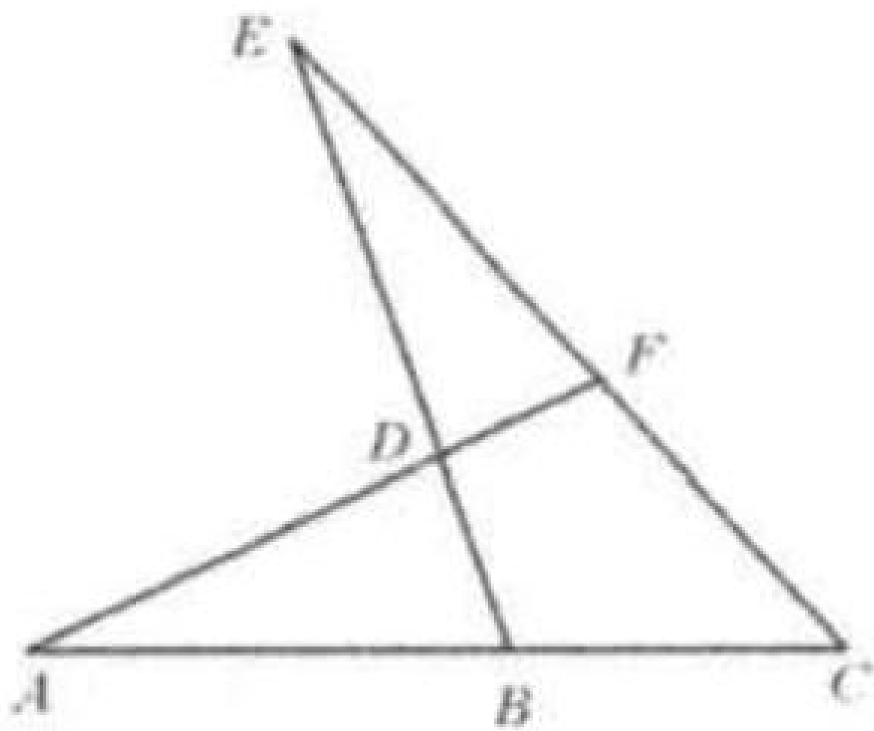
Example 11

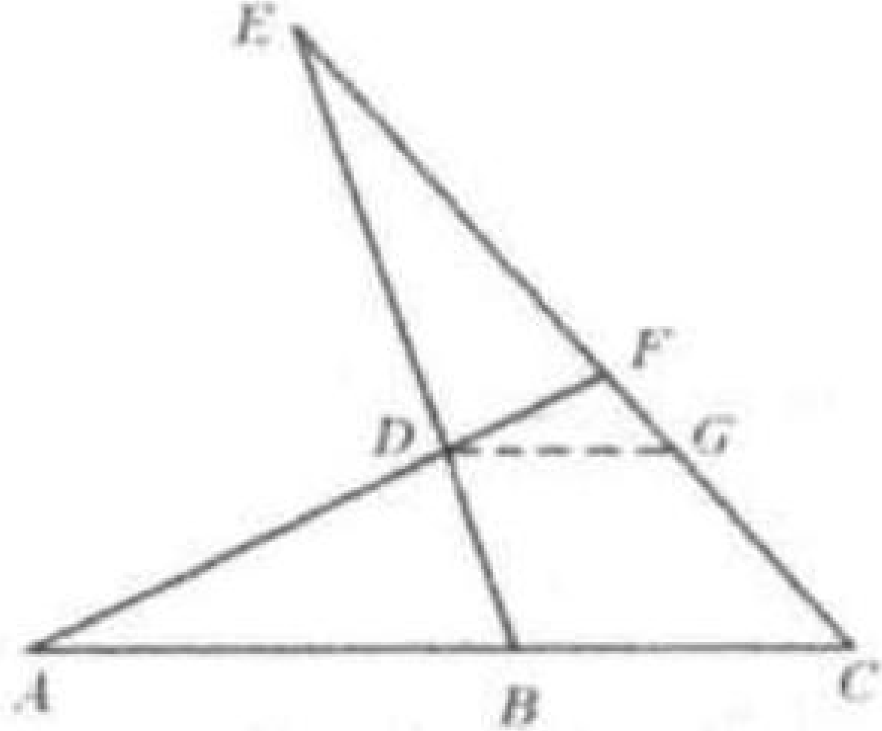
B is the trisection point of the side AC of $\triangle AFC$. Draw a line through B to meet the extension of CF at E, to meet AF at D such that $\frac{ED}{DB} = \frac{AB}{BC} = \frac{2}{1}$. Show that $\frac{AD}{DF} = \frac{7}{2}$.

Solution:



Method 1:
As shown in the figure to the right, draw $DG//AG$ through D to meet EC at G.

$$\text{In } \triangle FAC, \frac{FA}{FD} = \frac{AC}{DG} = \frac{3BC}{DG}$$



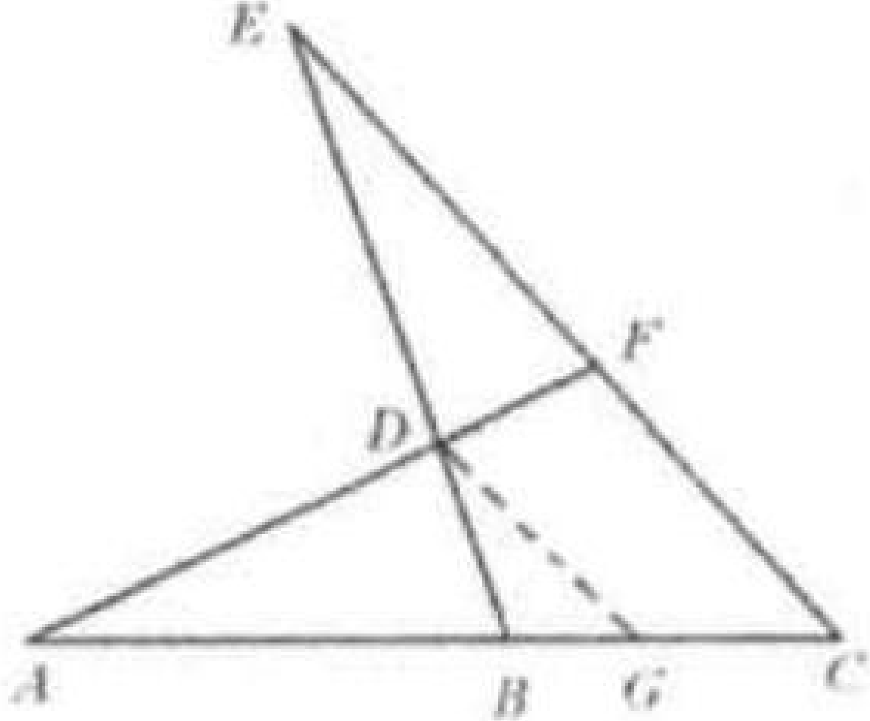
In $\triangle EBC$, $\frac{BC}{DG} = \frac{EB}{ED} = \frac{3}{2}$

Substituting (2) into (1) gives us: $\frac{AF}{FD} = \frac{9}{2} \Rightarrow \frac{AD}{DF} = \frac{7}{2}$

Method 2:

As shown in the figure to the right, draw $DG//CE$ through D to meet AC at G.

In $\triangle BEC$, $\frac{BG}{GC} = \frac{BD}{DE} = \frac{1}{2} \quad \Rightarrow \quad \frac{BC}{GC} = \frac{3}{2}$



Since $BC = \frac{1}{3}AC$, $\frac{AC}{GC} = \frac{9}{2}$

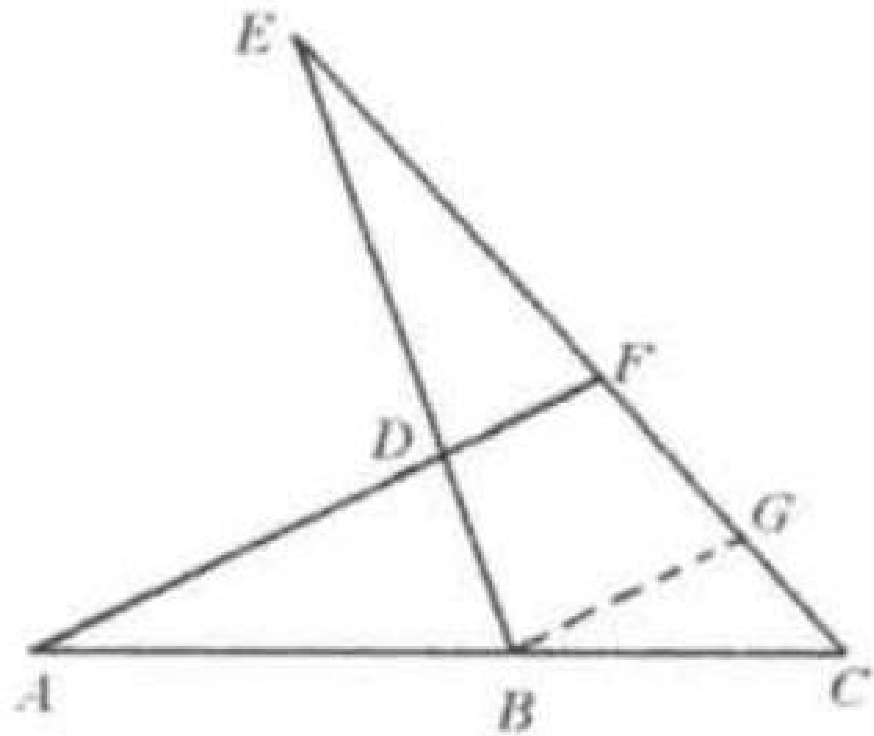
In $\triangle ACF$, $\frac{AF}{DF} = \frac{AC}{GC}$, $\therefore \frac{AF}{DF} = \frac{9}{2}$, $\therefore \frac{AD}{DF} = \frac{7}{2}$.

Method 3:

As shown in the figure to the right, draw $BG//AF$ through B to meet CE at G.

In $\triangle CFA$, $\frac{AF}{BG} = \frac{AC}{BC} = \frac{3}{1}$,

In $\triangle EBG$, $\frac{BG}{DF} = \frac{EB}{ED} = \frac{3}{2}$,



(1) × (2): $\frac{AF}{DF} = \frac{9}{2}, \quad \therefore \frac{AD}{DF} = \frac{7}{2}$.

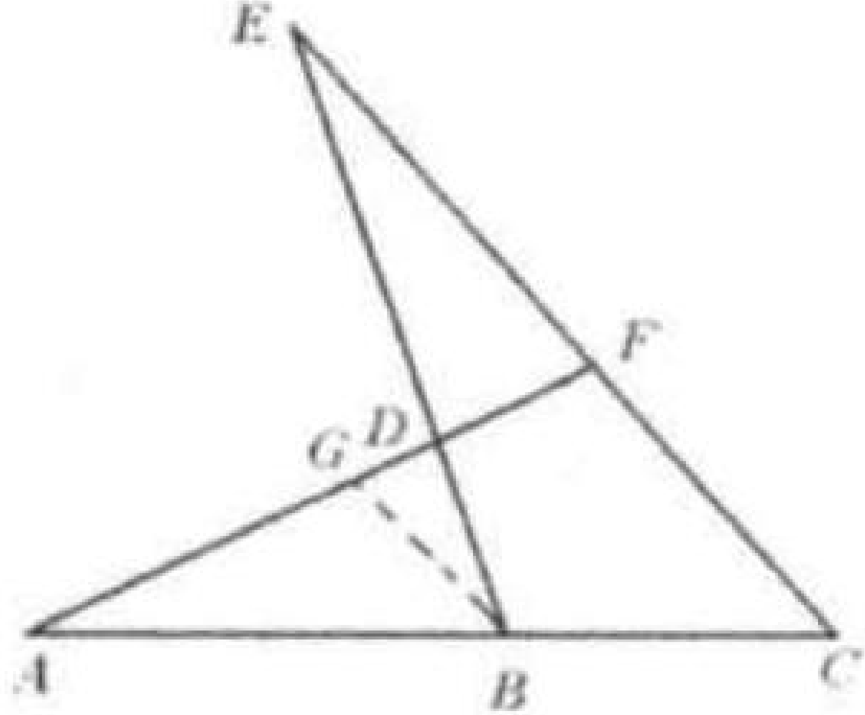
Method 4:

As shown in the figure to the right, draw $BG//CE$ through B to meet AF at G.

Since $\triangle EFD \sim \triangle BGD$, $\frac{DF}{DG} = \frac{ED}{DB} = \frac{2}{1}$.

Therefore $DG = \frac{1}{2}DF$

In $\triangle ACF$, $\frac{AG}{AF} = \frac{AB}{AC} = \frac{2}{3}$. In other words,



$\frac{AD-DG}{AD+DF}=\frac{2}{3}$. $\frac{AD-\frac{1}{2}DF}{AD+DF}=\frac{2}{3}$.

Simplifying yields: $\frac{AD}{DF}=\frac{7}{2}$.

Method 5:

As shown in the figure to the right, draw $AG//CE$ through A to meet the extension of CE at G.

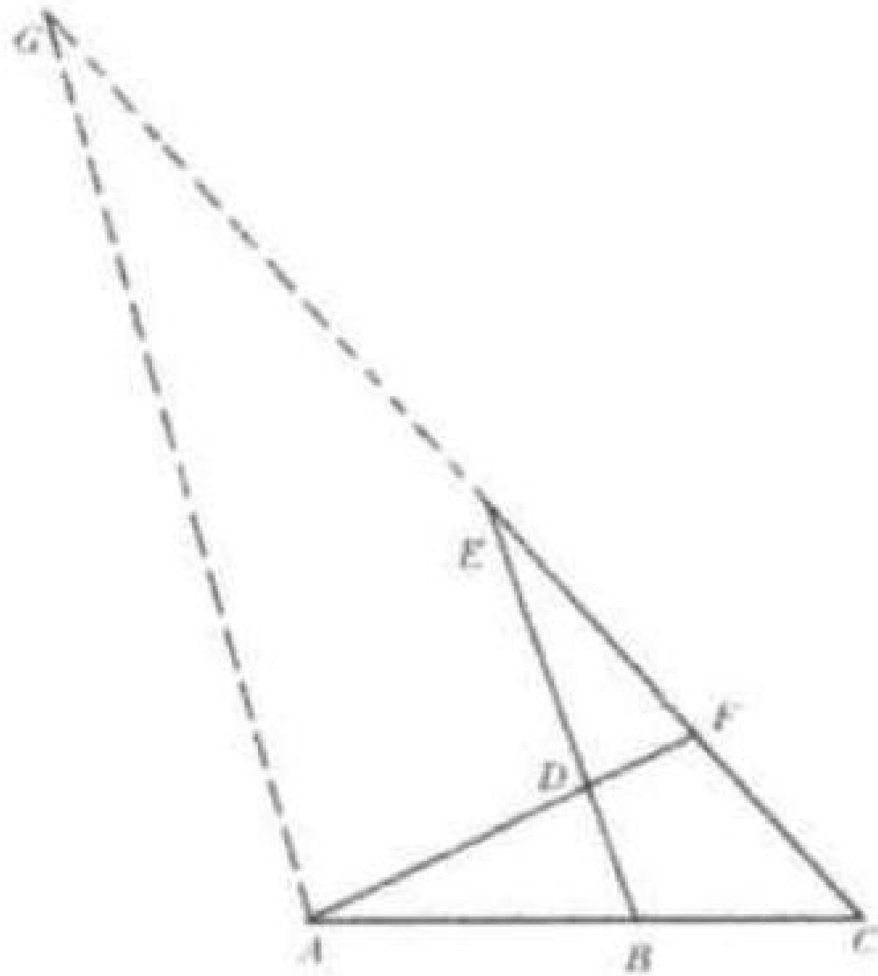
In $\triangle ACG$, $\frac{BE}{AG}=\frac{BC}{AC}=\frac{1}{3}$

In $\triangle AFG$, $\frac{AG}{DE}=\frac{AF}{DF}$

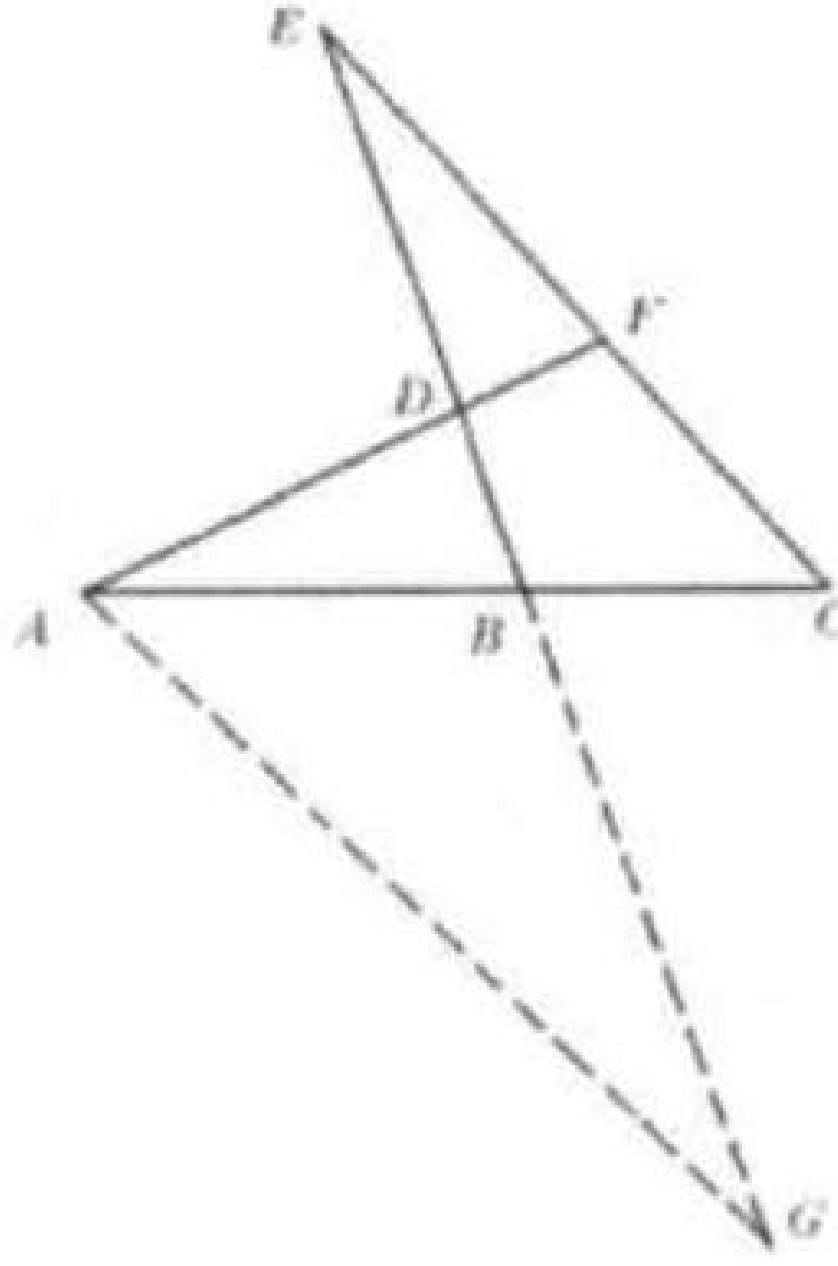
(1) $\times$ (2): $\frac{BE}{DE}=\frac{AF}{3DF}$

Since $\frac{BE}{DE}=\frac{3}{2}$, $\frac{3}{2}=\frac{AF}{3DF}$, $\frac{AF}{DF}=\frac{9}{2}$.

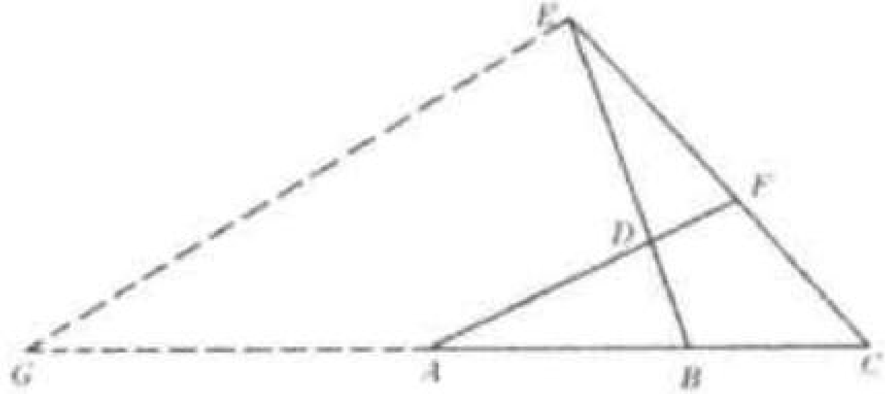
Hence $\frac{AD}{DF}=\frac{7}{2}$.



Method 6:
As shown in the figure to the right, draw $AG//CE$ through A to meet the extension of EB at G.
Since $\triangle ABG \sim \triangle CBE$, $\frac{BG}{BE} = \frac{AB}{BC} = \frac{2}{1}$ and $BG = 2EB$
Since $\triangle FED \sim \triangle AGD$,
$$\frac{DF}{AD} = \frac{ED}{DG} = \frac{ED}{DB+BG}$$
$$= \frac{ED}{DB+2EB} = \frac{2DB}{DB+6DB} = \frac{2}{7}.$$
Therefore $\frac{AD}{DF} = \frac{7}{2}$.



Method 7:
As shown in the figure to the right, draw $EG//FA$ through E to meet the extension of CA at G.
In $\triangle BEG : \frac{EG}{AD} = \frac{EB}{DB} = \frac{GB}{AB} = \frac{3}{1}$



Therefore we have:

$GB = 3AB$

$EG = 3AD$

In $\triangle CEG$:

$$\frac{EG}{AF} = \frac{GC}{AC} = \frac{GB+BC}{AB+BC} = \frac{3(2BC)+BC}{2BC+BC} = \frac{7}{3}$$

$$\Rightarrow \quad EG = \frac{7}{3}AF = \frac{7}{3}(AD+DF)$$

Substituting (2) into (3): $3AD = \frac{7}{3}(AD+DF) \quad \Rightarrow \quad 9AD = 7(AD+DF)$

Therefore $\frac{AD}{DF} = \frac{7}{2}$.

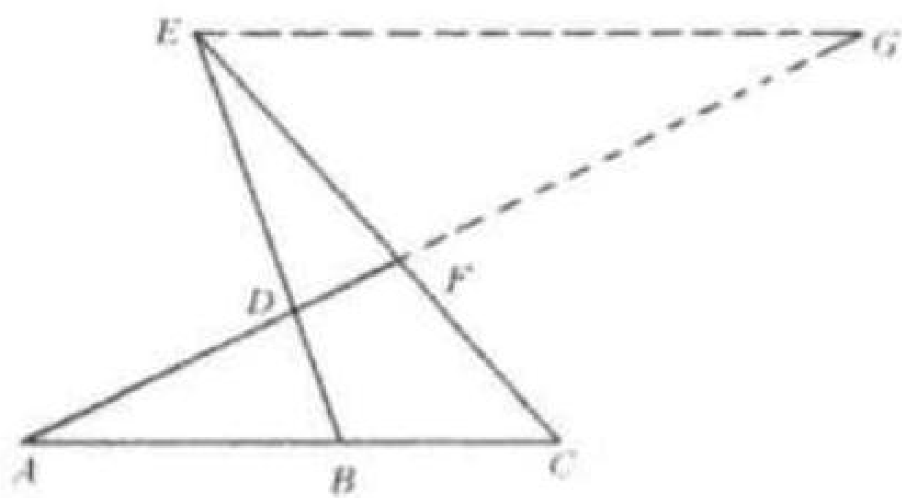
Method 8:

As shown in the figure to the right, draw $EG//AB$ through E to meet the extension of AF at G.

Since $\triangle DAB \sim \triangle DGE$, $\frac{EG}{AB} = \frac{ED}{DB} = \frac{DG}{AD} = \frac{2}{1}$.

We have $EG = 2AB$

$DG = 2AD$



Since $\triangle FAC \sim \triangle FGE$: $\frac{EG}{AC} = \frac{FG}{AF} = \frac{2AB}{AB+BC} = \frac{2AB}{AB+\frac{1}{2}AB} = \frac{4}{3}$.

We have $3FG = 4AF$

Or $3(DG - DF) = 4(AD + DF)$

Substituting (2) into (4):

$$3(2AD - DF) = 4(AD + DF) \Rightarrow$$

$$6AD - 3DF = 4AD + 4DF \quad \Rightarrow \quad 2AD = 7DF$$

Therefore $\frac{AD}{DF} = \frac{7}{2}$.