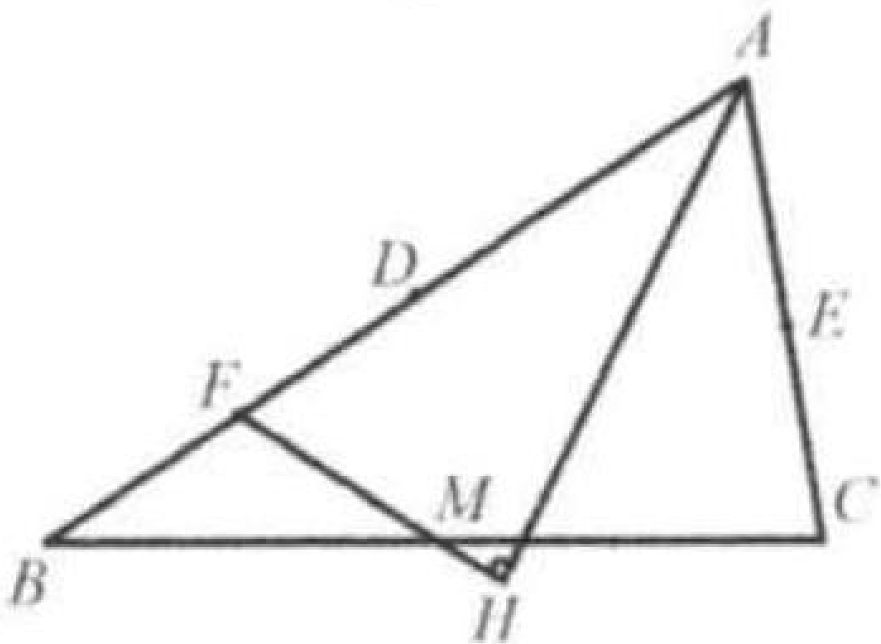
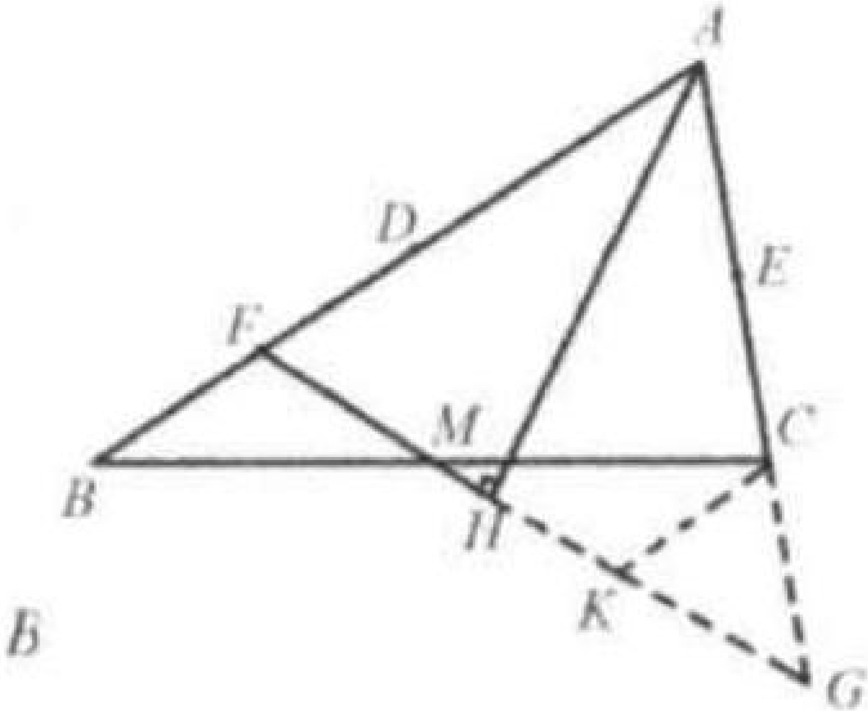
Problem

As shown in the figure below, in $\triangle ABC, D, E$ are the midpoints of AB, AC, respectively. $AB > AC$. Take F, a point between DB such that $DF = AE$. Draw $FH \perp AH, AH$ is the angle bisector of $\angle A$. Let H be the feet of the perpendicular to AH from F. FH meets BC at M. Show that $BM = MC$.



Solution

Extend FH to meet the extension of AC at G.
Draw $CK//AB$. CK meets FG at K.
Since $CK//AB, \angle B = \angle MCK$.



Since $FH \perp AH$, and AH is the angle bisector of $\angle A$,
$\angle AFH = \angle AGH, AF = AG$.
We also know that $\angle BMF = \angle CMK$ (vertical angles).
$AF = AG \quad \Rightarrow AD + DF = AE + EC + CG$
$\Rightarrow \quad (DF + BF) + DF = AE + AE + CG$
Since $DF = AE$, (1) becomes:
$AE + BF + AE = AE + AE + CG \quad \Rightarrow BF = CG$.
Since $CK//AB, \angle AFH = \angle AGH = \angle GKC$.
Thus $CG = CK = BF$.
Therefore, $\triangle BFM \cong \triangle CKM$ and $BM = MC$.