Statistical Data Analysis, Lecture 10

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Topics in this course

1. Summarizing data
2. Exploring distributions
3. Bootstrap methods
4. Robust estimators
5. Nonparametric tests
6. Analysis of categorical data
7. Multiple linear regression

Chapter 6: Nonparametric methods

Contents of Chapter 6:

1. One sample problems
2. Asymptotic efficiency
3. Two sample problems
4. Tests for correlation

Chapter 7: Analysis of categorical data

Contents of Chapter 7:

1. Fisher's exact test
2. Chisquare test
3. Extreme values
4. Bootstrap methods for contingency tables

tests for correlation

Ranks of the two samples

Given a paired sample $(X_1, Y_1), \ldots, (X_n, Y_n)$ we define

- $S_1, \ldots, S_n$ the ranks of $X_1, \ldots, X_n$ in the ordered sample $X_{(1)}, \ldots, X_{(n)}$
- $R_1, \ldots, R_n$ the ranks of $Y_1, \ldots, Y_n$ in the ordered sample $Y_{(1)}, \ldots, Y_{(n)}$

If the two samples are independent, then the ranks $S_1, \ldots, S_n$ are independent of $R_1, \ldots, R_n$.

If the two samples are positively dependent, then the ranks $S_1, \ldots, S_n$ will run approximately in parallel with $R_1, \ldots, R_n$.

If the two samples are negatively dependent, then the ranks $S_1, \ldots, S_n$ will run approximately in opposite order as $R_1, \ldots, R_n$.

Spearman's rank correlation test

Assumption Given a paired sample $(X_1, Y_1), \dots, (X_n, Y_n)$

Test rank correlation test of Spearman

Hypothesis $H_0 : X_i$ and Y_i are independent $i = 1, \dots, n$
vs. $H_1 : X_i$ and Y_i are dependent $i = 1, \dots, n$.

Test statistic $r_s = \frac{\sum_{i=1}^n (R_i - \bar{R})(S_i - \bar{S})}{\left[\sum_{i=1}^n (R_i - \bar{R})^2 \sum_{i=1}^n (S_i - \bar{S})^2\right]^{\frac{1}{2}}}$

Distribution Either exact distribution or a normal approximation

This is a nonparametric test.

Kendall's rank correlation test

Assumption Given a paired sample $(X_1, Y_1), \ldots, (X_n, Y_n)$

Test rank correlation test of Kendall

Hypothesis $H_0 : X_i$ and Y_i are independent $i = 1, \ldots, n$
vs. $H_1 : X_i$ and Y_i are dependent $i = 1, \ldots, n$.

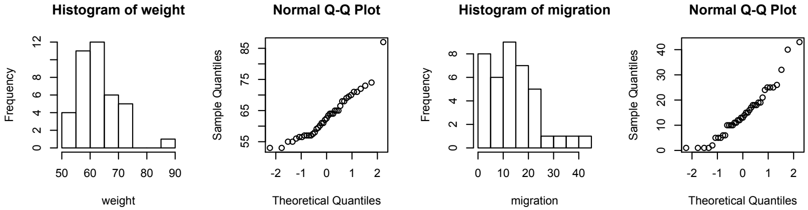
Test statistic $\tau = \frac{\sum\sum_{i \neq j} \mathrm{sgn}(R_i - R_j)\mathrm{sgn}(S_i - S_j)}{n(n-1)} = \frac{4N_\tau}{n(n-1)} - 1$, where the statistic N_τ is equal to the number of pairs (i, j) with $i < j$ for which either $X_i < X_j$ *and* $Y_i < Y_j$, or $X_i > X_j$ *and* $Y_i > Y_j$.

Distribution Either exact distribution or a normal approximation

This is a nonparametric test.

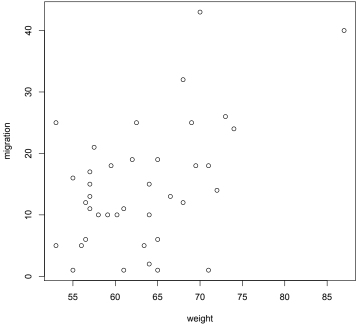
Example rank correlation test (1)

From a sample of 39 Peruvian men that had moved from a native culture to a modern society, the following variables were measured (amongst others): years since migration, systolic and diastolic blood pressure, heart rate, weight, length.



Example rank correlation test (2)

We investigate dependence between years since `migration` and `weight` using Spearman's test.




```
> cor.test(migration,weight,method="s")

Spearman's rank correlation rho

data:  migration and weight
S = 6415.128, p-value = 0.02861
alternative hypothesis: true rho is not equal to 0
sample estimates:
      rho
0.3506956

Warning message:
In cor.test.default(migration, weight, method = "s") :
  Cannot compute exact p-values with ties
```

R uses a normal approximation for the *p*-value because of ties.

Conclusion?

Example rank correlation test (3)

Kendall's rank correlation test yields:

```
> cor.test(migration,weight,method="k")

Kendall's rank correlation tau

data:  migration and weight
z = 2.1864, p-value = 0.02879
alternative hypothesis: true tau is not equal to 0
sample estimates:
      tau
0.2505268

Warning message:
In cor.test.default(migration, weight, method = "k") :
  Cannot compute exact p-value with ties
```

R uses a normal approximation for the *p*-value because of ties.

Conclusion?

Permutation test

Assumption Given a paired sample $(X_1, Y_1), \ldots, (X_n, Y_n)$

Test permutation test for paired samples

Hypothesis $H_0 : X_i$ and Y_i are independent $i = 1, \ldots, n$
vs. $H_1 : X_i$ and Y_i are dependent $i = 1, \ldots, n$.

Test statistic Some test statistic T that expresses dependence between the two samples

Distribution The right p-value is

$$P_{H_0}\left(T \geq t \middle| X_1, \ldots, X_n, Y_{(1)}, \ldots, Y_{(n)}\right)$$
$$= \frac{\#(\text{permutations } \pi \text{ with } T(X_1, \ldots, X_n, Y_{(\pi_1)}, \ldots, Y_{(\pi_n)}) \geq t)}{n!}$$

The left p-value is computed likewise.

This is a nonparametric test.

Example permutation test for paired samples

We can verify the p-value for Kendall's rank correlation test, using a permutation test, in a bootstrap fashion (i.e. considering B permutations, instead of all $n!$ permutations)

```
> B=1000
> t=cor.test(migration,weight,method="k")[[1]]
> permutationtval = numeric(B)
> for(i in 1:B)    ...
> pl= ...;  pr= ...
> p=2*min(pl,pr)
> p
[1] 0.029
```

Compare to p-value output by Kendall's test: $p = 0.02879$.

Remark Smaller values of B yield more variation in the p-value.

Remark Any test statistic that expresses dependence between the two samples can be used in a permutation test for paired samples.

categorical data

Idea (1)

Question Are kind of study and gender independent?
Consider the following data (numbers given are counts):

	exact	arts
men	23	17
women	7	13

Notation

	exact	arts	total
men	N_{11}	N_{12}	$N_{1\cdot}$
women	N_{21}	N_{22}	$N_{2\cdot}$
total	$N_{\cdot 1}$	$N_{\cdot 2}$	$n = N_{\cdot\cdot}$

Idea (2)

Question: Does frequency of nucleotides in DNA depend on its position in the DNA sequence?
Consider the following data of 100 DNA sequences of length 5:

position	1	2	3	4	5	total
A	33	34	19	20	21	127
G	22	27	23	24	21	117
C	31	18	34	30	25	138
T	14	21	24	26	33	118
total	100	100	100	100	100	500

Model I and Fisher's exact test

2 x 2 tables (Model I)

	exact	arts	total
men	23	17	$N_{1\cdot} = 40$
women	7	13	$N_{2\cdot} = 20$
total	$N_{\cdot 1} = 30$	$N_{\cdot 2} = 30$	$n = 60$

2×2 Model We assume the row and column totals are fixed.

Null hypothesis No dependence between row and column variable

Test statistic N_{11} — this number determines the entire table (given the values of $N_{\cdot 1}$, $N_{1\cdot}$, and n).

Distribution Under H_0 we have $N_{11} \sim \text{hypergeom}(n, N_{1\cdot}, N_{\cdot 1})$.

2 x 2 tables (Model I)

	exact	arts	total
men	23	17	$N_{1\cdot} = 40$
women	7	13	$N_{2\cdot} = 20$
total	$N_{\cdot 1} = 30$	$N_{\cdot 2} = 30$	$n = 60$

Null hypothesis No dependence between row and column variable

Test statistic N_{11} — this number determines the entire table

What if we want directed alternative? For instance:

Alternative hypothesis Studying arts is more common among women than men.

Fisher's exact test – directed alternative

If necessary, rephrase the alternative in terms of categories corresponding to N_{11}:

Alternative hypothesis Studying exact sciences is more common among men than women.

```
> study
     [,1] [,2]
[1,]   23    7
[2,]   17   13
> fisher.test(study,alternative="greater")

Fisher's Exact Test for Count Data

data:  study
p-value = 0.08511
alternative hypothesis: true odds ratio is greater than 1
95 percent confidence interval:
 0.8634461       Inf
sample estimates:
odds ratio
   2.47347
```

Result 2×2 Model

	exact	arts	total
men	23	17	40
women	7	13	20
total	30	30	60

Under H_0 we expect N_{11} to be $30 \times 40/60 = 20$. We have found a value of 23.

Under H_0 $N_{11} \sim \text{hypergeom}(60, 40, 30)$.
Note that `phyper` in R takes parameters $(N_{1\cdot}, n - N_{1\cdot}, N_{\cdot 1})$.

```
> pl=phyper(23,40,20,30)
> pr=1-phyper(23-1,40,20,30)
> c(pl,pr)
[1] 0.97305140 0.08511085
```

Conclusion? Neither is less than $2.5\% = \alpha/2$. H_0 is not rejected. We cannot conclude that exact studies are chosen more frequently by men.

Fisher's exact test

This exact test is called Fisher's exact test for 2 x 2 tables.

```
> study=matrix(c(23,17,7,13),nrow=2,ncol=2)
> study
     [,1] [,2]
[1,]   23    7
[2,]   17   13
> fisher.test(study)

Fisher's Exact Test for Count Data

data:  study
p-value = 0.1702
alternative hypothesis: true odds ratio is not equal to 1
95 percent confidence interval:
 0.7290557 9.0441625
sample estimates:
odds ratio
   2.47347
```

Remark This two-sided p-value is different from $2 \cdot \min\{P(N_{11} \geq 23), P(N_{11} \leq 23)\}$.

Model(s) II and χ^2-test

General contingency table – Models II

	B_1	$\cdots\cdots\cdots$	B_j	$\cdots$	B_c	total
A_1	N_{11}	$\cdots\cdots\cdots$	N_{1j}	$\cdots$	N_{1c}	$N_{1\cdot}$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
A_i	N_{i1}	$\cdots\cdots\cdots$	N_{ij}	$\cdots$	N_{ic}	$N_{i\cdot}$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
A_r	N_{r1}	$\cdots\cdots\cdots$	N_{rj}	$\cdots$	N_{rc}	$N_{r\cdot}$
total	$N_{\cdot 1}$	$\cdots\cdots\cdots$	$N_{\cdot j}$	$\cdots$	$N_{\cdot c}$	$n = N_{\cdot\cdot}$

The general form of a contingency table, with row variable A (r categories) and column variable B (c categories).

Model II A – more details next week!

1 sample of size n. 1 rc-nomial distribution with probabilities p_{ij},

$$\sum_{i=1}^{r}\sum_{j=1}^{c} p_{ij} = 1$$

Null hypothesis No dependence between row & column variable,
$p_{ij} = p_{i\cdot}p_{\cdot j}$ for $i = 1, \ldots, r, j = 1, \ldots, c$.
($p_{ij}, p_{i\cdot}, p_{\cdot j}$ all unspecified!)

Test statistic

$$X^2 = \sum_{i=1}^{r}\sum_{j=1}^{c} \frac{\left(N_{ij} - n\hat{p}_{ij}\right)^2}{n\hat{p}_{ij}} \qquad \text{with } \hat{p}_{ij} = \frac{N_{i\cdot}N_{\cdot j}}{n^2}$$

Distribution $X^2 \sim \chi^2_{(r-1)(c-1)}$ under H_0, approximately.

Model II B – more details next week!

r samples of size $N_{i\cdot}$ each. r c-nomial distributions with probabilities p_{ij},

$$\sum_{j=1}^{c} p_{ij} = 1 \qquad \text{for } i = 1, \ldots, r$$

Null hypothesis The r samples are homogeneous, i.e.
$p_{1j} = p_{2j} = \cdots = p_{rj} \equiv p_j$ for $j = 1, \ldots, c$.
(all p_j unspecified!)

Test statistic

$$X^2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{\left(N_{ij} - n\hat{p}_{ij}\right)^2}{n\hat{p}_{ij}} \qquad \text{with } \hat{p}_{ij} = \frac{N_{i\cdot} N_{\cdot j}}{n^2}$$

Distribution $X^2 \sim \chi^2_{(r-1)(c-1)}$ under H_0, approximately.

Model II C – more details next week!

c samples of size $N_{\cdot j}$ each. c r-nomial distributions with probabilities p_{ij},

$$\sum_{i=1}^{r} p_{ij} = 1 \qquad \text{for } j = 1, \ldots, c$$

Null hypothesis The c samples are homogeneous, i.e.
$p_{i1} = p_{i2} = \cdots = p_{ic} \equiv p_i$ for $i = 1, \ldots, r$.
(all p_i unspecified!)

Test statistic

$$X^2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{\left(N_{ij} - n\hat{p}_{ij}\right)^2}{n\hat{p}_{ij}} \qquad \text{with } \hat{p}_{ij} = \frac{N_{i\cdot} N_{\cdot j}}{n^2}$$

Distribution $X^2 \sim \chi^2_{(r-1)(c-1)}$ under H_0, approximately.

to finish

To wrap up

Today we discussed

1. Two sample problems: Tests for correlation
2. Fisher's exact test
3. Chisquare test (beginning)

Next week More on categorical data, and linear regression