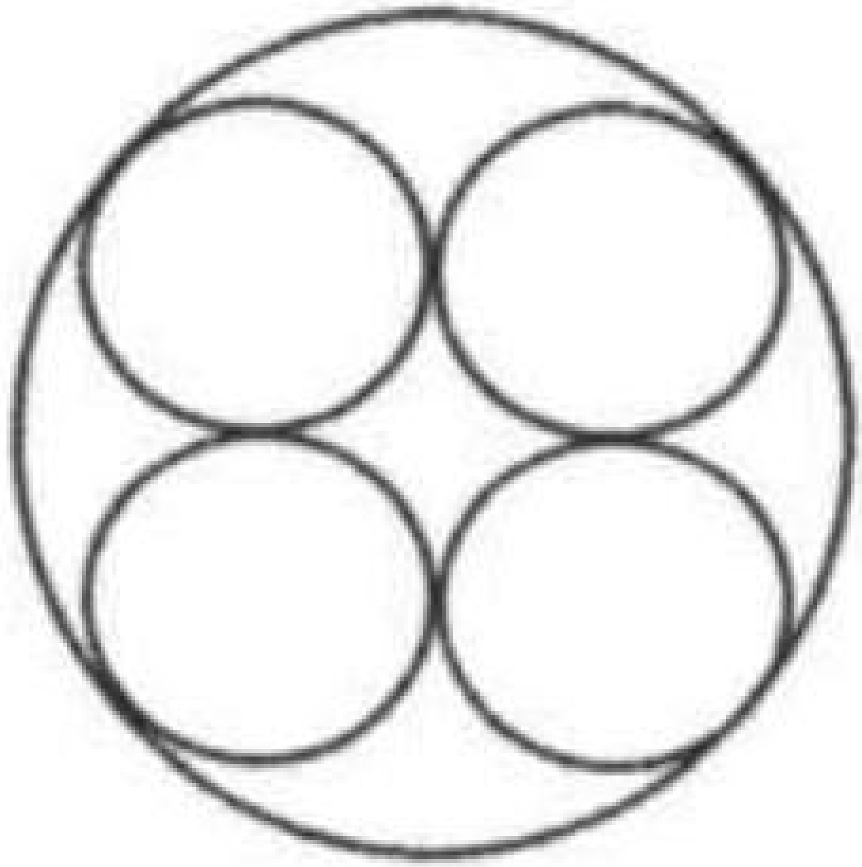
Problem

Four circles of radius r are mutually tangent inside a circle of radius one unit.
Find the radius r. Express your answer in simplified radical form.



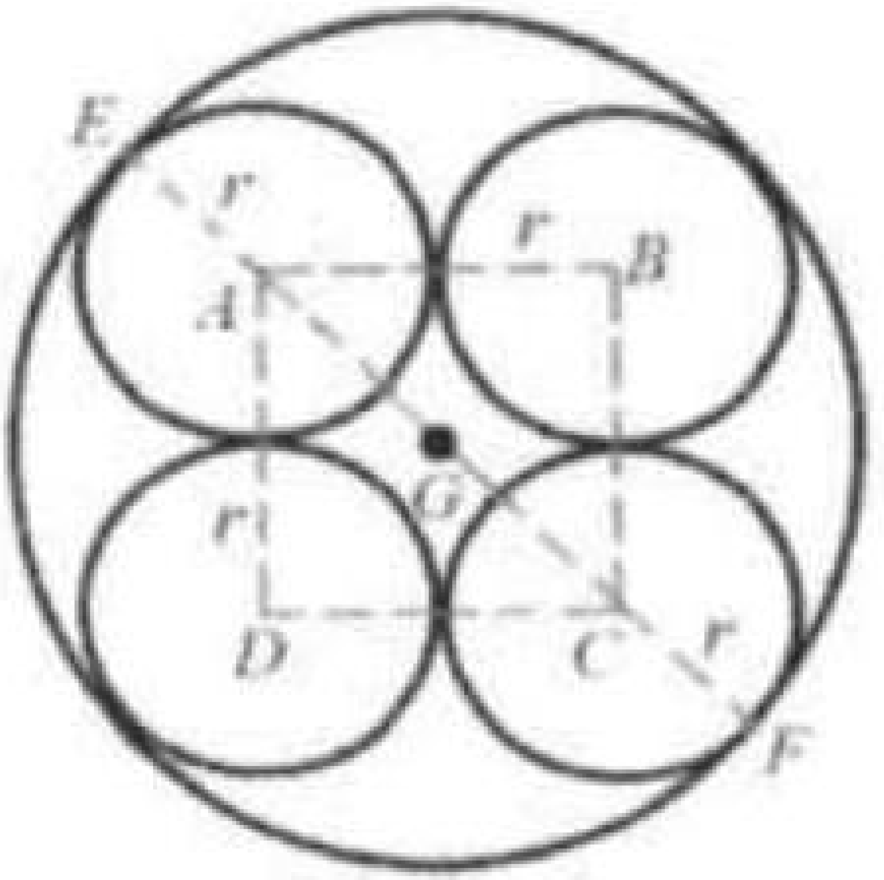
Solution

$\sqrt{2}-1.$

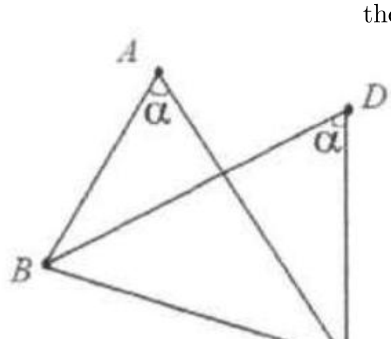
We draw the diameter EF as shown in the figure.

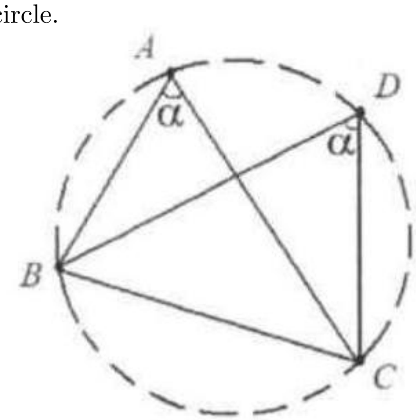
Applying Pythagorean Theorem to right triangles ADC :

$(2r)^2+(2r)^2=(2-2r)^2 \quad \Rightarrow \quad r=\sqrt{2}-1.$

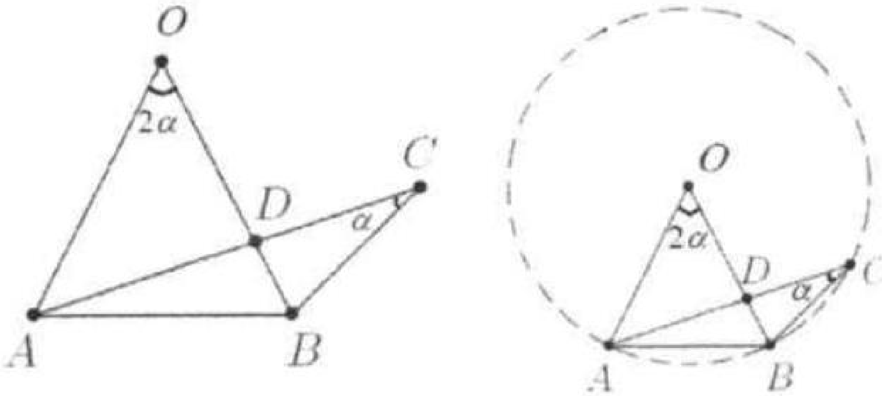


4. When four points are concyclic, draw the circle. 4.1. Triangles ABC and DBC share the side BC. If $\angle BAC = \angle BDC$, points A, B, C, and D lie on the same circle.

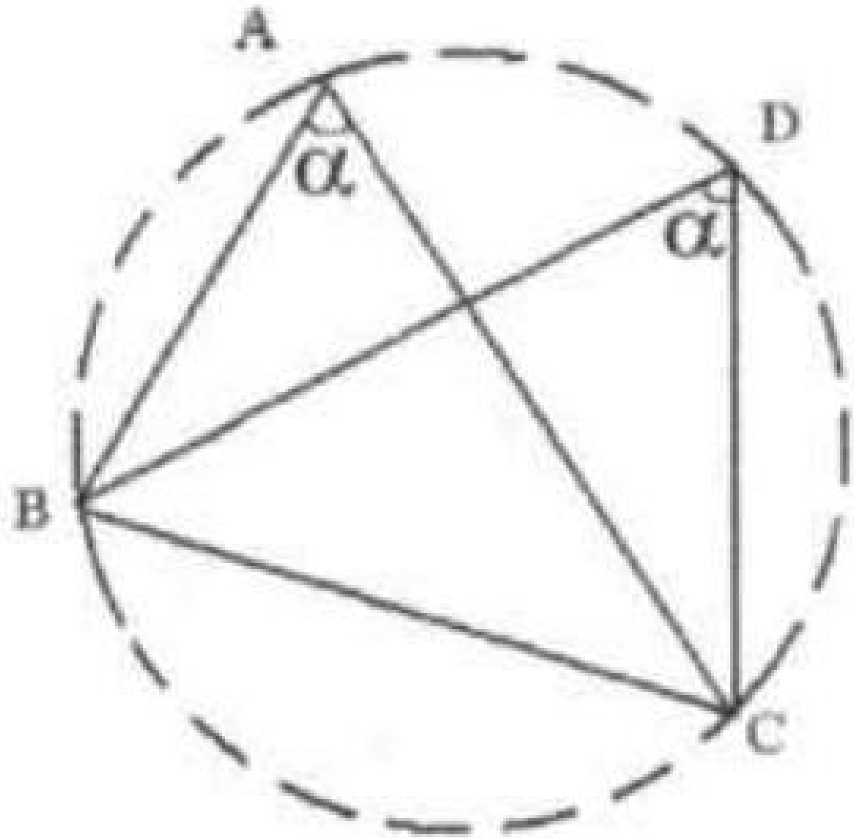




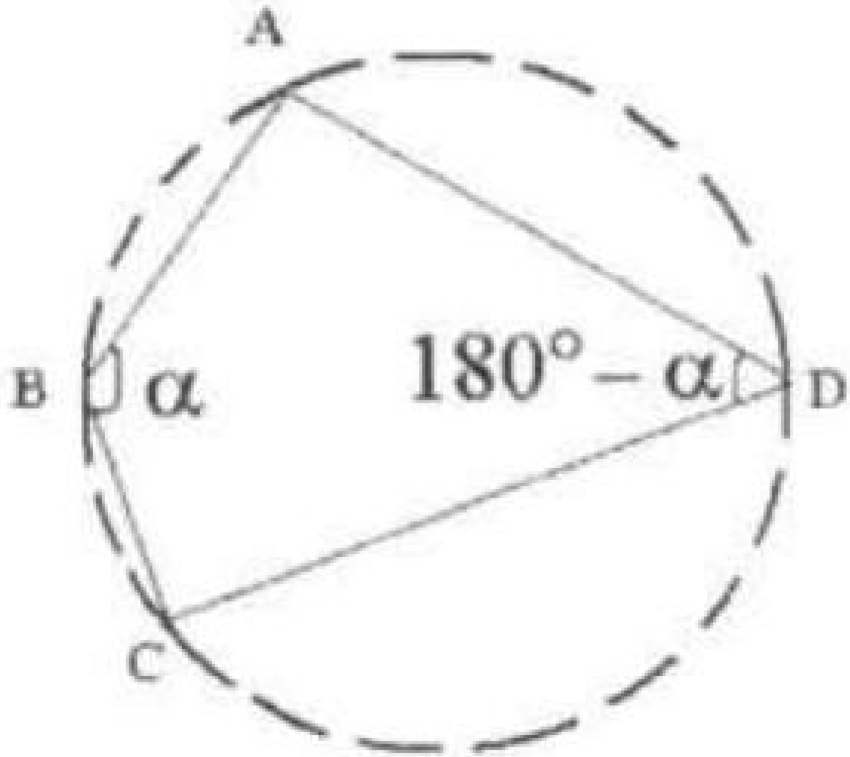
4.2. Triangles ABC and ABO share the side AB. If $\angle AOB = 2\angle ACB$, points A, B, C lie on the same circle O.



Theorem 6.10. A quadrilateral is cyclic (i.e. may be inscribed in a circle) if one side subtends congruent angles at the two opposite vertices.
If $\angle BAC = \angle BDC$, points A, B, C, and D are concyclic.



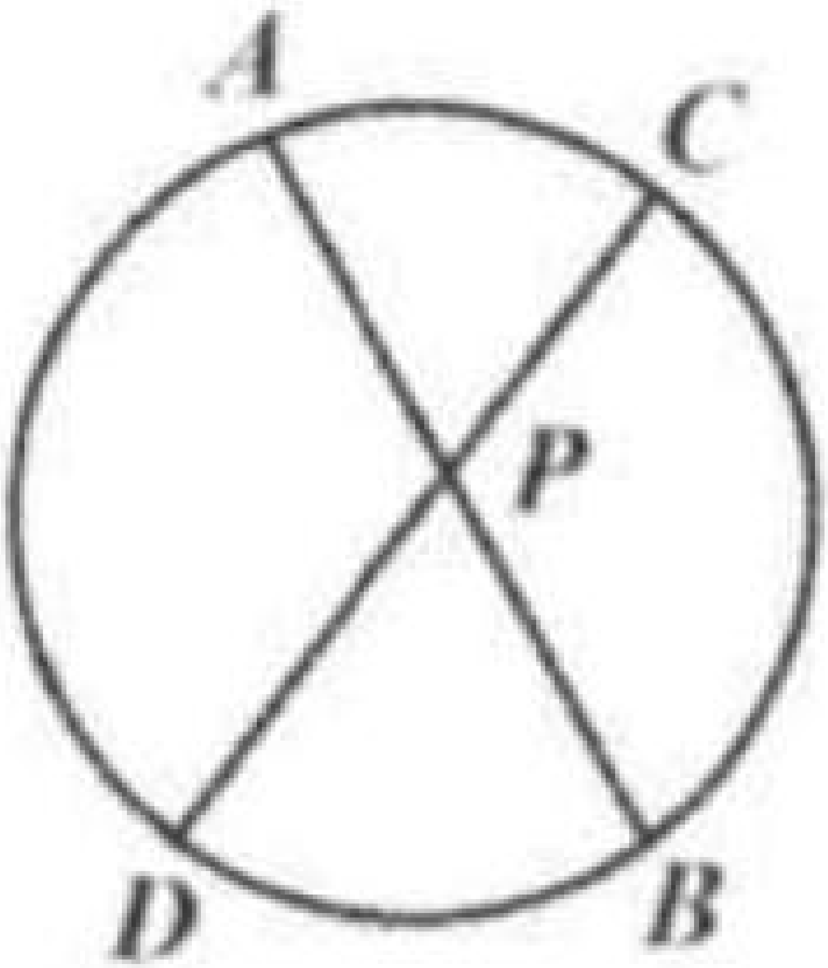
Theorem 6.11. The opposite angles of a cyclic (inscribed) quadrilateral are supplementary.
If the opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed by a circle.
If $\angle B + \angle D = 180°$, points A, B, C, and D are concyclic.



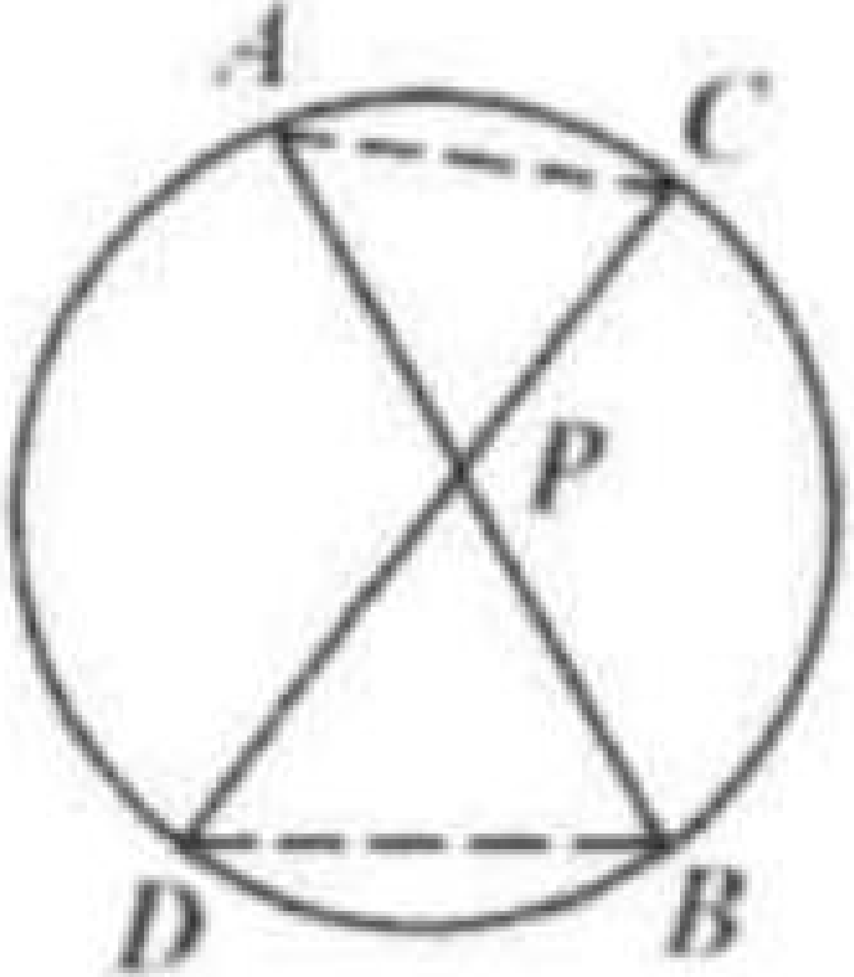
Theorem 6.12. Power of a Point formula (1): If two chords of a circle intersect, the product of the measures of the segments of one chord equals the product of the segments of the other chord.

$PA \times PB = PC \times PD$.

Proof: Connect AC and BD.$\angle PAC = \angle PDB$.



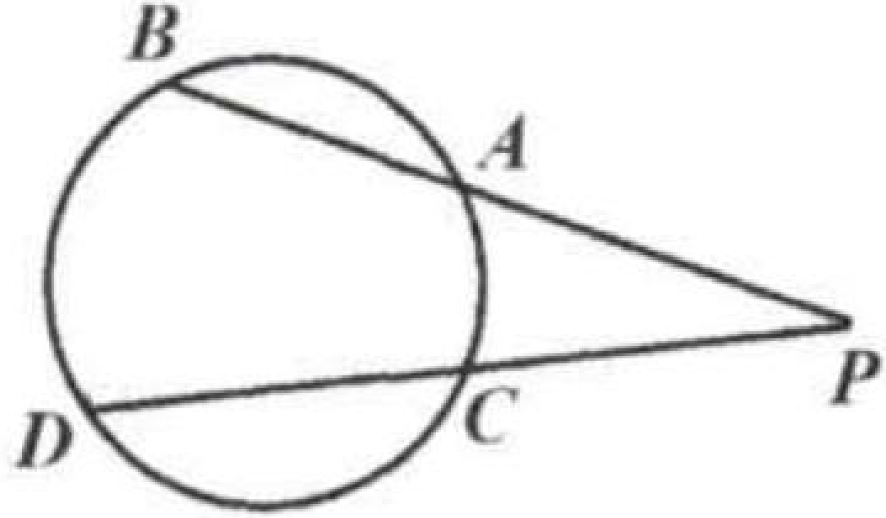
We know that $\angle APC = \angle BPD$. Thus $\triangle PAC \sim \triangle PDB$.
$\frac{PA}{PC} = \frac{PD}{PB} \quad \Rightarrow PA \cdot PB = PC \cdot PD$.
The converse of Theorem 6.12.



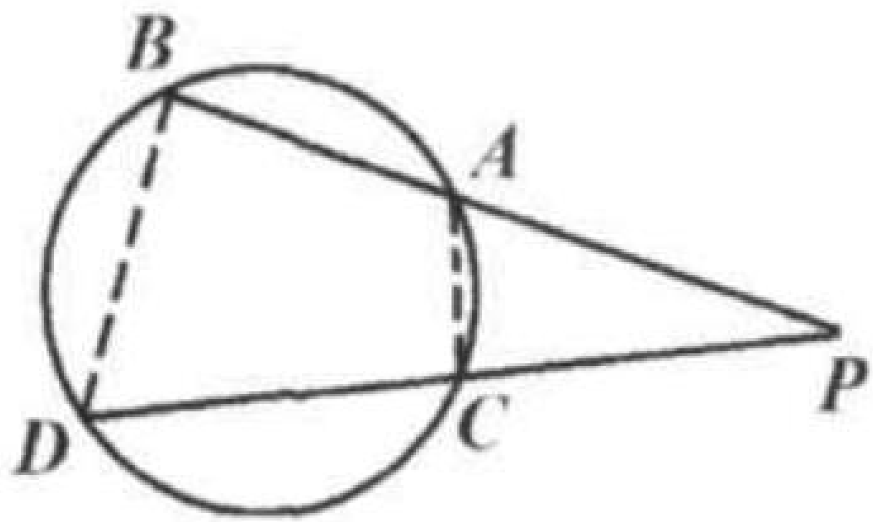
If $AP \times PB = CP \times PD$, points A, C, B, and D are concyclic.
Theorem 6.13. Power of a Point formula (2): If two secants intersect outside the circle, the product of the measures of one secant and its external segment equals the product of the measures of the other secant and its external segment.

$$PA \times PB = PC \times PD.$$

Proof:

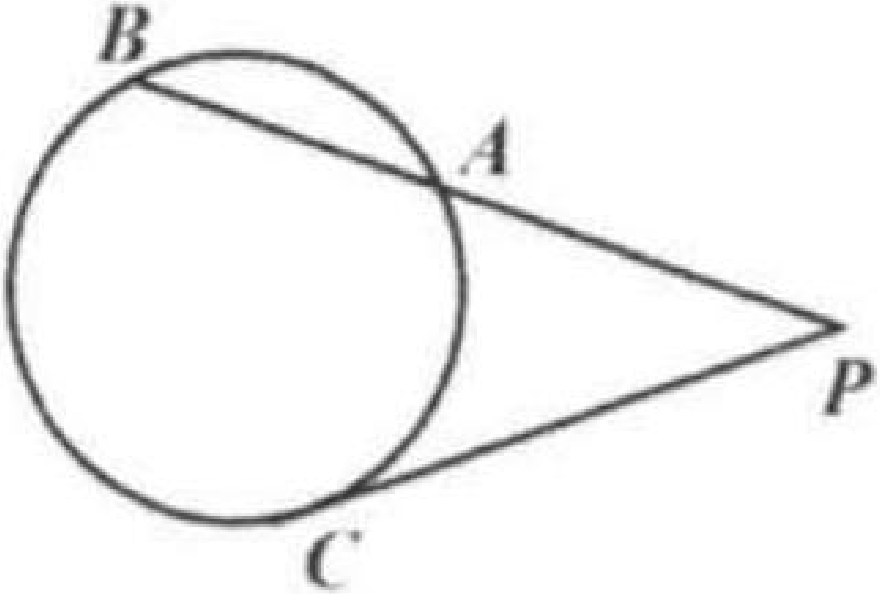


Connect AC and BD.$\angle PAC = \angle PDB$.
We know that $\angle APC = \angle BPD$. Thus $\triangle PAC \sim \triangle PDB$.
$\frac{PA}{PC} = \frac{PD}{PB} \quad \Rightarrow PA \cdot PB = PC \cdot PD$.

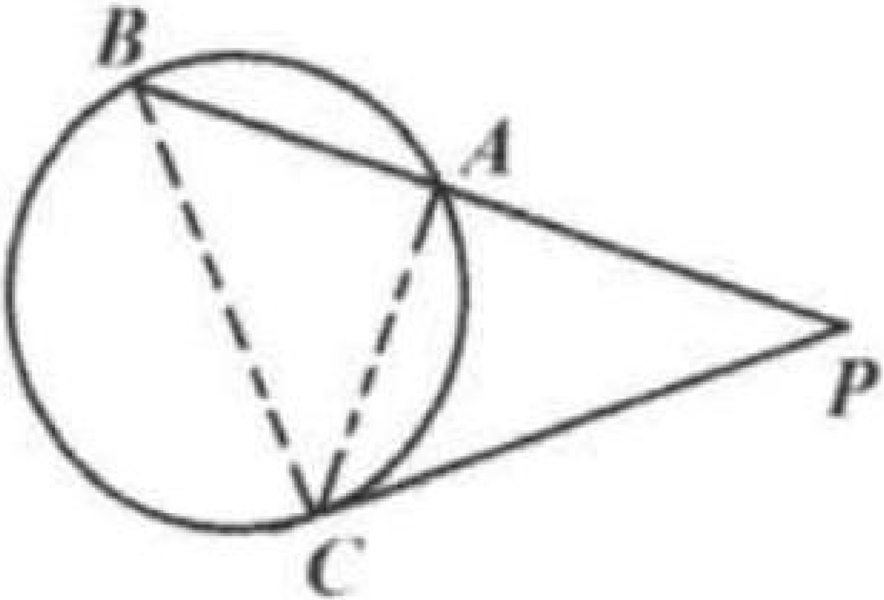


The converse of Theorem 6.13. If $PA \cdot PB = PC \cdot PD$, points A, B, D, and C are concyclic.

Theorem 6.14. Power of a Point formula (3): $PC^2 = PB \times PA$



Proof: Connect AC and BC.$\angle PCA = \angle PBC$ (both face the same arc AC).
We know that $\angle APC = \angle BPC$. Thus, $\triangle PAC \sim \triangle PCB$.
$\frac{PA}{PC} = \frac{PC}{PB} \quad \Rightarrow PC^2 = PA \times PB$

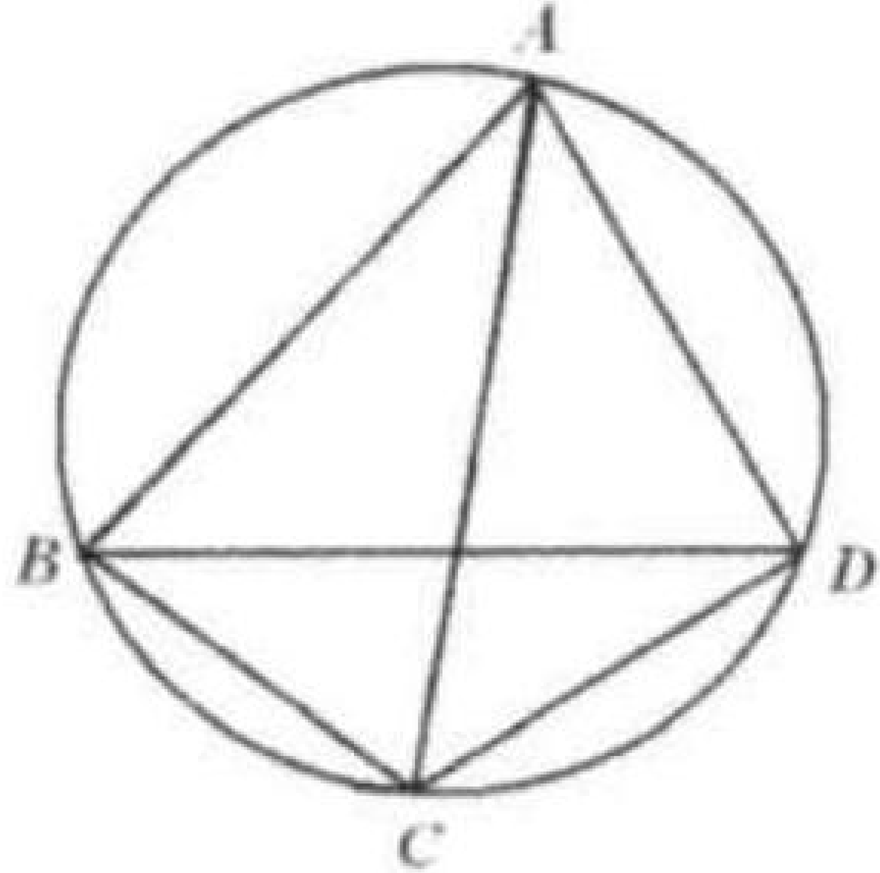


Theorem 6.15. Ptolemy's Theorem: If quadrilateral $ABCD$ is a cyclic quadrilateral, then $AC \times BD = AB \times CD + AD \times BC$.

Proof: Method 1:

Extend AB to P such that $\angle PCA = \angle DCB$.

Then $\triangle ACP \sim \triangle DCB. \frac{AC}{CD} = \frac{AP}{BD}$.



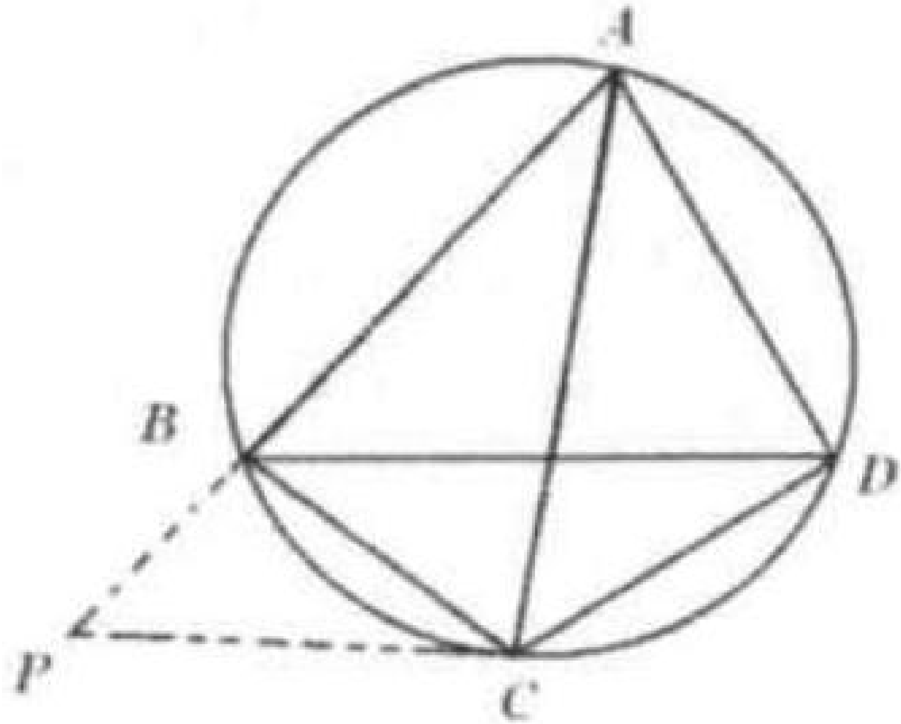
So $AC \cdot BD = CD \cdot AP$

We also have $\angle CBP = \angle ADC, \angle BPC = \angle CBD = \angle CAD$.

Then $\triangle ACD \sim \triangle PCB$. We have $\frac{AD}{PB} = \frac{CD}{BC}$ or

$AD \cdot BC = CD \cdot PB$

(1) - (2): $AC \cdot BD - AD \cdot BC = CD(AP - PB) = AB \cdot CD$.



That is, $AC \cdot BD = AB \cdot CD + BC \cdot AD$.