Statistical Data Analysis, Lecture 2

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Topics in this course

1. Summarizing data
2. Exploring distributions
3. Density estimation
4. Bootstrap methods
5. Nonparametric tests
6. Analysis of categorical data
7. Multiple linear regression

Chapter 3: Exploring distributions

Contents of Chapter 3:

1. Quantile function
2. Location-scale family
3. QQ-plots and symplots
4. Goodness-of-fit tests
   - Shapiro-Wilk test
   - Kolmogorov-Smirnov test
   - Chi-square test

distributions

Distribution functions

Random variable $X : \Omega \to \mathbb{R}$

Population distribution function $F(x) = P(X \leq x), \quad x \in \mathbb{R}$

Empirical distribution function given sample $x_1, \ldots, x_n$:

$$\widehat{F}_n(x) = \frac{1}{n} \sum_{j=1}^{n} 1_{\{x_j \leq x\}}, \quad x \in \mathbb{R}$$

In R, sample stored in vector `x`:

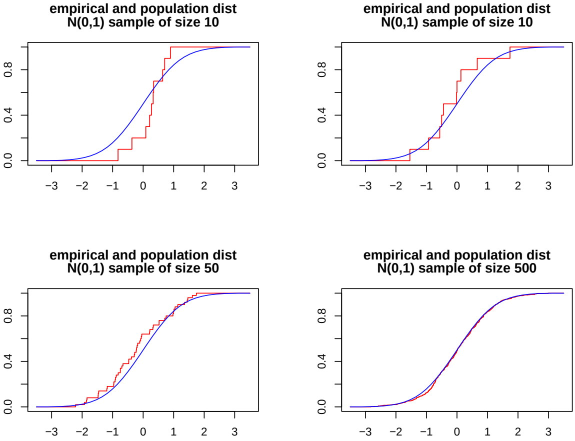
```
x <- rnorm(10, 0, 1)
plot(ecdf(x), col="red")
```

compare to e.g. $N(0,1)$ distribution function:

```
x.lattice <- seq(-3,3,0.001)
lines(x.lattice, pnorm(x.lattice,mean=0,sd=1), col="blue")
```

Example distribution functions

samples $x_1, \ldots, x_n$ from $N(0,1)$:



Goal (1)

Distribution: part of (parametric) statistical model.

Empirical distribution $\rightsquigarrow$ set up statistical model.

Goal: find underlying distribution

Goal (2)

Questions:

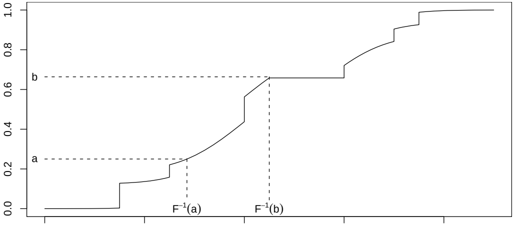
- One sample data from
  - specific distribution? (QQ-plot, goodness-of-fit tests)
  - symmetric distribution? (symplot)
- Two sample data from
  - same distribution? (QQ-plot)

quantile function and location-scale family

Quantile function

Quantile function F^{-1}: "inverse" of F.

Definition $F^{-1}(\alpha) = \inf\{x : F(x) \geq \alpha\}, \quad \alpha \in (0, 1).$



R: qnorm, qexp, qpois, etc.

Definition location-scale family

If $X \sim F$, denote $Y = a + bX \sim F_{a,b}, \quad a \in \mathbb{R},\ b > 0.$

$$\Rightarrow \quad F_{a,b}(x) = F\left(\frac{x-a}{b}\right)$$

Location-scale family (LSF) w.r.t. F: $\quad \{F_{a,b} : a \in \mathbb{R}, b > 0\}$

$\mathrm{E}(X) = 0\ \ \&\ \ \mathrm{var}(X) = 1 \Longrightarrow \mathrm{E}(Y) = a\ \ \&\ \ \mathrm{var}(Y) = b^2.$

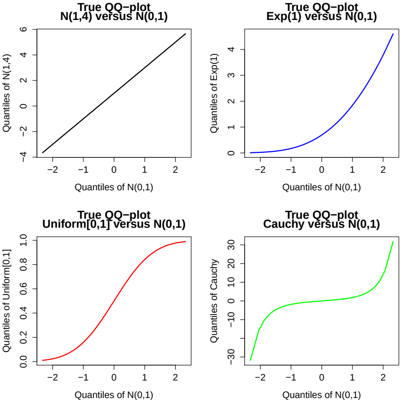
Quantiles of F and $F_{a,b}$ (1)

Claim $F_{a,b}^{-1}(\alpha) = a + bF^{-1}(\alpha)$.

Proof (for invertible F):

Quantiles of F and $F_{a,b}$ (2)

Hence, $\{(F^{-1}(\alpha), F_{a,b}^{-1}(\alpha)) : \alpha \in (0,1)\}$ forms line $y = a + bx$.



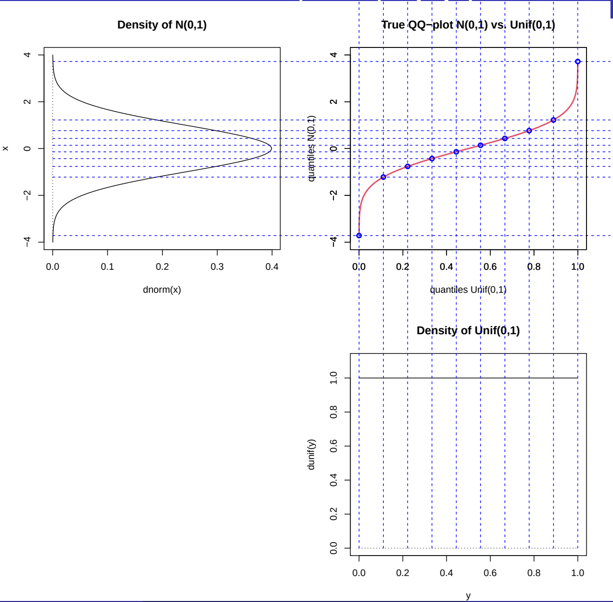
$N(0,1)$ & $N(1,4)$: same LSF.

$N(0,1)$ & $Exp(1)$: not same LSF.

$N(0,1)$ & $Unif(0,1)$: not same LSF.

$N(0,1)$ & $Cauchy(1)$: not same LSF.

Relative heaviness of tails in (true) QQ-plots



QQ-plots and symplots

Definition QQ-plot

$$X_1, \ldots, X_n \overset{i.i.d.}{\sim} F \text{ continuous:} \quad \mathrm{E}F(X_{(i)}) = \tfrac{i}{n+1} \quad \Rightarrow \quad X_{(i)} \approx F^{-1}(\tfrac{i}{n+1}).$$

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If $Y_i = a + bX_i, \ i = 1, \dots, n:$ $\mathrm{E}F_{a,b}(Y_{(i)}) = \frac{i}{n+1}$.

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$\Rightarrow \quad Y_{(i)} \approx F_{a,b}^{-1}(\frac{i}{n+1}) = a + bF^{-1}(\frac{i}{n+1})$.

$\Rightarrow \ \left\{\left(F^{-1}\left(\frac{i}{n+1}\right), y_{(i)}\right) : i = 1, \ldots, n\right\}$ approximates line $y = a + bx$.

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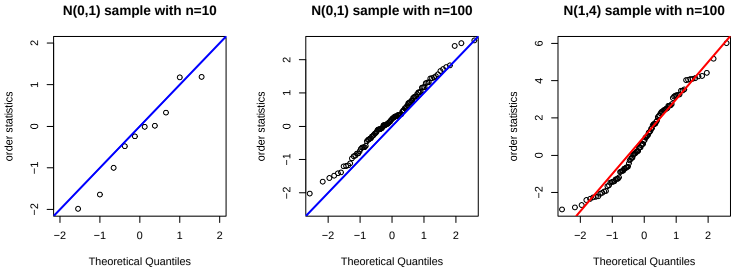
true F unknown... $\rightsquigarrow$ QQ-plots!

R: qqnorm, qqexp, qqunif, etc.

Example QQ-plot

of $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ vs $N(0,1)$; varying n, μ, σ^2.



$y = x$ $\quad$ $y = x$ $\quad$ $y = 1 + 2x$

In R:

```
par(pty="s")
qqnorm(rnorm(10))
```

Using QQ-plots — Example 1

- plot histogram
- plot different QQ-plots & choose most linear

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- plot different QQ-plots & choose most linear
- determine location (a) & scale (b) by fitting
  - straight line $y = a + bx$ visually
  - sample mean & variance
    to theoretical values (preferred!):
    $Y = a + bX \Rightarrow E(Y) = a + bE(X)$ & $var(Y) = b^2 var(X)$
    $\Rightarrow b = \sigma_Y / \sigma_X$ estimated by $\hat{b} = \hat{\sigma}_Y / \sigma_X$
    & $a = E(Y) - bE(X)$ estimated by $\hat{a} = \bar{Y} - \hat{b}E(X)$
    In R: use `abline(a = ...  , b = ...)` (different from qqline!)

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    & $a = E(Y) - bE(X)$ estimated by $\hat{a} = \bar{Y} - \hat{b}E(X)$

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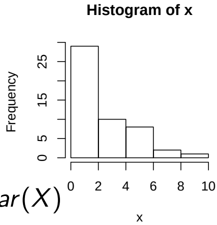


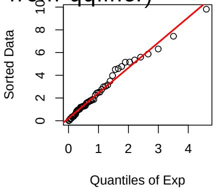
- Example $\bar{Y} = 1.98$, $\hat{\sigma}_Y^2 = 4.2$, $X \sim Exp(1)$

  $\Rightarrow$ $\hat{b} = 2.049$, $\hat{a} = 1.98 - \sqrt{4.2}/1 \approx -0.069$

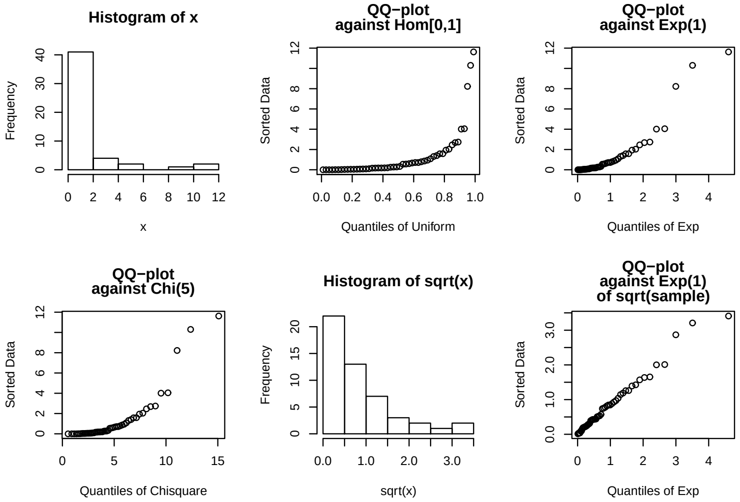
  $\Rightarrow$ model distribution $Y \sim -0.069 + 2.049 \cdot Exp(1)$

  $= -0.069 + Exp(1/2.049)$

  - Or round: $Y \sim Exp(0.5)$ (rather don't!)



Using QQ-plots — Example 2



Example: possibly transform.

Definition symmetry plot

Investigate symmetry/skewness of a distribution.

F symmetric around $\theta \;\Rightarrow\; F^{-1}(1-\alpha) - \theta = \theta - F^{-1}(\alpha), \quad \alpha \in (0,1).$

$\Rightarrow\; \{(\theta - F^{-1}(\alpha),\ F^{-1}(1-\alpha) - \theta) : \alpha \in (0,1)\}$ straight line $y = x$.

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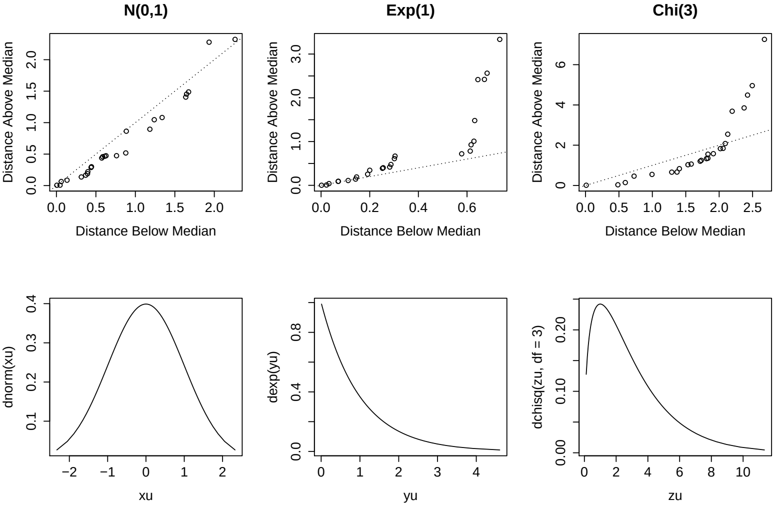
Symplot of $x_1, \ldots, x_n$: plot of

$$\left\{ (\mathrm{med}(x) - x_{(i)}, x_{(n-i+1)} - \mathrm{med}(x)) : i = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

R: symplot

Example symplot



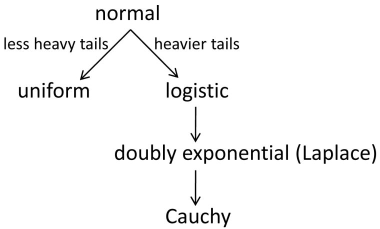
Sample size matters!

Other ways to investigate symmetry

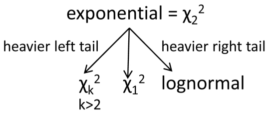
- histogram
- boxplot
- skewness parameter (be cautious, see syllabus)
- difference sample mean & sample median

Systematic search for underlying distribution

- Investigate symmetry plot & histogram
- Try several QQ-plots
  - if symmetric:



  - if not symmetric:



- Not satisfactory? Try transformations!

Two sample QQ-plot

$x_1, \ldots, x_m$ & $y_1, \ldots, y_n$: from same LSF?

If $m = n$: empirical QQ-plot plots $\{(x_{(i)}, y_{(i)}) : i = 1, 2, \ldots, n\}$.

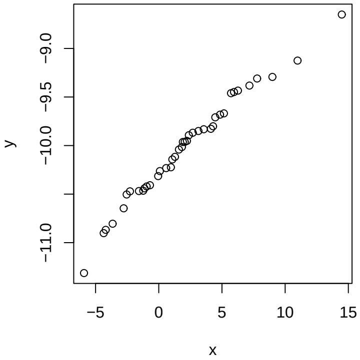
If $m < n$, it plots $\{(x_{(i)}, y^*_{(i)}) : i = 1, 2, \ldots, m\}$, where

$$y^*_{(i)} = \frac{1}{2}\left(y_{([i\frac{n+1}{m+1}])} + y_{([i\frac{n+1}{m+1} + \frac{m}{m+1}])}\right).$$

Idea: match $x_{(i)}$ with $y_{(j)}$ for which $\frac{i}{m+1} \approx \frac{j}{n+1}$.

R: qqplot

Two sample QQ-plot, example



Roughly straight line: possibly same LSF.

(GoF) tests

goodness-of-fit tests

Recap hypothesis testing:

- H_0, H_1, α,
- test statistic,
- its H_0-distribution,
- test score,
- p-value OR critical region,
- conclusion

Goodness-of-fit test

Idea: sample $x_1, \dots, x_n$ from unknown F. Test

$$H_0 : F \in \mathcal{F}_0$$
$$H_1 : F \notin \mathcal{F}_0$$

where $\mathcal{F}_0 = \{F_0\}$ (simple H_0)
or $\mathcal{F}_0 =$ collection of distributions (composite H_0), e.g. LSF.

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Aim: omnibus test with reasonable power.

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Aim: omnibus test with reasonable power.

Interpretation: is H_0 not too implausible?

Different tests we consider

- Shapiro-Wilk: $H_0 : F \in \{N(\mu, \sigma^2); \mu \in \mathbb{R}, \sigma^2 > 0\}$
- Kolmogorov-Smirnov: simple H_0 & adjusted (composite H_0)
- Chi-square test: simple H_0

different test statistics, with different distributions under H_0

to finish

To summarize

Today we discussed

- Quantile function
- Location-scale family
- QQ-plots and symplots
- Goodness-of-fit tests

Next week goodness-of-fit tests and kernel density estimators