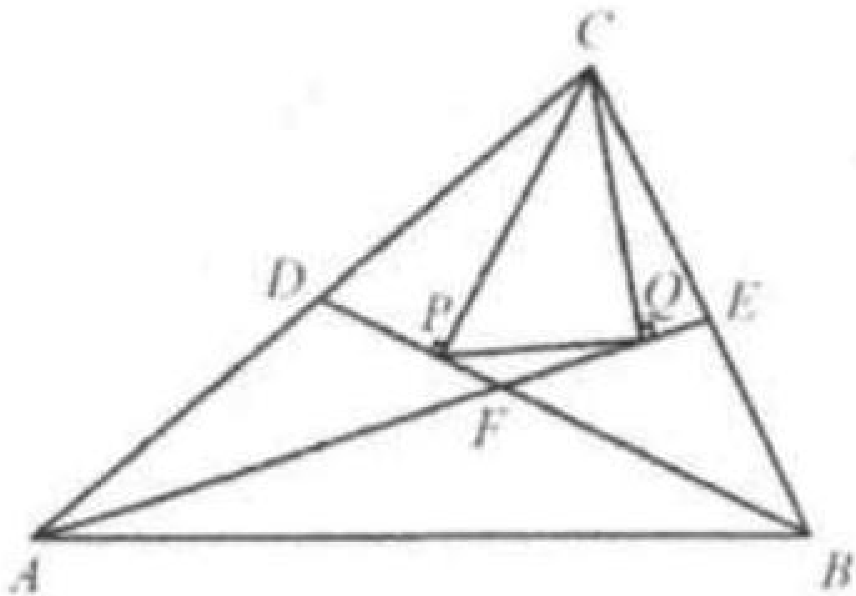
Example 7

Given any $\triangle ABC, AE$ bisects $\angle BAC, BD$ bisects $\angle ABC$, $CP \perp BD$, and $CQ \perp AE$, prove that PQ is parallel to AB.

Solution: Method 1:

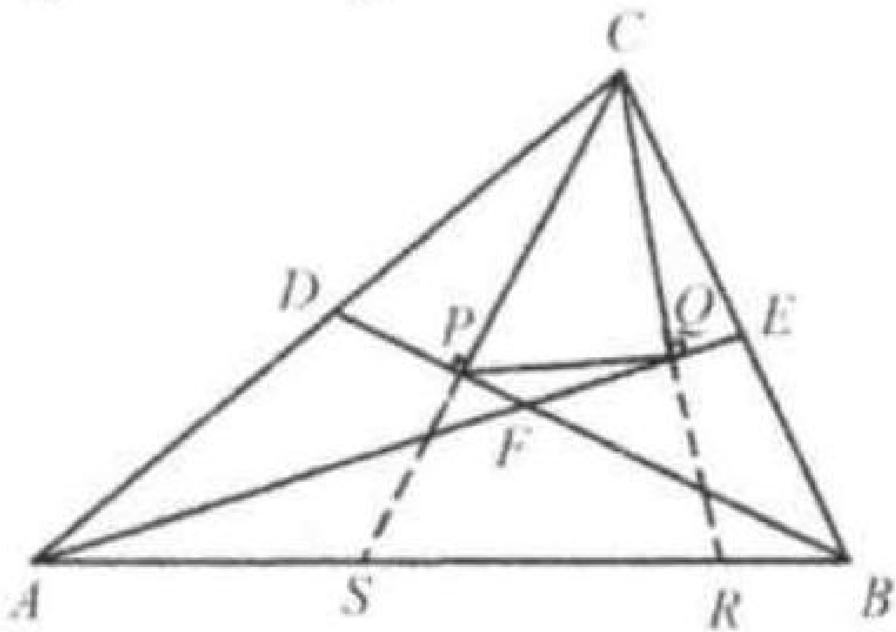


Extend CP and CQ to meet AB at S and R, respectively.

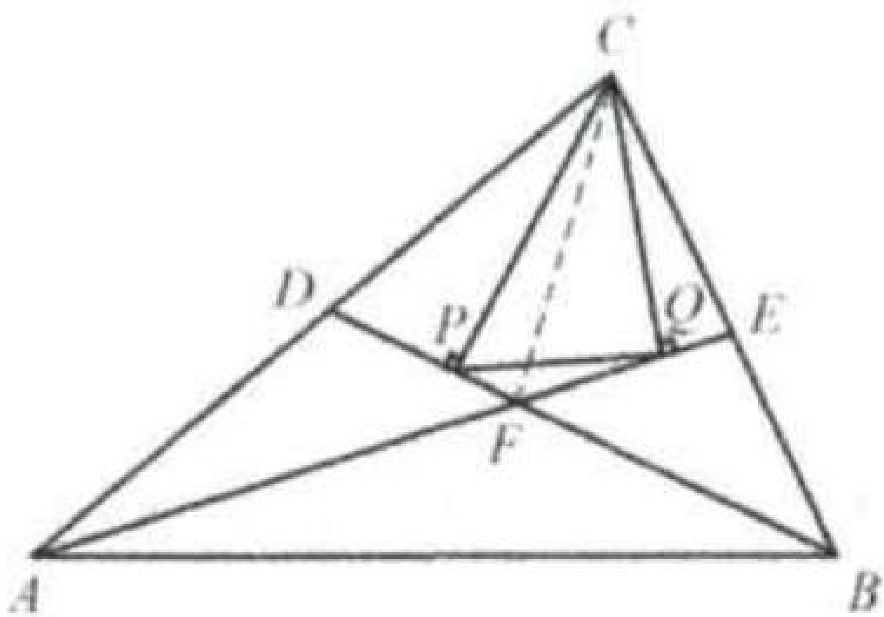
Since BP is the perpendicular bisector of CS, and AQ is the perpendicular bisector of CR, it shows that $\triangle CPB \cong \triangle SPB$, and $\triangle CQA \cong \triangle RQA$, respectively.

It then follows that $CP = SP$ and $CQ = RQ$ or P and Q are midpoints of CS and CR, respectively. Therefore, in $\triangle CSR, PQ//SR$. Thus, $PQ//AB$.

Method 2:



Connect CF. We know that CF bisects $\angle A$.
Since $\angle CPF = \angle CQF = 90^\circ$, CPFQ are concyclic.
Thus $\angle PCF = \angle PQF = \frac{\angle C}{2} - \angle PCD$
$= \frac{\angle C}{2} - (\angle C - \angle PCE) = \frac{\angle C}{2} - \left[\angle C - \left(90^\circ - \frac{\angle B}{2}\right)\right]$



$$= 90^\circ - \frac{\angle C}{2} - \frac{\angle B}{2} = \frac{\angle B}{2} = \frac{180^\circ - \angle C - \angle B}{2} = \frac{\angle A}{2} = \angle EAB.$$

Thus, $PQ//AB$.