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Real equivariant K-theory invariants for class AI topological insulators

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Abstract

Equivariant K-theory gives a mathematical framework for the classification of topological phases of matter in the presence of further symmetries, like crystal or time reversal symmetries [14]. Studying *Real* equivariant K-theory, the flavor relevant for symmorphic class A and class AI systems, we are able to prove generalizations of the family completion [9], localization [4], and Chern character isomorphism [11] theorems. The last of these shows, with fixed spatial symmetries and up to torsion, that there are always less topological phases in class AI than in class A.

Finally, we present a method to effectively calculate all integer K-theory invariants of a given class AI system. Applying it to an example of Wu and Hu [17], we find that it should exhibit edge states even when the sixfold rotational symmetry of their setup is broken down to a twofold one.

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Introduction

The theory of topological phases of matter, protected by symmetries, is a rather young subject, with many advances in the recent years. Though the details depend on the concrete setting, this theory is concerned with whether certain physical systems can be *continuously* deformed into each other, while preserving symmetries and other boundary conditions. Most prominently these are, in the area of many particle systems, *having a gap* and being *non interacting*, but also other setups are being studied. An overview of the subject, and the many directions people have taken it, can be found in [27].

Being a kind of mathematical subject, a lot of effort has been put into the systematic classification of topological phases. One framework for this is given by K-theory, a mathematical tool originally designed to study vector bundles on topological spaces [20]. Roughly speaking, one can, to a physical system, assign a vector bundle representing its topological phase. Details about this construction can be found in [14]. In this framework classifying the possible phases with respect to certain symmetries mean calculating the associated (twisted, equivariant) K-groups. A recent example of such calculations is [16].

On the other hand, the problem of determining the topological phase of a given physical system, that is calculating the element in the K-group it is represented by, has been studied less. In general this means giving a list of invariants, which can be calculated for a given system, such that two systems are in the same topological phase if these invariants give the same values for them, maybe up to integer multiples of a constant. Examples of such invariants are the Chern numbers, the Chern-Simons invariants, and the Fu-Kane invariants, an overview is given in [27].

In this thesis, I study *Real equivariant* K-theory, which is the flavor relevant for the classification of bosonic systems which might have spatial and time-reversing symmetries, with the goal of finding methods to compute the topological phase for a given system. Real and Real equivariant K-theory were first introduced by Atiyah in [25] and [6], though I will study a certain generalization of that notion. The main computational results concern the *integer* parts of the K-groups. These give, in principle, a rougher classification than the full K-groups, but in many cases the K-groups have no torsion to begin with (see [16]), so that one doesn't lose information. We will see that two bosonic time-reversal symmetric systems have the same integer topological phase if and only if they have the same integer topological phase after forgetting that they are time-reversal symmetric.

For the latter the relevant K-groups are given by the $\mathbb{C}$-linear equivariant K-theory of [4], for which there is a method to completely compute the integer part of a given system, using the so-called equivariant Chern character isomorphism [11]. Studying this in a bit more detail, we find that the list of integer invariants one needs to compute often drastically decreases in the presence of time-reversing symmetries.

The thesis is organized in three chapters. The first chapter is dedicated to the study of Real representations. These are representations of a group which may act $\mathbb{C}$-linearly or $\mathbb{C}$-antilinearly, reflecting the fact that time-reversing symmetries a generally considered to act $\mathbb{C}$-antilinearly on the Hilbert space on which one models a physical system. The main results here a several comparisons between Real representations of G and usual representations of $G_0 \subset G$, the subgroup of $\mathbb{C}$-linearly acting elements. One of these is that two Real representations of G are isomorphic if and only if their restrictions to G_0 are. Up to the last section, this is mostly a recollection of the material presented in [1]. The last section itself is a generalization of Segal's work [2] on the prime spectrum of representation rings to the Real case. This structural result plays a major role in the proofs of the localization and family completion theorems, so I present it in quite some detail.

In the second chapter, I introduce the notion of Real equivariant K-theory we want to study. One should note that the Real equivariant K-theory studied in the 60s is only a special case of the definition I study, corresponding to the case where the Real group of equivariance is *split* (see section 1.1). We will see that it has (almost) all the nice properties known from equivariant K-theory, enabling us to generalize several theorems from the equivariant to the Real equivariant case. These are the equivariant Bott periodicity [6], the equivariant Chern character isomorphism (see section 7 of [11],), the localization theorem of [4], and the family completion theorem treated in [9]. Especially that last one is interesting, as even the proof in the split case relies on the more general notion of Real equivariant K-group we study.

Lastly, in the third chapter, I try to motivate how Real equivariant K-theory is relevant for the classification of topological phases, and detail exactly in which cases it can be applied. The argumentation I present there is supposed to be only a conceptual overview, a more rigorous treatment can be found in [14]. After that, I sketch how one can use the Real equivariant Chern character as a computational tool with an example: Studying an example of light traveling in a grid of dielectric materials, proposed by Wu and Hu in [17], we find that it should be topologically non-trivial even after breaking the sixfold rotational symmetry their argument relies on. Finally, I briefly discuss how one could apply these computational methods also to the study of systems with different types of symmetry, like fermionic systems.

1 Real representation theory

In this chapter I will introduce some aspects of *Real* representations of *finite* groups. The material presented here, with slight modification in nomenclature, is mostly taken from [1]. Throughout note that the capitalization of 'Real' is intended.

The reason to consider this theory is that we want to study representations of groups on complex vector spaces, where each element acts either $\mathbb{C}$ linear or antilinear in a prespecified way. Such things show up naturally, for example, in the study of physical systems with so called *time-reversing symmetries.* As it will turn out this theory interpolates naturally between $\mathbb{C}$ linear and $\mathbb{R}$ linear representation theory, and behaves almost equally well. In particular it is possible to set these aspects all into a common framework, which I sketch below.

1.1 Real groups and Real representations

Definition 1.1.1.

1. A *Real group* is a pair (G, ϕ_G) of a group G and a morphism of groups $\phi_G : G \to C_2$. I will often leave out the subscript on ϕ and call just G the Real group, when things are clear from the context.
2. A morphism of Real groups $(G, \phi_G) \to (H, \phi_H)$ is a morphism of groups $f : G \to H$ such that $\phi_H \circ f = \phi_G$.
3. Call $G_0 := \ker \phi_G$ the *even* elements, and $G_1 := G \setminus G_0$ the *odd* elements. I will call a subgroup of G, which is contained in G_0, an *even subgroup.*
4. The associated *Real group ring* ${}^{\phi}\mathbb{C}G$ is the ring:

   $$\frac{\mathbb{R}[i, \{x_g | g \in G\}]}{(i^2 + 1, x_1 - 1, x_g x_h - x_{gh}, x_g i - \phi(g) i x_g)}$$

   The construction of Real group rings gives a functor from the category of Real groups to (noncommutative) rings.

Remark. The point is that $C_2 \cong \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, i.e. an element of C_2 can be interpreted as either complex conjugation or identity on the complex numbers. On the other hand, we have that $C_2 \cong \{\pm 1\}$. In the following I will use $\phi(g)$ to mean either a sign or an endomorphism of the complex numbers, depending on the context.

Here are a few example of Real groups and the associated Real group rings:

- For any group G the trivial morphism $G \xrightarrow{1} C_2$ lets us interpret it as a Real group (with no odd elements), ${}^1\mathbb{C}G \cong \mathbb{C}G$ is the usual (complex) group ring.
- For any group G the projection $G \times C_2 \xrightarrow{\pi_2} C_2$ lets us interpret it as a Real group (with a free odd element), ${}^{\pi_2}\mathbb{C}\,(G \times C_2) \cong \mathrm{Mat}_{2\times 2}(\mathbb{R}G)$ is 2 by 2 matrices in the $\mathbb{R}$ group ring of G. This will be proven later.
- For any subgroup $H \subset G$, where G is a Real group, we can make H into a Real group by restriction: $H \xrightarrow{\phi_G|_H} C_2$.

Since it will be important to distinguish them, let me give a certain class of Real groups a name:

Definition 1.1.2. Let (G, ϕ) be a Real group. We say that it is *split* if either $G = G_0$ or the short exact sequence

$$0 \to G_0 \to G \xrightarrow{\phi} C_2 \to 0$$

has a splitting.

Remark. First of all, there are, of course, nonsplit Real groups. The simplest example is given by the unique surjection $C_4 \to C_2$. Also notice that, as Real groups, subgroups of a split Real group are not necessarily split themselves. An example for that is the split Real group $(C_4 \times C_2, \pi_2)$ and the subgroup generated by the odd element $([1], [1])$. This subgroup is then, as a Real group, isomorphic to the previous example $C_4 \to C_2$. Finally, for any Real group (G, ϕ) the map

$$\mathrm{id} \times \phi : (G, \phi) \to (G \times C_2, \pi_2),\; g \mapsto (g, \phi(g))$$

is an inclusion of Real groups, so every Real group is a subgroup of a split one.

Definition 1.1.3. 1. Let V be a complex vector space. The *Real linear group* $\mathrm{RGL}(V)$ of V is the Real group given by the set of all linear or antilinear endomorphisms of V and with the group structure of composition. The map to C_2 indicates whether an element acts linearly or antilinearly, and the subgroup of even elements is exactly $\mathrm{GL}(V)$.

2. A *Real representation* of a Real group G is a complex vector space V and a morphism of Real groups $\rho : G \to \mathrm{RGL}(V)$. The category $\mathrm{RRep}(G)$ of Real representations of G is then defined by taking as morphisms $\mathbb{C}$-linear maps intertwining the action of G.

When it is clear from the context, I will sometimes just call these representations of G, and call the usual notions $\mathbb{C}$-linear or $\mathbb{R}$-linear representations.

This means that a Real representation is an action of G by either linear or antilinear maps, specified by the morphism to C_2. It is useful to notice the following: for H some group, the Real representations of $H \xrightarrow{1} C_2$ are exactly $\mathbb{C}$-linear representations of H. Thus, when talking of 'representations of H' we can always think of Real representations, since these notions coincide whenever there could be confusion.

On the other hand, any Real representation of G can be restricted to a $\mathbb{C}$-linear representation of G_0. These relations, and relations with $\mathbb{R}$-linear representations, will be investigated a bit later.

The canonical example of a Real representation is ${}^{\phi}\mathbb{C}$, which is the action of G on $\mathbb{C}$ via ϕ, interpreted as either conjugation or identity. Given two Real representations M, N there are several ways in which one can naturally construct new ones:

- The direct sum $M \oplus N$ is a Real representation through $\rho_M \oplus \rho_N$.
- The tensor product $M \otimes_{\mathbb{C}} N$ is a Real representation through $\rho(g)(m \otimes n) = \rho_M(g)(m) \otimes \rho_N(g)(n)$
- The space of $\mathbb{C}$-linear functions $\mathrm{Hom}_{\mathbb{C}}(M, N)$ is a representation through $\rho(g)(h) = \rho_N(g) h \rho_M(g^{-1})$.

A special case of the last construction is the dual representation $M^\vee := \mathrm{Hom}_{\mathbb{C}}(M, {}^{\phi}\mathbb{C})$. From now on we restrict our considerations to finite dimensional Real representations.

Proposition 1.1.4. $\mathrm{Hom}_{\mathbb{C}}(M, N) \cong N \otimes M^\vee$ as Real representations.

Proof. Let $\alpha : N \otimes M^\vee \to \mathrm{Hom}_{\mathbb{C}}(M, N)$, $n \otimes f \mapsto (m \mapsto f(m)n)$. This is intertwines the action of G:

$$\begin{aligned}\alpha(\rho(g)(n \otimes f)) &= \alpha(\rho_N(g)n \otimes \phi(g) f \rho_M(g^{-1})) = (m \mapsto \phi(g)(f(\rho_M(g^{-1})(m)))\rho_N(g)n) \\ &= (m \mapsto \rho_N(g)(f(\rho_M(g^{-1})(m))n)) = \rho_N(g)(m \mapsto f(m)n)\rho_M(g^{-1}) \\ &= \rho(g).(m \mapsto f(m)n)\end{aligned}$$

Given a basis m_i of M, denote by $\langle m_i|$ the elements of the dual basis. Then the inverse of α is given by $\beta : \mathrm{Hom}_{\mathbb{C}}(M, N) \to N \otimes M^\vee$, $h \mapsto \sum h(m_i) \otimes \langle m_i|$. $\square$

Definition 1.1.5. The *Real character* χ_M of a Real representation M of G is the map $\chi_M : G_0 \to \mathbb{C}$, $g \mapsto \mathrm{tr}(\rho(g))$.

Notice that we are taking the $\mathbb{C}$-trace and are thus forced to restrict to the even elements G_0. This turns out not to lose any information about the Real representation, which is shown later.

Real characters are compatible with the above constructions, just as in the usual case:

Proposition 1.1.6. For (finite dimensional) Real representations M, N one has $\chi_{M\otimes N} = \chi_M \chi_N$, $\chi_{M \oplus N} = \chi_M + \chi_N$, and $\chi_{M^\vee} = \chi_M^*$

Definition 1.1.7.

1. The *Real conjugation* by an element $n \in G$ is the map
$$\mathrm{Ad}_n^R : G \to G,\ g \mapsto n g^{\phi(n)} n^{-1}.$$
That is, we conjugate with n, but also take the inverse if n is odd. Notice two things: firstly, Real conjugation maps even to even, and odd to odd elements. Secondly, Real conjugation is not necessarily a group morphism.

2. Denote the set of *Real conjugacy classes* by $\mathrm{RC}(G) := G_0/G$. That is the quotient of the even elements by the Real conjugation action.
3. A *Real class function* of G is a map $f : G_0 \to \mathbb{C}$ such that $\forall n \in G : f \circ \mathrm{Ad}^R_n = f$. Equivalently, it is a function $\mathrm{RC}(G) \to \mathbb{C}$.
4. An *anti Real class function* on G is a map $f : G_0 \to \mathbb{C}$ such that $\forall n \in G : f \circ \mathrm{Ad}^R_n = \phi(n) f$, where $\phi(n)$ is to be interpreted as a sign.

While Real class functions are a natural notion, taking roughly the same place in the theory as class functions in the $\mathbb{C}$-linear case, anti Real class functions are a more exotic notion only appearing again much later.

Notice that an (anti) Real class function is always a class function of G_0.

Proposition 1.1.8. Real characters are Real class functions.

Proof. Let $g \in G_0$. Since g has finite order all eigenvalues of $\rho(g)$ are roots of unity, so lie in $U(1) \subset \mathbb{C}$. Let $v_i \in V$ be elements of a basis in which $\rho(g^{-1})$ is in Jordan normal form, λ_i the corresponding eigenvalue, denote by $\langle v_i|$ the corresponding elements of the dual basis, by $|v_i\rangle$ the linear map $\mathbb{C} \to V$, $1 \mapsto v_i$, and by K the antilinear complex conjugation map on $\mathbb{C}$. Then we have $\mathrm{id}_V = \sum |v_i\rangle \langle v_i|$ and $\mathrm{id}_{\mathbb{C}} = K^2$. I will now show $\chi_M(g) = \chi_M(\mathrm{Ad}^R_n(g))$ for the case $\phi(n) = -1$, the case $\phi(n) = 1$ is well known.

$$\begin{aligned}
\chi_M(\mathrm{Ad}^{\mathrm{R}}_n(g)) &= \mathrm{tr}\left(\rho(ng^{-1}n^{-1})\right) = \mathrm{tr}\left(\rho(n)\rho(g^{-1})\rho(n^{-1})\right) \\
&= \sum \mathrm{tr}\left(\rho(n)\, |v_i\rangle\, KK\, \langle v_i|\, \rho(g^{-1})\rho(n^{-1})\right) \\
&= \sum \mathrm{tr}\left(K\, \langle v_i|\, \rho(g^{-1})\rho(n^{-1})\rho(n)\, |v_i\rangle\, K\right) \\
&= \sum \mathrm{tr}\left(K\, \langle v_i|\, \rho(g^{-1})\, |v_i\rangle\, K\right) \\
&= \sum \mathrm{tr}\left(K(\text{times } \lambda_i^{-1})K\right) = \sum \mathrm{tr}\left(\text{times } (\lambda_i^{-1})^*\right) \\
&= \sum \lambda_i = \mathrm{tr}(\rho(g)).
\end{aligned}$$

Here I have used trace cyclicity and that all the $\lambda_i \in U(1)$. □

Let $\mathbb{K}$ be the largest subfield fixed by the action of G on $\mathbb{C}$ through ϕ, so $\mathbb{K} = \mathbb{C}$ if $G = G_0$, and $\mathbb{K} = \mathbb{R}$ else. In a Real representation all elements of G act at least $\mathbb{K}$ linear, so that the set of morphisms $\mathrm{Hom}_G(M, N)$ between two Real representations has a canonical structure of a $\mathbb{K}$ vector space.

Theorem 1.1.9. Let M, N be finite dimensional Real representations. Then

$$\dim_{\mathbb{K}} \mathrm{Hom}_G(M, N) = (\chi_M, \chi_N) := \frac{1}{|G_0|} \sum_{g \in G_0} \chi_M(g)^* \chi_N(g).$$

Proof. In the case of $\mathbb{K} = \mathbb{C}$ this is well known, since there Real representation are just complex representations. So, for $\mathbb{K} = \mathbb{R}$ consider the $\mathbb{R}$-linear representation of G

on $\operatorname{Hom}_{\mathbb{C}}(M,N)$ given by $\beta(g)(h) := \rho_N(g)h\rho_M(g^{-1})$. The set $\operatorname{Hom}_G(M,N)$ is then just the set of invariants $\operatorname{Hom}_{\mathbb{C}}(M,N)^G$, and also the $\mathbb{R}$ vector space structures match. Denote by $\psi(g) = \operatorname{tr}_{\mathbb{R}} \beta(g)$ the $\mathbb{R}$ character of β. The dimension of an invariant space, in this usual setting, is given by $\dim_{\mathbb{R}} \operatorname{Hom}_{\mathbb{C}}(M,N)^G = \frac{1}{|G|}\sum_{g\in G}\psi(g)$. But β is also a Real representation, and we have the following relation between ψ its Real character χ_β:

$$\psi(g) = \begin{cases} \chi_\beta(g) + \chi_\beta(g)^* & g \in G \\ 0 & \text{else} \end{cases}$$

This can be proven, actually for any Real representation and the associated $\mathbb{R}$ representation of G, by considering a concrete $\mathbb{C}$ basis $(v_1, ..., v_n)$ of the representation space and the associated $\mathbb{R}$ basis $(v_1, iv_1, ..., v_n, iv_n)$. Let $(\lambda_1, ..., \lambda_n)$ be the diagonal of the representation of g in the first basis. If g acts $\mathbb{C}$-linear, then its diagonal in the second basis will be

$$(\mathrm{Re}\lambda_1, \mathrm{Re}\lambda_1, ..., \mathrm{Re}\lambda_n, \mathrm{Re}\lambda_n).$$

If g acts antilinear, then its diagonal in the second basis will be

$$(\mathrm{Re}\lambda_1, -\mathrm{Re}\lambda_1, ..., \mathrm{Re}\lambda_n, -\mathrm{Re}\lambda_n).$$

We calculate:

$$\begin{aligned} \dim_{\mathbb{R}} \operatorname{Hom}_{\mathbb{C}}(M,N)^G &= \frac{1}{|G|}\sum_{g\in G}\psi(g) = \frac{1}{2|G_0|}\sum_{g\in G_0}\chi_\beta(g) + \chi_\beta(g)^* \\ &= \frac{1}{2|G_0|}\sum_{g\in G_0}\chi_N(g)\chi_M(g)^* + \chi_N(g)^*\chi_M(G) \\ &= \frac{1}{2}\left((\chi_M, \chi_N) + (\chi_N, \chi_M)\right) \end{aligned}$$

But, since χ_M, χ_N are also characters of complex representations of G_0 (one just forgets the action of the odd elememnts), (χ_M, χ_N) is an integer ≥ 0. The scalar product on class functions is hermitean, so the result follows. $\square$

The above is essentially a comparison between $\mathbb{R}$-dimensions of invariant spaces for G and G_0. I want to make this explicit since it will be useful later:

Proposition 1.1.10. Let V be a (finite dimensional) Real representation of G. Then we have

$$\dim_{\mathbb{R}} V^G = \frac{1}{|G/G_0|}\dim_{\mathbb{R}} V^{G_0}.$$

In particular we have $V^G = 0$ if and only if $V^{G_0} = 0$.

Proof. Again, let ψ be the character of V as an $\mathbb{R}$-linear representation of G. We have seen above that ψ is zero on $G \setminus G_0$, so that

$$\dim_{\mathbb{R}} V^G = \frac{1}{|G|}\sum_{g\in G}\psi(g) = \frac{1}{|G/G_0|}\frac{1}{|G_0|}\sum_{g\in G_0}\psi(g) = \frac{1}{|G/G_0|}\dim_{\mathbb{R}} V^{G_0}.$$

$\square$

1.2 Structure of Real representations

We now start to investigate the 'global' structure of Real representations, and how they relate to various other notions of representation. Most importantly, we will see in which sense the theory puts $\mathbb{R}$ and $\mathbb{C}$-linear representation on the same footing.

Just as usual (complex) representation theory can be viewed as the theory of $\mathbb{C}G$ modules, the Real representation theory is the theory of ${}^{\phi}\mathbb{C}G$ modules.

Proposition 1.2.1. The functor $F : {}^{\phi}\mathbb{C}G\mathbf{mod} \to \mathrm{RRep}(G)$, defined by $F(M) = (M, \rho(g) = (m \mapsto x_g m))$ and identity on morphisms, is an equivalence of categories.

The proof is formally the same as in the $\mathbb{C}$-linear case. Notice that M is finitely generated if and only if $F(M)$ has finite dimension. Sometimes the terminology of representations, other times the terminology of modules will be more useful to present the theory. Thus I will use the terminologies of modules and representations kind of interchangeably. Just as in the complex case, modules decompose into sums of nicely behaved indecomposables, which is the content of the following statement.

Theorem 1.2.2 (Maschke)**.** ${}^{\phi}\mathbb{C}G$ is a semisimple ring. That means that every module is the direct sum of irreducible ones.

Proof. We will proof that for every module $M \in {}^{\phi}\mathbb{C}G\mathbf{mod}$ and every submodule $A \subset M$, there is a module B such that $A \oplus B \cong M$. Let P be a linear projector onto A, that is $PM \subset A$ and $P|_A = \mathrm{id}_A$. The average $P' = \frac{1}{|G|}\sum_{g \in G} \rho(g) P \rho(g^{-1})$ then intertwines the G action and still is a projector onto A. Also note that it is still $\mathbb{C}$-linear, since each term contains either no or two antilinear operators. Thus we have $M \cong A \oplus \mathrm{im}(1 - P')$ given by $m \mapsto P'm \oplus (1 - P')m$. $\square$

The simple/irreducible modules have the following properties, making them especially easy to study.

Proposition 1.2.3. Let M be a simple module of ${}^{\phi}\mathbb{C}G$, then M is finitely generated. In particular, M is finite dimensional as a Real representation.

Lemma 1.2.4 (Schur)**.** Let M, N be simple Real representations of G. If they are not isomorphic, then $\mathrm{Hom}_G(M, N) = 0$. The endomorphism ring $\mathrm{Hom}_G(M, M)$ is a real division algebra, and thus isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ as an $\mathbb{R}$-algebra. We say that M is of real, complex, or quaternionic type, accordingly.

The proofs of these are basic results on modules of semisimple rings, but with added details for the case at hand. In particular, the last statement in Schur's lemma uses the classification of finite dimensional real division algebras.

I will now describe the standard decomposition for Real representations. For this, fix a representative S_i for every isomorphism class of simple Real representations, and let $E_i = \mathrm{Hom}_G(S_i, S_i)$ be their endomorphism rings.

Lemma 1.2.5 (Standard decomposition)**.** Let M be a finite dimensional Real representation of G. Then the evaluation gives a natural isomorphism

$$\oplus_i \operatorname{Hom}_G(S_i, M) \otimes_{E_i} S_i \to M$$

For clarification, we view $\operatorname{Hom}_G(S_i, M)$ as a right E_i module by precomposition, and S_i as a left E_i module by applying the endomorphism. The action of G is then by acting on S_i, which is well defined since we are tensoring over maps intertwining the G action. The appearance of E_i compared to $\mathbb{C}$ in the complex case comes from the following: in the complex case $E_i = \mathbb{C}\,\mathrm{id}_{S_i}$, which makes live and notation a lot easier. Finally, this really is a decomposition into sums of the S_i: All E_i modules are free, so the tensor product is a sum of $\dim_{E_i} \operatorname{Hom}_G(S_i, M)$ copies of S_i.

Proof. First we need to show that evaluation factors over the tensor product, so let $f \in \operatorname{Hom}_G(S_i, M)$, $s \in S_i$ and $e \in E_i$. Clearly $(f.e)(s) = f(e.s) = f(e(s))$, so the evaluation is E_i-bilinear.

Next, this is a morphism of Real representations: $g.(f \otimes s) = f \otimes g.s \mapsto f(g.s) = g.f(s)$.

Finally, this definition is clearly natural in M and takes direct sums in M into direct sums of morphisms, so by Maschke's theorem we can reduce to the case $M = S_j$. In this case we use Schur's lemma to see that we are reduced to the morphism $E_j \otimes_{E_j} S_j \to S_j$ sending $f \otimes s$ to $f(s)$. This has the inverse $s \mapsto \mathrm{id}_{S_i} \otimes s$, so it is an isomorphism. ☐

Let me also state an immediate result about the Real characters of simple representations:

Proposition 1.2.6. Let M, N be non isomorphic simple Real representations. Then $(\chi_M, \chi_N) = 0$.

The proof is an easy application of Schur's lemma and Theorem 1.1.9.

We will now study how Real representations of G relate to complex representations of G_0. One has a natural morphism of Real groups $G_0 \to G$, where one views G_0 as a Real with no odd elements. By functoriality of Real group rings we get a morphism $f : \mathbb{C}G_0 \cong {}^{1}\mathbb{C}G_0 \to {}^{\phi}\mathbb{C}G$, $i \mapsto i$, $x_g \mapsto x_g$. Then, as usual, we have the adjunctions $f^e \dashv f^* \dashv f_*$ between the associated extension, restriction, and corestriction functors. Also, with the choice of monoidal structure presented above, these are strictly monoidal and additive functors. The interaction of characters and the extension and restriction functors, for a general inclusion of (Real) subgroups is explained in the following

Lemma 1.2.7. Let $f : H \to G$ be the inclusion of a (Real) subgroup, M a finitely generated ${}^{\phi}\mathbb{C}G$ module and N a finitely generated ${}^{\phi}\mathbb{C}H$ module.

1. f^*M is finitely generated and $\chi_{f^*M} = \chi_M|_{H_0}$.
2. f^eN is finitely generated and we have $\chi_{f^eN}(g) = \sum_{[x]\in G/H} \chi_N(\mathrm{Ad}^R_{x^{-1}}(g))$ with the convention that χ_N is 0 outside of H.

Proof. 1. This is purely formal since the vector space structure and action of the groups stay the same.

2. By the induced ring morphism $f : {}^{\phi}\mathbb{C}H \to {}^{\phi}\mathbb{C}G$ we view ${}^{\phi}\mathbb{C}G$ as a right ${}^{\phi}\mathbb{C}H$ module. As such it is a free module of dimension $|G/H|$. For each class in G/H pick a representative t_i. We then have an isomorphism $\left({}^{\phi}\mathbb{C}H\right)^{G/H} \to {}^{\phi}\mathbb{C}G, u \mapsto \sum t_i u([t_i])$ of right modules.

Let's investigate the action of G on $f^e N = {}^{\phi}\mathbb{C}G \otimes_{{}^{\phi}\mathbb{C}H} N$ in a particular basis. Let n_j be a basis of N, then $t_i \otimes n_j$ is a basis of $f^e N$, and let $g \in G_0$. Then we have that

$$\begin{aligned} g.(t_i \otimes n_j) &= \sum_k \{t_k^{-1} g t_i \in H\} t_k \otimes t_k^{-1} g t_i n_j = \sum_{lk} \{t_k^{-1} g t_i \in H\} t_k \otimes (t_k^{-1} g t_i)_{jl} n_l \\ &= \sum_{lk} \{t_k^{-1} g t_i \in H\} \phi(t_k) \left((t_k^{-1} g t_i)_{jl}\right) t_k \otimes n_l \end{aligned}$$

where $\{t_k^{-1} g t_i \in H\}$ is 1 if the membership relation is true and 0 else, and $\phi(t_k)$ is to be interpreted as an endomorphism of $\mathbb{C}$. The sum of diagonal elements is the given by

$$\begin{aligned} \chi_{f^e N}(g) &= \sum_i \{t_i^{-1} g t_i \in H\} \sum_j \phi(t_i) \left((t_i^{-1} g t_i)_{jj}\right) \\ &= \sum_i \{t_i^{-1} g t_i \in H\} \phi(t_i) \left(\chi_N(t_i^{-1} g t_i)\right) = \sum_i \{t_i^{-1} g t_i \in H\} \chi_N(\mathrm{Ad}^R_{t_i^{-1}}(g)) \end{aligned}$$

where I again used that the eigenvalues of these actions all lie in $U(1)$, so that their conjugates are their inverses. This formula is independent of the chosen representatives since χ_N is a Real class function w.r.t. H. □

The interplay between f^* and f^e for f the inclusion $G_0 \to G$ is of great technical importance. The next two propositions capture this in the detail needed.

Proposition 1.2.8. Let f be the inclusion $G_0 \to G$, and assume $G_0 \neq G$, for example $w \in G \setminus G_0$. For $M \in \mathbb{C}G_0\mathbf{mod}$ we have an isomorphism $f^* f^e M \cong M \oplus w \star \overline{M}$. Here $w \star \overline{M}$ is the conjugate vector space of M, and $g \in G_0$ acts like wgw^{-1} does on M[1].

Proof. So we are comparing the $\mathbb{C}G_0$ modules $M \oplus w \star \overline{M}$ and ${}^{\phi}\mathbb{C}G \otimes_{\mathbb{C}G_0} M$. The isomorphism is given by

$$M \oplus w \star \overline{M} \to {}^{\phi}\mathbb{C}G \otimes_{\mathbb{C}G_0} M,\ a \oplus w \star \bar{b} \mapsto 1 \otimes a + w^{-1} \otimes b.$$

That this is $\mathbb{C}$-linear and G_0 equivariant follows directly from the definition of $w \star \overline{M}$.

An explicit inverse is given in the following way. An element of $G \setminus G_0$ is uniquely of the form $w^{-1}g$, $g \in G_0$. The inverse is then

$$g \otimes m \mapsto g.m \in M,\ w^{-1} g \otimes m \mapsto w \star \overline{g.m} \in w \star \overline{M},$$

where $g \in G_0$. That this is well defined and an inverse is left to the reader. □

[1] For clarity I sometimes denote the elements of $w \star \overline{M}$ by $w \star \bar{b}$ with $b \in M$. So the definition means $g.(w \star \bar{b}) = w \star \overline{wgw^{-1}.b}$.

Proposition 1.2.9. Let f be the inclusion $G_0 \to G$. For $M \in {}^{\phi}\mathbb{C}G\mathbf{mod}$ we have an isomorphism $f^e f^* M \cong \bigoplus_{G/G_0} M$.

Proof. If $G = G_0$, then the functors f^e, f^*, f_* are all naturally isomorphic to the identity functor $\mathrm{id}_{\mathbb{C}G\mathbf{mod}}$, so the statement is formal.

Else, pick an odd element $\omega \in G \backslash G_0$. The isomorphism $\alpha : f^e f^* M = {}^{\phi}\mathbb{C}G \otimes_{\mathbb{C}G_0} f^* M \to M \oplus M$ is given by $\alpha(a \otimes m) := a.m \oplus a.(im)$. This is well defined since, for $b \in \mathbb{C}G_0$, we have

$$\begin{aligned}\alpha(ab \otimes m) &= (ab).m \oplus (ab).(im) = a.(b.m) \oplus a.(i(b.m)) \\ &= \alpha(a \otimes b.m).\end{aligned}$$

Its inverse is given by $\beta(n \oplus m) := 1 \otimes \frac{n-im}{2} + \omega^{-1} \otimes \omega.\left(\frac{n+im}{2}\right)$. □

With this at hand we can prove the following key lemma:

Lemma 1.2.10. Let $f : G_0 \to G$ be the inclusion of the even elements, then $f^* M \cong f^* N$ implies $M \cong N$.

This is, to me, the most surprising result in this topic: The Real representations are already determined by their restriction to G_0!

Proof. Again the case $G = G_0$ is formal. Else, $f^* M \cong f^* N$ implies $M \oplus M \cong f^e f^* M \cong f^e f^* N \cong N \oplus N$. For S a simple ${}^{\phi}\mathbb{C}G$ module, let a_S, b_S be the cardinal numbers of copies of (the isomorphy class of) S in the simple decomposition of M, N. By Schur's lemma these are well defined. By the above, we have that $2a_S = 2b_S$, so that $a_S = b_S$. So the simple decompositions of M and N are equal, so $M \cong N$. □

Let G be just a group for now. We have already seen that ${}^{1}\mathbb{C}G \cong \mathbb{C}G$, so that the Real representation theory of $(G, \phi_G = 1)$ is the $\mathbb{C}$-linear representation theory of G. On the other hand, as mentioned earlier, we have ${}^{\pi_2}\mathbb{C}(G \times C_2) \cong \mathrm{Mat}_{2\times 2}(\mathbb{R}G)$. This last ring is *Morita-equivalent* to $\mathbb{R}G$, which means there is an additive equivalence $\mathbb{R}G\mathbf{mod} \to \mathrm{Mat}_{2\times 2}(\mathbb{R}G)\mathbf{mod}$. In this case it is given by sending M to $M \oplus M$ (with the obvious action of 2 by 2 matrices). To see how this works on the level of Real group rings, we need to understand the isomorphism:

Proposition 1.2.11. Denote, for now, the generator of C_2 as T, so that all elements of $G \times C_2$ are of the form g or gT, with $g \in G$. The isomorphism ${}^{\pi_2}\mathbb{C}(G \times C_2) \cong \mathrm{Mat}_{2\times 2}(\mathbb{R}G)$ is given by sending

$$x_g \mapsto \begin{pmatrix} x_g & 0 \\ 0 & x_g \end{pmatrix},\ ix_g \mapsto \begin{pmatrix} 0 & -x_g \\ x_g & 0 \end{pmatrix},\ x_{gT} \mapsto \begin{pmatrix} x_g & 0 \\ 0 & -x_g \end{pmatrix},\ ix_{gT} \mapsto \begin{pmatrix} 0 & x_g \\ x_g & 0 \end{pmatrix}$$

and extending $\mathbb{R}$-linearly

Proof. The proof reduces to the case of $G = \{ 1 \}$, where it is a standard result. To see this reduction, note that the Real group ring of $G \times C_2$ can also be written as

$$\begin{aligned} {}^{\pi_2}\mathbb{C}(G \times C_2) &\cong \frac{\mathbb{R}[i, T, \{ x_g \mid g \in G \}]}{(i^2 + 1, T^2 - 1, Ti + iT, ix_g - x_g i, Tx_g - x_g T, x_g x_h - x_{gh})} \\ &\cong \frac{\mathbb{R}[i, T]}{(i^2 + 1, T^2 - 1, Ti + iT)} \otimes_{\mathbb{R}} \mathbb{R}G \end{aligned}$$

It is a standard result that first factor above is the ring of 2 by 2 $\mathbb{R}$ matrices, with the isomorphism sending

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \, i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \, T \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \, iT \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and extending $\mathbb{R}$-linearly.

□

Unraveling things now, we see that the Morita equivalence between $\mathbb{R}G$**mod** and ${}^{\pi_2}\mathbb{C}(G \times C_2)$**mod** is, in terms of representations, given by sending M to $M \otimes_{\mathbb{R}} \mathbb{C}$ and letting $n \in G \times C_2$ act like

$$n.(m \otimes \lambda) := (\pi_1(n).m) \otimes \phi(n)(\lambda).$$

In particular, we can deduce that this equivalence is not only additive, but also monoidal: It maps tensor products of $\mathbb{R}$-linear representations to tensor products of Real representations! This shows that the Real representation theory of $(G \times C_2, \pi_2)$ is the $\mathbb{R}$-linear representation theory of G.

1.3 Real representation rings

One is often interested in assigning ring elements to representations in an additive and multiplicative manner. One such assignment, for example, is 'taking the character'

$$\chi : \mathrm{RRep}(G) \to \text{Real class functions}.$$

It turns out, just as in the $\mathbb{R}$ and $\mathbb{C}$-linear cases, that there is an universal such assignment, which takes values in the so-called Real representation ring:

Definition 1.3.1. The *Real representation ring RG* associated to a Real group G has as its elements formal differences of finite dimensional Real representations, that is pairs of isomorphism classes of finite dimensional Real representations divided by the equivalence relation

$$([A], [B]) \sim ([N], [M]) :\iff A \oplus M \cong N \oplus B.$$

One writes $[A] \in RG$ for the class of $([A], [0])$. This is a group by direct sum and a commutative ring by taking the tensor product: $[A]+[B] = [A \oplus B]$ and $[A][B] = [A \otimes B]$. The additive inverse of $[A]$ is the class of $([0], [A])$, so that the class of $([A], [B])$ is $[A]-[B]$. The unit of multiplication is the element $[{}^{\phi}\mathbb{C}]$.

Remark. One usually defines the above equivalence relation, when applying it to general rings, by asking only for an isomorphism after adding some other representation/module to it. In the case of Real representations of it is actually the same since, by counting simple summands, one finds

$$A \oplus T \cong B \oplus T \implies A \cong B.$$

Another reason, for us in particular, to study these rings is that the *Real equivariant K groups* introduced in chapter 2 are modules over it. Thus, knowing the structure of this ring can yield powerful tools to study these K groups.

In the case $G = G_0$ this is the usual representation ring of G_0, and I will not distinguish between them in my notation. Let $f : G \to H$ be a morphism of Real groups. As we have seen earlier, both extending and restricting representations along f are additive, monoidal, and preserve finite dimensionality. Thus these descend to morphisms between the representation rings

$$Rf^* : RH \to RG,\ Rf^e : RG \to RH.$$

In the special case of the inclusion $f : G_0 \to G$ we have seen that the restriction of representations creates/reflects isomorphisms (Lem. 1.2.10). Thus the induces morphism on Representation rings is injective!

These next two proposition will be of technical importance in the next section.

Proposition 1.3.2. For Real representation M one has $\chi_M = \chi_{f^*M}$, where the first is the Real and the second is the $\mathbb{C}$-linear character. Thus we have a commutative square of rings:

$$\begin{tikzcd}
RG \arrow[r, "\text{Real character}", "{[A]-[B]\mapsto\chi_A-\chi_B}"'] \arrow[d, hook, "Rf^*"'] & \text{Real class functions} \arrow[d, hook] \\
RG_0 : \arrow[r, hook, "\text{character}", "{[A]-[B]\mapsto\chi_A-\chi_B}"'] & \text{class functions}
\end{tikzcd}$$

This shows that the 'Real character' morphism in the above diagram is also injective.

With this one can view the Real representation ring as a subring of the Real class functions, and I will do so regularly.

Proposition 1.3.3. *RG is finitely generated as a $\mathbb{Z}$ module.*

Proof. Let S be the set of isomorphism classes of simple ${}^{\phi}\mathbb{C}G$ modules. By Maschke's Theorem 1.2.2 the morphism $\mathbb{Z}^S \to RG$, $f \mapsto \sum_{s\in S} f(s)s$ is surjective. But by Proposition 1.2.6 the characters of different simple modules are linearly independent vectors in $\mathrm{Set}(G, \mathbb{C})$, which is finite dimensional. So $|S| \leq +\infty$ and RG is finitely generated. $\square$

In particular, RG is finitely generated over any ring mapping into it. This shows that, as a ring, RG is a quotient of a polynomial ring over $\mathbb{Z}$ (with one polynomial variable for each element of S). Thus we get the following

Corollary. RG is noetherian.

1.4 The prime spectrum of Real representation rings

Having introduced the basics of Real representation theory, I will now generalize Segal's results on the representation ring of compact Lie groups [2]to the case of finite Real groups. I believe that most of the results could also be proven for an appropriate notion of compact 'Real Lie group', but I have not studied this direction too much. The results about the prime ideal structure of RG we obtain here will allow us to prove a powerful theorems about the (algebraic) localizations and completions of the K groups to be introduced later.

The main idea here is to study, given some prime ideal $\mathfrak{p} \subset RG$, the minimal (w.r.t. inclusion) subgroups S of G such that $\mathfrak{p}$ is the preimage of some prime ideal of RS along the restriction map $RG \to RS$[2]. These turn out to be cyclic subgroups of G_0, and any two are related by Real conjugation.

Let $\mathcal{S} : \mathrm{CRing} \to \mathrm{Sch}^{\mathrm{op}}$ denote the prime spectrum functor. I will regularly forget a lot of the structure of a scheme without mentioning it.

Definition 1.4.1. Let $H \subset G$ be some subgroup and $\mathfrak{p} \in \mathcal{S}RG$. I say that $\mathfrak{p}$ *comes from* H if and only if $\mathfrak{p} \in \mathrm{im}\,(\mathcal{S}RH \to \mathcal{S}RG)$.

The following characterization of 'coming from' will be useful:

Proposition 1.4.2. View RH as an RG module through the restriction. Then $\mathfrak{p}$ comes from $H \iff RH_{\mathfrak{p}} \neq 0$.

Proof. Denote the map of underlying sets induced by the morphism on spectra by $q : \mathcal{S}RH \to \mathcal{S}RG$. Then $\mathfrak{p}$ comes from H iff $q^{-1}(\mathfrak{p}) \neq \emptyset$. $q^{-1}(\mathfrak{p})$ is the underlying set of the following pullback of Schemes:

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{S}RH \arrow[d] \\ \mathcal{S}k_{\mathfrak{p}} \arrow[r] & \mathcal{S}RG \end{tikzcd}$$

Since only affine schemes are involved, we have that $A \cong \mathcal{S}\,(RH \otimes_{RG} k_{\mathfrak{p}})$, so $q^{-1}(\mathfrak{p}) \neq \emptyset \iff RH \otimes_{RG} k_{\mathfrak{p}} \neq 0$. By Nakayama's lemma, since RH is finitely generated as an RG module (Prop. 1.3.3), this is equivalent to $RH_{\mathfrak{p}} \neq 0$. $\square$

[2]That is the natural map of restricting a representation of G to one of $S \subset G$.

Lemma 1.4.3. Consider the morphism $RG \to \prod_S RS$ induced by the restrictions, where S runs over all cyclic even subgroups of G. It is finite and injective, so induces a surjective map on spectra.

Proof. Finiteness comes from Proposition 1.3.3 and that the number of cyclic even subgroups of G is finite. Injectivity follows from the injectivity of the Real character morphism (1.3.2): If $[A] - [B] \neq 0$, then $\chi_A - \chi_B \neq 0$, so $\exists g \in G_0 : \chi_A(g) - \chi_B(g) \neq 0$. This shows that the restriction of $[A] - [B]$ to the cyclic even subgroup generated by g is nonzero. Surjectivity on spectra now follows from the Cohen-Seidenberg theorems, see Theorem 5.10 in [28]. $\square$

So if a prime comes from H, it comes from some cyclic even subgroup of H, so the minimal subgroups, such that a prime comes from them, are cyclic even.

It is now useful to restrict our attention to one characteristic at a time. So let q be 0 or a prime number. Following usual convention I will say a prime ideal $\mathfrak{a}$ of any ring A is *above* q if it maps to (q) in the unique morphism of schemes $\mathcal{S}A \to \mathcal{S}\mathbb{Z}$. Also I shall denote by $\mathbb{F}_q$ the initial field of characteristic q. Then the primes of A above q are exactly the primes of $A \otimes \mathbb{F}_q$. From now on let $\mathfrak{p}$ be a prime above q.

Lemma 1.4.4. Let S be a minimal subgroup such that $\mathfrak{p}$ comes from it. Then $q \nmid |S|$.

Proof. The case $q = 0$ is trivial, so let $q > 0$. We know already that S is cyclic even, so assume $q \mid |S|$ for a proof of negation. Then by the Chinese remainder theorem there is some factorization $S \cong T \times \frac{\mathbb{Z}}{q^k\mathbb{Z}}$ $(k > 0)$ and $RS \cong \frac{RT[X]}{(X^{q^k}-1)}$. Since $\mathfrak{p}$ is above q, it must come from a prime of RS which is above q as well. This means it comes from a prime of

$$RS \otimes \mathbb{F}_q \cong \frac{(RT \otimes \mathbb{F}_q)[X]}{(X-1)^{q^k}}.$$

The restriction morphism $RS \to RT$ is given by sending $X \mapsto 1$. After tensoring with $\mathbb{F}_q$ the restriction is surjective and has nilpotent kernel, so it induces a bijection on spectra. This shows that every prime above q, which comes from S, also comes from the proper subgroup T, which contradicts minimality of S. $\square$

Following Segal I will call a cyclic group S q-regular if $q \nmid |S|$. Let $\mathfrak{p}_S \subset RS$ be the prime ideal of elements whose characters vanish at some generator (and thus all generators) of S. Denote the quotient by $\widetilde{RS} := \frac{RS}{\mathfrak{p}_S}$.

Lemma 1.4.5.

1. Via an isomorphism $RS \cong \frac{\mathbb{Z}[X]}{(X^{|S|}-1)}$ we have that $\widetilde{RS} \cong \frac{\mathbb{Z}[X]}{(\Phi_{|S|})}$, where Φ_n is the nth cyclotomic polynomial.
2. If S is q-regular we have that $RS \otimes \mathbb{F}_q \to \prod_T \widetilde{RT} \otimes \mathbb{F}_q$, where the product goes over all subgroups of S, is an isomorphism.

Proof. 1. Let $|S| = N$ and $t \in S$ be some generator. Looking at the morphism $\frac{\mathbb{Z}[X]}{(X^N-1)} \to RS \to \mathrm{Set}(S, \mathbb{C}) \xrightarrow{\mathrm{ev}_t} \mathbb{C}$, X is mapped to some primitive Nth root of unity ζ_N. So the kernel of this is the set of all polynomials with integer coefficients which vanish at ζ. This is exactly (Φ_N). This does not depend on the generator considered, so we conclude.

2. Again fixing some isomorphism $RS \cong \frac{\mathbb{Z}[X]}{(X^N-1)}$, the morphism we consider is given by $\frac{\mathbb{F}_q[X]}{(X^N-1)} \to \prod_{d|N} \frac{\mathbb{F}_q[X]}{(\Phi_d)}$ where $q \nmid N$. By the Chinese remainder theorem, since $X^N - 1 = \prod_{d|N} \Phi_d$, the above is an isomorphism iff $(\Phi_a) + (\Phi_b) = (1)$ in $\mathbb{F}_q[X]$ for $a, b \mid N$, $a < b$. By Theorem 2 in [3] there are integer polynomials u, v such that

$$u\Phi_a + v\Phi_b = \begin{cases} p & b/a = p^t \text{ for a prime } p \\ 1 & \text{else.} \end{cases}$$

Since q divides neither a nor b this will always be a unit in $\mathbb{F}_q$. □

Proposition 1.4.6. If S is q-regular the image of $\mathcal{S}(\widetilde{RS} \otimes \mathbb{F}_q)$ in $\mathcal{S}(RS \otimes \mathbb{F}_q)$ is exactly the set of primes of RS above q which do not come from any proper subgroup of S.

Proof. The proof goes by induction on the order of S. The base case is $|S| = 1$. The only group with this property is the trivial group (which is q-regular). Here we have, by Lemma 1.4.5, the isomorphism $RS \otimes \mathbb{F}_q \cong \widetilde{RS} \otimes \mathbb{F}_q$. Since the trivial group does not have any proper subgroups the claim follows.

For the induction step assume the statement for any cyclic group with order less than $|S|$. Again by Lemma 1.4.5, we have a bijection $\mathcal{S}(RS \otimes \mathbb{F}_q) \cong \amalg_T \mathcal{S}(\widetilde{RT} \otimes \mathbb{F}_q)$. Note that any subgroup T of S will again be q-regular, and if it is a proper subgroup one even has $|T| < |S|$. So the primes of $RS \otimes \mathbb{F}_q$ decompose as the primes of $\widetilde{RS} \otimes \mathbb{F}_q$ and the primes of proper subgroups which do not come from any 'second level' proper subgroups. So the primes of $\widetilde{RS} \otimes \mathbb{F}_q$ must be the ones of $RS \otimes \mathbb{F}_q$ which do not come from any proper subgroup. □

With this the statement of Lemma 1.4.3 can be refined to the following

Lemma 1.4.7. Consider the morphism $RG \otimes \mathbb{F}_q \to \prod_S \widetilde{RS} \otimes \mathbb{F}_q$, where S runs over all cyclic even q-regular subgroups of G. It induces a surjective map on spectra.

Proof. By the results of Lemma 1.4.4, 1.4.3, and Proposition 1.4.6 every prime of $RG \otimes \mathbb{F}_q$ comes from some minimal cyclic even q-regular subgroup S of G. Since it is minimal, the prime does not come from a subgroup of S, so it is in the image $\mathcal{S}(\widetilde{RS} \otimes \mathbb{F}_q)$. □

If $\mathfrak{p} \in \mathcal{S}(RG \otimes \mathbb{F}_q)$ is the image of both $\mathfrak{a} \in \mathcal{S}(\widetilde{RA} \otimes \mathbb{F}_q)$ and $\mathfrak{b} \in \mathcal{S}(\widetilde{RB} \otimes \mathbb{F}_q)$, with A and B minimal cyclic even q-regular subgroups, what is their relation? The claim is that this happens iff these primes are related by Real conjugation with some $n \in G$. The motivation for this can be seen in the classical (non-Real) case: for $H \subset G$ a pair of groups the restriction $RG \to RH$ factors over invariants RH^{W_H} of RH with respect to the Weyl group of H in G. This is of course since, in the language of characters, the characters of G restricted to H are not only invariant w.r.t. conjugation with elements of H, but also with elements of G which fix the subgroup H.

Definition 1.4.8.

1. The *Real centralizer* of $g \in G$ is $Z_g^R := \{n \in G | \operatorname{Ad}_n^R(g) = g\}$.
2. The *Real normalizer* of a (Real) subgroup H of G is the set
$$N_H^R := \left\{ n \in G \;\middle|\; \operatorname{Ad}_n^{\mathrm{R}}(H) = H \right\}$$
It is a (Real) subgroup of G, contains H, and H is normal in N_H^R (as plain groups).
3. The *Real Weyl group* is then the quotient $W_H^R := N_H^R/H$. Note that this is only (naturally) a Real group if H is an even subgroup of G.

Let S be a cyclic even subgroup of G. Then, for all $n \in N_S^R$, $\operatorname{Ad}_n^{\mathrm{R}} : S \to S$ is an automorphism of groups[3], and identity if $n \in S$. So we get a natural action of W_S^R on S, and thus on RS. In terms of characters, these send χ_M to $\chi_M \circ \operatorname{Ad}_{n^{-1}}^{\mathrm{R}}$. This action even descends onto an action on $\widetilde{RS} \otimes \mathbb{F}_q$, for S q-regular. The next lemma is a collection of similar technical results on this:

Lemma 1.4.9.

1. The kernel of the composition $RS \xrightarrow{\operatorname{Ad}_n^{\mathrm{R}}} RS \to \widetilde{RS}$ is $\mathfrak{p}_S$, so the action descends to one of W_S^R on $\widetilde{RS}$.
2. The natural morphism $RS^{W_S^R} \otimes \mathbb{F}_q \to (RS \otimes \mathbb{F}_q)^{W_S^R}$ is an isomorphism.
3. If S is q-regular then the natural morphism $\widetilde{RS}^{W_S^R} \otimes \mathbb{F}_q \to (\widetilde{RS} \otimes \mathbb{F}_q)^{W_S^R}$ is an isomorphism.
4. If S is q-regular then the natural morphism $RS^{W_S^R} \otimes \mathbb{F}_q \to \widetilde{RS}^{W_S^R} \otimes \mathbb{F}_q$ is surjective.

Proof. Let me give the proof in some detail, since in [2] this is only mentioned in a subsentence.

1. The kernel of the composition consists of those elements whose characters vanish on $\operatorname{Ad}_{n^{-1}}^R(t)$ for t some generator of S. Since $\operatorname{Ad}_{n^{-1}}^R : S \to S$ is an automorphism $\operatorname{Ad}_{n^{-1}}^R(t)$ is also a generator of S. So the kernel consists of elements vanishing on some generator of S, so it equals $\mathfrak{p}_S$.

2. It suffices to show that it is a bijection of sets. The key insight is that $RS \cong \mathbb{Z}^{|S|}$ and that W_S^R acts by permuting the different summands. Explicitly, let $\hat{S} = \operatorname{Hom}(S, U(1))$. Interpreting the elements of $\hat{S}$ as one dimensional representations of S, these are exactly the simple representations. So, as we have seen earlier, $RS \cong \operatorname{Set}(\hat{S}, \mathbb{Z})$. W_S^R acts on this

[3] Since S is abelian, taking inverses is a group morphism.

by permuting $\hat{S}$, so that the we have $RS^{W_S^R} \cong \mathrm{Set}(\hat{S}, \mathbb{Z})^{W_S^R} = \mathrm{Set}(\hat{S}/W_S^R, \mathbb{Z})$. Tensoring this with $\mathbb{F}_q$ gives the isomorphism $RS^{W_S^R} \otimes \mathbb{F}_q \cong \mathrm{Set}(\hat{S}/W_S^R, \mathbb{F}_q)$.

On the other hand, we also have $RS \otimes \mathbb{F}_q \cong \mathrm{Set}(\hat{S}, \mathbb{F}_q)$ and W_S^R still acts by permuting $\hat{S}$, so that $(RS \otimes \mathbb{F}_q)^{W_S^R} \cong \mathrm{Set}(\hat{S}/W_S^R, \mathbb{F}_q)$.

Under these isomorphisms the morphism in question turns into the identity, so that it is clearly bijective.

4. Elements of $\widetilde{RS}^{W_S^R}$ are, basically, elements of RS whose characters only change on non-generating elements when acting with W_S^R. The idea is now to construct an element $\theta \in RS$ which is invariant, vanishes on non-generating elements, and takes the constant value $|S|$ on generators. Then, for $[y] \otimes 1 \in \widetilde{RS}^{W_S^R} \otimes \mathbb{F}_q$, $y\theta$ is an element of $RS^{W_S^R}$ and $[y\theta] = |S|[y]$ in $\widetilde{RS}$. Since $|S|$ is a unit in $\mathbb{F}_q$ $y\theta \otimes \frac{1}{|S|}$ is a preimage of $[y] \otimes 1$. Such an element θ is given by

$$\theta = \prod_{n=1}^{|S|-1} (1 - X^n)$$

where X is a representation on $\mathbb{C}$ sending some generator of S to multiplication by $e^{\frac{2\pi i}{N}}$. Clearly it's character vanishes on non generators, and is constant on generators. To compute that constant value consider the polynomial

$$p(t) = \prod_{n=0}^{|S|-1} (1 - e^{\frac{2\pi i n}{|S|}} t) = 1 - t^{|S|}.$$

The value we are after is then $\lim_{t\to 1} \frac{p(t)}{1-t} = \lim_{t \to 1} \frac{-|S|t^{|S|-1}}{-1} = |S|$ where I used L'Hopital's rule.

3. Let us first show that the morphism is surjective. Recall from part 2 of Lemma 1.4.5 that

$$RS \otimes \mathbb{F}_q \to \prod_{T \subset S} \widetilde{RT} \otimes \mathbb{F}_q$$

is an isomorphism. The action of W_S^R will then leave the factor at $S \subset S$ fixed, so that we have an isomorphism

$$(RS \otimes \mathbb{F}_q)^{W_S^R} \to (\widetilde{RS} \otimes \mathbb{F}_q)^{W_S^R} \prod (\prod_{T \subsetneq S} \widetilde{RT} \otimes \mathbb{F}_q)^{W_S^R}.$$

Now consider this commutative diagram:

$$\begin{tikzcd}
(RS \otimes \mathbb{F}_q)^{W_S^R} \arrow[rr] & & (\widetilde{RS} \otimes \mathbb{F}_q)^{W_S^R} \prod (\prod_{T \subsetneq S} \widetilde{RT} \otimes \mathbb{F}_q)^{W_S^R} \arrow[d] \\
RS^{W_S^R} \otimes \mathbb{F}_q \arrow[u] \arrow[r] & \widetilde{RS}^{W_S^R} \otimes \mathbb{F}_q \arrow[r] & (\widetilde{RS} \otimes \mathbb{F}_q)^{W_S^R}
\end{tikzcd}$$

The top map is surjective by the above discussion, the left vertical map is an isomorphism by point 2 proven above, and the bottom left horizontal map is surjective by point 4 proven above. The right vertical map is the projection onto the first factor, so it is also surjective. This shows that the bottom right horizontal map must be surjective, too. But this is exactly the map in question.

For injectivity it suffices to prove that $\widetilde{RS}^{W_S^R} \otimes \mathbb{F}_q \to \widetilde{RS} \otimes \mathbb{F}_q$ is injective. Consider the short exact sequence of abelian groups

$$0 \to \widetilde{RS}^{W_S^R} \to \widetilde{RS} \to M \to 0.$$

Tensoring with $\mathbb{F}_q$ we get an exact sequence

$$\mathrm{Tor}(M, \mathbb{F}_q) \to \widetilde{RS}^{W_S^R} \otimes \mathbb{F}_q \to \widetilde{RS} \otimes \mathbb{F}_q \to M \otimes \mathbb{F}_q \to 0.$$

For $q = 0$ we have that $\mathbb{F}_q = \mathbb{Q}$ and thus the first term vanishes, so that the morphism is injective.

For $q > 0$ the Tor term is given by the q-torsion of M. That is, we are looking for elements $[f] \in \widetilde{RS}$ such that $[qf]$ is invariant under the action of W_S^R. This means that for all $t \in W_S^R$ we have $q[f - t.f] = 0$ in $\widetilde{RS}$.

I claim that $\widetilde{RS}$ has no q-torsion, so that already $[f]$ is W_S^R invariant, so that the Tor term also vanishes in this case.

So let $[f] \in \widetilde{RS}$ such that $q[f] = 0$. By the first point of Lemma 1.4.5 this is equivalent to ask for $f \in \mathbb{Z}[X]$ such that $\Phi_{|S|} \mid qf$. So assume $qf = g\Phi_{|S|}$ for some $g \in \mathbb{Z}[X]$. Since the cyclotomic polynomial is monic one can prove, by induction, that all coefficients of g are divisible by q, that is $g = qg'$ for some $g' \in \mathbb{Z}[X]$. But then, since $\mathbb{Z}$ is torsion free, $qf = qg'\Phi_{|S|}$ implies $f = g'\Phi_{|S|}$. So already $[f] = 0$ in $\widetilde{RS}$. □

The main statement of this section is

Proposition 1.4.10. For each Real conjugacy class of q-regular cyclic even subgroups of G fix one representative S_i. Then the product of restrictions $RG \otimes \mathbb{F}_q \to \prod (\widetilde{RS_i} \otimes \mathbb{F}_q)^{W_{S_i}^R}$ is surjective and its kernel is the nilradical. In particular it induces an homeomorphism on spectra.

To prove this statement we will need the following small lemma:

Lemma 1.4.11. Let Q be a (Real) q-group, S a q-regular cyclic even group, and an exact diagram $1 \to S \to H \to Q \to 1$. Then the restriction $RH \to RS^Q$ is surjective.

Proof. The action, in terms of characters, is given by $x.\alpha = \alpha \circ \mathrm{Ad}^{\mathrm{R}}_{x^{-1}}$. By assumption we have $\gcd(|S|, |Q|) = 1$, so by the Schur-Zassenhaus theorem the above extension is split. A $\mathbb{Z}$-basis of RS is given by the morphisms $\alpha : S \to \mathbb{C}^*$, and a $\mathbb{Z}$-basis of RS^Q is given by orbit sums $\sum_{[x] \in Q/\mathrm{Stab}(\alpha)} x.\alpha$.

So fix an α and let K be its stabilizer in H (w.r.t. the (Real) conjugation action, again). Then we have a split extension $S \to K \to K/S$, so $K \cong K/S \rtimes S$. We can find a Real representation of K on $\mathbb{C}$ whose character restricts to α: $([A], t) \in K/S \rtimes S$ acts as $([A], t).z := \alpha(t)\phi(A)(z)$, denote this by M. Notice that $\chi_M|_S = \alpha$. Let $f : K \to H$ be the inclusion. Then by Lemma 1.2.7 the character of $f^e M$ restricted to S is

$$\chi_{f^e M}(t) = \sum_{[x] \in H/K} \chi_M(\mathrm{Ad}^{\mathrm{R}}_{x^{-1}}(t)) = \sum_{[x] \in H/K} \alpha(\mathrm{Ad}^{\mathrm{R}}_{x^{-1}}(t))$$

where I used that $\mathrm{Ad}^{\mathrm{R}}_{x^{-1}}(t) \in S$ since S is normal in H. This shows that $\chi_{f^e M}|_S$ is one of the basis elements described earlier. Since this works for any $\alpha : S \to \mathbb{C}^*$ the claimed surjectivity follows. $\square$

Proof of Proposition 1.4.10. Already in Lemma 1.4.3 one could have restricted to a product of all cyclic even subgroups up to Real conjugacy, since the characters are Real class functions. So one can reduce to a product over the S_i in Lemma 1.4.7. But then the map there factors over the one presented here, so also $RG \otimes \mathbb{F}_q \to \prod (\widetilde{RS_i} \otimes \mathbb{F}_q)^{W^R_{S_i}}$ must induce a surjection on spectra. This implies that its kernel consists of nilpotent elements. All of the $(\widetilde{RS_i} \otimes \mathbb{F}_q)^{W^R_{S_i}}$ are reduced, since already the $\widetilde{RS_i} \otimes \mathbb{F}_q \cong \mathbb{F}_q[X]/(\Phi_{|S_i|})$ are reduced: this can be seen from the fact that, since $q \nmid |S_i|$, $\Phi_{|S_i|}$ factors into *distinct* irreducible polynomials in $\mathbb{F}_q[X]$, and this is a UFD. So the kernel also includes all nilpotent elements, so it is the nilradical.

To prove surjectivity let us, given some $[y] \otimes 1 \in (\widetilde{RS_i} \otimes \mathbb{F}_q)^{W^R_{S_i}}$, construct an $x \in RG \otimes \mathbb{F}_q$ which restricts to $[y] \otimes 1$ on S_i and to 0 on all other S_j. By 4. in Lemma 1.4.9 we know that $[y] \otimes 1$ is the image of $y\theta \otimes \frac{1}{|S_i|} \in RS_i^{W^R_{S_i}} \otimes \mathbb{F}_q$. Let Q be a Sylow q-subgroup of $W^R_{S_i}$ (for $q = 0$ it is trivial, for $q = 2$ it might be Real) and let H be its preimage in $N^R_{S_i}$. In particular $y\theta$ is in RS^Q so by Lemma 1.4.11 there is an element $y' \in RH$ restricting to $y\theta$. Denote by $i : H \to G$ the inclusion, then $x := i^e y' \otimes \frac{1}{|S_i|} |W^R_{S_i}/Q|^{-1}$ is the element we are looking for. Note that $|W^R_{S_i}/Q| = |N^R_{S_i}/H|$ is a unit in $\mathbb{F}_q$ since Q is a Sylow q-subgroup.

Its restriction to the other S_j are zero, which we will prove using Lemma 1.2.7. To see this let t be a generator of a q-regular even cyclic subgroup not Real conjugate to S_i. If $\mathrm{Ad}^{\mathrm{R}}_{n^{-1}}(t)$ is in H, it generates a p-regular cyclic even subgroup of H. These are all contained in S_i, so $\mathrm{Ad}^{\mathrm{R}}_{n^{-1}}(t) \in S_i$ but it is not a generator.

$$\chi_{i^e y'}(t) = \sum_{[x] \in G/H} \chi_{y'}(\mathrm{Ad}^{R}_{x^{-1}}(t)) = \sum_{[x] \in G/H} \chi_{y\theta}(\mathrm{Ad}^{\mathrm{R}}_{x^{-1}}(t)) = 0$$

since $\chi_{y\theta}$ vanishes outside of the generators of S_i.

On the other hand, let t be a generator of S_i. Then $\mathrm{Ad}^{\mathrm{R}}_{n^{-1}}(t) \in H$ iff $\mathrm{Ad}^{\mathrm{R}}_{n^{-1}}(t) \in S_i$ iff $n \in N^R_{S_i}$, so

$$\chi_{i^e y'}(t) = \sum_{[n]\in G/H} \chi_{y'}(\mathrm{Ad}^{\mathrm{R}}_{n^{-1}}(t)) = \sum_{[n]\in N^R_{S_i}/H} \chi_{y\theta}(\mathrm{Ad}^{\mathrm{R}}_{n^{-1}}(t)) = \sum_{[n]\in N^R_{S_i}/H} \chi_{y\theta}(t)$$
$$= \frac{|N^R_{S_i}|}{|H|}\chi_{y\theta}(t)$$

where I used that $y\theta$ is a $W^R_{S_i}$ invariant. This shows that the image of x in $(\widetilde{RS_i}\otimes\mathbb{F}_q)^{W^R_{S_i}}$ is $|N^R_{S_i}/H|[y\theta]\otimes\frac{1}{|S_i|}|W^R_{S_i}/Q|^{-1} = [y\theta]\otimes\frac{1}{|S_i|} = [y]\otimes 1$. □

Proposition 1.4.12. The minimal groups such that $\mathfrak{p}\subset RG$ comes from them are a whole Real conjugacy class of even cyclic subgroups of G. Any one of them is called the *support* of $\mathfrak{p}$.

Proof. Let S, T be Real conjugate cyclic even subgroups of G. Then S is q-regular if and only if T is and there is a $g\in G$ such that $\mathrm{Ad}^{\mathrm{R}}_g(S)=T$. The key observation is that

$$\begin{tikzcd} & RS \arrow[dd, "\mathrm{Ad}^{\mathrm{R}}_x"] \\ RG \arrow[ur] \arrow[dr] & \\ & RT \end{tikzcd}$$

is a commutative diagram and the vertical morphism is an isomorphism. Let $\mathfrak{p}$ be above q. By Proposition 1.4.10 exactly one of S_i is minimal such that $\mathfrak{p}$ comes from it. Recall that, since S_i is q-regular, being minimal means that $\mathfrak{p}$ is in the image of $\widetilde{RS_i}\otimes\mathbb{F}_q$. Let T be Real conjugate to S_i. It is an easy extension of 1. in Lemma 1.4.9 that the above isomorphism also descends to one involving $\widetilde{RT}\otimes\mathbb{F}_q$, so T is minimal such that $\mathfrak{p}$ comes from it.

On the other hand, if T is not Real conjugate to S_i, it is either not q-regular, in which case it is not minimal such that $\mathfrak{p}$ comes from it by Lemma 1.4.4, or Real conjugate to one of the other S_j. But then, since $\mathfrak{p}$ is not in the image $\widetilde{RS_j}\otimes\mathbb{F}_q$, it can't be in the image of $\widetilde{RT}\otimes\mathbb{F}_q$, so T is not minimal such that $\mathfrak{p}$ comes from it. □

With this the characterization in Proposition 1.4.2 can be refined to the following:

Proposition 1.4.13. $\mathfrak{p}$ comes from H if and only if H contains a subgroup Real conjugate to the support of $\mathfrak{p}$.

Finally, let me state a lemma that will be needed later:

Lemma 1.4.14. Let H be a subgroup of G, let $\mathfrak{q}\subset RH$ be a prime ideal with support $S\subset H$. Then $S\subset G$ is a support of $\mathfrak{p}=r^{-1}(\mathfrak{q})$, where $r: RG\to RH$ is the restriction.

Proof. Assume $\mathfrak{q}$ is above the prime q. Then also $\mathfrak{p}$ is and S is q-regular. Since S is the support of $\mathfrak{q}$, $\mathfrak{q}$ is in the image of $\mathcal{S}(\widetilde{RS} \otimes \mathbb{F}_q) \to \mathcal{S}(RH \otimes \mathbb{F}_q)$. But then $\mathfrak{p}$ also must be in the image of $\mathcal{S}(\widetilde{RS} \otimes \mathbb{F}_q) \to \mathcal{S}(RG \otimes \mathbb{F}_q)$, since the restriction from RG to RS factors over RH. This means that S is a support of $\mathfrak{p}$. $\square$

2 Real equivariant K-theory

In this chapter I will sketch the development of Real equivariant K-theory. To this end I will present a kind of manual of how to modify existing literature so it applies in this case.

The versions of Real equivariant K-theory which were studied already in the 60s are more restrictive than what I want to describe here. It could be said that only equivariance with respect to a *split* Real group was studied, and that was on different footing then the $\mathbb{C}$-linear equivariant case. To use tools from modern equivariant topology one needs to be able to pass to equivariance with respect to arbitrary subgroups. So, since a subgroup of a split Real group may not be split itself, we also need a description in the nonsplit case. This will be most apparent when we discuss the family completion theorem at the end of this chapter.

Let me state some conventions. Throughout, let (G, ϕ) be a finite Real group. As in the previous chapter, most of the notions could also be developed for some notion of compact Real Lie group, but we shall not need that.

Also, from now on we will make some use of notions from equivariant (algebraic) topology. A good general reference, while a bit unwieldy, is [22], and references there in. Most importantly note that when I write (topological, G-) space, I mean compactly generated and weak Hausdorff.

2.1 Real equivariant vector bundles and K groups

I will start with a parallel to Segal's [4], where the basic definitions and properties of equivariant K-theory are laid down.

Definition 2.1.1. Let (G, ϕ) be a Real group and X be a G-space. A *Real equivariant vector bundle* or *G-vector bundle* on X is a complex vector bundle $\pi : E \to X$ where E is a G-space, π is equivariant, the action is fiberwise $\mathbb{R}$-linear, and, for $\lambda \in \mathbb{C}$, $g \in G$, $e \in E$, we have $g.(\lambda.e) = \phi(g)(\lambda).(g.e)$. That is, g acts linear or antilinear depending on $\phi(g)$.

A morphism of such vector bundles is a morphism of the underlying complex vector bundles which, in addition, is G-equivariant.

Finally, given a Real representation M of G, let $\underline{M} := M \times X \to X$ be the associated *trivial* bundle.

With this terminology, all the usual notions and operations are as described in §1 of [4].

The proofs don't depend on G acting $\mathbb{C}$-linearly, or produce Real representations when necessary. Let me highlight some relevant constructions and results:

The basic constructions of new G-vector bundles are taking direct sums, tensor products, and Hom-bundles. These are all defined like in the $\mathbb{C}$-linear case and then given the "natural" G-action (compare section 1.1). Another important construction is the pullback bundle: given a map of G-spaces $f : X \to Y$ and a G-vector bundle E on Y, the pullback bundle f^*E on X is naturally a G-vector bundle, too. When X is compact, it turns out that the isomorphism class of f^*E only depends on the G-homotopy class of f. This will later enable us to build a cohomology theory from G-vector bundles.

The central object of study from now on will be:

Definition 2.1.2. Let X be a compact G-space. The *Real equivariant K-group* $K_G(X)$ is the group of formal differences (with respect to the direct sum) of G-vector bundles on X. The tensor product of bundles makes these into commutative rings.

The pullback of G-vector bundles makes this into a contravariant functor from the category of compact G-spaces to the category of commutative rings. Additionally, since the pullback of G-vector bundles (on compact spaces) is G-homotopy invariant, this functor factors through the appropriate homotopy category.

When X is pointed one can also consider the *reduced* K-group which is defined as $\tilde{K}_G(X) := \ker[K_G(X) \to K_G(*)]$. This, again, gives a contravariant functor from the appropriate homotopy category of compact pointed G-spaces.

A few remarks about these definitions:

- For $f, g : X \to Y$ pointed maps, already an *unpointed* homotopy between them implies that $\tilde{K}_G(f) = \tilde{K}_G(g)$.
- Given a morphism of Real groups $\alpha : H \to G$ one also has a morphism $\alpha^* : K_G(X) \to K_H(X)$ natural in X, given by pulling back the action. In a slight generalization, if Y is a G-space, X an H-space, and $f : X \to Y$ continuous such that $f(h.x) = \alpha(h).f(x)$, then the "combined pullback" gives a morphism $K_G(Y) \to K_H(X)$.
- I will often write $[E]$ to mean the formal difference $[E - \underline{0}]$.
- Comparing with section 1 we see that $K_G(*)$ is the Real representation ring RG. Since $*$ is final in the category of G-spaces we see that all K_G groups are naturally thought of as RG-algebras.

Let me compare this notion to other flavors of K-theory, and comment on notation. First of all, if $G = G_0$, so that there are no odd elements, this gives $\mathbb{C}$-linear equivariant K-theory of [4], $K_G = K\mathbb{C}_G$. For $(G, \phi) = (C_2, \mathrm{id})$ this is what Atiyah originally called Real K-theory in [25], $K_{(C_2,\mathrm{id})} = KR$. Finally, if $(G, \phi) \cong (H \times C_2, \pi_2)$, then, for spaces X on which the C_2 factor acts trivially, this produces $\mathbb{R}$-linear equivariant K-theory, $K_{(H \times C_2, \pi_2)}(X) = K\mathbb{R}_H(X)$[1].

[1]This follows from arguments similar to Proposition 1.2.11 and the discussion following it.

The Real equivariant K-theory introduced in section 5 of [6] is also captured: There the premise is that we are given a group H with an involution, and an H-space X with a compatible involution. Then $KR_H(X)$ is defined there to be formal differences of $\mathbb{C}$-linear H-equivariant vector bundles on X, which also have a compatible involution acting antilinearly on the vector bundle. If we let $(G, \phi) = (H \rtimes C_2, \pi_2)$, where C_2 acts on H through the involution, then X having a compatible involution means that X is a G-space. In this setting we then find that $KR_H(X) = K_G(X)$. This is what I mean when I say that the split case was already studied in the 60's.

Proposition 2.1.3. For $H \subset G$ and X a compact H-space on can form the *induced* G-space $G \times_H X := (G \times X)/H$. Here G acts on the first factor from the left, and one quotients by the diagonal action $(g, x) \to (g.h^{-1}, hx)$ of H. Then the inclusion $X \to G \times_H X$, $x \mapsto [1, x]$ intertwines the inclusion $H \subset G$ and the resulting pullback $K_G(G \times_H X) \to K_H(X)$ is an isomorphism.

Proof. The idea is that an inverse is given by applying the induction also to the total space of an H-vectorbundle $E \to X$. The space $G \times_H E$ is naturally a G equivariant $\mathbb{R}$ vector bundle on $G \times_H X$. We give it the structure of a $\mathbb{C}$ vector bundle by defining the scalar multiplication fiberwise as $\lambda.[g, e] = [g, \phi(g)(\lambda).e]$, which evidently makes it into a Real G-vector bundle.

Starting with a G-vector bundle $E \to G \times_H X$ there is a natural isomorphism $G \times_H E|_X \to E$, $[g, e] \mapsto g.e$. In the other direction there is an isomorphism $E \to (G \times_H X)|_X$, $e \mapsto [1, e]$. This gives already an equivalence on the level of categories of vector bundles, so an isomorphism of K-groups. $\square$

Remark. If X is already a G-spaces, then there is an equivariant isomorphism $G \times_H X \cong G/H \times X$ sending $[g, x] \mapsto ([g], g.x)$, thus we have isomorphisms $K_G(G/H \times X) \cong K_H(X)$.

Furthermore, notice that these isomorphisms hold as RG modules, where RG acts on $K_H(X)$ through $RG \to RH$.

Unfortunately the Real equivariant version is not as well behaved with respect to free actions as usual equivariant K-theory, so we only get the following weaker version of Proposition 2.1 in [4]:

Proposition 2.1.4 (2.1')**.** Let $N \subset G$ be an *even* normal subgroup whose action on the compact G-space X is free. Then we have a natural isomorphism $K_G(X) \to K_{G/N}(X/N)$ sending $[E] \mapsto [E/N]$.

The proof is as in [4]. The reason why this is not true for all normal subgroups is that, once one quotients by an element acting antilinearly, there isn't necessarily a global notion of multiplication with complex numbers on E/N. We will see an example of this at the end of section 2.4. In the case of $G = C_2$, so that we are talking about KR-theory, the bundles of the form E/C_2 have a canonical so called *twisted almost complex* *structure*, as defined in §3 of [10]. This is a feature that really sets Real equivariant K-theory apart from usual equivariant K-theory as equivariant cohomology theories (to which both extend by a version of Bott periodicity).

Since we will need it later, I want to discuss a convenient reformulation of Proposition 2.2 in [4]. This is basically the standard decomposition of 1.2.5, but now indexed over a (compact) space. In the following proposition $KE_i(X)$ will denote K-theory of bundles of *right* E_i modules.

Proposition 2.1.5 (2.2'). Fix a complete set $\{S_i\}_{i\in I}$ of irreducible Real representations of G, let E_i be their endomorphism rings, and let G act trivially on X. Then the map $\mu : \oplus_{i\in I} KE_i(X) \to K_G(X)$, $[A] \in KE_i(X) \mapsto [A \otimes_{E_i} \underline{S_i}]$, is a group isomorphism, and its inverse is given by $\epsilon : K_G(X) \to \oplus_{i\in I} KE_i(X)$, $[F] \mapsto \oplus_{i\in I}[\mathrm{Hom}_G(\underline{S_i}, F)]$.

Proof. First I want to explain how $A \otimes_{E_i} \underline{S_i}$ is to be interpreted as a Real equivariant vector bundle. Since X has the trivial G action, we just[2] need to show how the fibers of $A \otimes_{E_i} \underline{S_i}$ are Real representations of G. Note that S_i is actually an $(E_i, {}^{\phi}\mathbb{C}G)$ bimodule (since E_i, by definition, acts ${}^{\phi}\mathbb{C}G$ linearly on S_i), so $\left(A \otimes_{E_i} \underline{S_i}\right)_x = A_x \otimes_{E_i} S_i$ naturally inherits the ${}^{\phi}\mathbb{C}G$ module structure, that is the structure of a Real representation. Of course, this is given by $g.(a \otimes s) = a \otimes g.s$.

Next, $\mathrm{Hom}_G(\underline{S_i}, F)$, the subbundle of the Hom-bundle given by pointwise equivariant maps, becomes a bundle of right E_i modules by pointwise precomposition.

I will show that these constructions are inverses to each other already on the level of vector bundles.

$$\begin{aligned}\epsilon(\mu([A] \in KE_i(X)) &= \epsilon([A \otimes_{E_i} \underline{S_i}]) = \oplus_{j\in I}[\mathrm{Hom}_G(\underline{S_j}, A \otimes_{E_i} \underline{S_i})] \\ &= \oplus_{j\in I}[A \otimes_{E_i} \mathrm{Hom}_G(\underline{S_j}, \underline{S_i})]\end{aligned}$$

$\mathrm{Hom}(\underline{S_j}, \underline{S_i})$ is, as an (E_i, E_j) bimodule, zero when $i \neq j$, and E_i with the standard structure when $i = j$, so $\epsilon(\mu([A] \in KE_i(X)) = [A] \in KE_i(X)$.

$$\mu(\epsilon([F])) = \mu(\oplus_{i\in I}[\mathrm{Hom}_G(\underline{S_i}, F)]) = [\oplus_{i\in I} \mathrm{Hom}_G(\underline{S_i}, F) \otimes_{E_i} \underline{S_i}]$$

There is a well-defined Real G bundle morphism $\mathrm{Hom}_G(\underline{S_i}, F) \otimes_{E_{S_i}} \underline{S_i} \to F$ defined by sending $f \otimes s \mapsto f(s)$. Since this restricts fiberwise to the standard decomposition 1.2.5, which is an isomorphism, already the morphism on vector bundles is an isomorphism. $\square$

Remark. On should note that this also works when one puts reduced K groups everywhere.

The proofs of Proposition 2.3, 2.4 applies as stated, and the proof of 2.5 given in [7, p. 31] only uses properties the $\mathbb{R}$-representation $C^0(G, \mathbb{R})$ of G, which apply in the exact same way. For us, the most important consequence of these is the following

Proposition 2.1.6. Let X be a compact G-space. Then every class in $K_G(X)$ has a representative of the form $[E - \underline{M}]$.

[2] Continuity etc. can be easily checked over a trivialization.

2.2 Half-exactness and grading

Let us take the first steps towards giving the K-groups the structure of a cohomology. First we will establish a certain exactness property, relating the K-groups of X, A and X/A for X compact and $A \subset X$ closed.

Lemma 2.2.1. Let X/A be pointed in A/A. Then we get an exact sequence $\tilde{K}_G(X/A) \to K_G(X) \to K_G(A)$ of abelian groups.

Proof. First of all, notice that X/A is compact, so that our notation makes sense.

The composition $A \to X \to X/A$ is equal to the composition $A \to * \to X/A$ (where the last map is the inclusion of the base point). But since every class in $\tilde{K}_G(X/A)$, by definition, pulls back to 0 on $*$, they also pull back to 0 on A. Thus, the composition in K-groups is zero.

Consider $[E - \underline{M}] \in K_G(X)$ restricting to 0 on A, and let's try to construct a preimage. Spelling this out there is a G-vector bundle T on A such that $E|_A \oplus T \cong \underline{M} \oplus T$. We can, by [4, 2.4], assume T to be a trivial bundle $\underline{N}$. Thus we have the bundle $E \oplus \underline{N}$ and $\underline{M} \oplus \underline{N}$ on X, and an isomorphism between them on A. Then, by [4, 1.2], we can extend this isomorphism to some open neighborhood $A \subset U \subset X$.

Cover the space X/A by the opens $O_1 = (X/A) \setminus (A/A) = X \setminus A$ and $O_2 = U/A$. Since $O_1 \cap O_2 = U \setminus A$ we can use the isomorphism of the previous paragraph to glue the bundles $\underline{M} \oplus \underline{N}$ on O_2 and $E \oplus \underline{N}$ on O_1 together. The resulting bundle E' the pulls back to $E \oplus \underline{N}$ on X, as can be checked on the open cover U, $X \setminus A$.

By construction $[E' - \underline{M} \oplus \underline{N}] \in \tilde{K}_G(X/A)$ and pulls back to $[E \oplus \underline{N} - \underline{M} \oplus \underline{N}] = [E - \underline{M}] \in K_G(X)$. This shows that the sequence $\tilde{K}_G(X/A) \to K_G(X) \to K_G(A)$ is exact. □

Lemma 2.2.2. Additionally to the above, assume A is pointed and contractible, and let X be pointed in the point of A. Then the projection $\tilde{K}_G(X/A) \to \tilde{K}_G(X)$ is an isomorphism.

Proof. Surjectivity follows from the previous lemma and $\tilde{K}_G(A) = 0$. For injectivity the idea is to construct a left inverse already on the level of vector bundles. Since A is contractible, every bundle on it is isomorphic to a trivial one. So, analogous to the above proof, we can use this trivialization (extended to some neighborhood of A) to lift the bundle to X/A. The key insight is that this now does not depend on the trivialization chosen on A (or the extension), at least up to isomorphism. Details (in the nonequivariant case) can be found around Lemma 2.10 in [20]. □

Remark. This is actually much stronger than what we need for the K-groups to become a cohomology theory. We only need this to be true after modifying the spaces X and X/A to behave well homotopically. Through this lemma we will have more flexibility proving some important properties of the K-groups without resorting to put further conditions on our spaces (as if compact Hausdorff is not enough).

In particular, this shows that the functor $\tilde{K}_G$ is half-exact. Let me explain what this means and what it gives to us. Let $f : A \to X$ be a map of (well-pointed[3]) G-spaces and consider the pushout $M_f := \operatorname{colim} A \wedge I_+ \leftarrow A \to X$. Here I_+ is the interval with a disjoint basepoint and the map $A \to A \wedge I_+$ send $a \mapsto (a, 0)$. Then the defining map $p : X \to M_f$ is an acyclic cofibration, $p \circ f$ is homotopic to $j : A \to M_f$, $a \mapsto (a, 1)$, and this last map is a cofibration. Finally consider the sequence $A \to M_f \to M_f/j(A)$, where the last space is called the homotopy cofiber $C_f := M_f/j(A)$ of f. Notice that, when A, X are compact, then also M_f, C_f are.

One calls any sequence $U \to V \to W$ of pointed spaces a cofiber sequence if it is homotopy equivalent to a sequence of the form $A \to M_f \to C_f$ for some $f : A \to X$. For us it is important that, when U, V, W are compact, then A, X can be chosen compact as well. The canonical example is $U \to V$ being the inclusion of a subcomplex into a G-CW complex, and $V \to W = V/U$ the quotient map.

Topological half-exactness means that $\tilde{K}_G$ takes cofiber sequences to exact sequences of abelian groups:

Lemma 2.2.3. Let $U \to V \to W$ be a cofiber sequence of compact G-spaces. Then the sequence $\tilde{K}_G(W) \to \tilde{K}_G(V) \to \tilde{K}_G(U)$ of abelian groups is exact.

Proof. By the previous discussion there is a homotopy equivalent sequence of the form $A \to M_f \to C_f$ for some map $f : A \to X$ between well-pointed compact spaces. By the previous lemma, and since $j : A \to M_f$ is a closed inclusion [4], the sequence $\tilde{K}_G(C_f) \to K_G(M_f) \to K_G(A)$ exact. But, since the sequence $U \to V \to W$ is homotopy equivalent, we also have that $\tilde{K}_G(W) \to K_G(V) \to K_G(W)$ is exact. Playing around with diagrams a bit one concludes that then also the sequence $\tilde{K}_G(W) \to \tilde{K}_G(V) \to \tilde{K}_G(W)$ is exact. $\square$

A classical construction in homotopy theory is the *long cofiber sequence*. The essence of it is that one can extend any cofiber sequence $U \to V \to W$ arbitrarily far to the right, so that every three term piece is a cofiber sequence. In particular, this can be done in the concrete form

$$U \xrightarrow{f} V \xrightarrow{j} W \xrightarrow{\partial} \Sigma U \xrightarrow{-\Sigma f} \Sigma V \xrightarrow{-\Sigma j} \Sigma W \to ...$$

Here $\Sigma ? = S^1 \wedge ?$ is the pointed suspension, and G acts trivially on the S^1 factor. Applying $\tilde{K}_G$ to this we get a long exact sequence of abelian groups

$$... \to \tilde{K}_G(\Sigma W) \to \tilde{K}_G(\Sigma V) \to \tilde{K}_G(\Sigma U) \to \tilde{K}_G(W) \to \tilde{K}_G(V) \to \tilde{K}_G(U)$$

[3] The well-pointed hypothesis is needed to guarantee that these constructions yield cofibrations.

[4] Since it is a cofibration and we are working with CGWH spaces

Remark. If $i : A \to X$ is the inclusion of a closed subspace, notice that $X/A = C_i/CA$. Since CA is contractible we have an isomorphism $\tilde{K}_G(X/A) \cong \tilde{K}_G(C_i)$. This gives the more flexible exact sequence

$$... \to \tilde{K}_G(\Sigma(X/A)) \to \tilde{K}_G(\Sigma X) \to \tilde{K}_G(\Sigma A) \to \tilde{K}_G(X/A) \to \tilde{K}_G(X) \to \tilde{K}_G(A)$$

Using this we can generalize a previous proposition. We have seen in 2.1.3 that $K_G(G/H \times X) \cong K_H(X)$. This also holds for reduced K groups:

Proposition 2.2.4. Let X be a pointed compact G-space. The inclusion $X \to (G/H)_+ \wedge X$, $x \mapsto ([1], x)$ induces an isomorphism $\tilde{K}_G((G/H)_+ \wedge X) \to \tilde{K}_H(X)$. This is even an isomorphism of RG modules with RG acting on $\tilde{K}_H(X)$ through the restriction $RG \to RH$.

Proof. The space $(G/H)_+ \wedge X$ can be seen as $\frac{G/H \times X}{G/H}$. Thus, for the sequence $G/H \to G/H \times X \to \frac{G/H \times X}{G/H} = (G/H)_+ \wedge X$, where the first map is the inclusion of a closed subspace, we get the diagram with exact rows:

$$\begin{array}{ccccc} \tilde{K}_G((G/H)_+ \wedge X) & \longrightarrow & K_G(G/H \times X) & \longrightarrow & K_G(G/H) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{K}_H(X) & \longrightarrow & K_H(X) & \longrightarrow & K_H(*) \end{array}$$

Here the vertical morphisms are restrictions along the inclusions at $[1] \in G/H$. We can actually say that the rows form short exact sequences: For the bottom row that is obvious from the definition of $\tilde{K}_G$, for the top we use the long exact sequence. The inclusion $G/H \to G/H \times X$, $[g] \mapsto ([g], *)$ has a retract by the projection $\pi_1 : G/H \times X \to G/H$. This shows that the right map in the top row must be surjective. On the other hand, it also show that the map which would come two before the left map in the top row is also surjective, so that, by exactness, the map must be injective. Then, since the two vertical maps on the right are isomorphisms, also the one on the left is.

Tracing the action of RG through this shows that this gives an isomorphism of RG-modules, too. □

We can also use this argument to show that the reduced K groups are additive:

Proposition 2.2.5. Let X, Y be pointed compact G-spaces. Then the inclusions $i_1 : X \to X \vee Y$, $i_2 : Y \to X \vee Y$ induce an isomorphism $\tilde{K}_G(X \vee Y) \to \tilde{K}_G(X) \oplus \tilde{K}_G(Y)$.

Proof. Consider the closed inclusion/quotient sequence $X \to X \vee Y \to Y$, where I identified $\frac{X \vee Y}{X} = Y$. The quotient $X \vee Y \to Y$ has a section given by i_2, and the inclusion $i_1 : X \to X \vee Y$ has a retract given by the other quotient $X \vee Y \to X$. Thus we get a split short exact sequence

$$0 \to \tilde{K}_G(Y) \to \tilde{K}_G(X \vee Y) \to \tilde{K}_G(X) \to 0$$

where the splitting is given by the map $i_2^* : \tilde{K}_G(X \vee Y) \to \tilde{K}_G(Y)$. This shows that the direct sum of i_1^* and i_2^* gives an isomorphism $\tilde{K}_G(X \vee Y) \to \tilde{K}_G(X) \oplus \tilde{K}_G(Y)$. □

Another application of this sort of argument is the definition of the reduced external product. Given vector bundles E on X and F on Y we can form the external product bundle $E \times F$ on $X \times Y$, and this extends to an external product on K groups:

$$K_G(X) \times K_G(Y) \to K_G(X \times Y),\ (a, b) \mapsto \pi_1^* a \otimes \pi_2^* b.$$

If a, b lie in the reduced K groups, then their external product is naturally an element of the reduced K group of $X \wedge Y$:

Proposition 2.2.6. Let X, Y be pointed compact G-spaces, and let $a \in \tilde{K}_G(X), b \in \tilde{K}_G(Y)$, and consider the projection $p : X \times Y \to X \wedge Y := \frac{X \times Y}{X \vee Y}$. Then restriction $\tilde{K}_G(X \wedge Y) \to \tilde{K}_G(X \times Y)$ along p is injective, and the external product of a and b lies in its image. The unique preimage in $\tilde{K}_G(X \wedge Y)$ is called their *reduced external product* and is denoted $a \tilde{\otimes} b$.

Proof. First we show that $\pi_1^* a \otimes \pi_2^* b$ restricts to zero on the point, and thus is an element of $\tilde{K}_G(X \times Y)$: The inclusion of the basepoint into $X \times Y$ composed with either projection is equal to the inclusion of the basepoint in X or Y. Since, by assumption, a and b restrict to zero on the basepoint, and pulling back commutes with the tensor product, $\pi_1^* a \otimes \pi_2^* b$ also restricts to zero on the basepoint.

Now consider the closed inclusion/quotient sequence $X \vee Y \to X \times Y \to X \wedge Y$. We get the exact sequence

$$\tilde{K}_G(\Sigma(X \times Y)) \to \tilde{K}_G(\Sigma(X \vee Y)) \to \tilde{K}_G(X \wedge Y) \to \tilde{K}_G(X \times Y) \to \tilde{K}_G(X \vee Y).$$

We want to show that $\pi_1^* a \otimes \pi_2^* b$ restricts to zero on $X \vee Y$. Using the previous proposition this is equivalent to it restricting to zero on both X and Y: Let $e : * \to Y$ be the inclusion of the basepoint. Then the restriction to X is $a \otimes e^* b = 0$, and similar for Y.

So by exactness $\pi_1^* a \otimes \pi_2^* b$ is in the image of $\tilde{K}_G(X \wedge Y)$, but does it have a unique preimage? We now show that the map $\tilde{K}_G(X \wedge Y) \to \tilde{K}_G(X \times Y)$ is injective. By exactness this is equivalent to the map $\tilde{K}_G(\Sigma(X \times Y)) \to \tilde{K}_G(\Sigma(X \vee Y))$ being surjective. By the previous proposition, and since $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$, this is equivalent to the map $K_G(\Sigma(X \times Y) \to \tilde{K}_G(\Sigma X) \oplus \tilde{K}_G(\Sigma Y))$ induced by the suspensions of the inclusions $X \to X \times Y$ and $Y \to X \times Y$ being surjective. But these have retracts given by the suspensions of the projections, thus showing that the map on reduced K groups is surjective. $\square$

Remark. The reduced external product is associative. This is an direct consequence of the associativity of the smash product and the tensor product of vector bundles.

At this point I want to discuss graded K-groups, reduced and unreduced, making first steps toward realizing K-theory as an equivariant cohomology. Taking cues from equivariant stable topology, a G-equivariant cohomology theory should be graded in ROG, the representation ring for $\mathbb{R}$-linear representations of G, see chapter IX, section 5 of [22]. Furthermore, in the reduced cohomology, one should have suspension isomorphisms

$\tilde{K}_G^*(X) \cong \tilde{K}_G^{*+V}(S^V \wedge X)$, where $S^V = \{\infty\} \coprod V$ is the one point compactification of the $\mathbb{R}$-linear G-representation V. At this point we use this to *define* the negatively graded part of the K-groups:

Definition 2.2.7. Let X be a pointed compact G-space and V be an $\mathbb{R}$-linear representation of G. Then define the K groups in degree $-V$ to be $\tilde{K}_G^{-V}(X) := \tilde{K}_G(S^V \wedge X)$ and $K_G^{-V}(X) := \ker\big(K_G(S^V \times X) \to K_G(X)\big)$ (where the morphism is the restriction to $\{\infty\} \times X$). For a (pointed) G-map $f : X \to Y$ one has pullbacks along $\mathrm{id}_{S^V} \wedge f$ and $\mathrm{id}_{S^V} \times f$, making these into functors.

Remark. While the definition for the the reduced K-groups is kind of forced onto us, for the unreduced case one usually puts $K_G^*(X) := \tilde{K}_G^*(X_+)$, where X_+ is X with a disjoint basepoint. The definition above is a reformulation of this into a more directly usable definition. But, since this should hold at least as a natural isomorphism, let's do a little sanity check:

Proposition 2.2.8. For any $\mathbb{R}$-linear representation V there are isomorphisms $\tilde{K}_G^{-V}(X_+) \to K_G^{-V}(X)$, natural in X.

Proof. By definition $\tilde{K}_G^{-V}(X_+) = \tilde{K}_G(S^V \wedge X_+) = \tilde{K}_G(S^V \times X/\{\infty\} \times X)$. Since $\{\infty\} \times X \subset S^V \times X$ is closed we have an exact sequence

$$\tilde{K}_G(\Sigma(S^V \times X)) \to \tilde{K}_G(\Sigma X) \to \tilde{K}_G(S^V \wedge X_+) \to \tilde{K}_G(S^V \times X) \to \tilde{K}_G(X)$$

The inclusion $i : X \to S^V \times X$ has the retract $\pi_2 : S^V \times X \to X$. Thus both $\tilde{K}_G(i)$ and $\tilde{K}_G(-\Sigma i)$ are surjective, so the LES breaks down to a short exact sequence

$$0 \to \tilde{K}_G(S^V \wedge X_+) \to \tilde{K}_G(S^V \times X) \to \tilde{K}_G(X) \to 0$$

Playing with kernels a bit then gives that also this is exact, proving the proposition:

$$0 \to \tilde{K}_G(S^V \wedge X_+) \to K_G(S^V \times X) \to K_G(X) \to 0$$

$\square$

Corollary. There are natural maps $K_G^{-V-W}(X) \to K^{-V}(S^W \times X)$ identifying $K_G^{-V-W}(X)$ with the kernel of $K^{-V}(S^W \times X) \to K^{-V}(X)$.

Proof. The maps come from the following diagram

$$\begin{tikzcd}
0 \arrow[r] & K_G^{-V-W}(X) \arrow[r] \arrow[d, dotted] & K_G(S^V \wedge S^W \times X) \arrow[r] \arrow[d] & K_G(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_G^{-V}(S^W \times X) \arrow[r] & K_G(S^V \times S^W \times X) \arrow[r] & K_G(S^W \times X) \arrow[r] & 0
\end{tikzcd}$$

Using the above isomorphism these maps correspond to $\tilde{K}_G(S^V \wedge (S^W \times X)_+) \to \tilde{K}_G(S^V \wedge S^W \wedge X_+)$.

Since S^W is wellpointed the inclusion $X \to S^W \times X$ is a cofibration, so after adding disjoint basepoints it is a cofibration of well pointed spaces, showing that $X_+ \to (S^W \times X)_+ \to S^W \wedge X_+$ is a cofiber sequence. Then also $S^V \wedge X_+ \to S^V \wedge (S^W \times X)_+ \to S^V \wedge S^W \wedge X_+$ is a cofiber sequence. Since the inclusion $S^V \wedge X_+ \to S^V \wedge (S^W \times X)_+$ has a retract, the LES reduces to the following short exact sequence:

$$0 \to \tilde{K}_G(S^V \wedge S^W \wedge X_+) \to \tilde{K}_G(S^V \wedge (S^W \times X)_+) \to \tilde{K}_G(S^V \wedge X_+) \to 0$$

Piecing everything together we get the desired short exact sequence

$$0 \to K_G^{-V-W}(X) \to K^{-V}(S^W \times X) \to K^{-V}(X) \to 0$$

□

2.3 Bott periodicity

Bott periodicity is arguably the most important theorem in K-theory. Nonequivariantly, it is the statement that $\tilde{K}(X) \cong \tilde{K}(S^2 \wedge X)$. It was proven by Raoul Bott in 1959 using Morse theory on the infinite unitary group (one of the representing spaces of non equivariant K-theory), see [26]. Many different proofs have been given since then, but maybe the most versatile was given by Atiyah in [6]. In what follows I want to argue that the proofs given there for the various versions of K-theory actually apply, with minor adjustments, to our setting.

So let V be a Real representation of G. We can regard V as an $\mathbb{R}$ linear representation of G by forgetting its complex structure. In this setting we have the following two theorems:

Theorem 2.3.1 (Equivariant Bott periodicity)**.** Let V be a Real representation of G. There is a class $b_V \in K_G^{-V}(*) = \tilde{K}_G(S^V)$ such that the Bott map

$$\beta_X^V : K_G(X) \to K_G^{-V}(X),\ [E] \mapsto [\pi_1^* b_V \otimes \pi_2^* E]$$

is an isomorphism natural for all X.

Theorem 2.3.2 (Spin Bott periodicity)**.** Let $\mathbb{R}^8$ be the trivial $\mathbb{R}$-linear representation of G. There is a class $b_S \in K_G^{-\mathbb{R}^8}(*) = \tilde{K}_G(S^8)$ such that the Bott map

$$\beta_X^S : K_G(X) \to K_G^{-\mathbb{R}^8}(X),\ [E] \mapsto [\pi_1^* b_S \otimes \pi_2^* E]$$

is an isomorphism natural for all X.

In the case that G is split, these are proven in section 5 and 6 of [6]. To be concrete, the second theorem is only a special case of Theorem 6.2 there, but this case is more than enough for our purposes. In general, the two theorems above are very different, since

$\mathbb{R}^8$ with the trivial action is only a Real representation of G if $G = G_0$. In that case $\mathbb{R}^8 \cong \mathbb{C}^4$ and one can show that $b_S = \pm b_{\mathbb{C}^4}$.

Notice that in the special case $G = \{1\}$ we recover the usual Bott periodicity: The "smallest" Real representation of this is is $\mathbb{C}$, and $S^{\mathbb{C}} = S^2$, so that the first theorem (or rather, its reduced form discussed at the end of the section) provides an isomorphism $\tilde{K}(X) \cong \tilde{K}(S^2 \wedge X)$.

We will use these (actually only the second version) to define K-groups in arbitrary integer degree. This will enable us to view them as a sort of cohomology theory which puts us into a position where we can usefully work with K-theory. But for now, let me sketch how the proof given in [6] works.

The key insight is that, when one includes certain naturality conditions, the list of properties a natural inverse to β^V should have shrinks dramatically:

Lemma 2.3.3. Let $\alpha^V : K_G^{-V} \to K_G$ be a natural transformation such that:

- α^V_X is a $K_G(X)$ module morphism: $\alpha^V_X(p \otimes \pi_X^* q) = \alpha^V_X(p) \otimes q$ for all $p \in K_G^{-V}(X)$ and $q \in K(X)$
- $\alpha^V_*(b_V) = 1 \in K_G(*) \cong RG$

Then α^V is a natural inverse to β^V.

Proof. Let $q \in K_G(X)$, $p \in K^{-V}(X)$. Then

$$\alpha_X(\beta_X(q)) = \alpha_X(\pi_1^* b \otimes \pi_2^* q) = \alpha_X(\pi_1^* b) \otimes q = (X \to *)^* \alpha_*(b) \otimes q = 1 \otimes q = q.$$

showing that $\alpha_X \circ \beta_X = \mathrm{id}$.

For the different direction let me introduce some notation: We will be working on the space $S^V \times S^V \times X$. For $i, j, k \in \{0, 1\}$ let π_{ijk} denote map projecting out the spaces depending on the values of i, j, k. For example $\pi_{110} : S^V \times S^V \times X \to S^V \times S^V$ and $\pi_{001} : S^V \times S^V \times X \to X$.

We calculate:

$$\beta_X(\alpha_X(p)) = \pi_1^* b \otimes \pi_2^* \alpha_X(p) = \pi_1^* b \otimes \alpha_{S^V \times X}(\pi_{101}^* p) = \alpha_{S^V \times X}(\pi_{101}^* p \otimes \pi_{010}^* b)$$

We now use a trick to switch around the projections so that we can pull out p and use that $\alpha(b) = 1$. First note that $\pi_{101}^* p \otimes \pi_{010}^* b \in K_G^{-V}(S^V \times X)$ restricts to zero on $K^{-V}(X)$, because already $\pi_{010}^* b$ does: The composition $S^V \times X \to S^V \times S^V \times X \to S^V$ relevant here is equal to the constant map. But by definition b restricts to zero on the point.

Denote $f : S^V \times S^V \to S^V \wedge S^V$ the projection and $t : S^V \times S^V \to S^V \times S^V$, $(x, y) \mapsto (y, -x)$ with the convention that $-\infty = \infty$. Note that $S^V \wedge S^V = S^{2V}$ is the one point compactification of $V \oplus V$. On this direct sum we can also map $(x, y) \mapsto (y, -x)$. This is homotopic to the identity (it is a 'rotation' in the $V \oplus V$ plane) and extends to map $t' : S^V \wedge S^V \to S^V \wedge S^V$. t' is then also homotopic to the identity and we have $f \circ t = t' \circ f$. So up to homotopy we have $f \circ t = f$ and $(f \times \mathrm{id}_X) = (f \circ t \times \mathrm{id}_x)$

Since $\pi_{101}^* p \otimes \pi_{010}^* b$ is zero on $K_G^{-V}(X)$, it is equal to $(f \times \mathrm{id}_X)^* h$ for some $h \in K_G(S^V \wedge S^V \times X)$. Now $(f \times \mathrm{id}_X)^* h = (f \circ t \times \mathrm{id}_X)^* h = (t \times \mathrm{id}_X)^* (f \times \mathrm{id}_X)^* h$, so that

$$\pi_{101}^* p \otimes \pi_{010}^* b = (t \circ \mathrm{id}_X)^* (\pi_{101}^* p \otimes \pi_{010}^* b) = \pi_{011}^* (-\mathrm{id}_{S_V} \times \mathrm{id}_X)^* p \otimes \pi_{100}^* b$$

With this we get

$$\begin{aligned}\beta_X(\alpha_X(p)) &= \alpha_{S^V \times X}(\pi_{101}^* p \otimes \pi_{010}^* b) = \alpha_{S^V \times X}(\pi_{011}^* (-\mathrm{id}_{S_V} \times \mathrm{id}_X)^* p \otimes \pi_{100}^* b) \\ &= \alpha_{S^V \times X}(\pi_{100}^* b) \otimes (-\mathrm{id}_{S_V} \times \mathrm{id}_X)^* p = 1 \otimes (-\mathrm{id}_{S_V} \times \mathrm{id}_X)^* p \\ &= (-\mathrm{id}_{S_V} \times \mathrm{id}_X)^* p\end{aligned}$$

Finally, since $(-\mathrm{id}_{S_V} \times \mathrm{id}_X) \circ (-\mathrm{id}_{S_V} \times \mathrm{id}_X) = \mathrm{id}_{S^V \times X}$, $(-\mathrm{id}_{S_V} \times \mathrm{id}_X)^*$ is an automorphism of $K^{-V}(X)$, we get that β_X is surjective and α_X is injective. Together with $\alpha \circ \beta = \mathrm{id}$ this implies that these are actual inverses to each other. □

After establishing this, Atiyah uses the theory of indices of families of elliptic operators to construct such natural α. Let me summarize Proposition 2.2 in [6] and the remark following it.

Proposition 2.3.4. Let M be a compact manifold with an action of G, let E, F be smooth G vector bundles on M, and let $D : \Gamma(E) \to \Gamma(F)$ be equivariant elliptic operator. Then there is a natural transformation $\mathrm{ind}_D : K_G(M \times X) \to K_G(X)$ with the properties:

- ind_D is a $K_G(X)$ module morphism: $\mathrm{ind}_D(p \otimes \pi_2^* q) = \mathrm{ind}_D(p) \otimes q)$ for $p \in K_G(M \times X)$ and $q \in K_G(X)$
- If Q is a smooth G vector bundle on M, then $\mathrm{ind}_D([Q]) = [\ker D_Q - \operatorname{coker} D_Q] \in K_G(*) = RG$

Here $D_Q : \Gamma(E \otimes Q) \to \Gamma(F \otimes Q)$ is some extension of D. That is, locally there should be trivializations of Q such that $D_Q = D \otimes \mathrm{id}$.

Remark. For details of the definition of D_Q and ind_D see [6] and references therein. Also note that the arguments cited that make this work in the equivariant $\mathbb{C}$ linear case also work, with minor adjustments, in the Real equivariant case, but that discussion is out of scope for this thesis.

Given an equivariant elliptic operator D on M and a map $j : M \to S^V$, we can build the maps

$$\alpha(D, j)_X : K_G^{-V}(X) \subset K_G(S^V \times X) \xrightarrow{(j \times \mathrm{id}_X)^*} K_G(M \times X) \xrightarrow{\mathrm{ind}_D} K_G(X)$$

These are natural in X and $K_G(X)$ module morphisms, so it remains to find D, j, b_V such that $\alpha(D, j)_*(b_V) = 1$.

For the Real equivariant case this works out exactly as detailed in §4 of [6]. There are certainly some complicated checks to do (like the Dolbeault operator being Real equivariant), but since we won't need it to much I will leave it at that. I do want to mention though that, in the case $G = e$, it works out that $b_{\mathbb{C}} = [\underline{\mathbb{C}} - H]$ where H is the dual of the tautological line bundle on $S^2 \cong P\mathbb{C}^2$

The spinor case is a bit different. Here one chooses $M = S^8$ and views this as a homogeneous space

$$S^8 \cong SO(9)/SO(8) \cong Spin(9)/Spin(8) \cong Spin^R(9)/Spin^R(8)$$

Here $Spin^R(n) = Spin^c(n) \rtimes C_2$ is the Real spin group, where the action of C_2 on $Spin^c(n) = Spin(n) \times_{C_2} U(1)$ is by conjugation on the $U(1)$ factor. These groups are Real groups through the projection $Spin^c(n) \rtimes C_2 \xrightarrow{\pi_2} C_2$. Since the Real structure of these is split, they do fall into the setting already investigated in the 60's. To shorten the discussion here, notice that any Real group G has a canonical Real morphism to $Spin^R(8)$ given by $\chi : G \xrightarrow{\phi} C_2 \xrightarrow{i_2} Spin^c(8) \rtimes C_2 = Spin^R(8)$.

The differential operator D, the bundles it acts on, and the class u in §6 of [6] are Real equivariant with respect to the action of $Spin^R(8) \subset Spin^R(9)$. Since the 'antilinear' $C_2 \subset Spin^R(8)$ acts trivially on the homogeneous space S^8 and χ factors over its inclusion, these pull back to Real equivariant (with respect to G) gadgets on the trivial G space S^8

Lemma 2.3.5. The pullback of $u \in K_{Spin^R(8)}(S^8)$ via χ has index 1 in $K_G(*)$.

Proof. Let $D : \Gamma(E) \to \Gamma(F)$, $u = [A - B]$ for A, B smooth. As sketched in §2 of [6] choose some extension $D_A : \Gamma(E \otimes A) \to \Gamma(F \otimes A)$. Notice that $\chi^*\Gamma(E \otimes A) = \Gamma(\chi^*E \otimes \chi^*A)$, so that χ^*D_A is an extension of χ^*D. Thus $\mathrm{ind}_{\chi^*D}(\chi^*A) = [\ker \chi^*D_A - \operatorname{coker} \chi^*D_A] = [\chi^* \ker D_A - \chi^* \operatorname{coker} D_A] = \chi^*\mathrm{ind}_D(A)$. Applying this also to B we find that $\mathrm{ind}_{\chi^*D}(\chi^*u) = \chi^*\mathrm{ind}_D(u) = \chi^*1 = 1$. □

Remark. Clearly we have not used anything special about χ, D or u. This suggests that one could generalize this lemma to a statement like 'pulling back the group action commutes with taking the index'.

Notice that χ^*u restricts to zero on the point of S^8 since already u does. We set the spin Bott class $b_S = \chi^*u \in K_G^{-8}(*)$.

As mentioned earlier there are also reduced versions of the Bott isomorphisms. These are given by taking the reduced external product with the Bott classes:

Lemma 2.3.6. Let b_V be the Bott element in $\tilde{K}_G(S^V)$. Then taking reduced external products with it gives an isomorphism

$$\tilde{K}_G(X) \to \tilde{K}_G(S^V \wedge X) = \tilde{K}_G^{-V}(X),\ [E] \mapsto b_V \tilde{\otimes} [E]$$

and similar for the Spin case.

Proof. The proof is basically studying the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{K}_G(X) & \longrightarrow & K_G(X) & \longrightarrow & K_G(*) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{K}_G(S^V \wedge X) & \longrightarrow & K_G^{-V}(X) & \longrightarrow & K_G^{-V}(*) & \longrightarrow & 0
\end{array}$$

until one realizes it commutes. Then, since the middle and right vertical morphisms are isomorphisms (by Bott periodicity), the left one is, too. The question is: what is the bottom row and why is it exact? By the isomorphism $K_G^{-V}(X) \cong \tilde{K}_G(S^V \wedge X_+)$ and $S^V \wedge X_+ \cong S^V \times X/\{\infty\} \times X$ we rewrite it as

$$0 \to \tilde{K}_G(S^V \wedge X) \to \tilde{K}_G\left(\frac{S^V \times X}{\{\infty\} \times X}\right) \to \tilde{K}_G(S^V) \to 0.$$

This can be seen to be exact since the inclusion $S^V \to S^V \times X/\{\infty\} \times X$ has a retract (and by the isomoprhism $S^V \wedge X \cong S^V \times X/(S^V \vee X)$).

The right square commutes by the naturality of the Bott map. The left square commutes by the definition of the reduced external product: $K_G^{-V}(X)$ is defined as a subgroup of $K_G(S^V \times X)$, and by definition $b_V \tilde{\otimes} a \in \tilde{K}_G(S^V \wedge X)$ maps to $\pi_1^* b_V \otimes \pi_2^* a$ in $K_G(S^V \times X)$, for $a \in \tilde{K}_G(X)$. □

This version of Bott periodicity will be crucial in the next chapters, where we will try to avoid using unreduced K-groups since they are slightly harder to work with.

Finally, I want to demonstrate that there is also Bott periodicity for $\mathbb{R}^8$ with the sign representation through $\phi : G \to C_2$. This representation is important since an element $g \in G$ has fixed points in it if and only if it is even.

Lemma 2.3.7. Let $W = \mathbb{R}^8$ with the sign representation through $\phi : G \to C_2$ and let $V = \mathbb{C}^8$ be the eight dimensional trivial Real representation of G. Then there is a class $b_{\tilde{S}} \in \tilde{K}_G(S^W)$ such that

$$\beta_X^{\tilde{S}} : \tilde{K}_G(X) \to \tilde{K}_G(S^W \wedge X),\ [E] \mapsto b_{\tilde{S}} \tilde{\otimes} [E]$$

is a natural isomorphism.

Proof. As $\mathbb{R}$-linear representations we have that $V \cong \mathbb{R}^8 \oplus W$, so that the representation spheres $S^V \cong S^8 \wedge S^W$ are isomorphic, too.

By the reduced spin Bott periodicity $\tilde{K}_G(S^W) \cong \tilde{K}_G(S^8 \wedge S^W) \cong \tilde{K}_G(S^V)$, so that $b_V = b_S \tilde{\otimes} a$ for a unique $a \in \tilde{K}_G(S^W)$. Call this element $b_{\tilde{S}}$. By the associativity of the reduced external product, and since both $b_V \tilde{\otimes}?$ and $b_S \tilde{\otimes}?$ are isomorphisms, $b_{\tilde{S}} \tilde{\otimes}?$ is an isomorphism, too. □

Later we will be interested in how these Bott classes behave under taking the complex conjugate bundles. It turns out we will only need a special case:

Proposition 2.3.8. Let $b_{\mathbb{C}} = [\underline{\mathbb{C}} - H] \in \tilde{K}(S^2)$ be the Bott class. Then $\overline{b_{\mathbb{C}}} = -b_{\mathbb{C}}$.

Proof. It is well known that $H \otimes H \oplus \underline{\mathbb{C}} \cong H \oplus H$ (see example 1.13 in [20]). On the other hand, since $\tilde{K}(S^2) \cong \mathbb{Z}$ there is some $k \in \mathbb{Z}$ such that $\overline{b_{\mathbb{C}}} = kb_{\mathbb{C}}$. We calculate:

$$b_{\mathbb{C}} \otimes \overline{b_{\mathbb{C}}} = kb_{\mathbb{C}} \otimes b_{\mathbb{C}} = k[H \otimes H \oplus \underline{\mathbb{C}} - H \oplus H] = 0$$

and

$$b_{\mathbb{C}} \otimes \overline{b_{\mathbb{C}}} = [\underline{\mathbb{C}} \oplus H \otimes \overline{H} - H \oplus \overline{H}].$$

Since for any complex line bundle $H \otimes \overline{H} \cong \underline{\mathbb{C}}$ we get

$$b_{\mathbb{C}} + \overline{b_{\mathbb{C}}} = [\underline{\mathbb{C}} \oplus \underline{\mathbb{C}} - H \oplus \overline{H}] = 0.$$

□

2.4 K_G as an equivariant cohomology theory

Having the tool of Bott periodicity in our hands, we can now extend our definitions of the Real equivariant K-groups to equivariant cohomology theories. The idea is that, as discussed before, the negative degree part can be defined by suspensions, and positive degree parts are defined by taking the periodicity as an axiom.

I will show how this works for K_G and for a certain flavor of it, which could be called the *delocalized* K group $K_{del,G}$. Finally, I will comment on how these things work out if one wants true *ROG*-grading.

For us and for now, a *reduced cohomology theory* on compact G-spaces is a contravariant functor $h^* : \mathcal{T}^G_{c,*} \to Ab^*$ from compact pointed G-spaces to $\mathbb{Z}$-graded abelian groups together with a natural isomorphisms $\sigma^n_X : h^{n-1}(X) \to h^n(\Sigma X)$, called the suspension isomorphisms, such that

- when $f, g : X \to Y$ are pointed G-homotopic, then $h^*(f) = h^*(g)$, and
- when $X \to Y \to Z$ is a cofiber sequence, then $h^*(Z) \to h^*(Y) \to h^*(X)$ is degree wise exact.

This is certainly a very minimal list of requirements. For a nice discussion of this sort of thing in the nonequivariant case see section 21.1.1 and 21.1.2 of [24], the equivariant case is completely analogous. One often also asks for additivity with respect to infinite wedge sums (when defined also on noncompact spaces) and for *weak* homotopy equivalences to be carried to isomorphisms. We shall not use such strong axioms, so I will leave it at that.

As argued above definition 2.2.7 there is a natural way to define the negative degree part of a would be $\tilde{K}_G$ cohomology. Spin Bott periodicity shows that this is enough:

Definition 2.4.1. For each integer $n \in \mathbb{Z}$ let $\tilde{K}^n_G(X) = \tilde{K}_G(S^{q(n)} \wedge X)$ where $q(n) \in \{0, .., 7\}$ and $q(n) \equiv -n \bmod 8$. By setting $\tilde{K}^n_G(f) = \tilde{K}_G(S^{q(n)} \wedge f)$ this defines a contravariant functor $\tilde{K}^*_G$ from pointed compact G spaces to graded abelian groups.

First of all, from the pointed homotopy invariance of $\tilde{K}_G$ we can conclude the pointed homotopy invariance of $\tilde{K}_G^*$: When f, $g : X \to Y$ are pointed homotopic, say through $H : X \wedge I_+ \to Y$, then $S^n \wedge f$, $S^n \wedge g : S^n \wedge X \to S^n \wedge Y$ are pointed homotopic through $S^n \wedge H$.

As we have previously proven, $\tilde{K}_G$ takes cofiber sequences to exact sequences of abelian groups. Smashing a cofiber sequence with a well pointed space like S^n produces again a cofiber sequence, so that $\tilde{K}_G^*$ takes cofiber sequences to degree wise exact sequences.

Finally, through spinor Bott periodicity, we can give $\tilde{K}_G^*$ a natural suspension isomorphism $\tilde{K}_G^{n-1} \cong \tilde{K}_G^n \circ \Sigma$. For $q(n) = 0, .., 6$ one has $q(n-1) = q(n) + 1$ and we simply uses the natural isomorphisms $S^{q(n)+1} \wedge X \cong S^{q(n)} \wedge S^1 \wedge X$ and apply $\tilde{K}_G$ to them. For $q(n) = 7$ we need to specify an isomorphism $\tilde{K}_G(X) \to \tilde{K}_G(S^8 \wedge X)$. For this we will use reduced external product with the spin Bott class $b_S \in \tilde{K}_G(S^8)$.

Proposition 2.4.2. With this suspension isomorphism $\tilde{K}_G^*$ is a reduced equivariant cohomology on pointed compact G spaces.

While it is a bit unmotivated at this point, let me define the delocalized version of this now. Delocalization means something like decomposing the space into its fixed points with respect to all (even) elements of G, applying a non-equivariant cohomology to them, and finally taking invariants with respect to a certain action of G on it:

Definition 2.4.3. The *delocalized* K group is defined to be the set of invariants

$$K_{del,G}(X) := \left(\bigoplus_{g \in G_0} K(X^g) \otimes \mathbb{C} \right)^G$$

of the following action:

$n \in G$ maps the summand at $g \in G_0$ to the summand at $\mathrm{Ad}_n^R(g)$ by pulling back along the action of n^{-1}, and we also take the conjugate bundles depending on $\phi(n)$, all tensored with $\mathrm{id}_{\mathbb{C}}$:

$$\phi(n) \circ K(n^{-1}. : X^{\mathrm{Ad}_n^R(g)} \to X^g) \otimes \mathrm{id}_{\mathbb{C}} : K(X^g) \otimes \mathbb{C} \to K(X^{\mathrm{Ad}_n^R(g)}) \otimes \mathbb{C}.$$

Here X^g means the fixed points of g in X. These are again compact, so that this all makes sense.

If $f : X \to Y$ is an equivariant map, then $f(X^g) \subset Y^g$. Using this we find that applying $K(f|_{X^g}) \otimes \mathrm{id}_{\mathbb{C}}$ summand wise commutes with the action of G, so that this defines a contravariant functor from compact G spaces to abelian groups.

The reduced version $\tilde{K}_{del,G}$ is defined in the same way, just taking the reduced K-groups of the fixed points.

Now we give the (reduced) delocalized K groups the structure of a cohomology, too:

Definition 2.4.4. For each integer $n \in \mathbb{Z}$ let $\tilde{K}^n_{del,G}(X) = \tilde{K}_{del,G}(S^{q(n)} \wedge X)$ where $q(n) \in \{0,..,7\}$ and $q(n) \equiv -n \bmod 8$. By setting $\tilde{K}^n_{del,G}(f) = \tilde{K}_{del,G}(S^{q(n)} \wedge f)$ this defines a contravariant functor $\tilde{K}^*_{del,G}$ from pointed compact G spaces to graded abelian groups.

Suppose $f, g : X \to Y$ are pointed homotopic by $H : X \wedge I_+ \to Y$, then their restrictions to fixed point sets are homotopic by $H : (X \wedge I_+)^g = X^g \wedge I_+ \to Y^g$. This shows that $\tilde{K}_{del,G}$, and thus also $\tilde{K}^*_{del,G}$, is pointed homotopy invariant.

Proposition 2.4.5. $\tilde{K}_{del,G}$, and thus also $\tilde{K}^*_{del,G}$, takes cofiber sequences to exact sequences.

Proof. Given a cofiber sequence, we can, by the homotopy invariance of $\tilde{K}_{del,G}$, assume it is of the standard form $A \to M_f \to C_f$ for some map $f : A \to X$ of well pointed spaces. Consider the sequence of fixed point sets $A^g \to (M_f)^g \to (C_f)^g$. The first map is still a closed inclusion, and we want to see whether the second map is still the quotient map. By the universal properties of subspaces and quotients there is a natural map $(M_f)^g/A^g \to (M_f/A)^g = (C_f)^g$. This map is a bijection, so, since both spaces involved are compact Hausdorff, an isomorphism.

Since K groups give exact sequences for quotients of closed subspaces, we get an exact sequence

$$\bigoplus_{g \in G_0} \tilde{K}(A^g) \otimes \mathbb{C} \to \bigoplus_{g \in G_0} \tilde{K}((M_f)^g) \otimes \mathbb{C} \to \bigoplus_{g \in G_0} \tilde{K}((C_f)^g) \otimes \mathbb{C}$$

Since we have tensored with $\mathbb{C}$ this is an exact sequence of $\mathbb{C}[G]$ modules. But for those taking invariants is an exact functor, so also the sequence of invariants is exact. $\square$

Now we want to give $\tilde{K}^*_{del,G}$ a suspension isomorphism. As with $\tilde{K}^*_G$, for $q(n) = 0,..,6$ one has $q(n-1) = q(n)+1$ and we simply uses the natural isomorphisms $S^{q(n)+1} \wedge X \cong S^{q(n)} \wedge S^1 \wedge X$ and apply $\tilde{K}_{del,G}$ to them. For $q(n) = 7$ we need to specify an isomorphism $\tilde{K}_{del,G}(X) \to \tilde{K}_{del,G}(S^8 \wedge X)$. Let $b_S \in \tilde{K}(S^8)$ be the spin Bott class. Then taking reduced external products with it gives an isomorphism

$$\bigoplus_{g \in G_0} \tilde{K}(X^g) \otimes \mathbb{C} \to \bigoplus_{g \in G_0} \tilde{K}(S^8 \wedge X^g) \otimes \mathbb{C}$$

Does this commute with the action of G? Since S^8 has the trivial G action we have that $n.(b_S \tilde{\otimes} E) = \bar{b_S}^{\phi(n)} \tilde{\otimes} (n^{-1}.)^* \bar{E}^{\phi(n)}$. By Bott periodicity and $\tilde{K}(S^8) \cong \mathbb{Z}$ we know that there is an integer k such that $b_S = k b_{\mathbb{C}} \tilde{\otimes} b_{\mathbb{C}} \tilde{\otimes} b_{\mathbb{C}} \tilde{\otimes} b_{\mathbb{C}}$. We have see earlier (Prop. 2.3.8) that $\bar{b_{\mathbb{C}}} = -b_{\mathbb{C}}$, so that $\bar{b_S} = (-1)^4 b_S = b_S$. This shows that for all $n \in G$ we have $n.(b_S \tilde{\otimes} E) = \bar{b_S}^{\phi(n)} \tilde{\otimes} (n^{-1}.)^* \bar{E}^{\phi(n)} = b_S \tilde{\otimes} n.E$. So the above isomorphism restricts to an isomorphism of invariants.

Proposition 2.4.6. With this suspension isomorphism $\tilde{K}^*_{del,G}$ is a reduced cohomology on pointed compact G spaces

In the next sections we want to discuss a certain natural transformation $K_G(X) \to K_{del,G}(X)$ which turns into an isomorphism after tensoring the domain with $\mathbb{C}$. The proof of this will depend crucially on these things fitting into cohomology theories, since it will allow us to reduce to the case where $X = (G/H)_+$ for $H \subset G$ some subgroup.

Let me comment on how one could convert $\tilde{K}_G$ into an ROG graded cohomology theory. We have seen that we should define $\tilde{K}_G^{-V}(X) := \tilde{K}_G(S^V \wedge X)$ for V an $\mathbb{R}$-linear representation of G. In principle it should be possible to define $\tilde{K}_G^V(X)$ by Bott periodicity: by Proposition 2.12 in [1] there is a Real representation W of G such that, after forgetting the complex structure, $W \cong V \oplus V \otimes \sigma$, where $\sigma = \mathbb{R}$ and G acts on σ as a sign through $\phi : G \to C_2$.

Then, formally, we should have an isomorphism

$$\tilde{K}_G^V(X) \cong \tilde{K}_G^{V+V\otimes\sigma}(S^{V\otimes\sigma} \wedge X) \cong \tilde{K}_G(S^{V\otimes\sigma} \wedge X),$$

where the first step is the desired ROG suspension isomorphism, and the second step is Bott periodicity with respect to W. I think that, with some care, one could define an ROG graded Real equivariant K-cohomology by putting $\tilde{K}_G^{[V-U]}(X) = \tilde{K}_G(S^{U\oplus V\otimes\sigma} \wedge X)$. The precise definition of such gadgets requires a lot of coherent structure, having to do with different representatives of elements of ROG and how the various suspension isomorphisms work together, and I don't know if that does work out.

Finally, let me now give a counterexample to a hypothetical stronger version of Proposition 2.1.4. So we are looking for a Real group (G, ϕ) and a compact space X on which G acts freely, such that $K_G(X) \neq K(X/G)$.

An example of this is given by $(G, \phi) = (C_2, \mathrm{id})$ and taking for X the two-sphere S^2 with the antipodal action. On the one hand, $X/C_2 \cong \mathbb{RP}^2$, and $K(\mathbb{RP}^2) \otimes \mathbb{C} \cong \mathbb{C}$. This can be seen, for example, from the (nonequivariant) Chern character isomorphism (Prop. 4.5 in [20])) and explicit calculations of $H^{ev}(\mathbb{RP}^2, \mathbb{C})$.

On the other hand, figuring $K_{C_2}(X)$ out is a bit more tricky. It is imaginable that $S^1 \wedge (X_+) \cong S^{3\sigma}/S^0$, and the proof is left to the spatial visualization of the reader (the key being to imagine X as the unit-sphere in 3σ). Here 3σ means $\sigma \oplus \sigma \oplus \sigma$ with σ the one-dimensional sign representation from above. Using this we have $K_{C_2}(X) \cong \tilde{K}_{C_2}(X_+) \cong \tilde{K}^1_{C_2}(S^{3\sigma}/S^0)$. The calculation will use the values of $\widetilde{K\mathbb{R}}$ on spheres, see table 2.1 for a list. From the long exact sequence for $S^0 \to S^{3\sigma}$ we look at the following five piece fragment:

$$\tilde{K}^0_{C_2}(S^{3\sigma}) \to \tilde{K}^0_{C_2}(S^0) \to \tilde{K}^1_{C_2}(S^{3\sigma}/S^0) \to \tilde{K}^1_{C_2}(S^{3\sigma}) \to \tilde{K}^1_{C_2}(S^0)$$

The last term is zero: by definition this is $\tilde{K}_{C_2}(S^7)$. By the standard decomposition this is $\widetilde{K\mathbb{R}}(S^7) = 0$, since (C_2, id) has only one isomorphy class of simple representations, and that is an $\mathbb{R}$-type. Similarly the second term is $\widetilde{K\mathbb{R}}(S^0) \cong \mathbb{Z}$.

The first term is zero: by spinor Bott periodicity we have $\tilde{K}_{C_2}(S^{3\sigma}) \cong \tilde{K}_{C_2}(S^8 \wedge S^{3\sigma})$. But $S^8 \wedge S^{3\sigma} \cong S^{3^\phi\mathbb{C}} \wedge S^5$, and so by equivariant Bott periodicity w.r.t. $3^\phi\mathbb{C}$ and the standard decomposition we have $\tilde{K}_{C_2}(S^{3\sigma}) \cong \tilde{K}_{C_2}(S^5) = \widetilde{K\mathbb{R}}(S^5) = 0$. Similarly, we find that the fourth term is $\tilde{K}^1_{C_2}(S^{3\sigma}) \cong \widetilde{K\mathbb{R}}(S^4) \cong \mathbb{Z}$.

Putting this together we get the short exact sequence

$$0 \to \mathbb{Z} \to K_{C_2}(X) \to \mathbb{Z} \to 0$$

so that $K_{C_2}(X) \cong \mathbb{Z}^2$. This shows that $K_{C_2}(X) \neq K(X/C_2)$, even though C_2 acts freely on X.

2.5 The delocalization isomorphism

In this and the next section our efforts come to fruition with the construction of the Real Chern character map, and the proof that it is a rational/complex isomorphism:

Theorem 2.5.1. Let X be a compact G space. Then there is a natural map (constructed in the next section)

$$\mathrm{ch}^R : K_G(X) \to \left(\bigoplus_{g \in G_0} H^{ev}(X^g, \mathbb{C}) \right)^G$$

When X has the homotopy type of a finite G-CW complex, for example when X is a closed manifold, then this is an isomorphism after tensoring with $\mathbb{C}$ on the domain.

In the $\mathbb{C}$ linear setting this is an old result, see for example [12]. Clarifications about the inner workings of the proof, resulting in the idea to generalize it to the Real case, came greatly from §7 of Schlarmann's PhD thesis [11] which I recommend to read at this point.

We will first construct a *delocalization*[5]

$$K_G(X) \to K_{del,G}(X) = \left(\bigoplus_{g \in G_0} K(X^g) \otimes \mathbb{C} \right)^G$$

which is an isomorphism in the relevant cases. Then applying the non equivariant Chern character isomorphism to each summand will give the above theorem.

To prove that the delocalization is an an isomorphism we extend it to a natural transformation of cohomologies. Then, using machinery from algebraic topology, to show that it gives an isomorphism on finite G-CW complexes it suffices to show that it induces isomorphisms on 'equivariant points' G/H for all subgroups $H \subset G$. On these points we use the isomorphisms $\tilde{K}_G^{-n}((G/H)_+) = \tilde{K}_G((G/H)_+ \wedge S^n) \cong \tilde{K}_H(S^n)$. Then, since S^n has trivial H action, we can use the standard decomposition, together with well known results of the values of $K\mathbb{C}$, $K\mathbb{R}$, and $K\mathbb{H}$ on spheres, to reduce this to a representation

[5]It is called this in reference to the delocalized equivariant cohomology of [21].

theoretic calculation. By the spin Bott periodicity we can reduce these checks to the cases $0 \leq n \leq 7$, which we take care of one by one.

So what does the delocalization do? Let E be a G vector bundle on X, $g \in G_0$, and $C_g \subset G$ the even subgroup generated by g. Then we can restrict E to a C_g vector bundle on $X^g = \{x \in X | g.x = x\}$. Since C_g acts trivially on X^g we can use the standard decomposition to show that this splits up into a direct sum of 'eigenbundles' $E|_{X^g} = \oplus_\lambda E|_{g=\lambda}$. Here $E_{g=\lambda} = \{e \in E|_{X^g} | g.e = \lambda e\}$ and the direct sum goes over $|C_g|$th roots of unity.

Definition 2.5.2. The delocalization is defined to be the map

$$\delta_X : K_G(X) \to \bigoplus_{g \in G_0} K(X^g) \otimes \mathbb{C},\ [E] \mapsto \bigoplus_{g \in G_0} \sum_\lambda [E|_{g=\lambda}] \otimes \lambda$$

This morphism should remind of the definition of the Real character, but instead of dimensions one takes elements in K-groups. Just as the Real character is a Real class function, this does also have certain invariance properties under the Real conjugation.

Let $n \in G$, $g \in G_0$, and E a G vector bundles on X. The map

$$n^{-1}. : X^{\mathrm{Ad}_n^R(g)} \to X^g,\ x \mapsto n^{-1}.x$$

is an isomorphism which fits into a commutative square

$$\begin{array}{ccc} E|_{X^{\mathrm{Ad}_n^R(g)}} & \xrightarrow{n^{-1}} & E|_{X^g} \\ \downarrow & & \downarrow \\ X^{\mathrm{Ad}_n^R(g)} & \xrightarrow{n^{-1}} & X^g \end{array}$$

By definition of G vector bundles this gives an isomorphism of $E|_{X^{\mathrm{Ad}_n^R(g)}}$ and $(n^{-1}.)^*\overline{E|_{X^g}}^{\phi(n)}$. A direct computation shows that this actually restricts to an isomorphism of $E|_{\mathrm{Ad}_n^R(g)=\lambda}$ and $(n^{-1}.)^*\overline{E|_{g=\lambda}}^{\phi(n)}$ for each λ. For this consider $e \in E|_{\mathrm{Ad}_n^R(g)=\lambda} \subset E|_{X^{\mathrm{Ad}_n^R(g)}}$. We have the following line of implications:

$$\begin{aligned} \mathrm{Ad}_n^R(g).e = \lambda e &\implies n^{-1}.\,\mathrm{Ad}_n^R(g).e = n^{-1}.(\lambda e) \\ &\implies g^{\phi(n)} n^{-1} e = \phi(n)(\lambda) n^{-1} e \\ &\implies g.n^{-1} e = \lambda n^{-1} e \end{aligned}$$

This is exactly the action defining $K_{del,G}$:

Proposition 2.5.3. The delocalization defines a natural transformation

$$\delta_X : K_G(X) \to K_{del,G}(X).$$

Remark. Even though not worked out above, this all also works when one restricts to pointed compact G spaces and reduced K groups.

The (reduced) delocalization extends to a natural transformation $\delta^* : \tilde{K}^*_G \to \tilde{K}^*_{del,G}$ by setting $\delta^n_X = \delta_{S^{q(n)}\wedge X}$.

Proposition 2.5.4. The delocalization is a transformation of cohomologies. That is, it commutes with the suspension isomorphisms.

Proof. We need to show that for every $n \in \mathbb{Z}$ and every pointed compact G space the following square commutes:

$$\begin{tikzcd} \tilde{K}_G(S^{q(n-1)} \wedge X) \arrow[r, "\delta_{S^{q(n-1)}\wedge X}"] \arrow[d, "\text{suspension}"] & \tilde{K}_{del,G}(S^{q(n-1)} \wedge X) \arrow[d, "\text{suspension}"] \\ \tilde{K}_G(S^{q(n)+1} \wedge X) \arrow[r, "\delta_{S^{q(n)+1}\wedge X}"] & \tilde{K}_{del,G}(S^{q(n)+1} \wedge X) \end{tikzcd}$$

For $q(n) = 0, .., 6$ this easily follows from the naturality of δ. For $q(n) = 7$ this reduces to showing that, for an arbitrary G-vector bundle E on X, one has $b_S \tilde{\otimes} E|_{g=\lambda} = (b_S \tilde{\otimes} E)|_{g=\lambda}$ as bundles on $S^8 \wedge X^g$. Note that here we have used b_S to denote the spin Bott classes both in $\tilde{K}_G(S^8)$ and in $\tilde{K}(S^8)$. This is justified since forgetting the G action on the former Bott class gives the latter. Since we defined b_S by pulling back through ϕ it can be represented by bundles on which G_0 acts trivially.

By Proposition 2.2.6 we can view these as represented by $\pi_1^* b_S \otimes \pi_2^* E|_{g=\lambda}$ and $(\pi_1^* b_S \otimes \pi_2^* E)|_{g=\lambda}$ on $S^8 \times X^g$ [6]. But here they are obviously equal since g acts fiberwise as identity on the first factor. □

Theorem 2.5.5. Let X be a finite pointed G CW complex. Then the complexified delocalization

$$\delta^*_X : \tilde{K}^*_G(X) \otimes \mathbb{C} \to \tilde{K}^*_{del,G}(X)$$

is an isomorphism. It follows that it is an isomorphism on compact G spaces with the homotopy type of a finite pointed G CW complex.

Proof. First of all note that the complexified delocalization is again a transformation of cohomologies.

By definition X has a filtration $* = X^{-1} \subset X^0 \subset X^1 \subset ... \subset X^n = X$ such that every inclusion fits into a cofiber sequence $X^{k-1} \to X^k \to \bigvee_i S^k \wedge (G/H_i)_+$ where the last wedge has finitely many summands. By studying the long exact sequences induced by these one can show that $\delta^*_{X^k}$ is an isomorphism if both $\delta^*_{X^{k-1}}$ and $\delta^*_{\bigvee_i S^k \wedge (G/H_i)_+}$ are. Since any cohomology is finitely additive this last one is an isomorphism if $\delta^*_{S^k \wedge (G/H_i)}$ is, and this again, by suspension, is an isomorphism if $\delta^*_{(G/H_i)_+}$ is.

[6] To show that $(b_S \tilde{\otimes} E)|_{g=\lambda}$ is represented by $(\pi_1^* b_S \otimes \pi_2^* E)|_{g=\lambda}$ on really uses the inclusion $\tilde{K}_{C_g}(S^8 \wedge X^g) \subset \tilde{K}_{C_g}(S^8 \times X^g)$ and naturality of the standard decomposition

So we 'only' need to show that δ^*_* and $\delta^*_{(G/H)_+}$ for any $H \subset G$ are isomorphisms. Further more, since δ^*_X and δ^{*+8}_X are the same morphism, we only need to show it for δ^{-n}_* and $\delta^{-n}_{(G/H)_+}$ for any $H \subset G$ and $n = 0, ..., 7$.

δ^*_* is always the unique isomorphism $0 \to 0$. For $\delta^{-n}_{(G/H)_+}$, $n = 0, ..., 7$, lets first study the domain and the codomain to find more convenient formulations.

For the domain we have that $S^n \wedge (G/H)_+ \cong \bigvee_{G/H} S^n$ and restriction to the summand at $1/H$ and of the action to H gives an isomorphism $\tilde{K}^{-n}_G((G/H)_+) \otimes \mathbb{C} = \tilde{K}_G(S^n \wedge (G/H)_+) \otimes \mathbb{C} \cong \tilde{K}_H(S^n) \otimes \mathbb{C}$. We then apply the (reduced) standard decomposition 2.1.5 to this:

$$\tilde{K}^{-n}_G((G/H)_+) \otimes \mathbb{C} \cong \bigoplus_{S_i} \widetilde{KE}_i(S^n) \otimes \mathbb{C}.$$

n	0	1	2	3	4	5	6	7
$\widetilde{K\mathbb{R}}(S^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
$\widetilde{K\mathbb{C}}(S^n)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$\widetilde{K\mathbb{H}}(S^n)$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0

Table 2.1: Values of reduced K groups of real, complex, and quaternionic bundles.

Table 2.1 lists the values of the K groups possibly appearing in the standard decomposition. Notice that the domain of the delocalization is 0 for n odd. For n even there are basically 2 cases: for $0, 4$ all simple representations of H contribute, for $2, 6$ only the simple representations of $\mathbb{C}$ type do.

For the codomain notice that, since the action of G permutes the summands via the Real conjugation, we can restrict to a set of representatives of Real conjugacy classes of G and let only the Real centralizers $Z^R_g := \{n \in G | \operatorname{Ad}^R_n(g) = g\}$ act on them:

$$\tilde{K}_{del,G}(X) = \bigoplus_{[g]\in \mathrm{RC}(G)} \left(\tilde{K}(X^g) \otimes \mathbb{C}\right)^{Z^R_g}$$

The fixed points of $S^n \wedge (G/H)_+$ correspond to the sum $\bigvee_{(G/H)^g} S^n$, so for the codomain of $\delta^{-n}_{(G/H)_+}$ we find

$$\begin{aligned}\tilde{K}^{-n}_{del,G}((G/H)_+) &= \bigoplus_{[g]\in \mathrm{RC}(G)} \left(\tilde{K}\left(\bigvee_{(G/H)^g} S^n\right) \otimes \mathbb{C}\right)^{Z^R_g} \\ &= \bigoplus_{[g]\in \mathrm{RC}(G)} \left(\bigoplus_{(G/H)^g} \tilde{K}\left(S^n\right) \otimes \mathbb{C}\right)^{Z^R_g}\end{aligned}$$

In the last term Z_g^R only permutes the summands (and complex conjugates the bundles when the element is odd) since G acts trivially on S^n. Comparing with table 2.1 we see that this is zero for odd n. For even n we have that $\tilde{K}(S^n) = \mathbb{Z}$ is, by Bott periodicity, generated by $b_{\mathbb{C}}^{\tilde{\otimes} n/2}$. Since $\bar{b_{\mathbb{C}}} = -b_{\mathbb{C}}$ we see that the complex conjugation action is trivial for $n/2$ even and the sign action for $n/2$ odd.

We can condense the notation a bit by viewing the sum of $\mathbb{C}$'s over $(G/H)^g$ as the set $\mathrm{Hom}((G/H)^g, \mathbb{C})$ of complex valued functions on $(G/H)^g$. The action of $h \in Z_g^R$ is then given by $h.f(aH) = \phi(h)^{n/2} f(h^{-1}aH)$ where $\phi(h)$ is to be interpreted as a sign.

For $n/2$ even the action of Z_g^R is purely by permuting $(G/H)^g$, so we can quotient it out without losing information. Let us study the resulting set $Z_g^R \backslash (G/H)^g$. Consider the function of sets

$$\begin{aligned} f : X = \{a \in G | aH \in (G/H)^g\} &\to Y = \{h \in H_0 | [h] = [g] \text{ in } RC(G)\} \\ a &\mapsto \mathrm{Ad}^R_{a^{-1}}(g) \end{aligned}$$

This is well defined since

$$aH \in (G/H)^g \iff gaH = aH \iff a^{-1}ga \in H \iff \mathrm{Ad}^R_{a^{-1}}(g) \in H$$

and $\phi(\mathrm{Ad}^R_{a^{-1}}(g)) = \phi(g) = 1$. f is not injective, but very close:

$$\begin{aligned} f(a) = f(b) &\iff \mathrm{Ad}^R_{a^{-1}}(g) = \mathrm{Ad}^R_{b^{-1}}(g) \iff g = \mathrm{Ad}^R_{ab^{-1}}(g) \iff ab^{-1} \in Z_g^R \\ &\iff Z_g^R a = Z_g^R b \end{aligned}$$

On the other hand, f is surjective: by definition

$$h \in Y \iff \exists a \in G : \mathrm{Ad}^R_{a^{-1}}(g) = h,$$

and this a is automatically in X. So f descends to a bijection $Z_g^R \backslash X \to Y$. On $Z_g^R \backslash X$ we can act by multiplying with H from the right. We can use the bijection f to transport this to an action on Y, which can be seen to be the Real conjugation action. Quotienting this out we get a bijection

$$f : Z_g^R \backslash X / H = Z_g^R \backslash (G/H)^g \to Y/H = \{[h] \in RC(H) | [h] = [g] \text{ in } RC(G)\}$$

Connecting back to the delocalized K groups, we are looking at the direct sum over $[g] \in RC(G)$ of functions from $\{[h] \in RC(H) | [h] = [g] \text{ in } RC(G)\}$ to $\mathbb{C}$. Since every $[h] \in RC(H)$ appears exactly once, this is isomorphic to RClass(H), the Real class functions on H!

$$\tilde{K}^{-n}_{del,G}((G/H)_+) \cong \mathrm{RClass(H)}$$

For $n/2$ odd one also has to take care of signs. In fact, we will end up with anti Real class functions, but maybe in a kind of unexpected way. Let $\sigma\mathrm{Class}(H)$ be the set of anti Real class functions on H. We want to prove that

$$\begin{aligned}\sigma\mathrm{Class}(H) &\to \bigoplus_{[g]\in\mathrm{RC}(G)} \mathrm{Hom}((G/H)^g, \mathbb{C})^{Z_g^R} \\ f &\mapsto \bigoplus \left[aH \mapsto \phi(a) f(\mathrm{Ad}^R_{a^{-1}} g)\right]\end{aligned}$$

is an isomorphism, where $\phi(a)$ is a sign. First of all, is this well defined? If we took ah, for $h \in H$, to evaluate the function, we get

$$\phi(ah) f(\mathrm{Ad}^R_{h^{-1}} \mathrm{Ad}^R_{a^{-1}} g) = \phi(a) f(\mathrm{Ad}^R_{a^{-1}} g),$$

so it does not depend on the representative of the H coset. Acting with $n \in Z_g^R$ on $\left[aH \mapsto \phi(a) f(\mathrm{Ad}^R_{a^{-1}} g)\right]$ gives

$$\left[aH \mapsto \phi(n)\phi(n^{-1}a) f(\mathrm{Ad}^R_{a^{-1}n} g)\right] = \left[aH \mapsto \phi(a) f(\mathrm{Ad}^R_{a^{-1}} g)\right],$$

so these functions are Z_g^R invariant.

On the right hand side one evaluates f in at least one representative of every Real conjugacy class of H, so the kernel of the morphism is zero. For surjectivity, notice that, similar to the previous discussion, the right hand side is isomorphic to functions $RC(H) \to \mathbb{C}$ with the condition that they are zero on $[h]$ whenever there is an $\omega \in H_1$ such that $[h] = [\mathrm{Ad}^R_w h]$. By picking representatives for each Real conjugacy class of H these can be extended to anti Real class functions of H which can be used to verify surjectivity.

The above discussions show that the delocalizations $\delta^{-n}_{(G/H)_+}$ are the isomorphisms $0 \to 0$ for n odd. Looking closely, they also show that the delocalization for n even at least has a chance to be an isomorphism, but to prove it we do need to know what it does. So let us investigate, in the cases $n = 0, 2, 4, 6$, what morphism the delocalization induces from the standard decomposition of $\tilde{K}_H(S^n)$ to the (anti) Real class functions of H.

$n = 0$: In this case we don't even use the standard decomposition. Given an element of $[V] \in RH = K_H(*) = \tilde{K}_H(S^0)$, the corresponding element of $\tilde{K}_G((G/H)_+)$ is $G \times_H V$ on G/H (and 0 on $+$). To apply the delocalization and so on, we basically need to compute the following: Given $gaH = aH$, how does g act on $G \times_H V|_{aH}$? Given a basis v_i of V, we have a basis $[a, v_i]$ of $G \times_H V|_{aH}$. Acting with g gives, since $a^{-1}ga \in H$, $g[a, v_i] = [a, (a^{-1}ga)_{ji} v_j] = \phi(a)((a^{-1}ga)_{ji})[a, v_j]$. From this, and the usual trick that all eigenvalues are in $U(1)$, we see that the λ eigenspaces of g on $G \times_H V|_{aH}$ correspond directly to the λ eigenspaces of $\mathrm{Ad}^R_{a^{-1}} g$ on V! So making this into a Real class function, as discussed above, gives exactly the Real character of V. By Corollary 5.10 of [1] the Real characters of irreducible Real representations of H form a $\mathbb{C}$ basis of Real class functions on H, so this is an isomorphism.

$n = 2$: Only the direct summands of $\mathbb{C}$ type irreducibles in the standard decomposition of $\tilde{K}_H(S^2)$ survive being tensored with $\mathbb{C}$, so let us discuss only what the delocalization does to them. Let S be an irreducible Real representation of $\mathbb{C}$ type of H, and let E

be its endomorphism ring. Choosing an isomorphism $c : \mathbb{C} \cong E$ is the same as choosing only the image J of i under it. We can now split the irreducible S depending on whether the complex structures given by i and J match up or not: $S^{\pm} := \{s \in S | Js = \pm is\}$. These are both still complex vector spaces, and representations of H_0: For $h \in H_0$ and $s \in S^{\pm}$ one has $Jhs = hJs = h \pm is = \pm ihs$ since J is a morphism of H representations.

As for $\tilde{K\mathbb{C}}(S^2)$, $\tilde{KE}(S^2)$ is generated by $b_E = [\underline{E} - H_E]$, where H_E is a left E module version of the dual of the tautological line bundle on $S^2 = P\mathbb{C}^2$. Under the inverse of the standard decomposition this is mapped to $b_E \otimes_E \underline{S}$, which is now a complex vector bundle through the action on S. Using the above splitting, and up to a choice of what to call H_E, we find that this is equal to $b_{\mathbb{C}} \otimes \underline{S^+} \oplus \bar{b_{\mathbb{C}}} \otimes \underline{S^-}$ as an element of $\tilde{K}_{H_0}(S^2)$.

Extending $b_E \otimes_E \underline{S}$ to $(G/H)_+ \wedge S^2$ and then restricting to $(aH)_+ \wedge S^2$ for $gaH = aH$ we find, similar to $n = 0$, that this is isomorphic to $\overline{(b_E \otimes_E \underline{S})}^{\phi(a)}$ with g acting as $a^{-1}ga \in H_0$. Applying the above splitting, and recalling $\bar{b_{\mathbb{C}}} = -b_{\mathbb{C}}$, this is equal to $\phi(a) b_{\mathbb{C}} \otimes \overline{(\underline{S^+} - \underline{S^-})}^{\phi(a)}$. Again, making the complex conjugation into an inversion on eigenvalues, and putting $b_{\mathbb{C}} = 1$, this gives the element $\bigoplus_{[g]} \left[aH \mapsto \phi(a)(\chi_{S^+}(\mathrm{Ad}^R_{a^{-1}} g) - \chi_{S^-}(\mathrm{Ad}^R_{a^{-1}} g))\right]$. Here $\chi_{S^{\pm}}$ means the characters of the H_0 representations $S^{\pm}$.

On the other hand, we could make an anti Real class function on H for every irreducible of $\mathbb{C}$ type S, by sending it to $\chi_{S^+} - \chi_{S^-}$. We will check below that this is indeed an anti Real class function. If it was, it would exactly give the above element under the isomorphism discussed above. So the delocalization is an isomorphism if these *anti Real characters* form a basis for the anti Real class functions.

If $H = H_0$ then either S^+ or S^- is zero, the anti Real class functions are just the class functions, and the anti Real character is just the regular character. Also all irreducibles are of $\mathbb{C}$ type, so their characters do form a basis.

For the case $H \neq H_0$ let $w \in H_1$. Then, as H_0 representations, $wS^+ \subset S^-$ shows that both are non trivial subrepresentations of S. Theorem 5.4 of [1] then tells us that $S^- \cong w \star \overline{S^+}$. Here $w \star V$, for an H_0 representation V means the same vector space, but $h \in H_0$ acts like whw^{-1} originally did. From this we conclude the following formula on characters: $\chi_{S^-} = \chi_{S^+} \circ \mathrm{Ad}^R_w$. Not that this actually holds for an arbitrary $w \in H_1$.

Consider the endomorphism P on the class functions of H_0 given by precomposition by Ad^R_w. Then P^2 is precomposition by Ad_{w^2} with $w^2 \in H_0$, so equal to identity on class functions. One easily sees that the $+1$ eigenspace of P are the Real class functions, and the -1 eigenspace are the anti Real class functions. This also shows that $\chi_{S^+} - \chi_{S^-}$ is an anti Real class function.

Consider the projector $1 - P$ onto the anti Real class functions. It must send a basis of the class functions of H_0, given by characters of irreducibles of H_0, to a generating set of the anti Real class functions. By Theorem 5.4 and Corollary 5.5 of [1] this generating set is then $\{0, \pm(\chi_{S_1^+} - \chi_{S_1^-}), ..., \pm(\chi_{S_k^+} - \chi_{S_k^-})\}$ for S_i the $\mathbb{C}$ type irreducibles of H. On can now, of course drop the 0 and the signs and it will still be a generating set. The remaining elements are then orthogonal (with respect to the standard inner product of class functions of H_0), thus they form a basis of the anti Real class functions. But these are exactly the anti Real characters of $\mathbb{C}$ type irreducibles of H.

$n = 4$: In this dimension all summands of the standard decomposition of $\tilde{K}_H(S^4)$ survive being tensored with $\mathbb{C}$. Treating the $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ type irreducibles independently we show that their images produce a basis of Real class functions, showing that the delocalization is an isomorphism.

First let us get the case $H = H_0$ out of the way. Then we only have to deal with $\mathbb{C}$ types S and their endomorphism rings are canonically isomorphic to $\mathbb{C}$, so that $S = S^+$ and $S^- = 0$. In the standard decomposition then $\tilde{K}\mathbb{C}(S^4)$ is generated by $b_\mathbb{C}\tilde{\otimes}b_\mathbb{C}$. Following the $n = 2$ discussion we find that this generator at the summand corresponding to S is mapped to the character of S in the class functions of $H = H_0$. Since the characters of irreducibles form a basis of class functions we are done.

When $H \neq H_0$ let $w \in H_1$ and let us first consider a $\mathbb{C}$ type S and E its endomorphism ring. Then, as an element of $\tilde{K}_{H_0}(S^4)$, the generator at S is mapped to $b_\mathbb{C}\tilde{\otimes}b_\mathbb{C} \otimes (\underline{S^+} \oplus \underline{S^-})$, since $\overline{b_\mathbb{C}\tilde{\otimes}b_\mathbb{C}} = b_\mathbb{C}\tilde{\otimes}b_\mathbb{C}$. This is then mapped, by the delocalization, to $\chi_{S^+} + \chi_{S^-}$, which is exactly the Real character of S.

Let O be an $\mathbb{R}$ type irreducible of H. Its endomorphism ring is given by $E = \mathbb{R}\,\mathrm{id}_O$. What is a generator $b \in \tilde{K}\mathbb{R}(S^4)$? We actually only need to know what its complexification is, since $b \otimes_\mathbb{R} \underline{O} = (b \otimes_\mathbb{R} \underline{\mathbb{C}}) \otimes \underline{O}$. According to Theorem 5.12 in [13] this generator can be chosen such that $b \otimes_\mathbb{R} \underline{\mathbb{C}} = 2b_\mathbb{C}\tilde{\otimes}b_\mathbb{C}$. Thus the generator at O will be mapped, by the inverse of the standard decomposition and the delocalization, to twice the Real character of O.

Finally, let Q be an $\mathbb{H}$ type irreducible of H. Its endomorphism ring E is now very tricky to deal with since it is not commutative. Choosing an isomorphism $\mathbb{H} \cong E$ is the same as choosing elements $I, J, K \in E$ which behave like the standard $i, j, k \in \mathbb{H}$. Split Q into $Q^\pm = \{q \in Q | Iq = \pm iq\}$. As H_0 representations, these are subrepresentations of Q and J is an isomorphism between them. Thus if q_i is a $\mathbb{C}$ basis of Q^+, then q_i, Jq_i is a $\mathbb{C}$ basis of Q.

Consider a bundle F of right E modules. If f_i is a local E frame then f_i, f_iJ is a local $\mathbb{C}' = \mathbb{R}\,\mathrm{id} \oplus \mathbb{R}I \subset E$ frame. We will view F as a $\mathbb{C}$ bundle through the morphism $\mathbb{C} \to E\, 1 \mapsto \mathrm{id}, i \mapsto I$ when necessary. We want to show that $F \otimes_E Q \cong F \otimes_\mathbb{C} Q^+$ as $\mathbb{C}$ bundles. In the presence of a local frame we can make the (local) morphism of $\mathbb{C}$ bundles $F \otimes_\mathbb{C} Q^+ \to F \otimes_E Q$ defined by $f_i \otimes q_j \mapsto f_i \otimes q_j$ and $f_iJ \otimes q_j \mapsto f_iJ \otimes q_j$. This is a fiberwise isomorphism, but to make it globally well defined we need to show that it is independent of the choice of the local E frame f_i. If g_k is another local E frame then one can write $g_k = f_i(C^1_{ki} + JC^2_{ki})$ for $C^{1,2}_{ik}$ smooth local sections of $\mathbb{C}$. Using this one can check that both local frames yield the same local morphism, so they glue to a global isomorphism.

According to Theorem 5.16 in [13] $\tilde{K}E(S^4)$ now has a generator b which gives $b_\mathbb{C}\tilde{\otimes}b_\mathbb{C}$ when viewed as a $\mathbb{C}$ bundles. Applying the same yoga as above we see that the generator of the summand corresponding to Q is mapped to the character of Q^+. By Theorem 5.4 of [1] this is exactly half of the Real character of Q.

In conclusion the generators in the standard decomposition are mapped to non zero multiples of the Real characters of the irreducible representations of H. These are, of course, still a basis of the Real class functions.

$n = 6$: This case behaves exactly like the $n = 2$ case, one only needs to replace $b_{\mathbb{C}}$ by $b_{\mathbb{C}}\tilde{\otimes}b_{\mathbb{C}}\tilde{\otimes}b_{\mathbb{C}}$. This works since $b_{\mathbb{C}}\tilde{\otimes}b_{\mathbb{C}}\tilde{\otimes}b_{\mathbb{C}}$ is still odd under complex conjugation. $\square$

2.6 The Real Chern character

In this section we will use the delocalization and the well known theory of the (nonequivariant) Chern character, see Proposition 4.5 in [20], and the Chern-Weil theory, see appendix C of [23], to construct the Real Chern character isomorphism, and discuss some applications.

Proposition 2.6.1. When X is a compact G space with finite homotopy type, then the *Real Chern character*

$$\begin{aligned} \mathrm{ch}_G^R : K_G(X) \otimes \mathbb{C} &\to \bigoplus_{[g]\in \mathrm{RC}(G)} H^{ev}(X^g, \mathbb{C})^{Z_g^R} \\ [E] \otimes 1 &\mapsto \bigoplus_{[g]\in \mathrm{RC}(G)} \sum_{\lambda} \lambda \mathrm{ch}\left(E|_{g=\lambda}\right) \end{aligned}$$

is an isomorphism. Here $\mathrm{ch} : K(X) \to H^{ev}(X, \mathbb{C})$ denotes the usual Chern character, and on the right hand side $n \in Z_g^R$ acts as a pullback along $n^{-1}. : X^g \to X^g$ and as the sign $\phi(n)$ in degrees 2 mod 4.

Proof. By Theorem 2.5.5 we have that

$$\tilde{K}_G(X) \otimes \mathbb{C} \cong \bigoplus_{[g]\in \mathrm{RC}(G)} (\tilde{K}(X^g) \otimes \mathbb{C}))^{Z_g^R}$$

By the usual, non equivariant, Chern character isomorphism we have that $\tilde{K}(X^g) \otimes \mathbb{C} \cong \tilde{H}^{ev}(X^g, \mathbb{C})$. Transporting the action of Z_g^R along this isomorphism gives exactly the action described above. Finally, we apply this to X_+ to get the unreduced version above. $\square$

The Real Chern character is an extremely important tool to study the *torsion free part* of $K_G(X)$ computationally. For now, let me give a few important consequences:

Proposition 2.6.2. Consider the map $K_G(X) \to K_{G_0}(X)$ forgetting the action of the odd elements of G. If X has finite homotopy type this becomes an injection after tensoring with $\mathbb{C}$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} K_G(X)\otimes\mathbb{C} & \longrightarrow & \left(\bigoplus_{g\in G_0} H^{ev}(X^g,\mathbb{C})\right)^G \\ \big\downarrow & & \big\downarrow \\ K_{G_0}\otimes\mathbb{C} & \longrightarrow & \left(\bigoplus_{g\in G_0} H^{ev}(X^g,\mathbb{C})\right)^{G_0} \end{array}$$

The right vertical morphism is an inclusion of invariants with respect to G into invariants with respect to only $G_0 \subset G$. So, since the horizontal maps are isomorphisms, the left vertical map is an injection, too. $\square$

On the other hand, this is not necessarily true for the torsion part of K_G and K_{G_0}. For example, consider $G = (C_2, \phi = \mathrm{id}_{C_2})$ acting trivially on S^1. Then $G_0 = \{1\}$ is the trivial group and $K_{G_0}(S^1) \cong \mathbb{Z}$. By the standard decomposition $K_G(S^1) \cong K\mathbb{R}(S^1) \cong C_2 \oplus \mathbb{Z}$, so the forgetful map is not injective on the torsion part.

Lemma 2.6.3. If X is a pointed closed G *manifold* then the isomorphism above can be described using the Chern-Weil theory. Specifically, if E is a G bundle on X and F is the curvature from of some G equivariant linear connection on E then its image can be described as

$$\bigoplus_{[g]} \mathrm{tr}\left[ge^{\frac{F|_{X^g}}{2\pi i}}\right] \in \bigoplus_{[g]} \tilde{H}^{ev}_{dR}(X^g,\mathbb{C})^{Z^R_g}$$

Proof. Naturally the delocalization maps $[E]$ to $\bigoplus \sum_\lambda [E|_{g=\lambda}] \otimes \lambda$ which is then mapped, by the Chern character, to $\bigoplus \sum_\lambda \lambda \mathrm{ch}(E|_{g=\lambda})$. The connection and curvature form restrict to connections and curvature forms on the $E|_{g=\lambda}$ and the above Chern characters are equal to traces of exponentials of these curvatures. The sum of traces is the trace of the direct sum, and the same is true for the exponential. Finally, since we can replace $\lambda\,\mathrm{id}$ by g on $E|_{g=\lambda}$, we find that the previous expression equals $\bigoplus \mathrm{tr}\left[ge^{\frac{F|_{X^g}}{2\pi i}}\right]$. $\square$

Remark. Actually, by the previous proposition, one can get away with only asking for a G_0 equivariant connection.

In this form the Real Chern character yields itself to computations very nicely. In the light of Proposition 2.6.2 it must be said, though, that it gives no more information than its well known $\mathbb{C}$-linear counter part (to which it reduces for $G = G_0$). That is, if one wanted to compute whether $[E] = 0$ in $K_G(X) \otimes \mathbb{C}$, then this is true if and only if it is true in $K_{G_0}(X) \otimes \mathbb{C}$, and one could just forget about the odd elements.

On the other hand, invariance with respect to the odd elements does place more restrains on the possible values of the Real Chern character, reducing the number of necessary computations to conclude whether $[E] = 0$ or not in $K_G(X) \otimes \mathbb{C}$. In the next chapter we will see some examples of this in action.

2.7 The localization theorem

Comparing the delocalization isomorphism 2.5.5 and the main Proposition on the structure of Real representation rings 1.4.10 there is some similarity. In some sense, for S a cyclic even group, taking the quotient $RS \to \widetilde{RS}$ is like restricting the character to generators of S. Thus doing this for all (conjugacy classes of) even cyclic subgroups is like evaluating the character at all even elements, and then we take invariants with respect to the conjugation action. On the other hand, in the delocalization we also, in some sense, evaluate the characters on even elements of the group and then take invariants. The link between these is provided by the localization theorem of Segal, whose generalization to Real equivariant K theory I want to briefly discuss here.

So let $\mathfrak{p} \subset RG$ be a prime ideal, X a locally compact G-space.

Theorem 2.7.1 (Segal's localization theorem)**.** Let

$$X^{\mathfrak{p}} = \{\, x \in X \mid \mathfrak{p} \text{ comes from } G_x \,\},$$

where $G_x \subset G$ is the isotropy (or stabilizer) group of x. Then the localization at $\mathfrak{p}$ of the restriction $K_G^*(X)_{\mathfrak{p}} \to K_G^*(X^{\mathfrak{p}})_{\mathfrak{p}}$ is an isomorphism.

Before outlining its proof, let me show how this gives a bridge between the two results mentioned earlier. $K_G(X) \otimes \mathbb{C}$ is a module of $RG \otimes \mathbb{C} \cong \mathrm{RClass}(G)$, the ring of Real class functions. This ring is a product of copies of $\mathbb{C}$ for each Real conjugacy class in G. Its maximal ideals $\mathfrak{p}_{[g]}$ are given by the characters which vanish at a given Real conjugacy class $[g]$ (these are already prime ideals in RG), and every module splits into the direct sum of localizations at these. Thus we get

$$K_G(X) \otimes \mathbb{C} \cong \bigoplus_{[g]} K_G(X)_{\mathfrak{p}_{[g]}} \otimes \mathbb{C} \cong \bigoplus_{[g]} K_G(X^{\mathfrak{p}_{[g]}})_{\mathfrak{p}_{[g]}} \otimes \mathbb{C}.$$

The spaces appearing in the last expression are, very importantly, ambient unions of X^h, $h \in [g]$, and not disjoint unions. If it wasn't for that we could deduce the delocalization with not too much effort. On the other hand, it might be worthwhile to also study this in the case of other characteristics, maybe getting a route to mod q characters.

But what is $K_G(X)$ for X only locally compact?

Definition 2.7.2. Let X be a locally compact G-space, and A a closed sub-G-space. Then define

$$K_G^*(X) := \tilde{K}_G^*(X^+) \text{ and } K_G^*(X, A) := \tilde{K}_G^*(\mathrm{cofib}(A^+ \to X^+))$$

where X^+, A^+ are the one point compactifications and we take the added "point at infinity" as the basepoint.

It is a quick check that this coincides with the original definition if X is already compact, since then $X^+ = X \coprod *$.[7] This is now functorial only for proper maps. With this in place we see that the proofs of Proposition 2.9 through 2.11 in [4] work out word for word. Then the only obstacle in adapting Segal's proof of the localization theorem is the following:

Proposition 2.7.3. $X^{\mathfrak{p}}$ is exactly the points $x \in X$ such that $K_G^*(Gx)_{\mathfrak{p}} \neq 0$.

Proof. One direction is easy: if $x \in X^{\mathfrak{p}}$, then $K_G^*(Gx)_{\mathfrak{p}} \neq 0$:

$$x \in X^{\mathfrak{p}} \implies (RG_x)_{\mathfrak{p}} = K_G^0(Gx)_{\mathfrak{p}} \neq 0$$

by definition of "coming from", and I used that $Gx \cong G/G_X$ as G-spaces.

For the back direction we need a more sophisticated argument: We do have that

$$x \notin X^{\mathfrak{p}} \implies (RG_x)_{\mathfrak{p}} = K_G^0(Gx)_{\mathfrak{p}} = 0,$$

but, unlike in the $\mathbb{C}$-linear case, the inclusion $RG_x = K_{G_x}^0(*) \subset K_{G_x}^*(*) = K_G^*(Gx)$ might be a strict inclusion, so that this is not enough.

Luckily, there is a ($\mathbb{Z}$-graded) ring structure on $K_G^*(Gx)$ making it into a ($\mathbb{Z}$-graded) RG algebra such that the unit lies in $K_G^0(Gx)$. So, since the unit equals zero after localization, the whole algebra must be zero after localization. This structure works as follows: Elements of $K_G^a(X)$ and $K_G^b(X)$ are represented by vector bundles on $S^{q(a)} \wedge X^+$ and $S^{q(b)} \wedge X^+$. We take their reduced external product on $S^{q(a)} \wedge S^{q(b)} \wedge X^+ \wedge X^+$ and pull back via the (reduced) diagonal on X^+. This gives a vector bundle on $S^{q(a)+q(b)} \wedge X^+$. Finally we apply spinor Bott periodicity until we get an element of $K_G^{a+b}(X)$ ($q(a)+q(b)$ and $q(a+b)$ can only differ by a multiple of 8). $\square$

I don't want to give an adaption of the proof in full detail here, since it turned out that I didn't use this result for anything else and I find the proof not very elucidating.

2.8 The completion theorem

Above we have seen that certain maps, inclusions of fixed points, induce isomorphisms in Real equivariant K groups after we take the algebraic localization of them. In this section we will see that certain "almost" G-homotopy equivalences induces isomorphisms in Real equivariant K groups after we take an appropriate algebraic completion. In the $\mathbb{C}$-linear case this is known as the Atiyah-Segal completion theorem, and the proof I want to present is adapted from [9].

Consider the following:

[7] At least in zero degree, since we have not really defined graded unreduced K theory, though comparing with the reduced definition should give a good hint on how that works.

Theorem 2.8.1 (Equivariant J.H.C. Whitehead theorem)**.** Let $f : X \to Y$ be a G-map between G-CW complexes which induces homotopy equivalences between the fixed points $f^H : X^H \to Y^H$ for all subgroups $H \subset G$. Then f is already a G-homotopy equivalence.

This is one of the main tools in equivariant algebraic topology and part of what makes G-CW complexes so important there. If X and Y are finite, this theorem shows that a map f fulfilling the hypothesis of the theorem above induces isomorphisms on Real equivariant K groups.

Let us weaken the assumption in the following way:

Definition 2.8.2.

1. A *Real family of subgroups* $\mathcal{J}$ is a set of (Real) subgroups of G containing $\{1\}$, closed under taking subgroups and Real conjugation.
2. A G-map $f : X \to Y$ between G-CW complexes is called a $\mathcal{J}$-equivalence if $f^H : X^H \to Y^H$ is a homotopy equivalence for all $H \in \mathcal{J}$.

The idea is that such a $\mathcal{J}$-equivalence will induce an isomorphism after a suitable algebraic completion at $\mathcal{J}$. But to define this we will need to slightly adjust our definition of K groups.

Let's forget about the generalization we made in the previous section to define K groups of non-compact spaces and try something different. We could, for a G-CW complex X, take the limit of the K groups for all finite subcomplexes as the definition, giving us a functor defined for all G-CW complexes. Sadly this lacks, already in the non-equivariant case, the half-exactness property we used to make Real equivariant K-theory a cohomology. This has to do with the fact that taking limits of exact sequences is not necessarily an exact functor. One could now either really study the non-exactness of that, leading to the theory of $\lim^1$ terms and so on, or one could take the whole diagram, instead of its limit, as the definition of the "K group". Such objects are called *progroups* (since they are, in old terminology, projective systems of groups) and an exposition to them can be found in section 2 of [19].

Definition 2.8.3.

1. For X a G-CW complex, denote by $\mathcal{K}_G^*(X)$ the progroup
$$\mathcal{K}_G^*(X) := \{\, K_G^*(X_i) \mid X_i \subset X \text{ a finite subcomplex} \,\}\,.$$
2. The *procompletion* of $\mathcal{K}_G(X)$ at $\mathcal{J}$ is $\mathcal{K}_G(X)_{\hat{\mathcal{J}}} = \{K_G(X_i)/JK_G(X_i)\}$ where, again, X_i runs through finite subcomplexes of X, and J runs through finite products of ideals of the form $I_H^G = \ker RG \to RH$ for $H \in \mathcal{J}$.

It turns out, as stated in [9], that both of these define equivariant cohomologies on G-CW complexes, now with values in progroups (or, more precisely, in pro-RG-modules). This notion is now flexible enough to prove the completion theorem:

Theorem 2.8.4 (Completion theorem)**.** Let $f : X \to Y$ be a G-map between G-CW complexes, and let $\mathcal{J}$ be a Real family of subgroups. If f is a $\mathcal{J}$-equivalence, then f induces an isomorphism of progroups $\mathcal{K}_G(Y)_{\hat{\mathcal{J}}} \to \mathcal{K}_G(X)_{\hat{\mathcal{J}}}$.

Before I present the proof let me discuss this a bit. One particular case of this is the following: Let EG be a (nonequivariantly) contractible space with free G action. Then the unique map $EG \to *$ is a $\mathcal{J}$-equivalence for $\mathcal{J}$ only containing the trivial subgroup. Thus we get an isomorphism $\mathcal{K}_G^*(*)_{\hat{\mathcal{J}}} \to \mathcal{K}_G^*(EG)_{\hat{\mathcal{J}}}$. Since $*$ is a finite complex, we can do away with it being a projective system and replace it with the true algebraic completion of $K_G^*(*)$ at the ideal I_1^G. In the $\mathbb{C}$-linear case it turns out that the projective system $\mathcal{K}_G^*(EG)$ is then already $\mathcal{J}$ complete. Also, in the $\mathbb{C}$-linear case, we can quotient out free actions of the group, so that we end up with an isomorphism

$$K_G^*(*)_{\hat{I_1^G}} \to \mathcal{K}^*(BG)$$

computing the K progroups of classifying spaces as algebraic completions of representation rings. In the Real case though we can only quotient out free actions of even subgroups, so something of this form can't be deduced in general.

The proof given by [9] must be altered enough that I want to present it in some detail here. The idea is the following: First, by the long exact sequence for the cofiber sequences $X \to Y \to C_f$, we can reduce this to saying that the $\tilde{\mathcal{K}}_G^*(T)_{\hat{\mathcal{J}}}$[8] vanish for any pointed G-CW complex such that T^H is contractible for $H \in \mathcal{J}$. Then, by induction, we can assume that the theorem holds for all proper subgroups of G, and we may assume that $G \notin \mathcal{J}$ (that case follows from the J.H.C. Whitehead theorem)[9]. Then there is a special G-CW complex Y which has a lot of nice properties, one being that $Y^G = S^0$ and Y^H being contractible for all proper subgroups of G. We then smash the cofiber sequence $S^0 \to Y \to Y/S^0$ with T and apply the long exact sequence again to reduce to the statements that both

$$\tilde{\mathcal{K}}_G^*(T \wedge Y)_{\hat{\mathcal{J}}} \text{ and } \tilde{\mathcal{K}}_G^*(T \wedge (Y/S^0))_{\hat{\mathcal{J}}}$$

vanish (as progroups). The first does so because of the properties of Y, the second because of the properties of T.

Let me now define Y and prove that it has the necessary properties. For every proper subgroup $H \subset G$ for which $H_0 \subset G_0$ is a strict inclusion, there is a Real representation V_H such that $V_H^G = 0$ and $V_H^H \neq 0$. It is well known that there are such representations in the $\mathbb{C}$-linear case. In the Real case, that is $G_0 \neq G$, first choose a $\mathbb{C}$-linear representation of G_0 such that $W^{G_0} = 0$ and $W^{H_0} \neq 0$. Then the extension $i^e W$, for $i : G_0 \to G$ the inclusion, has the desired properties: By Proposition 1.1.10 we have $(i^e W)^G = 0$ if and only if $(i^e W)^{G_0} = 0$. This last one can be seen as the invariants of $i^* i^e W$, and so Proposition 1.2.8 gives (for $w \in G \setminus G_0$)

$$(i^e W)^{G_0} \cong (W \oplus w \star \bar{W})^{G_0} = W^{G_0} \oplus (w \star \bar{W})^{G_0} = 0 \oplus 0.$$

[8]Reduced K progroups are defined just as their unreduced counterpart by taking the same projective system, but now consisting of (quotients of) reduced K groups.

[9]The base case, $G = \{1\}$, is immediate: The only Real family is $\mathcal{J} = \{G\}$, and so a $\mathcal{J}$-equivalence is already a G-homotopy equivalence.

On the other hand, $(i^e W)^H \neq 0$ iff $(i^e W)^{H_0} \neq 0$, and we can use the same method to compute

$$(i^e W)^{H_0} \cong (W \oplus w \star \bar{W})^{H_0} = W^{H_0} \oplus (w \star \bar{W})^{H_0} \supset W^{H_0} \neq 0.$$

The only other possible proper subgroup is $G_0 \subset G$, if G does have odd elements, and in that case set $V_{G_0} = \mathbb{R}^8$ with the sign representation through $\phi : G \to C_2$. Then there too we have $V_{G_0}^G = 0$ and $V_{G_0}^{G_0} \neq 0$.

We define Y to be the colimit of S^U for U direct sums of finitely many copies of the V_H (and the morphisms of the diagram the natural inclusions of representation spheres $S^U \to S^{U \oplus V_H}$). Y is then a G-CW complex, the S^U as finite subcomplexes of Y, and every finite subcomplex is contained in some $S^U \subset Y$.

Lemma 2.8.5. *Y has the following properties:*

1. $Y^G = S^0$.
2. Y^H *is contractible for every proper subgroup* $H \subset G$ *(that is* $H \neq G$*).*
3. $\tilde{\mathcal{K}}_G^*(Y)_{\hat{\mathcal{J}}}$ *is zero as a progroup, if* $G \notin \mathcal{J}$*.*

Proof. Since every point of Y is contained in some S^U, and for every $U^G = 0$, we have

$$Y^G = \bigcup_U (S^U)^G = \bigcup_U S^0 = S^0,$$

proving the first statement.

For the second statement notice that every S^U in the diagram defining Y maps into a $S^{U \oplus V_H}$, and this map is H-homotopic to the constant map: View both representation spheres as the representations with a point ∞ at infinity added. Let $0 \neq x \in V_H^H$ and define $F : S^U \times I \to S^{U \oplus V_H}$ by

$$F(\infty, t) = \infty, \; F(u, t) = u \oplus \frac{x}{1-t}$$

with the convention that $u \oplus \frac{x}{1-1} = \infty$. Since x is H invariant this is an H-equivariant (pointed) homotopy from the inclusion to the constant. This shows that Y^H is contractible.

The third statement takes some work. By definition all the U are such that they have a Bott class $b_U \in \tilde{K}_G^0(S^U)$ such that $\tilde{K}_G^*(S^0) \to \tilde{K}_G^*(S^U), x \mapsto x b_U$ is an isomorphism[10]. This is since every U is the direct sum of a Real representations with copies of the eight dimensional sign representation, for which we have such Bott classes (we just take the appropriate reduced external product of the b_{V_H}).

[10] We have seen this without the grading. Degree wise we define these as $\tilde{K}_G^n(S^0) := \tilde{K}_G(S^{q(n)} \wedge S^0) \to \tilde{K}_G(S^U \wedge S^{q(n)} \wedge S^0) = \tilde{K}_G^n(S^U)$, where the map is the reduced external product with b_U. Then this is an isomorphism by Bott periodicity.

Given such a Bott class b_U define the Euler class $\chi_U \in RG = \tilde{K}^0_G(S^0)$ to be the pullback of b_U along $e_U : S^0 \to S^U$. If $U^H \neq 0$ then, as above, $S^0 \to S^U$ is H-homotopic to the constant, so that $\chi_U \in I^G_H = \ker RG \to RH$.

A general inclusion $S^U \to S^{U\oplus W}$ is equal to $\mathrm{id}_{S^U} \wedge e_W$, and pulling back along this maps $xb_{U\oplus W} = xb_U \tilde{\otimes} b_W$ to $x\chi_W b_U$.

To show that $\tilde{\mathcal{K}}^*_G(Y)_{\hat{\mathcal{J}}}$ is zero as a progroup it is enough to show that $\tilde{\mathcal{K}}^*_G(Y)/J\tilde{\mathcal{K}}^*_G(Y)$ is zero as a progroup for every $J = I^G_{H_1}...I^G_{H_n}$. This nomenclature is a bit unfortunate, but for now H_i has nothing to do with the even/odd elements of H but is just a list of proper subgroups of G.

Then, since every finite subcomplex of Y is contained in some S^U, we may restrict to these in the diagram defining the K progroup. Now, let $W = V_{H_1} \oplus ... \oplus V_{H_n}$. Then, as discussed above, the pull back along $S^U \to S^{U\oplus W}$ maps $xb_{U\oplus W}$ to $\chi_{V_{H_1}}...\chi_{V_{H_n}} xb_U$, and $\chi_{V_{H_1}}...\chi_{V_{H_n}} \in I^G_{H_1}...I^G_{H_n} = J$. So the induced map $\tilde{K}^*_G(S^{U\oplus W})/J\tilde{K}^*_G(S^{U\oplus W}) \to \tilde{K}^*_G(S^U)/J\tilde{K}^*_G(S^U)$ is zero.

Since this woks for every S^U in the diagram, we have that every "node" in the diagram is hit by a zero morphism, showing that this is zero as a progroup. □

Before we can begin the proof of the theorem, we need one more lemma about restricting the group in the completed K progroups:

Lemma 2.8.6. Let $H \subset G$ be a (proper) subgroup,

$$\mathcal{J}|H := \{Q \in \mathcal{J} | Q \subset H\}$$

be the restricted Real family of subgroups, and X be a pointed G-CW complex. Then there is a natural isomorphism

$$\tilde{\mathcal{K}}^*_G((G/H)_+ \wedge X)_{\hat{\mathcal{J}}} \cong \tilde{\mathcal{K}}^*_H(X)_{\hat{\mathcal{J}|H}}.$$

Proof. By Proposition 2.2.4 we now that $\tilde{\mathcal{K}}^*_G((G/H)_+ \wedge X) \cong \tilde{\mathcal{K}}^*_H(X)$ as pro-RG-modules, where RG acts on $\tilde{\mathcal{K}}^*_H(X)$ through the restriction $RG \to RH$. As stated in [9] it now suffices to show that the $\mathcal{J}$-adic and $\mathcal{J}|H$-adic topologies on RH are the same.

Fix some notation: $r^A_B : RA \to RB$ is the restriction, for A a subgroup of B Since we are really dealing with bases of these topologies, we need to show that both bases are finer than the other: For all $B_i \in \mathcal{J}$ there are $A_i \in \mathcal{J}|H$ such that $r^G_H(I^G_{B_1}...I^G_{B_m})RH \supset I^H_{A_1}...I^H_{A_n}$, and, conversely, for all $A_i \in \mathcal{J}|H$ there are $B_i \in \mathcal{J}$ such that $r^G_H(I^G_{B_1}...I^G_{B_m})RH \subset I^H_{A_1}...I^H_{A_n}$.

The latter is quick to do: For $A \in \mathcal{J}|H$ we have

$$r^H_A(r^G_H(I^G_A)RH) = r^G_A(I^G_A)r^H_A(RH) = 0,$$

so that $r^G_H(I^G_A)RH \subset I^H_A$. For products of $I^H_{A_i}$ one just takes the product of the $I^G_{A_i}$.

The former direction needs the structure of the prime spectrum of (Real) representation rings discussed in chapter 1. Again, as above, we can restrict to a single $B \in \mathcal{J}$, the general case follows from taking products. So we need $A_i \in \mathcal{J}|H$ such that $J = r_H^G(I_B^G)RH \supset I_{A_1}^H ... I_{A_n}^H$.

Since, by the corollary after Proposition 1.3.3, RH is noetherian, there is a finite list of prime ideals $\mathfrak{q}_i \subset RH$ which all contain J and whose product is contained in J. Let S_i ba a support of $\mathfrak{q}_i$, then $\mathfrak{q}_i \supset I_{S_i}^H$. So J also contains the product of the $I_{S_i}^H$, and we now show that $S_i \in \mathcal{J}|H$.

Let $\mathfrak{p}_i = (r_H^G)^{-1}(\mathfrak{q}_i)$, then by Lemma 1.4.14 S_i is a support of $\mathfrak{p}_i$. Furthermore, by construction $\mathfrak{p}_i$ contains I_B^G. Now, since $RG/I_B^G \to RB$ is injective and integral, there is a prime ideal $\mathfrak{u}_i \subset RB$ such that $\mathfrak{p}_i = (r_B^G)^{-1}(\mathfrak{u}_i)$. This shows that $\mathfrak{p}_i$ comes from B, and thus must have a support $S_i' \subset B$.

Now, since Real families of subgroups are closed under taking subgroups, and $B \in \mathcal{J}$ by assumption, we have $S_i' \in \mathcal{J}$. But any two supports of $\mathfrak{p}_i$ must be Real conjugate to each other by Proposition 1.4.12, and Real families are closed under Real conjugation, so that also $S_i \in \mathcal{J}$. Finally, since $S_i \subset H$, we have $S_i \in \mathcal{J}|H$. □

Proof of the completion theorem. As explained earlier, we assume the theorem for all proper subgroups $H \subset G$ and that $G \notin \mathcal{J}$, and need to show that, for a G-CW complex T such that T^H is contractible for $H \in \mathcal{J}$, both

$$\tilde{\mathcal{K}}_G^*(T \wedge Y)_{\hat{\mathcal{J}}} \text{ and } \tilde{\mathcal{K}}_G^*(T \wedge (Y/S^0))_{\hat{\mathcal{J}}}$$

are zero as progroups.

Let us start with $\tilde{\mathcal{K}}_G^*(T \wedge Y)_{\hat{\mathcal{J}}}$. We will actually show that $\tilde{K}_G^*(W \wedge Y)_{\hat{\mathcal{J}}} = 0$ for any finite G-CW complex W. Then the whole diagram defining the progroup consists of zeroes, and thus is zero itself. Like in the proof that the delocalization is an isomorphism, we can reduce to the case that $W = (G/H)_+$ for $H \subset G$ using the suspension and cofiber sequences. Then, by Lemma 2.8.6 , we have that

$$\tilde{\mathcal{K}}_G^*((G/H)_+ \wedge Y)_{\hat{\mathcal{J}}} \cong \tilde{\mathcal{K}}_H^*(Y)_{\mathcal{J}\hat{|}H}.$$

If $H = G$, then this is zero by the third point of Lemma 2.8.5. If $H \subset G$ is a strict subgroup, then Y is H-contractible by the second point of 2.8.5 and the J.H.C. Whitehead theorem. Thus $\tilde{\mathcal{K}}_H^*(Y) \cong \tilde{\mathcal{K}}_H^*(*) = 0$, and so also the completion vanishes.

Now for $\tilde{\mathcal{K}}_G^*(T \wedge (Y/S^0))_{\hat{\mathcal{J}}}$. Since $(Y/S^0)^G = *$ we now that it is build out of cells $(G/H)_+ \wedge D^n$ for $H \subset G$ a *proper* subgroup. Similar to the above reduction it suffices to prove $\tilde{\mathcal{K}}_G^*(T \wedge (G/H)_{\hat{\mathcal{J}}})$ vanishes. This is, by Lemma 2.8.6, isomorphic to $\tilde{\mathcal{K}}_H^*(T)_{\mathcal{J}\hat{|}H}$.

By assumption T^B is contractible for all $B \in \mathcal{J}$. In particular this is also true for all $B \in \mathcal{J}|H$, meaning that $T \to *$ is a $\mathcal{J}|H$-equivalence. Now, by our induction hypothesis, the completion theorem holds for H, so this induces an isomorphism $\tilde{\mathcal{K}}_H^*(T)_{\mathcal{J}\hat{|}H} \cong \tilde{\mathcal{K}}_H^*(*)_{\mathcal{J}\hat{|}H} = 0$. □

Remark. Let me comment on Remark 5.3 in [9]. There the authors outline some difficulties they had generalizing their proof to Real equivariant K theory. I think this is because they only knew how to define K_G in the cases that either $G = G_0$ or that $G_0 \to G \to C_2$ is a split extension: Their argument only works if one allows arbitrary (Real) subgroups of G. But, since a Real subgroup of a split Real group might not be split itself, it is crucial to have a notion of K_G also in the nonsplit case.

Let me give a, to me, surprising consequence of the completion theorem. Consider the Real family $\mathcal{J}_0$ given by all subgroups of G_0. Since $RG \to RG_0$ is injective, $I^G_{G_0} = 0$, and so the $\mathcal{J}$-adic topology is actually the (0)-adic topology! Since every module is already (0) complete the completion does nothing. On the other hand, a $\mathcal{J}_0$-equivalence is a G-equivariant map which is a G_0-homotopy equivalence.

Proposition 2.8.7. Let $f : X \to Y$ be a G-equivariant map between finite G-CW complexes which is a G_0-homotopy equivalence. Then $f^* : K^*_G(Y) \to K^*_G(X)$ is an isomorphism.

In particular, if $G = C_2$ and $\phi_G = \mathrm{id}_{C_2}$ this is a statement about the classical KR-theory: It induces isomorphisms already for non-equivariant homotopy equivalences (which are C_2-equivariant maps)!

Remark. In their original paper [5] Atiyah and Segal gave a proof of this theorem for the split case and for the trivial family $\mathcal{J} = \{1\}$. The improvement really lies in the generalization to arbitrary Real families of subgroups, which, to my knowledge, is a new result even in the split case.

3 K-theory invariants for topological insulators

I now want to outline the way in which K groups arise in the study of physical systems, and how one can use them to make predictions in concrete cases.

First, I want to discuss why one should even expect K groups to have anything to do with physics. We will see that, for systems which can be modeled as hermitean eigenvalue problems that exhibit translation symmetry, we can construct a vector bundle $\mathcal{B}^-$ which encodes (some) topological information of the system. This means that arbitrary continuous deformations of the system (with some boundary conditions) do not change the equivalence class $[\mathcal{B}^-]$ in the relevant K group.

Having established this correspondence I want to explain how one can, in some cases of interest, use the (Real) Chern character to determine whether the class $[\mathcal{B}^-]$ is trivial or not. Here we will see that because of the particular construction of the bundle $\mathcal{B}^-$ these calculations can be shortcut, so that the K groups are only appearing in the background.

Finally, by discussing an example, I will show how these types of calculations look like in practice, and what information about the system is really needed.

It could be said that I am focusing on the symmetry classes A and AI in the Altland-Zirnbauer classification. I will comment on how these things could be applied the other 8 symmetry classes at the end of this chapter.

3.1 The below gap Bloch bundle

Let's start with a big disclaimer: the discussion I want to present here is far from being rigorous. I aim to make clear the concepts used and the flow of argumentation, for an in-depth discussion of mathematical details one should read the foundational work by Freed and Moore [14].

A large chunk of models used in physics falls into the class of hermitean eigenvalue problems. That is, we are given a Hilbert space $\mathcal{H}$ so that vectors in the space correspond, in some way, to actual states of the physical system we want to describe. The dynamics of the system are then described by a hermitean operator O on the Hilbert space $\mathcal{H}$. One is the interested in determining the eigenstates and eigenvalues of O, that is in solving the eigenvalue problem

$$Ov = \lambda v \text{ for } v \in \mathcal{H},\ \lambda \in \mathbb{R}.$$

The main example is the description of quantum mechanical systems using the Schrödinger equation. For a single particle the space in question is $\mathcal{H} = L^2(\mathbb{R}^3)$ and the operator is $O = -\frac{\Delta}{2m} + V$ where V stands for the potential the particle is moving in. Here the vectors correspond to states in a rather peculiar way: states of the system are thought of as rays in the Hilbert space. That means that two non-zero vectors describe the same state if and only if one is a scalar multiple of the other, and the zero vector doesn't describe any state.

On the other hand, the dynamics are described more directly: an eigenstate of O is a fixed state of the physical system, and the eigenvalue is interpreted as the energy of the state. More generally, $v(t)$ describes the time evolution of a state of the system if $i\partial_t v(t) = Ov(t)$.

Another example of interest is the propagation of light in dielectric materials. Here the Hilbert space is the space of square integrable function with values in $\mathbb{C}^3$ and the operator is $Of(x) = \nabla \times (\epsilon(x)^{-1}\nabla \times f(x))$. This is hermitean if the dielectric tensor ϵ is. If a vector is divergence less, $\nabla \cdot f = 0$, its pointwise real part describes the magnetic field in the material.

The eigenvalue problem in question is $Of = (\omega/c)^2 f$, and the time evolution of the field represented by f is given by $H(x,t) = \Re\left(f(x)\exp(-i\omega t)\right)$. The time evolution of a general state can then be found by decomposing it into eigenstates (which are also called harmonic modes). For more details on this see the excellent book [15].

An extremely important property the operator O, and thus the dynamics, might have is having a *gap*. Since O is hermitean we have that its spectrum consists of real numbers, $\mathrm{Spec}(O) \subset \mathbb{R}$. Having a gap then means that the spectrum of O does not contain a small interval $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. For the physical system described this usually means something like that it does not support waves with a frequency $\simeq \lambda_0$ traveling through it. We will assume, from now on, that the gap is around $\lambda_0 = 0$, but this is only to make the following discussion easier notation wise.

Let's say we had two different dynamics on a system, represented by operators O_1, O_2. One might think of putting a quantum mechanical particle in different potentials, so that the states of the system are the same, but behave differently over time. If both are gapped one might ask whether it is possible to continuously deform the dynamics from O_1 to O_2 while maintaining the gap. If it is not possible one says that they are *topologically* distinct.

Of course, we could impose even more conditions on the operators and ask the same question again. A very important restriction is for the dynamics to have some kind of discrete translation symmetry. This might be the case for electrons moving in the background of a crystalline material, or for light propagating in periodically structured materials.

So lets say we have some unitary action of $\mathbb{Z}^n$ on the Hilbert space $\mathcal{H}$, and let us only consider operators O commuting with the action. In mathematically favorable situations (see [14]) this can be captured by considering the so called *Bloch bundle*:

Definition 3.1.1.

1. The space of group morphisms

$$\mathrm{BZ} = \mathrm{Hom}(\mathbb{Z}^n, U(1)) \cong T^n$$

is called the *Brillouin zone.*

2. Let $u \in \mathrm{BZ}$. An element $f \in \mathcal{H}$ is said to be a *Bloch state of phase* u if

$$b.f = u(b)f \text{ for all } b \in \mathbb{Z}^n.$$

3. The *Bloch bundle* is the complex vector bundle $\mathcal{B} \to \mathrm{BZ}$ whose fibers are given by

$$\mathcal{B}_u = \{ f \in \mathcal{H} \mid b.f = u(b)f \text{ for all } b \in \mathbb{Z}^n \}.$$

The scalarproduct on $\mathcal{H}$ makes this into a bundle of Hilbert spaces.

In the real world, this is all kind of ill defined. For example, usually there are no Bloch states in any physical system, since, for example, a translationally invariant state couldn't fulfill certain finiteness conditions. In quantum mechanics this is usually dealt with by considering systems on a large torus instead of real space. In photonic crystals one usually drops the finiteness condition, builds the Bloch bundle, and then takes integration over a unit cell of the translation symmetry for the inner product.

Be that as it may, in most cases there is *some* natural way to make sense of this. Then, if O commutes with the translation action, it maps the fibers of the Bloch bundle into them selves. Thus we get a whole bunch of hermitean eigenvalue problems $(\mathcal{B}_u, O_u)$. If O was gapped, all the O_u are gapped, and vice versa.

Incidentally, it happens that a gapped O can (almost) always be continuously deformed so that it only has the eigenvalues $\{ \pm 1 \}$. This also holds in the presence of translational symmetries, so that specifying O is, up to continuous deformation, the same as specifying a splitting of the Bloch bundle into above and below gap parts:

$$\mathcal{B}^{\pm} := \{ f \in \mathcal{B} \mid f \text{ is the sum of eigenstates with eigenvalue above/below the gap.} \}$$

Notice that this splitting depends on the operator O, but does not change under continuous deformations of O (which preserve the gap and translational symmetry). Thus two gapped, translationally symmetric dynamics of the described system can be deformed into each other only if they induce the same splitting on the Bloch bundle.

Now, to make this notion more robust, one might ask the splittings they induce not to be the same, but to be isomorphic, maybe after adding some trivial bundles to them. First of all, this might be necessary for example when the vectors correspond to states of the system in a degenerate way, or if one wants a notion which does not depend on "forgetting trivial high/low energy modes of the system".

In most cases it turns out that $\mathcal{B}^+$ is infinite dimensional and that $\mathcal{B}^-$ is finite dimensional. Then, as discussed in [14], this notion of equivalence is captured by asking the classes $[\mathcal{B}^-] \in \tilde{K}(\mathrm{BZ})$ to be the same. So, since this is a strictly weaker condition than the splittings to be equal, the dynamics can be deformed into each other only if they induce the same class in the K group of the Brillouin zone.

Definition 3.1.2. I will call the class $[\mathcal{B}^-] \in \tilde{K}(\mathrm{BZ})$ of the system its *topological phase.*

The enormous interest in these topological phases in the recent years can be, in part, credited to the phenomenon of edge states. As we have argued, two gapped dynamics on the same system can't be deformed into each other, while maintaining the gap, if they have different topological phases. So, if one did deform one into the other, one would expect to close the gap at some point. The idea now is that, if one had two systems of finite size, which, if they were infinitely extended, had different topological phases, and put them side by side, one would expect modes closing the gap right at the interface between them. While the above argumentation is a heuristic at best, the existence of these modes has been confirmed theoretically, numerically, and experimentally in many cases, maybe most notably with the discovery of the quantum Hall effect by von Klitzing.

The presence and robustness of these so called *edge states* is often taken as the defining property of topological phases, while here I use a more modest notion.

3.2 Further symmetries

Let's say we want to investigate a physical system that does have more than only translational symmetry. It is well known that, when expressing these as a group acting on the Hilbert space, there can arise some technical issues, since the vectors may correspond degeneratly to states of the system. In quantum mechanics, for example, one might need extend the group, fix whether some elements need to square to -1 instead of the identity, act $\mathbb{C}$ linearly or antilinearly, and whether they should commute or anticommute with the operator describing the dynamics.

For now we will only discuss the following case: suppose all the necessary extending of groups etc. has been done, and we are left with a split extension

$$0 \to \mathbb{Z}^n \to H \to G \to 0$$

where H is the total group of symmetries and G is finite. Further assume that all the elements of H are supposed to commute with the operator O, there are no additional phase factors to take into account, the elements of H act linear or antilinear depending on the value of some group morphism $\phi : H \to C_2$, and $\mathbb{Z}^n \subset \ker \phi$.[1]

This will often be the case, for example, when considering integer-spin bosonic systems with magneto spatial symmetries. In this setting ϕ factors over G, making it into a Real group, and we can actually upgrade the invariant $[\mathcal{B}^-]$ to an element of the Real equivariant K group $\tilde{K}_G(\mathrm{BZ})$.

To see this let $f \in \mathcal{B}_u$ and choose some splitting of the extension so that we can view $G \subset H$. The idea is that, for $g \in G$, $g.f$ will be still a Bloch state, but now for a different

[1] One could say that we are only considering class A or class AI systems, but that would be a bit less precise.

phase factor which we will determine now:

$$b.g.f = g.g^{-1}.b.g.f = g.u(g^{-1}bg)f = \phi(g)(u(g^{-1}bg))g.f$$

Here we used that $g^{-1}bg \in \mathbb{Z}^n$ if $b \in \mathbb{Z}^n$. So if we let G act on $\mathrm{BZ} = \mathrm{Hom}(\mathbb{Z}^n, U(1))$ as

$$g.u := \phi(g) \circ u \circ \mathrm{Ad}_{g^{-1}} \text{ for } g \in G,\ u \in \mathrm{BZ}$$

then $g. : \mathcal{B}_u \to \mathcal{B}_{g.u}$ is linear or antilinear depending on $\phi(g)$, thus makes $\mathcal{B} \to \mathrm{BZ}$ into a Real equivariant vector bundle.

Notice that the conjugation action of G on $\mathbb{Z}^n$ does not depend on the choice of splitting we made.

3.3 Calculating invariants using the Chern character

We have seen that some amount of topological information about a system is encoded in the class of the below gap Bloch bundle in the relevant K group. Without further information this is only useful if one knows that the K group is trivial, so that automatically the topological phase must be trivial (with respect to the symmetries considered).

But, of course and luckily, many of the relevant K groups are rather large, so that there may be many different topological phases. A very thorough calculation of many K groups of interest can be found in a recent article by Cornfeld and Carmeli [16]. A problem generally not solved[2] by these calculations, and many others alike, is the following: Let's say we had $\tilde{K}_G(\mathrm{BZ}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. How would one calculate, from information on the physical system, the integer and sign corresponding to its topological phase?

In many situations there are known formulas to calculate some part of these invariants, for example Chern numbers, Chern-Simons/FKMM invariants and what not. But there doesn't seem to be a good kind of overarching framework which produces a complete list of formulas for these invariants, dependent on the symmetries considered.

At least in the kind of limited setting detailed above, we can use the Real Chern character to calculate the integer parts. In particular, since the Brillouin zone is, mathematically, a compact manifold, we can use the incarnation presented in Lemma 2.6.3. If we have access to the states below the gap, for *enough* points in the Brillouin zone, and know how to 1) apply the action of the group to them and 2) calculate inner products of them, we are good to go. We can then build a connection on $\mathcal{B}^-$ out of this, usually called the Berry connection, and integrate the forms provided by the Real Chern character over the fixed point sets.

By some usual yoga on (co)homology on manifolds, the values of these integrals are enough to distinguish different differential forms, and thus, by Theorem 2.5.5, to decide

[2] While most of theses papers do provide explicit isomorphisms, it is often not so clear what to do with them. They might, for example, depend on the knowledge of some classifying map, which is hard to construct in practice.

whether two systems are in different topological phases or not, at least with respect to the integer parts of the K group.

Let me stress the following: since the Real Chern character only sees the integer part of topological phase, only the implication

$$\text{different integrals} \implies \text{different phase}$$

holds. To conclude in the other direction, one would need, for example, that the torsion part of the K group is zero. That is the case for many examples of interest, especially in 3 dimensions, see table II in [16].

To effectively use the Real Chern character we should follow these steps:

1. Determine the Real conjugacy classes of G, and the Real centralizers of their representatives.
2. Determine the fixed point manifolds of these representatives, their (even) cohomology, how the centralizers act on their cohomology, and what the invariant cohomology classes under that action look like.
3. For the conjugacy classes, and degrees of cohomology, where one can expect non-zero results, integrate the forms provided by Lemma 2.6.3 over the fixed point manifolds. This may also mean to integrate over sufficiently many submanifolds, for example if the degree is less than the dimension of the fixed point manifold.

Step 2 is really only to shortcut some computations. It may turn out that one can expect non-zero forms only in degree zero, so one does not need to determine the Berry connection, which often saves a big headache. On the other hand, one might find that one actually needs to compute less integrals to determine the forms than one naively would expect, due to the forms being invariants for the action of the centralizer. An example of both of these cases is discussed below.

3.4 The case of crystal symmetries p6 with time reversal symmetry

To connect a bit more concretely to physics we will now consider the important case of crystal symmetries in 2 dimensions, with or without time reversal. The above conditions are then met when, for example, we consider bosonic particles in a symmorphic[3] crystal.

So we have our two lattice basis vectors $b_1, b_2 \in \mathbb{R}^2$ and get the reciprocal lattice vectors a_1, a_2, with the normalization that $a_i \cdot b_j = \delta_{ij}$. The Brillouin zone can then be seen as the reciprocal space quotiented by the reciprocal lattice:

$$\mathrm{BZ} \cong \mathbb{R}^2/(k \sim k + a_1, k \sim k + a_2)$$

[3] Symmorphic here means that the magnetic space group should split, so it is a bit of a more general notion than usual.

The idea here is that a point k in the reciprocal space represents morphism $\mathbb{Z}^2 \to U(1)$ via $(n,m) \mapsto \exp(2\pi i k \cdot (nb_1 + mb_2))$, and adding integer multiples of a_1, a_2 to k doesn't change the morphism.

Now a general element of the magnetic point group is of the form TA or A, where $A \in O(2)$ is a rotation and/or reflection, and T is the time reversal operator. Elements of the form A act linearly, elements of the form TA act antilinearly. So how do these act on the Brillouin zone? A small calculation shows that A acts like itself, and TA acts like $-A$.

Specializing even further, consider a setup with wallpaper group p6 and time-reversal symmetry, for example a triangular lattice with basis vectors (see Figure 3.1)

$$b_1 = \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right), \; b_2 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

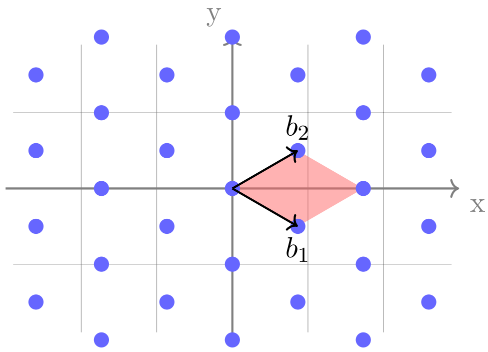


Figure 3.1: An example of a lattice with sixfold rotational symmetry. The red area is one possible choice of a unit cell. Of course, this particular lattice has many more symmetries, but we shall ignore them for now.

The magnetic point group is $C_6 \times C_2$, where C_6 is the sixfold rotation and C_2 is the time reversal. So, since rotations act linearly and time reversal acts antilinearly, the Real structure is given by the projection

$$\phi = \mathrm{pr}_2 : C_6 \times C_2 \to C_2.$$

Denoting the generator of the rotations by γ the Real conjugacy classes are $[1]$, $[\gamma]$, $[\gamma^2]$, and $[\gamma^3]$, and their Real centralizers are generated by $\{\gamma, T\}$, $\{\gamma\}$, $\{\gamma\}$, and $\{\gamma, T\}$.

In cartesian and lattice coordinates γ is given by the matrices

$$\gamma_C = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \; \gamma_L = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

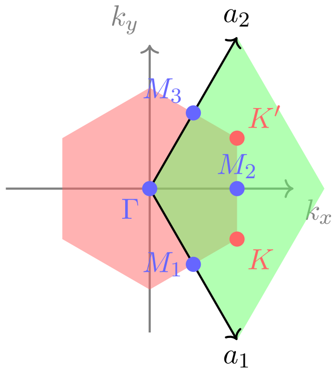


Figure 3.2: These are two different choices of unit cells in the reciprocal lattice. The red one is a very common choice since it makes it easy to visualize how the sixfold rotation acts on the Brillouin zone, while the green one has the advantage that it is simply a square in the reciprocal lattice coordinates. Note that for both one needs to identify opposing sides to get the Brillouin zone.

From the matrix in lattice coordinates we can calculate how γ acts on the reciprocal space in the reciprocal lattice basis:

$$\gamma_{RL} = (\gamma_L^{-1})^t = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

Let us now determine the fixed point manifolds, working in reciprocal lattice coordinates. That means that two vectors represent the same point in the Brillouin zone if and only if their entries differ by integers.

First of all, the fixed points of 1 are the whole Brillouin zone BZ. Assume (a, b) is a fixed point of γ, meaning that

$$\gamma_{RL}.(a, b) - (a, b) \in \mathbb{Z}^2 \implies (a, b - a) \in \mathbb{Z}^2$$

So only the point represented by $(0, 0)$, called Γ, is fixed by γ.

For γ^2 we find, using the same method, three fixed points. One is the point Γ, the other two are

$$K = (1/3, 2/3),\ K' = (2/3, 1/3).$$

Finally, for γ^3, we find four points. One is, again, Γ, the other three are

$$M_1 = (1/2, 0),\ M_2 = (1/2, 1/2),\ M_3 = (0, 1/2).$$

See Figure 3.2 for a graphical representation of the Brillouin zone and its fixed points. Since the fixed points are either the whole Brillouin zone or collections of disjoint points determining their cohomology is rather easy:

- $\tilde{H}^0_{dR}(\mathrm{BZ}^1,\mathbb{C}) = 0$, $\tilde{H}^2_{dR}(\mathrm{BZ}^1,\mathbb{C}) = \mathbb{C}\langle \mathrm{d}a_1 \wedge \mathrm{d}a_2\rangle$
- $\tilde{H}^0_{dR}(\mathrm{BZ}^\gamma,\mathbb{C}) = 0$
- $\tilde{H}^0_{dR}(\mathrm{BZ}^{\gamma^2},\mathbb{C}) = \mathbb{C}\langle K\rangle \oplus \mathbb{C}\langle K'\rangle$
- $\tilde{H}^0_{dR}(\mathrm{BZ}^{\gamma^3},\mathbb{C}) = \mathbb{C}\langle M_1\rangle \oplus \mathbb{C}\langle M_2\rangle \oplus \mathbb{C}\langle M_3\rangle$

where a_1, a_2 denote the coordinates in the directions of the reciprocal lattice vectors, $\mathbb{C}\langle K\rangle$ means a summand generated by a function only non zero at K, we used Γ as the basepoint, and the degrees left out are zero.

Now let us discuss the action of the centralizers on these groups. Acting on the fixed points of 1, T sends (a,b) to $(-a,-b)$. Additionally, on $\tilde{H}^2$, T acts with an extra sign, so that

$$T.(\mathrm{d}a_1 \wedge \mathrm{d}a_2) = -\mathrm{d}(-a_1) \wedge \mathrm{d}(-a_2) = -\mathrm{d}a_1 \wedge \mathrm{d}a_2.$$

Thus the only invariant cohomology class is 0, so that the whole summand at [1] in the delocalization is zero. This actually shows that, in time reversal symmetric settings in 2d, one can not expect any degree two integer invariants!

Considering the action of γ on $\mathrm{d}a_1 \wedge \mathrm{d}a_2$ shows that this maps to it self, so that all cohomology classes are invariant under it. So, if we did break time reversal symmetry in some way, one might expect further invariants.

For the fixed points of γ^2 we need to investigate the action of γ on them. A small calculation shows that $\gamma_{RL}.K = K'$, $\gamma_{RL}.K' = K$, so that the invariants in the cohomology are the 'diagonal' elements of the form $aK + aK'$, $a \in \mathbb{C}$.

For the fixed points of γ^3, γ induces cyclic permutations between them, and T leaves them fixed. So the invariants in the cohomology are, again, the 'diagonal' elements $aM_1 + aM_2 + aM_3$, $a \in \mathbb{C}$.

Since the invariant elements are of this diagonal form, it suffices to calculate their value on one of the points to determine them. What I mean is, since the fixed points of γ^2, γ^3 are disjoint points, calculating the integrals necessary to determine the forms given in 2.6.3 means evaluating them on each point. But the symmetries of the system force these to give equal values on K and K', and on M_1, M_2, and M_3 respectively.

To conclude, we have shown not only that $\tilde{K}_{C_6\times C_2}(\mathrm{BZ}) \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$, but that an isomorphism is given by

$$[E] \mapsto (\mathrm{tr}_{E_K}(\gamma^2)) \oplus (\mathrm{tr}_{E_{M_1}}(\gamma^3)).$$

So, if we wanted to determine the topological phase of a system with p6 and time reversal symmetry (up to torsion), we would need to know bases of eigenstates below the gap at the points K and M_1 and calculate these traces. That is, we need to calculate the trace of the operator representing rotation by 120° (180°) on the space of states below the gap with crystal momentum K (M_1). Actually, since we really are working in a reduced setting with basepoint Γ, we ought to subtract the traces of γ^2, γ^3 on $\mathcal{B}^-_\Gamma$, too.

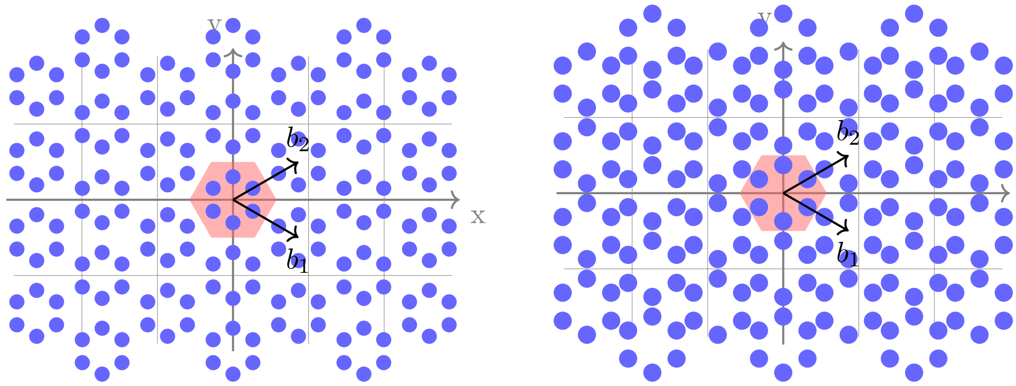


Figure 3.3: An xy-crosssection of the setup proposed by Wu and Hu in [17]. The blue areas have high dielectric constant, and the red area is a possible choice of unit cell. On the left is the *shrunken* phase, on the right the *extended* phase.

3.5 Example: deformed hexagonal lattice

In a paper from 2015 [17], Wu and Hu propose a setup of infinitely long parallel dielectric rods in a hexagonal lattice. By then extending or shrinking the distances in groups of six of these rods one ends up, again, with a hexagonal lattice, whose unit cell now contains six rods, see Figure 3.3.

They then study the propagation of light through this system, focusing on harmonic modes in which the magnetic field is restricted to the xy-plane, the so-called TM modes. They show, using numerical simulations, that in both the extended and shrunken setting the system is gapped, with three bands below the gap (see Figure 3.4), and then argue that these should actually be in different topological phases with respect to the p6 and time-reversal symmetry. Their argumentation is based around constructing, out of the rotation by 60° and time reversal, a new 'pseudo' time reversal which squares to -1, and then applying methods classifying 2d class AII topological insulators.

I claim that there is a more direct argumentation, based on the discussion in the previous section. One can, for example numerically, calculate the below gap states at Γ, K, and M_1, and then calculate the traces of γ^2, γ^3 as explained above.

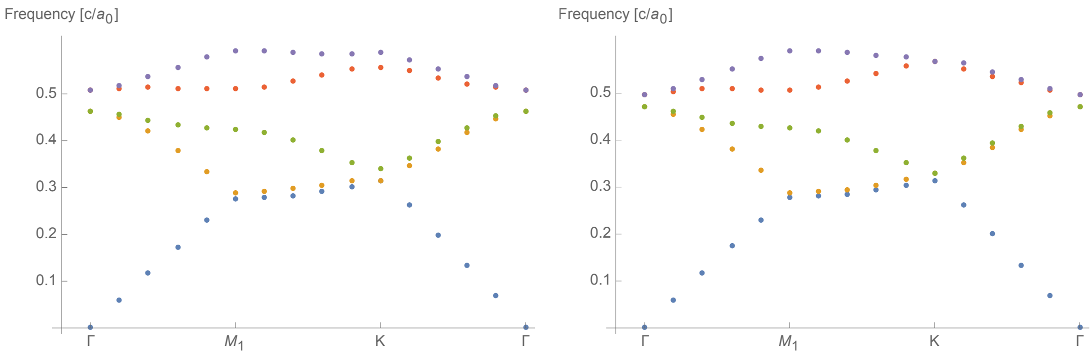
I have implemented the system in the MPB software [18] which can calculate not only the eigenfrequencies of the system at arbitrary points in the Brillouin zone, but also the corresponding eigenstates. In the shrunken phase these traces compute to

$$\mathrm{tr}_{\mathcal{B}_K^-}(\gamma^2) - \mathrm{tr}_{\mathcal{B}_\Gamma^-}(\gamma^2) = 1.5,\ \mathrm{tr}_{\mathcal{B}_{M_1}^-}(\gamma^3) - \mathrm{tr}_{\mathcal{B}_\Gamma^-}(\gamma^3) = -2,$$

in the extended phase they compute to

$$\mathrm{tr}_{\mathcal{B}_K^-}(\gamma^2) - \mathrm{tr}_{\mathcal{B}_\Gamma^-}(\gamma^2) = 1.5,\ \mathrm{tr}_{\mathcal{B}_{M_1}^-}(\gamma^3) - \mathrm{tr}_{\mathcal{B}_\Gamma^-}(\gamma^3) = 2,$$

up to numerical accuracy. The MPB configuration files can be found online under `https://github.com/leonard-[]tokic/Wu-[]Hu-[]Simulation` (mirror).



(a) A band diagram of the extended phase. (b) A Band diagram of the extended phase.

Figure 3.4: Here we see the frequencies for the first five TM-modes in the setup by Hu and Wu, as calculated by MPB at a representative set of points in reciprocal space (compare with Figure 3.2). Here a_0 is the unit of length we also specified our lattice basis vectors in. In (a) are the frequencies for the extended phase, in (b) the frequencies of the shrunken phase. For both the band diagram suggests a gap around $0.48\,c/a_0$ between the third and fourth band.

Note that, when examining this a bit more closely (that is not forming the differences, but looking at all these quantities separately), one sees that the behavior of the states really only changes at the Γ point, just as Wu and Hu find.

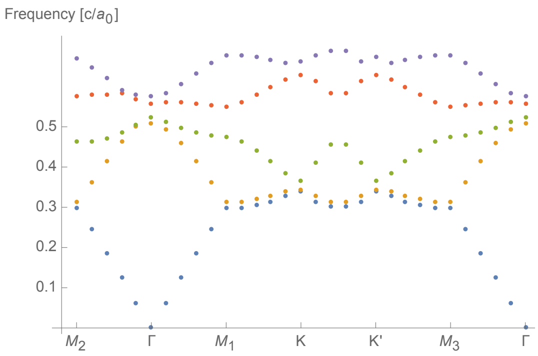
Remark. Even though we are calculating 'integer invariants', the values of these are not necessarily integers, but integer multiples of some constant depending only on the symmetries considered. Knowing this constant is not necessary to decide whether a topological phase is trivial or not, though.

Let me be explicit about what this computation is. By the theory laid out in [15], the states are represented by their (imaginary) in-plane H-field. So, for example, let H_1, H_2, H_3 be the fields of the lowest three states at the M_1 point. The inner product on these fields is given by

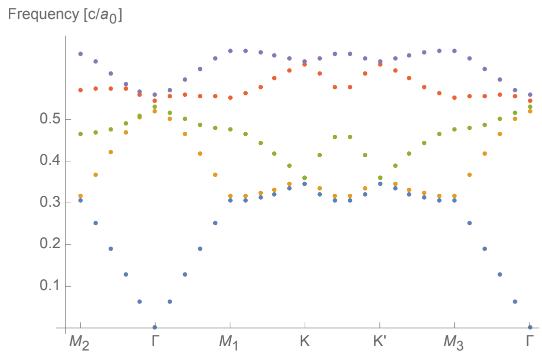
$$(H, H') := \frac{1}{A} \int_{\text{unit cell}} H^*(x, y) \cdot H'(x, y) \mathrm{d}x \mathrm{d}y$$

where x, y are cartesian coordinates in the plane, and A is the area of the unit cell. Assume H_1, H_2, H_3 to be normalized with respect to this. Since the magnetic field H is a (pseudo) vectorial quantity, acting with γ on it is defined as

$$(\gamma.H)(x, y) := \gamma_C.(H(\gamma_C^{-1}.(x, y))).$$



(a) A band diagram of the modified extended phase.



(b) A band diagram of the modified shrunken phase.

Figure 3.5: Here we see the frequencies for the first five TM-modes in the modified setup shown in Figure 3.6, as calculated by MPB at a representative set of points in reciprocal space (compare with Figure 3.2). a_0 here is the unit of length we also specified our lattice basis vectors in. In (a) are the frequencies for the extended phase, in (b) the frequencies of the shrunken phase. For both the band diagram suggests a gap around $0.53\,c/a_0$ between the third and fourth band.

Thus, the trace of γ^3 on $\mathcal{B}^-_{M_1}$ is given by

$$\begin{aligned}\mathrm{tr}_{\mathcal{B}^-_{M_1}}(\gamma^3) &= (H_1, \gamma^3.H_1) + (H_2, \gamma^3.H_2) + (H_3, \gamma^3.H_3) \\ &= \sum_{i=1}^{3} \frac{1}{A} \int_{\text{unit cell}} H_i^*(x,y) \cdot \gamma_C^3.H_i(\gamma_C^{-3}.(x,y))\mathrm{d}x\mathrm{d}y\end{aligned}$$

Of course, in a numerical setting, these integrals would then be approximated by sums over a large but finite number of points. I hope this example makes it clear what knowledge about the system is required to calculate these invariants, at least for the 'degree zero' case.

The above calculation suggests that the topological phases of these phases would be different even when a only rotation by 180° is taken into account as a symmetry, while the argumentation by Wu and Hu would not work in that case.

Replacing the circular cross-section of the rods by parallelograms, as in Figure 3.6, one is left with a system which now only has rotation by 180° and time reversal as symmetries. We find that both phases are still gapped (see Figure 3.5). Denoting the rotation by 180° still by γ^3, one finds that

$$\mathrm{tr}_{\mathcal{B}^-_{M_1}}(\gamma^3) - \mathrm{tr}_{\mathcal{B}^-_{\Gamma}}(\gamma^3)$$

is still an invariant of the topological phase.

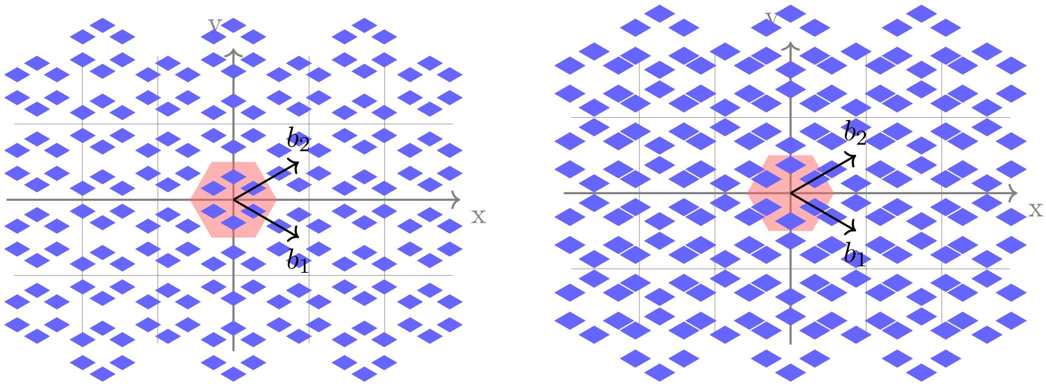


Figure 3.6: The modified setup: After replacing the round crosssections with parallelograms both the shrunken and extended phases are only symmetric with respect to rotations by 180°.

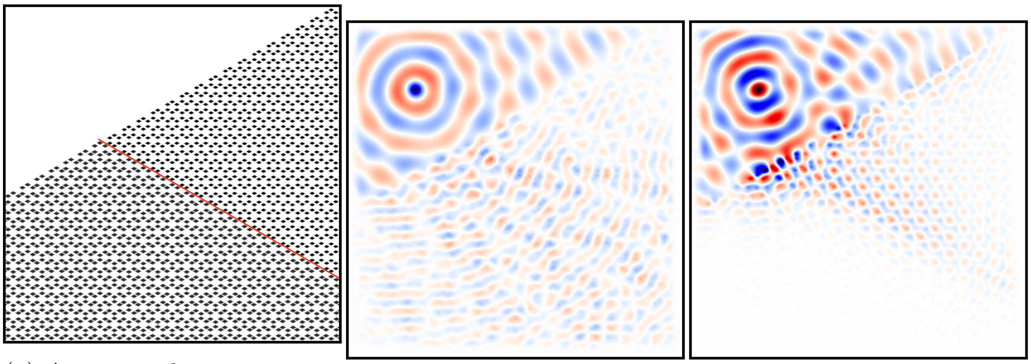
Extracting the eigenstates using MPB, one finds exactly the same results for this invariant as before (that is -2 in the shrunken and 2 in the extended phase). I conjecture that putting these phases next to each other still produces topologically protected edgestates. This is supported by some simulations I ran in MEEP [29], an FDTD solver for Maxwell's equations, see Figure 3.7.

3.6 Other Altland-Zirnbauer classes

As mentioned before the strategy above works for symmorphic crystals with AZ symmetry class A or AI, roughly speaking. It seems to be generally conjectured that the other Altland-Zirnbauer classes are classified by $\tilde{K}_G^{-n}(BZ)$, $n = 1, ..., 7$. This view is, for example, expressed in the paper [16].

Since we have the suspension isomorphism $\tilde{K}_G^{-n}(BZ) = \tilde{K}_G(S^n \wedge BZ)$ and the injection $\tilde{K}_G(S^n \wedge BZ) \to \tilde{K}_G(S^n \times \mathrm{BZ})$ we could also in this case use the Real equivariant Chern character to get information about the integer part of topological phases. There is a big step to overcome, though: how do systems in these symmetry classes correspond to Real equivariant bundles on $S^n \times BZ$?

In the classes A and AI this correspondence was rather direct (as outlined at the beginning of this chapter), giving us a good handle for computations. I think that finding a way to make this connection in a computable way also in the other classes would be a rewarding direction of research.



(a) A grayscale representation of the dielectric function in the xy-plane. The red line shows the position of the edge between the extended and shrunken phase.

(b) The z-component of the electric field at $t = 100\, a_0/c$. The source has a frequency of $0.4\, c/a_0$, which lies below the gap for both phases.

(c) The z-component of the electric field at $t = 100\, a_0/c$. The source has a frequency of $0.53\, c/a_0$, which lies in the gap for both phases.

Figure 3.7: Consider the setup shown above, a kind of 'T' crossing between the extended phase, the shrunken phase, and vacuum. We place a continuous source producing light polarized in the z direction in the vacuum at the top left, turning it on at $t = 0$. We find the light travels through the medium kind of evenly if the frequency of the source lies below the gap (b), but seems to be confined near the edge for frequencies in the gap (c). See `https://github.com/leonard-[]tokic/Wu-[]Hu-[]Simulation` for configuration files and animations.

Popular summary

Equivariant K-theory, in some sense, is the study of $\mathbb{C}$-linear representations of a group parameterized by a space. Given a group G acting on a space X it assigns to it an abelian group, called $K_G(X)$, whose elements are equivalence classes of such parameterized representations.

Studying *Real* equivariant K-theory, a generalization which also includes representations where the group may act $\mathbb{C}$-linearly or antilinearly, I am able to transfer many of the classical theorems of K-theory to it.

One of them, the Chern character isomorphism, can be used to translate calculations in K-theory to calculations of a couple of integrals.

It turns out that equivariant K-theory plays a role in the study of physical systems. Given a physical system with symmetries represented by a group G, we can assign to it an element of $K_G(T^n)$, where T^n is the n-dimensional torus. This element is a shadow of the physical system, invariant under certain *continuous* deformations.

Using the Chern character isomorphism I calculate the K-theory elements of a pair of systems proposed by Wu and Hu [17], and of a slight modification of these. I am able to prove that these have different 'shadows' in K-theory, even after the modification, while the argument given by Wu and Hu does not cover the modified setup.

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