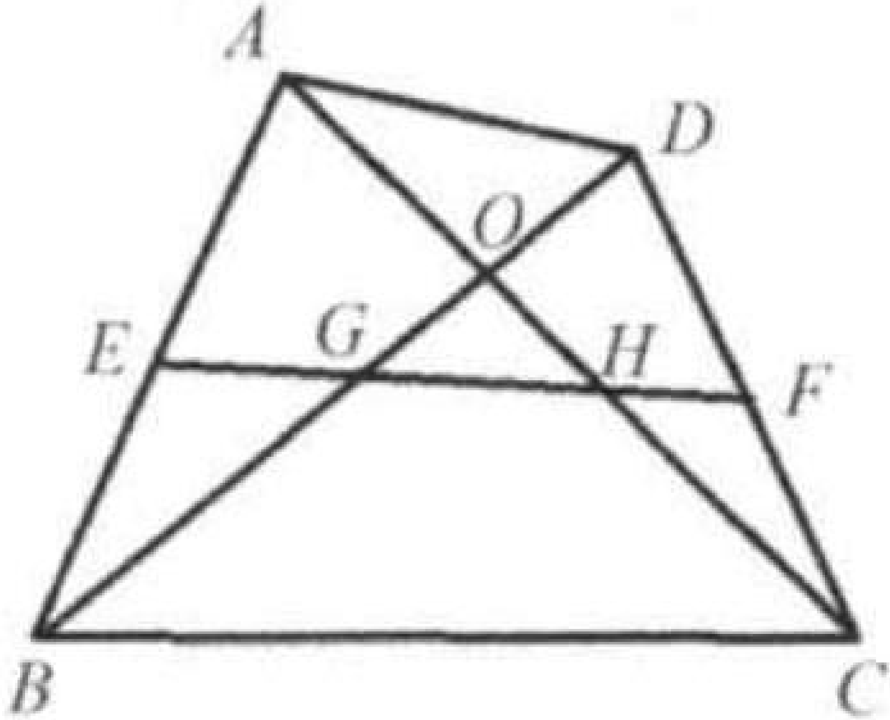
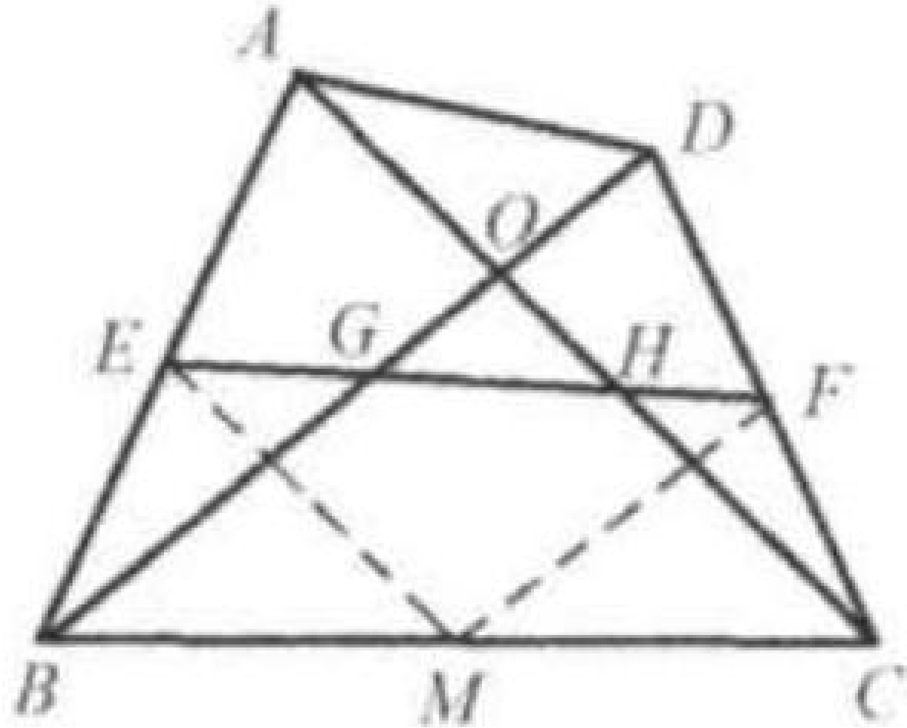
Problem

$ABCD$ is a convex quadrilateral. Diagonlas $AC = BD.AC$ and BD meet at O, E, F are midpoints of AB, CD, respectively. EF meets BD at G and AC at H. Show that $OG = OH$.



Solution

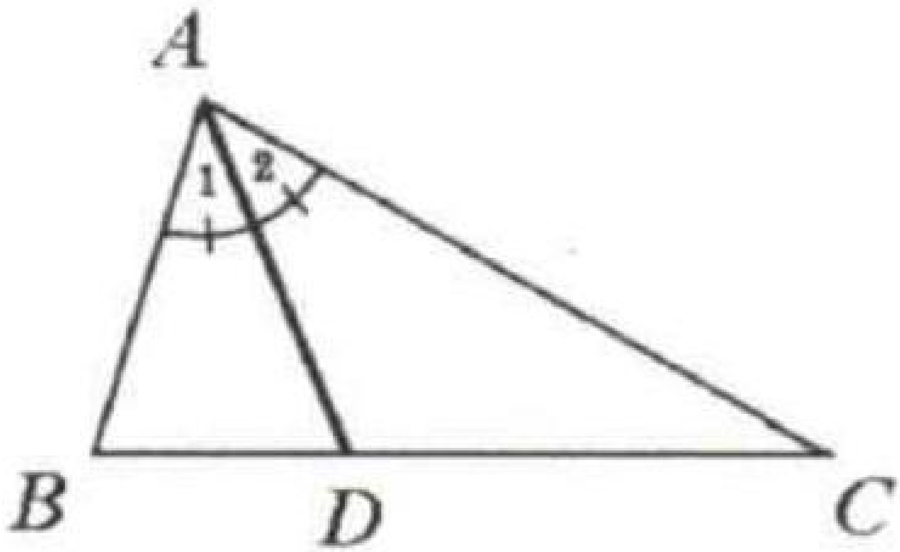
Take m , the midpoint of BC.
Connect EM and FM.
EM is the midline of $\triangle BAC.EM//AC$ and $EM = \frac{1}{2}AC$
FM is the midline of $\triangle CBD.FM//BD$ and $FM = \frac{1}{2}BD$.



Since $AC = BD, EM = FM$. So $\angle MEF = \angle MFE$.
Since $EM//AC, \angle MEF = \angle OHG$.
Since $FM//BD, \angle MFE = \angle OGH$.
Thus $\angle OHG = \angle OGH$, and $OG = OH$.

Chapter 3 Draw the Auxiliary lines with Angle Bisectors

Angle Bisector An angle bisector of a triangle is a segment or ray that bisects an angle and extends to the opposite side. As shown in the figure, AD is the angle bisector of $\angle A$.
$\angle 1 = \angle 2$.
Theorem 3.1. The Angle Bisector Theorem



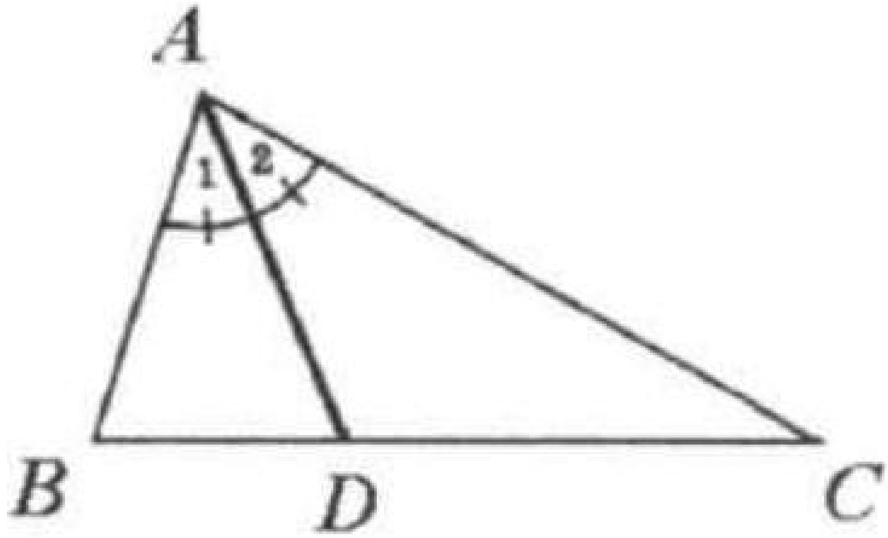
The angle bisector of a triangle divides the opposite side into segments that are proportional to the adjacent sides.

$$\frac{AB}{AC} = \frac{BD}{CD} \quad \text{or} \quad \frac{AB}{BD} = \frac{AC}{CD}$$

Proof: Since $\triangle ABD$ and ADC share the same vertex, the ratio of their areas is

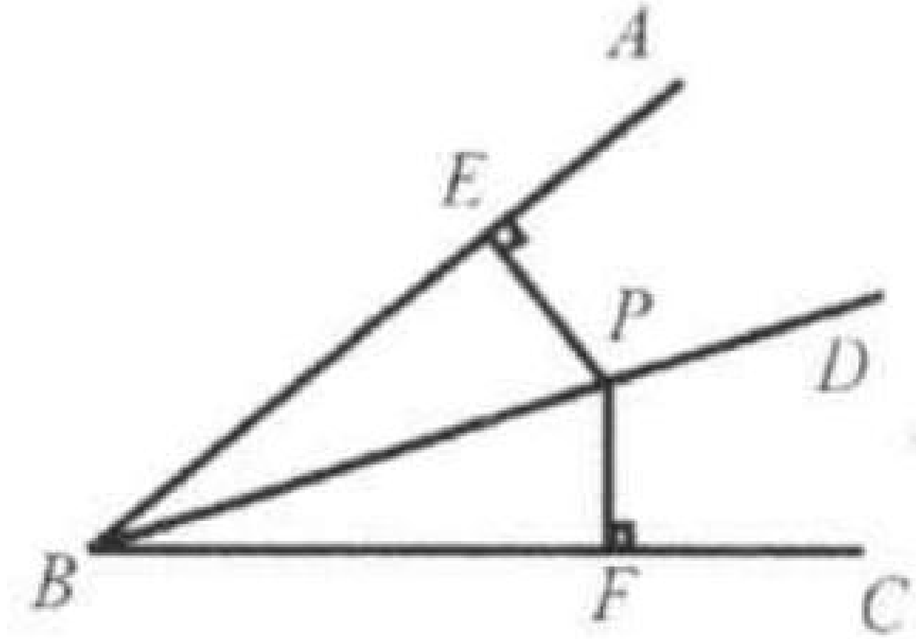
$$\frac{S_{\triangle ABD}}{S_{\triangle ADC}} = \frac{BD}{CD}.$$

We also know that $\frac{S_{\triangle ABD}}{S_{\triangle ADC}} = \frac{\frac{1}{2}AB \times AD \times \sin\angle 1}{\frac{1}{2}AD \times AC \times \sin\angle 2} = \frac{AB}{AC}$.

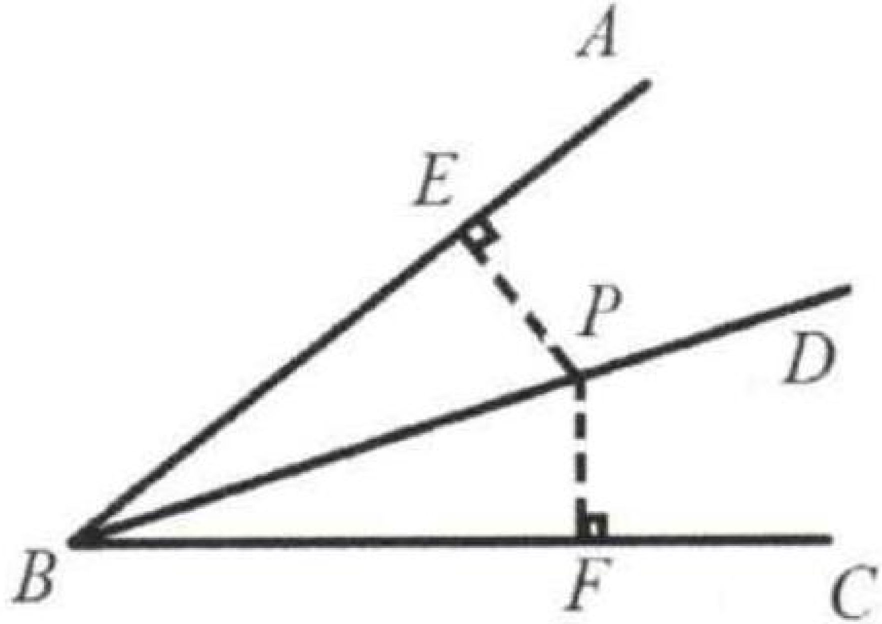


Therefore: $\frac{AB}{AC} = \frac{BD}{CD}$.

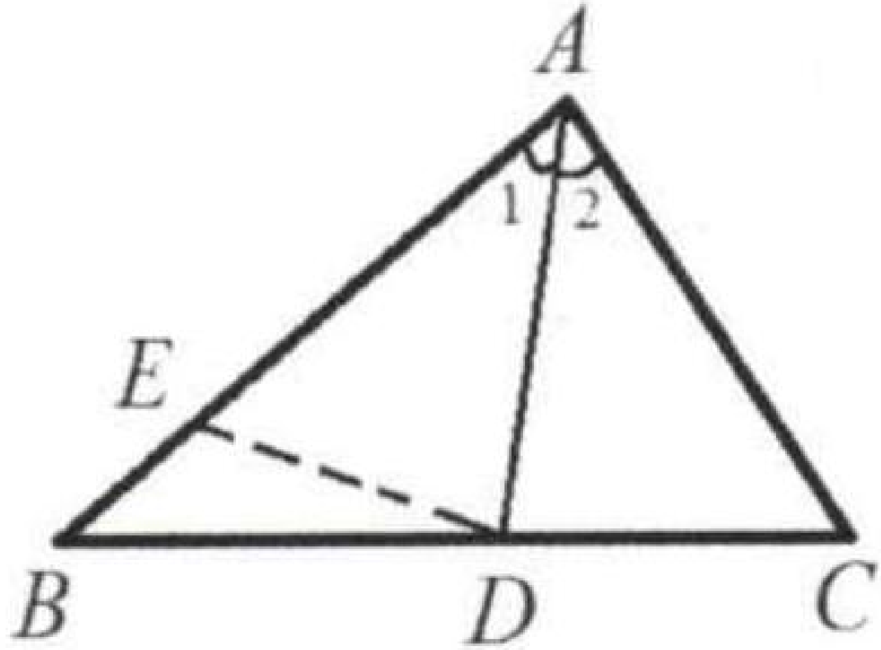
Theorem 3.2. Any point on the bisector of an angle is equidistant from the sides of the angle.



Construct congruent triangles using the angle bisector. (1). BD is the angle bisector of $\angle ABC$. P is the point on BD. Drawing $PE \perp AB$, $PF \perp BC$, we get $PE = PF.\triangle BPE \cong \triangle BPF$.



(2). $AB > AC, \angle 1 = \angle 2$. Find the point E on AB so that $AE = AC$, and connect $DE. \triangle ADE \cong \triangle ADC$.



(3). $\angle 1 = \angle 2$. Extend AC to F such that $AF = AB$. Connecting DF, we get

$$\triangle ADF \cong \triangle ADB.$$