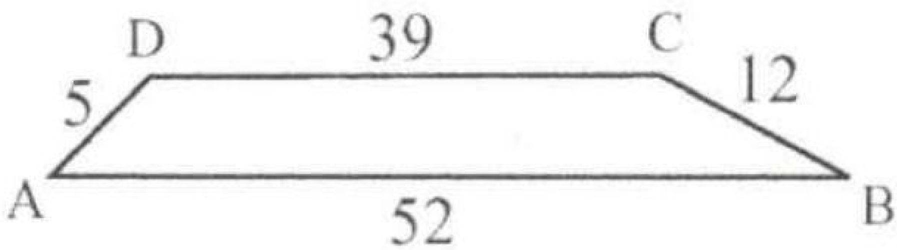
Example 5

(2002 AMC 10A Problem 25) In trapezoid $ABCD$ with bases AB and CD, we have $AB = 52, BC = 12, CD = 39$, and $DA = 5$. The area of $ABCD$ is

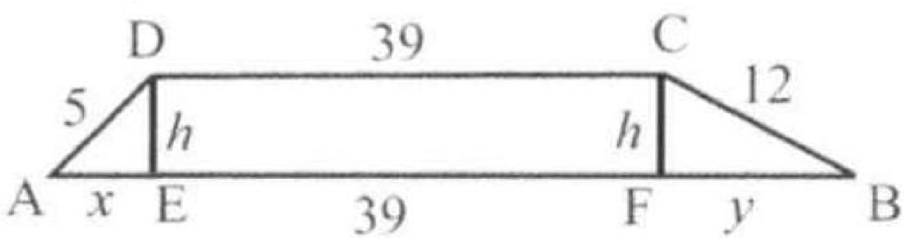
(A) 182
(B) 195
(C) 210
(D) 234
(E) 260



Solution: (C).

Method 1:

First drop perpendiculars from D and C to AB. Let E and F be the feet of the perpendiculars to AB from D and C, respectively, and let $h = DE = CF, x = AE$, and $y = FB$.



$25 = h^2 + x^2, 144 = h^2 + y^2$ and $13 = x + y$

So

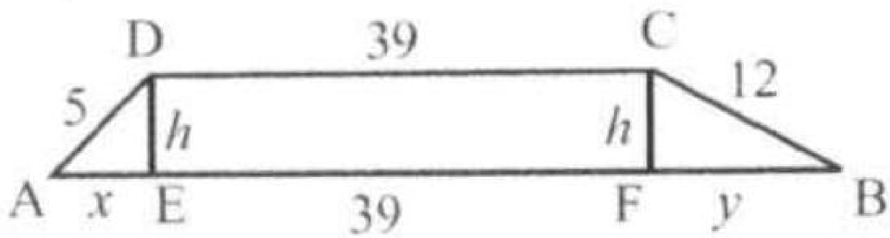
$144 = h^2 + y^2 = h^2 + (13 - x)^2 = h^2 + x^2 + 169 - 26x = 25 + 169 - 26x$, which gives $x = \frac{50}{26} = \frac{25}{13}$, and $h = \sqrt{5^2 - \left(\frac{25}{13}\right)^2} = 5\sqrt{1 - \frac{25}{169}} = 5\sqrt{\frac{144}{169}} = \frac{60}{13}$.

Hence Area of $ABCD = \frac{1}{2}(39 + 52) \times \frac{60}{13} = 210$.

Method 2:

First drop perpendiculars from D and C to AB. Let E and F be the feet of

the perpendiculars to AB from D and C, respectively, and let
$h = DE = CF, x = AE$, and $y = FB$.



By the Pythagorean Theorem, we have:
$h^2 = 5^2 - x^2 = 12^2 - y^2 \Rightarrow y^2 - x^2 = 12^2 - 5^2 = 119$
We know that $y + x = 52 - 39 = 13$.
Therefore (1) becomes
$(y - x)(y + x) = 119 \Rightarrow 13(y - x) = 119 \quad \Rightarrow y - x = \frac{119}{13}$
Solving the system of equations (2) and (3), we get $x = \frac{25}{13}$.
Therefore $h^2 = 5^2 - x^2 = 25 - \frac{5}{13} \quad \Rightarrow \quad h = \frac{60}{13}$.
The area of $ABCD$ is then $\frac{39+52}{2} \times \frac{60}{13} = 210$.