Modeling with Difference Equations

Suppose we have variables

$$\vec{y} = (y_1, y_2, \ldots, , y_n)$$

and there is a natural time step Δt to the problem so that it makes sense to model the values of these variables only at the discrete times

$$0, \Delta t, 2\Delta t, \ldots.$$

Then the model might take the form of a *difference equation*.

One-step Difference Equation:

- $\vec{y}_n$ contains the values of the state variables at time $n\Delta t$.
- $\vec{y}_n$ only depends on n and $\vec{y}_{n-1}$.

The difference equation has the form:

$$\begin{aligned} \vec{y}_{n+1} &= \vec{f}(n, \vec{y}_n) \\ \vec{y}_0 &= \text{initial data} \end{aligned}$$

You can *iterate* to obtain the values of the state variables at any point in time:

$$
\begin{aligned}
&\text{Start with } \vec{y}_0 \\
&\vec{y}_1 = \vec{f}(0, \vec{y}_0) \\
&\vec{y}_2 = \vec{f}(1, \vec{y}_1) \\
&\vdots
\end{aligned}
$$

Example: SIR model.

$$
\begin{aligned}
s_{n+1} &= s_n - \alpha s_n i_n \\
i_{n+1} &= i_n + \alpha s_n - \beta i_n \\
r_{n+1} &= r_n + \beta i_n
\end{aligned}
$$

So

$$
\vec{y}_n = \begin{pmatrix} s_n \\ i_n \\ r_n \end{pmatrix} \qquad \text{and} \qquad \vec{f}\left(n, \begin{pmatrix} s \\ i \\ r \end{pmatrix} \right) = \begin{pmatrix} s - \alpha s i \\ i + \alpha s i - \beta i \\ r + \beta i \end{pmatrix}
$$

m-Step Difference Equation:

More generally, $\vec{y}_n$ may depend not only on n and $\vec{y}_{n-1}$ but on the values of the state variables up to some number m of states in the past. In this case we have:

$$\vec{y}_{n+m} = \vec{f}(n, \vec{y}_n, \vec{y}_{n+1}, \ldots, \vec{y}_{n+m-1})$$

This is called an *m-step difference equation.* In this case, the initial conditions are the first m values of $\vec{y}$:

$$\vec{y}_0, \vec{y}_1, \ldots, \vec{y}_{m-1}.$$

You need all of these in order to start iterating.

An m-step difference equation can be turned into a 1-step difference equation by introducing more state variables.

Example: On homework 2 we had the example of annual plants that produced seeds some of which sprouted the next year but some of which only sprouted the year after that. We got the model:

$$p_n = (\alpha p_{n-1} + \beta(1-\alpha)\sigma p_{n-2})\sigma\gamma$$

with initial conditions p_0 and p_1.

Let

$$q_n = p_{n-1}$$

Then

$$\begin{aligned} p_n &= (\alpha p_{n-1} + \beta(1-\alpha) q_{n-1})\sigma\gamma \\ q_n &= p_{n-1} \end{aligned}$$

with initial conditions

$$\begin{pmatrix} p_1 \\ q_1 = p_0 \end{pmatrix}$$

Autonomous versus Non-autonomous

If $\vec{f}$ doesn't depend explicitly on n then the system is *autonomous*. In this case the state variables only change because of their interactions with each other.

If $\vec{f}$ depends explicitly on n then there is some external factor/force/variable that you are not including in your model, that causes the state variables to change.

Example: Consider the simple 1-step annual plant model from homework 2. There we had

$$p_{n+1} = \sigma \gamma p_n$$

This is autonomous; the population at time $(n+1)\Delta t$ depends *only* on the population at time n and not explicitly on time.

Now, suppose there's a fire every 10 years that kills 90% of the seeds.

The population at time $n+1$ now depends not only on the population the previous year but also on whether n is a multiple of 10.

$$p_{n+1} = \sigma(\gamma p_n - 0.9\delta_n \gamma p_n)$$

where

$$\delta_n = \begin{cases} 0 & 10 \text{ divides } n \\ 1 & 10 \text{ does not divide } n \end{cases}$$

Notice: the fire is an external factor; we are not attempting to model all the atmospheric variables and environmental conditions that cause the fire to occur.

Remember: Your system should be non-autonomous only if there is something *external* to the system that causes the state variables to change.

- Most models are autonomous.
- Non-autonomous models can be made autonomous by including time as another state variable.

Example: The annual plant with fires every 10 years above. Let $q_n = n$.

Then the system becomes

$$\begin{aligned} p_{n+1} &= \sigma(\gamma p_n - 0.9\delta(q_n)\gamma p_n) \\ q_{n+1} &= q_n + 1 \end{aligned}$$

where

$$\delta(q) = \begin{cases} 0 & 10 \text{ divides } q \\ 1 & 10 \text{ does not divide } q \end{cases}$$

The Dynamics

Consider an autonomous system

$$\vec{y}_{n+1} = \vec{f}(\vec{y}_n).$$

The state variable $\vec{y}_n$ jumps around to different points $\mathbb{R}^n$ as time evolves. In some cases it may settle down to some *limiting value* $\vec{y}^*$. If this is the case, it's not hard to see that $\vec{y}^*$ must be an *equilibrium* of the system, i.e.

$$\vec{f}(\vec{y}^*) = \vec{y}^*$$

A *stable equilibrium* is an equilibrium $\vec{y}^*$ where, if you start the system close to that equilibrium, then the iterates remain close to that equilibrium for all time (or maybe even converge to it).

An *unstable equilibrium* is an equilibrium where, if you start the system close to that equilibrium, then the iterates get far away from that equilibrium.

In simulations we will only ever see stable equilibria.

Cobweb Diagrams for a Single State Variable

Consider a difference equation with a single state variable:

$$x_{n+1} = f(x_n)$$
$$x_0 \text{ initial conditions}$$

There is a single state variable, so the iterates are points on the real line $\mathbb{R}$.

To understand the dynamics, we can draw a *cobweb diagram* of the system. For this you draw the graph of f along with the graph of $y = x$ and you can see how the iterates evolve starting from different initial conditions.

Example:

$$x_{n+1} = -0.3x_n + 2$$

There is a single equilibrium at $x = 2/1.3$ which is (globally) attracting; from any starting point, the iterates converge to this equilibrium.

Example:

$$x_{n+1} = 1.5x_n - 2$$

There is a single equilibrium at $x = 4$ which is unstable; the iterates either go off to $+\infty$ or $-\infty$ depending on whether they start greater or less than this equilibrium.

Even first-order difference equations with a single state variable can exhibit chaotic behavior as the following example illustrates.

Example: Populations evolving in an environment with limited resources are often modeled using logistic growth. If we nondimensionalize the system so that the maximum population the environment can support is 1 unit then this takes the form of the differential equation:

$$x' = kx(1-x)$$

We'll take a closer look at this when we study differential equations. The solutions are very well-behaved; if x starts greater than 1 then the solution decays exponentially to 1 and if x starts greater than 1 then it increases to 1.

If you discretize time (use Euler's method to solve this equation) this becomes

$$\begin{aligned}
\frac{x_{n+1} - x_n}{\Delta t} &= k x_n (1 - x_n) \\
x_{n+1} &= x_n + k\Delta t x_n - k\Delta t x_n^2 \\
x_{n+1} &= k\Delta t x_n \left[\left(1 + \frac{1}{k\Delta t} \right) - x_n \right]
\end{aligned}$$

If we nondimensionalize again we can write this in the form

$$x_{n+1} = \alpha x_n (1 - x_n)$$

This is called discrete time logistic growth.

If $x_n > 1$, then this model doesn't make sense for a population, since $x_{n+1} < 0$.

We have $f(x) = \alpha x(1-x)$.

The graph of f is an upside down quadratic with zeros at $x = 0$ and $x = 1$ and a maximum value of $\alpha/4$ (at the vertex where $x = 0.5$). So if $0 < \alpha < 4$ then populations that start between 0 and 1 will remain there forever.

The fixed points are:

$$\begin{aligned} x &= \alpha x(1-x) \\ \alpha x^2 - (\alpha - 1)x &= 0 \\ [(\alpha x - (\alpha - 1)]\, x &= 0 \\ x &= 0, \frac{\alpha - 1}{\alpha} \end{aligned}$$

Here are cobweb diagrams when $\alpha < 1$, $1 < \alpha < 2$ and $\alpha > 2$.

In the first case, there is only one fixed point at the origin (the other occurs when $x < 0$). This is globally stable; the population rapidly decays to nothing wherever it starts.

In the second case, the origin is unstable and the other fixed point is stable. All orbits (except those with $x_0 = 0$ or $x_0 = 1$) converge to the non-zero fixed point.

In the third case, both fixed points are unstable and the orbits are chaotic.

Example: Because of some of the weirdnesses of discrete time logistic growth (negative populations when the population is too large and chaotic behavior when the parameter is large) people sometimes use the Beverton-Holt equation to model populations in discrete time when there are limited resources. This has the form

$$x_{n+1} = \frac{\mu x_n}{1 + kx_n}$$

The graph of $f(x) = \mu x/(1 + kx)$ is increasing and concave down with a slope of μ at the origin and a horizontal asymptote at $K = (\mu - 1)/k$. When $\mu < 1$ the population decays to 0 and when $\mu > 1$, all positive populations converge monotonically to the carrying capacity K.

Linear Difference Equations

A first-order autonomous difference equation is *linear* (and homogeneous) if it has the form

$$\vec{y}_{n+1} = A\vec{y}_n$$

for some matrix A.

If there is only one state variable we have

$$x_{n+1} = \lambda x_n$$

By iterating we can see easily that

$$x_n = \lambda^n x_0$$

If $|\lambda| > 1$ then these grow exponentially in size. If $|\lambda| < 1$ then they decay exponentially to 0.

In higher dimensions, suppose λ is an eigenvalue of the matrix A with eigenvector $\vec{y^*}$. Then, if $\vec{y_0} = \vec{y^*}$,

$$\vec{y_n} = \lambda^n \vec{y^*}$$

In other words the state variable just grows or decays (and possibly rotates if $\lambda \in \mathbb{C}$) in the direction of $\vec{y^*}$ (depending on whether $|\lambda| > 1$ or $|\lambda| < 1$.

Suppose $\vec{y}_1^*$, $\vec{y}_2^*$, …, $\vec{y}_n^*$ is a basis of eigenvectors with respective eigenvalues λ_1, λ_2, …, λ_n where $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$. Then the initial condition can be written

$$\vec{y}_0 = \sum_{i=1}^{n} \alpha_i \vec{y}_i^*$$

so

$$\vec{y}_n = \sum_{i=1}^{n} \alpha_i \lambda_i^n \vec{y}_i^*$$

When n is large $|\lambda_1^n| >> |\lambda_i^n|$ for all $i > 1$, so $\vec{y}_n$ behaves like $\alpha_1 \lambda_1^n \vec{y}_1^*$. In other words, *linear difference equations exhibit exponential growth/decay at the rate given by the magnitude of the largest eigenvalue.* (It's not hard to see that this is also true even when A does not have a full basis of eigenvectors.)

Non-linear Difference Equations

The orbits of non-linear difference equations with many state variables can be hard to study and very complex.

However, we can understand the dynamics near an equilibrium by linearizing the system.

Suppose $\vec{y}^*$ is an equilibrium of the system and J is the Jacobian of $\vec{f}$ evaluated at $\vec{y}^*$, namely,

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \dots & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \dots & \frac{\partial f_2}{\partial y_n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \dots & \frac{\partial f_n}{\partial y_n} \end{pmatrix}$$

Then, if $\vec{y}_0$ is close to $\vec{y}^*$,

$$\begin{aligned}\vec{y}_1 = \vec{f}(\vec{y}_0) &\approx \vec{f}(\vec{y}^*) + J(\vec{y}_0 - \vec{y}^*) \\ &= \vec{y}^* + J(\vec{y}_0 - \vec{y}^*)\end{aligned}$$

Thus,

$$\vec{y}_1 - \vec{y}^* \approx J(\vec{y}_0 - \vec{y}^*)$$

Iterating, we get

$$\vec{y}_n - \vec{y}^* \approx J^n(\vec{y}_n - \vec{y}^*)$$

as long as $\vec{y}_n$ remains close to $\vec{y}^*$.

In other words, near $\vec{y}^*$, the system behaves like a linear system with matrix J. In particular, if the largest eigenvalue of J has magnitude greater than 1 then the equilibrium is unstable and if it has magnitude less than 1 then it is stable.

Example: We can refine the discrete logistic growth equation for the growth of a population by introducing a class of youth which don't reproduce in one time-step but then after that can reproduce.

Let

- x_n be the number of youth at time n and
- y_n be the number of adults at time n.

- Some proportion of the youth die and the rest grow into adults in one time-step.
- A fixed proportion of the adults reproduce each time-step; their new-borns are youth at the next time-step.
- In addition, some of the adults die each time-step. To account for the limited resources of the environment and mimic logistic growth, we have a density-dependent death rate; the proportion that die is proportional to the size of the population.

In other words we have

$$x_{n+1} = ay_n$$
$$y_{n+1} = bx_n - cy_n^2$$

where $b < 1$.

The fixed points are where

$$x = ay$$

$$y = bx - cy^2$$

Substituting the expression of x into the second equation we have

$$y = aby - cy^2$$

$$\text{or} \qquad 0 = [(ab-1) - cy]y$$

$$\text{so} \qquad y = 0 \text{ or } \frac{ab-1}{c}$$

Thus, the fixed points are: $(0,0)$ and $\left(\frac{a}{c}(ab-1), \frac{1}{c}(ab-1)\right)$.

Of course, the latter is only physcially meaningful when $ab > 1$.

To examine the stability of the fixed points we find the Jacobian of the map.

$$J = \begin{pmatrix} 0 & a \\ b & -2cy \end{pmatrix}$$

Evaluating at $(0, 0)$ we get

$$J(0,0) = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

with characteristic equation

$$\lambda^2 - ab = 0$$

so $\lambda = \pm\sqrt{ab}$. The origin is stable when $ab < 1$ and unstable when $ab > 1$.

Evaluating at the other fixed point we get

$$J = \begin{pmatrix} 0 & a \\ b & -2(ab-1) \end{pmatrix}$$

with characteristic equation

$$\begin{aligned} (-\lambda)[-2(ab-1)-\lambda] - ab &= 0 \\ \lambda^2 + 2(ab-1)\lambda - ab &= 0 \\ \lambda = -(ab-1) \pm \sqrt{(ab-1)^2 + ab} \end{aligned}$$

Recall that we are in the case where $ab > 1$, so the eigenvalue with the largest magnitude is the negative one, so

$$|\lambda_1| = (ab-1) + \sqrt{(ab-1)^2 + ab}$$

We have $|\lambda| > 1$ if and only if

$$ab - 1 + \sqrt{ab - 1)^2 + ab} > 1$$

$$\sqrt{ab - 1} > 2 - ab$$

This is definitely satisfied if $ab \geq 2$. If $ab < 2$ then it's satisfied provided

$$(ab - 1)^2 + ab > (2 - ab)^2$$

$$\text{or} \qquad a^2b^2 - ab + 1 > 4 - 4ab + a^2b^2$$

$$\text{or} \qquad ab > 1.$$

In other words, it is always stable when physically relevant.