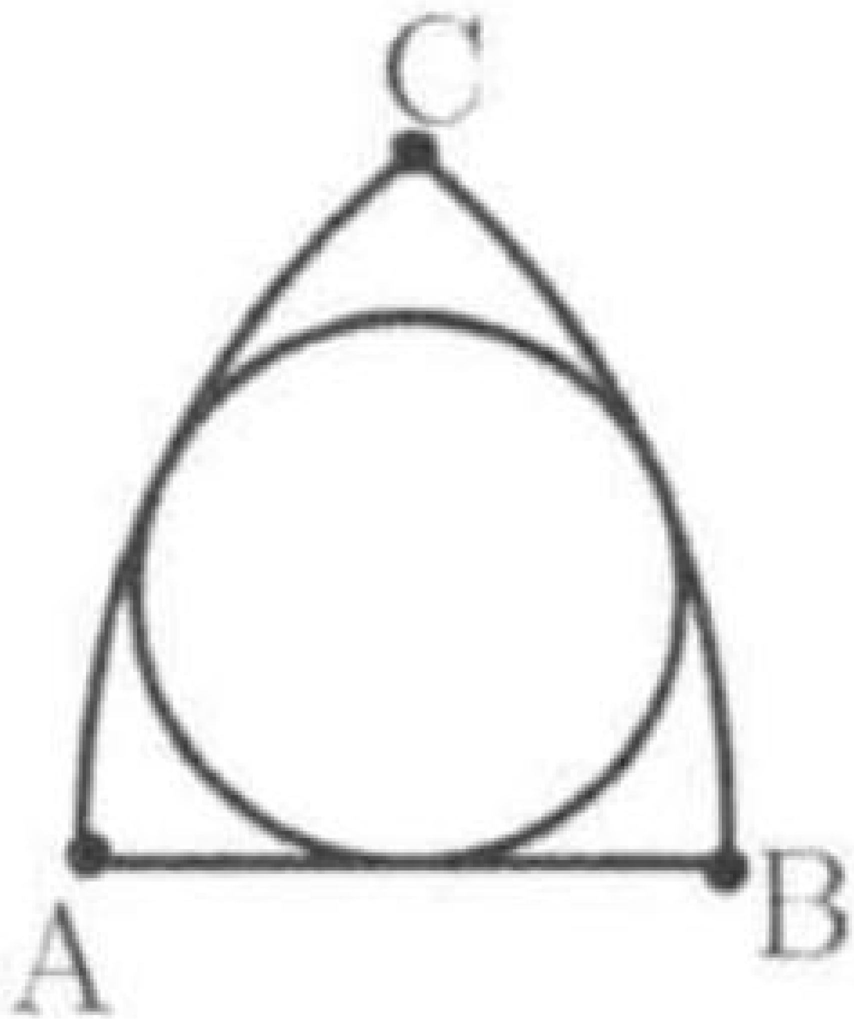
Problem

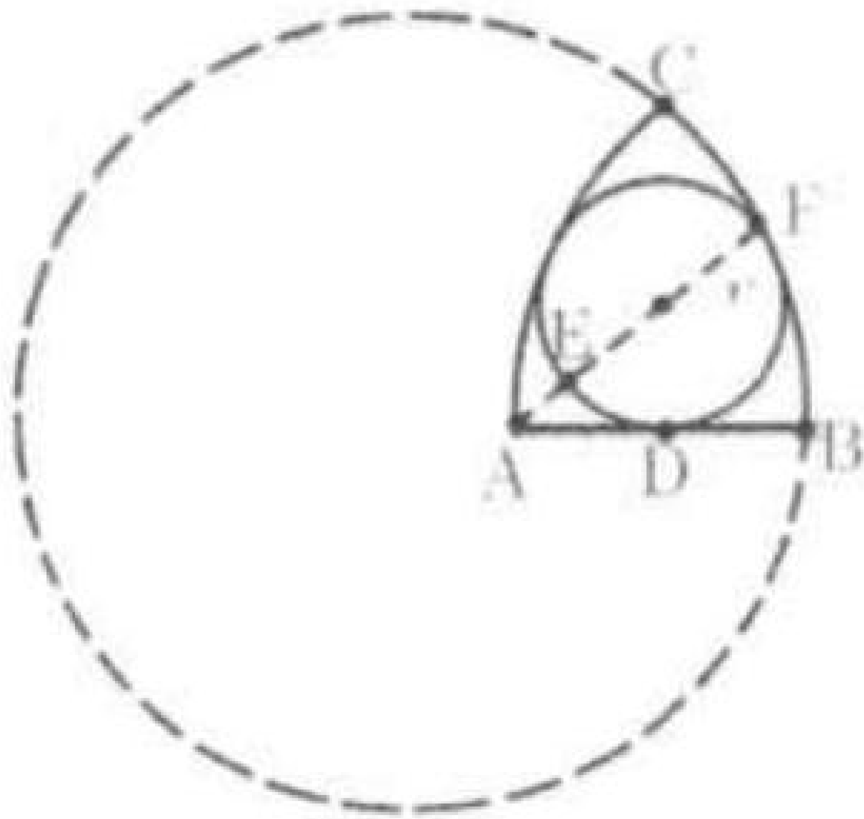
(AMC) If circular arcs AC and BC have centers at B and A, respectively, then there exists a circle tangent to both arcs AC and BC, and to AB. If the length of arc BC is 12 , then the circumference of the circle is

(A) 24
(B) 25
(C) 26
(D) 27
(E) 28



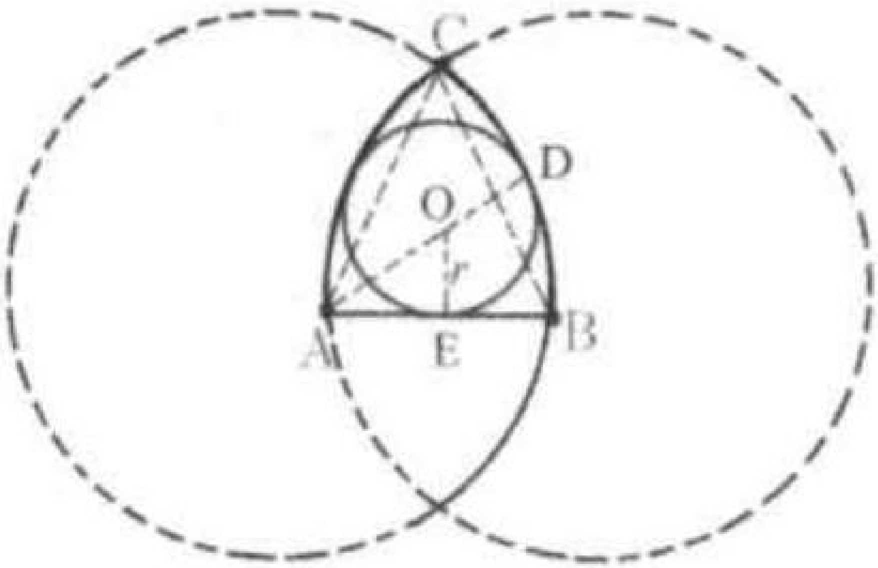
Solution

(D). Method 1 (official solution):
Construct the circle with center A and radius AB. Let F be the point of tangency of the two circles. Draw AF, and let E be the point of intersection of AF and the given circle.
By the Power of a Point Theorem, $AD^2 = AF \cdot AE$. Let r be the radius of the smaller circle. Since AF and AB are radii of the larger circle, $AF = AB$ and $AE = AF - EF = AB - 2r$. Because $AD = AB/2$, substitution into the first equation yields $(AB/2)^2 = AB \cdot (AB - 2r)$; or, equivalently, $r/AB = 3/8$.
Points A, B, and C



are equidistant from each other, so arc $BC = 60^\circ$ and thus the circumference of the larger circle is $6 \cdot$ (length of arc BC) $= 6 \cdot 12$.
Let c be the circumference of the smaller circle. Since the circumferences of the two circles are in the same ratio as their radii, $c/72 = r/AB = 3/8$.
Therefore $c = (3/8) \cdot 72 = 27$.
Method 2 (our solution):
Construct the circles with centers A and B with radius AB. Triangle ABC is an equilateral triangle.

For circle A, we have $\frac{2\pi r}{360} = \frac{AB}{60} \Rightarrow AB = \frac{36}{\pi}$.
Extend AO to $D. AD = AB$. Draw $OE \perp AB$ at E. $AE = 6$.
Applying Pythagorean Theorem to triangle AOE,



$$r^2 = AO^2 - AE^2 = (AD - r)^2 - \left(\tfrac{1}{2}AB\right)^2$$
$$= (AB - r)^2 - \left(\tfrac{1}{2}AB\right)^2$$

$r = \frac{3AB}{8} = \frac{3\times 36}{8\pi} = \frac{27}{2\pi}$. The circumference is $2\pi r = 2\pi \times \frac{27}{2\pi} = 27$.