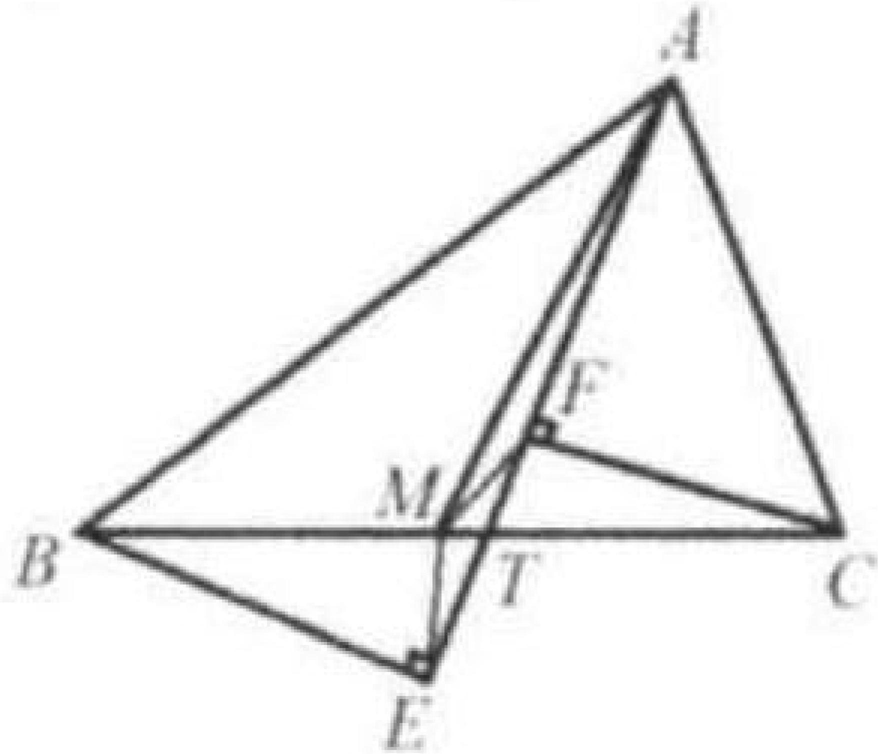
Problem

In $\triangle ABC, AM$ is the median. AT is the angle bisector of $\angle A . BE \perp AT$ at T and $CF \perp AT$ at F. Show that $ME = MF$.



Solution

Method 1:

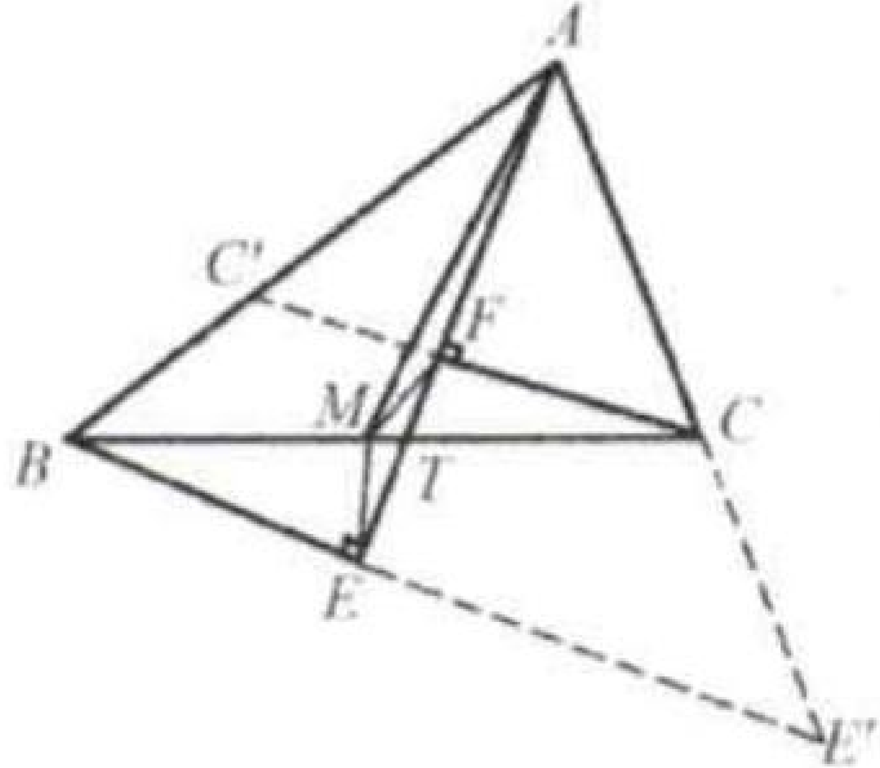
Extend CF to meet AB at C'.

Since AT is the angle bisector of $\angle A, AF$ is the angle bisector of $\angle A . AF$ is the perpendicular bisector of $C'C$ in $\triangle AC'C$, so $\triangle AC'F \cong \triangle ACF, CF = FC'$.

Since F is the midpoint of CC', M is the midpoint of BC,

$FM // BC' // AB . \angle MFE = \angle BAT$.

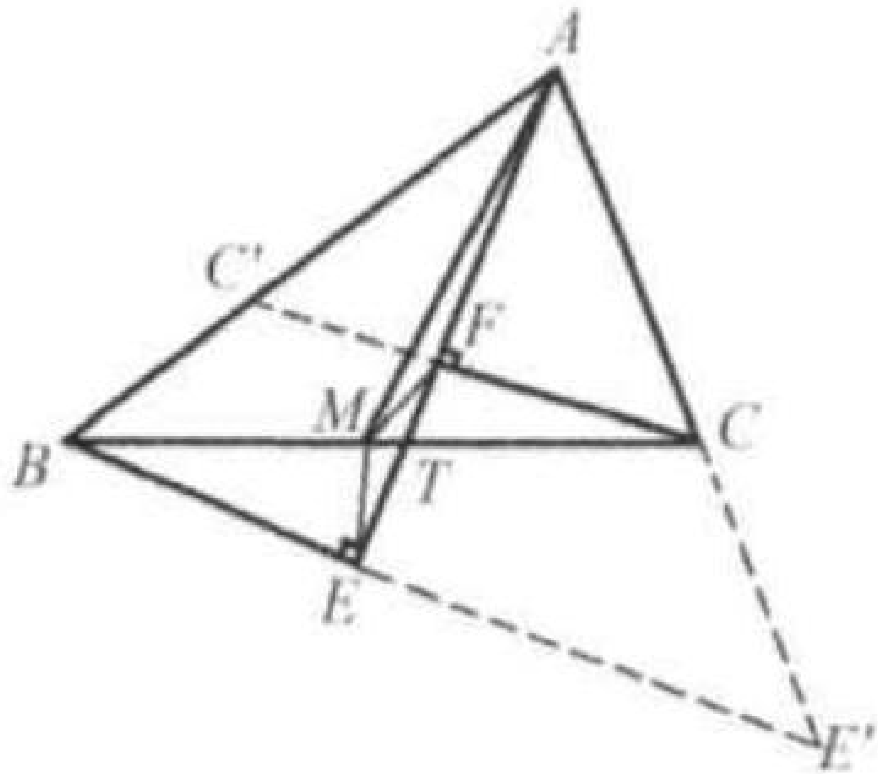
Extend BE to meet the extension of AC at E'.



Similarly, we get $ME//CE\prime, \angle MEF = \angle CAT$.
Since $\angle BAT = \angle CAT, \angle MFE = \angle MEF$. Thus $ME = MF$.

Method 2:

Extend CF to meet AB at C'.
Since AT is the angle bisector of $\angle A, AF$ is the angle bisector of $\angle A.AF$ is the perpendicular bisector of $C'C$ in $\triangle AC'C$, so $\triangle AC'F \cong \triangle ACF, CF = FC'$.
Since F is the midpoint of CC', M is the midpoint of BC, $MF = \frac{1}{2}BC$.



Extend BE to meet the extension of AC at E'.
Similarly, we get $ME = \frac{1}{2}CE$.
Since $CC'//BE', CB = CE'$. Thus $ME = MF$.