Gruppo 2318
Guillermo Benito   Claudia Lorenzetti   Leon Mengoni   Filippo Pra Floriani

Laboratory of Computational Physics - mod.B
SOLAR DYNAMO

1 Intro

The objective of this project is to conduct Bayesian inference on sunspot data, which follow the 11-year solar cycle. Given the solar dynamo model, we want to estimate the best values for the parameters of said model.
In order to test the theory, we first of all construct fake data by simulating the solar cycle, given fixed values for the parameters of the generative model. Next, we set a prior and we compute the likelihood. Once we get the expression of the posterior, through a MAP estimate, we obtain the best values for the parameters. Then, we test the validity of our setup, by checking if our original parameters reside within the credibility interval of our MAP parameters.
Subsequently, we conduct inference on the real data.

2 Model

The sun's magnetic field is generated by an inner magnetic dynamo. It is well known that it changes in a cyclic manner with a period of almost 11 years: the existence of such a stable period indicates that the magnetic field continues to be generated within the Sun. The amplitude of the oscillator is subject to a long-term modulation, including periods of very low activity, known as Grand Minima. It is believed that Grand Minima are induced by the tiny planetary tidal forcing.
The zero-dimensional dynamo model takes into account the fact that the final ODE is obtained by merging two coupled delay ODEs: one for the so-called Ω-effect (1) and the other for the α-effect (2). Both equations are written in toroidal coordinates:

$$\frac{dB(t)}{dt} = \frac{\Omega_0}{L} A(t - T_0) - \frac{B(t)}{\tau} \tag{1}$$

$$\frac{dA(t)}{dt} = \alpha_0 f(B(t - T_1))B(t - T_1) - \frac{A(t)}{\tau} \tag{2}$$

The first equation (1) is the effect of a differential rotation, a change in the rotation rate as a function of latitude and radius: the inner magnetic field is stretched out and wound around the Sun. The second equation (2), the one that describes the poloidal field, is due to the twist of the magnetic field lines caused by the effects of the Sun's rotation. Where Ω_0 is a characteristic value of the angular velocity distribution $\Omega(r, \theta)$ in the convection zone, L is a characteristic length of variation, α_0 is a characteristic value of the α-effect, and τ is the diffusion timescale.

By coupling the two delay ODEs, we obtain a second order delay ODE which exhibits some interesting dynamic phenomena, that completely disappear if the time delay is set to zero:

$$\left(\tau\frac{d}{dt}+1\right)^2 B(t) = -\mathcal{N}(1+\epsilon\cos(\omega_d t))f(B(t-q))B(t-q)+\sigma B_{max}\sqrt{\tau}\eta(t) \tag{3}$$

where:

- $B(t)$ is the magnetic field, our observable;
- $f(B(t))$ is a non-linear function of $B(t)$;
- $\eta(t)$ is gaussian white noise ($\langle\eta(t)\rangle = 0$ and $\langle\eta(t)\eta(t')\rangle = \delta(t-t')$);
- $\epsilon\cos(\omega_d t)$ is a weak external modulation induced by planetary tidal forcing, of amplitude ϵ;
- $\mathcal{N} = \frac{\alpha_0\Omega_0\tau^2}{L}$ is the strength of the dynamo, a dimensionless parameter that depends on a combination of the parameters used in (1) and in (2);
- q is the delay;
- σ is the dimensionless noise amplitude;
- τ is the diffusion-time constant;
- T is the observation time;
- B_{max} defines a critical field strength [1].

Following the Wilmot-Smith model [2], a symmetric box-shaped function has been chosen to express the non-linear function $f(B(t))$, which limits the α-effect to a range of $0 \leq B(t) \leq B_{max}$:

$$f(B(t)) = \frac{1}{2}(1-\mathrm{erf}(B^2(t)-B^2_{max})) \tag{4}$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{5}$$

3 Bayesian Inference

As introduced in section 1, we aim to find the best parameters that fit (3). To achieve this, we want to construct the posterior for our parameters $\boldsymbol{\theta} = (q, \mathcal{N}, \sigma, B_{max}, \tau)$:

$$p(\boldsymbol{\theta}|B) \propto f(B|\boldsymbol{\theta})p(\boldsymbol{\theta}) \tag{6}$$

given the likelihood $f(B|\boldsymbol{\theta})$ and our parameter prior $p(\boldsymbol{\theta})$, i.e. our previous knowledge. In this case we require $q, \mathcal{N}, B_{max}$ and τ to be positive, while we use a Jeffrey's prior for $\sigma : p(\sigma) = \frac{1}{\sqrt{\sigma}}$.

In our case, instead of working with our time-dependent signal, we convert our model equation to Fourier space, and we sample the posterior in Fourier space by using a Markov Chain Monte Carlo algorithm:

$$p(\boldsymbol{\theta}|\widehat{B}) \propto f(\widehat{B}|\boldsymbol{\theta})p(\boldsymbol{\theta}) \tag{7}$$

However, instead of sampling from the distribution of $\widehat{B}$, we want to sample from the distribution of $\widehat{\eta}$, because η is a gaussian white noise and its Fourier transform is also distributed as a gaussian. We therefore express $\widehat{\eta}$ as a function of $\widehat{B}$.
After having sampled our posterior, we obtain the best estimates through the use of a minimization algorithm.

3.1 Model equation to Fourier space

First of all we want to convert (3) to the frequency domain by applying a Fourier transform. In this way it is possible to focus just on the frequency modes which are more interesting to be analyzed, by filtering out the negligible ones.
We define the Fourier transform of our signal $B(t)$ as the n-th coefficient of the complex Fourier series:

$$\mathcal{F}[B(t)]_n = \widehat{B}_n = \frac{1}{T}\int_0^T dt e^{-2\pi i n \frac{t}{T}} B(t) \tag{8}$$

In order to prevent *spectral-leakage*, a window function will be used. In this case we will use the Hann window function:

$$w(t) = \cos^2\left(\pi\left(\frac{t}{T} - \frac{1}{2}\right)\right) \tag{9}$$

The application of $w(t)$ to our signal helps modulate the trend of the ODE, by suppressing the function near $t = 0$ and $t = T$, whose period is forced to be exactly T.

Our equation, therefore, is multiplied by the window and we can then "pull" $w(t)$ inside the differential operator, since the derivatives of the window are negligible for most of our observation period T. The equation simplifies to

$$\left(\tau\frac{d}{dt} + 1\right)^2 B^w(t) = -\mathcal{N}(1 + \epsilon\cos(\omega_d t))\widetilde{f}(B(t-q))w(t) + \sigma B_{max}\sqrt{\tau}\eta^w(t) \tag{10}$$

where the windowed signal and noise are:

$$B^w(t) = w(t)B(t) \qquad \eta^w(t) = w(t)\eta(t) \tag{11}$$

and the non-linear function has been rewritten as:

$$\widetilde{f}(B(t)) = f(B(t))B(t) \tag{12}$$

The Fourier transform of our windowed signal is, therefore:

$$\mathcal{F}[B^w(t)]_n = \widehat{B}_n^w = \frac{1}{T}\int_0^T dt e^{-2\pi i n \frac{t}{T}} B^w(t) \tag{13}$$

We can now convert (10) to Fourier space.

3.1.1 Left hand term

First we focus on the left hand term, without the window function:

$$\mathcal{F}\left[\left(\tau\frac{d}{dt}+1\right)^2 B(t)\right]_n \tag{14}$$

where

$$\left(\tau\frac{d}{dt}+1\right)^2 B(t) = \tau^2\ddot{B}(t) + 2\tau\dot{B}(t) + B(t) \tag{15}$$

So, for each term of the above equation we have:

1.
$$\mathcal{F}[B(t)]_n = \frac{1}{T}\int_0^T dt B(t) e^{-2\pi i n \frac{t}{T}} = \widehat{B}_n \tag{16}$$

   With the window:

$$\mathcal{F}[B^w(t)]_n = \widehat{B}_n^w \tag{17}$$

2.
$$\begin{aligned}\mathcal{F}[\dot{B}(t)]_n &= \frac{1}{T}\int_0^T dt \dot{B}(t) e^{-2\pi i n \frac{t}{T}} \\ &= \frac{1}{T}\left\{\left[B(t)e^{-2\pi i n \frac{t}{T}}\right]_0^T - \left(-\frac{2\pi i n}{T}\right)\int_0^T dt B(t) e^{-2\pi i n \frac{t}{T}}\right\} \\ &= \frac{B(T)-B(0)}{T} + \frac{2\pi i n}{T}\widehat{B}_n\end{aligned} \tag{18}$$

   The application of the window leads to $B^w(T) = B^w(0)$ and we obtain:

$$\mathcal{F}[\dot{B}^w(t)]_n = \frac{2\pi i n}{T}\widehat{B}_n^w \tag{19}$$

3.

$$\begin{aligned}
\mathcal{F}[\ddot{B}(t)]_n &= \frac{1}{T}\int_0^T dt\ddot{B}(t)e^{-2\pi in\frac{t}{T}} \\
&= \frac{1}{T}\left\{\left[\dot{B}(t)e^{-2\pi in\frac{t}{T}}\right]_0^T - \left(-\frac{2\pi in}{T}\right)\int_0^T dt\dot{B}(t)e^{-2\pi in\frac{t}{T}}\right\} \\
&= \frac{\dot{B}(T)-\dot{B}(0)}{T} + \frac{2\pi in}{T}\mathcal{F}[\dot{B}(t)]
\end{aligned} \tag{20}$$

For the same reason as before, we obtain $\dot{B}^w(T) = \dot{B}^w(0)$. Therefore:

$$\mathcal{F}[\ddot{B}^w(t)]_n = \frac{2\pi in}{T}\mathcal{F}[\dot{B}^w(t)] = \left(\frac{2\pi in}{T}\right)^2 \widehat{B}^w_n \tag{21}$$

In summary, the left hand term in the frequency domain is:

$$\mathcal{F}\left[\left(\tau\frac{d}{dt}+1\right)^2 B^w(t)\right]_n = \left(\frac{2\pi in\tau}{T}\right)^2 \widehat{B}^w_n + \frac{4\pi in\tau}{T}\widehat{B}^w_n + \widehat{B}^w_n = \left(\frac{2\pi in\tau}{T}+1\right)^2 \widehat{B}^w_n \tag{22}$$

3.1.2 Right hand term

Next, we analyze the right hand term, with the $w(t)$:

$$\mathcal{F}\left[-\mathcal{N}(1+\epsilon\cos(\omega_d t))\widetilde{f}(B(t-q))w(t) + \sigma B_{max}\sqrt{\tau}\eta^w(t)\right]_n \tag{23}$$

For each term we get:

1.

$$\begin{aligned}
&\mathcal{F}\left[(1+\epsilon\cos(\omega_d t))\widetilde{f}(B(t-q))w(t)\right]_n = \frac{1}{T}\int_0^T dt(1+\epsilon\cos(\omega_d t))\widetilde{f}(B(t-q))w(t)e^{-2\pi in\frac{t}{T}} \\
&= e^{-2\pi in\frac{q}{T}}\frac{1}{T}\int_{-q}^{T-q} dt'[1+\epsilon\cos(\omega_d(t'+q))]\widetilde{f}(B(t'))w(t'+q)e^{-2\pi in\frac{t'}{T}} \\
&= e^{-2\pi in\frac{q}{T}}\frac{1}{T}\int_{-q}^{T-q} dt'[1+\epsilon\cos(\omega_d(t'+q))]\widetilde{f}(B(t'))\times \\
&\quad\times\left[\cos\left(\pi\left(\frac{t'}{T}-\frac{1}{2}\right)\right)\cos\left(\pi\frac{q}{T}\right) - \sin\left(\pi\left(\frac{t'}{T}-\frac{1}{2}\right)\right)\sin\left(\pi\frac{q}{T}\right)\right]^2 e^{-2\pi in\frac{t'}{T}} \\
&\simeq e^{-2\pi in\frac{q}{T}}\cos^2\left(\pi\frac{q}{T}\right)\frac{1}{T}\int_{-q}^{T-q} dt'[1+\epsilon\cos(\omega_d(t'+q))]\widetilde{f}(B(t'))\times \\
&\quad\times\cos^2\left(\pi\left(\frac{t'}{T}-\frac{1}{2}\right)\right)e^{-2\pi in\frac{t'}{T}} \\
&= e^{-2\pi in\frac{q}{T}}\cos^2\left(\pi\frac{q}{T}\right)\frac{1}{T}\int_{-q}^{T-q} dt'[1+\epsilon\cos(\omega_d(t'+q))]\widetilde{f}^w(B(t'))e^{-2\pi in\frac{t'}{T}} \\
&= e^{-2\pi in\frac{q}{T}}\cos^2\left(\pi\frac{q}{T}\right)\mathcal{F}\left\{[1+\epsilon\cos(\omega_d(t+q))]\widetilde{f}^w\right\}_n
\end{aligned} \tag{24}$$

where $\widetilde{f}^w(B(t)) = \widetilde{f}(B(t))w(t)$, and where the approximation arises from ignoring the terms that depend on $\sin\left(\pi\left(\frac{t}{T}-\frac{1}{2}\right)\right)$.

2.
$$\mathcal{F}[\eta^w(t)]_n = \widehat{\eta}_n^w \tag{25}$$

So the full transformed right hand term, with the driving external perturbation, is:

$$\begin{aligned}&\mathcal{F}\left[-\mathcal{N}(1+\epsilon\cos(\omega_d t))\widetilde{f}(B(t-q))w(t) + \sigma B_{max}\sqrt{\tau}\eta^w(t)\right]_n = \\ &= -\mathcal{N}e^{-2\pi i n\frac{q}{T}}\cos^2\left(\pi\frac{q}{T}\right)\mathcal{F}\left\{[1+\epsilon\cos(\omega_d(t+q))]\widetilde{f}^w\right\}_n + \sigma B_{max}\sqrt{\tau}\widehat{\eta}_n^w\end{aligned} \tag{26}$$

3.1.3 Full equation transformed

Finally, the full equation with window function transformed to Fourier space is:

$$\left(\frac{2\pi i n\tau}{T}+1\right)^2\widehat{B}_n^w = -\mathcal{N}e^{-2\pi i n\frac{q}{T}}\cos^2\left(\pi\frac{q}{T}\right)\mathcal{F}\left\{[1+\epsilon\cos(\omega_d(t+q))]\widetilde{f}^w\right\}_n + \sigma B_{max}\sqrt{\tau}\widehat{\eta}_n^w \tag{27}$$

To proceed with the inference, we have to isolate $\widehat{\eta}_n^w$ by writing it as a function of $\widehat{B}_n^w$:

$$\widehat{\eta}_n^w = \frac{\left(\frac{2\pi i n\tau}{T}+1\right)^2\widehat{B}_n^w + \mathcal{N}e^{-2\pi i n\frac{q}{T}}\cos^2\left(\pi\frac{q}{T}\right)\mathcal{F}\left\{[1+\epsilon\cos(\omega_d(t+q))]\widetilde{f}^w\right\}_n}{\sigma B_{max}\sqrt{\tau}} \tag{28}$$

With or without windowing, the formal expression of the equation is the same.

Without the external modulation ($\epsilon = 0$), $\widehat{\eta}_n^w$ is written as:

$$\widehat{\eta}_n^w = \frac{\left(\frac{2\pi i n\tau}{T}+1\right)^2\widehat{B}_n^w + \mathcal{N}e^{-2\pi i n\frac{q}{T}}\cos^2\left(\pi\frac{q}{T}\right)\mathcal{F}[\widetilde{f}^w]_n}{\sigma B_{max}\sqrt{\tau}} \tag{29}$$

3.2 Likelihood function

We now want to define the likelihood function for the transformed signal in $\widehat{\eta^w}$.
The transformed noise $\widehat{\eta}_n$ is distributed following a normal distribution:

$$f(\widehat{\eta}_n|\boldsymbol{\theta}) \propto \exp\left(-\frac{1}{2}|\widehat{\eta}_n|^2\right) \tag{30}$$

Therefore, given $\widehat{\boldsymbol{\eta}} = (..., \widehat{\eta}_{-1}, \widehat{\eta}_0, \widehat{\eta}_1, ...)$, the distribution of $\widehat{\boldsymbol{\eta}}$ is given by the product

$$f(\widehat{\boldsymbol{\eta}}|\boldsymbol{\theta}) = \prod_n f(\widehat{\eta}_n) \propto \exp\left(-\frac{1}{2}\sum_n|\widehat{\eta}_n|^2\right) = \exp\left(-\frac{1}{2}\widehat{\boldsymbol{\eta}}^T\widehat{\boldsymbol{\eta}}\right) \tag{31}$$

The likelihood function for $\widehat{\boldsymbol{B}}$ in Fourier space is obtained from the change of variables $\widehat{\boldsymbol{\eta}} \longrightarrow \widehat{\boldsymbol{B}}$:

$$\mathcal{L} = f(\widehat{\boldsymbol{B}}|\boldsymbol{\theta}) = f(\widehat{\boldsymbol{\eta}}|\boldsymbol{\theta})\Big|_{\widehat{\boldsymbol{\eta}}=\widehat{\boldsymbol{\eta}}(\widehat{\boldsymbol{B}})} \cdot \left|\frac{d\widehat{\boldsymbol{\eta}}}{d\widehat{\boldsymbol{B}}}\right| \tag{32}$$

However, when we deal with a windowed signal, such as $B^w(t)$, correlations arise between different modes of the Fourier transform. Therefore, the probability distribution $f(\widehat{\boldsymbol{\eta}}^{\boldsymbol{w}}|\boldsymbol{\theta})$ is not a simple product of independent gaussian distributions anymore, and the covariance matrix $\Sigma_{n,m} = \langle \widehat{\eta}^w_n, \widehat{\eta}^w_m \rangle$ must be included:

$$f(\widehat{\boldsymbol{\eta}}^{\boldsymbol{w}}|\boldsymbol{\theta}) \propto \exp\left(-\frac{1}{2}\left(\widehat{\boldsymbol{\eta}}^{\boldsymbol{w}}\right)^T \Sigma^{-1}\widehat{\boldsymbol{\eta}}^{\boldsymbol{w}}\right) \tag{33}$$

3.2.1 Computing the Jacobian

We now would like to compute the jacobian $J^w = \frac{d\widehat{\boldsymbol{\eta}}^w}{d\widehat{\boldsymbol{B}}^w}$ and its determinant:

$$\det(J^w) = \left|\frac{d\widehat{\boldsymbol{\eta}}^w}{d\widehat{\boldsymbol{B}}^w}\right| \tag{34}$$

The jacobian element (n, m) is given by:

$$\begin{aligned}
J^w_{n,m} &= \frac{d\widehat{\eta}^w_n}{d\widehat{B}^w_m} \\
&= \frac{d}{d\widehat{B}^w_m}\left\{\frac{1}{\sigma B_{max}\sqrt{\tau}}\left[\left(\frac{2\pi i n\tau}{T}+1\right)^2 \widehat{B}^w_n + \mathcal{N}e^{-2\pi i n\frac{q}{T}}\cos^2\left(\pi\frac{q}{T}\right) \times \right.\right. \\
&\qquad \left.\left. \times\, \mathcal{F}\big\{[1+\epsilon\cos(\omega_d(t+q))]\widetilde{f}^w\big\}_n\right]\right\} \\
&= \frac{1}{\sigma B_{max}\sqrt{\tau}}\left[\left(\frac{2\pi i n\tau}{T}+1\right)^2 \delta_{nm} + \mathcal{N}e^{-2\pi i n\frac{q}{T}}\cos^2\left(\pi\frac{q}{T}\right) \times \right. \\
&\qquad \left. \times\, \frac{d}{d\widehat{B}^w_m}\mathcal{F}\big\{[1+\epsilon\cos(\omega_d(t+q))]\widetilde{f}^w\big\}_n\right]
\end{aligned} \tag{35}$$

Now, we want to compute $\frac{d}{d\widehat{B}^w_m}\mathcal{F}\big\{g(t)\big\}_n$, where $g(t) = [1+\epsilon\cos(\omega_d(t+q))]\widetilde{f}^w$.
In order to do this, we observe that, since we are working with the discrete signal $\boldsymbol{B} = (B(t_0), B(t_1), ..., B(t_{N-1}))$, we will not have the integral as in (8), but we will be using the Discrete Fourier Transform (DFT):

$$\mathcal{F}[\boldsymbol{B}]_n = \widehat{B}_n = \sum_{i=0}^{N-1} e^{-2\pi i n\frac{t_i}{T}} B(t_i) \tag{36}$$

and its inverse:

$$\mathcal{F}^{-1}[\widehat{\boldsymbol{B}}](t_i) = B(t_i) = \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i n\frac{t_i}{T}}\widehat{B}_n \tag{37}$$

(The equations obtained in the previous section still hold.)

We can now see that we can rewrite $\mathcal{F}\big\{g(t)\big\}_n$ as:

$$
\begin{aligned}
&\mathcal{F}\{[1+\epsilon\cos(\omega_d(t+q))]\widetilde{f}^w\}_n = \\
&\quad= \sum_t e^{-2\pi i n \frac{t}{T}}[1+\epsilon\cos(\omega_d(t+q))]\widetilde{f}^w(B(t)) \\
&\quad= \sum_t e^{-2\pi i n \frac{t}{T}}[1+\epsilon\cos(\omega_d(t+q))]w(t)\widetilde{f}\left(w^{-1}(t)\frac{1}{N}\sum_k e^{2\pi i k\frac{t}{T}}\widehat{B}_k^w\right)
\end{aligned} \tag{38}
$$

so that $\frac{d}{d\widehat{B}_m^w}\mathcal{F}\{g(t)\}_n$ can be expanded in the following way:

$$
\begin{aligned}
&\frac{d}{d\widehat{B}_m^w}\mathcal{F}\{[1+\epsilon\cos(\omega_d(t+q))]\widetilde{f}^w\}_n = \\
&\quad= \frac{d}{d\widehat{B}_m^w}\left[\sum_t e^{-2\pi i n\frac{t}{T}}[1+\epsilon\cos(\omega_d(t+q))]w(t)\widetilde{f}\left(w^{-1}(t)\frac{1}{N}\sum_k e^{2\pi i k\frac{t}{T}}\widehat{B}_k^w\right)\right] \\
&\quad= \sum_t e^{-2\pi i n\frac{t}{T}}[1+\epsilon\cos(\omega_d(t+q))]w(t)\frac{d\widetilde{f}(B)}{dB}\frac{d}{d\widehat{B}_m^w}\left(w^{-1}(t)\frac{1}{N}\sum_k e^{2\pi i k\frac{t}{T}}\widehat{B}_k^w\right) \\
&\quad= \sum_t e^{-2\pi i n\frac{t}{T}}[1+\epsilon\cos(\omega_d(t+q))]w(t)\frac{d\widetilde{f}(B)}{dB}w^{-1}(t)\frac{1}{N}\sum_k e^{2\pi i k\frac{t}{T}}\delta_{km} \\
&\quad= \sum_t e^{-2\pi i(n-m)\frac{t}{T}}[1+\epsilon\cos(\omega_d(t+q))]\frac{d\widetilde{f}(B)}{dB}\frac{1}{N} \\
&\quad= \mathcal{F}\left[[1+\epsilon\cos(\omega_d(t+q))]\frac{1}{N}\frac{d\widetilde{f}(B)}{dB}\right]_{n-m}
\end{aligned} \tag{39}
$$

Therefore, we obtain:

$$
\begin{aligned}
J_{n,m}^w &= \frac{d\widehat{\eta}_n}{d\widehat{B}_m} \\
&= \frac{1}{\sigma B_{max}\sqrt{\tau}}\Bigg[\left(\frac{2\pi i n\tau}{T}+1\right)^2\delta_{nm} + \mathcal{N}e^{-2\pi i n\frac{q}{T}}\cos^2\left(\pi\frac{q}{T}\right)\times \\
&\qquad\times\mathcal{F}\left[[1+\epsilon\cos(\omega_d(t+q))]\frac{1}{N}\frac{d\widetilde{f}(B)}{dB}\right]_{n-m}
\end{aligned} \tag{40}
$$

Now, given that

$$
f(\widehat{B}^w|\boldsymbol{\theta}) \propto |\det J^w| \exp\left(-\frac{1}{2}\left(\widehat{\eta}^w\right)^T\Sigma^{-1}\widehat{\eta}^w\right) \tag{41}
$$

we would have to calculate the covariance matrix. However, as can be seen in the appendix, the covariance matrix is actually composed of mostly zeros, except three diagonal bands. We can therefore ignore the covariance matrix by sampling every 3 nodes: this is how we will lighten the computations in our code. Therefore, we can write our sampling posterior as:

$$
\begin{aligned}
f(\widehat{B}^w|\boldsymbol{\theta}) &\propto |\det J^w| \exp\left(-\frac{1}{2}|\widehat{\eta}^w|^2\right) \\
&\propto |\det J^w| \exp\Bigg(- \frac{1}{2\sigma^2 B_{max}^2 \tau} \sum_n \Bigg| \left(\frac{2\pi i n \tau}{T} + 1\right)^2 \widehat{B}_n^w + \\
&\quad + \mathcal{N} e^{-2\pi i n \frac{q}{T}} \cos^2\left(\pi \frac{q}{T}\right) \mathcal{F}\{[1 + \epsilon \cos(\omega_d (t+q))] \widetilde{f}^w\}_n \Bigg|^2 \Bigg)
\end{aligned} \tag{42}
$$

Without the external modulation ($\epsilon = 0$), the sampling posterior is written as:

$$
\begin{aligned}
f(\widehat{B}^w|\boldsymbol{\theta}) \propto |\det J^w| \exp\Bigg(- \frac{1}{2\sigma^2 B_{max}^2 \tau} \sum_n \Bigg| \left(\frac{2\pi i n \tau}{T} + 1\right)^2 \widehat{B}_n^w + \\
+ \mathcal{N} e^{-2\pi i n \frac{q}{T}} \cos^2\left(\pi \frac{q}{T}\right) \mathcal{F}[\widetilde{f}^w]_n \Bigg|^2 \Bigg)
\end{aligned} \tag{43}
$$

4 Results

First of all, we generate fake data by setting the parameters as defined in [1] and by means of Julia's Stochastic Delay Differential Equation (SDDE) package. To test the model we computed the posterior both as defined in (42) and in (43). Then we did the sampling from the posterior using the EMCEE algorithm with Python.
Subsequently, we conducted Bayesian inference on real data using the same EMCEE algorithm. First, we did the inference without considering the noise modulation (setting $\epsilon = 0$).

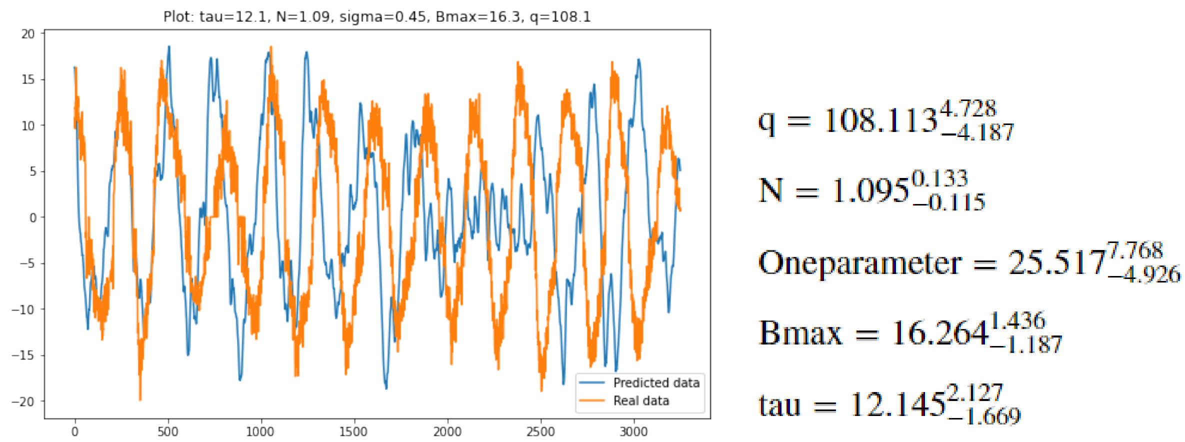
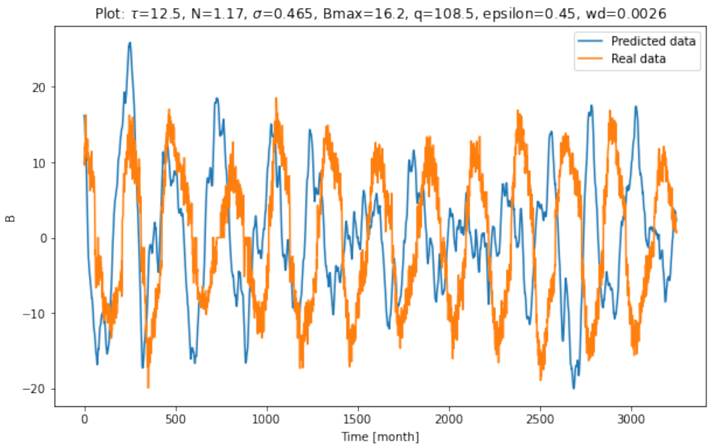


Figure 1: Comparison between generated and real data and best values for the parameters. $\epsilon = 0$

Then, we did the same by including the external modulation ($\epsilon \neq 0$).



$$q = 108.461^{6.031}_{-4.005}$$

$$N = 1.167^{0.801}_{-0.195}$$

$$\text{Oneparameter} = 26.575^{9.919}_{-5.855}$$

$$\text{Bmax} = 16.176^{1.639}_{-1.805}$$

$$\text{tau} = 12.499^{2.144}_{-1.890}$$

$$\text{epsilon} = 0.450^{0.365}_{-0.357}$$

$$\text{wd} = 0.003^{0.024}_{-0.001}$$

Figure 2: Comparison between generated and real data and best values for the parameters. Here we consider $\epsilon \neq 0$

As we can see in figures 1 and in 2, the simulation works well only for the first few periods (especially in figure 1). In conclusion, we used the parameters estimated on the model with no external modulation to simulate a few future solar cycles to see whether they could predict the future behaviour of the solar dynamo.

5 Appendix

Covariance matrix

The covariance matrix will be reduced to solely the positive quadrant, so that it becomes a $N \times N$ band matrix.
Without windowing, the (n, m) element of the covariance matrix is calculated as:

$$\begin{aligned}
\Sigma_{n,m} &= \langle \widehat{\eta}_n \widehat{\eta}_m^* \rangle \\
&= \left\langle \frac{1}{T^2} \iint_0^T dt dt' \eta(t) \eta(t') e^{-2\pi i \frac{(nt - mt')}{T}} \right\rangle \\
&= \frac{1}{T^2} \iint_0^T dt dt' \left\langle \eta(t) \eta(t') \right\rangle e^{-2\pi i \frac{(nt - mt')}{T}} \\
&= \frac{1}{T^2} \iint_0^T dt dt' \delta(t - t') e^{-2\pi i \frac{(nt - mt')}{T}} \\
&= \frac{1}{T} \int_0^T dt e^{-2\pi i (n-m) \frac{t}{T}} \\
&= \mathcal{F}\left[1\right]_{n-m} \\
&= \delta_{n-m,0}
\end{aligned} \tag{44}$$

The (n, m) element of the covariance matrix is calculated as:

$$\begin{aligned}
\Sigma_{n,m} &= \langle \widehat{\eta}_n^w \widehat{\eta}_m^{w*} \rangle \\
&= \left\langle \frac{1}{T^2} \iint_0^T dt dt' w(t) w(t') \eta(t) \eta(t') e^{-2\pi i \frac{(nt - mt')}{T}} \right\rangle \\
&= \frac{1}{T^2} \iint_0^T dt dt' w(t) w(t') \left\langle \eta(t) \eta(t') \right\rangle e^{-2\pi i \frac{(nt - mt')}{T}} \\
&= \frac{1}{T^2} \iint_0^T dt dt' w(t) w(t') \delta(t - t') e^{-2\pi i \frac{(nt - mt')}{T}} \\
&= \int_0^T dt w^2(t) e^{-2\pi i (n-m) \frac{t}{T}} \\
&= \mathcal{F}\left[w^2(t)\right]_{n-m}
\end{aligned} \tag{45}$$

$$\begin{aligned}
\mathcal{F}\left[w^2(t)\right]_n &= \frac{1}{T} \int_0^T dt \cos^4\left(\pi \left(\frac{t}{T} - \frac{1}{2}\right)\right) e^{-2\pi i n \frac{t}{T}} \\
&= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \cos^4(x) e^{-2\pi i n \left(\frac{x}{\pi} + \frac{1}{2}\right)} \\
&= e^{-\pi i n} \mathcal{F}\left[\cos^4(x)\right]_n
\end{aligned} \tag{46}$$

The Fourier coefficient of $\cos^4(x)$ can be calculated as:

$$\begin{aligned}
\mathcal{F}\left[\cos^4(x)\right]_n &= \mathcal{F}\left[\left(\frac{e^{ix} + e^{-ix}}{2}\right)^4\right]_n \\
&= \frac{1}{16} \mathcal{F}\left[e^{4ix} + 4e^{2ix} + 6 + 4e^{-2ix} + e^{-4ix}\right]_n
\end{aligned} \tag{47}$$

Recalling that

$$\mathcal{F}\left[e^{aix}\right]_n = \delta_{n,\frac{a}{2}} \tag{48}$$

we obtain:

$$\begin{aligned}\mathcal{F}\left[w^2(t)\right]_n &= \frac{e^{-\pi in}}{16}\left(\delta_{n,2} + 4\delta_{n,1} + 6\delta_{n,0} + 4\delta_{n,-1} + \delta_{n,-2}\right) \\ &= \frac{1}{16}\left(\delta_{n,2} - 4\delta_{n,1} + 6\delta_{n,0} - 4\delta_{n,-1} + \delta_{n,-2}\right)\end{aligned} \tag{49}$$

Therefore, the covariance matrix elements are:

$$\Sigma_{n,m} = \mathcal{F}\left[w^2(t)\right]_{n-m} = \frac{1}{16}\left(\delta_{n-m,2} - 4\delta_{n-m,1} + 6\delta_{n-m,0} - 4\delta_{n-m,-1} + \delta_{n-m,-2}\right) \tag{50}$$

and the matrix is an infinite matrix of the following kind :

$$\Sigma = \frac{1}{16}\begin{bmatrix} \ddots & \ddots & \ddots & & & \\ \ddots & 6 & -4 & 1 & & \\ \ddots & -4 & 6 & -4 & 1 & \\ & 1 & -4 & 6 & -4 & \ddots \\ & & 1 & -4 & 6 & \ddots \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

For the case where we take the windowed Fourier transform, we can consider the covariance matrix negligible by sampling every 3 modes.

References

[1] Carlo Albert et al. "Can Stochastic Resonance Explain Recurrence of Grand Minima?" In: *The Astrophysical Journal Letters* (2021).

[2] A.L. Wilmot-Smith et al. "A Time Delay Model for Solar and Stellar Dynamos". In: *The Astrophysical Journal Letters* (2006). DOI: 10.1086/508013.