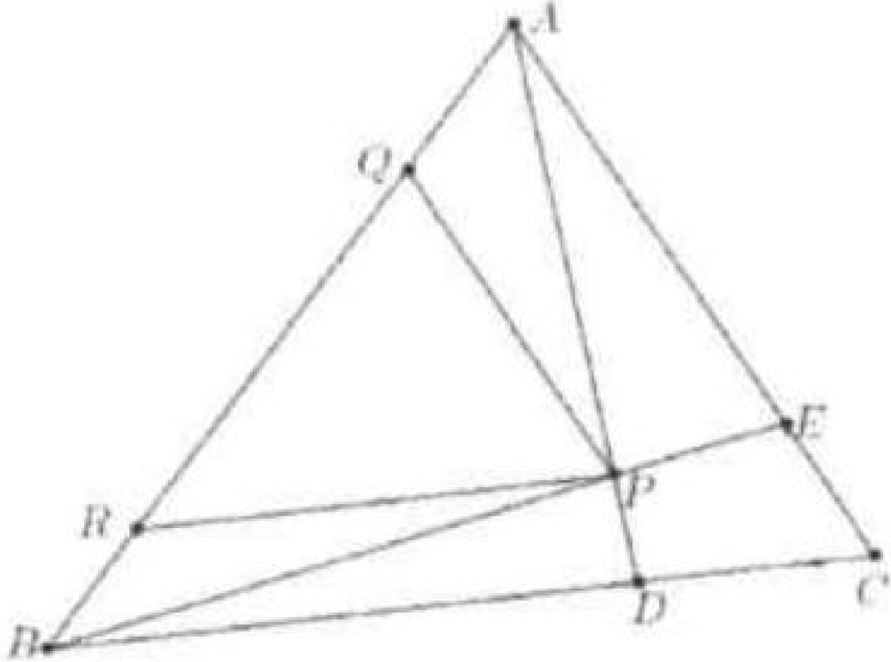
Example 19

(2002 AIME II) In triangle ABC, point D is on BC with $CD = 2$ and $DB = 5$, point E is on AC with $CE = 1$ and $EA = 3, AB = 8$, and AD and BE intersect at P. Points Q and R lie on AB so that PQ is parallel to CA and PR is parallel to CB. It is given that the ratio of the area of triangle PQR to the area



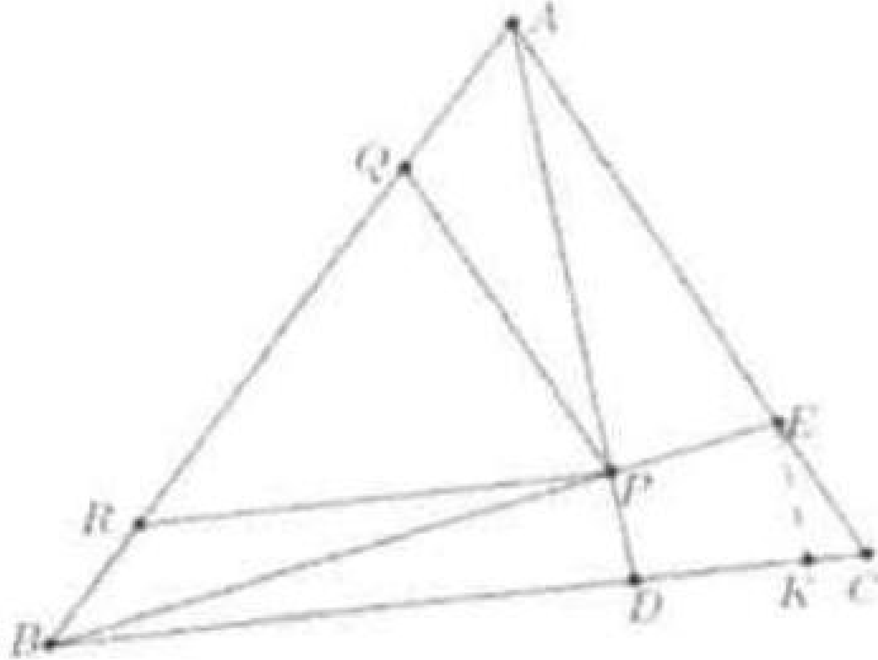
of triangle ABC is m/n, where m and n are relatively prime positive integers. Find $m + n$.

Solution: 901.

Method 1 (official solution):

Draw the line through E parallel to AD, and let K be its intersection with BC.

Because $CD = 2$ and $KC : KD = EC : EA = 1 : 3$, it follows that $KD = 3/2$.



Therefore, $\frac{QP}{AE} = \frac{BP}{BE} = \frac{BD}{BK} = \frac{5}{5+\frac{3}{2}} = \frac{10}{13}$. Thus

$\frac{QP}{AC} = \frac{3}{4} \cdot \frac{10}{13} = \frac{15}{26}$.

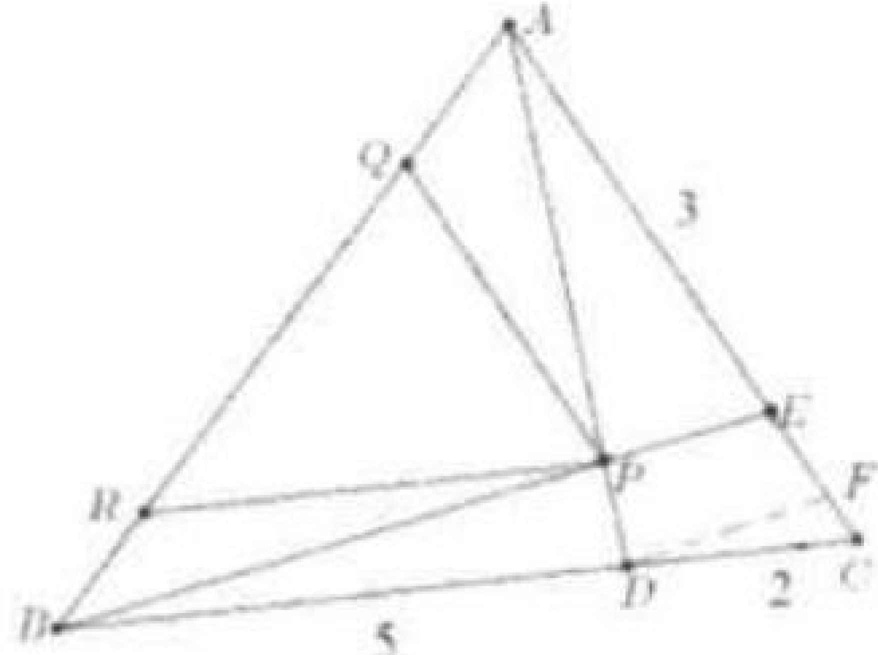
Since triangles PQR and CAB are similar, the ratio of their areas is $(15/26)^2 = 225/676$. Thus $m + n = 901$.

Method 2 (our solution):

Draw the line through D parallel to BE, and let F be its intersection with AC.

Observe that triangles BEC and DFC are similar.

$$\frac{BC}{DC} = \frac{EC}{FC} \quad \Rightarrow \quad \frac{7}{2} = \frac{1}{FC} \quad \Rightarrow \quad FC = \frac{2}{7}$$

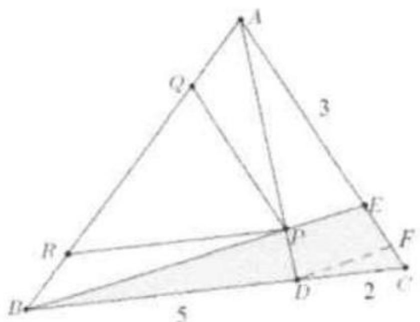


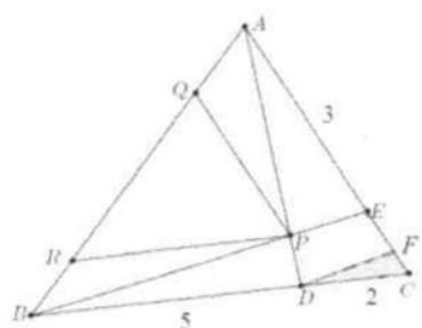
So $EF = \frac{5}{7}$.

Triangles RPA and BDA are similar.

It follows that $\frac{RP}{BD} = \frac{AP}{AD}$.

Since triangles ADF and APE are similar, so $\frac{AE}{AF} = \frac{AP}{AD}$.





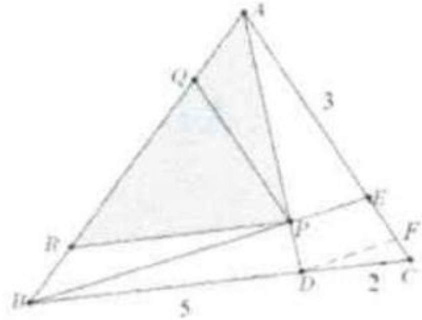
Thus, $\frac{RP}{BD} = \frac{AP}{AD} = \frac{AE}{AF} = \frac{3}{3+\frac{5}{7}} = \frac{21}{26} \Rightarrow RP = \frac{21}{26} \times 5$.

Since triangles PQR and CAB are similar, the ratio of their areas is

$$\frac{S_{\triangle PQR}}{S_{\triangle ABC}} = \left(\frac{RP}{BC}\right)^2 = \left(\frac{\frac{21}{26}\times 5}{7}\right)^2 = \left(\frac{15}{26}\right)^2 = \frac{225}{676}$$

Thus $m + n = 901$.

This is the problem 13 in 2002 AIME II.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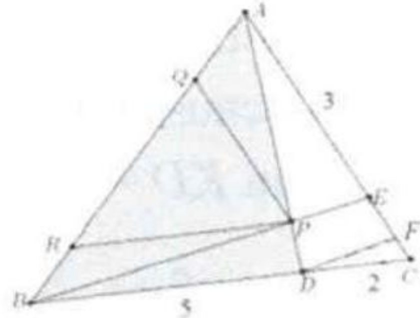
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