Inverted Pendulum Controller Design Project
EL6243: System Theory and Feedback Control

Eric Mauro

N10683188
elm459@nyu.edu

Abstract

This project covers the design of two single-loop feedback controllers for an inverted pendulum on a cart. First, the system dynamics were used to construct a transfer function plant model. Then, a controller was designed based on the set of all stabilizing controllers and certain properties of the sensitivity function. The second controller was based on minimizing an integral cost based on integral squares of errors and inputs. Both system designs were then simulated with several different reference inputs and disturbances using MATLAB software. Finally, they were compared in terms of general performance. Overall, each performed similarly, but the optimal design was more suited to handle disturbances.

1. System Dynamics

The inverted pendulum on a cart is a commonly studied control systems problem. It consists of a long pendulum mounted to the top of a motorized cart that moves back and forth to keep the pendulum upright. This is a challenging task because the equilibrium point is naturally unstable due to gravity. Without control, the pendulum will simply fall.

The equations of motion that describe the system are derived from the sum of forces and torques. The free-body diagram is shown below where F is the applied force (input), x is the position of the cart, N and P represent the the horizontal and vertical normal forces between the cart and pendulum, mg is the force of gravity, b is the coefficient of friction, and θ is the angle of the pendulum from the lower vertical position [1].

The first relevant equation of motion is found from combining the equations for the sum of forces in the horizontal direction of both the cart and the pendulum:

$$(M+m)\ddot{x} + b\dot{x} + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2 sin\theta = F \tag{1}$$

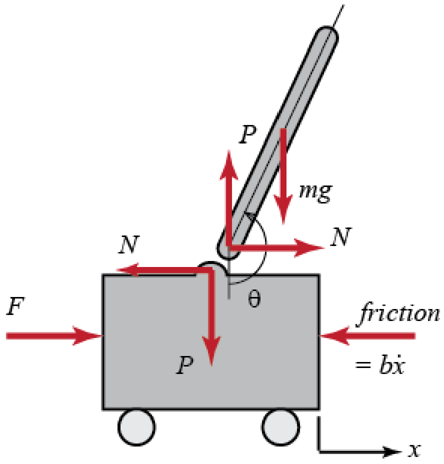

Figure 1: Free-Body Diagram [1]

The second equation is found from summing the forces perpendicular to the pendulum and from the sum of the moments about the centroid of the pendulum:

$$(I + ml^2)\ddot{\theta} + mglsin\theta = -ml\ddot{x}cos\theta \tag{2}$$

However, these equations are nonlinear due to the sines and cosines in the rotational motion. Since the controller should be designed to keep the pendulum upright, the equations can be linearized about the upward equilibrium position, $\theta = \pi$. Denoting ϕ as a small deviation from this angle, the approximations are given as

$$cos\theta = cos(\pi + \phi) \approx -1 \tag{3}$$

$$sin\theta = sin(\pi + \phi) \approx -\phi \tag{4}$$

$$\dot{\theta}^2 = \dot{\phi}^2 \approx 0 \tag{5}$$

and the linearized equations of motion, with F replaced with u to show that it's the input, are given as

$$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = u \tag{6}$$

$$(I + ml^2)\ddot{\phi} - mgl\phi = ml\ddot{x} \tag{7}$$

The transfer function is found by taking the Laplace transform of these two equations (6 and 7), assuming zero initial conditions. This project only focuses on the transfer function for the pendulum position, but the transfer function for the cart position can also be derived. After taking the Laplace transform and rearranging the equation, the transfer function is

$$G_p(s) = \frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{q}s}{s^3 + \frac{b(I+ml^2)}{q}s^2 - \frac{(M+m)mgl}{q}s - \frac{bmgl}{q}} \quad (rad/N) \tag{8}$$

where q is the coefficient divided out to normalize the denominator and is equal to

$$q = [(M+m)(I+ml^2) - (ml)^2] \tag{9}$$

A more thorough derivation of this transfer function can be found at [1] .

2. Controller Design

This project includes two controller designs: the first is a simple controller based on the sensitivity function while the second is based on minimizing the integral error cost of the system. For the purpose of solving the following equations and simulating the system, the parameters were set to the same values given at [1] with the plant model equal to

$$G_p(s) = \frac{4.545s}{s^3 + 0.1818s^2 - 31.18s - 4.455} \tag{10}$$

2.1. Set of all Stabilizing Controllers

The set of all stabilizing controllers is given by

$$G_c = \frac{YL + AM}{XL - BM} \tag{11}$$

where L is any strict Hurwitz polynomial, M is any polynomial such that $XL - BM$ is not zero, and

$$G_pH = \frac{B}{A} \tag{12}$$

For this case, H was set to 1 to model a perfect feedback sensor and simplify equations. B and A were therefore simply the numerator and denominator, respectively, of the plant model derived in Section 1. X and Y were solved through the Bezout equation

$$AX + BY \equiv 1 \tag{13}$$

and found to be

$$X = -0.2245, \qquad Y = 0.0494s^2 + 0.00898s - 1.54 \tag{14}$$

2.2. Controller Design based on Sensitivity Function

The sensitivity function has several properties that can be used to design L and M. It should be equal to zero at low frequencies and equal to one at high frequencies. For high frequencies, it is also desirable to have $1 - S(s) = O(\frac{1}{s^{\mu}})$ such that $\mu \geq 1$, and if $G_p H = O(\frac{1}{s^{v}})$, then $u \geq v$ ensures that the controller is proper.

For this design, μ was simply set equal to v, in this case, 2. L could be any strict Hurwitz polynomial, but based on the plant and μ, it needed to be at least a fifth-order polynomial. Along those lines, $YL + AM$ needed to be a second-order polynomial which implied that the higher order terms needed to cancel out. The coefficients of M were solved from the cancellations, and then the controller was completed.

L was set to

$$L(s) = (s+1)^3(s^2 + s + 1) \tag{15}$$

M was solved to be

$$M(s) = -0.0494s^4 - 0.0198s^3 - 0.346s^2 - 0.566s - 1.04 \tag{16}$$

and Gc was found to be

$$G_c(s) = \frac{8.297s^2 + 28.72s + 3.082}{s^2 + 3.818s - 0.2245} \tag{17}$$

2.3. Controller Design to Minimize Cost

The integral cost based on the inputs and errors is given by

$$J = \int_0^{\infty} (e^2(t) + k\hat{u}^2(t))\mathrm{dt} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} [E(s)E_*(s) + k\hat{U}(s)\hat{U}_*(s)]\mathrm{ds} \tag{18}$$

The controller that minimizes this cost was found through the Wiener-Hopf optimal controller design technique [2]. For further simplification purposes, H was kept equal to 1, $G_u G_{u*}$ was set to 1 as well, and R was set to a step input, ie: $\frac{1}{s}$. The first step in this process was to set the zeros of χ_r equal to the poles and zeros in the right-half plane of B and A. For this system, the zeros of χ_r were 0 and 5.565. The next step was to calculate Φ_1 and Φ_2 where

$$\Phi_1 = \frac{RR_*}{HH_*} + k\frac{G_u G_{u*} RR_*}{G_p G_{p*} HH_*} = \frac{-s^6 + 62.4s^4 - 994.6s^2 + 19.84}{20.66s^4} \tag{19}$$

and

$$\Phi_2 = \frac{(H-1)RR_*}{H} - k\frac{G_u G_{u*} RR_*}{G_p G_{p*} HH_*} = \frac{s^6 - 62.4s^4 + 973.9s^2 - 19.84}{20.66s^4} \tag{20}$$

The next step was to find a rational function Ω with no poles or zeros in $Re(s) \geq 0$ such that

$$\Phi_1 \chi_r \chi_{r*} = \Omega \Omega_* \tag{21}$$

and for this system

$$\Omega = \frac{s^4 + 16.91s^3 + 96.21s^2 + 188.7s + 24.79}{4.545s} \tag{22}$$

Evaluating and using partial fraction expansion on the term

$$\frac{\Phi_2 \chi_r \chi_{r*}}{\Omega_*} = \Gamma_p + \Gamma_r + \Gamma_l \tag{23}$$

results in Γ_p, a polynomial, Γ_r with higher degree in the denominator and poles only in $Re(s) \geq 0$, and Γ_l with higher degree in the denominator and poles only in $Re(s) < 0$. The only one used to find the sensitivity function was Γ_l, and in this case, Γ_l had three poles and was approximately equal to

$$\Gamma_l \approx \frac{4.545s - 25.3}{s^3 + 11.34s^2 + 33.1s + 4.455} \tag{24}$$

The sensitivity function is described by

$$S_o = \frac{f(s) - \Gamma_l}{\Omega} \tag{25}$$

where f(s) must be solved from several properties of the sensitivity function. First, from assumptions made about the integral error cost, $1 - S_o$ needed to have a denominator at least two orders higher than its numerator. To achieve this, $f(s)$ became a third order polynomial where the 3 coefficients of s canceled out the higher order parts of the numerator. The poles of G_pH in $Re(s) \geq 0$ needed to be represented as zeros of S_o, and the zeros of G_pH in $Re(s) \geq 0$ needed to be represented as zeros of $1 - S_o$. The zero at $s = 0$ was already represented in $1 - S_o$ due to the structure of Ω and Γ_l, but the pole at $s \approx 5.57$ was used to find the constant in $f(s)$. $f(s)$ was then solved to be about $0.22s^3 + 3.719s^2 + 21.17s - 270.9$, and S_o was found to be about

$$S_o \approx \frac{s^7 + 28.25s^6 + 321s^5 + 423.7s^4 - 10700s^3 - 40350s^2 - 5371s}{s^7 + 28.25s^6 + 321s^5 + 1844s^4 + 5424s^3 + 6954s^2 + 1661s + 110.4} \tag{26}$$

and

$$1 - S_o \approx \frac{1420s^4 + 16130s^3 + 47300s^2 + 7032s + 110.4}{s^7 + 28.25s^6 + 321s^5 + 1844s^4 + 5424s^3 + 6954s^2 + 1661s + 110.4} \tag{27}$$

Finally, the controller for this sensitivity was found from

$$G_{co} = \frac{1 - S_o}{S_o G_p H} \tag{28}$$

where G_{co} was a transfer function with numerator of degree 13 and denominator of degree 14 (ie, strictly proper).

3. Simulation Results

Using MATLAB and Simulink, both controller designs were simulated and tested with several different reference inputs and disturbances. The following figures show plots for responses of the system using each controller implementation but using the same input for comparison.

3.1. Open-Loop Response

Figure 2 shows the open-loop response of this system. As stated before, this system is naturally unstable because the pendulum will fall without any external control. While the real-world system would be constrained, the simulation is of the linearized system and does not account for limits. In this figure, the output simply moves toward infinity.

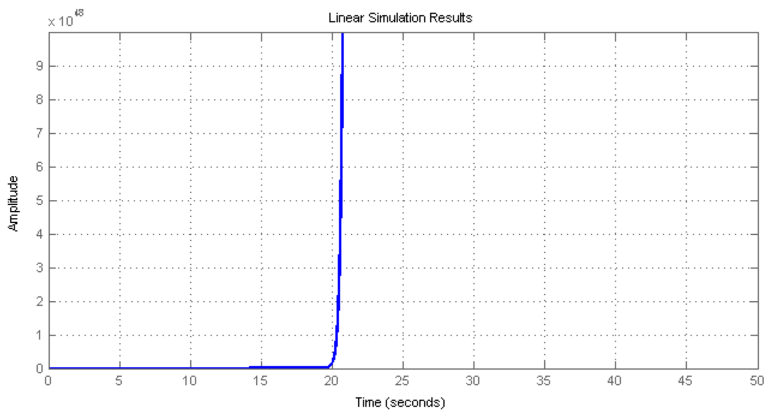

Figure 2: Unstable Open-Loop Response

3.2. Response to Different Reference Inputs

Figure 3 shows the step response of both systems. In the real-world scenario, this would correspond to the cart moving at a constant velocity. The simple controller design has very large overshoot and some ripple in the response, but it stabilizes fairly quickly. Depending on the actual real-world constraints, the overshoot could be a problem, but for the purposes of this project, implementation is not the most important concern. The optimal controller design has a much better looking response with very little overshoot and no ripple. However, the systems settles just above the equilibrium point.

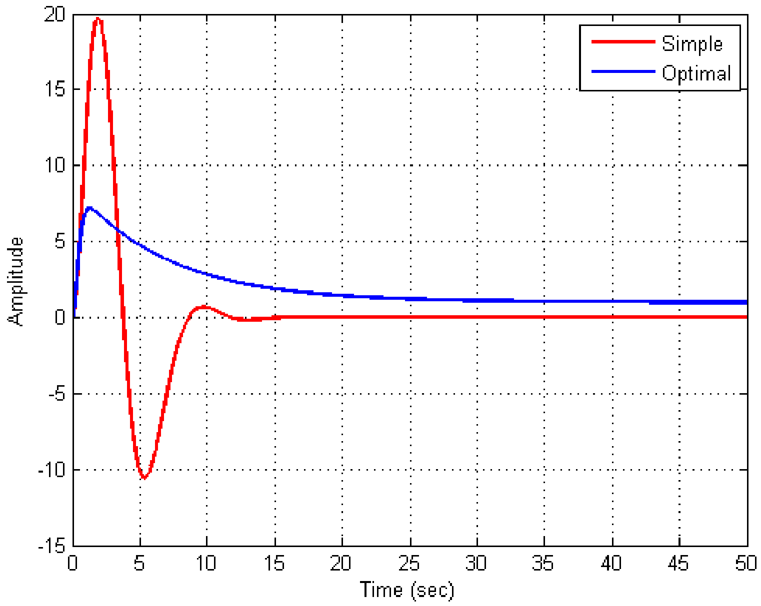

Figure 3: Step Response

Figure 4 shows the impulse response of the system. This response gives a better idea of how the system will react to quick pushes on the cart or pendulum which would be a more common input to the system if the cart is designed to become stationary as well. Like the step response, the simple design has some overshoot and ripple, but it settles fairly quickly. The optimal design again has little overshoot and almost no ripple, but settles at about the same rate as the simple design.

Figure 5 shows the response to a short square pulse. This would be the kind of response expected from a quick push and stop to the cart. It is a combination of two short steps, and in fact, looks like two step responses back to back. The simple design has three large peaks in its motion as it moves back and forth, but the optimal design is barely affected by the cart stopping. Both seem to settle to the equilibrium point at the same time.

Figure 6 shows the response to a low amplitude (0.2), low frequency ($\frac{\pi}{10}$ rad/s) sine wave. In this case, the pendulum has trouble settling while the cart is moving back and forth. The simple controller overcompensates for the oscillatory motion and outputs another sine wave with ten times

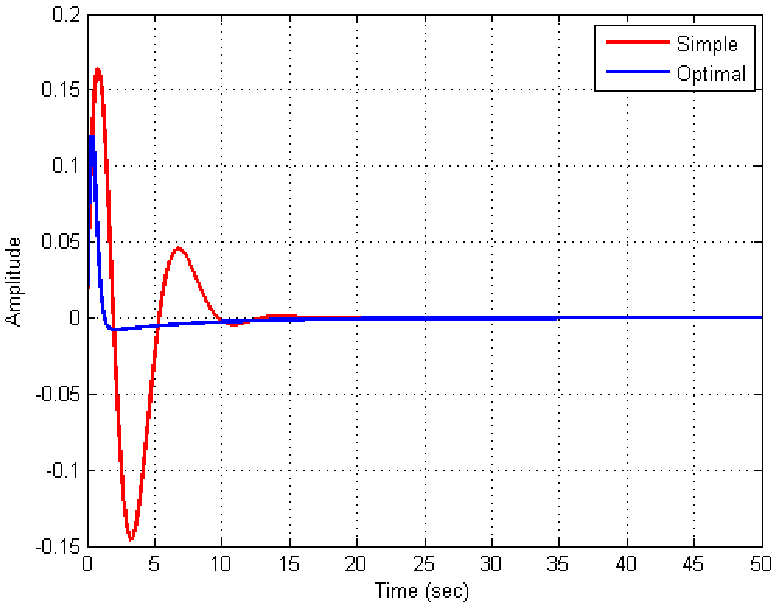

Figure 4: Impulse Response

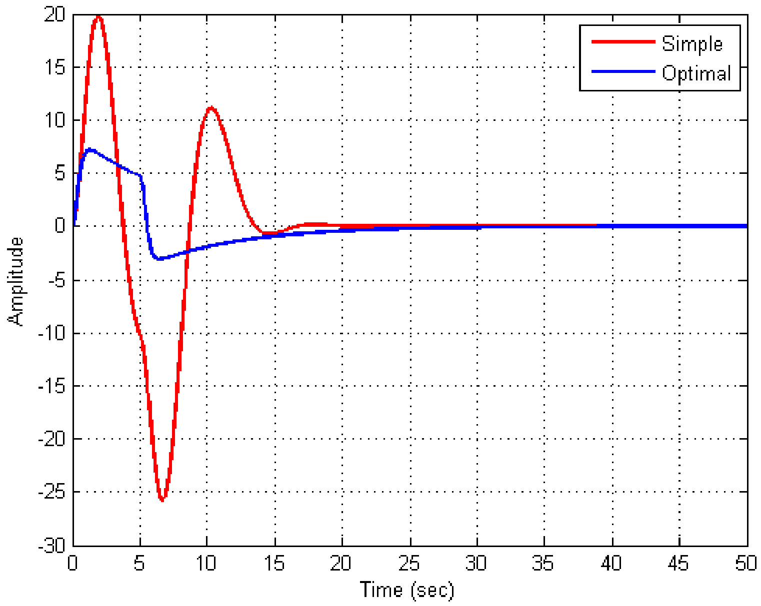

Figure 5: Response to Square Pulse

the amplitude and twice the frequency. While the optimal controller has better performance, it still oscillates around the center point and fails to stabilize the system

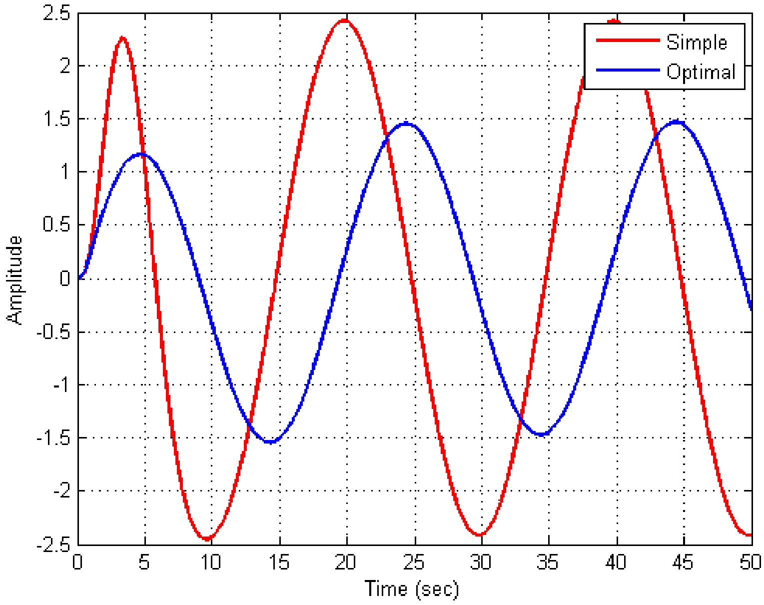

Figure 6: Response to Low Frequency Sine Wave

Figure 7 shows the response to a high frequency ($10\pi \quad rad/s$) sine wave with amplitude equal to 1. In contrast with the previous sine wave, both responses look very similar to the step response of Figure 3 but with very small oscillations about the curve and with a much smaller overall amplitude. Both also have a smaller amplitude response compared to the sine wave input.

Finally, the sensitivity function was plotted for this controller and is shown in Figure 8. The sensitivity function represents how well the controller adapts to variations in the process dynamics. The simple controller design has a peak of about 30 dB around 1 rad/s with minimums settling to 0 dB while the optimal design has a peak just under 20 dB around the same point but with minimums well below 0 dB. Overall, the optimal design should handle disturbances much better than the simple design.

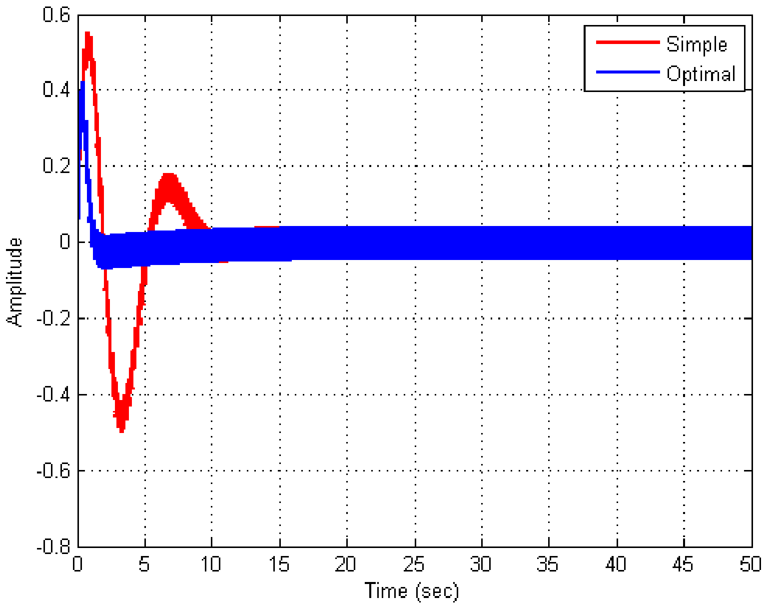

Figure 7: Response to High Frequency Sine Wave

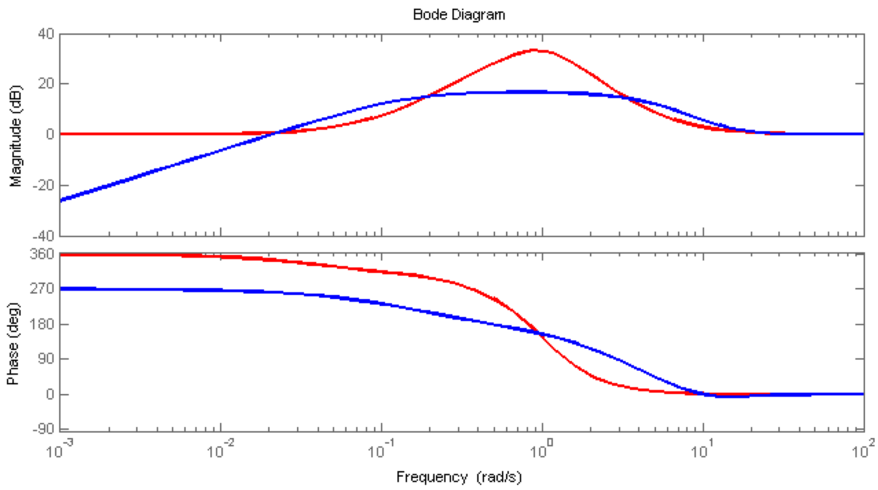

Figure 8: Sensitivity Function vs. Frequency

3.3. System Response with Different Disturbances

Each system was simulated with different disturbances at the output with a step input. Figure 9 shows the step response with a short square pulse disturbance. The simple design seems to almost repeat the same response from the original step input while the optimal design springs back very quickly. It barely affects the optimal design in comparison.

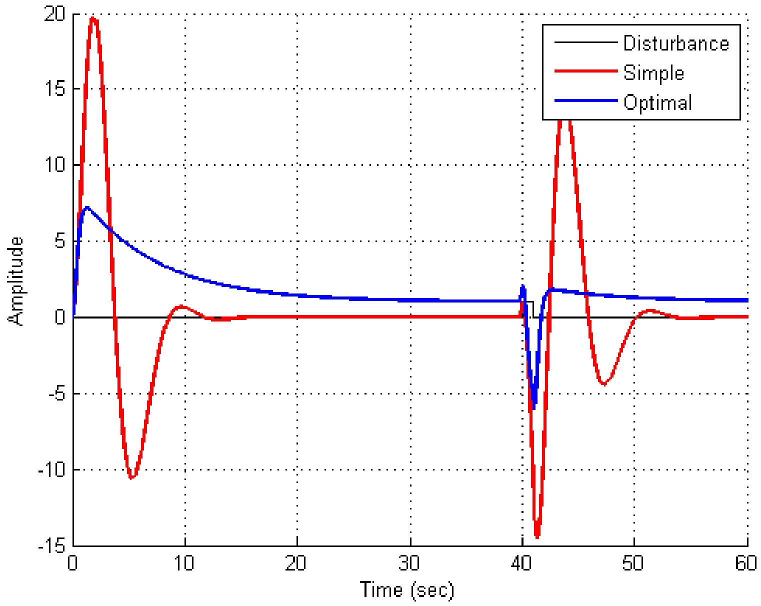

Figure 9: Step Response with Square Pulse Disturbance at t=40

Figure 10 shows the step response with a low frequency sine wave disturbance at the output. Like the previous low frequency sine wave responses, both controllers oscillate around the equilibrium point. However, the optimal design oscillates with less than half the amplitude of the simple design. While it does not completely settle, the optimal design still handles the disturbance much better.

Figure 11 shows the step response to a high frequency sine wave disturbance. Both systems are barely affected by the disturbance. They both show very small oscillations about their normal step responses with about the same variance. It is not apparent whether or not one systems performs better than the other in this case, but both handle the disturbance very well.

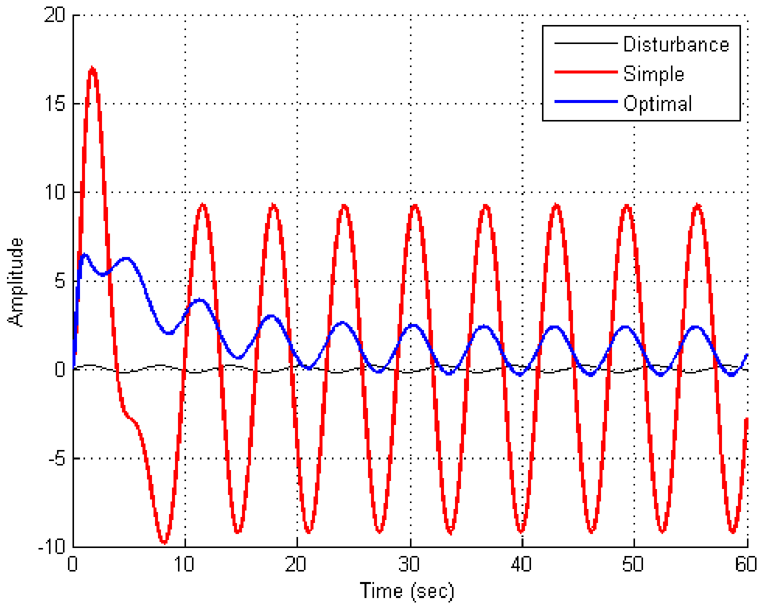

Figure 10: Step Response with Low Frequency Sine Wave Disturbance

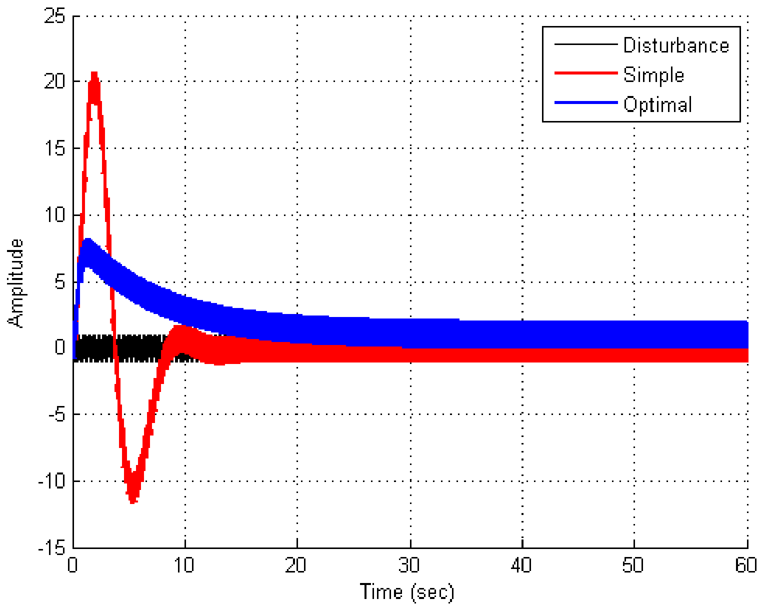

Figure 11: Step Response to High Frequency Sine Wave Disturbance

4. Conclusions

Overall, both systems have similar responses to different reference inputs. However, with the way the system was designed with linearization, the bigger overshoot and ripple in the first design could be a problem. The optimal design seems to perform better, and it also seems to handle disturbances much better. While the simulations are a good way to get an idea of how the system will perform with different controller design, the best way to test the designs is with a real system. A better simulation can also be achieved through using the linearized controller designs with the original nonlinear model.

References

[1] "Inverted Pendulum: System Modeling." Control Tutorials for MATLAB and Simulink. University of Michigan, Web. 7 Dec. 2016. http://ctms.engin.umich.edu/CTMS/index.php?example=InvertedPendulumion=SystemModeling.

[2] D. Youla, J. Bongiorno and H. Jabr, "Modern Wiener–Hopf design of optimal controllers Part I: The single-input-output case," in IEEE Transactions on Automatic Control, vol. 21, no. 1, pp. 3-13, Feb 1976. doi: 10.1109/TAC.1976.1101139