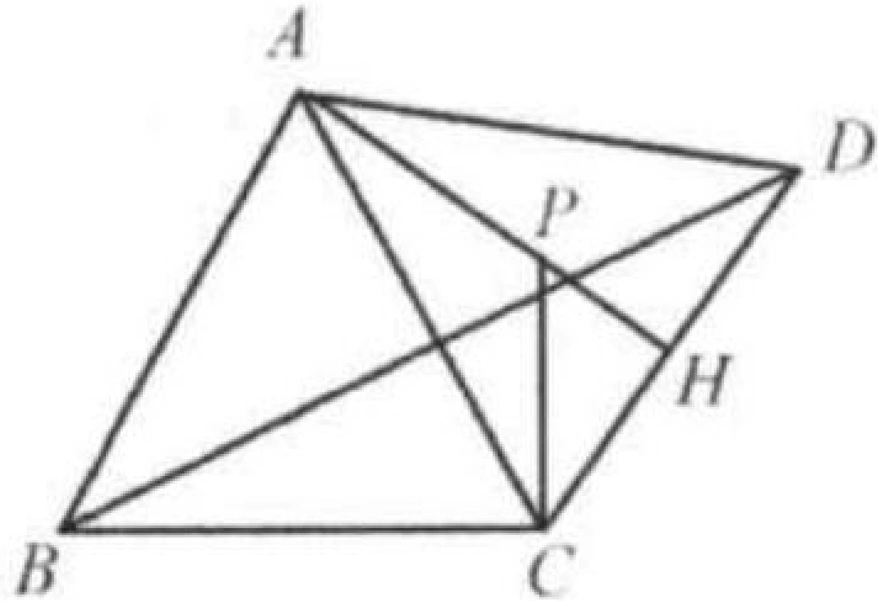
Example 6

(1984 China Middle School Math Contest) As shown in the figure, $AB = BC = CA = AD. AH \perp CD$ at $H, CP \perp BC$ at C and meets AH at P. Prove that $S = \frac{\sqrt{3}}{4} AP \times BD$, where S is the area of $\triangle ABC$.



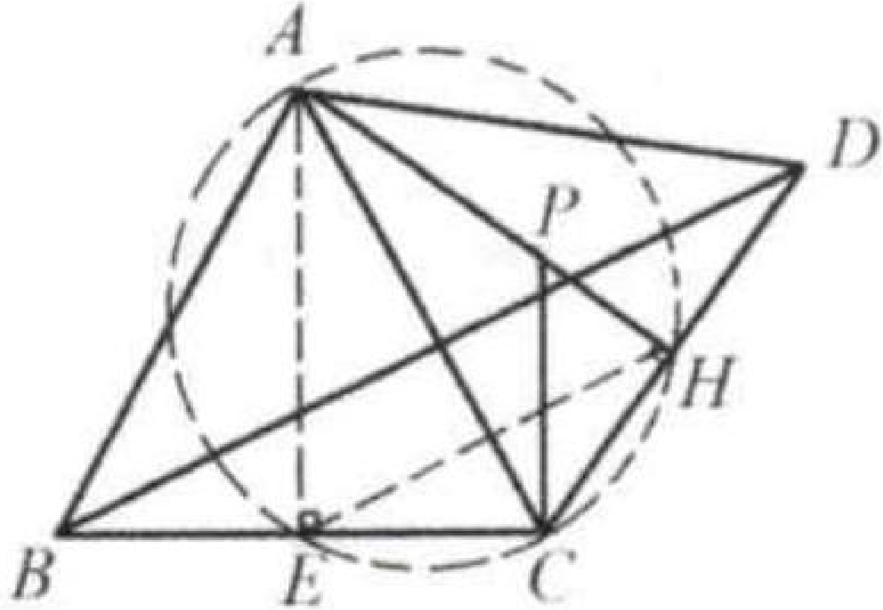
Solution: Method 1:

Draw $AE \perp BC$ at E. E is the midpoint of BC. H is the midpoint of DC.

Connect EH. $EH // BD$.

Then $\angle HEC = \angle DBC$.

Since $AH \perp CD, AE \perp BC$, points A, H, C, and E are concyclic. Therefore,



$\angle HAC = \angle HEC = \angle DBC$.

We know that $\angle EAC = \angle EHC = \angle BDC = 30^\circ$.

$\angle PCA = 90^\circ - 60^\circ = 30^\circ$. So $\angle PCA = \angle BDC$.

Thus $\triangle ACP \sim \triangle BDC$.

So $\frac{AP}{BC} = \frac{AC}{BD} \Rightarrow AP \times BD = BC \times AC \Rightarrow$

$S_{ABC} = \frac{\sqrt{3}}{4}AC^2 = \frac{\sqrt{3}}{4}BC \times AC = \frac{\sqrt{3}}{4}AP \times BD$.

Method 2:

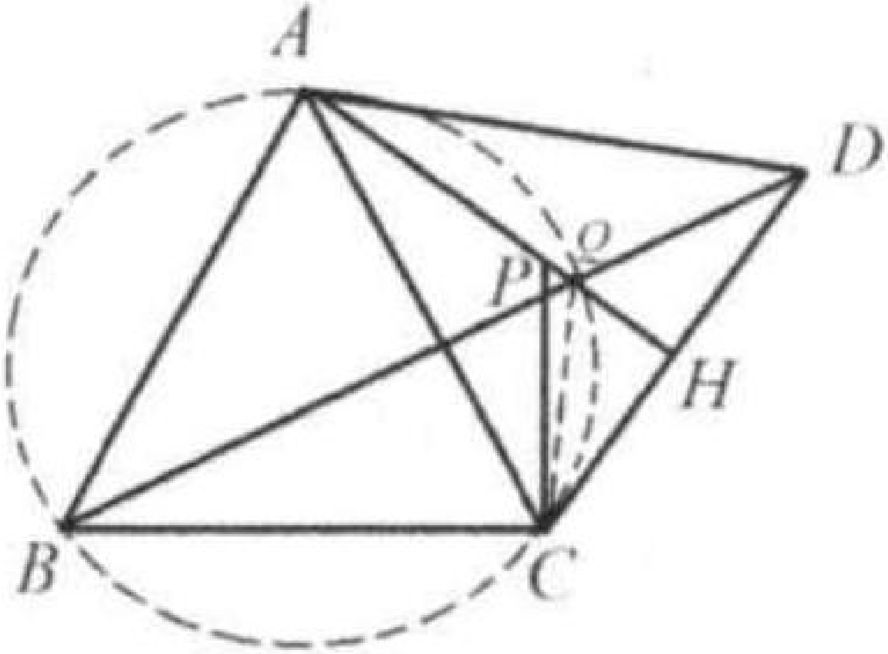
Let the point of intersection of BD and AH be Q.

Since $AH \perp CD, AC = AD, \angle ACQ = \angle ADQ$.

Since $AB = AD, \angle ADQ = \angle ABQ$.

$\angle ABQ = \angle ACQ$.

Points A, B, C, and Q are concyclic.



Therefore, $\angle AQB = \angle ACB = 60^\circ$.

$\angle DQ = 60^\circ$.

Since $\angle QHD = 90^\circ, \angle BDC = 90^\circ - 60^\circ = 30^\circ$.

Since $\angle ACP = 90^\circ - 60^\circ = 30^\circ, \angle ACP = \angle BDC$

Since $\angle APC = 90^\circ + \angle PCH, \angle BCD = 90^\circ + \angle PCH, \angle APC = \angle BCD$

From (1) and (2): $\triangle APC \sim \triangle BCD$.

So $\frac{AC}{BD} = \frac{AP}{BC} \Rightarrow AP \times BD = BC \times AC \Rightarrow$

$S_{ABC} = \frac{\sqrt{3}}{4} AC^2 = \frac{\sqrt{3}}{4} BC \times AC = \frac{\sqrt{3}}{4} AP \times BD$.

Method 3:

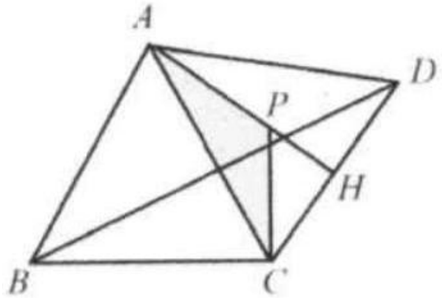
We want to get: $S = \frac{\sqrt{3}}{4} AP \times BD$.

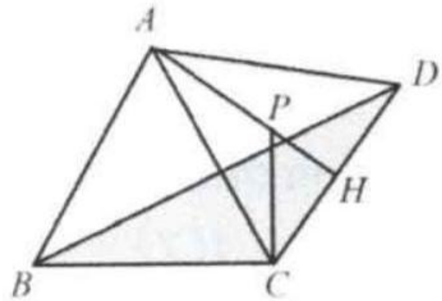
We know that the area of any equilateral triangle with side length of a is:

$S = \frac{\sqrt{3}}{4} a^2$, in our case, $S = \frac{\sqrt{3}}{4} AB^2$.

We must have $AB^2 = AP \times BD \quad \Rightarrow \quad \frac{AB}{BD} = \frac{AP}{AB}$ or $\frac{AC}{AP} = \frac{BD}{BC}$.

This is the ratio of two sides of two similar triangles $\triangle APC$ and $\triangle BCD$ as shown. So we only need to prove that $\triangle APC \sim \triangle BCD$.





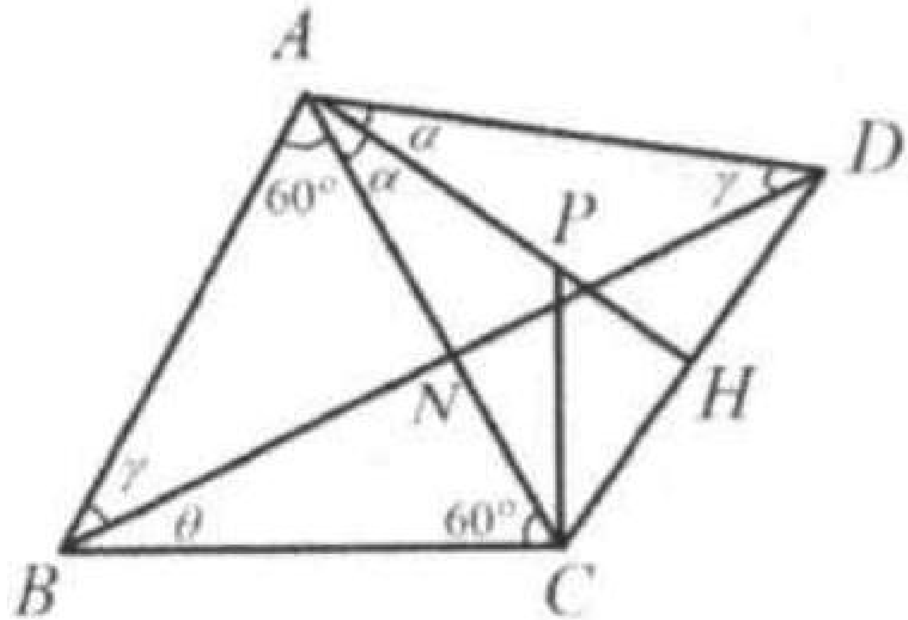
We label each angle as shown below. We name the point of intersection of AC and BD be N.

From triangle BAD, We have

$$60^\circ + \alpha + \alpha + \gamma + \gamma = 180^\circ$$

From triangles BNC, AND, We have

$$60^\circ + \theta = 2\alpha + \gamma$$



(1) can be written as

$$\alpha + \gamma = 60^\circ$$

From (2) and (3): $\theta = \alpha$

So we get $\angle CAP = \angle CBD$. We need one more pair of congruent angles.

In right triangle $AHD, \alpha + \gamma + \angle BDC = 90^\circ(5)$

Substituting (3) into (5): $\angle BDC = 30^\circ$.

Since $CP \perp BC$ at C and $\angle ACB = 60^\circ . \angle APC = 30^\circ$.

Thus $\triangle APC \sim \triangle BCD$ and we are done.

Note: the first two solutions are official solution and the third one is our solution.

We know that $\angle EAC = \angle EHC = \angle BDC = 30°$.

$\angle PCA = 90° - 60° = 30°$. So $\angle PCA = \angle BDC$.

Thus $\triangle ACP \sim \triangle BDC$.

So $\frac{AP}{BC} = \frac{AC}{BD} \Rightarrow AP \times BD = BC \times AC \Rightarrow$

$S_{ABC} = \frac{\sqrt{3}}{4}AC^2 = \frac{\sqrt{3}}{4}BC \times AC = \frac{\sqrt{3}}{4}AP \times BD$.

Method 2:

Let the point of intersection of BD and AH be Q.

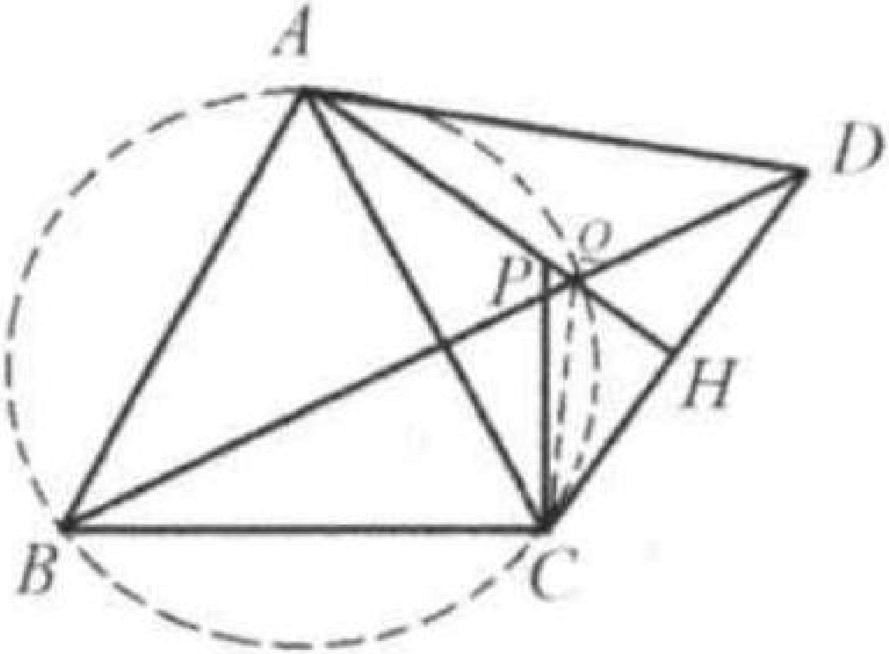
Since $AH \perp CD, AC = AD, \angle ACQ = \angle ADQ$.

Since $AB = AD, \angle ADQ = \angle ABQ$.

$\angle ABQ = \angle ACQ$.

Points A, B, C, and Q are concyclic.

Therefore, $\angle AQB = \angle ACB = 60°$.



$\angle DQ = 60°$.

Since $\angle QHD = 90°, \angle BDC = 90° - 60° = 30°$.

Since $\angle ACP = 90° - 60° = 30°, \angle ACP = \angle BDC$

Since $\angle APC = 90° + \angle PCH, \angle BCD = 90° + \angle PCH, \angle APC = \angle BCD$

From (1) and (2): $\triangle APC \sim \triangle BCD$.

So $\frac{AC}{BD} = \frac{AP}{BC} \Rightarrow AP \times BD = BC \times AC \Rightarrow$

$S_{ABC} = \frac{\sqrt{3}}{4}AC^2 = \frac{\sqrt{3}}{4}BC \times AC = \frac{\sqrt{3}}{4}AP \times BD$.

Method 3:

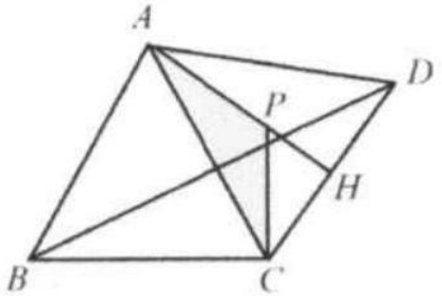
We want to get: $S = \frac{\sqrt{3}}{4}AP \times BD$.

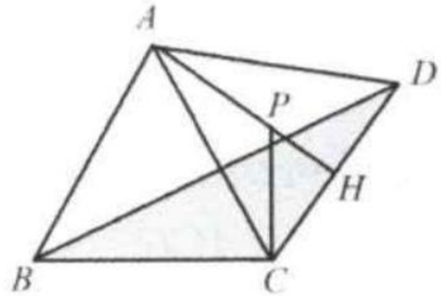
We know that the area of any equilateral triangle with side length of a is:

$S = \frac{\sqrt{3}}{4}a^2$, in our case, $S = \frac{\sqrt{3}}{4}AB^2$.

We must have $AB^2 = AP \times BD \quad \Rightarrow \quad \frac{AB}{BD} = \frac{AP}{AB}$ or $\frac{AC}{AP} = \frac{BD}{BC}$.

This is the ratio of two sides of two similar triangles $\triangle APC$ and $\triangle BCD$ as shown. So we only need to prove that $\triangle APC \sim \triangle BCD$.





We label each angle as shown below. We name the point of intersection of AC and BD be N.

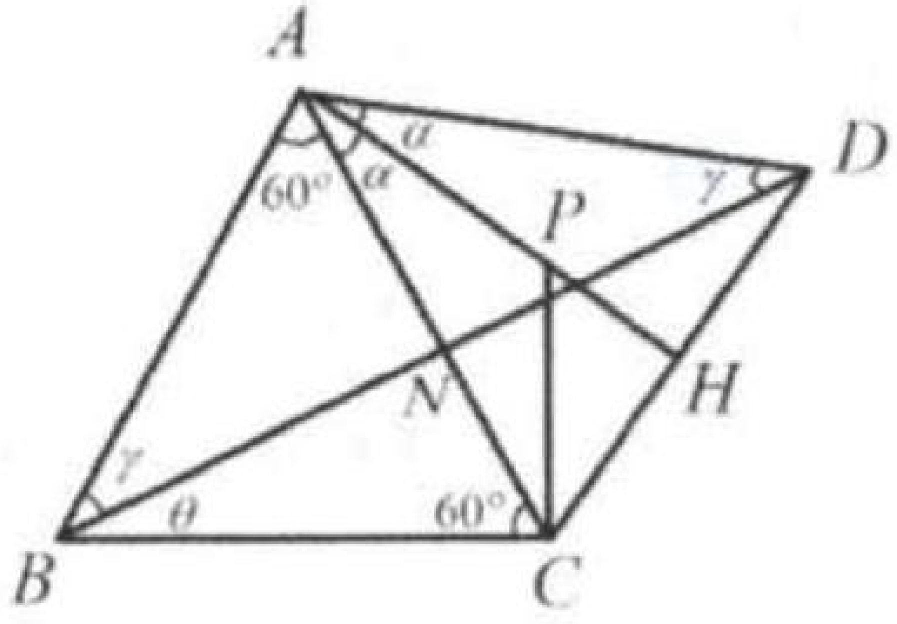
From triangle BAD, We have

$$60^\circ + \alpha + \alpha + \gamma + \gamma = 180^\circ$$

From triangles BNC, AND, We have

$$60^\circ + \theta = 2\alpha + \gamma$$

(1) can be written as



$$\alpha + \gamma = 60^\circ$$

From (2) and (3): $\theta = \alpha$

So we get $\angle CAP = \angle CBD$. We need one more pair of congruent angles.

In right triangle $AHD, \alpha + \gamma + \angle BDC = 90^\circ (5)$

Substituting (3) into (5): $\angle BDC = 30°$.
Since $CP \perp BC$ at C and $\angle ACB = 60°.\angle APC = 30°$.
Thus $\triangle APC \sim \triangle BCD$ and we are done.
Note: the first two solutions are official solution and the third one is our solution.