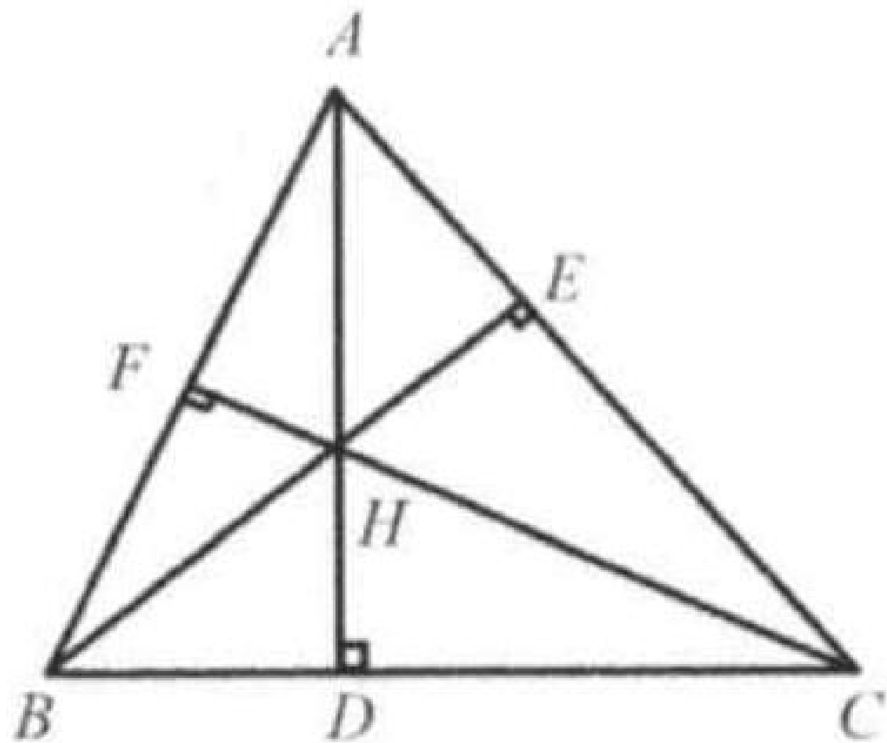
Problem 7

Problem

(1994 China Middle School Math Contest) $\triangle ABC$ is an acute triangle. Three altitudes AD, BE, CF meet at point $H.C$. If $BC = a, AC = b, AB = c$, then the value of $AH \cdot AD + BH \cdot BE + CH \cdot CF$ is

(A) $\frac{1}{2}(ab + bc + ca)$
(B) $\frac{1}{2}\left(a^2 + b^2 + c^2\right)$
(C) $\frac{3}{2}(ab + bc + ca)$
(D) $\frac{3}{2}\left(a^2 + b^2 + c^2\right)$



Solution

(B).

We know that points H, D, C, and E are concyclic (Figure 1).

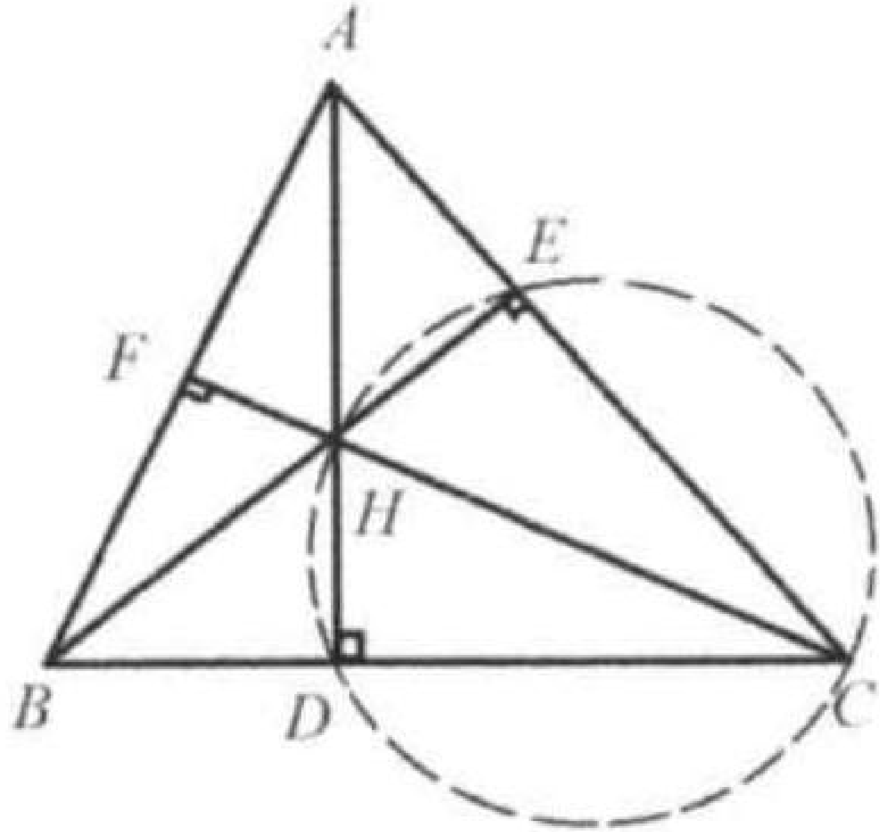


Figure 1

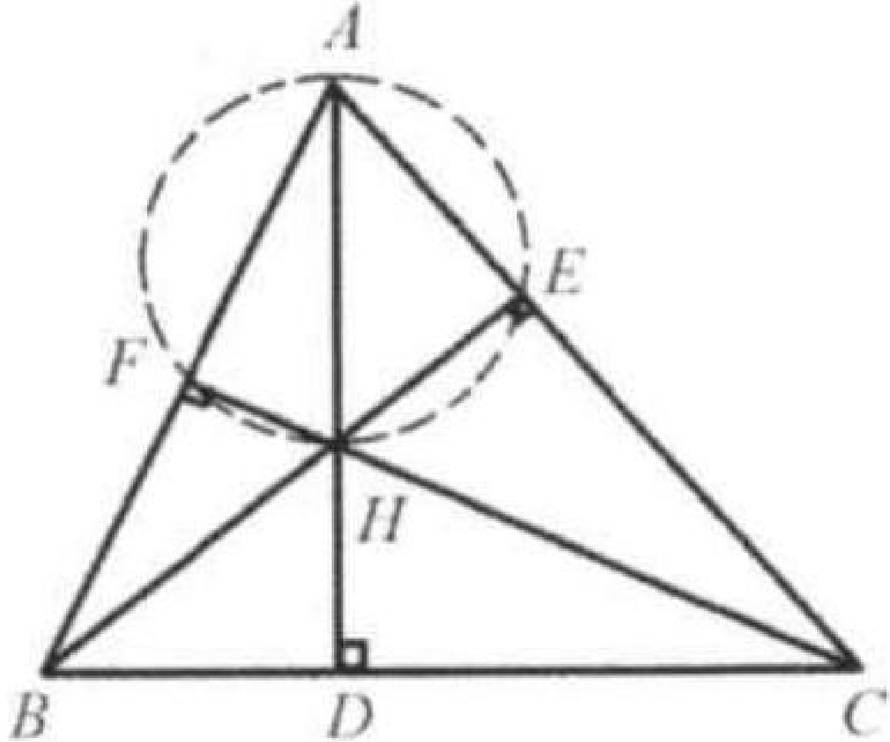


Figure 2

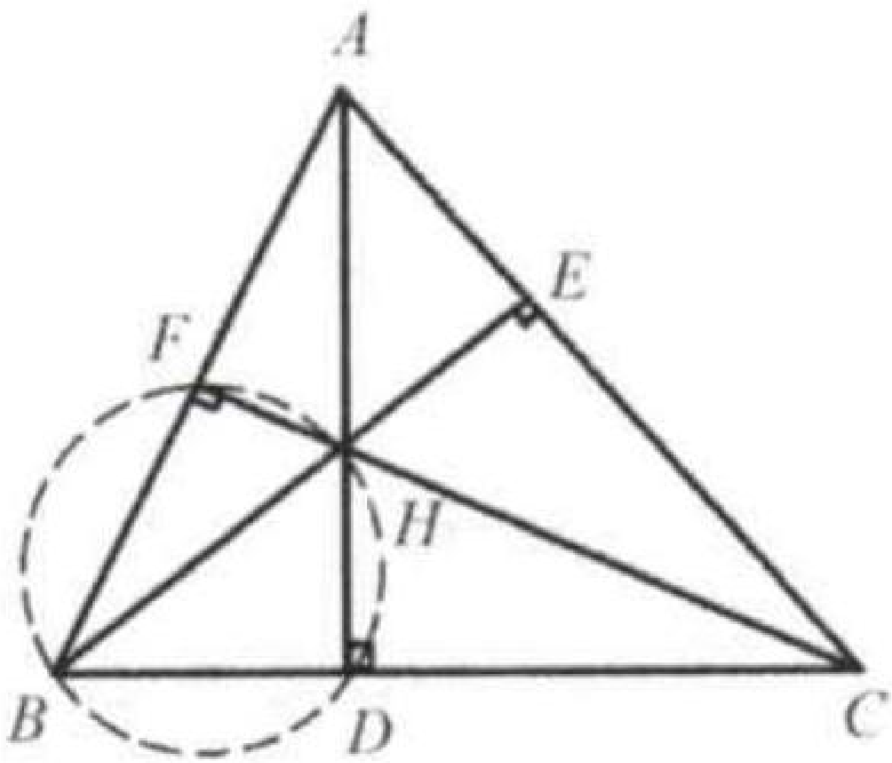


Figure 3

So we have

$$AH \cdot AD = AE \cdot AC = AC \cdot AB \cdot \cos \angle BAE$$

By the law of cosine, $\cos \angle BAE = \frac{AB^2+AC^2-BC^2}{2AB \cdot AC}$. So

$$AH \cdot AD = \frac{1}{2}\left(AC^2 + AB^2 - BC^2\right) = \frac{1}{2}\left(b^2 + c^2 - a^2\right)$$

Similarly we have:

$$BH \cdot BE = BF \cdot BA = BA \cdot BC \cdot \cos \angle CBF$$
$$= \frac{1}{2}\left(AB^2 + BC^2 - AC^2\right) = \frac{1}{2}\left(c^2 + a^2 - b^2\right)$$
$$CH \cdot CF = CD \cdot CB = CB \cdot AC \cdot \cos \angle ACD$$
$$= \frac{1}{2}\left(AC^2 + BC^2 - AB^2\right) = \frac{1}{2}\left(b^2 + a^2 - c^2\right)$$
$$(1) + (2) + (3) : AH \cdot AD + BH \cdot BE + CH \cdot CF = \frac{1}{2}\left(a^2 + b^2 + c^2\right).$$

The answer is (B).