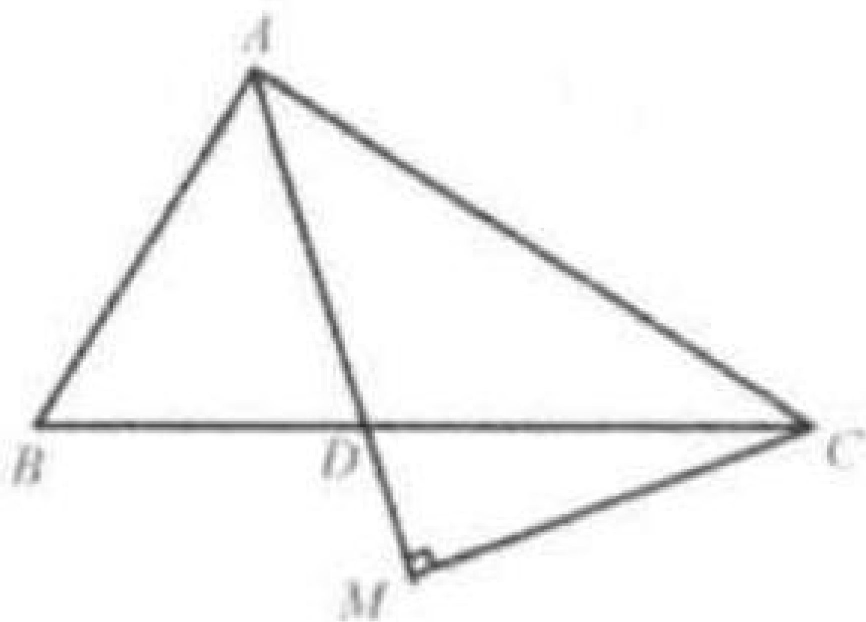
Example 6

As shown in the figure below, in $\triangle ABC, AD = AB.AD$ is the angle bisector of $\angle A$. $CM \perp AM$ at the extension of AD. Show that $AM = \frac{1}{2}(AB + AC)$.

Solution:



Method 1:
Extend AM to E such that $AM = ME$.
$AE = 2AM = AD + DE$.
Now we prove that $EC = DE$.

Since AD is the angle bisector of $\angle A, \angle BAD = \angle CAD = \alpha$.

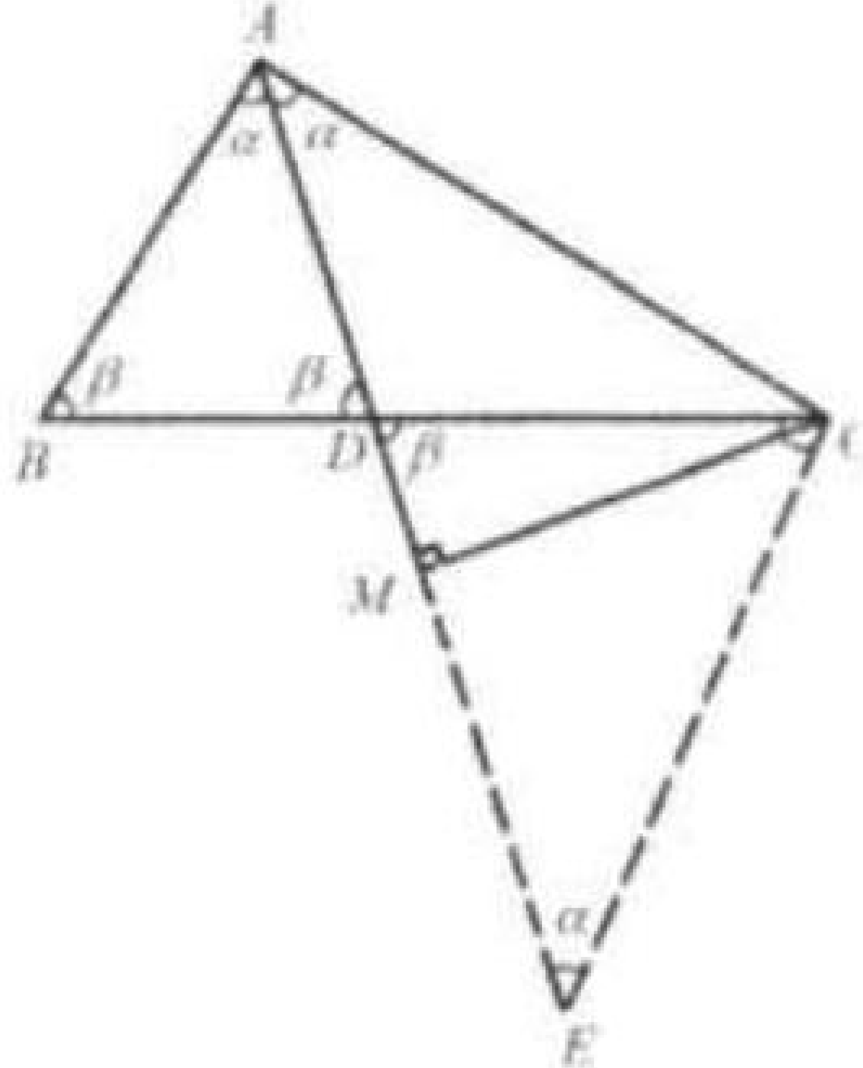
Since $AC = CE, \angle CEA = \angle CAE = \alpha$.

Since $AB = AD, \angle ABD = \angle ADB = \beta$.

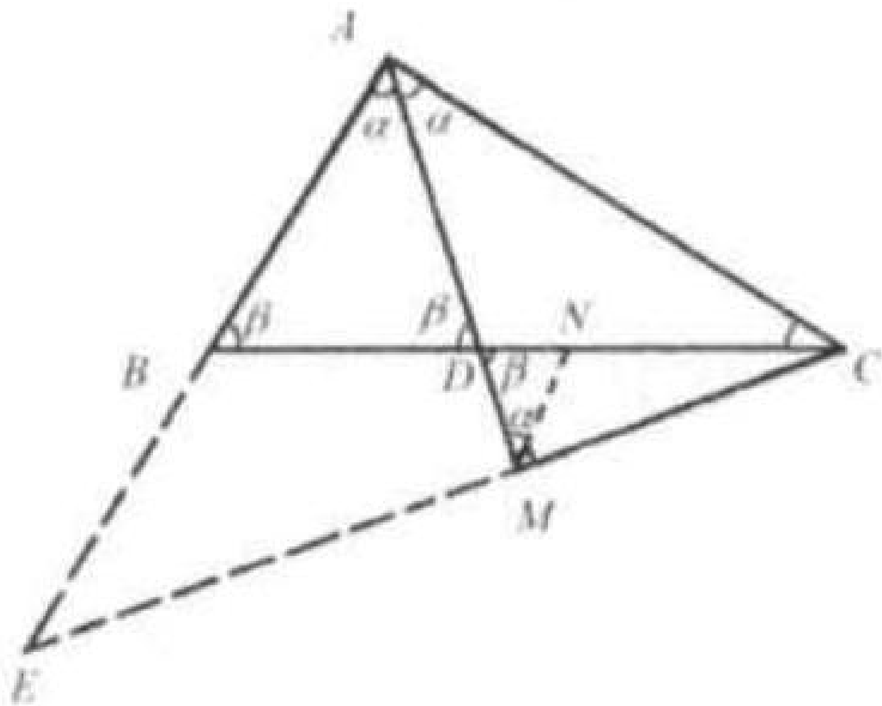
We also know that $\angle ADB = \angle CDE = \alpha$. (vertical angles).

Thus $\angle ECD = \angle EDC = \beta$. So $EC = DE$.

$2AM = AD + DE = AD + EC \Rightarrow AM = \frac{1}{2}(AB + AC)$.



Method 2: Extend CM to E to meet the extension of AB at E. So $AE = AC$.
Draw $MN//EA$ to meet BC at N.
Since AD is the angle bisector of $\angle A, AM$ is the angle bisector of $\angle A$. So
$\angle EAM = \angle CAM = \alpha \cdot AE = AC$



Since $MN//EA, \angle BAD = \angle NMD = \alpha$.
Since $AB = AD, \angle ABD = \angle ADB = \beta$.
We also know that M is the midpoint of $CE.MN//EB$, so $MN = \frac{1}{2}BE$.
In $\triangle MDN, \angle NMD = \alpha, \angle NDM = \beta$. So $\angle MND = \beta$.
Thus $MN = DM$.
$AM = AD+DM = AB+MN = AB+\frac{1}{2}BE = AB+\frac{1}{2}(AE-AB) = AB+\frac{1}{2}(AC$
$-AB) = \frac{1}{2}(AB + AC)$.