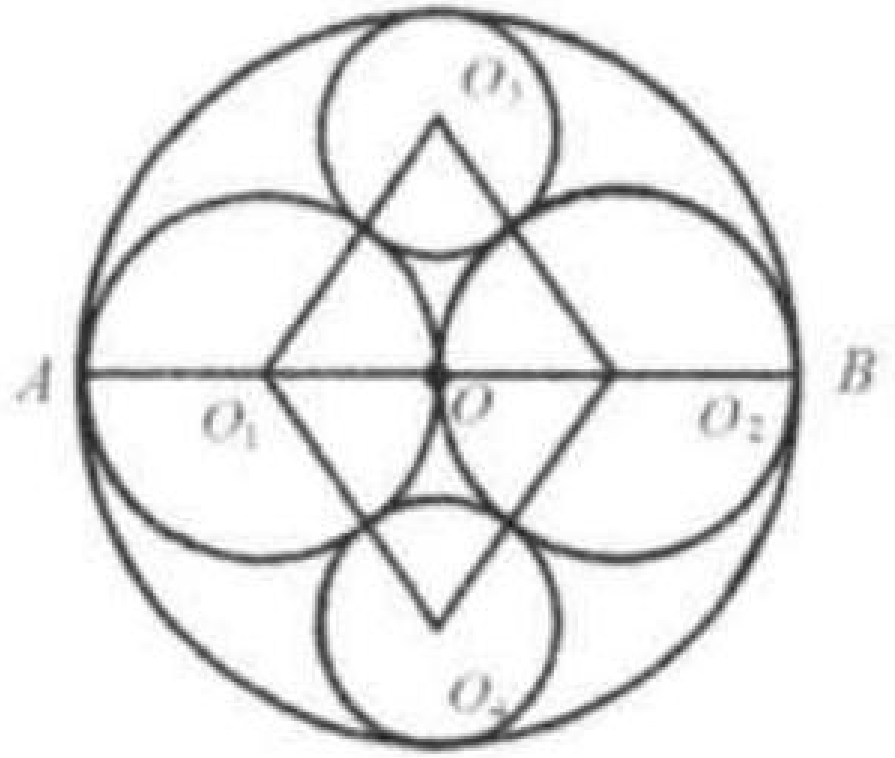
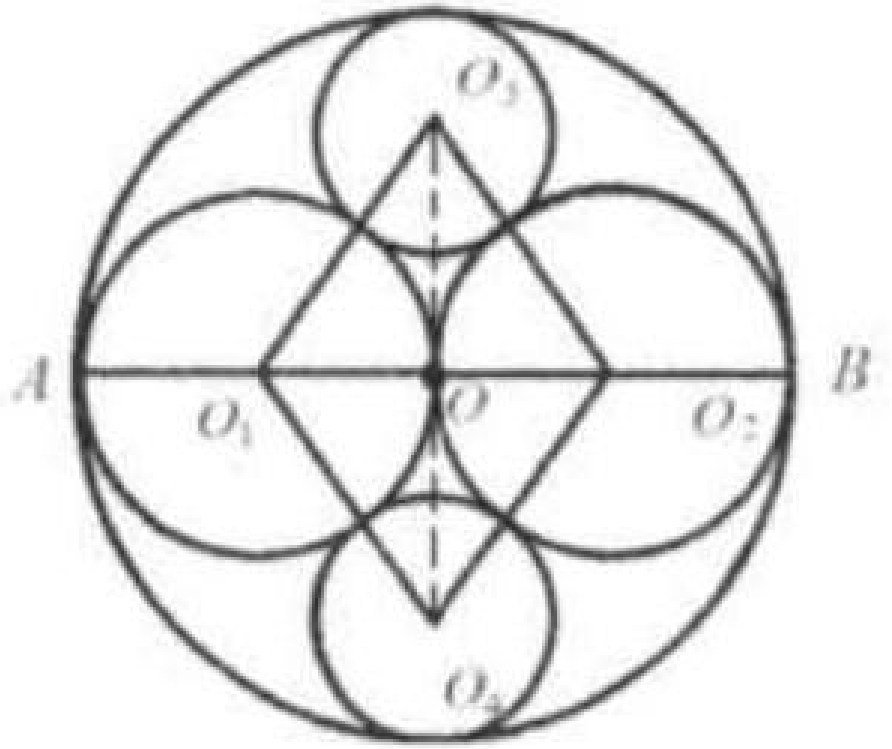
Example 10

(2002 China Middle School Math Contest) In the diagram, $AB = a$ cm is the diameter of circle O. Circle O_1 has the diameter of AO and is congruent to circle O_2, and both are tangent to circle O and to each other. Circle O_3 and circle O_4 are congruent and are tangent to circle O, to circle O_1 and to circle O_2. Find the area of quadrilateral $O_1O_2O_3O_4$.



Solution: 1: 9.
Let the radius of the circle O be R, the radius of the circle Q be r_1, and the radius of the circle S be r_2. We know that $r_1 = \frac{R}{2}$.
By the Pythagorean Theorem,
$(R - r_2)^2 = r_1^2 + (r_1 + r_2)^2 \Rightarrow R^2 - 2Rr_2 = 2r_1r_2 \Rightarrow$
$R^2 - 2Rr_2 = 2r_1r_2 \Rightarrow r_2 = \frac{R}{3}$.



The ratio of the areas of the smallest circle and largest circle is $\frac{\pi {r_2}^2}{\pi R^2} = \frac{{r_2}^2}{R^2} = \frac{\left(\frac{R}{3}\right)^2}{R^2} = \frac{1}{9}$.