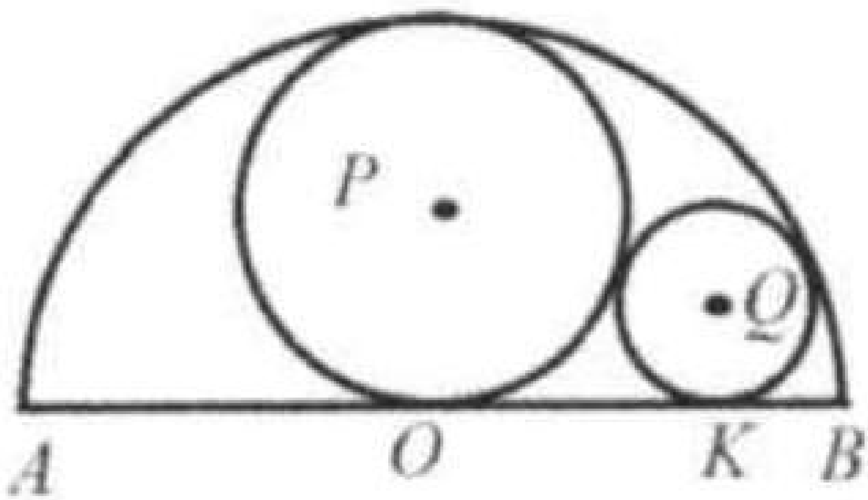
Problem 6

Problem

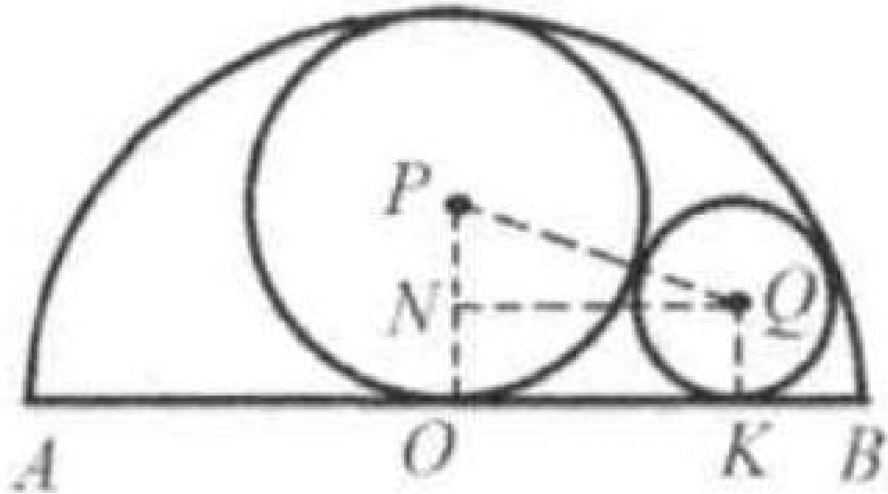
$AB = 2R$ is the diameter of the semicircle as shown in the figure. Two circles $P\,(r_1)$ and $Q\,(r_2)$ inscribed in the semicircle and are tangent to each other and to AB at O and K, respectively. Find the length of r_2.



Solution

Connect OP, KQ, PQ. Draw $QN//AB$ to meet PO at N.

$PO = \frac{1}{2}R, PQ = \frac{1}{2}R + r_2,$



Applying Pythagorean Theorem to $\triangle PON$:

$$\begin{gathered}OK^2 = PQ^2 - PN^2 \\ = \left(\tfrac{1}{2}R + r_2\right)^2 - (PO - NO)^2 \\ = \left(\tfrac{1}{2}R + r_2\right)^2 - \left(\tfrac{1}{2}R - r_2\right)^2 \\ = 2Rr_2.\end{gathered}$$

So $OK = \sqrt{2Rr_2}$.

We also know that $\frac{1}{AK} + \frac{1}{KB} = \frac{1}{QK} \Rightarrow \frac{1}{R+\sqrt{2Rr_2}} + \frac{1}{R-\sqrt{2Rr_2}} = \frac{1}{r_2}$

$$\Rightarrow \quad r_2 = \frac{1}{4}R.$$