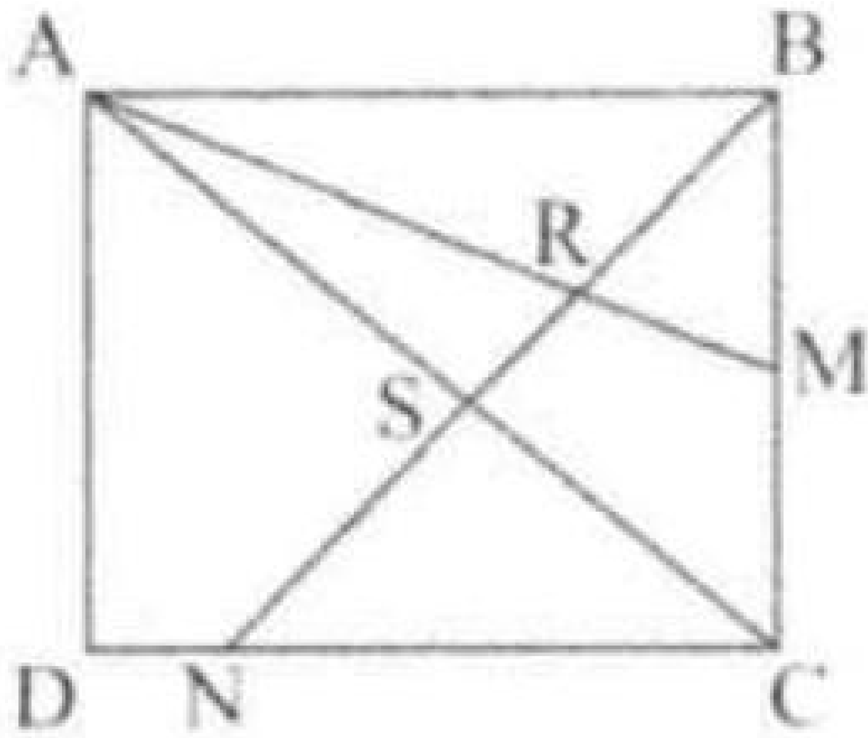
Problem 9

Problem

(2012 Mathcounts State Sprint 30) In rectangle $ABCD$, shown here, point M is the midpoint of side BC, and point N lies on CD such that $DN : NC = 1 : 4$. Segment BN intersects AM and AC at points R and S, respectively. If $NS : SR : RB = x : y : z$, where x, y and z are positive integers, what is the minimum possible value of $x + y + z$?



Solution

126. Draw $ME//NC$ to meet NB at $E. \triangle ABS \sim \triangle CNS, \triangle ABR \sim \triangle MER$.
Since $DN : NC = 1 : 4, NC = \frac{4}{5}AB$. So $EM = \frac{1}{2}NC = \frac{1}{2} \times \frac{4}{5}AB = \frac{2}{5}AB$.
So $BE = EN = \frac{x+y+z}{2}$, and $ER = EB - RB = \frac{x+y+z}{2} - z = \frac{x+y-z}{2}$.

$$\frac{AB}{NC} = \frac{BS}{NS} = \frac{5}{4} \quad \Rightarrow \quad \frac{y+z}{x} = \frac{5}{4} \quad \Rightarrow \quad 5x = 4y + 4z$$

$$\frac{AB}{EM} = \frac{BR}{ER} = \frac{5}{2} \quad \Rightarrow \quad \frac{z}{\frac{x+y-z}{2}} = \frac{5}{2} \quad \Rightarrow$$

$$4z = 5x + 5y - 5z \quad \Rightarrow \quad 5x + 5y = 9z$$

Solving the system of equations (1) and (2): $\frac{x}{y} = \frac{56}{25}$, and $\frac{y}{z} = \frac{5}{9}$.
Thus $x : y : z = 56 : 25 : 45$. The smallest value of $x + y + z$ is
$56 + 25 + 45 = 126$.