## Problem

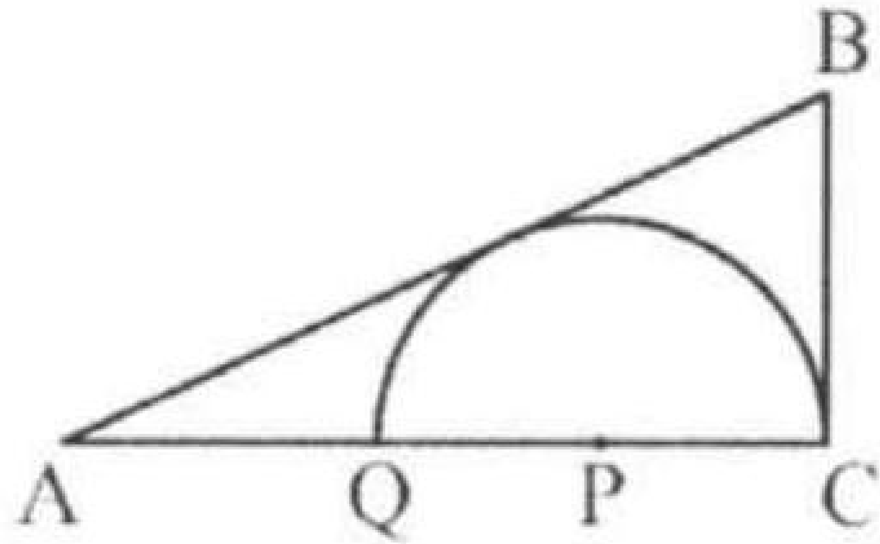
(2017 Mathcounts National) In right triangle $ABC$ with right angle at vertex $C$, a semicircle is constructed, as shown, with center $P$ on leg $AC$, so that the semicircle is tangent to leg $BC$ at $C$, tangent to the hypotenuse AB , and intersects leg $AC$ at $Q$ between $A$ and $C$. The ratio of $AQ$ to $QC$ is $2:3$. If $BC = 12$, then what is the value of $AC$ ? Express your answer in simplest radical form.

## Solution

$8\sqrt{10}$.

We know that triangle $ABC$ is a right triangle with right angle at vertex $C$.

The semicircle centered at $P$ and is



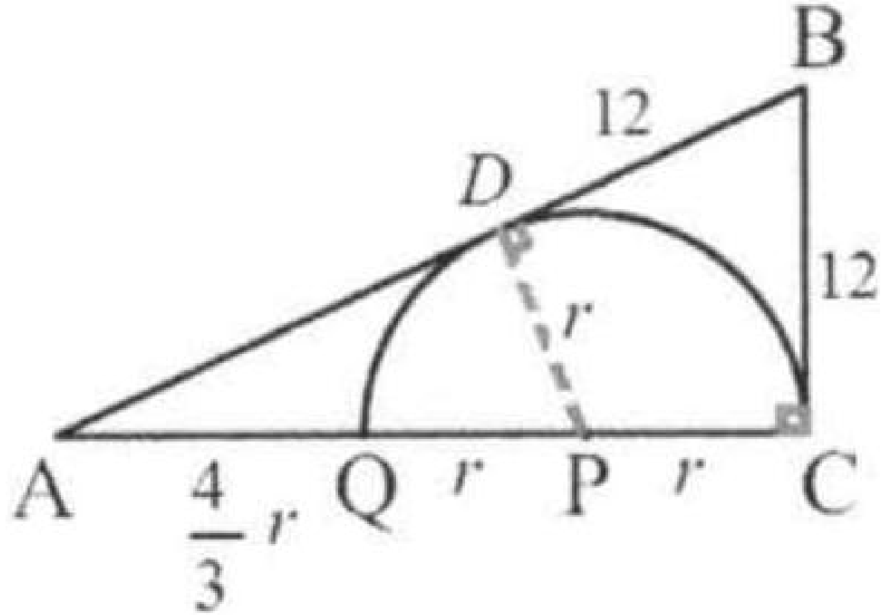

tangent to leg $BC$ at $C$, tangent to the hypotenuse $AB$. So $BC = 12$ and $BD = 12$.

Connect $DP$. $D$ is the tangent point as shown.

Since $\frac{AQ}{QC} = \frac{2}{3}$, $AQ = \frac{2}{3}QC = \frac{2}{3} \times 2r = \frac{4}{3}r$.

We want to find $\frac{4}{3}r + r + r$.

Applying Pythagorean theorem to triangle $ADP$ :




$$AD = \sqrt{\left(\frac{4}{3}r + r\right)^2 - r^2} = \sqrt{\left(\frac{4}{3}r + r\right)^2 - r^2} = \frac{2\sqrt{10}}{3}r.$$

Since $\triangle ABC \sim \triangle APD$, $\frac{BC}{AC} = \frac{DP}{AD} \Rightarrow \frac{12}{\frac{4}{3}r+r+r} = \frac{r}{\frac{2\sqrt{10}}{3}r}$

$$\Rightarrow \quad \frac{12}{\frac{4}{3}r+r+r} = \frac{1}{\frac{2\sqrt{10}}{3}} \quad \Rightarrow \quad \frac{4}{3}r + r + r = 12 \times \frac{2\sqrt{10}}{3} = 8\sqrt{10}.$$