Math 467. Homework #1

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page 77: 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, 5.1 p22: 2.1, 2.2, 2.3, 2.4

5.3 $f(x) = (a^T x)(b^T x)$ where a, b, x are n-dimensional vectors

a. Find $\nabla f(x)$: product rule $\rightarrow \nabla f(x) = \underbrace{a \cdot (b^T x)} + \underbrace{(a^T x)(b)}$

b. Hessian: $F(x) = D^2 f(x) \rightarrow a \cdot b^T + a^T b$

5.4 $f: \mathbb{R}^2 \to \mathbb{R}$ $g: \mathbb{R} \to \mathbb{R}^2$ $f(x) = [x_1^2/6, \; x_2^2/4]$

$g(t) = [3t+5, \; 2t-6]^T$

$F: \mathbb{R} \to \mathbb{R}$ $F(t) = f(g(t))$

Find $\frac{dF(t)}{dt}$ $F'(t) = f'(g(t)) \cdot g'(t)$, $f'(x) = [x_1/3, \; x_2/2]$

$g'(t) = [3, 2]^T$

$F'(t) = \left[\frac{3t+5}{3}, \frac{2t-6}{2}\right] \cdot [3, 2]^T$

$= 5t - 1$

5.5 $f(x) = 1/2 [x_1 \cdot x_2]$ $g(s,t) = [4s+3t, \; 2s+t]^T$

$\frac{\partial f(g(s,t))}{\partial s} \Rightarrow f(s,t) = 1/2 [(4s+3t)(2s+t)] = \frac{1}{2}(8s^2 + 10st + 3t^2)$

$\frac{\partial f(s,t)}{\partial s} = \frac{1}{2}(16s + 10t) = 8s + 5t$

$\frac{\partial f(g(s,t))}{\partial t} \Rightarrow f(s,t) = 1/2 (8s^2 + 10st + 3t^2)$

$\frac{\partial f(s,t)}{\partial t} = 1/2 (10s + 6t) = 5s + 3t$

5.6 $x(t) = [e^t + t^3, t^2, t+1]^T$, $t \in \mathbb{R}$ $f(x) = x_1^3 x_2 x_3^2 + x_1 x_2 + x_3$

$x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$

Find: $\frac{df(x(t))}{dt}$: $f'(x(t)) \cdot x'(t)$: $f(x(t)) = (e^t + t^3)^3 (t^2)(t+1)^2 + (e^t + t^3)(t^2) + (t+1)$

$= t^9 + 3e^t t^6 + 3e^{2t} t^3 + e^{3t} + e^t t^2 + t^5 + t + 1$

$f'(x(t)) = 9t^8 + 3e^t t^6 + 18 e^t t^5 + 6e^{2t} t^3 + 9 e^{2t} t^2 + 3e^{3t} + e^t t^2 + 2e^t t + 5t^4 + 1$

$f'(x) = [3x_1^2 x_2 x_3^2 + x_2, \; x_1^3 x_3^2 + x_1, \; x_1^3 x_2 2x_3 + 1]$ $x'(t) = [e^t + 3t^2, \; 2t, \; 1]$

$f'(x(t)) \cdot x'(t) = 12t(e^t + 3t^2)^3 + 2te^t + 6t^2 + 2t + 1$

5.7 $f(x) = o(g(x))$, show that for any $\varepsilon > 0$, $\exists \; \delta > 0$ s.t. if $\|x\| < \delta$, $\|f(x)\| < \varepsilon |g(x)|$

$\lim_{x \to 0} \frac{\|f(x)\|}{g(x)} = 0$ if $\|x\| < \delta$, then

$\frac{\|f(x)\|}{g(x)} < \varepsilon \Rightarrow \|f(x)\| \leq \varepsilon g(x)$

5.8 $f: \mathbb{R}^n \to \mathbb{R}$ $g: \mathbb{R}^n \to \mathbb{R}$ $f(x) = -g(x) + o(g(x))$, $g(x) > 0 \quad \forall \; x \neq 0$ $f(x) < 0$

$\rightarrow |o(g(x))| < g(x)/2$, $f(x) \leq -g(x) + |o(g(x))| < -g(x) + g(x)/2 = -1/2 g(x) < 0$