Problem 13

Problem

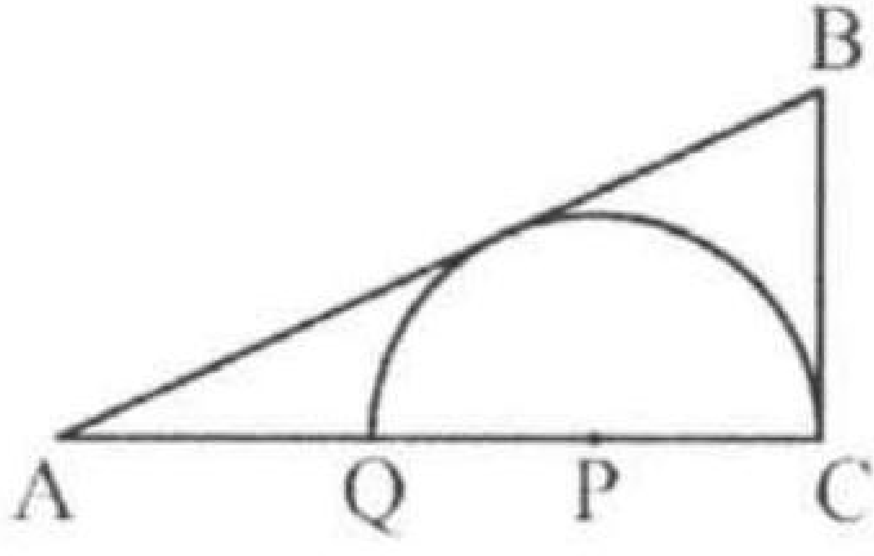
(2017 Mathcounts National) In right triangle ABC with right angle at vertex C, a semicircle is constructed, as shown, with center P on leg AC, so that the semicircle is tangent to leg BC at C, tangent to the hypotenuse AB , and intersects leg AC at Q between A and C. The ratio of AQ to QC is $2:3$. If $BC = 12$, then what is the value of AC ? Express your answer in simplest radical form.

Solution

$8\sqrt{10}$.

We know that triangle ABC is a right triangle with right angle at vertex C.

The semicircle centered at P and is



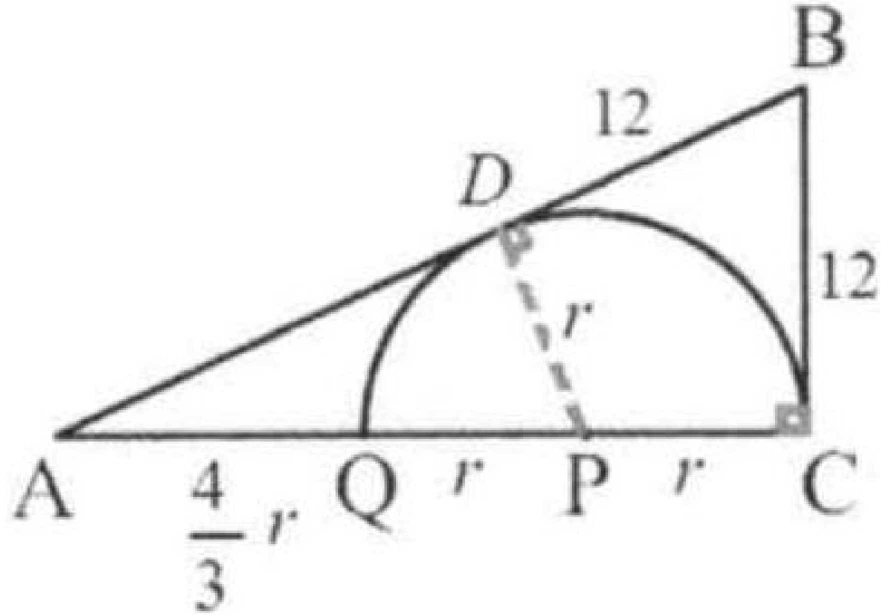
tangent to leg BC at C, tangent to the hypotenuse AB. So $BC = 12$ and $BD = 12$.

Connect DP. D is the tangent point as shown.

Since $\frac{AQ}{QC} = \frac{2}{3}, AQ = \frac{2}{3}QC = \frac{2}{3} \times 2r = \frac{4}{3}r.$

We want to find $\frac{4}{3}r + r + r.$

Applying Pythagorean theorem to triangle ADP :



$AD = \sqrt{\left(\frac{4}{3}r + r\right)^2 - r^2} = \sqrt{\left(\frac{4}{3}r + r\right)^2 - r^2} = \frac{2\sqrt{10}}{3}r.$

Since $\triangle ABC \sim \triangle APD, \frac{BC}{AC} = \frac{DP}{AD} \Rightarrow \frac{12}{\frac{4}{3}r+r+r} = \frac{r}{\frac{2\sqrt{10}}{3}r}$

$\Rightarrow \quad \frac{12}{\frac{4}{3}r+r+r} = \frac{1}{\frac{2\sqrt{10}}{3}} \quad \Rightarrow \quad \frac{4}{3}r + r + r = 12 \times \frac{2\sqrt{10}}{3} = 8\sqrt{10}.$