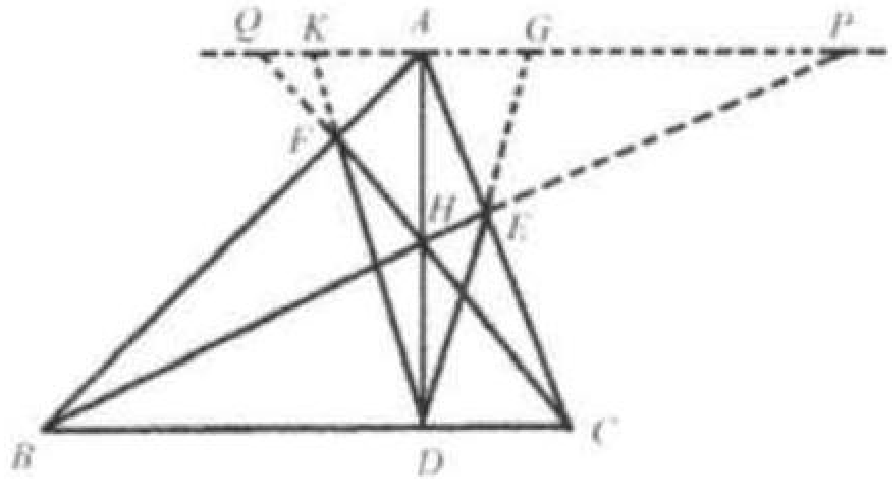
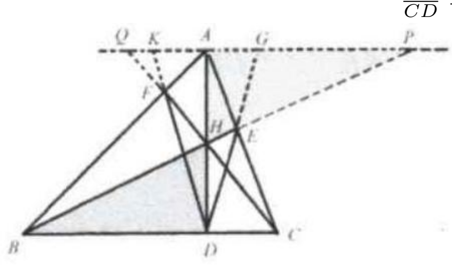
Example 21

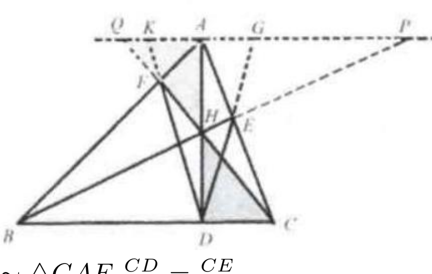
(1985 Yangzhou Math Contest, 1994 Canadian Mathematical Olympiad) Let ABC be an acute angled triangle. Let AD be the altitude on BC, and let H be any interior point on AD. Lines BH and CH, when extended, intersect AC and AB at E and F, respectively. Prove that $\angle EDH = \angle FDH$.

Solution: From A draw a line parallel to BC. Extend CH, DF, BE, and DE to meet the line at Q, K, P, and G, respectively.

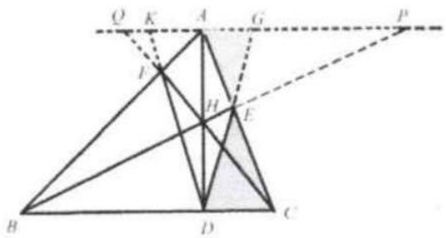


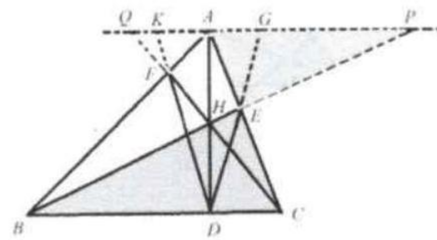
We know that $\triangle BDH \sim \triangle PAH . \frac{BD}{AP} = \frac{DH}{AH}$
We know that $\triangle CDH \sim \triangle QAH . \frac{CD}{AQ} = \frac{DH}{AH}$
From (1) and (2), we have: $\frac{BD}{AP} = \frac{CD}{AQ}$ or
$\frac{BD}{CD} = \frac{AP}{AQ}$





We know that $\triangle DCE \sim \triangle GAE . \frac{CD}{AG} = \frac{CE}{AE}$
We know that $\triangle BCE \sim \triangle PAE \cdot \frac{BC}{AP} = \frac{CE}{AE}$

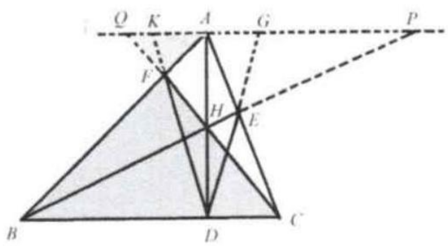


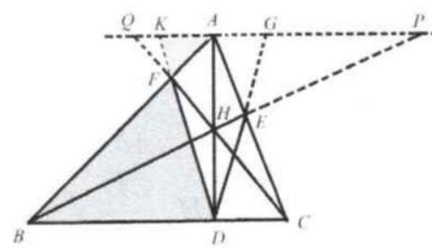


From (4) and (5), we have: $\frac{CD}{AG} = \frac{BC}{AP}$ or $\frac{CD}{BC} = \frac{AG}{AP}$

We know that $\triangle BCF \sim \triangle AQF. \frac{BC}{AQ} = \frac{BF}{AF}$

We know that $\triangle BDF \sim \triangle AKF. \frac{BD}{AK} = \frac{BF}{AF}$





From (7) and (8), we have: $\frac{BC}{AQ} = \frac{BD}{AK}$ or $\frac{BC}{BD} = \frac{AQ}{AK}$

$(3) \times (6) \times (9) : \frac{AG}{AK} = 1 \quad \Rightarrow \quad AG = AK.$

Since AD is the altitude, $\triangle ADK$ and $\triangle ADG$ are right triangles.

$\triangle ADK \cong \triangle ADG (AG = AK. \angle DAK = \angle DAG, AD = AD)$

Thus $\angle EDH = \angle FDH$.