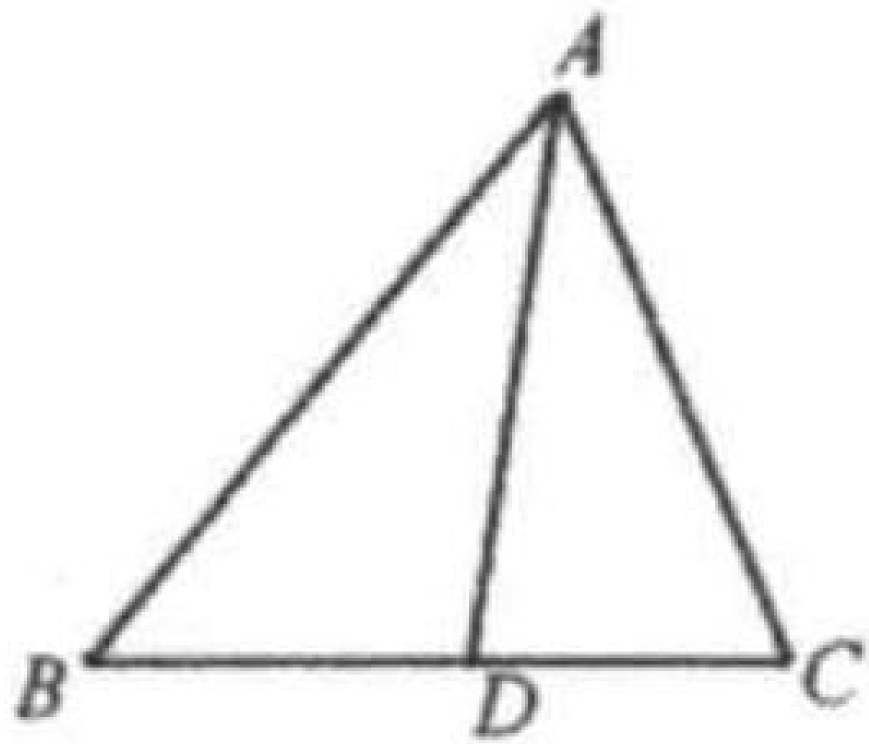
Example 4

In $\triangle ABC$, the angle bisector of $\angle A$ meets BC at D. Show that $AD^2 = AB \cdot AC - BD \cdot CD$.

Solution: Construct a circle that circumscribes the triangle as shown in the figure. Extend AD to meet the circle at E and connect BE.

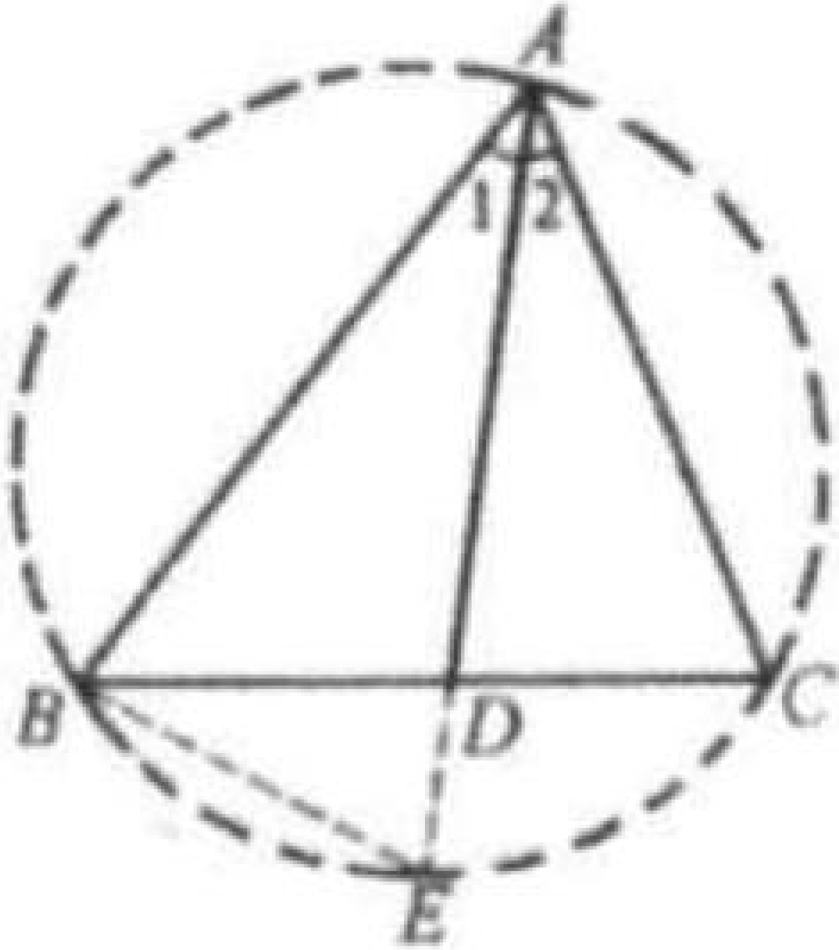


Since $\angle E = \angle C$ (both face the same arc AB).$\angle 1 = \angle 2$, $\triangle ABE \sim \triangle ADC$

$$AB \cdot AC = AD \cdot AE$$

$$BD \cdot DC = AD \cdot DE$$

$$(1) - (2) :$$



$$AB \cdot AC - BD \cdot CD = AD \cdot AE - AD \cdot DE$$
$$= AD(AE - DE) = AD \cdot AD = AD^2$$

Therefore $AD^2 = AB \cdot AC - BD \cdot CD$.