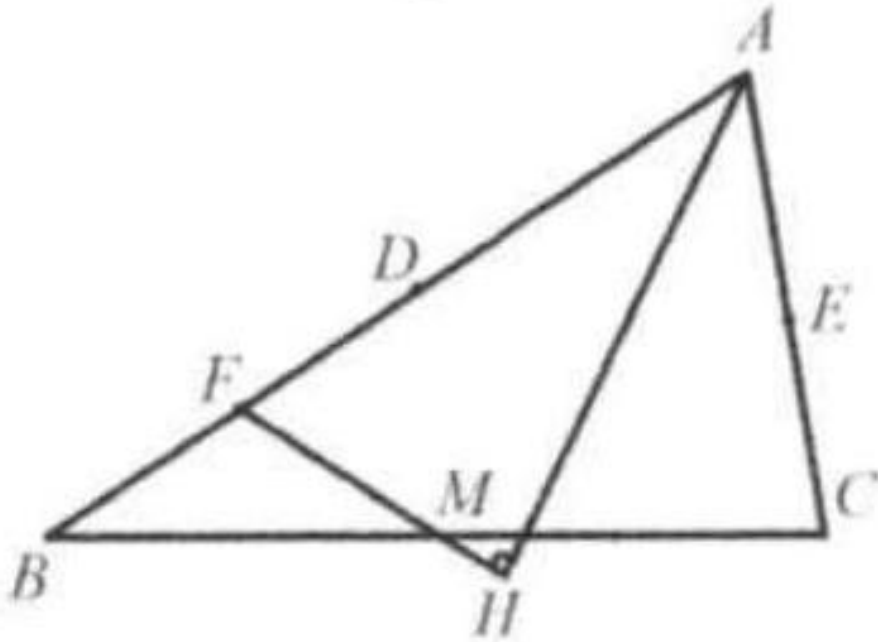# Problem 4

## Problem

As shown in the figure below, in $\triangle ABC, D, E$ are the midpoints of $AB, AC$, respectively. $AB > AC$. Take $F$, a point between $DB$ such that $DF = AE$. Draw $FH \perp AH, AH$ is the angle bisector of $\angle A$. Let $H$ be the feet of the perpendicular to $AH$ from $F$. FH meets $BC$ at $M$. Show that $BM = MC$.



## Solution

Extend $FH$ to meet the extension of $AC$ at $G$.
Draw $CK // AB$. $CK$ meets $FG$ at $K$.
Since $CK // AB, \angle B = \angle MCK$.

Since $FH \perp AH$, and $AH$ is the angle bisector of $\angle A$,
$$\angle AFH = \angle AGH, AF = AG.$$
We also know that $\angle BMF = \angle CMK$ (vertical angles).
$$AF = AG \quad \Rightarrow AD + DF = AE + EC + CG$$
$$\Rightarrow \quad (DF + BF) + DF = AE + AE + CG$$
Since $DF = AE$, (1) becomes:
$$AE + BF + AE = AE + AE + CG \quad \Rightarrow BF = CG.$$
Since $CK//AB, \angle AFH = \angle AGH = \angle GKC.$
Thus $CG = CK = BF.$
Therefore, $\triangle BFM \cong \triangle CKM$ and $BM = MC.$