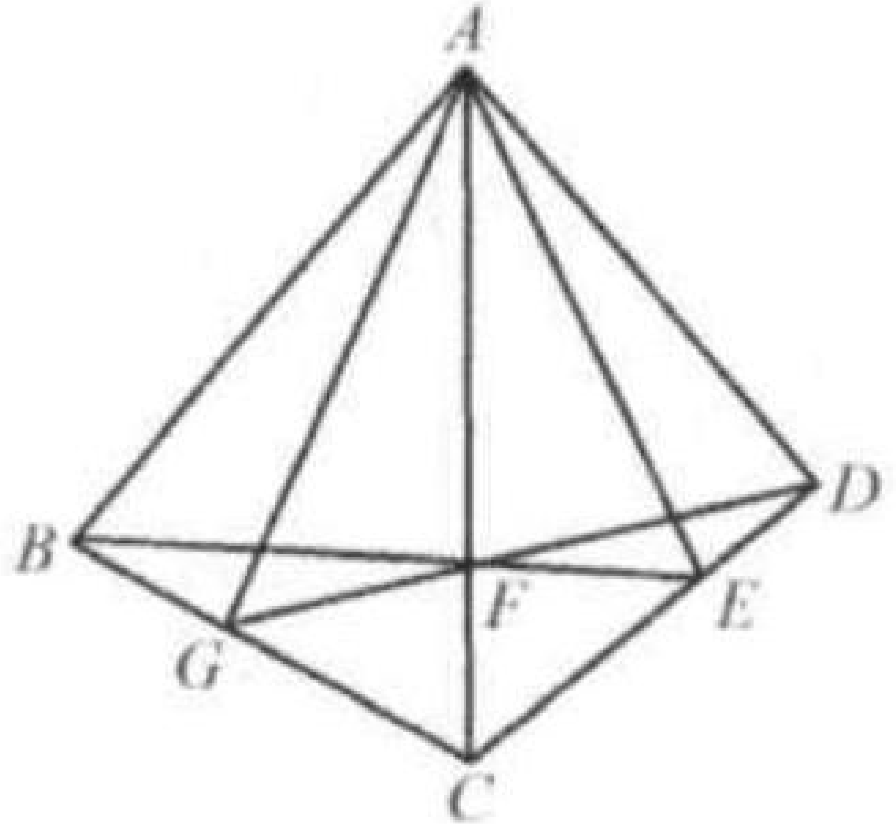
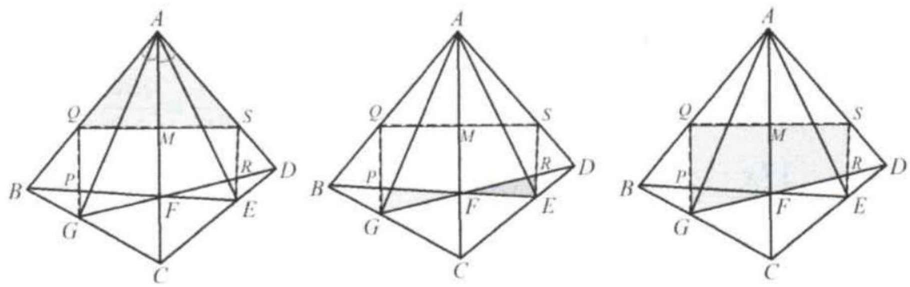
Problem

In quadrilateral $ABCD, AC$ is the angle bisector of $\angle BAD$. Take E, a point on CD. Connect BE. BE meets AC at F. Extend DF to meet BC at G. Prove that $\angle GAC = \angle EAC$.

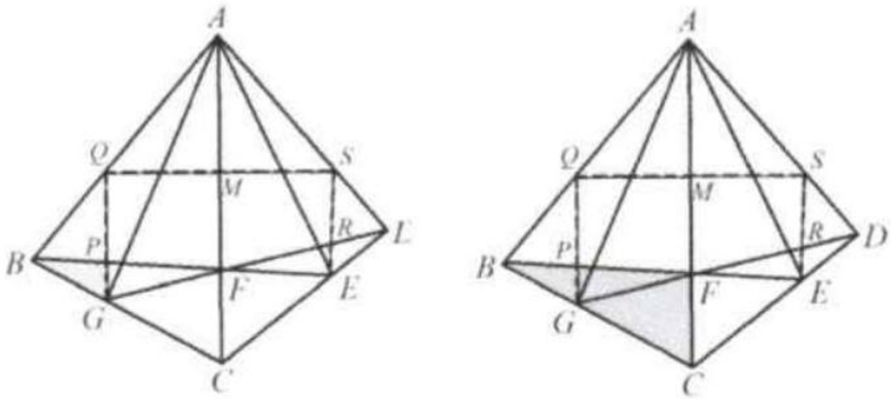


Solution

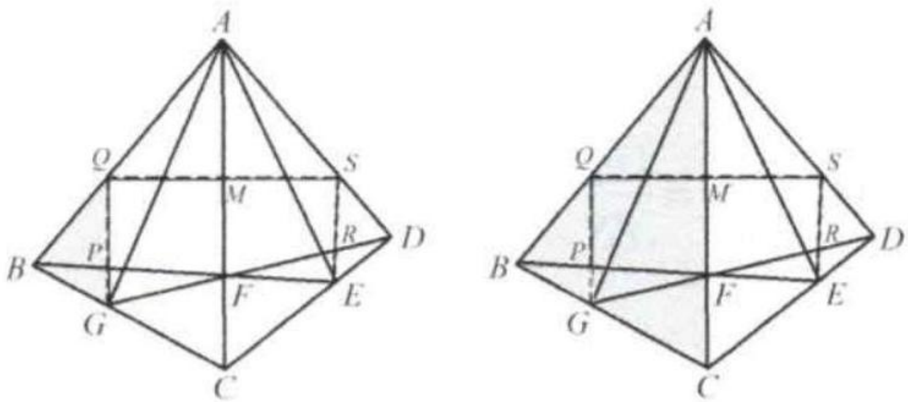
Draw $GQ//CA$ to meet BE, BA at P and Q, respectively.
Draw $ES//CA$ to meet DG, DA at R and S, respectively.
Connect QS to meet AC at M.
Since $\angle QAM = \angle SAM, GQ//CA//ES$,
By the angle bisector theorem, $\frac{AQ}{AS} = \frac{QM}{MS}$
We know that $\triangle PGF \sim \triangle ERF \cdot \frac{PE}{FE} = \frac{GP}{ER} = \frac{GF}{FR}$
We also know that in trapezoid $GQSR$, $\frac{QM}{MS} = \frac{GF}{FR}$
Thus $\frac{AQ}{AS} = \frac{QM}{MS} = \frac{PE}{FE} = \frac{GP}{ER}$



We know that $\triangle BGP \sim \triangle BCF \cdot \frac{GP}{CF} = \frac{BG}{BC}$

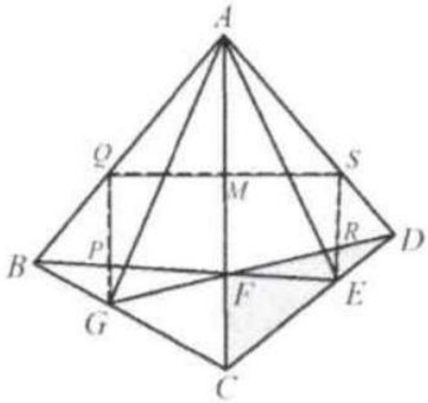


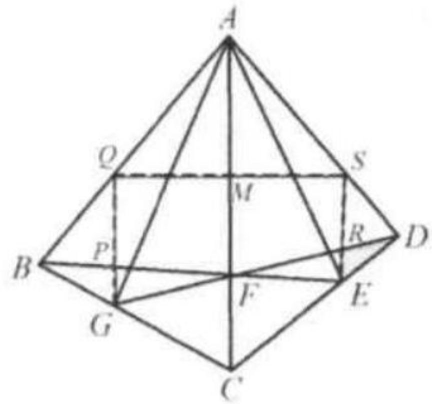
We know that $\triangle BGQ \sim \triangle BCA \cdot \frac{QG}{CA} = \frac{BG}{BC}$



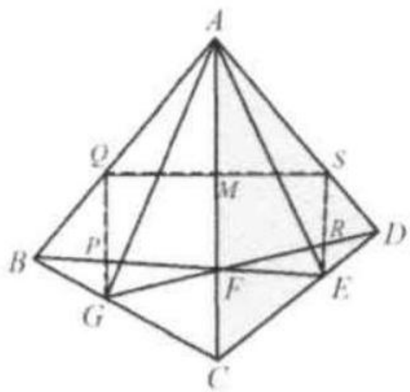
From (5) and (6), we have $\frac{GP}{CF} = \frac{QG}{CA} \Rightarrow \frac{GP}{QG} = \frac{CF}{CA}$

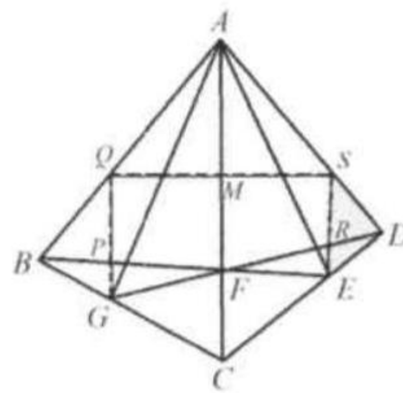
We know that $\triangle DFC \sim \triangle DRE . \frac{CF}{ER} = \frac{CD}{ED}$





We know that $\triangle DAC \sim \triangle DSE. \frac{CA}{ES} = \frac{CD}{ED}$



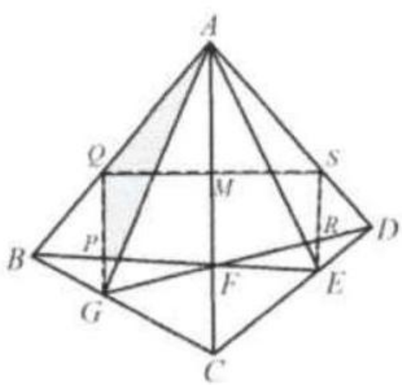


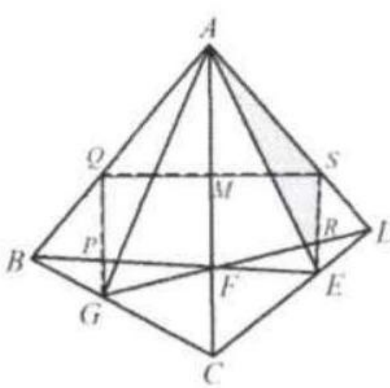
Combining (8) and (9), $\frac{CF}{ER} = \frac{CD}{ED} = \frac{CA}{ES} \Rightarrow \frac{CF}{CA} = \frac{ER}{ES}$

Combining (7) and (10), $\frac{GP}{GQ} = \frac{CF}{CA} = \frac{ER}{ES} \quad \Rightarrow \quad \frac{GP}{GQ} = \frac{ER}{ES}$

$\Rightarrow \quad \frac{GP}{ER} = \frac{GQ}{ES}$

Combining (4) and (11), $\frac{AQ}{AS} = \frac{GQ}{ES}$.



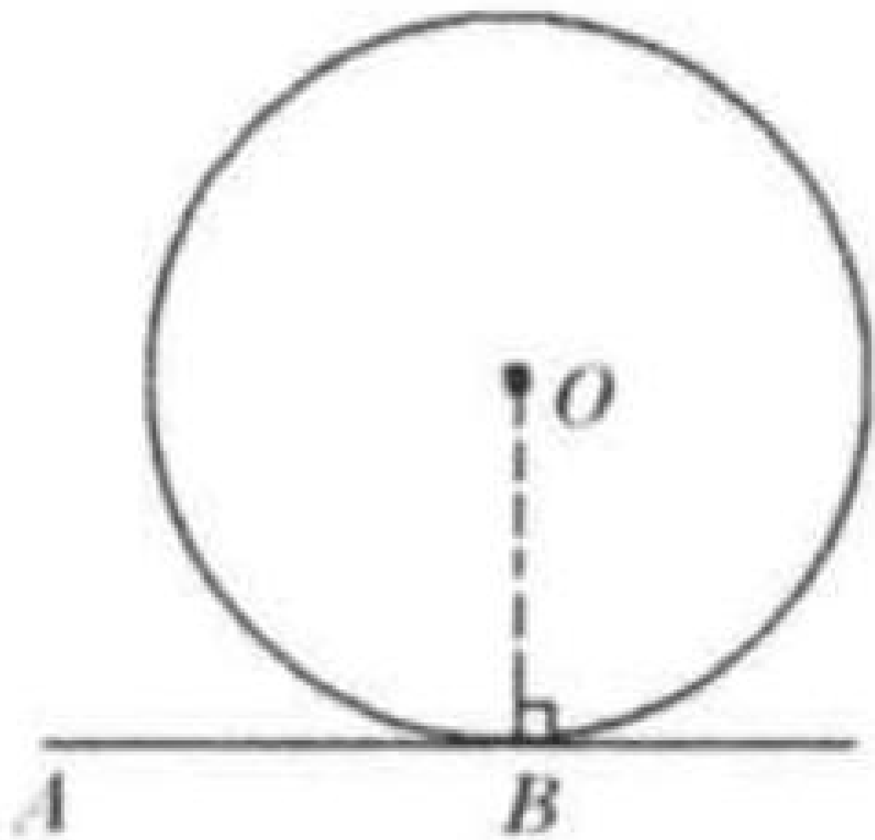


Since $AC//QG, \angle BAC = \angle BQG$.

Since $AC//SE, \angle DAC = \angle DSE$.
Since $\angle BAC = \angle DAC, \angle BQG = \angle DSE$.
$\angle AQG = 180° - \angle BQG = 180° - \angle DSE = \angle ASE$,
That is, $\angle AQG = \angle ASE$.
Therefore, $\triangle AQG \sim \triangle ASE \quad \Rightarrow \quad \angle QAG = \angle SAE$.
So $\angle GAC = \angle EAC$.

Chapter 6 Draw the Auxiliary Lines with Circles

1. Connect the center of the circle and points on the circumference B is the tangent point and O is the center. Connect OB. We have $OB \perp AB$.



Theorem 6.1. The radius of a circle is only perpendicular to a tangent line at the point of tangency.
Theorem 6.2. If a line is tangent to a circle, it is perpendicular to a radius at the point of tangency.
Theorem 6.3. A line perpendicular to a radius at a point on the circle is tangent to the circle at that point.

Theorem 6.4. A line perpendicular to a tangent line at the point of tangency with a circle contains the center of the circle.

Theorem 6.5. A line perpendicular to a chord of a circle and containing the center of the circle, bisects the chord and its major and minor arcs.

Theorem 6.6. The perpendicular bisector of a chord of a circle contains the center of the circle.