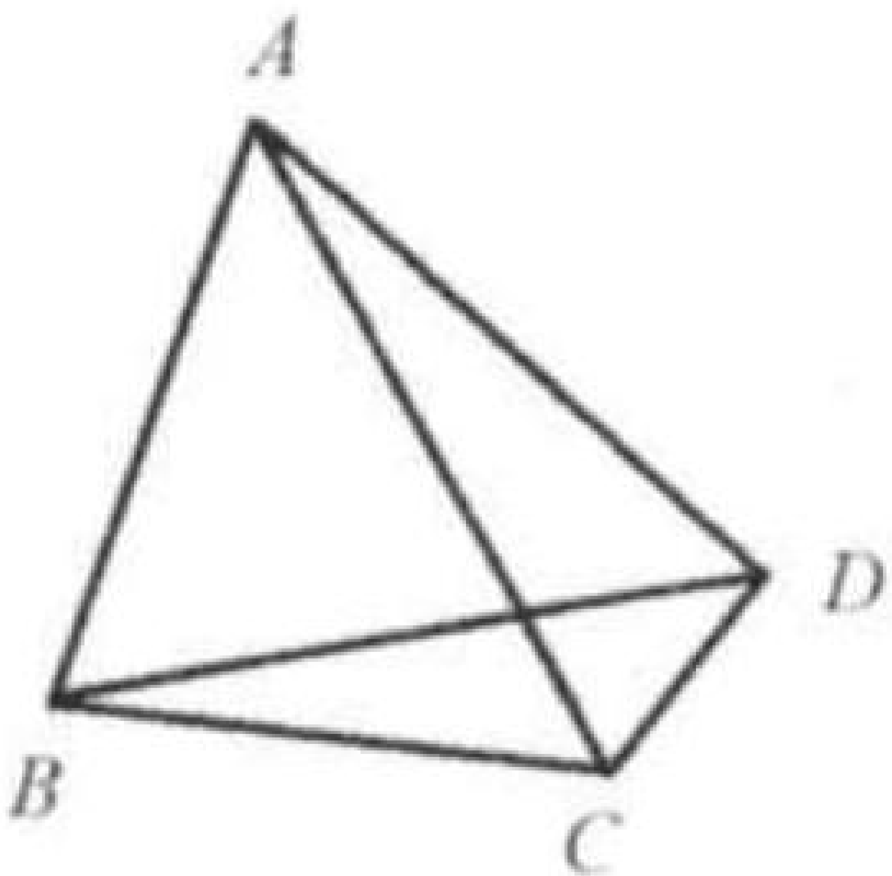
Example 5

As shown in the figure, $\angle ABD = \angle ACD = 60^\circ . \angle ADB = 90^\circ - \frac{1}{2}\ \angle BDC$. Show that triangle ABC is an isosceles triangle.

Solution: Method 1:

Since $\angle ABD = \angle ACD = 60^\circ$, points A, B, C, and D are concyclic. Therefore,



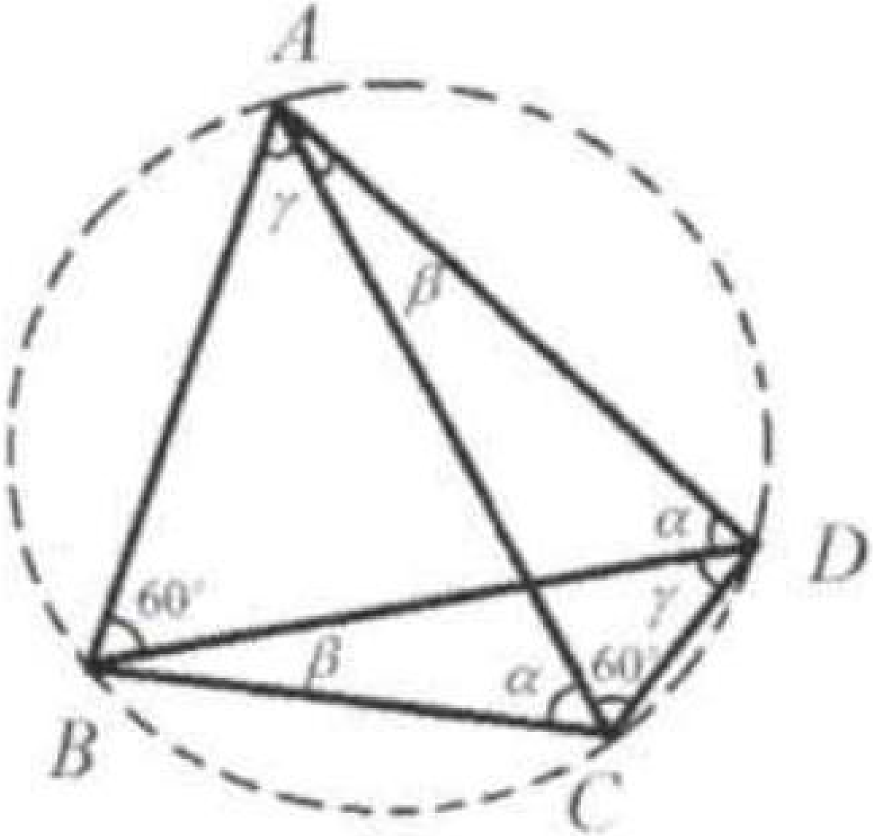
$\angle ADB = \angle ACB = \alpha ; \angle DBA = \angle DAC = \beta ; \angle BAC = \angle BDC = \gamma .$

$$\beta + \gamma + 60^\circ + \alpha = 180^\circ$$

We know that $\angle ADB = 90^\circ - \frac{1}{2}\angle BDC$ or

$$\alpha = 90^\circ - \frac{1}{2}\gamma \Rightarrow \quad 2\alpha + \gamma = 180^\circ$$

$$(1) + \alpha : \beta + \gamma + 60^\circ + 2\alpha = 180^\circ + \alpha$$



Substituting (2) into (3): $\beta + 60^\circ + 180^\circ = 180^\circ + \alpha \quad \Rightarrow \quad \beta + 60^\circ = \alpha$. That is $\angle ABC = \angle ACB$, triangle ABC is an isosceles triangle.

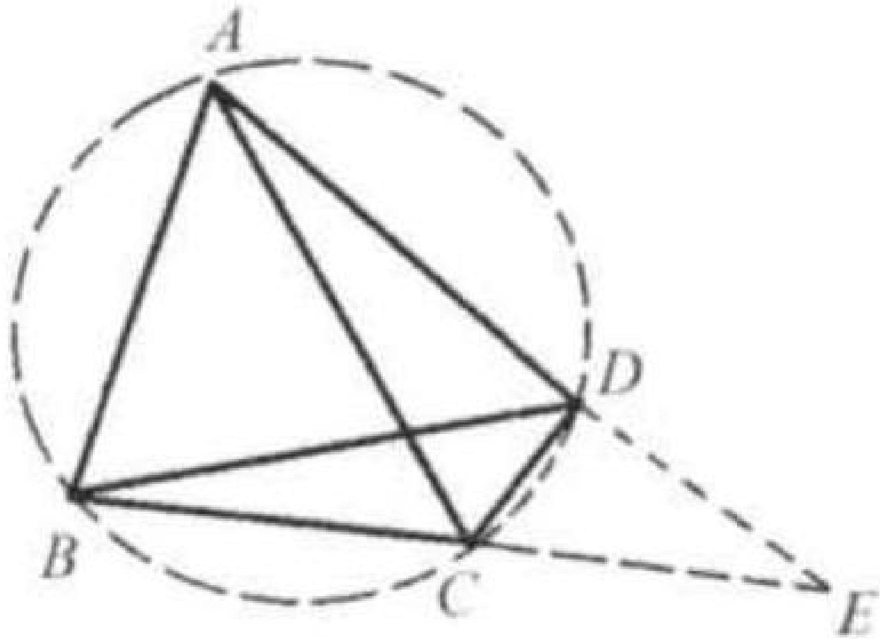
Method 2:

Extend AD and BC to meet at E.

Since $\angle ABD = \angle ACD = 60^\circ$, points A, B, C, and D are concyclic.

Therefore,

$$\angle CDE = \angle ABC, \angle ACB = \angle ADB.$$



We know that $\angle ADB = 90^\circ - \frac{1}{2}\angle BDC$ or

$$2\angle ADB = 180^\circ - \angle BDC \quad \text{or} \quad \angle ADB + \angle ADB + \angle BDC = 180^\circ$$

But

$$\angle CDE + \angle ADB + \angle BDC = 180^\circ$$

So $\angle ADB = \angle EDC$

That is $\angle ABC = \angle ACB$, triangle ABC is an isosceles triangle