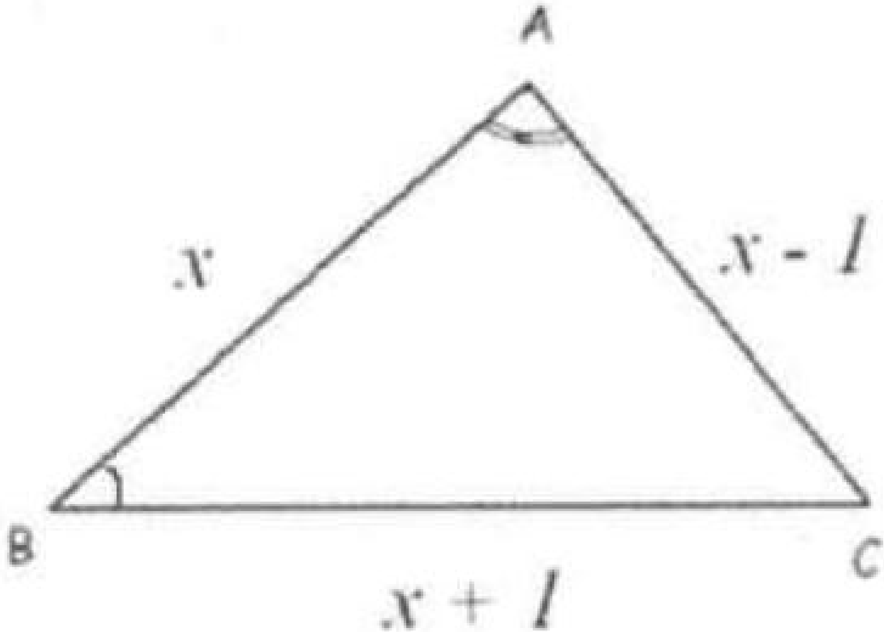
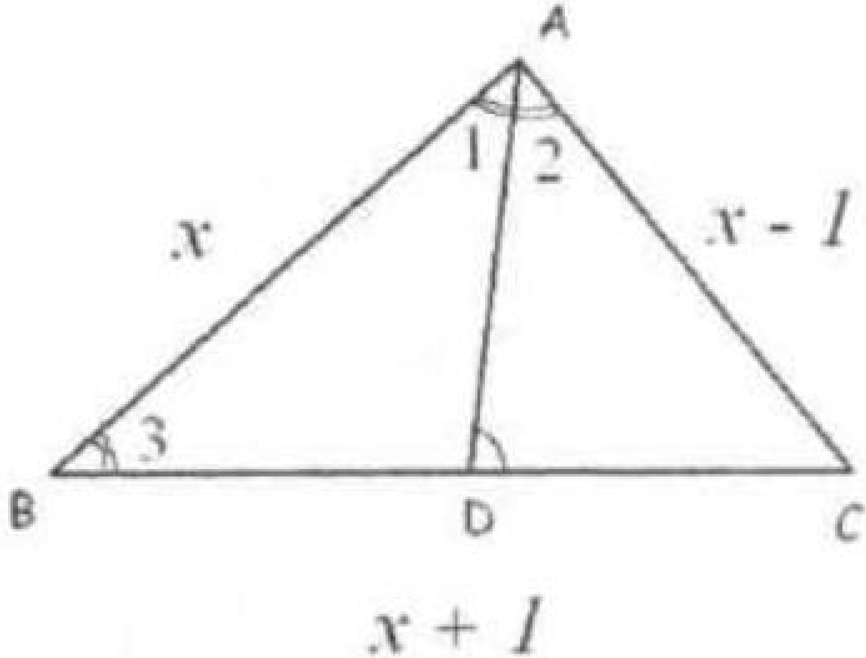
Problem

The lengths of the three sides of a triangle are consecutive positive integers. The largest angle of the triangle is two times of the smallest angle. What is the largest side of the triangle?



Solution

6. Method 1: Let $\angle A$ be the largest angle and $\angle B$ be the smallest angle. Draw the angle bisector of $\angle A$ to meet BC at D.$\angle$A is twice $\angle B$. We know that $\angle 2 = \angle 3, \angle ADC = \angle 1 + \angle 3 = \angle A$. $\triangle CAD \sim \triangle CBA$.



We have: $\frac{CD}{AC} = \frac{AC}{BC} \quad \Rightarrow \quad CD = \frac{AC^2}{BC}$

According to the angle bisector theorem, we get:

$\frac{AB}{BD} = \frac{AC}{CD} \quad \Rightarrow \quad \frac{CD}{BC-CD} = \frac{AC}{AB}$

Separate CD to get: $CD = \frac{AC \times BC}{AB+AC}$

Substitute (2) into (1), $\frac{AC^2}{BC} = \frac{AC \times BC}{AB+AC} \quad \Rightarrow \quad \frac{AC}{BC} = \frac{BC}{AB+AC}$

$\Rightarrow \quad \frac{x-1}{x+1} = \frac{x+1}{2x-1} \quad \Rightarrow \quad x^2 - 5x = 0 \quad \Rightarrow \quad x = 5$

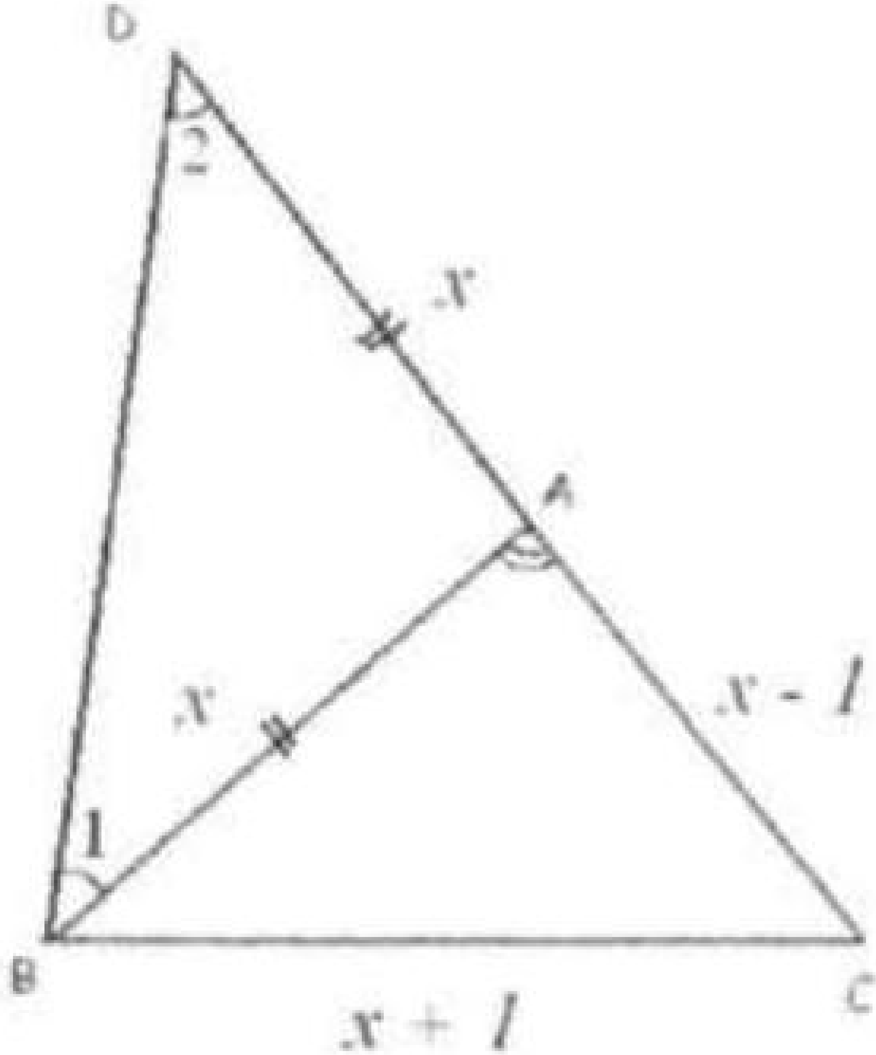
The largest side is $x + 1 = 6$.

Method 2: Extend CA to D such that $AD = AB$. Then $\angle 1 = \angle 2$ and $\angle CAB = 2\angle 2$. We are given that $\angle CAB = 2\angle ABC$, so $\angle 2 = \angle ABC$ and $\triangle ABC \sim \triangle BDC$.

$\frac{x-1}{x+1} = \frac{x+1}{x-1+x} \quad \Rightarrow \quad (x-1)(2x-1) = (x+1)^2$

$\Rightarrow \quad x^2 - 5x = 0 \quad \Rightarrow \quad x = 5 \quad \Rightarrow \quad x + 1 = 6.$

Method 3: We have the following theorem: In $\triangle ABC$, if $\angle A =$

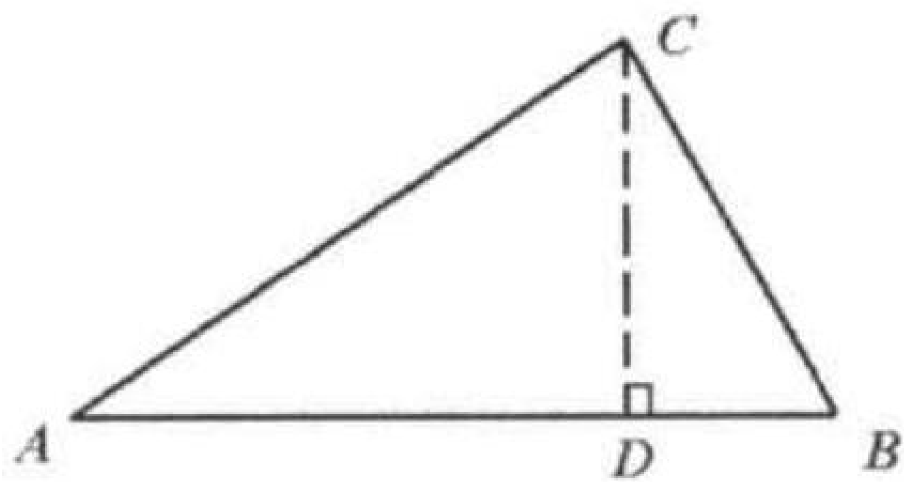


$2\angle B$, then $a^2 = b^2 + bc$
$a = (x+1), b = (x-1)$, and $c = x$

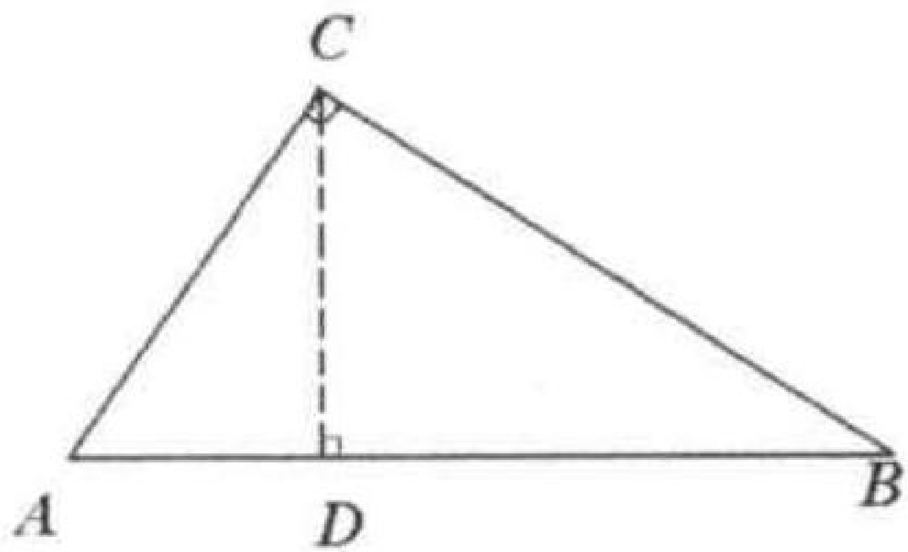
$$(x+1)^2 = (x-1)^2 + (x-1)x \Rightarrow x^2 - 5x = 0 \Rightarrow x = 5$$
$$\Rightarrow \quad x + 1 = 6.$$

Draw the height of the figure (especially when area calculation is involved).

$\longmapsto$

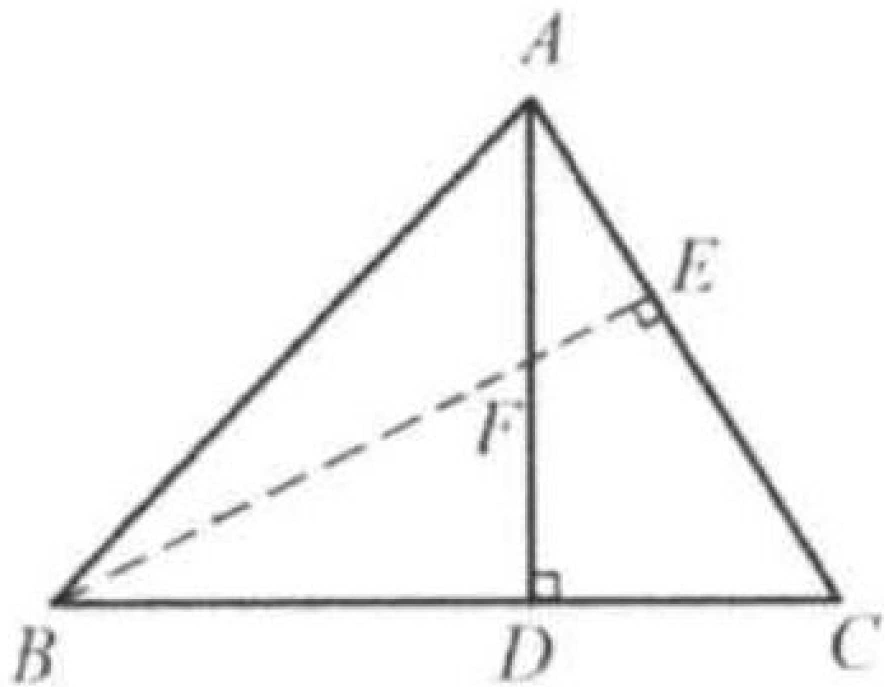


Draw the height to hypotenuse of a right triangle In triangle $ABC, \angle ACD = 90°$, Draw $CD \perp AB$. D is the feet of the perpendiculars to AB from C.
Then $\triangle ABC \sim \triangle ACD \sim \triangle CBD$.



Draw the second height of the figure when one height is shown. In triangle $ABC, AD \perp BC$. D is the feet of the perpendiculars to BC from A. Draw $BE \perp AC$. E is the feet of the perpendiculars to AC from B. AD meets BC at F.

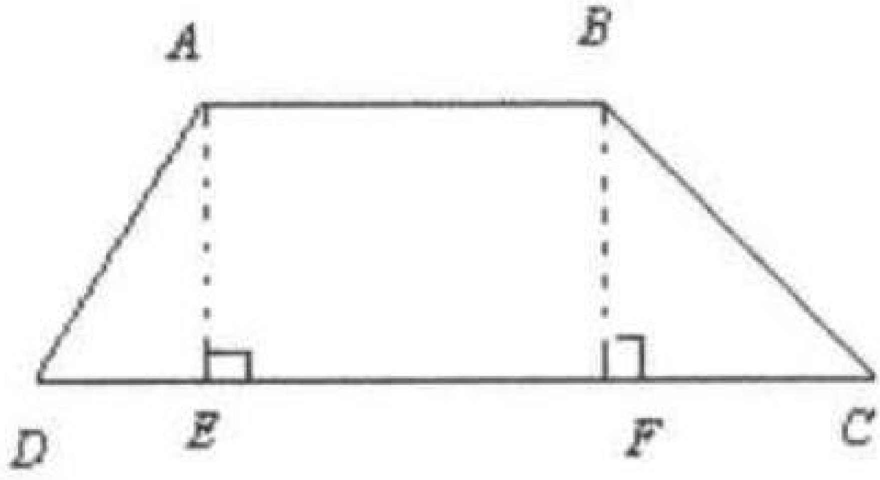
$$\angle CBE = \angle CAD, \angle AFE = \angle C = \angle BFD.$$



Draw two heights of trapezoid from the short base to the long base. In trapezoid $ABCD, AB//DC$. Draw AE and BF such that $AE \perp DC, BF \perp DC$. As shown in the figure to the right,

$AE = BF, AB = EF$.

$DF + CE = DC + EF = DC + AB$.



Chapter 4 Draw the Auxiliary lines with Perpendicular Lines