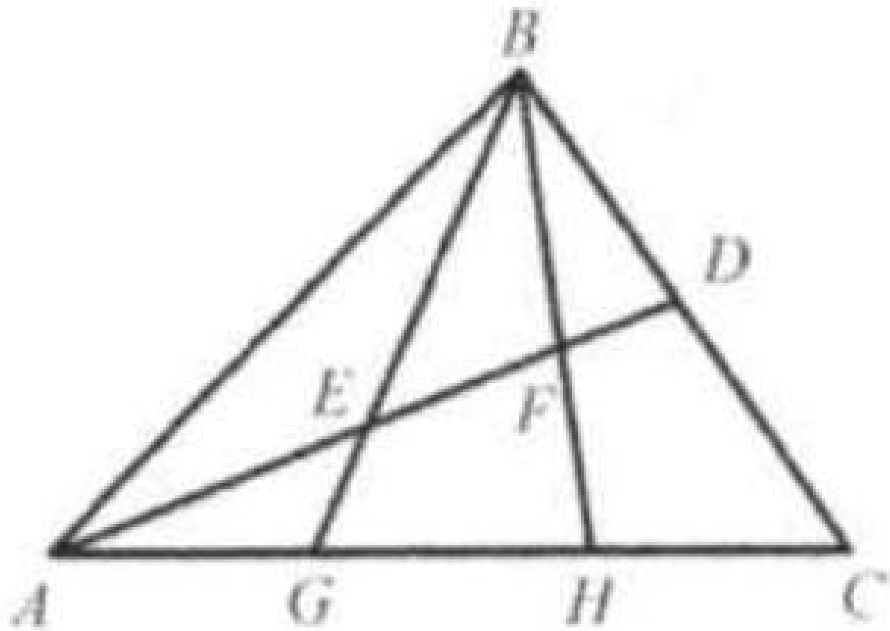
Problem 18

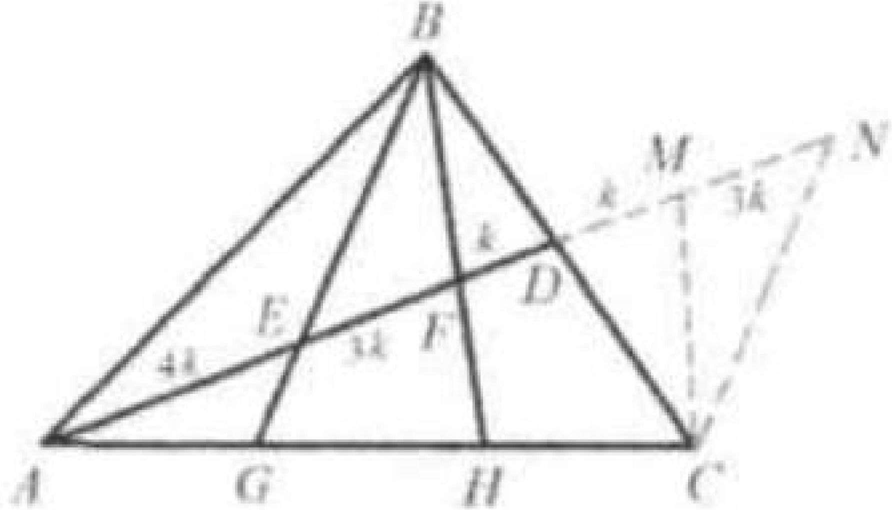
Problem

In triangle ABC, AD is the median on $BC.BG$ and BH divide AD into three parts such that $AE : EF : FD = 4 : 3 : 1.AG : GH : HC = x : y : z$. Find the value of $x + y + z$, where x and y are positive integers relatively prime.

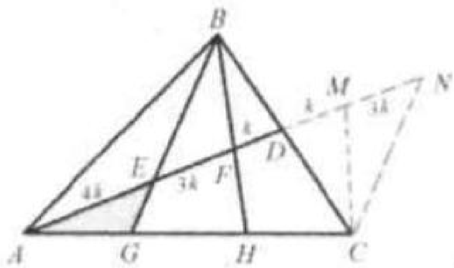


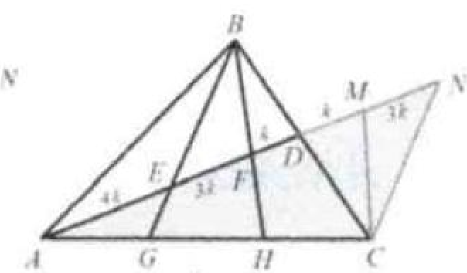
Solution

9. Extend AD to M and N and connect CM and CN such that $DM = DF$ and $DN = DE$. So $BFCM$ and $BECN$ are parallelograms since the diagonals bisect each other.
Thus $BE//NC, BF//MC$.

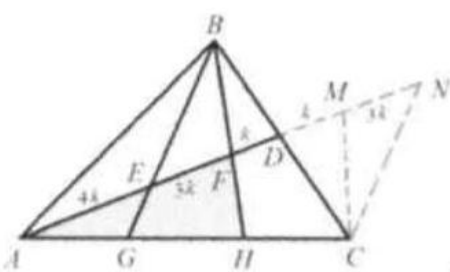


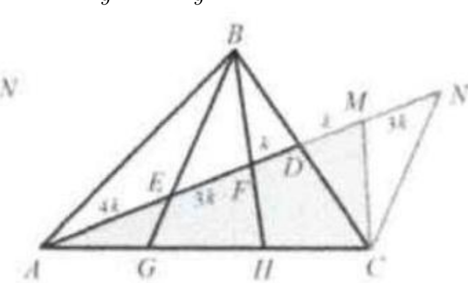
Let $AE = 4k, FE = 3k$, and $FD = k$. So $FMN = 3k$, and $DM = k$.
Since $BE//NC, \triangle AGE \sim \triangle ACN. \frac{AE}{AN} = \frac{AG}{AC} = \frac{4k}{12k} = \frac{1}{3} \quad \Rightarrow$
$AG = \frac{1}{3}AC = \frac{3}{9}AC$
Since $BH//MC. \triangle AHF \sim \triangle ACM$.





$\frac{AF}{AM} = \frac{AH}{AC} = \frac{7k}{9k} = \frac{7}{9} \Rightarrow AH = \frac{7}{9}AC$.
$GH = AH - AG = \frac{7}{9}AC - \frac{3}{9}AC = \frac{4}{9}AC$.
$HC = AC - AH = AC - \frac{7}{9}AC = \frac{2}{9}AC$.





$AG : GH : HC = 3 : 4 : 2$. The answer is $3 + 4 + 2 = 9$.