Example 18

(2005 AIME II Problem 14) In $\triangle ABC, AB = 13, BC = 15$, and $CA = 14$. Point D is on BC with $CD = 6$. Point E is on BC such that $\angle BAE = \angle CAD$.

Given that $BE = p/q$, where p and q are relatively prime positive integers, find q.
Solution: 463.
Draw $BF//AC$ to meet the extension of AE at G and AD at F.
We know that $AC//BF$. So $\triangle ADC \sim \triangle FDB$ (figure 1). $\frac{AC}{BF} = \frac{DC}{BD} \Rightarrow$

$$\frac{14}{BF} = \frac{6}{9} \quad \Rightarrow \quad BF = \frac{14 \times 9}{6} = 21$$

We know that $\angle BAG = \angle BFA = \alpha$ and $\angle ABG = \angle ABF$ (figures 2 and 3). So $\triangle ABG \sim \triangle ABF.\frac{AB}{BF} = \frac{BG}{AB} \Rightarrow \quad \frac{13}{21} = \frac{BG}{13} \Rightarrow \quad BG = \frac{169}{21}$
We know that $AC//BF$. So $\triangle BGE \sim \triangle CAE$ (figure 4).
$\frac{BG}{AC} = \frac{BE}{CE}$
$\Rightarrow \quad \frac{\frac{169}{21}}{14} = \frac{x}{15-x}$
$\Rightarrow \quad x = \frac{2535}{463}$.

The answer is 463 .

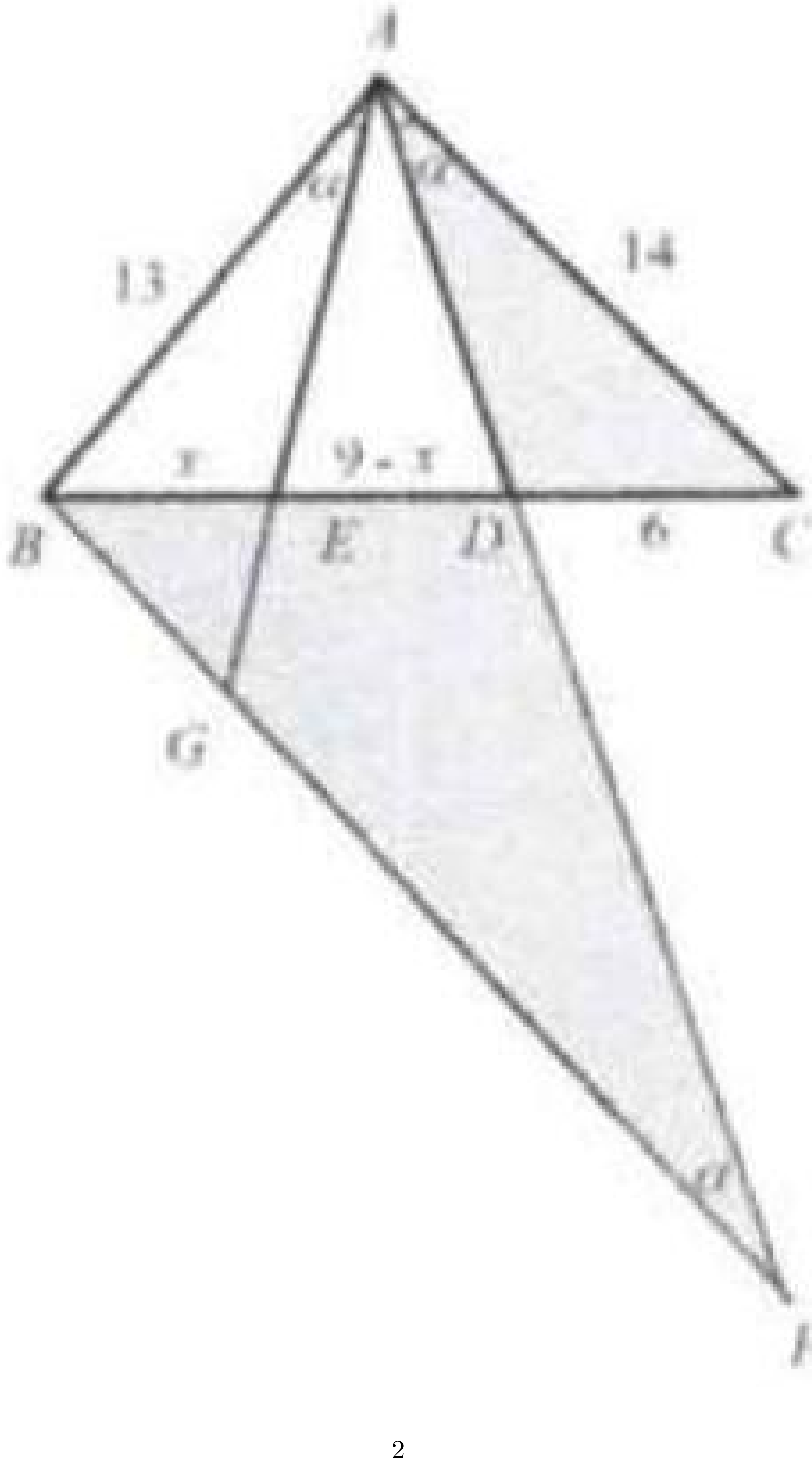
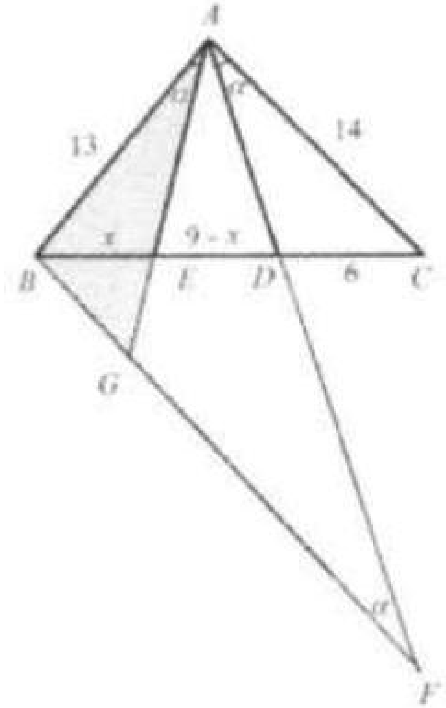
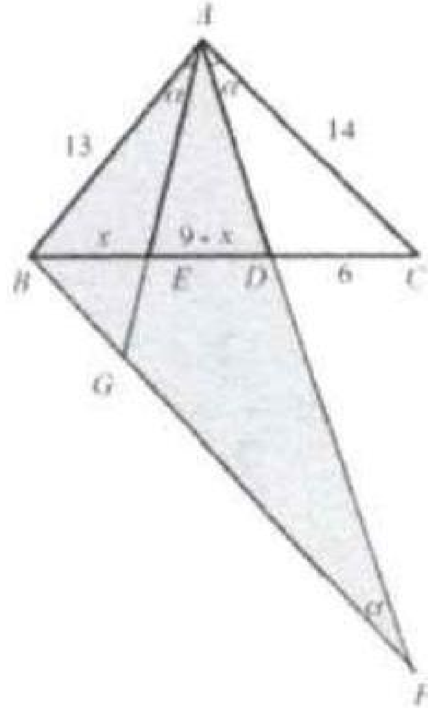

A
α
α
13
14
x
9 - x
B
E
D
6
C
G
α
F

Figure 1





Figures 2 and 3

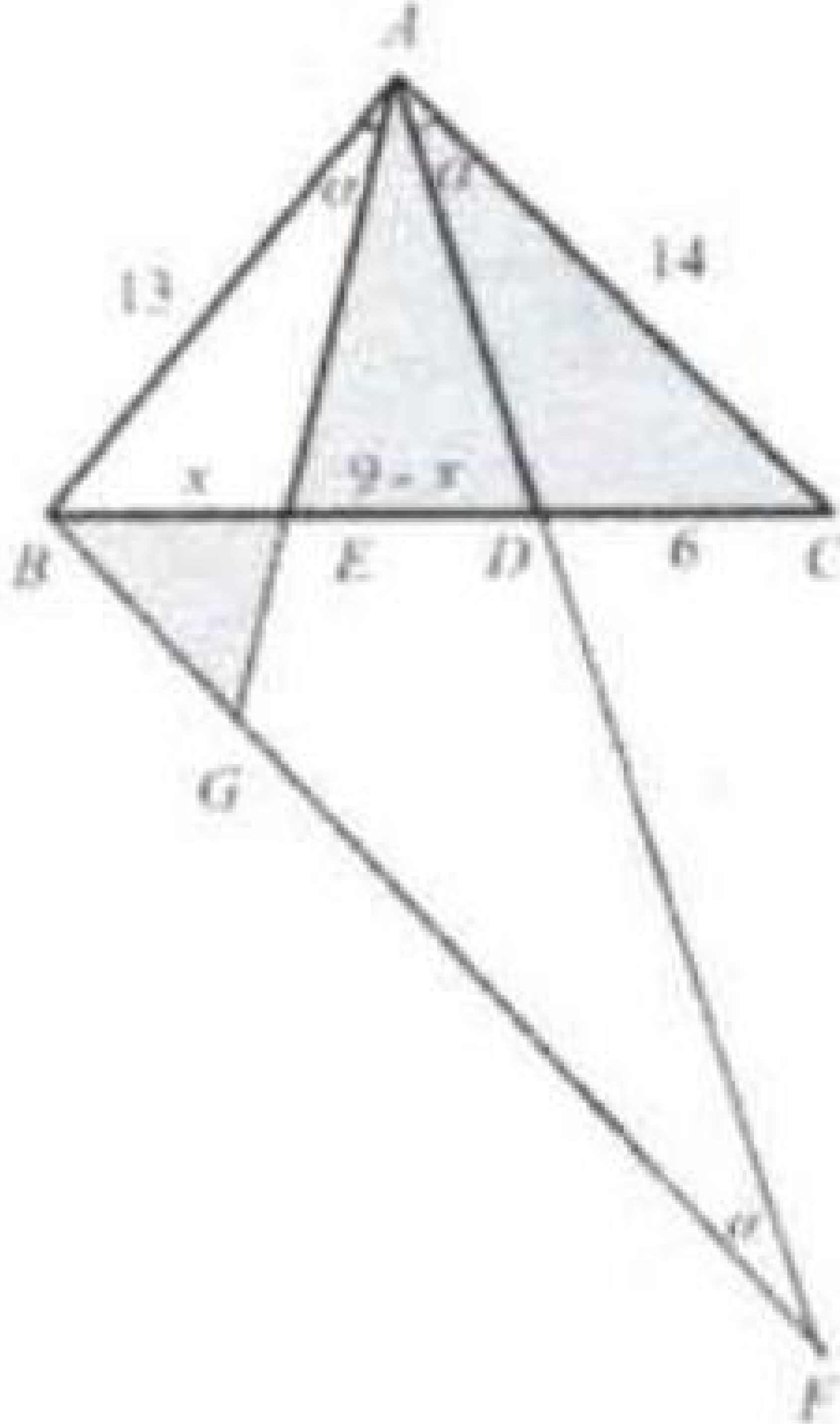

A
α
α
13
14
x
9 - x
B
E
D
6
C
G
α
F

Figure 4