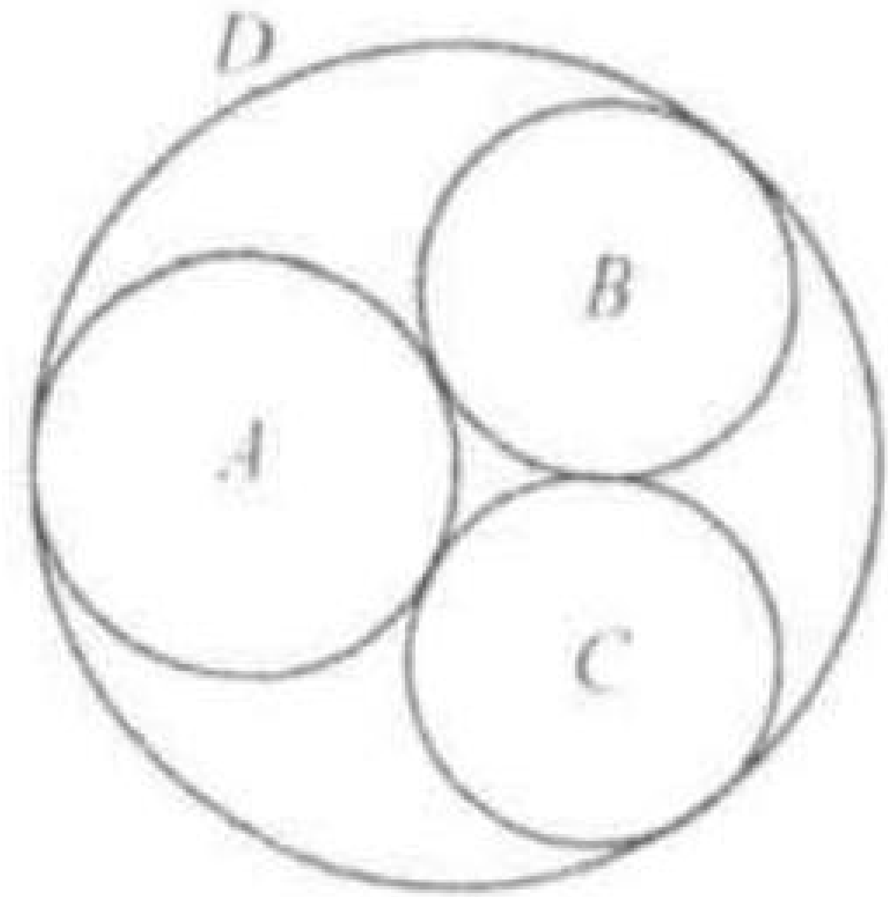
Example 9

(2004 AMC 10 A Problem 23) Circles A, B, and C are externally tangent to each other and internally tangent to circle D.
Circles B and C are congruent. Circle A has radius 1 and passes through the center of circle D. What is the radius of circle B ?

(A) $\frac{2}{3}$
(B) $\frac{\sqrt{3}}{2}$
(C) $\frac{7}{8}$
(D) $\frac{8}{9}$
(E) $\frac{1+\sqrt{3}}{3}$

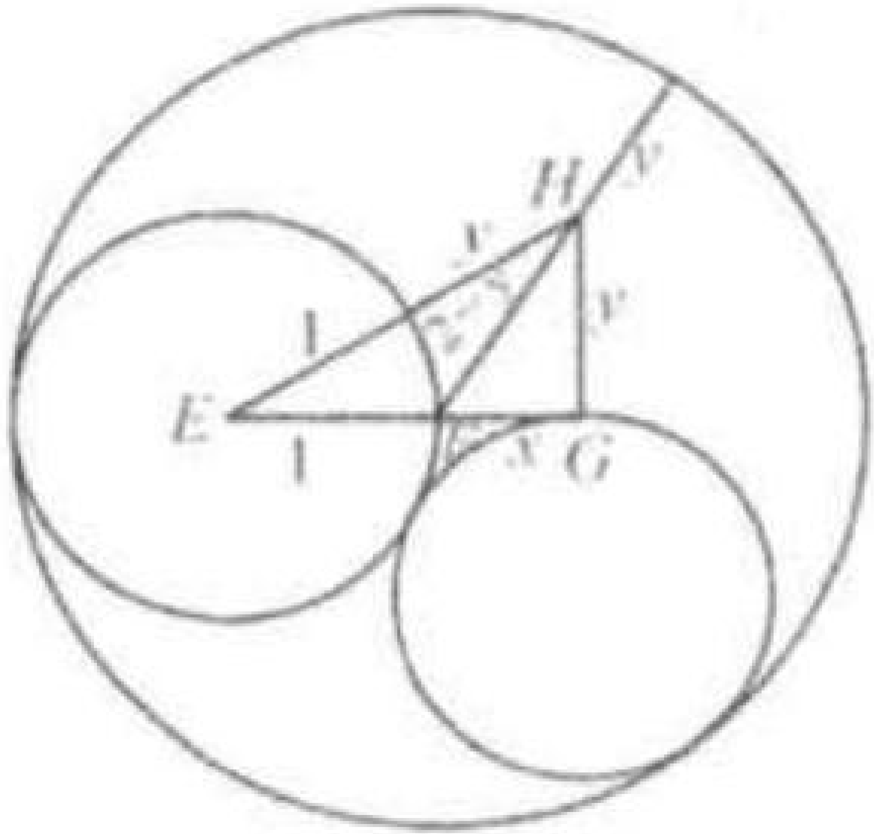


Solution: (D).

Method 1:

Let E, H, and F be the centers of circles A, B, and D, respectively, and let G be the point of tangency of circles B and C. Let $x = FG$ and $y = GH$.

Because circle A has radius 1 and passes through the center



of circle D, the radius of circle D is 2 . Applying the Pythagorean Theorem to right triangles EGH and FGH gives $(1+y)^2 = (1+x)^2 + y^2$, so $2y = 2x + x^2$,

And $(2-y)^2 = x^2 + y^2$, so $4 - 4y = x^2$.

From this it follows that

$$x^2 + \frac{x^2}{2} = y = 1 - \frac{x^2}{4}, \text{ so } 0 = \frac{3}{4}x^2 + x - 1 = \left(\frac{3}{2}x - 1\right)\left(\frac{1}{2}x + 1\right)$$

As a consequence, the two solutions are

$$x = \frac{2}{3}, y = 1 - \frac{(2/3)^2}{4} = \frac{8}{9} \text{ and } x = -2, y = 1 - \frac{(-2)^2}{4} = 0$$

The radius of circle B is the positive solution for y, which is 8/9.

Method 2:

We connect the centers of all the circles as shown. l is the tangent to the circle D. We know that G, B, and E lie in a straight line.

By Heron's formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$

$= \frac{1}{2}(1+r+1+r+r+r) = 2r+1.$

$$A = \sqrt{(2r+1)(2r+1-1-r)(2r+1-r-1)(2r+1-2r)} = r\sqrt{2r+1}$$

We also have $A = \frac{1}{2}BC \times (1+x) = \frac{1}{2} \times (2r) \times (1+x) = r(1+x)$

So we get $r(1+x) = r\sqrt{2r+1}$. Since $r \neq 0$, we have: $1 + x = \sqrt{2r+1}$

Applying the Pythagorean Theorem to right triangles ABF and EBF gives

$$(1+r)^2 - (1+x)^2 = r^2 = (2-r)^2 - x^2$$

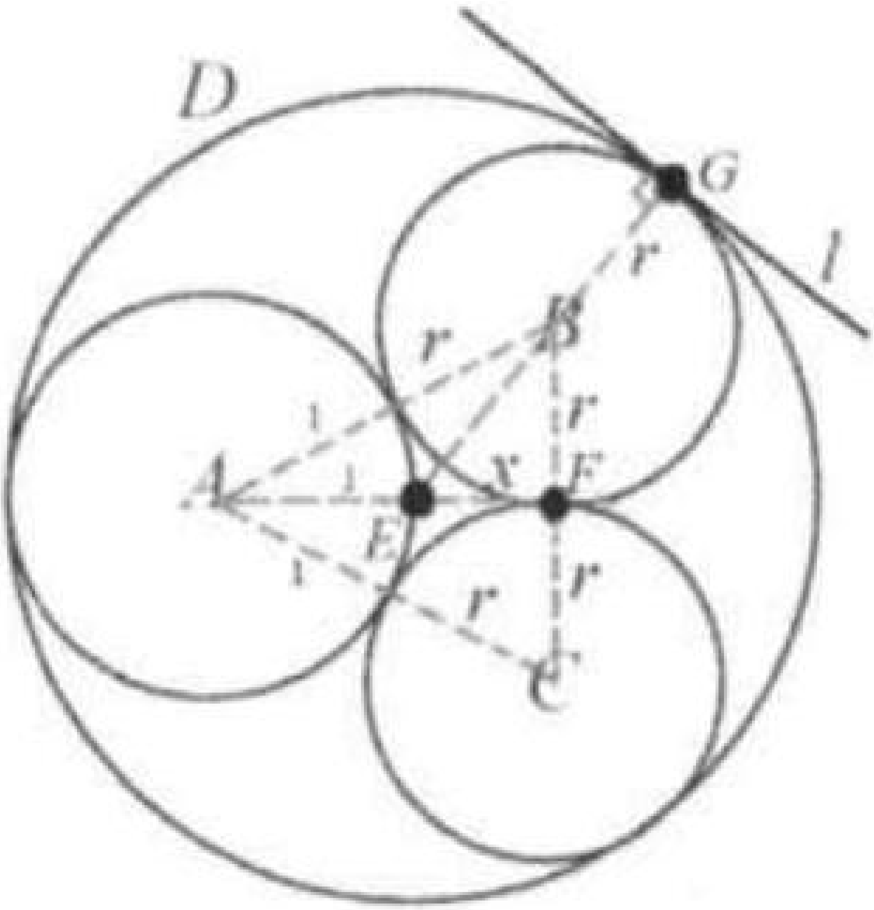
$$\Rightarrow 1 + x = 3r - 1$$

Substituting (1) into (2):

$$3r - 1 = \sqrt{2r+1} \quad \Rightarrow \quad (3r-1)^2 = 2r + 1$$

$$\Rightarrow \quad 9r^2 - 6r + 1 = 2r + 1 \quad \Rightarrow \quad 9r^2 - 8r = 0$$

$$\Rightarrow \quad r(9r - 8) = 0.$$



Since $r \neq 0$, we have $r = \frac{8}{9}$.