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Convergence of Approximate and Packet Routing Equilibria to Nash Flows Over Time

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***Abstract*—We consider a dynamic model of traffic that has received a lot of attention in the past few years. Infinitesimally small agents aim to travel from a source to a destination as quickly as possible. Flow patterns vary over time, and congestion effects are modeled via queues, which form based on the deterministic queueing model whenever the inflow into a link exceeds its capacity.**

Are equilibria in this model meaningful as a prediction of traffic behavior? For this to be the case, a certain notion of stability under ongoing perturbations is needed. Real traffic consists of discrete, atomic "packets", rather than being a continuous flow of non-atomic agents. Users may not choose an absolutely quickest route available, if there are multiple routes with very similar travel times. We would hope that in both these situations — a discrete packet model, with packet size going to 0, and ε-equilibria, with ε going to 0 — equilibria converge to dynamic equilibria in the flow over time model. No such convergence results were known.

We show that such a convergence result does hold in single-commodity instances for both of these settings, in a unified way. More precisely, we introduce a notion of "strict" ε-equilibria, and show that these must converge to the exact dynamic equilibrium in the limit as $\varepsilon \to 0$. We then show that results for the two settings mentioned can be deduced from this with only moderate further technical effort.

***Index Terms*—Nash flows over time, dynamic traffic models, continuity, convergence**

The full version of this extended abstract can be found at https://nolver.net/home/pubs/convergence-nash-flows/.

I. Introduction

Telecommunications networks and transportation networks are two settings where the natural description involves tracking users or packets as they traverse the network. These users arrive at different nodes in the network at different moments in time. In some situations, this temporal aspect can be to some extent ignored, and modeled through *static* models. This is reasonable if we anticipate that over the timescale being modeled, the solution of interest can reasonably be approximated by a temporally repeated flow.

We will be interested in the game-theoretic perspective, considering that the network traffic consists of *self-interested* users, each aiming to optimize their own objective (generally, travel time) subject to the environment induced by the other users. The interaction between agents is mediated through some form of congestion in the network. With static flow models, this leads to the very well-studied area of network congestion games [Rou05].

Static models do not always suffice, however. For example, in telecommunications networks with demands changing over short time scales; or modeling morning or evening rush hour traffic. Here, there is no plausible static approximation, and the variation on congestion over time must be considered.

In both telecommunications networks and transportation networks, many different dynamic models have been studied. Our focus in this work will be on two related models, one continuous and the other discrete (in some sense).

The deterministic queueing model: This model goes back to Vickrey [Vic69], who studied this model for a single link under departure time choice. As well as the deterministic queueing model, it goes variously by the names of the *fluid queueing model*, and the *Vickrey bottleneck model*. In this model, each link has a *capacity* and a *transit time*. If the inflow rate into the link always remains below its capacity, then the time taken to traverse the link is constant, as given by the transit time. However, if the inflow rate exceeds the link capacity for some period, a queue grows on the entrance of the link. The delay experienced by a user is then equal to the transit time, plus whatever time is spent waiting in the queue. As long as there is a queue present it will empty at rate given by the link capacity; depending on whether the inflow rate is smaller or larger than the capacity, this queue will decrease or increase size. Note that this model is nonatomic, in the sense that individual users are infinitesimally small.

There are many works investigating properties of equilibria in this model [BFA15], [CCL15], [CCO21], [CCO19], [Kai22], [Koc12], [KS11], [OSVK22], [SS18] and in generalized models [IS20], [PS20], [SVK19], [Ser20]. We will discuss some of these later in Section II-E.

Packet-routing models: We will use "packet-routing" or "packet-based" to refer to models of a similar form to the Vickrey bottleneck model, but with atomic, unsplittable agents (or packets). As one simple example of such a model, suppose

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all links in the network have an integer capacity and an integer travel time. Packets have unit size, and the capacity of a link represents the number of packets that can simultaneously be processed by the link in unit time (or equivalently, in a single time step; the model can be considered to be in discrete time). If more packets than the capacity of a link need to be processed in a time step, the excess packets wait in a first-in-first-out (FIFO) buffer. Various models of this type, varying in the details, have been considered, both in the telecommunications and transportation context, e.g., [CCCW17], [HMRT11], [Ism17], [KM15], [LMR94], [LMR99], [SST18], [TVK21], [WHK14]. The traffic simulator MATSim [HNA16] uses an atomic model; each "packet" represents a single vehicle.

So broadly speaking, we have described nonatomic and atomic variants of the same underlying dynamic model. The nonatomic nature of the deterministic queueing model is motivated primarily by better mathematical properties rather than as a reflection of reality. Individual vehicles are of course not really infinitesimal; though it seems reasonable to represent them as such, as long as traffic volumes are large enough that each individual road user alone is insignificant.

But while it seems *reasonable* to expect the (nonatomic) deterministic queueing model to be a good approximation to a corresponding (atomic) packet-routing model, is this actually true? Can this approximation be justified? Formally, consider the following question. Fix a network, including arc transit times and capacities, and the inflow rate at the source. Now consider a sequence $\beta_1, \beta_2, \ldots > 0$, with $\beta_i \to 0$ as $i \to \infty$. For each β_i, consider an instantiation of a specific packet-routing model with packets having size β_i. We maintain the inflow rate, measured as the product of packet size with the number of packets entering the network per unit time. An equilibrium solution can be determined for this packet model, and if we fix any link e in the network, we can observe how the length of the queue on this arc behaves in this equilibrium (a time-varying quantity). As $i \to \infty$, does this function converge (say in the uniform norm) to the corresponding queue delay function for e in the equilibrium of the deterministic queueing model? If this is *not* true, then one has to seriously question the relevance of the deterministic queueing model.

Positive experimental evidence for convergence was found in [ZSV+21]. In [SVZ21], convergence was shown for a *fixed* choice of paths for all packets (in an appropriate sense). This already involves some significant technicalities, but their result does not say anything about the relationship between *equilibria* in the two settings. A key difficulty is that we do not know a priori that the paths chosen in the equilibrium of the packet model will resemble those chosen in the equilibria of the deterministic queueing model.

There are a number of other distinct but similar questions one can ask, all concerning the stability of the deterministic queueing model and its equilibria. In exact equilibria, users choose exactly quickest paths to the sink. That is, given the strategies chosen by the other users, they choose a path that in hindsight yields the earliest possible arrival time. This is quite a strong property; note that users are taking into account queues that they will see "in the future", not the queues as they are on entry into the network. One motivation for this is that we view, for example, morning rush hour traffic as a repeated game, with the expectation that behavior converges to a Nash equilibrium.[1] Still, it seems implausible to expect that this process always achieves *exactly* a Nash equilibrium. It seems much more plausible to hope that we obtain an approximate equilibrium; no user is taking a path that is very far from their quickest option, but might be choosing a route that is close to quickest, but not quite. So it is natural to consider ε-equilibria in this model (or in the packet model), and ask if their behavior is similar to that of exact equilibria. Again as a precise question: do ε-equilibria of the deterministic queueing model converge to the exact equilibrium, in the same sense as above?

Other natural types of "perturbations" can be considered. For instance: arc travel times and capacities might vary slightly over time, in some predictable or unpredictable way, or the demand might vary slightly over time instead of being precisely constant. The question in each case is the same: for sufficiently small perturbations of whatever form, can we say that equilibria in the perturbed system are close to equilibria in the original unperturbed system?

Our results: We give a positive answer to the following two convergence questions in the single-commodity setting. We show that equilibria in a particular packet model converge to that of the deterministic queueing model, as the size of the packets goes to zero; and that ε-equilibria converge to the exact equilibrium as $\varepsilon \to 0$. Moreover, we do this in a unified way. We will prove a single main convergence result, and then show that both of these specific results follow.

Our convergence result is based around the notion of a *strict δ-equilibrium*, which is a fairly natural strengthening of an ε-equilibrium. It asks that for *every* node in the network that an agent uses in their path, not just the sink, the agent's departure time from that node is at most δ later than its earliest possible arrival time (considering all possible routes to the node). This is stronger than asking for an ε-equilibrium with $\varepsilon = \delta$, which only requires this at the sink. It is not the case that every ε-equilibrium is a strict δ-equilibrium with $\delta = \varepsilon$, but we are able to show that every ε-equilibrium is a strict δ-equilibrium with $\delta = O(\varepsilon)$ (here, the big-O notation hides network-dependent constants). So convergence results for strict δ-equilibria hold for ε-equilibria as well.

To obtain results for packet routing, we "embed" an equilibrium of the packet-routing model into our continuous framework, viewing each packet as consisting of a continuum of particles.

[1] Other equilibria notions distinct from Nash equilibria have been considered in the literature, in which agents make decisions without full information of the overall traffic situation. We refer to Graf, Harks and Sering [GHS20] and references therein for a discussion of instantaneous dynamic equilibria (see also [GH23a], [GH23b]), where agents make decisions as they traverse the network based on current queues; and to Graf, Harks, Kollias and Merkl [GHKM22] for a very interesting approach to a much more general information framework where users use predictions of future congestion patterns.

While we focus on these two specific implications (and also use a specific packet-routing model), our convergence result can certainly be used to derive other stability results. For example, convergence results for other packet-routing variants, and for perturbed transit times and inflow rates. We will not consider this further in the current paper however, in order to focus on our central results.

One challenge in proving such a result is that the perturbations we are considering are ongoing throughout the evolution of the equilibrium. A much weaker notion of stability would be that if we slightly perturb the equilibrium at a single moment in time, or some bounded number of moments in time, by (say) perturbing some queue lengths or transit times by a small amount, that the equilibrium in the once-perturbed instance stays close to the unperturbed equilibrium. This was demonstrated quite recently for the deterministic queueing model by Olver, Sering and Vargas Koch [OSVK22]. Their result can be seen as a precursor to this one, and we will rely on it in a number of places in our proof. However, their result is not strong enough to handle the convergence results we are interested in here. In some sense, we need to show not only that the perturbations that occur in our perturbed equilibria at a particular moment do not lead to vastly different behaviors, but rather that there is a tendency to "revert to the mean": if some queue gets a little longer than it should in the perturbed situation, future perturbations might push this back towards the unperturbed value, but not even further away.

A smaller technical issue we have to address involves the foundations of the definition of the model. Especially for the implications to convergence of packet-routing models, it is necessary for us to allow waiting in our nonatomic model; an agent (that is, an infinitesimal flow particle) is allowed to wait at a node before entering the next arc of its path to the sink. This expansion of the strategy space, from a finite set of paths to a finite *dimensional* space of paths along with waiting times at nodes, requires some technical care, and we make some efforts to handle this in a precise and clean way. As one example of a complication that arises, it is now possible for a positive measure of agents to enter an arc at precisely the same moment in time. Previous works in this area had no particular need to consider waiting, and did not face this issue.

A different application of our results is in the other direction, to port results on the deterministic queueing model to packet models. This allows us to profit from the cleaner and more analytically tractable setup of the continuous model. Here we briefly discuss two implications for packet models; we expect there will be more.

Suppose we are considering an instance of the packet model, in which packets enter the network at s at a constant rate, and wish to reach a sink t. Suppose further that the number of packets entering the network per unit of time, multiplied by their size, is not larger than the minimum capacity of an s-t-cut. Is it then true that queues in the network remain bounded for all time? For the deterministic queueing model analog of the instance, this is known to be true [CCO21]. The proof uses a very delicate potential function, obtained from the dual of a linear program that describes so called "steady-state" conditions. It is not clear how this argument could be directly ported to discrete packet models. But our convergence result implies that indeed queues do remain bounded in the packet model — at least as long as δ is sufficiently small. We expect that with some further technical effort, our convergence result can be strengthened so that this restriction can be bypassed.

A second question that can be attacked with this approach is that of the price of anarchy. Here, one needs to specify the objective to compare with some care. It's known that the ratio between the average journey time of agents in an equilibrium, compared to the global optimum, can be unbounded even on very simple examples [Koc12]. However, if one considers the average arrival time objective (equivalently, viewing packets as all being at the source at time 0), this becomes an interesting question. It remains open in the deterministic queueing model, but it is conjecture that the price of anarchy for this measure is precisely $\frac{e}{e-1}$, and it is known that this holds if another natural conjecture is true [CCO19]. If this conjecture is demonstrated, it will immediately imply (through [CCO19] and our result) that the price of anarchy is bounded in the packet model, at least if δ is sufficiently small.

The question of whether an equilibrium is "stable" under some form of perturbation is a rather natural one, also in other non-traffic settings. Aswathi, Balcan, Blum, Sheffet and Vempala [ABB+10] and Balcan and Braverman [BB17] (see also [RJLM06]) explicitly introduce and investigate a related notion in the context of bimatrix games. In that context, they say that a Nash equilibrium is (δ, ε)-*perturbation stable* if whenever all payoffs in the bimatrix game are adjusted by at most δ, any equilibrium in the resulting game is within distance ε (in variation distance) of an equilibrium of the unperturbed game. These papers study various properties (especially computational properties) of perturbation stable games.

Dynamic traffic modeling is a huge, multidisciplinary area, and we will not attempt to do it justice in this brief literature review. In particular, our discussions have focused on the work done by the algorithmic game theory community. We refer the reader to the survey by Friesz and Han [FH19] for a different perspective on the topic, considering a more general class of link dynamics through the lens of differential variational inequalities.

Outline: We give a precise definition of the model we study in Section II, discussing also properties of exact dynamic equilibria, and the definition of strict δ-equilibria. Section III gives a high-level overview of the main results and the proofs. The details are omitted in this extended abstract; we refer the reader to the full version. We briefly conclude with some future directions in Section IV.

II. Model and preliminaries

An instance is described by a directed network $G = (V, E)$, with arc capacities $\nu_e > 0$ and free-flow travel times $\tau_e > 0$

for all arcs $e \in E$[2]. In addition, there is a specified source node $s \in V$ and sink node $t \in V$, and a constant *network inflow rate* u_0. We may assume that every node in G is both reachable from s, and can reach t.

We use the notation $\delta^-(v)$ and $\delta^+(v)$ to denote the set of incoming and outgoing arcs at v, respectively, and similarly $\delta^-(S)$ and $\delta^+(S)$ for arcs entering or leaving a set $S \subseteq V$.

Whenever not specified, we will use $\|\cdot\|$ to refer to the infinity norm, which will be our main measure of distance. Given a point $x \in \mathbb{R}^m$ and a set $S \subseteq \mathbb{R}^m$, we use $\mathrm{d}(x, S)$ to denote the distance (with respect to the infinity norm) between x and S, that is: $\mathrm{d}(x, S) := \inf_{y \in S} \|y - x\|$. Similarly, given two sets $S, T \subseteq \mathbb{R}^m$, $\mathrm{d}(S, T) := \inf_{x \in S} \mathrm{d}(x, T) = \inf_{x \in S, y \in T} \|y - x\|$. We will use $B_r(x)$ to denote the ball of radius r around $x \in \mathbb{R}^m$ and $B_r(S) = \{\, x \in \mathbb{R}^m \mid \mathrm{d}(x, S) \leq r \,\}$ for any $S \subseteq \mathbb{R}^m$, both with respect to the infinity norm,.

A. Flows over time with waiting

In the literature (e.g., [CCL15]), flows over time are typically denoted by a family of functions $(f_e^+, f_e^-)_{e \in E}$, where $f_e^+(\xi)$ denotes the inflow rate into arc e at time ξ and $f_e^-(\xi)$ the flow rate out of arc e at time ξ. As we want to allow particles to wait at nodes, this choice becomes less convenient, as it is possible that particles wait at a node in such a way that an atom of particles enters an arc at the same moment in time. In this case the inflow rate would be infinite. Instead, we define flows in terms of *cumulative* flow functions which are essentially the integrals of f_e^+ and f_e^-.

A *flow over time with waiting* consists of a pair (F^+, F^-), where F^+ is a vector of functions $F_e^+ : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ for arcs $e \in E$, and similarly for F^-. For each arc e and $\xi \in \mathbb{R}_{\geq 0}$, $F_e^+(\xi)$ denotes the total amount of flow that has entered arc e up to time ξ and $F_e^-(\xi)$ denotes the total flow amount that has left arc e up to time ξ. Each F_e^+ and F_e^- should be a nondecreasing and right-continuous function. These functions must satisfy the following two conditions.

Relaxed flow conservation: For all times $\xi \in \mathbb{R}_{\geq 0}$ it must hold that

$$\textstyle\sum_{e \in \delta^-(v)} F_e^-(\xi) - \sum_{e \in \delta^+(v)} F_e^+(\xi) \geq \begin{cases} 0 & \forall v \in V \setminus \{\, s, t \,\}, \\ -u_0 \xi & \text{for } v = s. \end{cases} \quad (1)$$

Note that we require the flow to enter the network at s with constant inflow rate of u_0.

Queues operate at capacity: We assume that arcs always operate at capacity (waiting is allowed at nodes, but there is no waiting on arcs in our model). Let $z_e(\xi)$ be the *queue volume* on e at time ξ; that is, the total measure of particles in the queue at time ξ. We have

$$z_e(\xi) := F_e^+(\xi) - F_e^-(\xi + \tau_e);$$

particles that enter by time ξ, but have not left the queue by time ξ (and hence have not left the tail of the arc by time $\xi + \tau_e$) contribute to the queue volume.

[2]Excluding arcs with $\tau_e = 0$ is convenient for technical reasons; it should be possible with some additional effort to extend to at least the setting where there are no directed cycles of 0-length arcs, but we will not discuss this here.

For all $e \in E$ and all times $\xi \in \mathbb{R}_{\geq 0}$ we require that

$$z_e(\xi) = \max_{0 \leq \psi \leq \xi} \Big(F_e^+(\xi) - F_e^+(\psi) - \nu_e(\xi - \psi) \Big). \quad (2)$$

The interpretation of this is that for any $\psi \leq \xi$, $z_e(\xi)$ is at least the mass of particles entering in the interval $[\psi, \xi]$, minus the upper bound $\nu_e(\xi - \psi)$ on the mass of particles that can leave the queue in this time. Further, if ψ is chosen so that $z_e(\psi) = 0$ but $z_e(\xi') > 0$ for all $\xi' \in (\psi, \xi)$, then we do not merely have a lower bound on $z_e(\xi)$, but must have equality, since the queue must operate at capacity on the interval (ψ, ξ).

B. Agent perspective

A flow over time with waiting does not identify a path or flow corresponding to a given particle. For exact dynamic equilibria, this is not a concern; a flow over time that corresponds to a dynamic equilibrium provides sufficient information to reconstruct the flow attributable to departures from the source at any moment in time. This is no longer the case for our setting however, and we need additional direct information about particle behavior.

We will denote our set of agents (equivalently, particles) by $\mathcal{A} := \mathbb{R}_{\geq 0} \times [0, 1]$. We let μ denote the Lebesgue measure on $\mathcal{A}$. For each $a \in \mathcal{A}$, we use $\vartheta(a)$ to denote the first coordinate of a divided by u_0, which we will interpret as the *entry time* of agent a into the system, i.e., the time it arrives at the source. (Put differently, the first coordinate of $a \in \mathcal{A}$ indicates the measure of particles that arrive at the source before a.) Previous works on Nash flows over time generally took the set of particles to be indexed by $\mathbb{R}_{\geq 0}$, identifying an agent with its entry time. The strategy of a flow particle was then described by a unit flow. This approach turns out to be inconvenient for our more general setting, however.

A *strategy* for an agent consists of a pair (P, w), where P is an s-t-path, and $w \in \mathbb{R}_{\geq 0}^{V(P)}$ denotes the amount of time that the agent will wait at each node in the path. Let $\mathcal{S}$ denote the set of all possible strategies. We view $\mathcal{S}$ as a measurable space, where a set $Q \subseteq \mathcal{S}$ is measurable if $\{w \in \mathbb{R}_{\geq 0}^{V(P)} : (P, w) \in Q\}$ is Lebesgue measurable for every s-t-path P. A *strategy profile* φ is a measurable map from $\mathcal{A}$ to $\mathcal{S}$. We use $\mathsf{P}^\varphi(a)$ to denote the first component of $\varphi(a)$, i.e., the path that agent a chooses. For each $v \in V$, we define w_v^φ to be the partial function that defines $\mathsf{w}_v^\varphi(a)$ to be the time agent a waits at v, if $v \in \mathsf{P}(a)$. We may write $\mathsf{w}^\varphi(a)$ for the vector $(\mathsf{w}_v^\varphi(a))_{v \in V(\mathsf{P}^\varphi(a))}$. We will typically omit the explicit dependence on φ in our notation whenever it is unambiguous.

The measurability condition on φ implies that for any s-t-path P, any $\theta_1 \leq \theta_2$, and any Lebesgue measurable set $R \subseteq \mathbb{R}_{\geq 0}^{V(P)}$,

$$\{a \in \mathcal{A} : \vartheta(a) \in [\theta_1, \theta_2], \mathsf{P}(a) = P \text{ and } \mathsf{w}(a) \in R\}$$

is a measurable set.

Note that $\mu(\{a \in \mathcal{A} : \theta_1 \leq \vartheta(a) \leq \theta_2\}) = u_0(\theta_2 - \theta_1)$ for all $\theta_1 \leq \theta_2$, given the network inflow rate of u_0. In particular, the set of particles entering the network at some time θ is always a null set.

An *outcome* of the game for a given strategy profile φ specifies, for each particle a, their precise departure time from each node v on their path $\mathsf{P}(a)$. This must correspond to a flow over time with waiting as described above. We now make this precise.

We specify an outcome by a flow over time with waiting (F^+, F^-), and partial functions $d_v : \mathcal{A} \to \mathbb{R}_{\geq 0}$ for each $v \in V$. The value $d_v(a)$ is defined only when $v \in V(\mathsf{P}(a))$, and in that case, it describes the time at which agent a departs v and enters the arc $e = vw \in \mathsf{P}(a)$ that follows, or the time that the agent departs the network if $v = t$. We call each d_v a *departure time function*.

In order for (F^+, F^-, d) to represent a valid outcome of a given strategy profile φ, we require the following to hold. For each arc e, let z_e be the queue volume for e associated with (F^+, F^-).

- Departure times must be consistent with queue delays and node waiting times. Consider any agent a, and arc $e = vw \in \mathsf{P}(a)$. We must have that
$$d_w(a) = d_v(a) + q_e^d(a) + \tau_e + \mathsf{w}_w(a),$$
where $q_e^d(a)$ is the amount of time that a waits on the queue on arc e.
The value of $q_e^d(a)$ is essentially determined by the queue volume at the time $d_v(a)$ that agent a enters e, with the additional complication that if an atom of particles enters e at this same moment, a tiebreaking rule is required. We tiebreak according to entry time into the network. So we have
$$\begin{aligned} q_e^d(a) := \tfrac{1}{\nu_e}(z_e(d_v(a)) - \mu(\{a' \in \mathcal{A} : e \in \mathsf{P}(a'),\\ d_v(a') = d_v(a) \text{ and } \vartheta(a') > \vartheta(a)\})). \end{aligned} \tag{3}$$
- The cumulative flow $F_e^+(\xi)$ entering an arc $e = vw$ by some time ξ matches with φ and d_v. That is,
$$F_e^+(\xi) = \mu(\{a \in \mathcal{A} : e \in \mathsf{P}(a) \text{ and } d_v(a) \leq \xi\}).$$

C. Network loading

It is not immediately obvious how to construct the outcome (F^+, F^-, d), nor even that they exist or are unique. The demonstration of this is via a *network loading* procedure. This is fairly standard, and there are no major conceptual issues, but previous discussions of network loading that we are aware of do not allow for waiting, and this does introduce some minor technical complications. We defer the proof to the appendix.

Theorem II.1. *Given any strategy profile φ, there is a unique associated outcome (F^+, F^-, d).*

D. A form of approximate dynamic equilibria

We now recall the notion of *earliest arrival labels*, ubiquities in the study of Nash flows over time (see [CCL15], [CCO21], [KS11] among others). Let φ be a strategy profile, with outcome (F^+, F^-, d), and let $(z_e)_{e\in E}$ be the queue volume functions associated with this. Then for any $v \in V$, the earliest arrival label $\ell_v : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ maps an entry time θ to an earliest possible time a hypothetical particle departing at time θ could arrive at v, taking into account queueing delays induced by other agents using the current strategy profile. They can be defined via the Bellman equations
$$\ell_w(\theta) = \begin{cases} \theta & \text{if } w = s \\ \min_{e=vw} \ell_v(\theta) + \tau_e + z_e(\ell_v(\theta))/\nu_e & \text{otherwise.} \end{cases} \tag{4}$$
Note that $z_e(\ell_v(\theta))/\nu_e$ is the queue waiting time a hypothetical particle departing the source at time θ and arriving at the earliest possible time $\ell_v(\theta)$ experiences on edge $e = vw$. There is no issue to worry about in terms of tiebreaking, since all particles with $d_v(a) = \ell_v(\theta)$ will have entry time at most θ, and so do delay our hypothetical particle.

"Exact" dynamic equilibria: A dynamic equilibrium has a simple definition in our notation. It is that $d_v(a) = \ell_v(\vartheta(a))$ for all $a \in \mathcal{A}$ and $v \in \mathsf{P}(a)$. That is, each agent arrives and departs at each node on its path at an earliest possible time (in particular, the agent arrives at the sink at the earliest possible time) taking into account queueing delays.

Given the vector ℓ of earliest arrival labels of a dynamic equilibrium, we will follow [OSVK22] in calling ℓ an *equilibrium trajectory*. We will discuss properties of dynamic equilibria and equilibrium trajectories in more detail in Section II-E.

ε-equilibria: We can easily interpret the general notion of an ε-approximate Nash equilibrium (more briefly, an ε-equilibrium) in our model. Every agent should have a travel time that is at most ε larger than the best travel time they could achieve, taking into account the actions of all other agents. In other words, a strategy profile is an ε-equilibrium for some $\varepsilon > 0$ if the outcome satisfies
$$d_t(a) \leq \ell_t(\vartheta(a)) + \varepsilon \quad \text{for all } a \in \mathcal{A}. \tag{5}$$

Strict δ-equilibria: If we consider some arbitrary node v in an ε-equilibrium, it needs not be the case that every agent a that uses v in their path arrives at v within ε of the earliest possible arrival time. The reason is that the agent may be able to "catch up" by the time it reaches the sink. For example, if an arc entering the sink has large capacity, but at some point in time has a large queue, then agents could join the back of this queue over a larger interval of time, but exit the queue over a shorter interval.

It will be useful for our purposes to consider the stronger notion where this property does hold. Define a *strict δ-equilibrium* as a strategy profile where the outcome satisfies
$$d_v(a) \leq \ell_v(\vartheta(a)) + \delta \quad \text{for all } a \in \mathcal{A} \text{ and } v \in \mathsf{P}(a). \tag{6}$$

Given a strict δ-equilibrium, the corresponding earliest arrival labels ℓ will be of particular importance for us (as they were in the case of exact dynamic equilibria). If ℓ arises from a strict δ-equilibrium, we will say simply that ℓ is a *δ-trajectory*.

E. Properties of exact equilibria

We now briefly summarize some useful facts about the structure of (exact) equilibria. For more details, we refer to [CCL15] and [KS11] on thin flows and the piecewise-linear structure; to [CCO21] and [OSVK22] for long-term behavior;

and to [OSVK22] for the vector-field view and uniqueness and continuity of equilibria.

In most previous discussions of dynamic equilibria in networks of Vickrey bottlenecks, there is no strategy profile in the sense we have defined it for our model, where each particle chooses a single path. Rather, an equilibrium is described by a flow over time (F^+, F^-) (without waiting), which induces the earliest arrival labels $\ell(\theta)$ and associated queue volumes z_e. Since there is no waiting, z_e is continuous for each e. Let $q_e(\theta) = z_e(\ell_v(\theta))/\nu_e$ for each $e = vw \in E$. An arc $e = vw$ is called *active* at entry time θ if $\ell_w(\theta) = \ell_v(\theta) + \tau_e + q_e(\theta)$. This means that a particle departing the source at time θ has a shortest path to w that uses arc e, and that e defines $\ell_w(\theta)$ in the Bellman equations (4). Then one definition of a dynamic equilibrium is that $(F_e^+)'(\ell_v(\theta)) = 0$ whenever $e = vw$ is not active at entry time θ. This matches our earlier definition in Section II-D for our model where waiting is allowed: agents must arrive at the earliest possible time at each node on their path. It turns out that if one defines $x_e(\theta) := F_e^+(\ell_v(\theta))$ for all $e = vw$ and θ, then for a dynamic equilibrium, $x(\theta)$ is an s-t-flow of value $u_0\theta$, for each θ. From our perspective, $x_e(\theta)$ can be viewed as the measure of agents with entry time at most θ which choose arc e in their strategy.

Conveniently, as we describe below, ℓ alone, without reference to the defining flow over time, suffices to describe an exact dynamic equilibrium (which is not the case for approximate equilibrium concepts). Consider some $e = vw$. If $\ell_w(\theta) < \ell_v(\theta) + \tau_e$, then even without a queue, e is not active at entry time θ. Further, it can be argued that in a dynamic equilibrium, $q_e(\theta) = \max\{\ell_w(\theta) - \ell_v(\theta) - \tau_e, 0\}$. So information about whether arc $e = vw$ is active or not, whether it has a queue or not (from the perspective of a particle entering the network at time θ), and the length of that queue, is completely determined by $\ell(\theta)$.

Active and resetting arcs: For any $l^\circ \in \mathbb{R}^V$, let

$$\begin{aligned} E'_{l^\circ} &:= \{e = vw \in E : l^\circ_w \geq l^\circ_v + \tau_e\}, \quad \text{and} \\ E^*_{l^\circ} &:= \{e = vw \in E : l^\circ_w > l^\circ_v + \tau_e\}. \end{aligned} \tag{7}$$

So e is active at entrance time θ if $e \in E'_{\ell(\theta)}$, and has a queue if $e \in E^*_{\ell(\theta)}$. We call the arcs with a queue also resetting arcs.

Thin flows: It has been shown [CCL15], [KS11] that a flow over time is in equilibrium if and only if the resulting pair (x, ℓ) satisfies the following *thin flow conditions* for almost every θ: setting $x' = x'(\theta)$, $\ell' = \ell'(\theta)$, $E' = E'_{\ell(\theta)}$ and $E^* = E^*_{\ell(\theta)}$,

$$\begin{aligned} &x' \text{ is a static } s\text{-}t \text{ flow of value } u_0, \\ &\ell'_s = 1, \\ &\ell'_w = \min_{e=vw\in E'} \rho_e(\ell'_v, x'_e) && \forall w \in V \setminus \{\, s \,\}, \\ &\ell'_w = \rho_e(\ell'_v, x'_e) && \forall e = vw \in E' \text{ with } x'_e > 0, \end{aligned} \tag{8}$$

$$\text{where } \rho_e(\ell'_v, x'_e) := \begin{cases} \frac{x'_e}{\nu_e} & \text{if } e = vw \in E^*, \\ \max\{\ell'_v, \frac{x'_e}{\nu_e}\} & \text{if } e = vw \in E' \setminus E^*. \end{cases}$$

Note that the conditions are fully determined by the pair (E', E^*), with $E^* \subseteq E'$. As long as (i) each node v is reachable from s in (V, E'), (ii) each arc $e \in E^*$ lies on an s-t-path in (V, E'), and (iii) no arc of E^* lies on a directed cycle in (V, E'), these equations always have a solution, and ℓ' is uniquely determined [CCL15], [Koc12]. We will sometimes call this unique ℓ' (leaving out x') the *thin flow direction*.

We call a pair (E', E^*) satisfying (i)-(iii) above a *valid configuration*. Furthermore, we call a vector $l^\diamond \in \mathbb{R}^V$ *valid* if $(E'_{l^\diamond}, E^*_{l^\diamond})$ is a valid configuration. We define $\Omega \subseteq \mathbb{R}^V$ to be the subset of valid labels.

Define $X : \Omega \to \mathbb{R}^V$ be the vector field for which $X(l^\circ)$ is the unique solution to the thin flow equations for $(E'_{l^\circ}, E^*_{l^\circ})$, for all $l^\circ \in \Omega$. Then put differently $\ell'(\theta) = X(\ell(\theta))$ for almost every θ. Since $X(l^\circ)$ depends only on E'_{l° and $E^*_{l^\circ}$, it is piecewise constant, and indeed with a very specific structure. Each arc $e = vw$ divides Ω into two open halfspaces separated by the hyperplane $\{l^\circ \in \Omega : l^\circ_w - l^\circ_v = \tau_e\}$. So an equilibrium trajectory ℓ has a piecewise linear structure, with its direction only changing upon hitting a hyperplane. Each maximal piecewise-linear segment of ℓ is called a *phase*.

We can define an equilibrium trajectory starting from any initial point $l^\circ \in \Omega$, not necessarily an empty network. This can be interpreted, with some care, as starting with some initial queues present; if $\ell_w(0) - \ell_v(0) - \tau_e > 0$, this value represents a queue delay that an agent starting at time 0 and traversing e via a shortest path would experience on the arc, not the queue length at time 0.

A further generalization we will need in our arguments is the notion of a *generalized subnetwork*, as in [OSVK22]. A generalized subnetwork is defined by a valid configuration $(\tilde{E}, E^\infty)$. Given such a pair, we can define a new vector field $X^{(\tilde{E}, E^\infty)}(\cdot)$, by defining its value at position l° to be the solution to the thin flow equations determined by the pair $(\tilde{E} \cap E'_{l^\circ}, E^\infty \cup E^*_{l^\circ})$ (as opposed to $(E'_{l^\circ}, E^*_{l^\circ})$). Only arcs in $\tilde{E} \setminus E^\infty$ will have a corresponding hyperplane; arcs in $E \setminus \tilde{E}$ act always as being inactive, and arcs in E^∞ are viewed as always having a queue. We can define an equilibrium trajectory in this generalized subnetwork in the same way as for the full network; a trajectory ℓ that follows $X^{(\tilde{E}, E^\infty)}$ almost everywhere. We will sometimes refer to a "generalized network", by which we mean a network along with a choice of $(\tilde{E}, E^\infty)$.

Long-term behavior: Given an equilibrium trajectory ℓ in some generalized network, we say that ℓ has reached *steady state* by time T if $\ell'(\theta)$ is constant for all $\theta \geq T$. This means that queues change linearly from time T forward (in particular, if $E^\infty = \emptyset$ and the network has sufficient capacity, then in a steady state queues will remain constant [CCO21]).

Say that a label $l^\circ \in \Omega$ is a steady-state label if the equilibrium trajectory starting from l° is immediately at steady state. We denote the set of steady-state labels by I. It can be shown that there is a unique "steady-state direction" λ so that for every $l^\circ \in I$, the equilibrium trajectory starting from l° is $\ell(\theta) = l^\circ + \lambda\theta$. See [OSVK22] for details.

We will need the following result from [OSVK22], which builds on an earlier result by [CCO21], and shows that equilibrium trajectories always reach a steady state.[3]

Theorem II.2. ([OSVK22]) *Let G be a generalized network. Then there exists some T such that for any starting point l° within distance r to the set of steady state-labels I, the equilibrium trajectory ℓ starting from l° reaches steady state in time at most $T \cdot r$.*

Continuity: In [OSVK22], it was shown that there is a *unique* equilibrium trajectory for any given starting point $\ell(0) \in \Omega$, and that the trajectory ℓ depends *continuously* on the starting point $\ell(0)$. We will make crucial use of this.

Theorem II.3 ([OSVK22, Theorem 3.2])**.** *Let $\Psi : \Omega \to L^\infty([0,\infty))$ be the map that takes $l^\circ \in \Omega$ to the unique equilibrium trajectory ℓ satisfying $\ell(0) = l^\circ$. Then Ψ is a continuous map, where we imbue $L^\infty([0,\infty))$ with the supremum norm.*

III. Technical overview

A. The main convergence result

We are now ready to state our main result in its precise form.

Theorem III.1. *Fix a network G. Let $l^\circ \in \Omega$ be the labeling corresponding to the empty network, and let ℓ^* be the equilibrium trajectory starting from l°. Then for every $\varepsilon > 0$, there is a $\delta > 0$ such that every δ-trajectory ℓ starting from l° stays within ε distance to ℓ^*, i.e.,*

$$\|\ell(\theta) - \ell^*(\theta)\| \leq \varepsilon \quad \text{for all } \theta \in \mathbb{R}_{\geq 0}.$$

B. Implications for ε-equilibria and packet models

To show that we can use the above theorem to obtain convergence for our two applications ε-approximate equilibria and packet routings, we prove that both these concepts can be modeled as strict δ-equilibria, where δ depends on ε in the first case and on the packet size in the latter case.

Theorem III.2. *Let φ be an ε-equilibrium for some $\varepsilon > 0$. Then φ is a strict δ-equilibrium for $\delta = O(\varepsilon)$.*

(The big-O hides network-dependent constants.) To prove this, we show that for an ε-equilibrium, (i) the mass of particles that could in principle overtake a fixed agent at a given node is $O(\varepsilon)$, and (ii) the earliest arrival labels fulfill an approximate Lipschitz-property. If an agent a were to arrive at some node much later than the quickest path would allow, then by approximate Lipschitz-continuity, the measure of other agents that would be able to overtake a would be too large.

We will fix a packet-routing model that is similar to the one discussed in [HMRT11]. In this model, a packet enters the next arc of its path only once it has been fully processed by the previous arc. Given an equilibrium in this packet model, say with packets of size β, we can view each packet as consisting of a measure β of infinitesimal flow particles, each taking the same path. In order to maintain the temporal integrity of a packet, we exploit the flexibility of waiting at nodes in our model. If a packet is being processed by some arc $e = vw$, we hold all particles of the packet at w as long as any of the particles are still being processed by arc e. Once all these particles arrive, they depart all at once onto the next arc of the path. This results in a joint strategy choice for all particles, with waiting at nodes.

Theorem III.3. *Suppose we are given an equilibrium of the packet model with packet size β, and consider the corresponding flow over time strategy profile φ. Then φ is a strict δ-equilibrium for $\delta = O(\beta)$.*

We show that φ is an ε-equilibrium for some $\varepsilon = O(\delta)$; the claim then follows from Theorem III.2. The intuition for why this holds is simply that the "last" particle in each packet takes an earliest arrival path[4]; and for other particles, assuming some Lipschitzness, things cannot go too badly wrong.

C. Proof overview of the main convergence result

It will be somewhat convenient for our purposes to invert the dependence between ε and δ. We will think of $\delta > 0$ as being given, and we must choose ε depending on δ so that any δ-trajectory ℓ remains ε-close to the equilibrium trajectory ℓ^*, and moreover, the dependence of ε on δ must be such that $\varepsilon \to 0$ as $\delta \to 0$. From this perspective, δ is some "small" quantity, and ε will be some typically much larger quantity — but nonetheless still "quite small" in the sense that it goes to 0 as δ goes to 0. Our arguments will involve producing a sequence of parameters that are "quite small" in the same sense, eventually leading to our choice of ε. We make this precise with the following definition.

Definition III.4. We call a function $r\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ a *small parameter* if $r(\delta) \to 0$ as $\delta \to 0$, and its definition only depends on the network G and previously defined small parameters.

We will typically omit the dependence on δ if there is no ambiguity, and write simply r rather than $r(\delta)$.

Our main convergence theorem comes as a consequence of the following technical theorem for which T_{ss} is defined to be the time an equilibrium trajectory requires to reach steady state from the empty network.

Theorem III.5. *There exists a family of small parameters $(\varepsilon_j)_{j \in \{0,\ldots,|E|\}}$ such that the following holds, for any δ small enough. Let $(\tilde{E}, E^\infty)$ be a valid configuration and $j := |\tilde{E} \setminus E^\infty|$. Suppose we are given an interval $[\theta_0, \theta_1]$ and a δ-trajectory ℓ. Let ℓ^* be the equilibrium trajectory for the generalized subnetwork defined by $(\tilde{E}, E^\infty)$ starting with $\ell^*(\theta_0)$ being a valid labeling closest to $\ell(\theta_0)$ and T be the time required for ℓ^* to reach steady state.*

[3] This theorem is more refined than the main continuity theorem of [OSVK22] in terms of the dependence on $\|l(0) - l^\circ\|$, but it can easily be deduced from more technical results in their paper; we defer detailed discussion to the full version.

[4] "Last" rather than "first" because of details of the specific packet model; for other natural packet models this might hold for the first particle instead.

Then supposing that (i) each hyperplane intersecting $B_{2\delta}(\ell([\theta_0, \theta_1]))$ corresponds to an arc in $\tilde{E} \setminus E^\infty$, and (ii) $\min\{\theta_1 - \theta_0, T\} \leq T_{ss} + 1$, we have

$$\|\ell(\theta) - \ell^*(\theta)\| \leq \varepsilon_j \quad \textit{for all } \theta \in [\theta_0, \theta_1].$$

A few remarks:

- While $\ell(\theta)$ needs not be a valid labeling, we can show that it always remains close to Ω. So in this theorem, $\|\ell^*(\theta_0) - \ell(\theta_0)\| = O(\delta)$.
- Condition (ii), while slightly awkward, will be convenient for inductive purposes. In some cases, we will apply the theorem inductively to an interval of length at most $T_{\text{ss}}+1$, and in other cases, to a potentially unbounded interval but where the time T is guaranteed to be small.
- The equilibrium trajectory ℓ^* in this theorem is not (in general) an equilibrium of the original network, but rather of the generalized subnetwork determined by $(\tilde{E}, E^\infty)$. This is again for inductive purposes; the arcs in E^∞ will be "far away", and can be ignored. If we are able to focus on a smaller number of hyperplanes, we can proceed inductively. It may initially seem paradoxical that we show that ℓ stays close to ℓ^*, if ℓ^* is not not the equilibrium trajectory in the full network that, in the end, we are showing that ℓ remains close to. The resolution is in condition (i), which is very strong. At the end of the day, this condition will only hold for intervals where ℓ^* is close to the equilibrium trajectory on the full network.
- This technical theorem does imply our main theorem, Theorem III.1, fairly immediately. Simply take $(\tilde{E}, E^\infty) = (E, \emptyset)$, $\theta_0 = 0$ and θ_1 arbitrarily large. The trajectory ℓ^* is the equilibrium in the original network, starting from the empty network, and so condition (ii) is satisfied by the definition of T_{ss}. Condition (i) is vacuous, and so we obtain the desired claim, with $\varepsilon = \varepsilon_{|E|}$.

The inductive proof of Theorem III.5 can be broken into two main parts. Unless otherwise indicated, any reference to an equilibrium trajectory (in particular the steady-state direction λ) refers to such a trajectory in the generalized subnetwork defined by $(\tilde{E}, E^\infty)$, and Ω refers to the set of valid labels in this generalized subnetwork.

Part I: Before reaching (near to) steady state: This first part is heavily inductive, and makes little direct use of the properties of δ-trajectories. The induction is on $j = |\tilde{E} \setminus E^\infty|$, that is, the number of hyperplanes determining our vector field X; see Figures 1 to 3 for an illustration of these vector fields and some key features of the proof.

Let $I \subseteq \Omega$ be the set of steady-state labels, and λ the steady-state direction of the subnetwork.

We first consider the behavior when sufficiently far from I. A geometric argument shows that this means that $\ell(\theta)$ is reasonably far from some hyperplane. The argument is as follow, roughly speaking. If all j hyperplanes do not have a common intersection, then necessarily there is a (network-dependent) lower bound on the distance between hyperplanes, and so $\ell(\theta)$ must be "far" from at least one hyperplane. Otherwise, if all hyperplanes have a nonempty intersection, all points in this common intersection can be shown to be part of I. One can always find a constant Γ such that the distance between a given point and the intersection is at most Γ times the distance to the farthest hyperplane. So being "far" from the common intersection of all hyperplanes means being (relatively) far from some hyperplane.

However, we may have a situation where over an interval $\ell(\theta)$ remains far from I, but not far from any single hyperplane; rather, we are always far from some hyperplane, but this hyperplane changes over time. So we divide the interval into "periods", where in each period we are far from a single particular hyperplane; we do this in such a way that each period is not too short. We then apply the theorem inductively for each period, one after the other. This is a somewhat lossy process; we can control the distance that ℓ deviates from ℓ^* over the period in terms of the distance they are apart at the beginning of the period (here the continuity of equilibrium trajectories as stated in Theorem II.3 is central), but these bounds get worse as we consider more and more periods. Fortunately, we can bound the number of periods, because of the fact that ℓ^* converges to steady state (along with our assumption (i)) gives a bound on the amount of time ℓ^* (and then inductively, ℓ) stays away from I. Since we also argue that each period is not too short, this suffices.

In slightly more detail, suppose that $\ell(\theta)$ is far from some hyperplane, say the one associated with arc $e' = v'w'$, on an interval $[\theta'_0, \theta'_1]$ of length at most $T_{\text{ss}} + 1$. Therefore, we can proceed inductively on this interval. If e' is inactive at $\ell(\theta'_0)$, we consider the generalized subnetwork defined by $(\tilde{E} \setminus \{e'\}, E^\infty)$; if e' is active (hence has a queue) at $\ell(\theta'_0)$, we consider the generalized subnetwork defined by $(\tilde{E}, E^\infty \cup \{e'\})$. We apply the theorem inductively to deduce that ℓ remains close to ℓ^*_{ind}, the exact equilibrium of the generalized subnetwork starting from a closest valid point to $\ell(\theta'_0)$. As long as $\ell(\theta)$ is further than ε_{j-1} from the hyperplane, then we can in addition deduce that ℓ^*_{ind} does not hit the hyperplane either, and so due to continuity of equilibrium trajectories (Theorem II.3) ℓ^*_{ind} stays close to ℓ^* on this interval, and we have what we want for this particular interval.

Denote the first point in time that ℓ gets within distance r_2 of I (for some suitable small parameter r_2) by θ_{ss}. The next claim is that ℓ remains (somewhat) close to I for the remainder of the evolution: for some small parameter r_3 (which may be much larger than r_2), $\mathrm{d}(\ell(\theta), I) < r_3$ for all $\theta \in [\theta_{\text{ss}}, \theta_1]$. In order to reach a distance r_3 from I, there will need to be an interval $[\theta_{\text{start}}, \theta_{\text{end}}]$ where $\mathrm{d}(\ell(\theta_{\text{start}}), I) \leq r_2$, $\mathrm{d}(\ell(\theta_{\text{end}}), I) \geq r_3$, and $\mathrm{d}(\ell(\theta), I) \geq r_2$ for all $\theta \in [\theta_{\text{start}}, \theta_{\text{end}}]$. Since ℓ remains far from I in this interval, we can apply what we have already shown to deduce that ℓ remains close to the equilibrium trajectory ℓ^* starting from a point $\ell^*(\theta_{\text{start}})$ close to $\ell(\theta_{\text{start}})$. But this equilibrium trajectory will reach steady state very quickly, by Theorem II.2. By choosing r_3 large enough compared to r_2 (but still with $r_3 \to 0$ as $\delta \to 0$), we can ensure that this happens by some time $\theta' < \theta_{\text{end}}$. This exploits that δ-trajectories can be

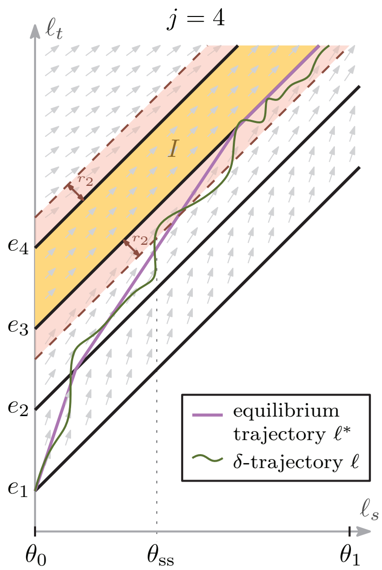

Figure 1. As an example consider a network with only two nodes s and t but four parallel arcs e_1 to e_4 with transit time $\tau_{e_i} = 2i + 1$ and capacities 1. The network inflow rate is $u_0 = 3$. First all hyperplanes are present and the equilibrium trajectory reaches steady state as soon as arcs e_1, e_2 and e_3 are active. To prove Theorem III.5 we consider inductively also generalized networks with fewer arcs.

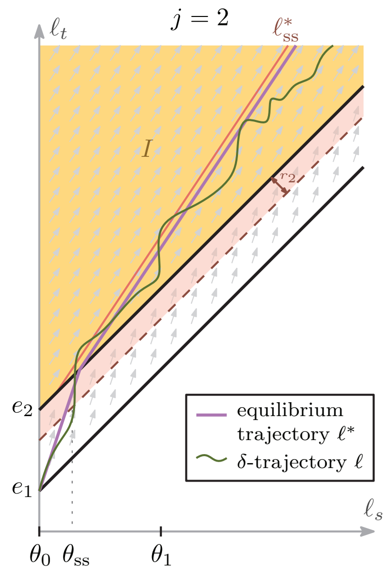

Figure 2. Here, e_3 and e_4 are removed and therefore $\tilde{E} = \{ e_1, e_2 \}$ and $E^\infty = \emptyset$. We consider an interval $[\theta_0, \theta_1]$ such that all other hyperplanes keep distance to ℓ. We split the interval at θ_{ss} which is the first point in time ℓ comes r_2-close to the set of steady-state labels I. Here, we also illustrated the equilibrium trajectory ℓ^*_{ss}, which starts within $B_{r_2}(\ell(\theta_{\text{ss}})) \cap I$ and therefore stays in steady state. ℓ^* and ℓ^*_{ss} are close due to the continuity of equilibrium trajectories; see Theorem II.3.

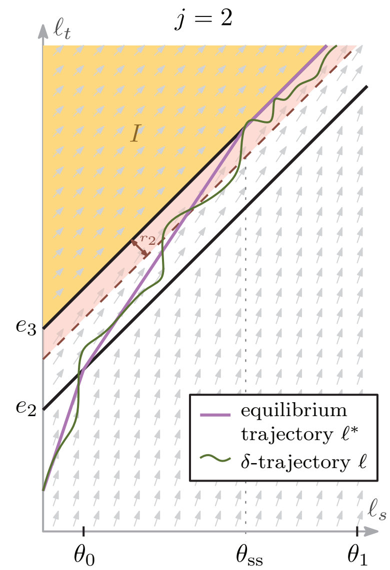

Figure 3. Here, we choose the hyperplanes of e_2 and e_3, which means that e_4 is removed and e_1 is promoted to a free arc. Hence $\tilde{E} = \{ e_1, e_2, e_3 \}$ and $E^\infty = \{ e_1 \}$.

shown to be approximately Lipschitz in a certain sense, and so the interval $[\theta_{\text{start}}, \theta_{\text{end}}]$ cannot be too short if r_3 is large. But this means that $\ell(\theta')$, being close to $\ell^*(\theta')$, is close to I — a contradiction.

Part II: While close to steady state: At this point, we have deduced that $\ell(\theta)$ is close to $\ell^*(\theta)$ until some time θ_{ss}, and that ℓ remains close to I from time θ_{ss} forwards. It remains to argue that $\ell(\theta)$ remains within distance ε_j of $\ell^*(\theta)$ for all $\theta \geq \theta_{\text{ss}}$ (for some small parameter ε_j). The main part is to prove that ℓ stays within ε_j distance to an equilibrium trajectory $\ell^*_{\text{ss}}(\theta) = \ell^*_{\text{ss}}(\theta_{\text{ss}}) + (\theta - \theta_{\text{ss}})\lambda$ with $\ell^*_{\text{ss}}(\theta_{\text{ss}}) \in I$ close to $\ell(\theta_{\text{ss}})$ (and therefore close to $\ell^*(\theta_{\text{ss}})$). Notice that we are asking for something quite strong. It's not enough to show that ℓ moves in "roughly the right direction"; even a small error in the direction, if maintained, would lead to a large error after a long enough period of time, and we might be considering an arbitrarily long interval. Some amount of self-correction is required; if ℓ deviates from ℓ^* in some direction by a significant amount, it should not deviate further in this direction (but could drift away in some other direction).

Consider some possible large $\theta \in (\theta_{\text{ss}}, \theta_1]$, and let $\Delta\theta := \theta - \theta_{\text{ss}}$, and $\Delta\ell := \ell(\theta) - \ell(\theta_{\text{ss}})$. Let us also define Δx_e to be the measure of particles that use arc $e = vw$, and enter the arc at some time in the interval $(\ell_v(\theta_{\text{ss}}), \ell_v(\theta)]$. Observe that if ℓ was an exact equilibrium, then $\Delta\ell = \lambda\Delta\theta$ and $\Delta x = y\Delta\theta$, where (y, λ) is the solution to the thin flow equations (8). This is a consequence of the uniqueness of equilibrium trajectories, or more precisely, a consequence of the earlier result by [CCL15] that thin flows are unique (in terms of λ). If there was in fact a second distinct solution $(\tilde{y}, \tilde{\lambda})$ with $\tilde{\lambda} \neq \lambda$, then $\tilde{\ell}(\theta) = l^\circ + \tilde{\lambda}\theta$ would be an equilibrium trajectory receding from ℓ^* at a linear rate.

Our approach can be viewed as taking the proof of [CCL15] on the uniqueness of thin flows, and making it "more robust" in certain ways. To explain this, we begin by sketching the basic idea of this proof (modified slightly to suit our present purposes). Suppose for a contradiction that $(\tilde{y}, \tilde{\lambda})$ is a second solution to (8) for configuration $(\tilde{E}, E^\infty)$, with $\tilde{\lambda} \neq \lambda$. Suppose that $S := \{v \in V : \tilde{\lambda}_v / \lambda_v < 1\}$ is nonempty and proper (if it is not, meaning that $\tilde{\lambda}_v \geq \lambda_v$ for all v, swap the role of λ and $\tilde{\lambda}$, after which S must be proper). One can then make the following key observations, as a consequence of the thin flow equations:

- All arcs $e = vw$ entering S either have $\tilde{y}_e = 0$, or have $\tilde{y}_e \leq y_e$, and the inequality is strict if $\tilde{y}_e > 0$. (Briefly: if $\tilde{y}_e > 0$, then the thin flow equations require that $\tilde{\lambda}_w \geq \tilde{\lambda}_v$; since e enters S, it follows that $\lambda_w > \lambda_v$, and then the thin flow equations require that $y_e = \lambda_w \nu_e$ and $\tilde{y}_e \leq \tilde{\lambda}_w \nu_e < \lambda_w \nu_e = y_e$.)
- All arcs $e = vw$ leaving S have $\tilde{y}_e \geq y_e$, and the inequality is strict if $y_e > 0$.

Since y and $\tilde{y}$ are both s-t-flows of the same value, $y(\delta^+(S)) -$

$y(\delta^-(S)) = \tilde{y}(\delta^+(S)) - \tilde{y}(\delta^-(S))$. This yields an immediate contradiction if $\tilde{y}(\delta^+(S)) > 0$ or $y(\delta^-(S)) > 0$. A small further argument rules out the case that these crossing flows are both zero.

We proceed with a similar cut-based argument in order to reach a contradiction if $\|\Delta\ell - \lambda\Delta\theta\|$ is very large. In order to do this, we first demonstrate that *some* of the thin flow conditions in (8) hold *approximately*. Here we directly invoke properties of strict δ-equilibria. For instance: we are able to show the following:

- $\Delta x/\Delta\theta$ is approximately an s-t-flow of value u_0; the appropriate flow conservation constraints hold at each node, up to an $O(\delta)$ error.
- For $e = vw \in E^\infty$, $|\Delta x_e - \nu_e \Delta\ell_w| \geq \nu_e\delta$.
- For $e = vw \in \tilde{E}\setminus E^\infty$, we can show that $\Delta x_e \leq \nu_e\Delta\ell_w + \nu_e\delta$. The thin flow equations imply the exact version of this (without the $\nu_e\delta$ error term), though this is a somewhat weak implication. In particular, we cannot directly show an approximate version of the statement that for $(\tilde{y}, \tilde{\lambda})$ a thin flow, and $e = vw \in \tilde{E}$ with $\tilde{\lambda}_w > \tilde{\lambda}_v$, $\tilde{y}_e = \nu_e\tilde{\lambda}_w$.

We then aim to define a cut S based on the ratio $\frac{\Delta\ell}{\lambda}$, with the intention of showing that $\frac{\Delta x}{\Delta\theta}(\delta^+(S))$ is significantly larger than $y(\delta^+(S))$ and $\frac{\Delta x}{\Delta\theta}(\delta^-(S))$ is significantly smaller than $y(\delta^-(S))$, thus deducing a contradiction to the fact that $\Delta x/\Delta\theta$ is approximately an s-t-flow of the same value as y. In order to obtain the desired contradiction, we need to use the above properties, and also some conclusions that can be drawn from induction. Significant technical complications arise due to the approximate nature of the information we have on ℓ.

IV. Conclusion

We have demonstrated that strict δ-equilibria converge to exact dynamic equilibria in the deterministic queueing model, and as two specific consequences, derived the convergence of ε-equilibria and of equilibria in a specific packet-routing model. But we emphasize that these are merely two consequences, and others can surely be obtained. Convergence of other packet models can certainly be demonstrated, as well as stability with respect to small perturbations of network parameters such as transit times and capacities. We leave detailed investigation of this to future work.

It must be admitted that our bounds are not very effective: we do not explicitly compute how ε depends on δ, but our dependence is certainly (at least) exponential, and very dependent on the specific network being considered. This issue arises even in the setting of continuity. One could hope that the correct dependence is much better, perhaps linear. But it is not clear how this can be approached with the current techniques of this paper. A related issue already mentioned is that our results only apply for δ sufficiently small (where "sufficiently small" depends on the instance). This seems like a potentially much easier issue to resolve.

As already mentioned, our result potentially allows for the transfer of results from the deterministic queueing model to atomic packet-routing models. This would especially be the case if the restriction to sufficiently small δ can be removed. There may be further applications of this, now or in the future, as a better understanding of the deterministic queueing model is obtained.

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