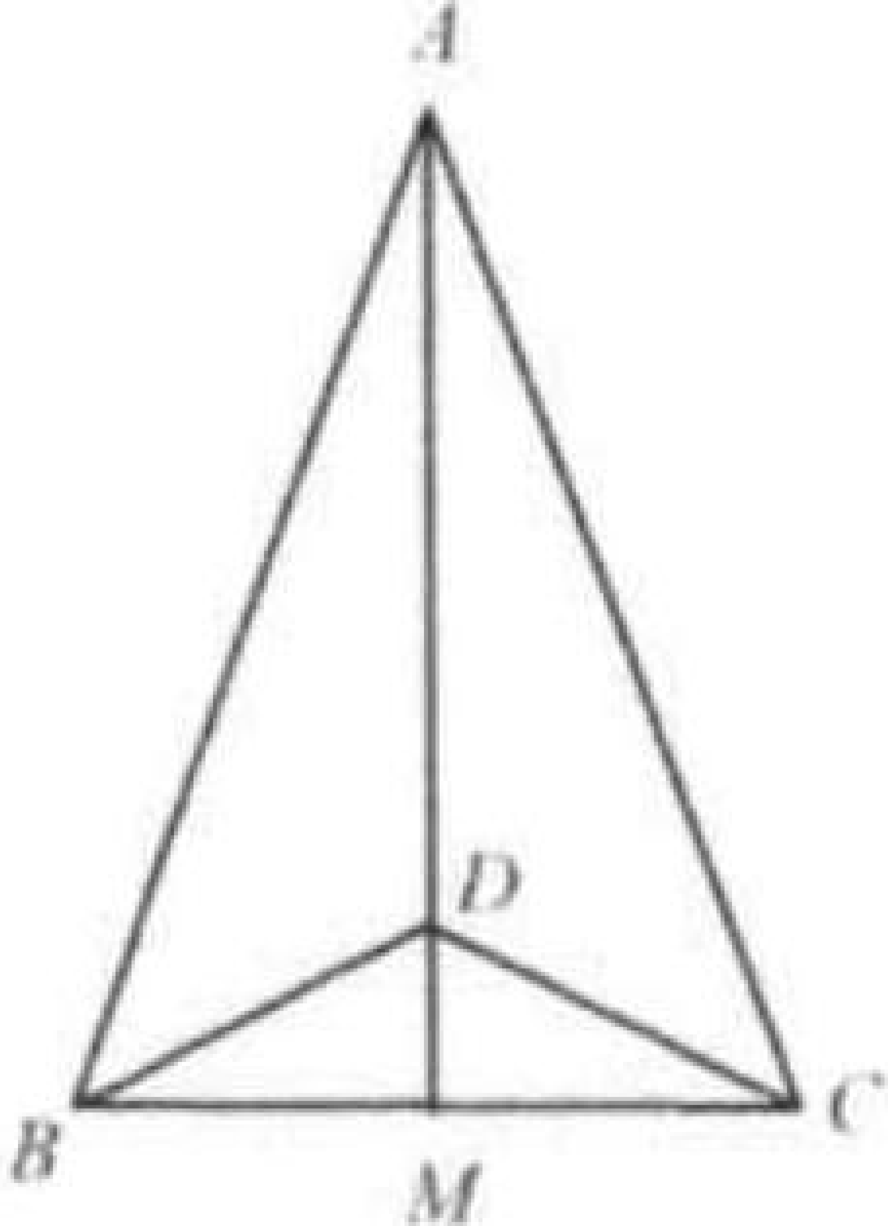
Problem 9

Problem

(AIME) Triangle ABC is isosceles, with $AB = AC$ and altitude $AM = 11$. Suppose that there is a point D on AM with $AD = 10$ and $\angle BDC = 3\angle BAC$. Then the perimeter of triangle ABC may be written in the form $a + \sqrt{b}$, where a and b are integers. Find $a + b$.


A
D
B
M
C

Solution

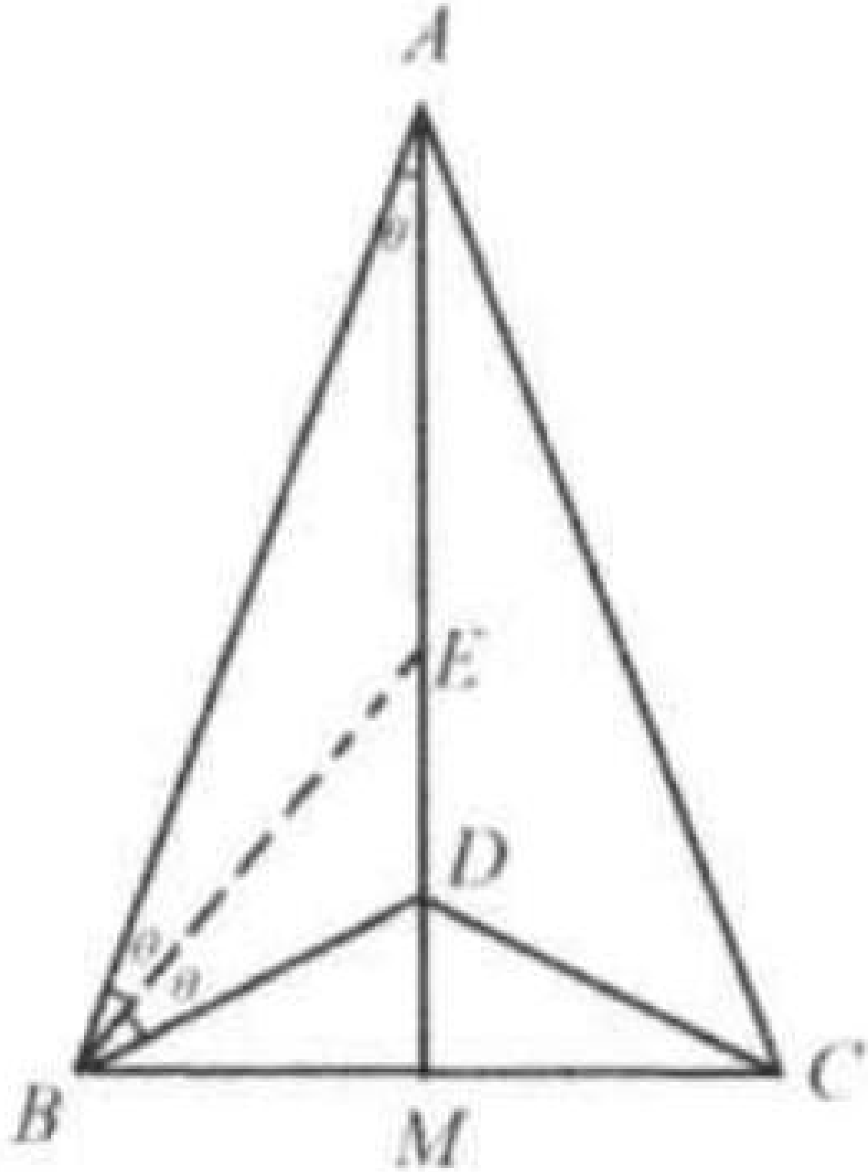
616. Let the bisector of $\angle ABD$ intersect AD at E, and let $x = BE = AE$. By the Pythagorean Theorem,

$$BM = \sqrt{BE^2 - EM^2} = \sqrt{x^2 - (11 - x)^2} = \sqrt{22x^2 - 121}$$

By applying the Pythagorean Theorem two more times, we find that
$AB = \sqrt{BM^2 + AM^2} = \sqrt{22x}$

$$BD = \sqrt{BM^2 + DM^2} = \sqrt{22x - 120}.$$

By the angle-bisector theorem, we have that $\frac{AB}{BD} = \frac{AE}{DE}$



from which $\frac{\sqrt{22x}}{\sqrt{22x-120}} = \frac{x}{10-x}$.

By squaring both sides of this equation and solving for x, we find that $x = 55/8$. Hence $BM = 11/2$ and $AB = (11/2)\sqrt{5}$. The perimeter of the triangle is $2(AB + BM) =$
$11\sqrt{5} + 11 = \sqrt{605} + 11$, so $a + b = 616$.