

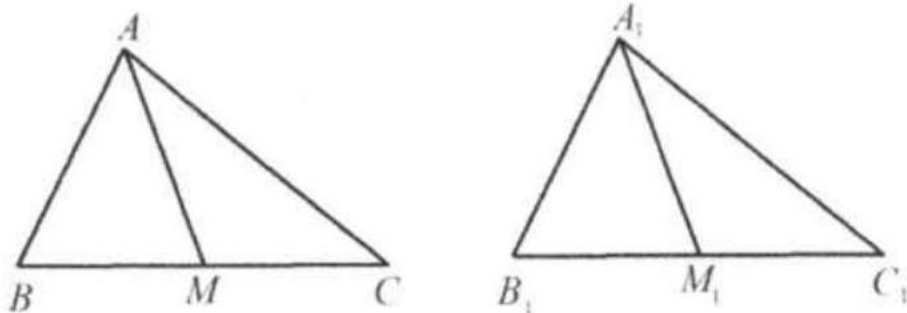
## Problem

Show that if any two sides and the median on the third side of one triangle are equal to the corresponding sides and the median of the other triangle, the two triangles are congruent.

## Solution

As shown in the figure below, in  $\triangle ABC$  and  $\triangle A_1B_1C_1$ ,  $AB = A_1B_1$ ,  $AC = A_1C_1$ , and  $AM = A_1M_1$ .

Extend  $AM$  to  $N$ ,  $A_1M_1$  to  $N_1$  such that  $AM = MN$ ,  $A_1M_1 = M_1N_1$ , respectively.

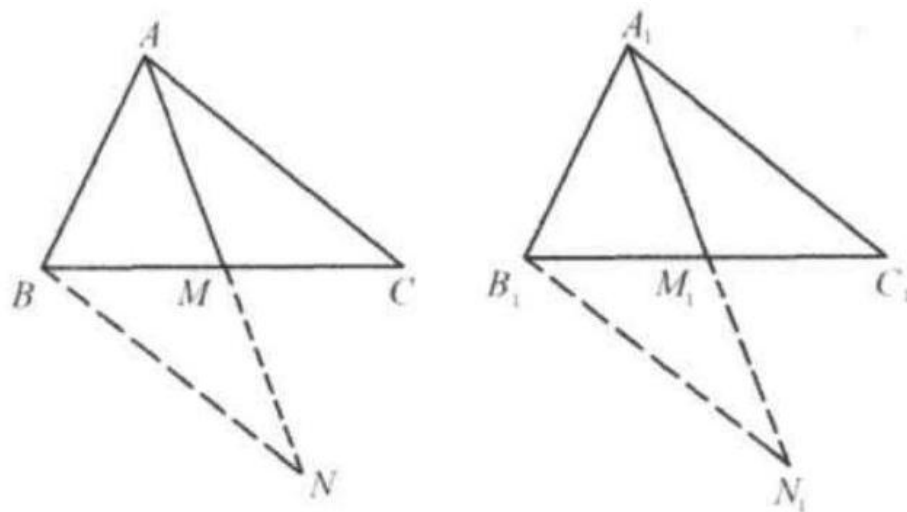


Connect  $BN$  and  $B_1N_1$ .

Since  $BM = MC$ ,  $AM = MN$ ,  $\angle AMC = \angle NMB$ ,  $\triangle AMC \cong \triangle NMB$ .

Similarly,  $\triangle A_1M_1C_1 \cong \triangle N_1M_1B_1$ .

Thus  $BN = AC = A_1C_1 = B_1N_1$ ,



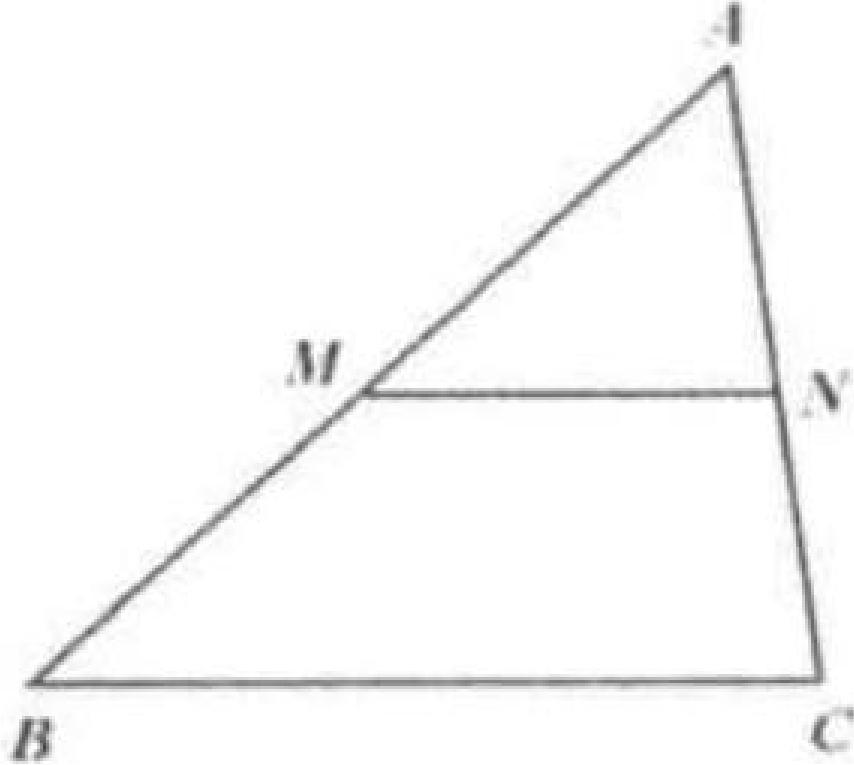
In  $\triangle ABN$  and  $\triangle A_1B_1N_1$ ,  $AB = A_1B_1$ ,  $BN = B_1N_1$ , and  $AN = A_1N_1$ .

Thus  $\triangle ABN \cong \triangle A_1B_1N_1$ .

Draw a line connecting the midpoints of triangle or trapezoid. Theorem 2.1.  
 The line segment whose endpoints are the midpoints of two sides of a triangle  
 is parallel to the third side of the triangle and has a measure equal to one-half  
 of the measure of the third side.

$$MN \parallel BC.$$

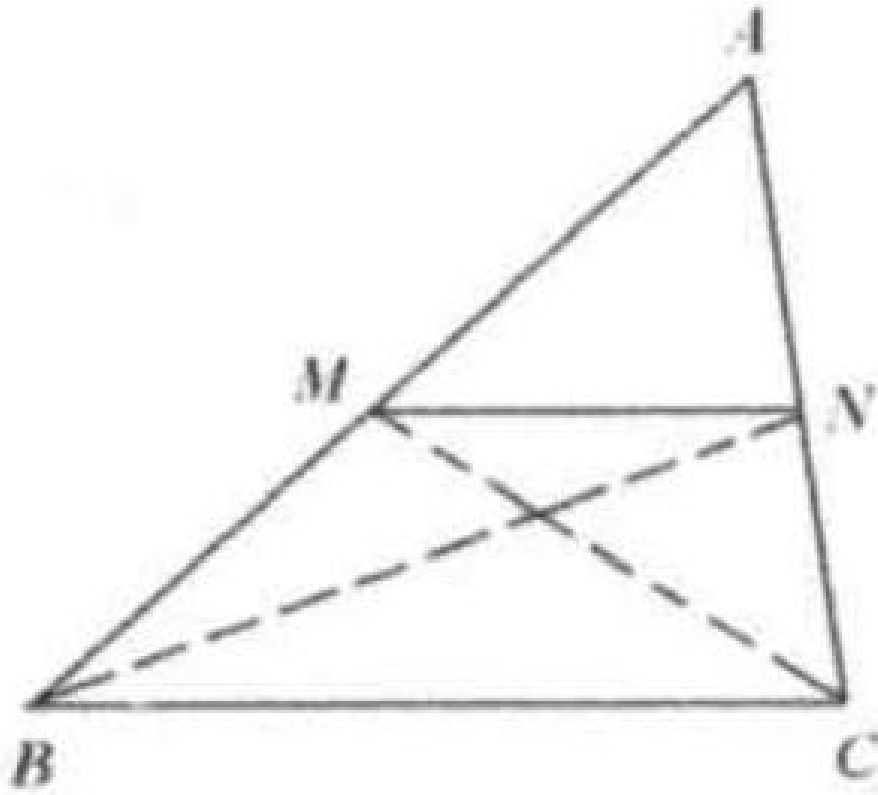
$$MN = \frac{1}{2}BC$$



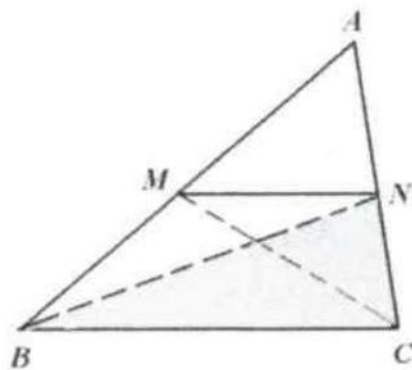
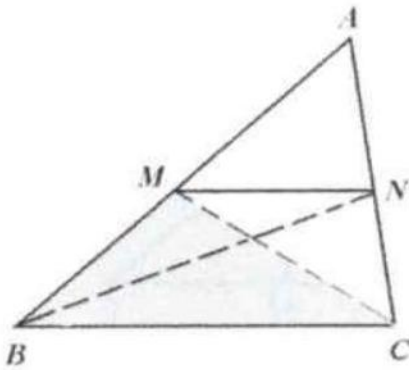
$MN$  is called the midline of  $\triangle ABC$ .

Proof: Let  $M, N$  be the midpoints of  $AB$  and  $AC$ , respectively. Connect  $BN$   
 and  $CM$ .

Since  $M$  is the midpoints of  $AB$ ,  $S_{\triangle CBM} = \frac{1}{2}S_{\triangle ABC}$ .



Since  $N$  is the midpoint of  $AC$ ,  $S_{\triangle BCN} = \frac{1}{2}S_{\triangle ABC}$ .

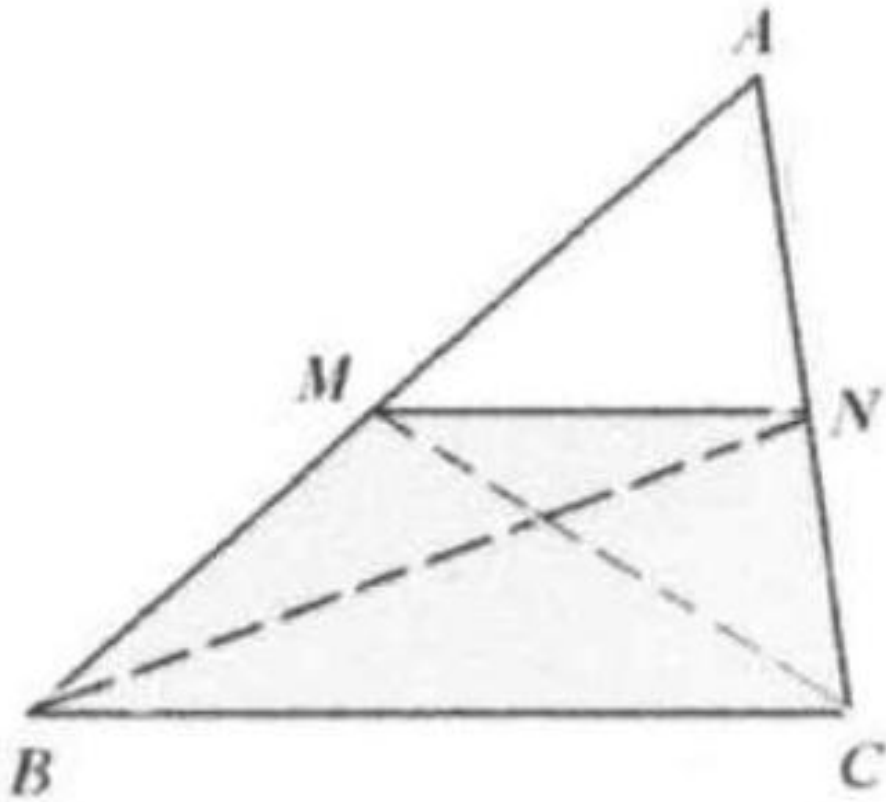


So  $S_{\triangle CBM} = S_{\triangle BCN}$

Thus  $MN \parallel BC$ .

$$\frac{S_{\triangle NMB}}{S_{\triangle BNC}} = \frac{\frac{1}{2}MN \times BN}{\frac{1}{2}BC \times BN} = \frac{MN}{BC}$$

## Chapter 2 Draw the Auxiliary Lines with the Midlines



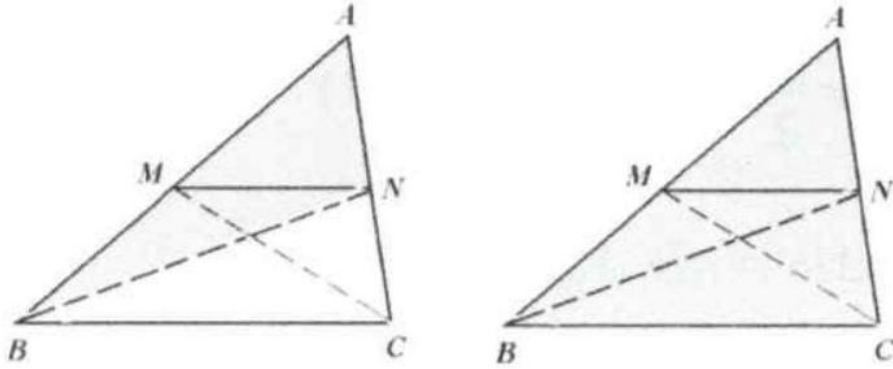
We see from the figures below that

$$S_{\triangle NMB} = S_{\triangle AMN} = \frac{1}{4}S_{\triangle ABC}$$

$$S_{\triangle BNC} = S_{\triangle BNA} = \frac{1}{2}S_{\triangle ABC}$$

Substituting (2) and (3) into (1):

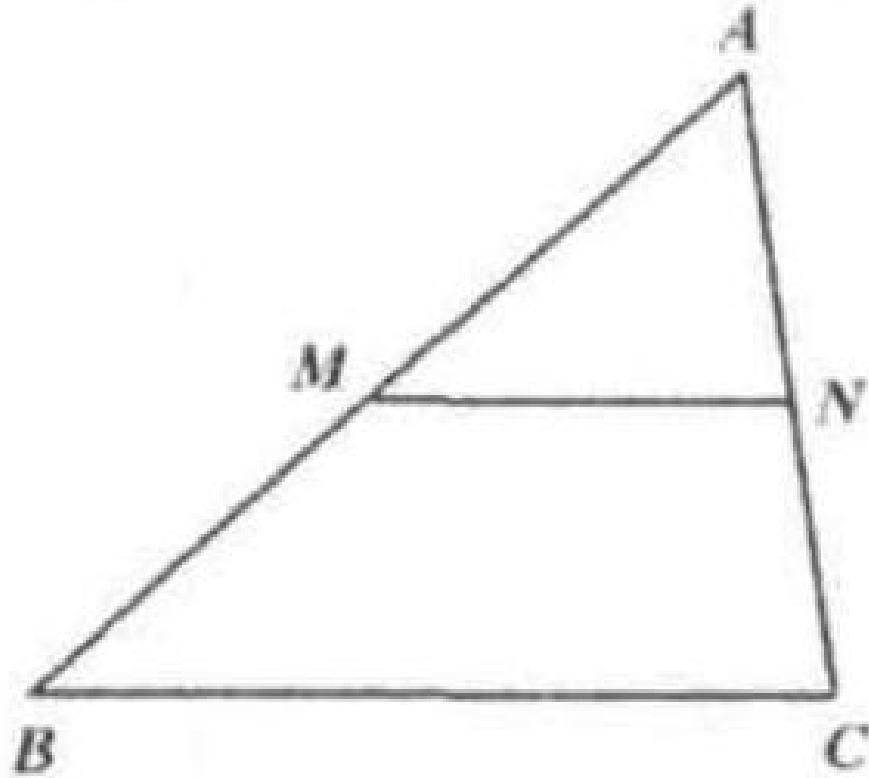
$$\frac{\frac{1}{4}S_{\triangle ABC}}{\frac{1}{2}S_{\triangle ABC}} = \frac{MN}{BC} \Rightarrow \frac{1}{2} = \frac{MN}{BC} \Rightarrow MN = \frac{1}{2}BC.$$



Theorem 2.2. If a line contains the midpoint of one side of a triangle ( $AB$ ) and is parallel to a second side ( $AC$ ) of the triangle, then it will bisect the third side of the triangle.

$$AN = NC.$$

Proof: Let  $M$  be the midpoint of  $AB$  and  $AMN \parallel BC$ .



Connect  $BN$  and  $CM$ .

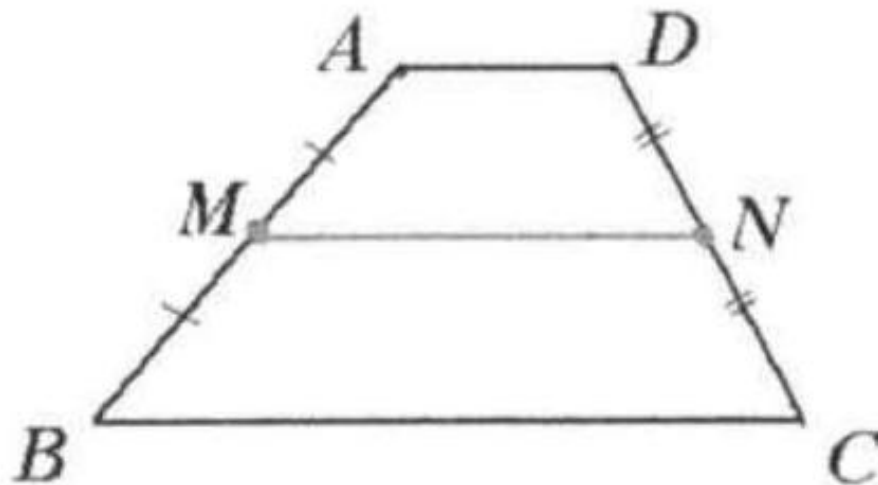
$$\triangle AMN \sim \triangle ABC.$$

$$\text{Thus } \frac{AM}{AN} = \frac{MB}{NC} \Rightarrow \frac{\frac{1}{2}AB}{AN} = \frac{\frac{1}{2}AB}{NC} \Rightarrow AN = NC.$$

Theorem 2.3. For any trapezoid  $ABCD$ , the following relationship is true:

$$MN = \frac{1}{2}(AD + BC)$$

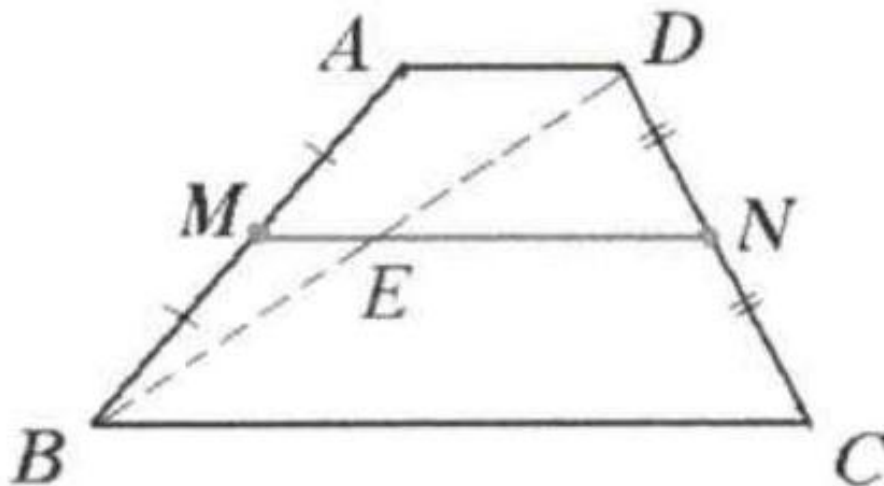
$M$  and  $N$  are the midpoints of  $AB$  and  $BC$ , respectively.  $MN$  is the median of the trapezoid.



Proof: Connect  $DB$  and  $DB$  meets  $MN$  at  $E$ .

In triangle  $BCD$ , since  $MN \parallel BC$ ,  $EN \parallel BC$ .

By Theorem 2.2,  $E$  is the midpoint of  $BD$ .



By Theorem 2.1,  $EN = \frac{1}{2}BC$

In triangle  $ADB$ , since  $MN \parallel AD$ ,  $ME \parallel AD$ .

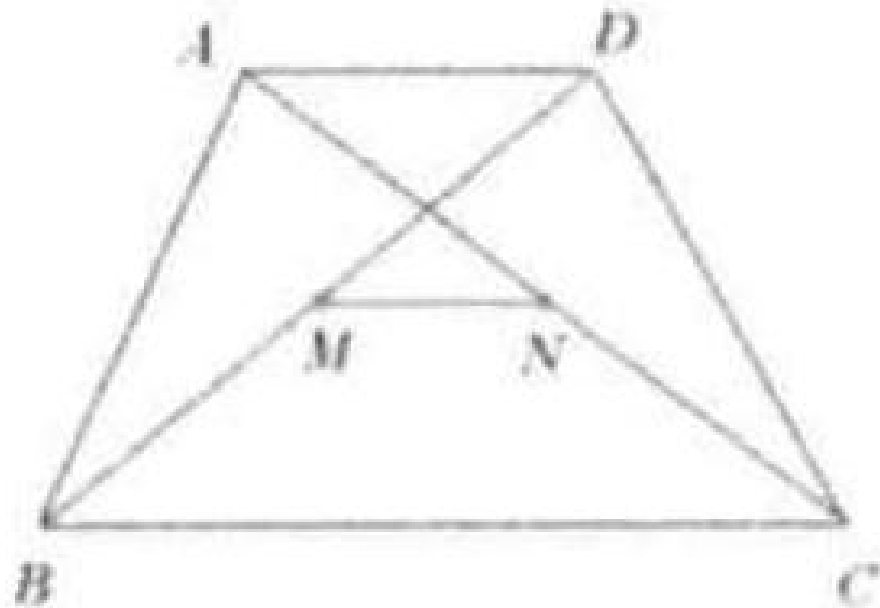
By Theorem 2.2,  $E$  is the midpoint of  $BD$ .

By Theorem 2.1,  $ME = \frac{1}{2}AD$

(2) + (1) :  $MN = \frac{1}{2}(AD + BC)$

Theorem 2.4. For any trapezoid  $ABCD$ , the following relationship is true:

$$MN = \frac{1}{2}(BC - AD)$$

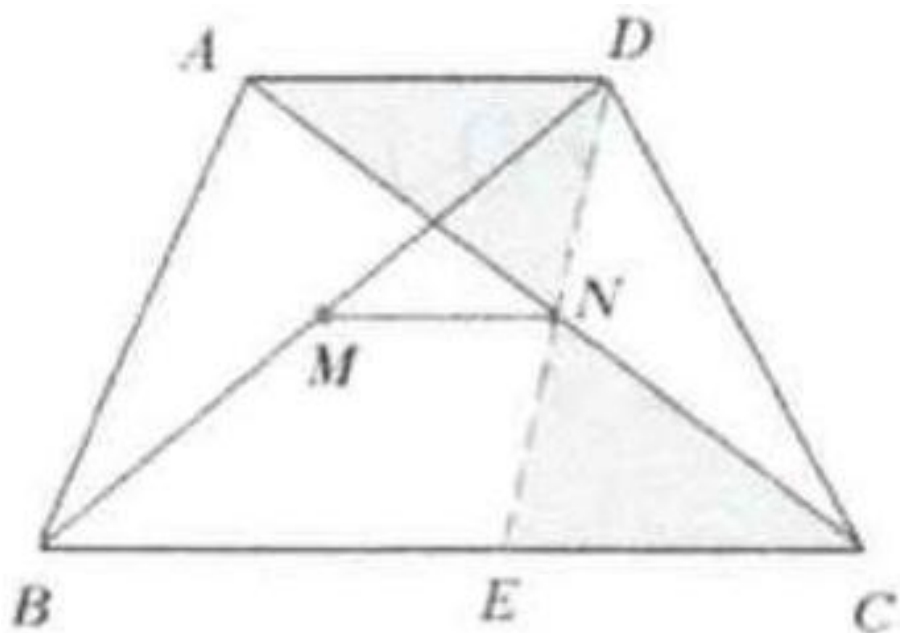


$M$  and  $N$  are the midpoints of the diagonals  $AC$  and  $BD$ , respectively.

Proof: Connect  $DN$  and extend  $DN$  to meet  $BC$  at  $E$ .

Since  $AD \parallel BC$ ,  $\angle DAN = \angle ECN$ ,  $\angle ADN = \angle CEN$ , and  $AN = NC$ ,

Thus  $\triangle ADN \cong \triangle ECN$ ,  $DN = NE$ ,  $AD = CE$ .



We also know that  $DM = MB$ .  
 By Theorem 2.1, we have  $MN \parallel BE$  and  
 $MN = \frac{1}{2}BE = \frac{1}{2}(BC - CE) = \frac{1}{2}(BC - AD)$