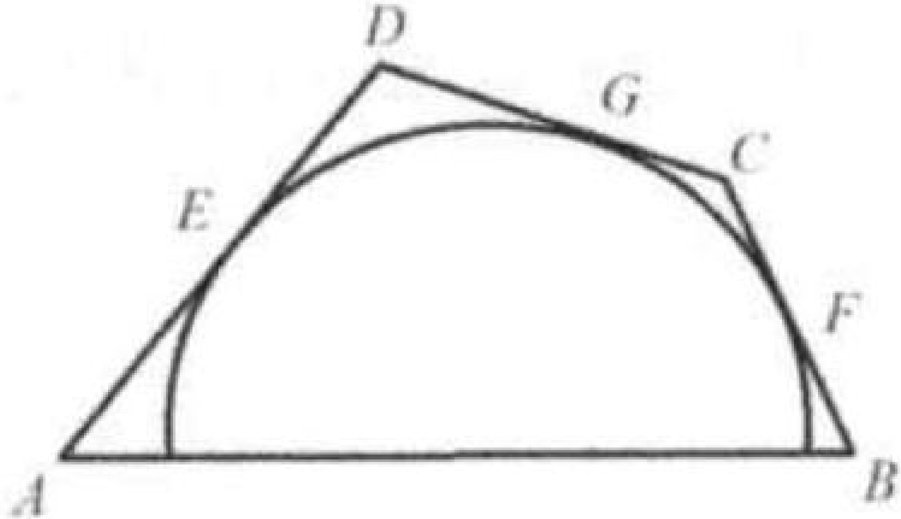
Example 6

(IMO) A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

Solution: Connect OE, OG, OF. Extend OF to meet the extension of GC to N.

Let the radius of the circle be r. Let $\angle NCF = \alpha$.



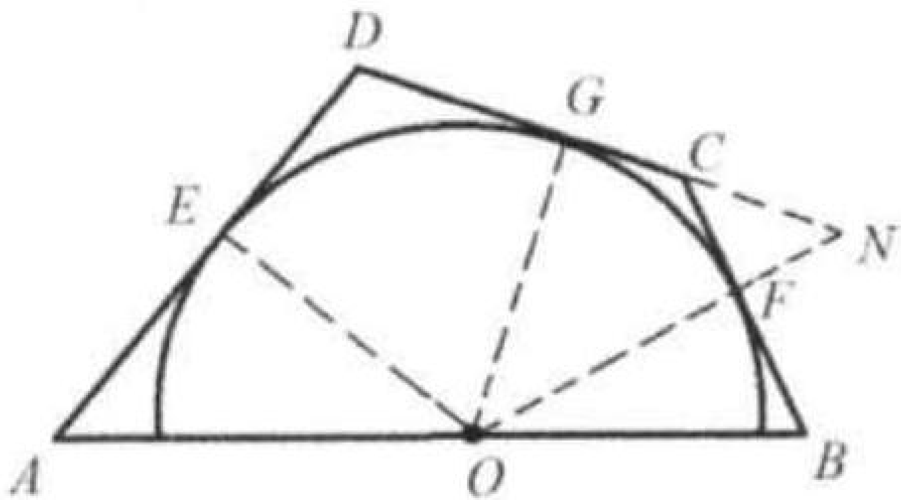
$\angle CNF = \beta . \alpha + \beta = 90°$.

Since quadrilateral $ABCD$ is cyclic, $\angle NCF = \angle A = \alpha$.

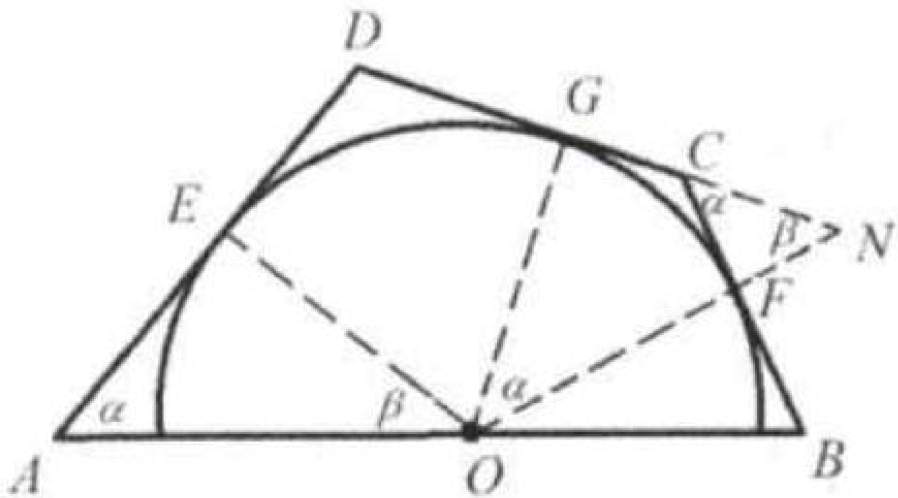
Thus Rt $\triangle AEO \sim$ Rt $\triangle CFN$.

So $CF = \frac{AE \times FN}{r}$.

We know that $\angle OGN = 90°$. So $\angle GON = \alpha$.



We know that $\angle OEA = 90°$. So $\angle EOA = \beta$.
Thus Rt $\triangle OGN \sim$ Rt $\triangle AEO$.
So $OA = \frac{AE \times ON}{OG} = \frac{AE(r+FN)}{r}$
$= AE + \frac{AE(FN) = AE + CF}{r}$.



Similarly, $OB = BF + DE$.
Thus $AB = OA + OB = AE + CF + BF + DE = AD + BC$.