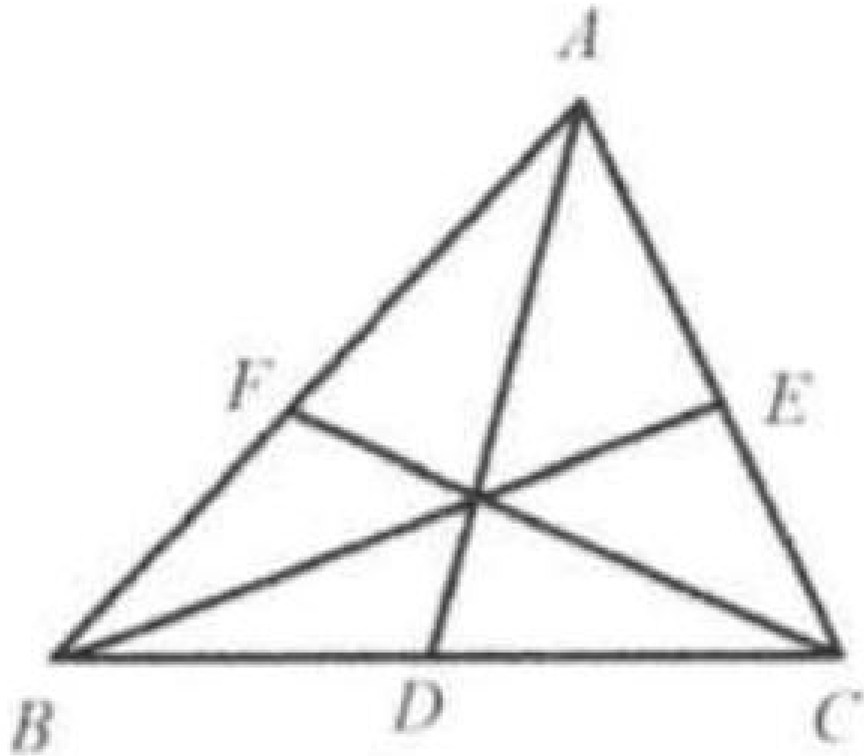
Example 10

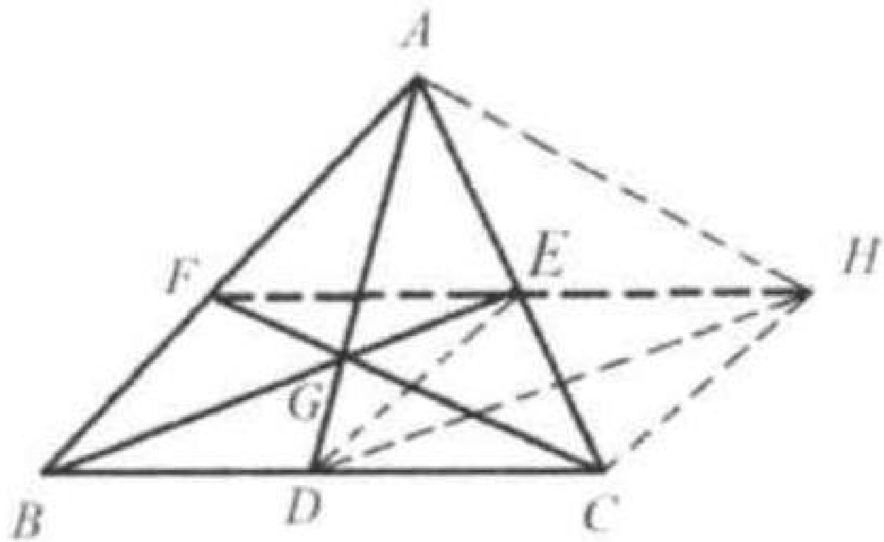
Given a triangle ABC and its medians AD, BE, and CF, construct a triangle with sides of length AD, BE, and CF. Show that the area of the triangle of medians is three-fourths of the area of the given triangle.



Solution: Extend FE to H such that $EH = DC = \frac{1}{2}BC$. Connect HD, HA, HC.

Since EF is the midline of triangle $ABC, EF//BC$, and $EF = \frac{1}{2}BC$. Thus $FH//BC$, and $FH = BC$, then $HCBF$ is a parallelogram. So $HC//BF$, and $HC = BF$. So $HC//AF$, and $HC = AF$. Thus $CHAF$ is a parallelogram. So $AH = CF$.

Since $EH//BD$, and $EH = BD, HDBE$ is a parallelogram.



So $HD = BE$.

This tells us that triangle ADH is a triangle with three sides of $AD, BE.CF$.

Now connect DE.

Since ED is the midline of triangle $ABC, ED//AB$, and $ED = \frac{1}{2}AB$ and $S_{\triangle EDA} = \frac{1}{2}S_{\triangle ADC} = \frac{1}{4}S_{\triangle ABC}$.

We know that $HDBE$ is a parallelogram, so

$S_{\triangle EDH} = S_{\triangle EBD} = \frac{1}{2}S_{\triangle EBC} = \frac{1}{4}S_{\triangle ABC}$

We also know that $HCFA$ is a parallelogram, so

$S_{\triangle EAH} = S_{\triangle EFC} = \frac{1}{2}S_{\triangle AFC} = \frac{1}{4}S_{\triangle ABC}$.

Therefore, $S_{\triangle ADH} = S_{\triangle EDH} + S_{\triangle EHA} + S_{\triangle EAD} = \frac{3}{4}S_{\triangle ABC}$.