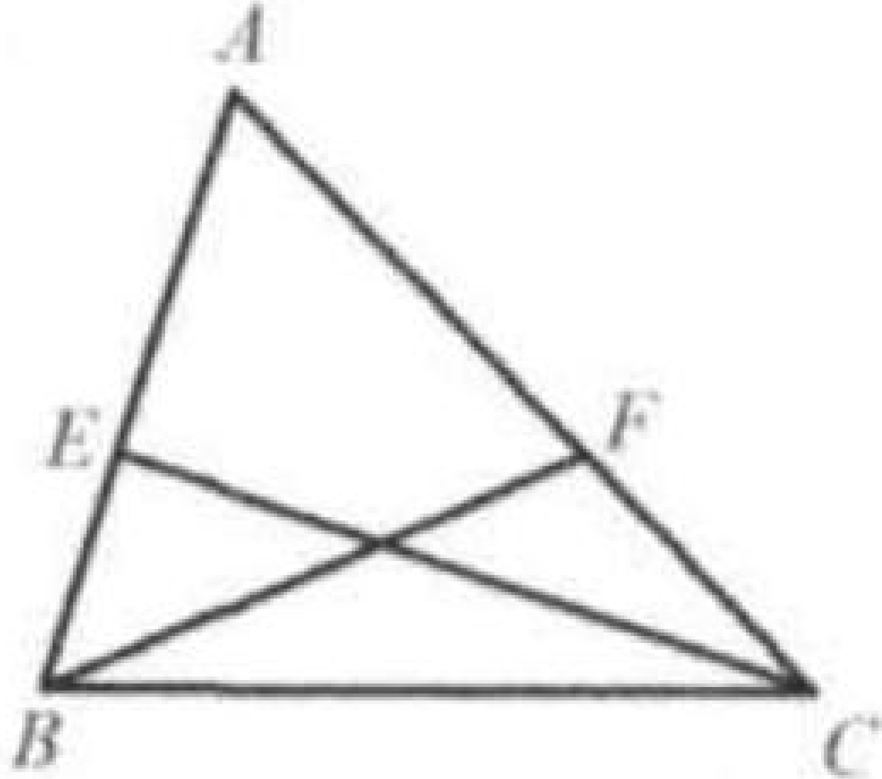
Example 18

In ABC, E is a point on AB and F is a point on AC such that $\angle FBC = \angle ECB = \frac{1}{2}\angle A$. Prove: $BE = CF$.

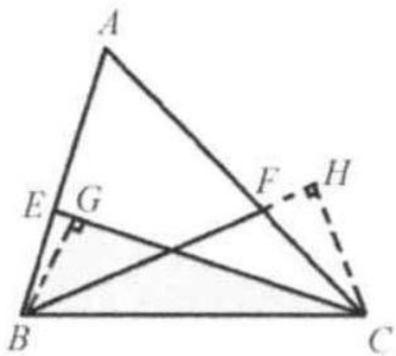
Proof: Draw $BG \perp CE$ to meet CE at G.

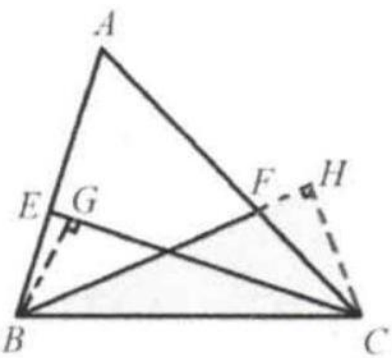


Draw $CH \perp BF$ to meet the extension of BF at H.

$\triangle BGC \cong \triangle CHB\,(BC = BC, \angle BGC = \angle CHB = 90°, \angle HBC = \angle GCB)$,

then $BG =$ CH.





$$\angle BEG = \angle A + \angle ACE = \angle FBC + \angle ACE + \angle ECB$$

$$\angle CFH = \angle AFB = \angle FBC + \angle ACE + \angle ECB$$

Thus $\angle BEG = \angle CFH$.

So we get

$$\triangle BEG \cong \triangle CFH \, (\angle BEG = \angle CFH, \angle BGE = \angle CHF = 90^\circ, BG = CH),$$

and $BE = CF$.