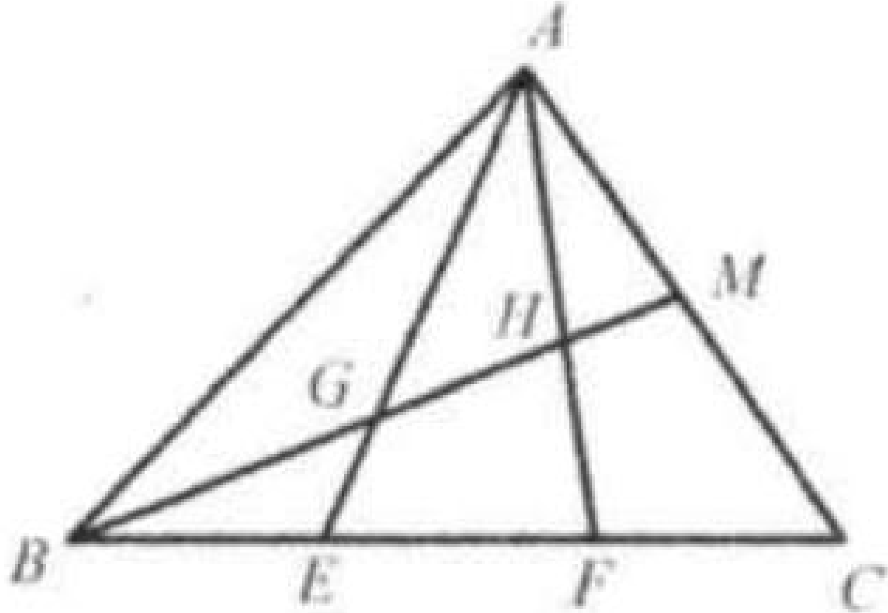
Example 17

In triangle ABC, BM is the median on AC. AE and AF trisect BC and meet BM at G and H, respectively. $BG : GH : HM = x : y : z$. Find the value of $x + y + z$, where x, y, and z are positive integers relatively prime.

Solution: 10 .

Method 1:

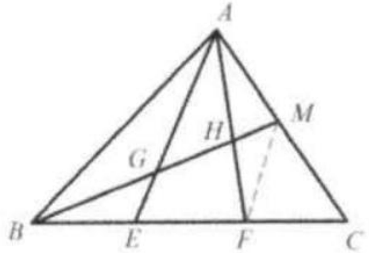
Connect FM. We see that $FM//AE$ since M is the midpoint of AC and F is the midpoint of EC.

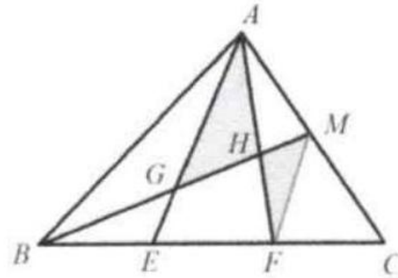


$MF = \frac{1}{2}AE = \frac{1}{2}(GE + AG) = 2GE$. So $AG = 3GE$ and $MF = \frac{2}{3}AG$.

We know that $AG//MF$. So $\triangle AGH \sim \triangle FMH$. $\frac{AG}{MF} = \frac{GH}{HM} = \frac{3}{2}$.

We also know that $BG = GM$. So $BG : GH : HM = 5 : 3 : 2$. The answer is $5 + 3 + 2 = 10$.





Method 2:
Draw $CP//AF$ and $CQ//AE$ through point C and to meet the extension of BM at P and Q, respectively. We see that $\triangle AGM \equiv \triangle CQM$
$(\angle GAM = \angle QCM, AM = MC, \angle AMG = \angle CMQ)$.
So $GM = MQ$. Similarly, we get $HM = MP$.
Let $BG = x, GH = y, HM = z$.

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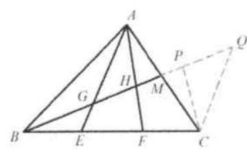
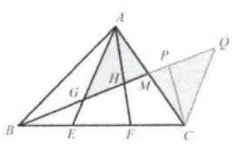
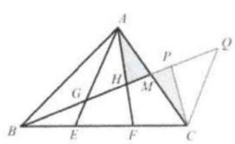
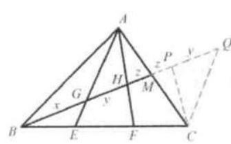
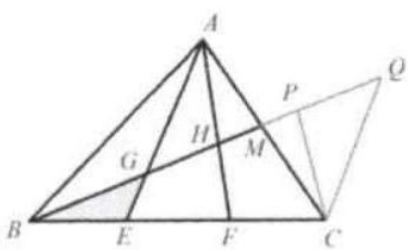


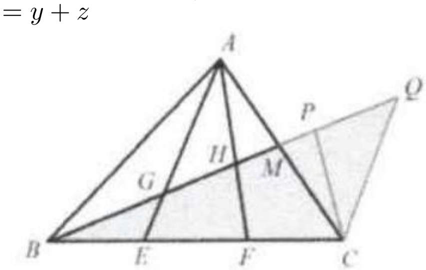




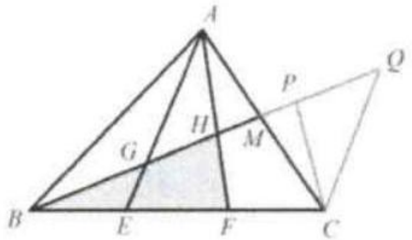


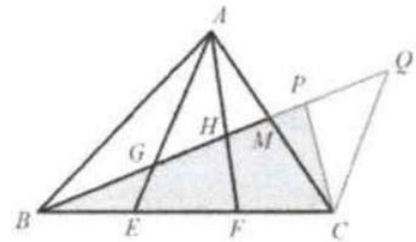
We know that $GE//CQ$. So $\triangle BEG \sim \triangle BCQ. \frac{BG}{BQ} = \frac{BE}{BC} = \frac{1}{3}$.
Therefore $x = y + z$





We know that $HF//PC$. So $\triangle BFH \sim \triangle BCP$. $\frac{BH}{BP} = \frac{BF}{BC} = \frac{2}{3}$
$\Rightarrow \frac{x+y}{2x+z} = \frac{2}{3}$.
Therefore $3y = 2z + x$
Substituting (1) into (2): $\frac{y}{z} = \frac{2}{3}$. So $x : y : z = 5 : 3 : 2$. The answer is $5 + 3 + 2 = 10$.



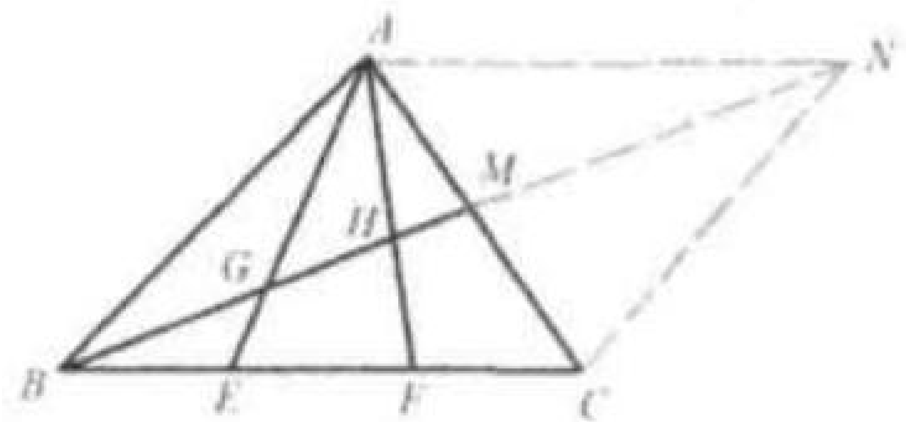


Method 3:

Extend BM to N such that $MN = BM$. Connect $AN, CN. ABCN$ is a parallelogram because the diagonals bisect each other. Thus $AF//BC. AN = BC = 3BE$.

We know that $AN//BC$. So $\triangle BEG \sim \triangle NAG$.

$\frac{NG}{BG} = \frac{AN}{BE} = 3$. So $NG = 3BG$.



$BN = BG + GN = 4BG$.

$BG = BN/4$.

We know that $AN//BC$. So $\triangle ANH \sim \triangle FBH$. $\frac{NH}{BH} = \frac{AN}{BF} = \frac{3}{2}$. So $NH = 3BH/2$.

$BN = BH + HN = 5BH/2$.

So $BH = 2BN/5$.

Thus $GH = BH - BG = \frac{2}{5}BN - \frac{1}{4}BN = \frac{3}{20}BN$.

$HM = BM - BH = \frac{1}{2}BN - \frac{2}{5}BN = \frac{1}{10}BN$.

$BG : GH : HM = \frac{1}{4} : \frac{3}{20} : \frac{1}{10} = 5 : 3 : 2$

The answer is $5 + 3 + 2 = 10$.