Problem 7

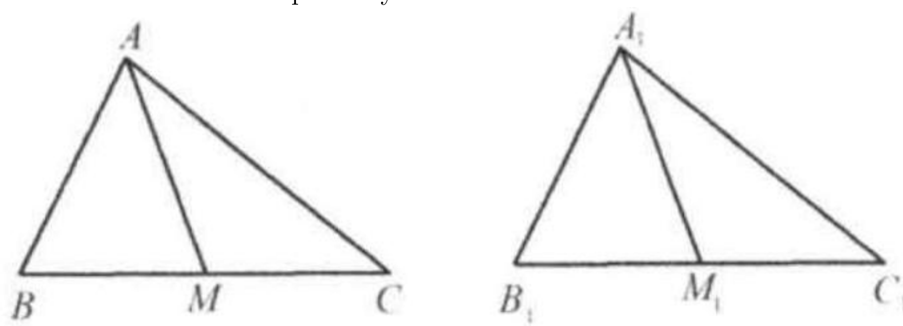
Problem

Show that if any two sides and the median on the third side of one triangle are equal to the corresponding sides and the median of the other triangle, the two triangles are congruent.

Solution

As shown in the figure below, in $\triangle ABC$ and $\triangle A_1B_1C_1, AB = A_1B_1, AC = A_1C_1$, and $AM = A_1M_1$.

Extend AM to N, A_1M_1.to N_1 such that $AM = MN, A_1M_1 = M_1N_1$, respectively.

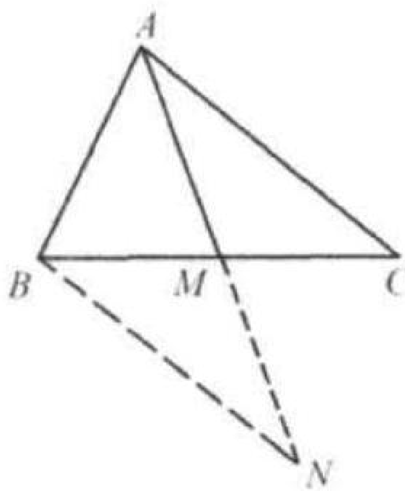


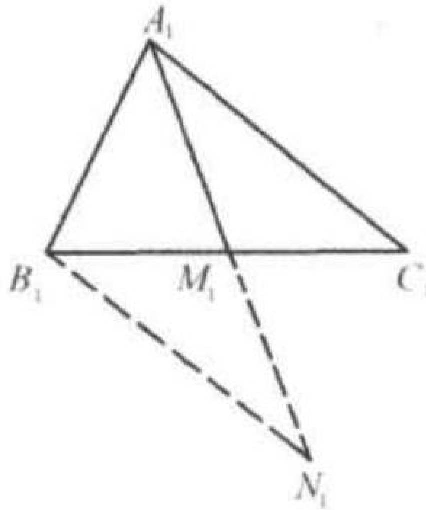
Connect BN and B_1N_1.

Since $BM = MC, AM = MN, \angle AMC = \angle NMB, \triangle AMC \cong \triangle NMB$.

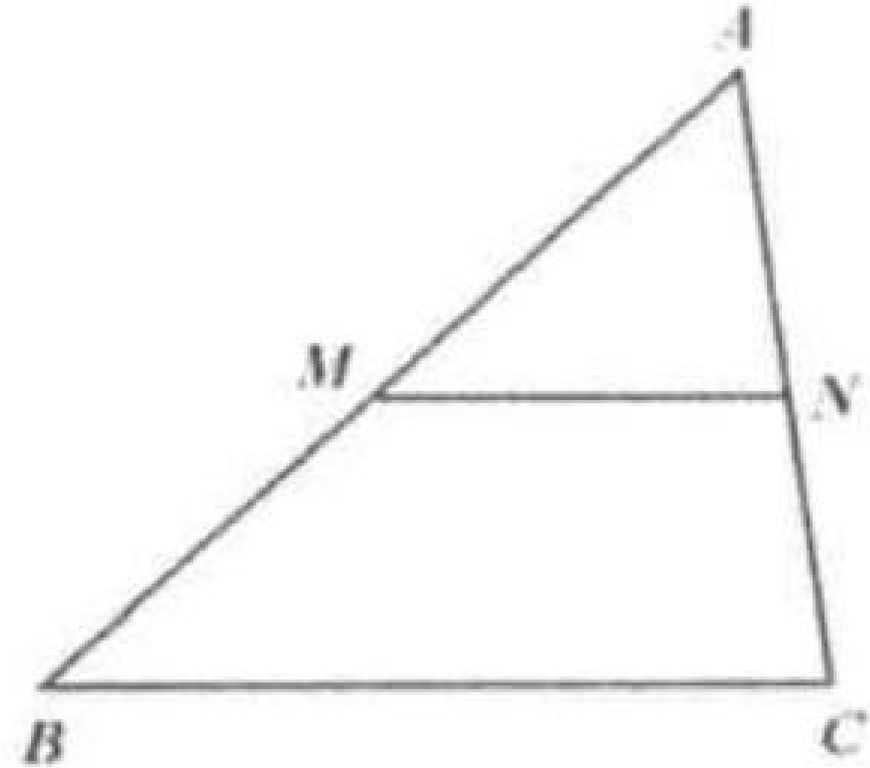
Similarly, $\Delta A_1M_1C_1 \cong \Delta N_1M_1B_1$.

Thus $BN = AC = A_1C_1 = B_1N_1$,





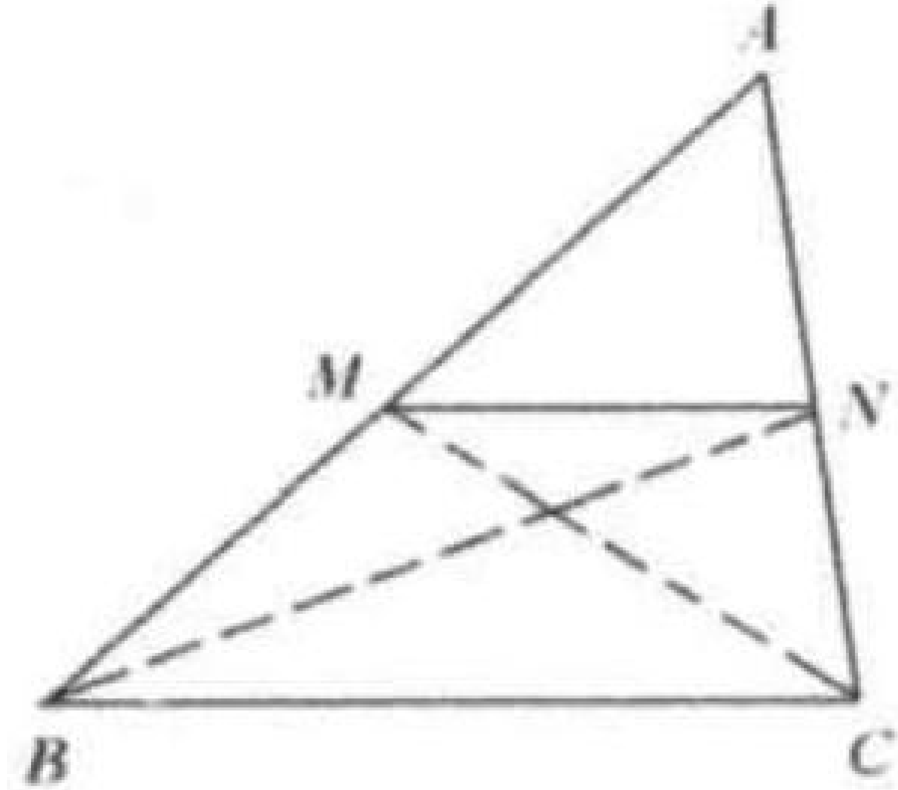
In $\triangle ABN$ and $\triangle A_1B_1N_1, AB = A_1B_1, BN = B_1N_1$, and $AN = A_1N_1$.
Thus $\triangle ABN \cong \triangle A_1B_1N_1$.
Draw a line connecting the midpoints of triangle or trapezoid. Theorem 2.1. The line segment whose endpoints are the midpoints of two sides of a triangle is parallel to the third side of the triangle and has a measure equal to one-half of the measure of the third side.
$MN//BC$.
$MN = \frac{1}{2}BC$



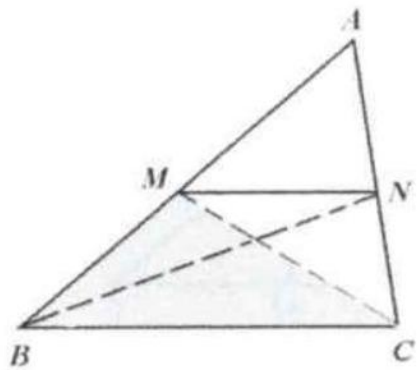
MN is called the midline of $\triangle ABC$.

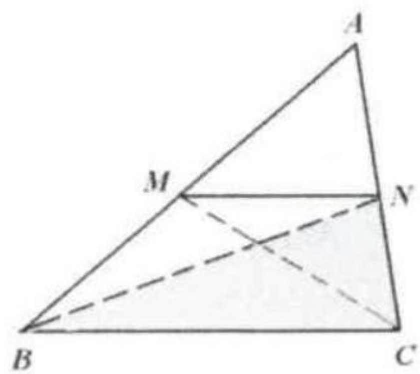
Proof: Let M, N be the midpoints of AB and AC, respectively. Connect BN and CM.

Since M is the midpoints of $AB, S_{\triangle CBM} = \frac{1}{2} S_{\triangle ABC}$.



Since N is the midpoints of $AC, S_{\triangle BCN} = \frac{1}{2}S_{\triangle ABC}$.



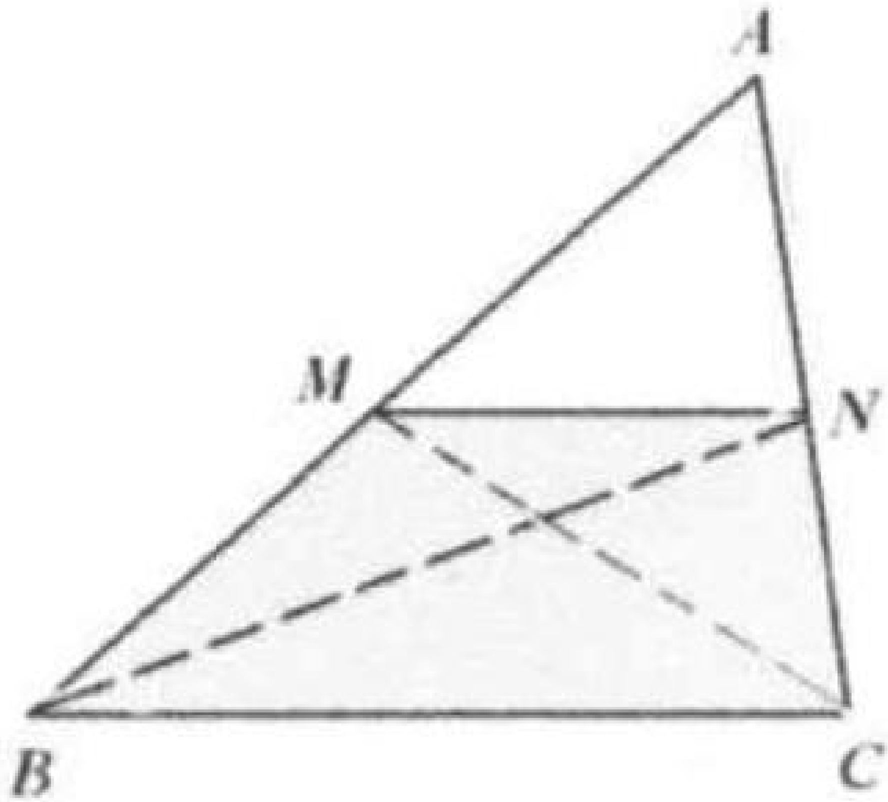


So $S_{\triangle CBM} = S_{\triangle BCN}$

Thus $MN//BC$.

$$\frac{S_{\triangle NMB}}{S_{\triangle BNC}} = \frac{\frac{1}{2}MN \times BN}{\frac{1}{2}BC \times BN} = \frac{MN}{BC}$$

Chapter 2 Draw the Auxiliary Lines with the Midlines



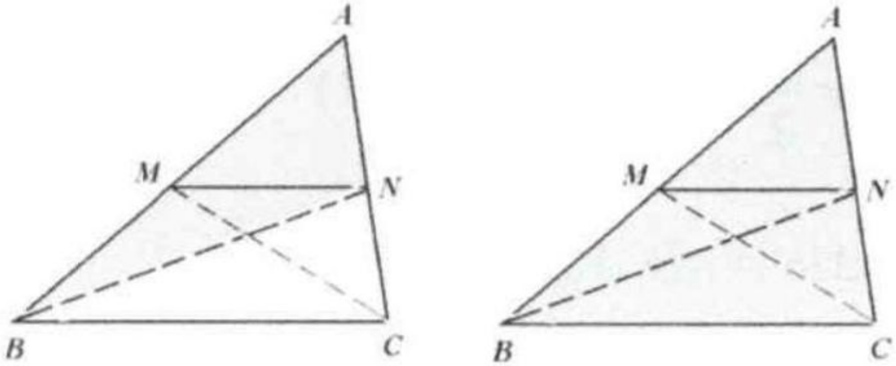
We see from the figures below that

$S_{\triangle NMB} = S_{\triangle AMN} = \frac{1}{4} S_{\triangle ABC}$

$S_{\triangle BNC} = S_{\triangle BNA} = \frac{1}{2} S_{\triangle ABC}$

Substituting (2) and (3) into (1):

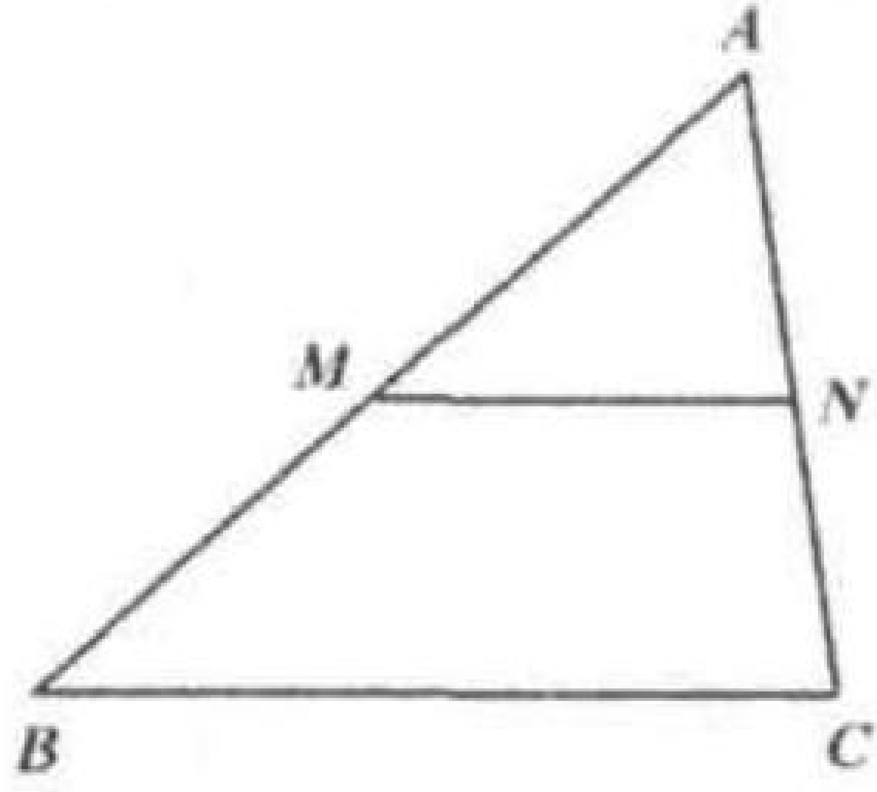
$\frac{\frac{1}{4} S_{\triangle ABC}}{\frac{1}{2} S_{\triangle ABC}} = \frac{MN}{BC} \quad \Rightarrow \quad \frac{1}{2} = \frac{MN}{BC} \quad \Rightarrow MN = \frac{1}{2} BC.$



Theorem 2.2. If a line contains the midpoint of one side of a triangle (AB) and is parallel to a second side (AC) of the triangle, then it will bisect the third side of the triangle.

$AN = NC$.

Proof: Let M be the midpoint of AB and $AMN//BC$.

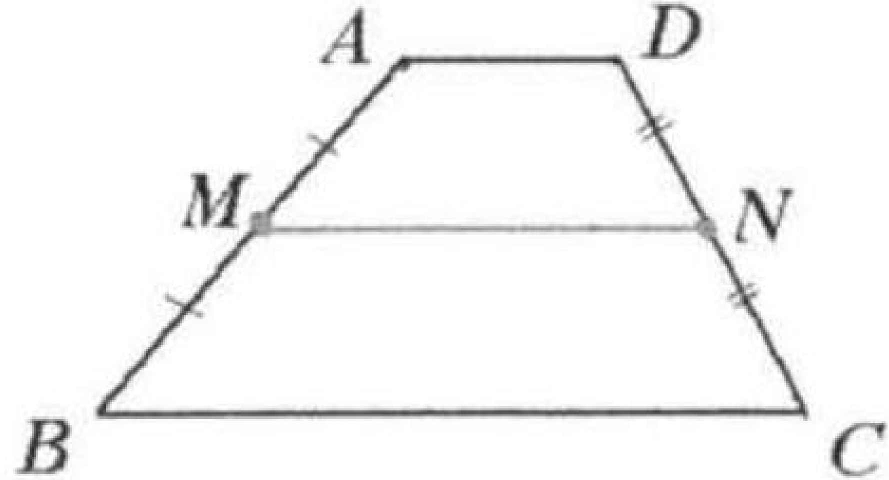


Connect BN and CM.

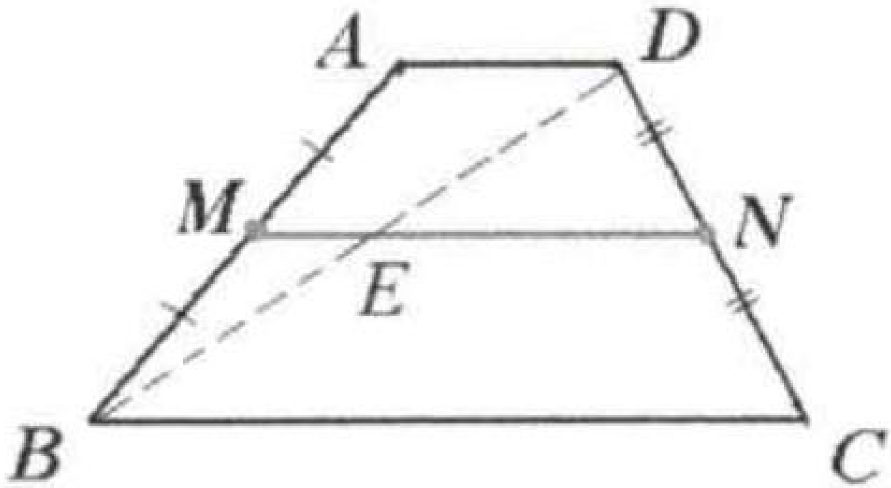
$\triangle AMN \sim \triangle ABC$.

Thus $\frac{AM}{AN} = \frac{MB}{NC} \quad \Rightarrow \quad \frac{\frac{1}{2}AB}{AN} = \frac{\frac{1}{2}AB}{NC} \Rightarrow \quad AN = NC.$

Theorem 2.3. For any trapezoid $ABCD$, the following relationship is true:
$MN = \frac{1}{2}(AD + BC)$
M and N are the midpoints of AB and BC, respectively. MN is the median of the trtapezoid.

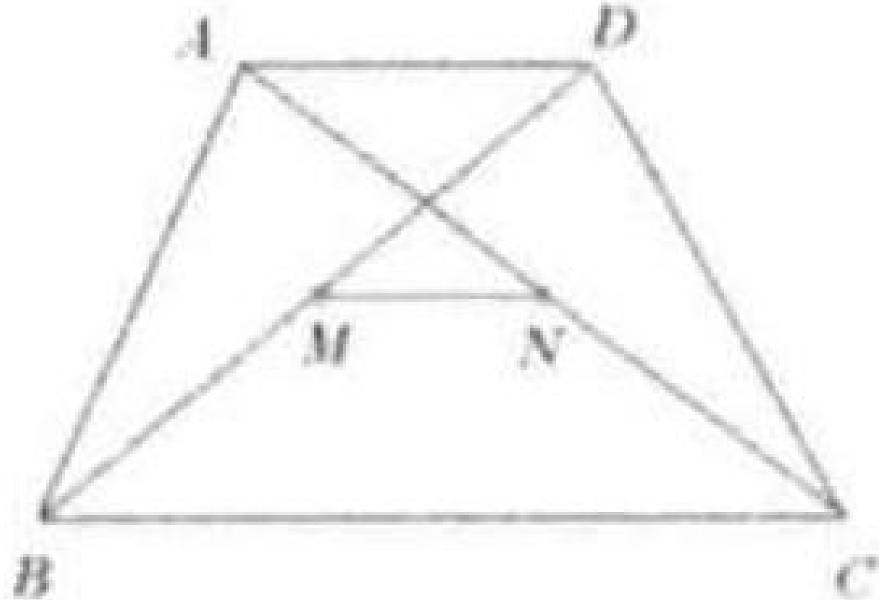


Proof: Connect DB and DB meets MN at E.
In triangle BCD, since $MN//BC, EN//BC$.
By Theorem 2.2, E is the midpoint of BD.

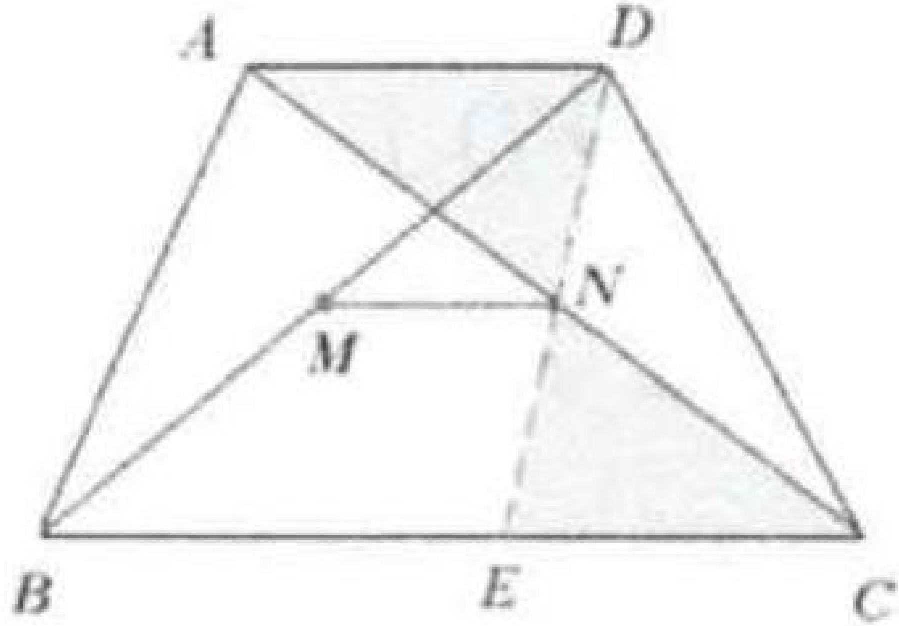


By Theorem 2.1, $EN = \frac{1}{2}BC$
In triangle ADB, since $MN//AD, ME//AD$.
By Theorem 2.2, E is the midpoint of BD.
By Theorem 2.1, $ME = \frac{1}{2}AD$
(2) +(1) : $MN = \frac{1}{2}(AD + BC)$
Theorem 2.4. For any trapezoid $ABCD$, the following relationship is true:

$MN = \frac{1}{2}(BC - AD)$



M and N are the midpoints of the diagonals AC and BD, respectively.
Proof: Connect DN and extend DN to meet BC at E.
Since $AD//BC, \angle DAN = \angle ECN, \angle ADN = \angle CEN$, and $AN = NC$,
Thus $\triangle ADN \cong \triangle ECN, DN = NE, AD = CE$.



We also know that $DM = MB$.
By Theorem 2.1, we have $MN//BE$ and
$MN = \frac{1}{2}BE = \frac{1}{2}(BC - CE) = \frac{1}{2}(BC - AD)$