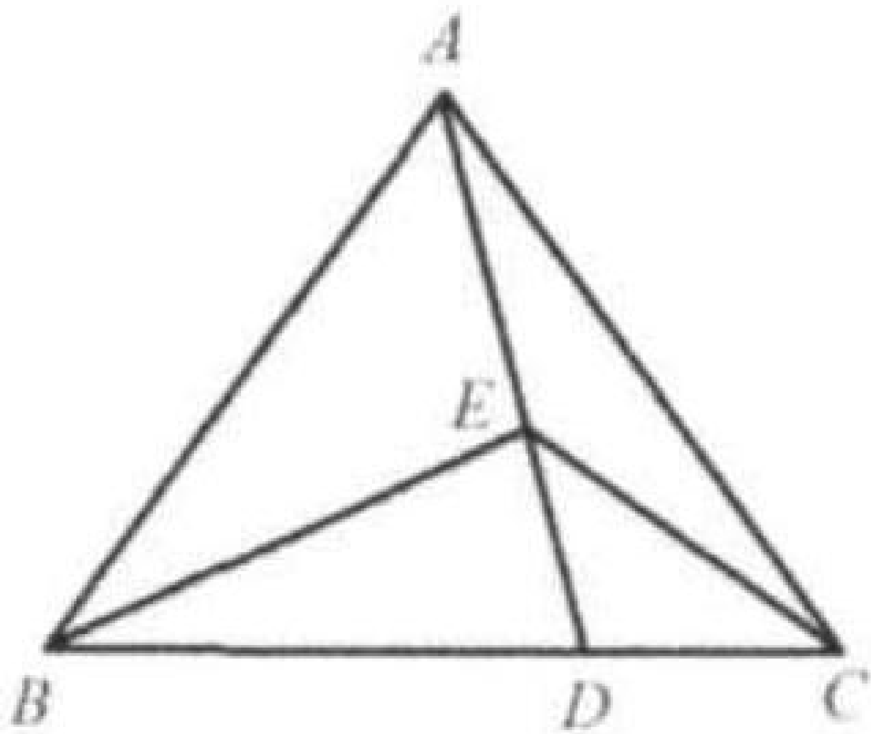
Example 7

(1992 China Middle School Math Contest) As shown in the figure, in $\triangle ABC$, $AB = AC.D$ is a point on $BC.E$ is a point on AD. $\angle BED = 2\angle CED = \angle A$. Show that $BD = 2CD$.

Solution: We draw the circumcircle of $\triangle ABC$. Extend AD to meet



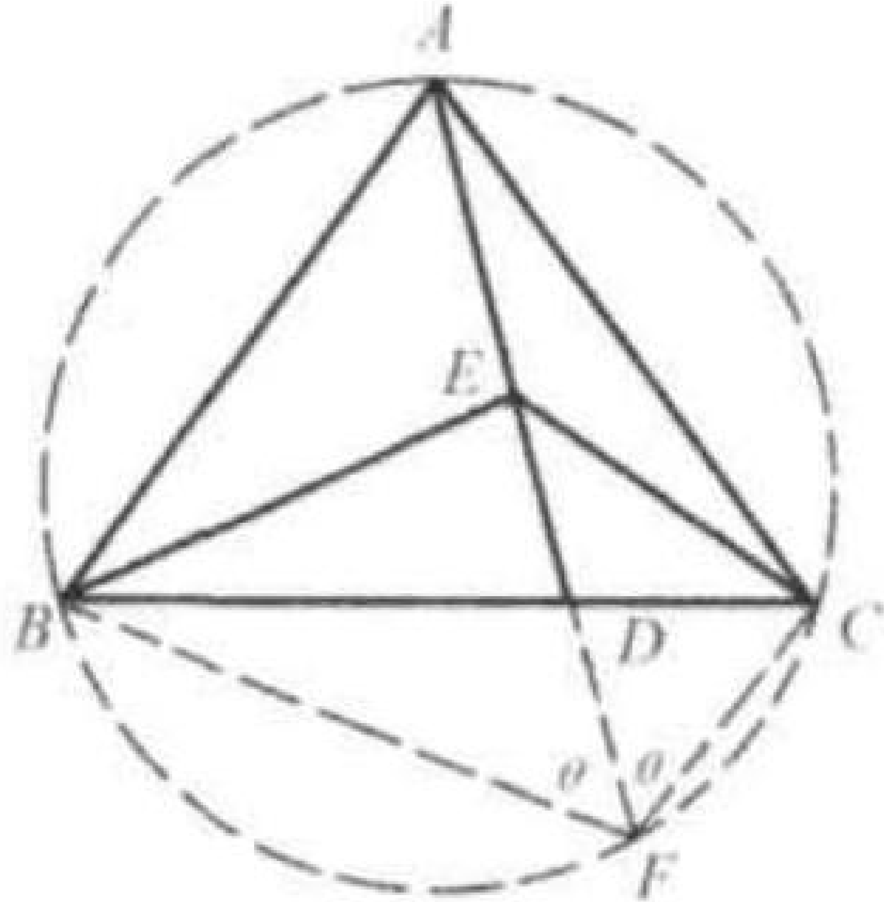
the circle at F. Connect BF, CF (figure 1).

Since both angles $\angle AFB$ and $\angle AFC$ face the arcs($AB = AC$) of the same length, $\angle AFB = \angle AFC = \angle ABC = \theta$.

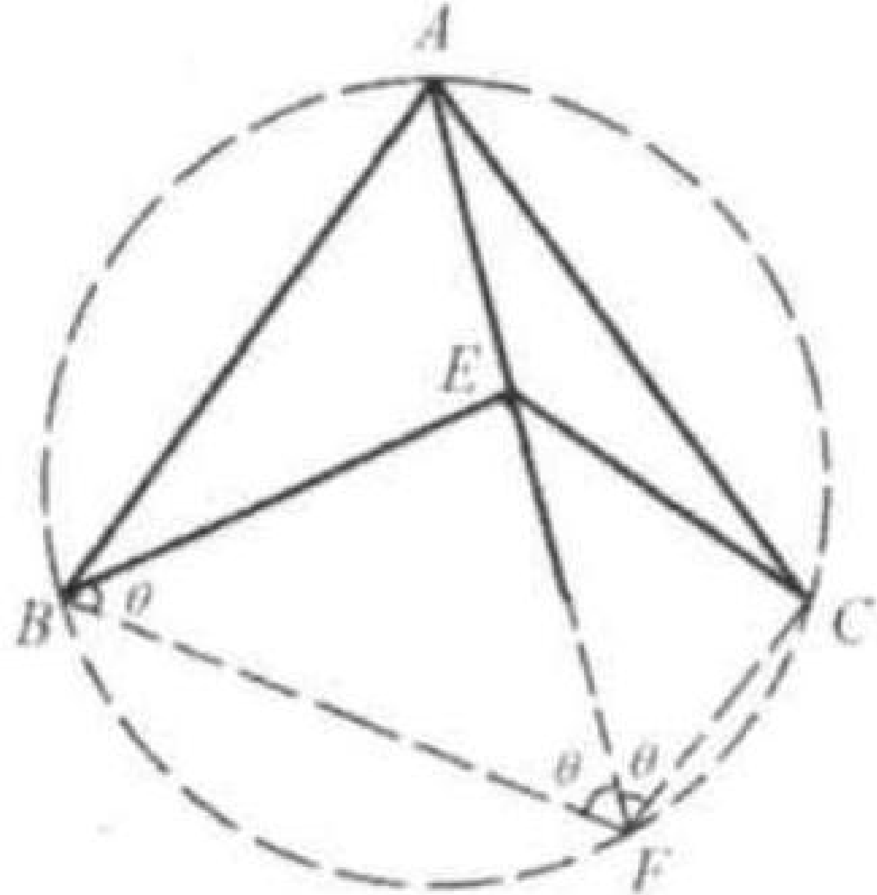
Thus, DF is the angle bisector of $\angle BFC$.

So by the angle bisector theorem, $\frac{FB}{FC} = \frac{BD}{CD}$.

Now we only need to prove that $\frac{FB}{FC} = 2$ or $BF = 2CF$.



We know that in $\triangle ABC, \angle A + \angle B + \angle C = 180°$, or $\angle A + \theta + \theta = 180°$.
We also know that in $\triangle EBF, \angle BEF + \angle BFE + \angle EBF = 180°$, or
$\angle BEF + \theta + \angle EBF = 180°$.
Since $\angle BED = \angle A, \angle EBF = \theta$.
We take the line segment BD out of the figure and redraw



the figure (2).

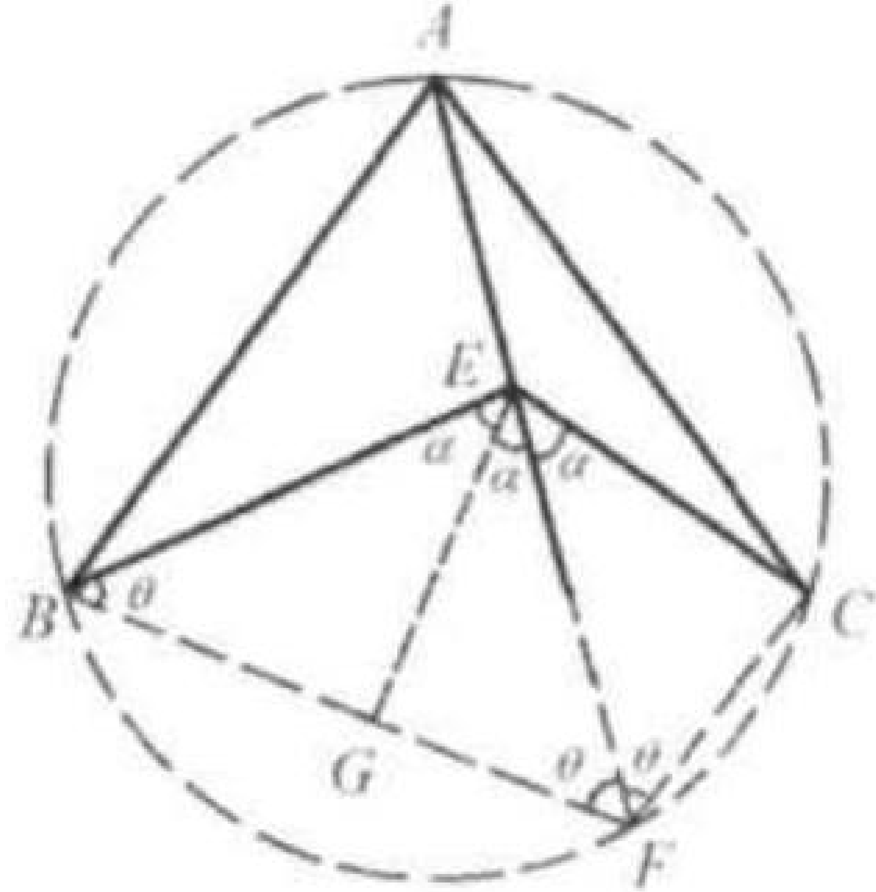
Since $EB = EF$, we draw EG, the perpendicular bisector of BF.

So $BG = GF, \angle GEB = \angle GEF = CEF = \alpha$ (figure 3).

Since $\angle GEF = \angle CEF = \alpha, EF = EF, \angle EFG = \angle EFC = \theta$,

$\triangle EFG \cong \triangle EFC$. So $GF = CF$.

So we proved that $BF = 2CF$ and we are done.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