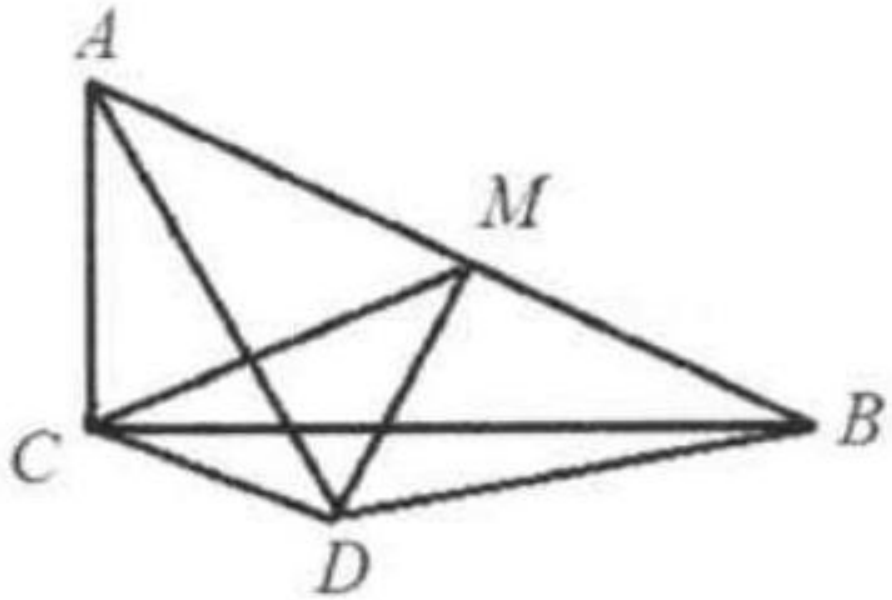
Example 19

(2003 AIME 2 Problem 11) Triangle ABC is a right triangle with $AC = 7, BC = 24$, and right angle at C. Point M is the midpoint of AB, and D is on the same side of line AB as C so that $AD = BD = 15$.
Given that the area of $\triangle CDM$ can be expressed as $\frac{m\sqrt{n}}{p}$, where m, n, and p are positive integers, m and p are



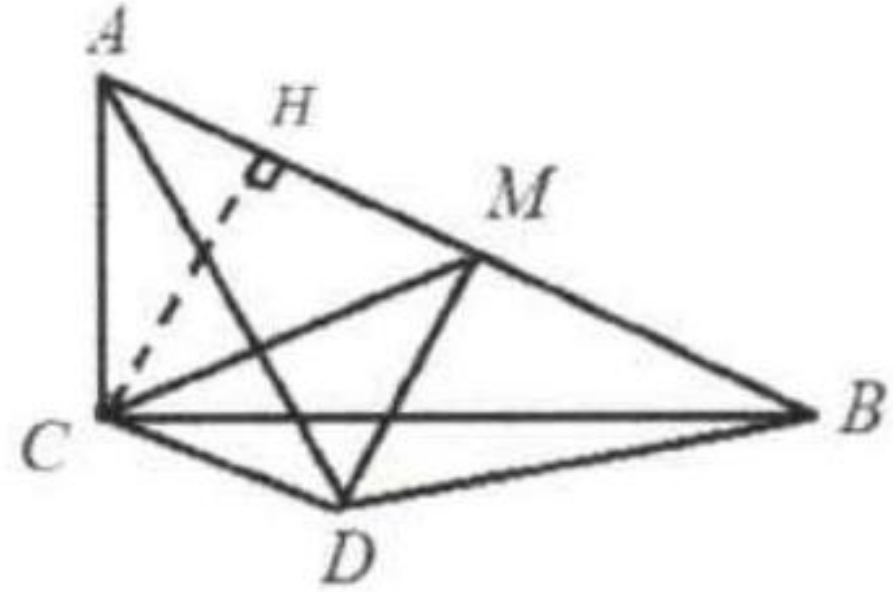
relatively prime, and n is not divisible by the square of any prime, find $m+n+p$.

Solution: 578. Draw $CH \perp AB$ to meet AB at H. Since $DA = DB, DM$ is the perpendicular bisector of triangle DAB. So $DM \perp AB$.

Thus $CH//DM$. Connect DH. We have

$S_{\triangle CDM} = S_{\triangle HDM} = \frac{1}{2}HM \times DM$.

$DM = \sqrt{AD^2 - AM^2} = \frac{5\sqrt{11}}{2}$,



$HM = AM - AH = \frac{1}{2}AB - \frac{AC^2}{AB} = \frac{527}{50}$.
Thus $CH//DM$ and
$S_{\triangle CDM} = S_{\triangle HDM} = \frac{1}{2}HM \times DM = \frac{1}{2} \times \frac{5\sqrt{11}}{2} \times \frac{527}{50} = \frac{527\sqrt{11}}{40}$.
So $m + n + p = 527 + 11 + 40 = 578$.