BOOTSTRAP FOR PANEL DATA*

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Preliminary

Abstract

This paper considers bootstrap methods for panel data. Theoretical results are given for the sample mean. It is shown that simple resampling methods (i.i.d., individual only or temporal only) are not always valid in simple cases of interest, while a double resampling that combines resampling in both individual and temporal dimensions is valid. This approach also permits to avoid multiples asymptotic theories that may arise in large panel models. In particular, it is shown that this resampling method provides valid inference in the one-way and two-way error component models and in the factor models. Simulations confirm these theoretical results.

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1 Introduction

The true probability distribution of a test statistic is rarely known. Generally, its asymptotic law is used as approximation of the true law. If the sample size is not large enough, the asymptotic behaviour of the statistics could lead to a poor approximation of the true one. Using bootstrap methods, under some regularity conditions, it is possible to obtain a more accurate approximation of the distribution of the test statistic. Original bootstrap procedure has been proposed by Efron (1979) for statistical analysis of independent and identically distributed (i.i.d.) observations. It is a powerful tool for approximating the distribution of complicated statistics based on i.i.d. data. There is an extensive literature for the case of i.i.d. observations. Bickel & Freedman (1981) established some asymptotic properties for bootstrap. Freedman (1981) analyzed the use of bootstrap for least squares estimator in linear regression models.

In practice, observations are not i.i.d. Since Efron (1979) there is an extensive research to extend bootstrap to statistical analysis of non i.i.d. data. Wild bootstrap is developed in Liu (1988) following suggestions in Wu (1986) and Beran (1986) for independent but not identically distributed data. Several bootstrap procedures have been proposed for time series. The two most popular approaches are sieve bootstrap and block bootstrap. Sieve bootstrap attempts to model the dependence using a parametric model. The idea behind it is to postulate a parametric form for the data generating process, to estimate the parameters and to transform the model in order to have i.i.d. elements to resample. The weakness of this approach is that results are sensitive to model misspecification and the attractive nonparametric feature of bootstrap is lost. On the other hand, block bootstrap resamples blocks of consecutive observations. In this case, the user is not obliged to specify a particular parametric model. For an overview of bootstrap methods for dependent data, see Lahiri (2003). Application of bootstrap methods to several indices data is an embryonic research field. The expression *"several indices data"* regroups : clustered data, multilevel data, and panel data.

The term *"panel data"* refers to the pooling of observations on a cross-section of statistical units over several periods. Because of their two dimensions (individual -or cross-sectional- and temporal), panel data have the important advantage to allow to control for unobservable heterogeneity, which is a systematic difference across individuals or periods. For an overview about panel data models, see for example Baltagi (1995) or Hsiao (2003). There is an abounding literature about asymptotic theory for panel data models. Some recent developments treat of large panels, when temporal and cross-sectional dimensions are both important. Paradoxically, literature about bootstrap for panel data is rather restricted. In general, simulation results suggest that some resampling methods work well in practice but theoretical results are rather limited or exposed with strong assumptions. As references of bootstrap methods for panel models, it can be quoted Bellman et al. (1989), Andersson & Karlsson (2001), Carvajal (2000), Kapetanios (2004), Focarelli (2005), Everaert & Pozzi (2007) and Herwartz (2006; 2007). In error component models, Bellman et al. (1989) uses bootstrap to correct bias after feasible generalised least squares. Andersson & Karlsson (2001) presents bootstrap resampling methods for one-way error component model. For two-way error component models, Carvajal (2000) evaluates by simulations, different bootstrap resampling methods. Kapetanios (2004) presents theoretical results when cross-sectional dimension goes to infinity, under the assumption that cross-sectional vectors of regressors and errors terms are i.i.d.. This assumption does not permit time varying regressors or temporal aggregate shocks in errors terms. Focarelli (2005) and Everaert & Pozzi (2007) uses bootstrap to reduce bias in dynamic panel models with fixed effects when T is fixed and N goes to infinity, bias quoted by Nickell (1981). Herwartz (2006; 2007) deliver a bootstrap version to Breusch-Pagan test in panel data models under cross-sectional dependence.

This paper aims to expose some theoretical results on various bootstrap methods for panel data. Theoretical results and simulation exercises concern the sample mean. The paper is organized as follows. In the second section, different panel data models are presented. Section 3 presents five bootstrap resampling methods for panel data. The fourth section presents theoretical results, analyzing validity of each resampling method. In section 5, simulation results are presented and confirm theoretical results. The sixth section concludes.

2 Panel Data Models

It is practical to represent panel data as a rectangle. By convention, in this document, rows correspond to the individuals and columns represent time periods. A panel dataset with N individuals and T time periods is represented by a matrix Y of N rows and T columns. Y contains thus NT elements. *y_{it} is $i's$ observation at period t.*

$$\underset{(N,T)}{Y} = \begin{pmatrix} y_{11} & y_{12} & \cdots & \cdots & y_{1T} \\ y_{21} & y_{22} & \cdots & \cdots & y_{2T} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & .. & \cdots \\ y_{N1} & y_{N2} & .. & \cdots & y_{NT} \end{pmatrix}$$

Consider a panel model without regressor.

$$y_{it} = \theta + \nu_{it} \qquad (2.1)$$

θ is an unknown parameter and ν_{it} a random variable. Inference is about the parameter θ and its estimator is the sample mean. It is usual in a first step to establish the validity of bootstrap for the sample mean, before investigating more complicated statistics. Assumptions about ν_{it} define different panel data models. The specifications commonly used can be summarized by *(2.2)* under assumptions below.

$$\nu_{it} = \mu_i + f_t + \lambda_i F_t + \varepsilon_{it} \qquad (2.2)$$

Assumptions

A1 : $(\mu_1, \mu_2, ..., \mu_N) \sim i.i.d.\left(0, \sigma_\mu^2\right),\ \sigma_\mu^2 \in (0, \infty)$

A2 : $\{f_t\}$ *is a stationary and strong α-mixing process*[1] *with* $E(f_t) = 0,\ \exists\ \delta \in (0,\infty) :$ $E|f_t|^{2+\delta} < \infty, \sum_{j=1}^{\infty} \alpha(j)^{\delta/(2+\delta)} < \infty$ *, and* $V_f^\infty = \sum_{h=-\infty}^{\infty} Cov(f_t, f_{t+h}) \in (0,\infty)$

A3 : $(\lambda_1, \lambda_2, ..., \lambda_N) \sim i.i.d.\left(0, \sigma_\lambda^2\right),\ \sigma_\lambda^2 \in (0, \infty)$

A4 : $\{\varepsilon_{it}\}_{i=1...N,\ t=1..T} \sim i.i.d.\left(0, \sigma_\varepsilon^2\right),\ \sigma_\varepsilon^2 \in (0, \infty)$

A5 : $\{F_t\}$ *is a stationary and strong α-mixing process with* $E(F_t) = 0,\ \exists\ \delta \in (0,\infty) :$ $E|F_t|^{2+\delta} < \infty, \sum_{j=1}^{\infty} \alpha(j)^{\delta/(2+\delta)} < \infty$ *, and* $V_F^\infty = \sum_{h=-\infty}^{\infty} Cov(F_t, F_{t+h}) \in (0,\infty)$

A6 : The five series are independent.

These assumptions can seem strong. They are made in order to have strong convergence and to simplify demonstrations. Considering special cases with different combinations of processes in

[1]See Appendix 7 for definition of an α-mixing process.

(2.2), gives the following panel data models : i.i.d. panel model, one-way error component model, two-way error component model and factor model.

Iid panel model

$$y_{it} = \theta + \varepsilon_{it} \tag{2.3}$$

This specification is the most simple panel model. Even if observations have a panel structure, they exhibit no dependence in cross-sectional or temporal dimension.

One-way ECM

$$\begin{aligned} y_{it} &= \theta + \mu_i + \varepsilon_{it} && (2.4) \\ y_{it} &= \theta + f_t + \varepsilon_{it} && (2.5) \end{aligned}$$

Two specifications are considered. The term, one-way error component model (ECM), comes from the structure of error terms : only one kind of heterogeneity, that is systematic differences across cross-sectional units or time periods, is taken into account. The specification *2.4 (resp. 2.5)* allows to control unobservable individual (resp. temporal) heterogeneity. The specification *(2.4)* is called *individual one-way ECM*, *(2.5)* is *temporal one-way ECM*. It is important to emphasize that here, unobservable heterogeneity is a random variable, not a parameter to be estimated. The alternative is to use *fixed effects model* in which heterogeneity must be estimated.

Two-way ECM

$$y_{it} = \theta + \mu_i + f_t + \varepsilon_{it} \tag{2.6}$$

Two-way error component model allows to control for individual and temporal heterogeneity, hence the term *two-way*. Like in one-way ECM, individual and temporal heterogeneities are random variables. Classical papers on error component models include Balestra & Nerlove (1966), Fuller & Battese (1974) and Mundlak (1978).

Factor Model

$$\begin{aligned} y_{it} &= \theta + \mu_i + \lambda_i F_t + \varepsilon_{it} && (2.7) \\ y_{it} &= \theta + \lambda_i F_t && (2.8) \end{aligned}$$

Two specifications are considered. In *(2.7),* the difference with one-way ECM, is the term $\lambda_i F_t$. The product allows the common factor F_t to have differential effects on cross-section units. This specification is used by Bai & Ng (2004), Moon & Perron (2004) and Phillips & Sul (2003). It is a way to introduce dependence among cross-sectional units. An other way is to use spatial model in which, the structure of the dependence can be related to geographic, economic or social distance (see Anselin (1988))[2]. The second specification, *(2.8)* allows to anylize properly the factor effect.

[2] Bootstrap methods studied in this paper do not take into account spatial dependence. Reader insterested by resampling methods for spatial data, can see for example Lahiri (2003), chap. 12.

3 Resampling Methods

This section presents the bootstrap methodology and five ways to resample panel data.

Bootstrap Methodology

From initial $N \times T$ data matrix Y, create a new matrix Y^* by resampling with replacement elements of Y. This operation must be repeated B times in order to have $B+1$ pseudo-samples : $\{Y_b^*\}_{b=1..B+1}$. Statistics are computed with these pseudo-samples in order to make inference. The probability measure induced by the resampling method conditionally on Y is noted P^* . $E^*()$ and $Var^*()$ are respectively expectation and variance associated to P^*. In this paper, inference is about θ and consists in building confidence intervals and testing hypothesis. There is close link between confidence interval and hypthosis tests : each can be seen as the dual program of the other**.** The resampling methods used to compute pseudo-samples are exposed below.

I.i.d. Bootstrap

In this document i.i.d bootstrap refers to original bootstrap as defined by Efron (1979). It was designed for one dimensional data, but it's easy to adapt it to panel data. For a $N \times T$ matrix Y, i.i.d. resampling is the operation of constructing a $N \times T$ matrix Y^* where each element y_{it}^* is selected with replacement from Y. Conditionally on Y, all the elements of Y^* are independent and identically distributed. There is a probability $1/NT$ that each y_{it}^* is one of the NT elements y_{it} of Y. The mean of Y^* obtained by i.i.d. bootstrap is noted $\overline{\overline{y}}_{iid}^*$.

Individual Bootstrap

For a $N \times T$ matrix Y, *individual resampling* is the operation of constructing a $N \times T$ matrix Y^* with rows obtained by resampling with replacement rows of Y. Conditionally on Y, the rows of Y^* are independent and identically distributed. Contrary to i.i.d. bootstrap case, y_{it}^* cannot take any value. y_{it}^* can just take one of the N values $\{y_{it}\}_{i=1,\dots N}$. The mean of Y^* obtained by individual bootstrap is noted $\overline{\overline{y}}_{ind}^*$.

Temporal Bootstrap

For $N \times T$ matrix Y, temporal resampling is the operation of constructing a $N \times T$ matrix Y^* with columns obtained by resampling with replacement colums of Y. Conditionally on Y, the columns of Y^* are independent and identically distributed. y_{it}^* can just take one of the T values $\{y_{it}\}_{t=1,\dots T}$. The mean of Y^* obtained by temporal bootstrap is noted $\overline{\overline{y}}_{temp}^*$.

Block Bootstrap

Block bootstrap for panel data is a direct accommodation of non-overlapping block bootstrap for time series, due to Carlstein (1986). The idea is to resample in temporal dimension, not single period like in temporal bootstrap case, but blocks of consecutive periods in order to capture temporal dependence. Assume that $T = Kl$, with l the length of a block, then there are K non-overlapping blocks. For $N \times T$ matrix Y, block bootstrap resampling is the operation of constructing a $N \times T$ matrix Y^* with columns obtained by resampling with replacement, the K blocks of columns of Y. The mean of Y^* obtained by block bootstrap is noted $\overline{\overline{y}}^*_{bl}$. Note that temporal bootstrap is a special case of block bootstrap, when $l = 1$. Moving block bootstrap (Kunsch (1989), Liu & Singh (1992)), circular block bootstrap (Politis & Romano (1992)) and stationary block bootstrap (Politis & Romano (1994)) can also be accommodated to panel data.

Double Resampling Bootstrap

For a $N \times T$ matrix Y, double resampling is the operation of constructing a $N \times T$ matrix Y^{**} with columns and rows obtained by resampling columns and rows of Y. The mean of Y^{**} is noted $\overline{\overline{y}}^{**}$. Two schemes are explored. The first scheme is a *combination of individual and temporal bootstrap.* The second scheme is a *combination of individual and block bootstrap.* The term *double* comes from the fact that the resampling can be made in two steps. In a first step, on dimension is taken into account : from Y, an intermediate matrix Y^* is obtained. An other resampling is made in the second dimension : from Y^* the final matrix Y^{**} is obtained. Carvajal (2000) and Kapetanios (2004) improve this resampling method by Monte Carlo simulations, but give no theoretical support. Double stars $**$ are used to distinguish estimator, probability measure, expectation and variance induced by double resampling.

4 Theoretical Results

This section presents theoretical results about resampling methods exposed in section 3, using models specified in section 2.

Multiple Asymptotics

In the study of asymptotic distributions for panel data, there are many possibilities. One index can be fixed and the other goes to infinity or N and T go to infinity. In the second case, how N and T go to infinity, is not always without consequence. Hsiao (2003 p. 296) distinguishes three approaches : *sequential limit, diagonal path limit* and *joint limit.* A sequential limit is obtained when an index is fixed and the other passes to infinity, to have intermediate result. The final result is obtained by passing the fixed index to infinity. In case of diagonal path limit, N and T pass to infinity along a specific path, for example $T = T(N)$ and $N \to \infty$. With joint limit, N and T pass to infinity simultaneously without a specific restrictions. In some cases, it can be necessary to control relative expansion rate of N and T. It is obvious that joint limit implies diagonal path limit. For equivalence conditions between sequential and joint limits, see Phillips & Moon (1999). In practice, it is not always clear how to choose among these multiple asymptotic distributions, which may be different. Table 1 summarizes asymptotic distributions for the different panel models.

For i.i.d. panel model, $NT \to \infty$ summarizes three cases of asymptotic : N is fixed and T goes to infinity, T is fixed and N goes to infinity, and finally N and T pass to infinity simultaneously. Two asymptotic theories are available for one-way ECM. In the case of two-way ECM, N and T must go to infinity. The relative convergence rate between the two indexes, δ defines a continuum of asymptotic distributions. Factor model *(1)* has a unique asymptotic distribution, when the two dimensions go to infinity. Finally, at our knowledge, there is no theory to derivate an asymptotic distribution for factor model (2). Details about convergences are exposed in appendix 1. It is important to mention that several demonstrations take advantage of the difference of convergence rates among elements in the specifications.

$Model$	$Asymptotic\ distribution$	$Variance\,(\omega)$
$I.i.d.\ panel$	$\sqrt{NT}\left(\overline{\overline{y}}-\theta\right) \underset{NT\to\infty}{\Longrightarrow} N\left(0,\omega\right)$	σ_{ε}^{2}
$Individual$ $One-way$ ECM	$\sqrt{N}\left(\overline{\overline{y}}-\theta\right) \underset{N\to\infty}{\Longrightarrow} N\left(0,\omega\right)$	$\sigma_{\mu}^{2}+\frac{\sigma_{\varepsilon}^{2}}{T}$
	$\sqrt{N}\left(\overline{\overline{y}}-\theta\right) \underset{N,T\to\infty}{\Longrightarrow} N\left(0,\omega\right)$	σ_{μ}^{2}
$Temporal$ $One-way$ ECM	$\sqrt{N}\left(\overline{\overline{y}}-\theta\right) \underset{T\to\infty}{\Longrightarrow} N\left(0,\omega\right)$	$\sigma_{f}^{2}+\frac{\sigma_{\varepsilon}^{2}}{N}$
	$\sqrt{T}\left(\overline{\overline{y}}-\theta\right) \underset{N,T\to\infty}{\Longrightarrow} N\left(0,\omega\right)$	σ_{f}^{2}
$Two-way$ ECM	$\sqrt{N}\left(\overline{\overline{y}}-\theta\right) \underset{\substack{N,T\to\infty \\ \frac{N}{T}\to\delta\in[0,\infty)}}{\Longrightarrow} N\left(0,\omega\right)$	$\sigma_{\mu}^{2}+\delta.V_{f}^{\infty}$
	$\sqrt{T}\left(\overline{\overline{y}}-\theta\right) \underset{\substack{N,T\to\infty \\ \frac{N}{T}\to\infty}}{\Longrightarrow} N\left(0,\omega\right)$	V_{f}^{∞}
$Factor$ $model\ (1)$	$\sqrt{N}\left(\overline{\overline{y}}-\theta\right) \underset{N,T\to\infty}{\Longrightarrow} N\left(0,\omega\right)$	σ_{μ}^{2}
$Factor$ $model\ (2)$	$Unknown$	

Table 1 : Classical asymptotic distributions

Bootstrap Consistency

There are several ways to prove consistency of a resampling method. For an overview, see Shao & Tu (1995, chap. 3). The method commonly used is to show that the distance between the cumulative distribution function on the classical estimator and the bootstrap estimator goes to zero when the sample grows-up. Different notions of distance can be used : sup-norm, Mallow's distance.... Sup-norm is the commonly used. The notations used for one dimension data must be to panel data, in order to be more formal. Because of multiple asymptotic distributions, there are several consistency definitions. The sample mean is noted $\overline{\overline{y}}$ and the bootstrap-sample mean $\overline{\overline{y}}^{*}$.

A bootstrap method is said *consistent* for sample mean if :

$$\sup_{x\in\mathbb{R}}\left|P^*\left(\sqrt{NT}\left(\overline{\overline{y}}^*-\overline{\overline{y}}\right)\leq x\right)-P\left(\sqrt{NT}\left(\overline{\overline{y}}-\theta\right)\leq x\right)\right|\underset{NT\to\infty}{\overset{P}{\to}}0 \qquad (4.1)$$

or

$$\sup_{x\in\mathbb{R}}\left|P^*\left(\sqrt{N}\left(\overline{\overline{y}}^*-\overline{\overline{y}}\right)\leq x\right)-P\left(\sqrt{N}\left(\overline{\overline{y}}-\theta\right)\leq x\right)\right|\underset{NT\to\infty}{\overset{P}{\to}}0 \qquad (4.2)$$

or

$$\sup_{x\in\mathbb{R}}\left|P^*\left(\sqrt{T}\left(\overline{\overline{y}}^*-\overline{\overline{y}}\right)\leq x\right)-P\left(\sqrt{T}\left(\overline{\overline{y}}-\theta\right)\leq x\right)\right|\underset{NT\to\infty}{\overset{P}{\to}}0 \qquad (4.3)$$

Definitions 4.1, 4.2 and 4.3 are given with convergence in probability ($\overset{P}{\to}$). This case implies a *weak consistency.* The case of almost surely ($a.s.$) convergence provides a *strong consistency.* These definitions of consistency does not require that the bootstrap estimator or the classical estimator has asymptotic distribution. The idea behind it, is the mimic analysis : when the sample grows, the bootstrap estimator *mimics* very well the behaviour of the classical estimator. This point is useful in the case of *factor model (2).* In the special when the sample mean asymptotic distribution is available, consistency can be established by showing that bootstrap-sample mean has the same distribution. The next proposition expresses this idea.

Proposition 1
Assume that $\sqrt{NT}\left(\overline{\overline{y}}-\theta\right)\Longrightarrow L$ *and* $\sqrt{NT}\left(\overline{\overline{y}}^*-\overline{\overline{y}}\right)\overset{*}{\Longrightarrow}L^*$. *If* L^**and* L *are identical and continuous, then*

$$\sup_{x\in\mathbb{R}}\left|P^*\left(\sqrt{NT}\left(\overline{\overline{y}}^*-\overline{\overline{y}}\right)\leq x\right)-P\left(\sqrt{NT}\left(\overline{\overline{y}}-\theta\right)\leq x\right)\right|\underset{NT\to\infty}{\overset{P}{\to}}0$$

where "$\overset{*}{\Longrightarrow}$*" means "converge in distribution conditionally on* Y*.*

Proof. $\overline{\overline{y}}$ *and* $\overline{\overline{y}}^*$ *having the same asymptotic distribution, implies that* $|P^*(..)-P(..)|$ *converges to zero. Under continuity assumption, the uniform convergence is given by the Pólya theorem (Pólya (1920) or Serfling (1980), p. 18)* ■

Similar propositions similar can be formulated for definitions *4.2* and *4.3.* Using Proposition 1, the methodology adopted in this document is, for each resampling method, to found bootstrap-mean asymptotic distribution. Comparing theses distributions with those in Table 1, permits to find consistent and inconsistent bootstrap resampling methods for each panel model. Consistent resampling methods can be used to make inference : confidence intervals and hypothesis tests.

Bootstrap Percentile-t Interval

In the literature, there are several bootstrap confidence interval. The method commonly used is *percentile-t interval* because its allows theoretical results for asymptotic refinements. With each pseudo-sample Y_b^*, compute the statististic

$$t_b^* = \frac{\overline{\overline{y}}_b^* - \overline{\overline{y}}}{\sqrt{\widehat{V}ar^*\left(\overline{\overline{y}}^*\right)}} \tag{4.4}$$

The empirical distribution of these $(B+1)$ realizations is :

$$\widehat{F}^*(x) = \frac{1}{B+1}\sum_{b=1}^{B+1} I\left(t_b^* \leq x\right) \tag{4.5}$$

The *percentile-t* confidence interval of level $(1-\alpha)$ is :

$$CI_{1-\alpha}^* = \left[\overline{\overline{y}} - \sqrt{Var^*\left(\overline{\overline{y}}^*\right)}.t_{1-\frac{\alpha}{2}}^*; \overline{\overline{y}} - \sqrt{Var^*\left(\overline{\overline{y}}^*\right)}.t_{\frac{\alpha}{2}}^*\right] \tag{4.6}$$

where $t_{\frac{\alpha}{2}}^*$ and $t_{1-\frac{\alpha}{2}}^*$ are respectively is the lower empirical $\frac{\alpha}{2}$-percentage point and $\left(1-\frac{\alpha}{2}\right)$-percentage point of $\widehat{F}^*$. B must be chosen so that $\alpha(B+1)/2$ is an integer. In the denominator of *(4.4)*, there is an estimator the bootstrap-variance. For validity of the confidence interval, this estimator must converge in probability to the bootstrap-variance.

$$\widehat{V}ar^*\left(\overline{\overline{y}}^*\right) \xrightarrow{P^*} Var^*\left(\overline{\overline{y}}^*\right) \tag{4.7}$$

Hypothesis Test

When testing hypothesis, Davidson (2007) quotes that a bootstrapping procedure must respect two golden rules. The first golden rule : the bootstrap Data Generating Process (DGP) must respect the null hypothesis. The second golden rule : the bootstrap DGP should be an estimate of the true DGP as possible. This means that the bootstrap data must *micmic* as possible the behaviour of the original data. To understand this approach, it must be taken in mind that bootstrap procedure has been originally designed for small samples.

I.i.d. Bootstrap

I.i.d. bootstrap treats observations as if there are independent. It does not take care of dependence structure. When the structure of panel data is not taken into account, the observations can be renumerate. Panel data sample can be represented by $\{y_1, y_2, ...y_n\}$ with $n = NT$. $\overline{y}_n$ is sample mean, $\overline{y}_n^*$ bootstrap-sample mean. In practice, what happens when use i.i.d. bootstrap with non i.i.d. data ? Proposition 2 answers this question.

Proposition 2

Assume that $Var^{*}(y_i^{*}) = S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \overline{y}_n)^2 \underset{n\to\infty}{\overset{a.s.}{\to}} \sigma^2 \in (0,\infty)$ *, then :*

$$\sqrt{n}\left(\overline{y}_n^{*} - \overline{y}_n\right) \underset{n\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\sigma^2\right) \quad a.s. \tag{4.8}$$

Proof . *In the particular case of* $\{y_1, y_2, ...y_n\}$ *independent and identically distribued, a proof of this proposition can be see in Freedman & Beckel (1981), Singh (1981) or Yang (1988). Note that if* y_i *are i.i.d, with finite variance,* $S_n^2 \underset{n\to\infty}{\overset{a.s.}{\to}} var(y_i)$ *.*

This proof follows the structure of Freedman & Beckel (1981). From the sample $\{y_1, y_2, ...y_n\}$*, construct a resample* $\{y_1^{*}, y_2^{*}, ...y_m^{*}\}$ *. Firstly, consider* $m \neq n$*. Conditionally on* $\{y_1, y_2, ..., y_n\}$*,* $\{y_1^{*}, y_2^{*}, ..., y_m^{*}\} \sim i.i.d.\left(\overline{y}, S_n^2\right)$*. Maintain n fixed and let m go to infinity and apply CLT :*

$\sqrt{m}\left(\frac{\overline{y}_m^{*} - \overline{y}_n}{\sqrt{S_n^2}}\right) \underset{m\to\infty}{\overset{*}{\Longrightarrow}} N(0,1).$

Secondly, let m and n go to infinity. $\sqrt{m}\left(\frac{\overline{y}_m^{*} - \overline{y}_n}{\sqrt{\sigma^2}}\right) = \sqrt{m}\left(\frac{\overline{y}_m^{*} - \overline{y}_n}{\sqrt{S_n^2}}\right)\left(\sqrt{\frac{S_n^2}{\sigma^2}}\right)$*. By assumption* $\sqrt{\frac{S_n^2}{\sigma^2}} \underset{n\to\infty}{\overset{a.s.}{\to}} 1$ *then :*$\sqrt{m}\left(\frac{\overline{y}_m^{*} - \overline{y}_n}{\sqrt{\sigma^2}}\right) \underset{n,m\to\infty}{\overset{*}{\Longrightarrow}} N(0,1).$

Finally, consider the special case with $m = n$ *:*$\sqrt{n}\left(\frac{\overline{y}_n^{*} - \overline{y}_n}{\sqrt{\sigma^2}}\right) \underset{n\to\infty}{\overset{*}{\Longrightarrow}} N(0,1)$ *, thus* $\sqrt{n}\left(\overline{y}_n^{*} - \overline{y}_n\right) \underset{n\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\sigma^2\right)$ *a.s.* ■

In *(4.4) a.s.* is added because bootstrap-variance converges almost surely to σ^2. In the continuation *a.s.* is ommitted in order to make notations less heavy. Almost surely convergence for bootstrap-variance, under assumptions of Proposition 1, implies *strong consistency*. Instead of *a.s.* convergence in Proposition 2, if there is convergence in probability, under assumptions of Proposition 1, *weak consistency* holds. Proposition 2 is a preliminary result that will be used to prove next propositions. If you apply i.i.d. bootstrap to dependent process, the idea to remember is that the asymptotic behaviour of the bootstrap mean does not take into account the structure of dependence in original data. Proposition 3 allows to deal with each panel data model.

Proposition 3

Assume that $Var^{*}(y_{it}^{*}) \underset{N,T\to\infty}{\overset{a.s.}{\to}} \omega$*, then :*

$$\sqrt{NT}\left(\overline{\overline{y}}_{iid}^{*} - \overline{\overline{y}}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\omega) \tag{4.9}$$

Proof . *Direct application of Proposition 2.* ■

According to Proposition 3, for each panel model specification, convergence of each bootstrap-variance must evaluated. Table 2 presents results for the four panel models. Algebraic details are presented in appendix 2. Naturally, i.i.d bootstrap is only consistent with i.i.d. panel model.

$Model$	$Asymptotic\ distribution$	$Variance\,(\omega)$	$Consistency$
$I.i.d.\ panel$	$\sqrt{NT}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{NT\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{ε}^{2}	Yes
$Individual$ $One-way$	$\sqrt{NT}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	$\sigma_{\mu}^{2}+\sigma_{\varepsilon}^{2}$	No
ECM	$\sqrt{NT}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	$\sigma_{\mu}^{2}+\sigma_{\varepsilon}^{2}$	
$Temporal$ $One-way$	$\sqrt{NT}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	$\sigma_{f}^{2}+\sigma_{\varepsilon}^{2}$	No
ECM	$\sqrt{NT}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	$\sigma_{f}^{2}+\sigma_{\varepsilon}^{2}$	
$Two-way$ ECM	$\sqrt{NT}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	$\sigma_{\mu}^{2}+\sigma_{f}^{2}+\sigma_{\varepsilon}^{2}$	No
$Factor\ model$ (1)	$\sqrt{NT}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	$\sigma_{\mu}^{2}+\sigma_{\lambda}^{2}\sigma_{F}^{2}+\sigma_{\varepsilon}^{2}$	No

Table 2 : Asymptotic distributions with i.i.d. bootstrap

Individual Bootstrap

In a conceptual manner, individual bootstrap can be perceived as equivalent to i.i.d. bootstrap on $\{\overline{y}_{1.},\overline{y}_{2.},.....\overline{y}_{N.}\}$ where $\overline{y}_{i.}$ is intertemporal average for cross-sectional unit i. Equality *(4.10)* allows to analyze the impact of individual bootstrap with the general specification (*2.1, 2.2)*.

$$\left(\overline{\overline{y}}_{ind}^{*}-\overline{\overline{y}}\right)=\left(\overline{\mu}_{iid}^{*}-\overline{\mu}\right)+\left(\overline{\lambda}_{iid}^{*}\overline{F}-\overline{\lambda F}\right)+\left(\left[\overline{\varepsilon}_{inter}\right]_{iid}^{*}-\overline{\overline{\varepsilon}}\right) \tag{4.10}$$

In a mimic analysis, individual bootstrap does not take care of random process in temporal dimension : the centering drop $\{f_t\}$ and $\{F_t\}$ is treated as a constant. This resampling method is appropriate for $\{\mu_i\}$ and $\{\lambda_i\}$. The impact on $\{\varepsilon_{it}\}$ seems ambiguous at first view. Under assumption *A5,* intertemporal averages are i.i.d., thus there is a hope. Proposition 4 allows to be more formal.

Proposition 4 *: CLT for individual bootstrap*
Assume that $Var^{*}\left(\overline{y}_{i.}^{*}\right)\underset{N\to\infty}{\overset{a.s.}{\to}}\omega$, *then :*

$$\sqrt{N}\left(\overline{\overline{y}}_{ind}^{*}-\overline{\overline{y}}\right)\underset{N\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right) \tag{4.11}$$

Proof . *Apply Proposition 2 to* $\{\overline{y}_{1.},\overline{y}_{2.},.....\overline{y}_{N.}\}$. ■

Table 3 presents asymptotic distribution for each panel. See details in appendix 3.

$Model$	$Asymptotic\ distribution$	$Variance\,(\omega)$	$Consistency$
$I.i.d.$	$\sqrt{NT}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{ε}^{2}	Yes
$panel$	$\sqrt{NT}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{ε}^{2}	
$Individual$ $One-way$	$\sqrt{N}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	$\sigma_{\mu}^{2}+\frac{\sigma_{\varepsilon}^{2}}{T}$	Yes
ECM	$\sqrt{N}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{μ}^{2}	
$Temporal$ $One-way$	$\sqrt{N}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{ε}^{2}	No
ECM	$\sqrt{N}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{ε}^{2}	
$Two-way$	$\sqrt{N}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	$\sigma_{\mu}^{2}+\frac{\sigma_{\varepsilon}^{2}}{T}$	No
ECM	$\sqrt{N}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{μ}^{2}	
$Factor$ $model$ (1)	$\sqrt{N}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{μ}^{2}	Yes

Table 3 : Asymptotic distributions with individual bootstrap

The guess about $\{\varepsilon_{it}\}$ is true. Individual bootstrap is consistent for iid panel model, one-way ECM and factor model. Bootstrap consistency requires N to go to infinity because resampling is only in this dimension. There is something sequential in the analysis when N and T go to infinity. Convergence is given with fixed T firstly and secondly T goes to infinity. Equivalence condition betwen sequential and joint limit in bootstrap context is to be developed.

Temporal Bootstrap

Heuristically, temporal resampling is equivalent to i.i.d bootstrap on $\{\overline{y}_{.1}, \overline{y}_{.2}, ...\overline{y}_{.T}\}$ where $\overline{y}_{.t}$ is cross-sectional average for period t. Equality *(4.12)* allows to analyze the impact of individual bootstrap with the general specification *(2.2)*.

$$\left(\overline{\overline{y}}^*_{temp} - \overline{\overline{y}}\right) = \left(\overline{f}^*_{iid} - \overline{f}\right) + \left(\overline{\lambda F}^*_{iid} - \overline{\lambda F}\right) + \left([\overline{\varepsilon}_{cross}]^*_{iid} - \overline{\overline{\varepsilon}}\right) \qquad (4.12)$$

Temporal boostrap DGP does not take care of random process in cross-sectional dimension. In a mimic analysis, this resampling method is appropriate for $\{f_t\}$ only if this process is iid. With $\{\varepsilon_{it}\}$ the situation is similar to individual bootstrap case. Proposition 5 permits to be more formal.

Proposition 5 *: CLT for temporal bootstrap*
Assume that $Var^*\left(\overline{y}^*_{.t}\right) \underset{T\to\infty}{\overset{a.s.}{\to}} \omega$, *then :*

$$\sqrt{T}\left(\overline{\overline{y}}^*_{temp} - \overline{\overline{y}}\right) \underset{T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\omega) \qquad (4.13)$$

Proof . *Apply Proposition 2 to* $\{\overline{y}_{.1}, \overline{y}_{.2}, ..., \overline{y}_{.T}\}$. ■

Table 4 presents asymptotic distribution for each model. For details, see appendix 4. Temporal bootstrap is consistent with i.i.d. panel model and temporal one-way ECM under *i.i.d.* assumption for $\{f_t\}$. The remark about sequential limit in individual bootstrap case, holds also here.

Model	*Asymptotic distribution*	$Variance\,(\omega)$	*Consistency*
I.i.d.	$\sqrt{NT}\left(\overline{\overline{y}}^* - \overline{\overline{y}}\right) \underset{T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\omega)$	σ^2_ε	*Yes*
panel	$\sqrt{NT}\left(\overline{\overline{y}}^* - \overline{\overline{y}}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\omega)$	σ^2_ε	
Individual One − way	$\sqrt{T}\left(\overline{\overline{y}}^* - \overline{\overline{y}}\right) \underset{T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\omega)$	σ^2_ε	*No*
ECM	$\sqrt{T}\left(\overline{\overline{y}}^* - \overline{\overline{y}}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\omega)$	σ^2_ε	

Temporal One − way ECM	$\sqrt{T}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right) \underset{T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\omega\right)$	$\sigma_f^2+\frac{\sigma_\varepsilon^2}{T}$	*Yes if* f_t *is iid*
	$\sqrt{T}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\omega\right)$	σ_f^2	
Two − way ECM	$\sqrt{T}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right) \underset{T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\omega\right)$	$\sigma_f^2+\frac{\sigma_\varepsilon^2}{N}$	*No*
	$\sqrt{T}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\omega\right)$	σ_f^2	
Factor model (1)	$\sqrt{T}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right) \underset{N,T\to\infty}{\overset{m.s.*}{\rightarrow}} 0$		*No*

Table 4 : Asymptotic distributions with temporal bootstrap

Block Bootstrap

Conceptually, block bootstrap is equivalent to i.i.d. bootstrap on $\left\{\overline{y}_{.1}^{bl},\overline{y}_{.2}^{bl},.....\overline{y}_{.K}^{bl}\right\}$ where

$$\overline{y}_{.k}^{bl}=\frac{1}{lN}\sum_{i=1}^{N}\sum_{t=kl-k+1}^{kl}y_{it}$$

Equality *(4.14)* presents the impact of block bootstrap with the general specification *(2.1-2)*.

$$\left(\overline{\overline{y}}_{bl}^{*}-\overline{\overline{y}}\right)=\left(\overline{f}_{bl}^{*}-\overline{f}\right)+\left(\overline{\lambda F}_{bl}^{*}-\overline{\lambda F}\right)+\left(\overline{\overline{\varepsilon}}_{bl}^{*}-\overline{\overline{\varepsilon}}\right) \tag{4.14}$$

Block boostrap DGP does not take care of random process in individual dimension. In a mimic analysis, it can be said that this resampling method is appropriate for $\{f_t\}$, $\{F_t\}$ and $\{\varepsilon_{it}\}$.

Proposition 6
Under assumption A2 and $l^{-1}+lT^{-1}=o\left(1\right)$ *as* $T\to\infty$

$$\sqrt{T}\left(\overline{f}_{bl}^{*}\right) \underset{T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,V_f^{\infty}\right) \tag{4.15}$$

Proof . *Under assumption A2 and the convergence rate imposed to l, a demonstration of the consistency of block bootstrap for time series, can be seen for example in Lahiri (2003), p. 55 .* ■

The condition about the convergence of l has a heuristic interpretation. If l is bounded, the block bootstrap method fail to capture the real dependence among the data. In other side, if l goes to infinity at the same rate that T, there are not enough blocks to resample. Block bootstrap is consistent with i.i.d. panel model and temporal one-way ECM.

$Model$	$Asymptotic\ distribution$	$Variance\,(\omega)$	$Consistency$
$I.i.d.$	$\sqrt{NT}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{ε}^{2}	Yes
$panel$	$\sqrt{NT}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{ε}^{2}	
$Individual$ $One-way$	$\sqrt{T}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{ε}^{2}	No
ECM	$\sqrt{T}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{ε}^{2}	
$Temporal$ $One-way$	$\sqrt{T}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	$V_{f}^{\infty}+\frac{\sigma_{\varepsilon}^{2}}{T}$	Yes
ECM	$\sqrt{T}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	V_{f}^{∞}	
$Two-way$	$\sqrt{T}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	$V_{f}^{\infty}+\frac{\sigma_{\varepsilon}^{2}}{N}$	No
ECM	$\sqrt{T}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	V_{f}^{∞}	
$Factor$ $model$ (1)	$\sqrt{T}\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{m.s.*}{\to}}0$		No

Table 5 : Asymptotic distributions with block bootstrap

Double Resampling Bootstrap

Two schemes of double resampling double bootstrap are explored. In a conceptual manner , the second scheme can be viewed as an application of first scheme, not directly on original data, but on a transformed matrix Y^{bl} that is obtained and by average of periods in each block.

$$Y^{bl}=\left\{\overline{y}_{ik}^{bl}\right\}_{i=1,\ldots,N}^{k=1,\ldots,K}\quad with\ \ \overline{y}_{ik}^{bl}=\frac{1}{l}\sum_{t=kl-k+1}^{kl}y_{it}$$

In the following, the first scheme properties are given. Similar results can be easily deduced for the second scheme. Like in i.i.d bootstrap case, y_{it}^{**} can take any of the NT values of elements of Y, with probability $1/NT$. $E^{**}\left(\right)$ and $Var^{**}\left(\right)$ are respectively expectation and variance conditionally on Y, with respect of the double resampling method.

$$E^{**}\left(y_{it}^{**}\right) = E^{*}\left(y_{it}^{*}\right) \quad (4.16)$$
$$Var^{**}\left(y_{it}^{**}\right) = Var^{*}\left(y_{it}^{*}\right) \quad (4.17)$$

Expectation and variance are identical to those obtained with *i.i.d.* bootstrap. The difference is that conditionally on Y, elements of Y^{**} are not all independent. Each element has a link with all the elements in the same column or on the same line with it. This link exists because elements in the same line belong to the same unit i and elements in the same column refer to the same period t.

Proposition 7 *: Double resampling bootstrap variance*
$\forall$ N, T, the double resampling bootstrap-variance is greater or equal to iid bootstrap-variance :

$$Var^{**}\left(\overline{\overline{y}}^{**}\right) \geq Var^{*}\left(\overline{\overline{y}}^{*}\right) \quad (4.18)$$

Proof . *An analysis of variance gives* :

$$Var^{**}\left(\overline{\overline{y}}^{**}\right) = \frac{1}{NT}Var^{**}\left(y_{it}^{**}\right) + \frac{1}{(NT)^2}\sum_{(i,t)\neq}^{N}\sum_{(j,s)}^{T}Cov^{**}\left(y_{it}^{**}, y_{js}^{**}\right)$$

$$For\ i \neq j\ and\ t \neq s,\ Cov^{**}\left(y_{it}^{**}, y_{js}^{**}\right) = 0$$

$$For\ i \neq j,\ Cov^{**}\left(y_{it}^{**}, y_{jt}^{**}\right) = Var^{*}\left(\overline{y}_{.t}^{*}\right)\ ,\ for\ t \neq s,\ Cov^{**}\left(y_{it}^{**}, y_{is}^{**}\right) = Var^{*}\left(\overline{y}_{i.}^{*}\right)$$

then

$$Var^{**}\left(\overline{\overline{y}}^{**}\right) = Var^{*}\left(\overline{\overline{y}}^{*}\right) + \left(1 - \frac{1}{T}\right)\frac{Var^{*}\left(\overline{y}_{i.}^{*}\right)}{N} + \left(1 - \frac{1}{N}\right)\frac{Var^{*}\left(\overline{y}_{.t}^{*}\right)}{T}$$

the result follows. ■

It is important to mention two things about inequality *(4.18)*. Firstly, no particular assumptions have been made about $\{y_{it}^{*}\}$. Secondly, *(4.18)* is a finite sample property : it holds for any sample size. The equality holds in *(4.18)* when $T = 1$ (cross-section data) , $N = 1$ (time series), or $\left[Var^{*}\left(\overline{y}_{i.}^{*}\right), Var^{*}\left(\overline{y}_{.t}^{*}\right)\right] = (0,0)$ (in cross-sectional averages and temporal averages are constant).

Equalities *(4.19)* and *(4.19)* permits to analyze the impact of double resampling bootstrap with the general specification (*2.1, 2.2)*

$$\left(\overline{\overline{y}}^{**} - \overline{\overline{y}}\right) = \left(\overline{\mu}_{iid}^{*} - \overline{\mu}\right) + \left(\overline{f}_{iid}^{*} - \overline{f}\right) + \left(\overline{\lambda}_{iid}^{*}\overline{F}_{iid}^{*} - \overline{\lambda F}\right) + \left(\overline{\overline{\varepsilon}}^{**} - \overline{\overline{\varepsilon}}\right) \quad (4.19)$$

$$\left(\overline{\overline{y}}_{bl}^{**} - \overline{\overline{y}}\right) = \left(\overline{\mu}_{iid}^{*} - \overline{\mu}\right) + \left(\overline{f}_{bl}^{*} - \overline{f}\right) + \left(\overline{\lambda}_{iid}^{*}\overline{F}_{bl}^{*} - \overline{\lambda F}\right) + \left(\overline{\overline{\varepsilon}}_{bl}^{**} - \overline{\overline{\varepsilon}}\right) \quad (4.20)$$

The first schme is equivalent to i.i.d. resampling on $\{\mu_i\}$, $\{f_t\}$, $\{F_t\}$, $\{\lambda_i\}$ and first scheme double resampling on $\{\varepsilon_{it}\}$. The second scheme is equivalent to i.i.d. resampling on $\{\mu_i\}$, $\{\lambda_i\}$ block resampling on $\{f_t\}$, $\{F_t\}$ and second scheme double resampling on $\{\varepsilon_{it}\}$. It is important to quote that the second scheme *mimics* very well the behaviour of $\{\lambda_i F_t\}$. This will permit applications with dynamic factors models. At this point, impact of double resampling is known for all processes except $\{\varepsilon_{it}\}$. Then, it is important to take care of the distribution of $\overline{\overline{\varepsilon}}^{**}$. The next proposition considers its asymptotic variance with the appropriate scaling factor.

Proposition 8 *: Double resampling bootstrap-variance for iid observations.*
Under assumption A4 :

$$Var^{**}\left(\sqrt{NT}\,\overline{\overline{\varepsilon}}^{**}\right) \underset{N,T\to\infty}{\overset{a.s.}{\to}} 3.\sigma_{\varepsilon}^{2} \tag{4.21}$$

Proof . *Variance decomposition in the proof of Proposition 7 gives*

$$Var^{**}\left(\sqrt{NT}\,\overline{\overline{\varepsilon}}^{**}\right) = Var^{*}\left(\varepsilon_{it}^{*}\right) + \left(1-\frac{1}{T}\right)\left[T.Var^{*}\left(\overline{\varepsilon}_{i.}^{*}\right)\right] + \left(1-\frac{1}{N}\right)\left[N.Var^{*}\left(\overline{\varepsilon}_{.t}^{*}\right)\right]$$

$$Var^{*}\left(\varepsilon_{it}^{*}\right) \underset{NT\to\infty}{\overset{a.s.}{\to}} \sigma_{\varepsilon}^{2}\ ,\ \left[T.Var^{*}\left(\overline{\varepsilon}_{i.}^{*}\right)\right] \underset{N\to\infty}{\overset{a.s.}{\to}} \sigma_{\varepsilon}^{2}\ ,\ \left[N.Var^{*}\left(\overline{\varepsilon}_{.t}^{*}\right)\right] \underset{T\to\infty}{\overset{a.s.}{\to}} \sigma_{\varepsilon}^{2}$$

then

$$Var^{**}\left(\sqrt{NT}\,\overline{\overline{\varepsilon}}^{**}\right) \underset{N,T\to\infty}{\overset{a.s.}{\to}} 3.\sigma_{\varepsilon}^{2}$$

■

Double resampling induces bootstrap-variance three times larger. This may imply a confidence interval, larger than the appropriate in case of normality. The question is whether properties of $\{y_{it}^{**}\}_{i,t\in\mathbb{N}}$ provide asymptotic normality. The next proposition considers with this issue.

Proposition 9 *: Normality*
Under assumption A4, , conditionally on Y, $\sqrt{NT}\,\overline{\overline{\varepsilon}}^{**}$ converges in distribution to a normal law.

Proof . *See Appendix 7.* ■

Tables 7 and 8 present asymptotic distribution for each panel model. For details about convergence, see appendix 6. The result for i.i.d. panel model is due to Proposition 8 and 9. In pratice, if there is a doubt about existence of a dependence in temporal dimension, the best choice is to use the second scheme.

Why does double resampling work well ?

The fact to resample in one dimension has an immediate drawback : processes that are not in the resampling dimension are dropped out by the centering $\left(\overline{\overline{y}}^{*}-\overline{\overline{y}}\right)$. The resampling in two dimensions avoids this drawback.

$Model$	$Asymptotic\ distribution$	$Variance\,(\omega)$	$Consistency$
$I.i.d.\ panel$	$\sqrt{NT}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	$3\ \sigma_{\varepsilon}^{2}$	$Conservative$
$Individual\ One-way\ ECM$	$\sqrt{N}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{μ}^{2}	Yes
$Temporal\ One-way\ ECM$	$\sqrt{T}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{f}^{2}	$Yes\ if\ f_t\ is\ iid$
$Two-way$	$\sqrt{N}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)\underset{\substack{N,T\to\infty\\ \frac{N}{T}\to\delta\in[0,\infty)}}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	$\sigma_{\mu}^{2}+\delta.\sigma_{f}^{2}$	$Yes\ if\ f_t\ is\ iid$
ECM	$\sqrt{T}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)\underset{\substack{N,T\to\infty\\ \frac{N}{T}\to\infty}}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{f}^{2}	
$Factor\ model\ (1)$	$\sqrt{N}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{μ}^{2}	Yes

Table 6 : Asymptotic distributions with double resampling bootstrap : scheme 1

$Model$	$Asymptotic\ distribution$	$Variance\,(\omega)$	$Consistency$
$I.i.d.\ panel$	$\sqrt{NT}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	$3\ \sigma_{\varepsilon}^{2}$	$Conservative$
$Individual\ One-way\ ECM$	$\sqrt{N}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{μ}^{2}	Yes
$Temporal\ One-way\ ECM$	$\sqrt{T}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	V_{f}^{∞}	Yes
$Two-way$	$\sqrt{N}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)\underset{\substack{N,T\to\infty\\ \frac{N}{T}\to\delta\in[0,\infty)}}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	$\sigma_{\mu}^{2}+\delta.V_{f}^{\infty}$	Yes
ECM	$\sqrt{T}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)\underset{\substack{N,T\to\infty\\ \frac{N}{T}\to\infty}}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	V_{f}^{∞}	
$Factor\ model\ (1)$	$\sqrt{N}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\omega\right)$	σ_{μ}^{2}	Yes

Table 7 : Asymptotic distributions with double resampling bootstrap : scheme 2.

5 Simulations

This section presents simulation design and results in order to confirm theorical results. Data Generating Process is the following : $\theta = 0$, $\mu_i \sim i.i.d.N(0,1)$, $\lambda_i \sim i.i.d.\ N(0,1)$, $\varepsilon_{it} \sim i.i.d.N(0,1)$, f_t and F_t follow the same process an $AR(1)$: $F_t = \rho F_{t-1} + \eta_t$, $\eta_t \sim i.i.d.N\left(0, \left(1-\rho^2\right)\right)$ *with* $\rho = 0$ or $\rho = 0.5$. For each bootstrap resampling scheme, B *is equal to 999 replications and the number of simulations is 1000.* Six sample sizes are considered : $(N,T) = (20,20)$, $(30,30)$, $(60,60)$, $(100,100)$, $(60,10)$ and $(10,60)$. For block bootstrap, the block length is $l = 5$ when $T = 100$ and $l = 2$ for other sizes. Tables 8 gives rejection rates with theoretical level 5%.

Simulations confirm theoretical results. I.i.d bootstrap, individual and temporal bootstrap give good results for i.i.d panel model. Individual bootstrap performs well with one-way ECM and factor model. Block bootstrap performs well with i.i.d. model and temporal one-way ECM. Double resampling performs well for individual and temporal one-way ECM, two-way ECM and factor model. Double resampling is conservative for i.i.d. panel model because it induces bootstrap-variance three times larger. In agreement with Proposition 7, under normality, the rejection rate with double resampling is always smaller than the one observed with *i.i.d.* resampling, for all sample sizes and all specifications. It would be useful to compare simulation results with classical asymptotical results. For this, a non parametric covariance matrix estimator is needed. Driscoll & Kraay (1998) extends Newey & West (1987)'s covariance matrix estimator to spatially dependent panel data when N is fixed and T goes to infinity. Non parametric variance estimation for large panels is to be developed.

6 Conclusion

This paper considers the issue of bootstrap resampling for panel data. Four specifications of panel data have been considered, namely i.i.d. panel data, one-way error component model, two-way error component model and lastly factor model. Five bootstrap resampling methods are explored and theoretical results are presented for the sample mean. It is demonstrated that simple resampling methods (i.i.d., individual only or temporal only) are not valid in simple cases of interest, while double resampling that combines resampling in both individual and temporal dimensions is valid in these situations. Simulations confirm these results. The logical follow-up of this paper is to extend results to more complicated statistics. Future research will extend theoretical results to linear regression model. Resampling methods which are not valid with the sample mean, are of course not valid for general linear regression model, because a mean can be perceived as a special case of linear regression.

$Models$	(N;T)	I.i.d.	Ind.	Temp.	Block	2Res-1	2Res-2
$I.i.d.$	(20;20)	**0.042**	**0.052**	**0.061**	0.096	*0.003*	*0.002*
	(30;30)	**0.050**	**0.067**	**0.068**	**0.062**	*0.001*	*0.002*
$panel$	(60;60)	**0.046**	**0.053**	**0.053**	0.074	*0.001*	*0.001*
	(100;100)	**0.052**	**0.053**	**0.051**	0.104	*0.002*	*0.004*
$model$	(60;10)	**0.050**	**0.055**	0.103	0.161	*0.002*	*0.002*
	(10;60)	**0.052**	0.098	**0.056**	**0.067**	*0.003*	*0.006*
	(20;20)	0.563	**0.059**	0.689	0.714	**0.043**	**0.040**
$Individual$	(30;30)	0.621	**0.051**	0.724	0.742	**0.041**	**0.039**
	(60;60)	0.743	**0.059**	0.810	0.842	**0.050**	**0.048**
$One-way$	(100;100)	0.782	**0.061**	0.852	0.843	**0.054**	**0.049**
	(60;10)	0.391	**0.054**	0.591	0.650	**0.040**	**0.038**
ECM	(10;60)	0.725	0.091	0.817	0.826	0.082	**0.050**
	(20;20)	0.736	0.822	0.299	0.090	0.268	**0.068**
$Temporal$	(30;30)	0.762	0.835	0.255	**0.067**	0.236	**0.053**
	(60;60)	0.813	0.870	0.263	**0.063**	0.253	**0.054**
$One-way$	(100;100)	0.886	0.919	0.268	**0.062**	0.268	**0.054**
ECM	(60;10)	0.841	0.880	0.304	0.146	0.299	0.132
$(\rho = 0.5)$	(10;60)	0.621	0.750	0.244	**0.066**	0.215	**0.040**
	(20;20)	0.581	0.164	0.158	0.216	**0.041**	**0.048**
$Two-way$	(30;30)	0.663	0.166	0.158	0.178	**0.048**	**0.056**
	(60;60)	0.743	0.177	0.176	0.186	**0.042**	**0.048**
ECM	(100;100)	0.830	0.161	0.174	0.190	**0.053**	**0.047**
	(60;10)	0.692	0.449	0.125	0.164	0.090	0.090
$\rho = 0$	(10;60)	0.680	0.110	0.431	0.060	0.077	0.430
	(20;20)	0.687	0.327	0.355	0.203	0.140	**0.042**
$Two-way$	(30;30)	0.767	0.324	0.342	0.194	0.143	**0.051**
	(60;60)	0.837	0.328	0.329	0.202	0.159	**0.047**
ECM	(100;100)	0.872	0.360	0.355	0.185	0.168	**0.051**
	(60;10)	0.781	0.600	0.297	0.152	0.226	0.094
$\rho = 0.5$	(10;60)	0.733	0.161	0.491	0.418	0.096	**0.050**
	(20;20)	0.4830	**0.056**	0.625	0.598	**0.034**	**0.024**
$Factor$	(30;30)	0.524	**0.062**	0.644	0.644	**0.044**	**0.039**
$model$	(60;60)	0.677	**0.063**	0.747	0.764	**0.053**	**0.054**
(1)	(100;100)	0.755	**0.057**	0.823	0.818	**0.048**	**0.056**
	(60;10)	0.364	**0.067**	0.543	0.508	**0.033**	**0.020**
	(10;60)	0.692	0.096	0.756	0.754	0.081	**0.064**
	(20;20)	0.119	0.086	0.173	0.176	*0.002*	*0.002*
$Factor$	(30;30)	0.121	**0.067**	0.158	**0.034**	*0.002*	*0.002*
$model$	(60;60)	0.140	**0.063**	0.163	0.098	*0.003*	*0.002*
(2)	(100;100)	0.120	**0.049**	0.118	0.104	*0.001*	*0.001*
	(60;10)	0.116	**0.067**	0.194	0.282	*0.002*	*0.004*
	(10;60)	0.133	0.112	0.151	**0.068**	*0.007*	*0.002*

Table 8 : Simulation results

References

[1] Andersson M. K. & Karlsson S. (2001) *Bootstrapping Error Component Models*, Computational Statistics, Vol. 16, No. 2, 221-231.

[2] Anselin L. (1988) *Spatial Econometrics: Methods and Mode*ls, Kluwer Academic Publishers, Dordrecht.

[3] Bai J. & Ng S. (2004) *A Panic attack on unit roots and cointegration*, Econometrica, Vol. 72, No 4, 1127-1177.

[4] Balestra P. & Nerlove M. (1966) *Pooling Cross Section and Time Series data in the estimation of a Dynamic Model : The Demand for Natural Gas*, Econometrica, Vol. 34, No 3, 585-612.

[5] Baltagi B. (1995) *Econometrics Analysis of Panel Data,* John Wiley & Sons.

[6] Bellman L., Breitung J. & Wagner J. (1989) *Bias Correction and Bootstrapping of Error Component Models for Panel Data : Theory and Applications,* Empirical Economics, Vol. 14, 329-342.

[7] Beran R. (1988) *Prepivoting test statistics: a bootstrap view of asymptotic refinements,* Journal of the American Statistical Association, Vol. 83, No 403, 687–697.

[8] Bickel P. J. & Freedman D. A. (1981) *Some Asymptotic Theory for the Bootstrap*, The Annals of Statistics, Vol. 9, No 6, 1196-1217.

[9] Carvajal A. (2000) *BC Bootstrap Confidence Intervals for Random Effects Panel Data Models*, Working Paper Brown University, Department of Economics.

[10] Davidson R. (2007) *Bootstrapping Econometric Models,* This paper has appeared, translated into Russian, in Quantile, Vol. 3, 13-36.

[11] Driscoll J. C. & . Kraay A. C. (1998) *Consistent Covariance Matrix Estimation with Spatially Dependent Panel Data*, The Review of Economics and Statistics, Vol. 80, No 4 , 549–560.

[12] Efron B. (1979) *Bootstrap Methods : Another Look at the Jackknife*, The Annals of Statistics, Vol. 7, No 1, 1-26.

[13] Everaert G. & Pozzi L. (2007) *Bootstrap-Based Bias Correction for Dynamic Panels*, Journal of Economic Dynamics & Control, Vol. 31, 1160-1184.

[14] Focarelli D. (2005) *Bootstrap bias-correction procedure in estimating long-run relationships from dynamic panels, with an application to money demand in the euro area,* Economic Modelling, Vol. 22, 305–325.

[15] Freedman D.A. (1981) *Bootstrapping Regression Models*, The Annals of Statistics, Vol. 9, No 6, 1218-1228.

[16] Fuller W. A. & Battese G. E. (1974) *Estimation of linear models with crossed-error structure*, Journal of Econometrics, Vol. 2, No 1, 67-78.

[17] Herwartz H. (2007) *Testing for random effects in panel models with spatially correlated disturbances*, Statistica Neerlandica, Vol. 61, No 4, 466–487.

[18] Herwartz H. (2006) *Testing for random effects in panel data under cross-sectional error correlation – a bootstrap approach to the Breusch Pagan Tes*t, Computational Statistics and Data Analysis, Vol. 50, 3567–3591.

[19] Hsiao C. (2003) *Analysis of Panel Data*, 2nd Edition. Cambridge University Press.

[20] Ibragimov I.A. (1962) *Some limit theorems for stationary processes*, Theory of Probability and Its Applications Vol. 7, 349–382.

[21] Kapetanios G. (2004) *A bootstrap procedure for panel datasets with many cross-sectional units,* Working Paper No. 523, Department of Economics, Queen Mary, London. Forthcoming in Econometrics Journal.

[22] Künsch H. R. (1989) *The jackknife and the bootstrap for general stationary observations*, Annals of Statistics, Vol. 17, 1217–1241.

[23] Lahiri S. N. (2003) *Resampling Methods for Dependent Data*, Springer Series in Statistics, Sinpring-Verlag, New York.

[24] Liu R. Y. (1988) *Bootstrap Procedure Under Nome Non-i.i.d. Models*, Annals of Statistics, Vol. 16, 1696–1708.

[25] Liu R. Y. & Singh K. (1992) *Moving Block Jackknife and bootstrap capture weak dependence*, in Exploring the Limits of Bootstrap , eds, by Lepage and Billard, John Wiley, NewYork.

[26] Moon H. R. & Perron B. (2004) *Testing for unit root in panels with dynamic factors*, Journal of Econometrics, Vol. 122, 81-126.

[27] Mundlak Y. (1978) *On the Pooling of Time Series and Cross Section Data,* Econometrica, Vol. 46, 1, 69-85.

[28] Newey W. K. & West K. D. (1987) *A Simple, Positive Semi-Definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix*, Econometrica, Vol. 55, No 3, 703-708.

[29] Phillips P.C.B. & Sul D. (2003) *Dynamic panel estimation and homogeneity testing under cross-section dependence*, Econometrics Journal, Vol. 6, 217-259.

[30] Phillips P.C.B. & Moon H.R. (1999) *Linear Regression Limit Theory for Nonstationary Panel Data,* Econometrica, Vol. 67, No 5, 1057-1112.

[31] Politis D. & Romano J. (1994) *The stationary bootstrap,* Journal of the American Statistical Association, Vol. 89, 1303-1313.

[32] Politis D. & Romano J. (1992) *A circular block resampling procedure for stationary data,* in Exploring the Limits of Bootstrap eds. by Lepage and Billard, John Wiley, NewYork.

[33] Pólya G. (1920) *Uber den zentralen Grenzwertsatz der Wahrscheinlinchkeitsrechnug und Momenten problem*, Math. Zeitschrift, 8.

[34] Serfling R. J. (1980) *Approximation Theorems of Mathematical Statistics,* New York: Wiley.

[35] Shao J. & Tu D. (1995) *The Jackknife and bootstrap*, Sinpring-Verlag, New York.

[36] Singh K. (1981) *On the Asymptotic Accuracy of Efron's Bootstrap*, The Annals of Statistics, Vol. 9, No. 6, 1187-1195.

[37] Wu C. F. J. (1986) *Jackknife, bootstrap and other resampling methods in regression analysis*, Annals of Statistics, Vol. 14, 1261–1295.

[38] Yang S.S. (1988) *A Central Limit Theorem for the Bootstrap Mean*, The American Statistician, Vol. 42, 202-203.

7 APPENDIX

Appendix 1 : Classical Asymptotic Distributions

Iid panel model

Under assumption A4, apply standard CLT.

One-way ECM

a) T is fixed. $\overline{y}_{i.}$ are i.i.d. with $E\left(\overline{y}_{i.}\right)=\theta$ and $Var\left(\overline{y}_{i.}\right)=\sigma_{\mu}^{2}+\frac{\sigma_{\varepsilon}^{2}}{T}$. *Applying a standard CLT, the result follows.*

b)

$$\sqrt{N}\left(\overline{\overline{y}}-\theta\right)=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mu_{i}+\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\varepsilon_{it}\right)$$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mu_{i}\underset{N\rightarrow\infty}{\Longrightarrow}N\left(0,\sigma_{\mu}^{2}\right)\quad\text{and}\quad\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\varepsilon_{it}\right)\underset{T\rightarrow\infty}{\overset{m.s.}{\rightarrow}}0$$

where" $\overset{m.s.}{\rightarrow}$" means *converge in mean square. The result follows using Slutsky's theorem.*

Two-way ECM

a)$\frac{N}{T}\rightarrow\delta\in[0,\infty)$

$$\sqrt{N}\left(\overline{\overline{y}}-\theta\right)=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mu_{i}+\frac{\sqrt{N}}{\sqrt{T}}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}f_{t}\right)+\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{i=1}^{T}\varepsilon_{it}\right)$$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mu_{i}\underset{N\rightarrow\infty}{\Longrightarrow}N\left(0,\sigma_{\mu}^{2}\right)\ ;\quad\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}f_{t}\right)\underset{N,T\rightarrow\infty}{\Longrightarrow}N\left(0,V_{f}^{\infty}\right)$$

CLT for $\{f_t\}$ *under assumption A2, is due to Ibragimov (1962).*

$$\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\varepsilon_{it}\right)\underset{N,T\to\infty}{\overset{m.s.}{\longrightarrow}}0$$

The result follows[3]*.*

b) $\frac{N}{T}\to\infty$

$$\sqrt{T}\left(\overline{\overline{y}}-\theta\right)=\frac{\sqrt{T}}{\sqrt{N}}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mu_i\right)+\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}f_t\right)+\frac{1}{\sqrt{N}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{i=1}^{T}\varepsilon_{it}\right)$$

$$\frac{1}{\sqrt{N}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\varepsilon_{it}\right)\underset{N,T\to\infty}{\overset{m.s.}{\longrightarrow}}0;\quad\frac{\sqrt{T}}{\sqrt{N}}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mu_i\right)\underset{N,T\to\infty}{\overset{m.s.}{\longrightarrow}}0$$

$$\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}f_t\right)\underset{T\to\infty}{\Longrightarrow}N\left(0,V_f^{\infty}\right)$$

The result follows.

Factor Models

First specification : $y_{it}=\theta+\mu_i+\lambda_iF_t+\varepsilon_{it}$

$$\sqrt{N}\left(\overline{\overline{y}}-\theta\right)=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mu_i+\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i\right)\left(\frac{1}{T}\sum_{t=1}^{T}F_t\right)+\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\varepsilon_{it}\right)$$

$$\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\varepsilon_{it}\right)\underset{N,T\to\infty}{\overset{m.s.}{\longrightarrow}}0\quad;\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i\right)\left(\frac{1}{T}\sum_{t=1}^{T}F_t\right)\underset{N,T\to\infty}{\overset{m.s.}{\longrightarrow}}0$$

The result follows.

Second specification : $y_{it}=\theta+\lambda_iF_t$

$$\sqrt{NT}\left(\overline{\overline{y}}-\theta\right)=\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t\right)$$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i\Rightarrow N\left(0,\sigma_\lambda^2\right)\qquad\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t\Rightarrow N\left(0,V_F^{\infty}\right)$$

To my knowledge, there is no theory about the convergence of the product. For Slutsky theorem, a convergence in probabilty at least, is needed.

[3] When the vector$(X_n,Y_n)^{'}$ converges to a normal distribution, the asymptotic distribution of any linear combination of the elements of the vector (in particular the sum) can be deduced. The fact that X_n and Y_n are independent and converge to a normal distribution, implies that their sum converge to the sum of their asymptotic normal distributions.

Appendix 2 : I.i.d. Bootstrap

	$Var^*\left(y_{it}^*\right)$	ω
Iid panel model	$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\varepsilon_{it}\right)^2-\left(\overline{\overline{\varepsilon}}\right)^2$	$\underset{NT\to\infty}{\to}\sigma_\varepsilon^2$
Ind. 1-way ECM	$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\mu_i+\varepsilon_{it}\right)^2-\left(\overline{\mu}+\overline{\overline{\varepsilon}}\right)^2$	$\underset{N\to\infty}{\to}\sigma_\mu^2+\sigma_\varepsilon^2$; $\underset{N,T\to\infty}{\overset{a.s.}{\to}}\sigma_\mu^2+\sigma_\varepsilon^2$
Temp. 1-way ECM	$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(f_t+\varepsilon_{it}\right)^2-\left(\overline{\mu}+\overline{\overline{\varepsilon}}\right)^2$	$\underset{T\to\infty}{\to}\sigma_f^2+\sigma_\varepsilon^2$; $\underset{N,T\to\infty}{\overset{a.s.}{\to}}\sigma_f^2+\sigma_\varepsilon^2$
Two-way ECM	$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\mu_i+f_t+\varepsilon_{it}\right)^2-\left(\overline{\mu}+\overline{f}+\overline{\overline{\varepsilon}}\right)^2$	$\underset{N,T\to\infty}{\to}\sigma_\mu^2+\sigma_f^2+\sigma_\varepsilon^2$
Factor Model (1)	$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\mu_i+\lambda_iF_t+\varepsilon_{it}\right)^2-\left(\overline{\mu}+\overline{\lambda F}+\overline{\overline{\varepsilon}}\right)^2$	$\underset{N,T\to\infty}{\to}\sigma_\mu^2+\sigma_\lambda^2*\sigma_F^2+\sigma_\varepsilon^2$

Appendix 3 : Individual Bootstrap

	$Var^*\left(\overline{y}_{i.}\right)$	ω
Iid panel model	$\frac{1}{N}\sum_{i=1}^{N}\left(\overline{\varepsilon}_{i.}\right)^2-\left(\overline{\overline{\varepsilon}}\right)^2$	$\underset{N\to\infty}{\to}\frac{\sigma_\varepsilon^2}{T}$
Ind. 1-way ECM	$\frac{1}{N}\sum_{i=1}^{N}\left(\mu_i+\overline{\varepsilon}_{i.}\right)^2-\left(\overline{\mu}+\overline{\overline{\varepsilon}}\right)^2$	$\underset{N\to\infty}{\to}\sigma_\mu^2+\frac{\sigma_\varepsilon^2}{T}$; $\underset{N,T\to\infty}{\to}\sigma_\mu^2$
Temp. 1-way ECM	$\frac{1}{N}\sum_{i=1}^{N}\left(\overline{\varepsilon}_{i.}\right)^2-\left(\overline{\overline{\varepsilon}}\right)^2$	$\underset{N\to\infty}{\to}\frac{\sigma_\varepsilon^2}{T}$
Two-way ECM	$\frac{1}{N}\sum_{i=1}^{N}\left(\mu_i+\overline{\varepsilon}_{i.}\right)^2-\left(\overline{\mu}+\overline{\overline{\varepsilon}}\right)^2$	$\underset{N\to\infty}{\to}\sigma_\mu^2+\frac{\sigma_\varepsilon^2}{T}$; $\underset{N,T\to\infty}{\to}\sigma_\mu^2$
Factor Model (1)	$\frac{1}{N}\sum_{i=1}^{N}\left(\mu_i+\lambda_i\overline{F}+\overline{\varepsilon}_{i.}\right)^2-\left(\overline{\mu}+\overline{\lambda F}+\overline{\overline{\varepsilon}}\right)^2$	$\underset{N,T\to\infty}{\to}\sigma_\mu^2$

Appendix 4 : Temporal Bootstrap

	$Var^*\left(\overline{y}_{.t}\right)$	ω
Iid panel model	$\frac{1}{T}\sum_{t=1}^{T}\left(\overline{\varepsilon}_{.t}\right)^2-\left(\overline{\overline{\varepsilon}}\right)^2$	$\underset{T\to\infty}{\longrightarrow}\ \frac{\sigma_\varepsilon^2}{N}$
Ind. 1-way ECM	$\frac{1}{T}\sum_{t=1}^{T}\left(\overline{\varepsilon}_{.t.}\right)^2-\left(\overline{\overline{\varepsilon}}\right)^2$	$\underset{T\to\infty}{\longrightarrow}\ \frac{\sigma_\varepsilon^2}{N}$
Temp. 1-way ECM	$\frac{1}{T}\sum_{t=1}^{T}\left(f_t+\overline{\varepsilon}_{.t}\right)^2-\left(\overline{\overline{\varepsilon}}\right)^2$	$\underset{T\to\infty}{\longrightarrow}\ \sigma_f^2+\frac{\sigma_\varepsilon^2}{N}$; $\underset{N,T\to\infty}{\longrightarrow}\ \sigma_f^2$
Two-way ECM	$\frac{1}{T}\sum_{t=1}^{T}\left(f_t+\overline{\varepsilon}_{.t}\right)^2-\left(\overline{\overline{\varepsilon}}\right)^2$	$\underset{T\to\infty}{\longrightarrow}\ \sigma_f^2+\frac{\sigma_\varepsilon^2}{N}$; $\underset{N,T\to\infty}{\longrightarrow}\ \sigma_f^2$
Factor Model (1)	$\frac{1}{T}\sum_{t=1}^{T}\left(\overline{\lambda}F_t+\overline{\varepsilon}_{.t}\right)^2-\left(\overline{\lambda F}+\overline{\overline{\varepsilon}}\right)^2$	$\underset{N,T\to\infty}{\overset{m.s.}{\longrightarrow}}\ 0$

Appendix 5 : Block Bootstrap

	$Var^*\left(\overline{y}_{.k}^{bl}\right)$	ω
Iid panel model	$\frac{1}{K}\sum_{k=1}^{K}\left(\overline{\varepsilon}_{.k}^{bl}\right)^2-\left(\overline{\overline{\varepsilon}}\right)^2$	$\underset{T\to\infty}{\longrightarrow}\ \frac{\sigma_\varepsilon^2}{N}$
Ind. 1-way ECM	$\frac{1}{K}\sum_{t=1}^{K}\left(\overline{\varepsilon}_{.k}^{bl}\right)^2-\left(\overline{\overline{\varepsilon}}\right)^2$	$\underset{T\to\infty}{\longrightarrow}\ \frac{\sigma_\varepsilon^2}{N}$
Temp. 1-way ECM	$\frac{1}{K}\sum_{k=1}^{K}\left(\overline{f}_{.k}^{bl}+\overline{\varepsilon}_{.k}^{bl}\right)^2-\left(\overline{\mu}+\overline{\overline{\varepsilon}}\right)^2$	$\underset{T\to\infty}{\longrightarrow}\ V_f^\infty+\frac{\sigma_\varepsilon^2}{N}$; $\underset{N,T\to\infty}{\longrightarrow}\ V_f^\infty$
Two-way ECM	$\frac{1}{K}\sum_{k=1}^{K}\left(\overline{f}_{.k}^{bl}+\overline{\varepsilon}_{.k}^{bl}\right)^2-\left(\overline{\mu}+\overline{\overline{\varepsilon}}\right)^2$	$\underset{T\to\infty}{\longrightarrow}\ V_f^\infty+\frac{\sigma_\varepsilon^2}{N}$; $\underset{N,T\to\infty}{\longrightarrow}\ V_f^\infty$
Factor Model (1)	$\frac{1}{K}\sum_{k=1}^{K}\left(\overline{\lambda F}_{.k}^{bl}+\overline{\varepsilon}_{.k}^{bl}\right)^2-\left(\overline{\mu}+\overline{\lambda F}+\overline{\overline{\varepsilon}}\right)^2$	$\underset{N,T\to\infty}{\overset{m.s.}{\longrightarrow}}\ 0$

Appendix 6 : Double Resampling Bootstrap

I.i.d. panel model

Use Propositions 8 and 9.

One-way ECM

$$\sqrt{N}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)=\sqrt{N}\left(\overline{\mu}^{*}_{iid}-\overline{\mu}\right)+\sqrt{N}\left(\overline{\overline{\varepsilon}}^{**}-\overline{\overline{\varepsilon}}\right)$$

$$\sqrt{N}\left(\overline{\mu}^{*}-\overline{\mu}\right)\underset{N\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\sigma^2_{\mu}\right)\quad;\quad\left[\sqrt{N}\left(\overline{\overline{\varepsilon}}^{**}-\overline{\overline{\varepsilon}}\right)\right]\underset{N,T\to\infty}{\overset{m.s.*}{\longrightarrow}}0$$

The result follows.

Two-way ECM

a) $\frac{N}{T}\to\delta\in[0,\infty)$:

$$\sqrt{N}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)=\sqrt{N}\left(\overline{\mu}^{*}-\overline{\mu}\right)+\frac{\sqrt{N}}{\sqrt{T}}\sqrt{T}\left(\overline{f}^{*}-\overline{f}\right)+\frac{1}{\sqrt{T}}\left[\sqrt{NT}\left(\overline{\overline{\varepsilon}}^{**}-\overline{\overline{\varepsilon}}\right)\right]$$

$$\frac{\sqrt{N}}{\sqrt{T}}\sqrt{T}\left(\overline{f}^{*}-\overline{f}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\delta.\sigma^2_{f}\right)\quad,\quad\frac{\sqrt{N}}{\sqrt{T}}\sqrt{T}\left(\overline{f}^{*}_{bl}-\overline{f}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\delta.V^{\infty}_{f}\right)$$

$$\frac{1}{\sqrt{T}}\left[\sqrt{NT}\left(\overline{\overline{\varepsilon}}^{**}-\overline{\overline{\varepsilon}}\right)\right]\underset{N,T\to\infty}{\overset{m.s.*}{\longrightarrow}}0$$

There result follows.

b) $\frac{N}{T}\to\infty$:

$$\sqrt{T}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)=\frac{\sqrt{T}}{\sqrt{N}}\left[\sqrt{N}\left(\overline{\mu}^{*}-\overline{\mu}\right)\right]+\sqrt{T}\left(\overline{f}^{*}-\overline{f}\right)+\frac{\sqrt{T}}{\sqrt{N}}\left(\frac{1}{\sqrt{T}}\left[\sqrt{NT}\left(\overline{\overline{\varepsilon}}^{**}-\overline{\overline{\varepsilon}}\right)\right]\right)$$

$$\sqrt{T}\left(\overline{f}^{*}-\overline{f}\right)\underset{T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\sigma^2_{f}\right)\qquad\sqrt{T}\left(\overline{f}^{*}_{bl}-\overline{f}\right)\underset{T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,V^{\infty}_{f}\right)$$

$$\frac{1}{\sqrt{N}}\left[\sqrt{NT}\left(\overline{\overline{\varepsilon}}^{**}-\overline{\overline{\varepsilon}}\right)\right]\underset{N,T\to\infty}{\overset{m.s.*}{\longrightarrow}}0\quad,\quad\frac{\sqrt{T}}{\sqrt{N}}\left[\sqrt{N}\left(\overline{\mu}^{*}-\overline{\mu}\right)\right]\underset{N,T\to\infty}{\overset{m.s.*}{\longrightarrow}}0$$

The result follows.

Factor Model

$$\sqrt{N}\left(\overline{\overline{y}}^{**}-\overline{\overline{y}}\right)=\sqrt{N}\left(\overline{\mu}^{*}_{iid}-\overline{\mu}\right)+\sqrt{N}\left(\overline{\lambda}^{*}_{iid}\overline{F}^{*}_{iid}-\overline{\lambda F}\right)+\left[\sqrt{N}\left(\overline{\overline{\varepsilon}}^{**}-\overline{\overline{\varepsilon}}\right)\right]$$

$$\left[\sqrt{N}\left(\overline{\overline{\varepsilon}}^{**}-\overline{\overline{\varepsilon}}\right)\right]\underset{N,T\to\infty}{\overset{m.s.*}{\longrightarrow}}0\quad,\quad\sqrt{N}\left(\overline{\lambda}^{*}_{iid}\overline{F}^{*}_{iid}-\overline{\lambda F}\right)\underset{N,T\to\infty}{\overset{m.s.*}{\longrightarrow}}0$$

the results follows.

Appendix 7 : $\alpha-$Mixing Process

Let $\{f_t\}_{t\in\mathbb{Z}}$ be a sequence of random variables. The strong mixing or α-mixing coefficient of $\{f_t\}_{t\in\mathbb{Z}}$ is defined as :

$$\begin{aligned} \alpha(j) &= \sup\{|P(A\cap B) - P(A)P(B)|\},\ j\in\mathbb{N} \\ with\ A &\in \sigma\langle\{f_t : t\le k\}\rangle, B\in\sigma\langle\{f_t : t\ge k+j+1\}\rangle,\ k\in\mathbb{Z} \end{aligned}$$

$\{f_t\}_{t\in\mathbb{Z}}$ is called strongly mixing (or α-mixing) if $\alpha(j)\to 0$ as $j\to\infty$.

Proof of Proposition 7 . (To be completed). ∎