Splittings of Thom spectra up to finite Galois extensions at heights 1 and 2

Leonard Vinko Tokić

The Ten Largest No. 7 – Adulthood, Hilma af Klint, 1907.

Contents

Acknowledgments

I want to thank Gerd Laures and Björn Schuster for helping me set foot in chromatic homotopy theory, always having an open ear for my questions, and encouraging me to keep pushing. Also, I want to thank Haynes Miller for inviting me to MIT for the fall term of 2023, during which time many of the ideas for the present thesis started to form. Over the course of this project, I enjoyed clarifying discussions with Keita Allen, Sanath Devalapurkar, Daniel Garrido, Mike Hopkins, Ishan Levy, Stephen McKean, Carl McTague, Piotr Pstrągowski, and Laurent Smits.

I also want to thank the algebraic topology community in general. During my time there so far, I have had countless heartwarming, funny, and informative encounters with its members, and I feel very grateful to be part of it.

Last but not least, my greatest thanks are directed at my family and friends! Their presence in my life is invaluable to me, and I am astonished by the warmth I encounter in them every day. Without them, I would not have found the endurance to finish this project.

This work was supported by the Deutsche Forschungsgemeinschaft - 510812084.

Introduction

The problem of splitting bordism spectra into indecomposable summands has a long history, starting with Thom's calculation of the unoriented and rationalized oriented bordism rings [Tho54]:

$$\mathcal{N}_* \simeq \mathbb{F}_2[x_i \mid i \neq 2^k - 1] \text{ and } \Omega_* \otimes \mathbb{Q} \simeq \mathbb{Q}[y_{4i}].$$

Thom constructs an isomorphism between the bordism rings and the homotopy groups of spectra associated to the universal vector bundles over BO and BSO, which we now call *Thom spectra*. The homotopy groups of these spectra can then be studied using methods from algebraic topology. His work sparked many exciting developments at the intersection of algebraic topology and differential geometry, notably the Hirzebruch signature theorem [Hir53] and the Atiyah-Singer index theorem [AS63].

An idea already present in Thom's work is that these isomorphisms of groups lift to the topological level: his calculations of $\mathcal{N}_*$ and $\Omega_* \otimes \mathbb{Q}$ lift to equivalences of spectra! Specifically, Thom, and later Wall [Wal60], proved that there are equivalences

$$MO \xrightarrow{\sim} \bigoplus_{\lambda \in P'} \Sigma^{|\lambda|} H\mathbb{F}_2 \text{ and } MSO_{(2)} \xrightarrow{\sim} \bigoplus_{\lambda \in P} \Sigma^{4|\lambda|} H\mathbb{Z}_{(2)} \oplus \bigoplus \Sigma^{?} H\mathbb{F}_2,$$

where the projections are induced by pairing with Stiefel-Whitney and Pontryagin classes.

Some years later, Anderson, Brown, and Peterson [ABP67] constructed similar splittings

$$MSpin_{(2)} \xrightarrow{\sim} \bigoplus_{\lambda \in P'', |\lambda| \text{ even}} \Sigma^{4|\lambda|} kO_{(2)} \oplus \bigoplus_{\lambda \in P'', |\lambda| \text{ odd}} \Sigma^{4|\lambda|-4} kO_{(2)}\langle 2 \rangle \oplus \bigoplus \Sigma^{?} H\mathbb{F}_2$$

and

$$MSpin^c_{(2)} \xrightarrow{\sim} \bigoplus_{\lambda \in P} \Sigma^{4|\lambda|} kU_{(2)} \oplus \bigoplus \Sigma^{?} H\mathbb{F}_2.$$

Here, P is the set of all partitions, $P' = \{\lambda \in P \mid 2^k - 1 \notin \lambda\}$, and $P'' = \{\lambda \in P \mid 1 \notin \lambda\}$.

These results share several connecting features:

- They are proved by studying the $H\mathbb{F}_2$-based Adams spectral sequence.
- Since their $\mathbb{F}_2$-cohomologies are cyclic $\mathcal{A}$-modules, the summands are indecomposable,
- Apart from MO, the projections onto the $H\mathbb{F}_2$ summands and the degrees in which these appear are not explicitly described. Rather, they can be inductively deduced by comparing the Hilbert-Poincaré series of the $H\mathbb{F}_2$-homologies.
- The projections onto the bottom summands admit structures of $\mathbb{E}_\infty$-ring maps and lift certain genera to maps of spectra.
- The $\mathcal{A}$-modules $H\mathbb{F}_2^* H\mathbb{F}_2$, $H\mathbb{F}_2^* H\mathbb{Z}$, and $H\mathbb{F}_2^* kO$ are isomorphic to $\mathcal{A}$, $\mathcal{A}//\mathcal{A}(0)$, and $\mathcal{A}//\mathcal{A}(1)$, respectively.

This suggests a pattern: as we ascend the Whitehead tower of BO, we retain some summands of the form $H\mathbb{F}_2$, while the other summands have $H\mathbb{F}_2$-cohomology similar to $\mathcal{A}//\mathcal{A}(n)$.

In the early 1980s, it was suspected that this pattern would not continue. While Pengelley [Pen83] showed that $H\mathbb{F}_2^* MString$ (then called $MO\langle 8 \rangle$) was an extended $\mathcal{A}(2)$-module, Davis [Dav82] was able to determine the $\mathcal{A}(2)$-module structure to some degree, showing that it was not simply a direct sum of $\mathbb{F}_2$s and $\mathcal{A}(2)$s (though the bottom summand is an $\mathbb{F}_2$). Moreover, Davis and Mahowald had just announced [DM82] that the $\mathcal{A}$-module $\mathcal{A}//\mathcal{A}(2)$ was not realizable as the cohomology of a spectrum.

Almost a decade later, Mahowald realized that there was an error in the charts of $\pi_*\mathbb{S}$ he and Davis used in their paper. In 1993, Mahowald and Hopkins succeeded in constructing the spectrum tmf (then called eo_2), which realized $\mathcal{A}//\mathcal{A}(2)$. The history of what follows is, from an outsider's perspective like mine, quite opaque. Many results regarding $MString$ and tmf were being produced and communicated around that time, but either published only much later or never at all.[1] I can only recommend reading Hopkins' account of this story in [Pet19, § B.1].

In the (unpublished) document [AHR10], Ando, Hopkins, and Rezk construct an $\mathbb{E}_\infty$-ring map

$$\tau_W : MString \longrightarrow tmf$$

which is a projection onto the bottom summand in $H\mathbb{F}_2$-homology, using a lot of hard chromatic homotopy theory and number theory, and the construction of tmf as the 'universal' elliptic ring spectrum due to Goerss, Hopkins, and Miller. It was only during their search for a ring map $MO\langle 8\rangle \to eo_2$ that the people involved realized they were lifting the Witten genus [Wit88] to the world of spectra. This genus associates modular forms to manifolds with *Spin*-structure and vanishing first Pontryagin class using string theory (in the physics sense). I guess this is also the reason for the renaming of these objects.

Since then, the question of decomposing $MString$ into a direct sum of indecomposables (like tmf and its variants), just like the results of Anderson-Brown-Peterson, Wall, and Thom, has remained wide open. At primes $p \geq 5$ $MString$ is known to split into a sum of BPs by work of Hovey [Hov08b], while at the prime 3 ongoing work by Lorman, McTague, and Ravenel promises to provide a splitting analogous to Pengelley's 2-local splitting of MSU [Pen82]. At the prime 2 a candidate for the next summand occurring in $MString$ is $\Sigma^{16} tmf_0(3)$, see [MR09, § 7], but not much is known in general.

A question that might be easier to answer is whether the Ando-Hopkins-Rezk orientation admits a section. It is known to be surjective in homotopy by unpublished work of Hopkins and Mahowald, a proof being presented by Devalapurkar in [Dev20]. In a more recent paper, Devalapurkar also reduces the question of a section, at $p = 2$, to more classical conjectures in homotopy theory and centrality statements about Ravenel's $X(n)$ spectra [Dev24]. As in the Anderson-Brown-Peterson splitting, at least after inverting 6, it seems unlikely that a section could have more structure than being unital, by work of McTague [McT18].

At $p = 2$, one might gain some understanding by focusing on one chromatic layer at a time. Using the $K(1)$-local equivalence $MString \to MSpin$ one could use the known decompositions of $L_{K(1)}MSpin$ and $L_{K(1)}tmf$ into copies of KO to study this question at height 1. By work of Laures [Lau03] and Hopkins [Hop14] we know explicit presentations of $L_{K(1)}MSpin$, $L_{K(1)}tmf$, and τ_W in the category of $K(1)$-local $\mathbb{E}_\infty$-rings. These show that a section can not be made $\mathbb{E}_\infty$.

One of the major hurdles is the lack of a *geometric* description of the cohomology theory represented by TMF. In contrast, KO was originally constructed as a cohomology theory described in terms of vector bundles, and a lot of our detailed knowledge about it is informed by that perspective. Conjecturally, TMF should be related to quantum field theories in 2 dimensions, and making that concrete is the content of much ongoing research, see [Ber24].

This forces one to work with the algebro-geometric description in terms of the spectral moduli stack of oriented elliptic curves. Giving a more conceptual construction of τ_W in these terms is to this day an open problem, too, though much progress has been made on realizing the program sketched by Lurie in his *Survey of Elliptic Cohomology* [Lur09a].

In this thesis, I make some progress towards understanding the situation in the $K(2)$-local setting. Using results of Hovey, Ravenel, and Sadofsky ([HR95], [HS99a]) on the topological side, and a variant of the Milnor-Moore argument by Laures and Schuster [LS19] which works for ungraded Hopf algebras, I am able to prove the following:

[1] It is interesting to note that the *published* accounts of the calculation of $\pi_* tmf$, on which many great computational results of the recent past were built, have only quite recently been freed from a circularity by Carrick-Davies-van-Nightevecht, see [CDN24].

Theorem 0.1. *Let $E = E(\mathbb{F}_4, \widehat{C_0})$ be the Morava E-theory associated to the formal group of the supersingular elliptic curve*

$$C_0 : y^2 + y = x^3,$$

and let $F_{3/2}$ be the open normal subgroup of automorphisms of the formal group law which are the identity up to order 8. Then there is a map

$$D : MU\langle 6\rangle \to L_{K(2)} \bigoplus_I E$$

such that its free $E^{hF_{3/2}}$-linearization induces a $K(2)$-local equivalence

$$E^{hF_{3/2}} \otimes MU\langle 6\rangle \xrightarrow{\sim} L_{K(2)} \bigoplus_I E.$$

In the case of $MString$, by some kind of miracle[2], one can shrink the involved Galois extension of $L_{K(2)}\mathbb{S}$ down quite a lot[3], and determine some of the projections.

Theorem 0.2. *Consider the open subgroup $H = \overline{\langle F_{3/2}, \alpha, -1, \omega, \sigma\rangle} \subset \mathbb{G}$ of the Morava stabilizer group (see Section 11.1 for the definition of these elements). There is a map*

$$Q : BSpin_+ \to L_{K(2)} \bigoplus_{j\in J} \Sigma^{n_j} TMF_0(3),$$

with $n_j \in \{0, 16, 32\}$, so that the E^{hH}-linear map

$$E^{hH} \otimes MString \xrightarrow{\tau_W pr^* Q} L_{K(2)} \bigoplus_{j\in J} \Sigma^{n_j} TMF_0(3)$$

is a $K(2)$-local equivalence.
Furthermore, one can choose Q such that the map

$$BSpin \xrightarrow{1\oplus p} TMF \oplus \Sigma^{16} TMF_0(3)$$

composed with the unit $\mathbb{S} \to E^{hH}$ appears in the splitting, where

$$p = \pi_4/\Delta^6 \in TMF_0(3)^{16} BSpin$$

and we implicitly fix an E^{hH}-linear splitting of $E^{hH} \otimes (TMF \oplus \Sigma^{16} TMF_0(3))$ as a sum of shifts of $TMF_0(3)$. Thus, the map

$$E^{hH} \otimes MString \xrightarrow{\tau_W \oplus \tau_W pr^* p} E^{hH} \otimes (TMF \oplus \Sigma^{16} TMF_0(3))$$

admits an E^{hH}-linear section, and restricted to $E^{hH} \otimes TMF$ this section can be made unital.

There are two directions in which one could try to push this result: firstly, one could try to make the section equivariant for the action of the Galois group, which would immediately give a section of the $K(2)$-local Ando-Hopkins-Rezk orientation.

Secondly, if one could show that all the $Q_j = pr_j \circ Q$, in certain clusters, lift to spectra M_j such that $E^{hH} \otimes M_j \simeq \oplus \Sigma^? TMF_0(3)$, one would have produced an additive $K(2)$-local splitting of $MString$ into copies of the M_js. That is, one could try to 'divide' both the target and the map by E^{hH}, and the 'furthermore' part of Theorem 0.2 is a first step in this direction. The main challenge in both approaches is to say anything meaningful about the Q_j. As we will see, one way of determining

[2]Having to do with the fact that, in $K(2)$-homology, the map $K(\mathbb{Z}, 3) \to BString$ *looks like* it should factor over the multiplication-by-2 map on $K(\mathbb{Z}, 3)$.

[3]Strictly speaking, E^{hH} is not a Galois extension of $L_{K(2)}\mathbb{S}$ since H is not a normal subgroup of $\mathbb{G}$.

the Q_j involves the formal group law of the elliptic curve, which we know well, but the resulting formulas seem too complicated to be used directly. Additionally, apart from [LO16] for $TMF_0(3)$, there are not many calculations of the cohomology of $BSpin$ for even the most natural candidates for the M_j, such as TMF or $TMF_0(5)$, making the second approach seem even harder.

Let us give a rough overview of the proof of Theorem 0.2. It is well known ([HR95]) that there is a finite even spectrum Z such that $Z \otimes MString$ splits as a sum of BPs 2-locally, so it splits as a sum of $BP\langle 2\rangle$s $K(2)$-locally ([HS99a]). This allows us to construct a unital map $E \to Z' \otimes MString$, where Z' is a finite sum of shifts of Z. Z' comes with a preferred unital map to E, and a concrete calculation shows that, in K-homology, this factors uniquely over $E^{hF_{3/2}\mathbb{S}} \to E$.

Putting this together we obtain a unital map of K_0E-comodules

$$K_0E \to K_0\left(E^{hF_{3/2}\mathbb{S}} \otimes MString\right).$$

The Milnor-Moore argument then tells us that the target of this map must be a cofree comodule, from which we deduce that there is a $K(2)$-local equivalence

$$E^{hF_{3/2}\mathbb{S}} \otimes MString \simeq \bigoplus E.$$

We then show, for suitable choices made in the Milnor-Moore argument, that we can force the above equivalence to be $E^{hF_{3/2}\mathbb{S}}$-linear! This is a key step that gives us a lot of control over the splitting. In particular, the splitting is determined by its restriction to $MString$, and thus by a set of classes in $E^0BString$.

Studying the K_0E-comodule structure of $K_0MString$ in more detail, we are then able to prove that these characteristic classes already factor over suspensions of $TMF_0(3)$ and $BSpin$, which leads us to the first statement of Theorem 0.2. Finally, we prove the 'furthermore' part of Theorem 0.2 by concrete calculations using the theory of cubical structures and cannibalistic classes.

Outline

In Part I, we introduce the various tools we will use to study $K(n)$-local spectra.

- In Section 1, we recall some basic definitions about complex oriented cohomology theories and discuss the concept of k-structures. This will be our main tool to study the complex oriented (co)homology of the spaces in the Whitehead towers of BU and BO.
- The Morava stabilizer group is a compact p-adic analytic Lie group, and we recall some useful results about these in Section 2.
- Section 3 contains most of what we need to know about Morava E-theories, including a discussion of complex orientations, completed E-homology, and continuous homotopy fixed points.
- Section 4 explores the structure of the Hopf algebroid K_0E, and we introduce the Milnor-Moore argument.

Part II, as an example, applies these tools to study the spectra $MSpin^c$, $MSpin$, and MSU at height 1. The results we obtain are not new, but they demonstrate the kind of arguments we will use to study $MString$. For this reason, the formal structure of this Part parallels that of Part III.

We use 2-completed KU as our model of E, discuss various orientations of it, introduce the relevant characteristic classes, and study how the Adams operations interact with the orientations in Sections 5 and 6. In Sections 7, 8, 9, and 10, we apply the program sketched in the introduction to deduce various $K(1)$-local splittings. In the case of $MSpin^c$, we are able to completely reprove the $K(1)$-local variant of the Anderson-Brown-Peterson splitting, while for $MSpin$ and MSU, we don't get quite as far without running into hard combinatorial questions.

Then, in Part III, we tackle the $K(2)$-local structure of $MString$. The strategy we use here is the same as in the previous part, but the technical details become more complicated. This Part is mostly contained in my upcoming paper [Tok24].

- In Section 11, we introduce a convenient choice of Morava E-theory at height 2 based on an elliptic curve, discuss the concrete structure of the stabilizer group, and relate it to topological modular forms with level structure.

- Section 12 introduces various orientations and characteristic classes for our choice of Morava E-theory. For this, it is important that we work with an elliptic Morava E-theory so that we have the Ando-Hopkins-Rezk orientation available. We also calculate approximations of some cannibalistic classes using the theory of cubical structures.
- In Sections 13 and 14 we start the program sketched in the introduction by proving cofreeness of $K_0\left(E^{hF_{3/2}\mathbb{S}} \otimes MString\right)$ and proving a first splitting statement at the spectrum level.
- To conclude, we prove the statements of Theorem 0.2 in Sections 15 and 16 as Theorem 15.6 and Proposition 16.9.

This is as far as we get without seriously studying the action of $\mathbb{G}$ on $K_0 MString$ and *Spin*-characteristic classes for TMF with other level structures.

Finally, we have banished several more technical discussions and lemmas to the Appendix.

- Appendix A introduces and proves the variant of the Milnor-Moore argument we will use.
- Appendix B studies coalgebras associated to certain p-adic analytic Lie groups, which include the restricted Morava stabilizer groups, and proves that they meet the requirements to apply the Milnor-Moore argument.
- Appendix C contains some duality statements about Morava E-(co)homology, which we felt would only clutter the main part of this thesis.

Notations and conventions

For R a homotopy ring spectrum, X some other spectrum, and classes $a \in R_m X$, $b \in R^n X$ we denote by

$$\langle a, b \rangle \in \pi_{m-n} R$$

their Kronecker pairing. If X is a suspension spectrum, we also denote by

$$a \frown b \in R_{m-n} X$$

their cap product. In the case that $R = E$ is a Morava E-theory, we will also use this notation for the analogous pairings between $E^* X$ and $E^{\vee}_* X$.

Also, note that we often abbreviate the pointed suspension spectrum $\Sigma^{\infty}_{+} X$ of a space X as X_+ when there is little room for confusion.

Finally, while most parts of this thesis take place in hSp, the homotopy 1-category of spectra, we sometimes employ constructions and concepts which are most naturally interpreted in the language of ∞-categories. The standard references for this are [Lur09b] and [Lur17], and we will use their content freely.

Part I

Background

In this Part we shall recall some important background about Morava E-theories, K-theories, and stabilizer groups, and fix some choices and terminology. We will discuss as much as we can for general heights and primes, and introduce our specific choices at height 1 and 2 at the prime 2 only in the corresponding parts.

Most of the statements in this Part are surely known to experts, and we only include proof when we either couldn't find the result directly stated in the literature, or some details of the proof will become important later on.

1 Complex periodic cohomology and k-structures

We begin with some basic notions in complex oriented cohomology theory, mainly to fix some notation. The material we present can mostly be found in [Ada95, ch. 2], for a review see also [Lur18, § 5.3].

Fix an equivalence $\pi_2 BU(1) \simeq \mathbb{Z}$, exhibiting $BU(1)$ as a form of $K(\mathbb{Z}, 2)$. This gives us a preferred pointed map $S^2 \to BU(1)$. We also identify $S^{\mathbb{C}}$ with S^2 by giving $\mathbb{C}$ the ordered $\mathbb{R}$-basis $(1, i)$. Let L be the universal complex line bundle over $BU(1)$. The zero section induces a canonical map of pointed spaces

$$s_0 : BU(1)_+ \to Th(L).$$

Since $S(L) \simeq EU(1) \simeq *$, composing this with the natural inclusion yields an equivalence

$$BU(1) \simeq Th(L)$$

of spaces. Finally, the following square

$$\begin{tikzcd} Th(L) \arrow[r, "\tilde{\Delta}"] & Th(L) \otimes BU(1)_+ \\ BU(1)_+ \arrow[u, "s_0"] \arrow[r, "\Delta"'] & BU(1)_+ \otimes BU(1)_+ \arrow[u, "s_0 \otimes \mathrm{id}"'] \end{tikzcd}$$

commutes, where $\tilde{\Delta}$ denotes the Thom diagonal.

Definition 1.1.

- Let E be a multiplicative cohomology theory. A *complex orientation* of E is a class $x_E \in \tilde{E}^2 BU(1)$ such that its pullback along $S^2 \to BU(1)$ is the image under the suspension isomorphism of $1 \in \tilde{E}^0 S^0 = E^0$. If a complex orientation exists, we call E *complex orientable*. When E is given as a commutative algebra in hSp, then this is the same datum as an extension of the unit $\mathbb{S} \to E$ across $\mathbb{S} = \Sigma^{-2} S^2 \to \Sigma^{-2} BU(1)$.
- We call a commutative algebra E in hSp *complex periodic* if E is complex orientable, $\pi_2 E$ is a locally free $\pi_0 E$-module of rank 1, and the multiplication map

$$\pi_2 E \otimes_{\pi_0 E} \pi_{-2} E \to \pi_0 E$$

is an equivalence.
- We call E *even periodic* if $\pi_* E$ is concentrated in even degrees and $\pi_{-2} E$ contains a unit of $\pi_* E$.
- For E complex periodic, the tensor product of line bundles equips $\mathrm{Spf}(E^0 BU(1))$ with the structure of a formal group over $\mathrm{Spec}(\pi_0 E)$, which we call the *Quillen formal group* of E and denote G_E^Q.

Proposition 1.2 ([Lur18, ex. 5.3.7]). *Let E be a complex periodic commutative algebra in hSp. Then a section $z \in \mathcal{O}_{G_E^Q}(-e) = \tilde{E}^0 BU(1)$ is a coordinate on G_E^Q iff its restriction to $\tilde{E}^0 S^2 = \pi_2 E$ is a unit in $\pi_* E$. In that case, we denote its inverse by $u \in \pi_{-2}E$, and $uz \in \tilde{E}^2 BU(1)$ is a complex orientation of E. Moreover, this construction gives a bijection of sets*

$$\{\textit{Coordinates on } G_E^Q\} \simeq \{\textit{Complex orientations of } E\} \times \{\textit{Units in } \pi_2 E\}.$$

Let MU and MUP be the Thom spectra associated to BU and $BU \times \mathbb{Z}$. The natural inclusion $f : BU(1) \to BU$ classifies the virtual rank 0 vector bundle $[L-1]$. Functoriality of the Thom construction then yields a map

$$x_{MU} : \Sigma^{-2} BU(1) \simeq Th(L-1) \xrightarrow{Th(f)} MU$$

which is a complex orientation of MU, and composing with the canonical map $MU \to MUP$ yields a complex orientation x_{MUP} of MUP. Furthermore, applying the Thom construction to the map

$$* \xrightarrow{(e,-1)} BU \times \mathbb{Z}$$

gives a unit $u_{MUP} \in \pi_{-2} MUP$. Thus, MUP is complex periodic, and $z_{MUP} = x_{MUP}/u_{MUP}$ is a coordinate on G^Q_{MUP}. These are universal in the following sense:

Proposition 1.3. *Let E be a commutative algebra in hSp equipped with a complex orientation x_E. Then there is a unique map $MU \to E$ of commutative algebras in hSp which sends x_{MU} to x_E.*
If E is complex periodic and z_E is a coordinate on G_E^Q there exists a unique map $MUP \to E$ of commutative algebras in hSp which sends z_{MUP} to z_E.
Finally, the formal group law over $\pi_0 MUP$ of G^Q_{MUP} in the coordinate z_{MUP} is the universal formal group law.

Remark 1.4. When E is an $\mathbb{E}_\infty$-ring, it is an interesting problem to lift the above maps to maps of $\mathbb{E}_\infty$-rings. We will discuss what is known about this for Morava E-theories in Section 3.1.

Proposition 1.5. *Let E be a commutative algebra in hSp equipped with a complex orientation x_E. Then we have*

- $E^* BU(1) = E^*[\![x_E]\!]$.
- $E_* BU(1) = E_*\langle \beta_0, \beta_1, \ldots \rangle$ *with β_i dual to x_E^i.*
- $E^* BU = E^*[\![c_1, c_2, \ldots]\!]$ *where c_i restricts, on each maximal torus $U(1)^n \subset U(n) \subset U$ to the i-th elementary symmetric polynomial in the classes $pr_1^* x_E, \ldots, pr_n^* x_E$.*
- $E_* BU = E_*[b_1, b_2, \ldots]$ *with $b_i = f_* \beta_i$ and $b_0 = 1$.*

Remark 1.6. When E is complex periodic with a given coordinate z, we will use the canonical unit in $\pi_{-2}E$ to shift the natural classes discussed above to degree 0.

Remark 1.7. We will also want to talk about complex orientations of Morava K-theories at the prime 2, which are *not* commutative algebras in hSp. This is not really a problem, though: The difference between the two multiplications is controlled by an operation $\beta : K \to \Sigma K$ squaring to 0 such that

$$\mu \circ \mathrm{swap} = \mu + u \circ \mu \circ (\beta \otimes \beta),$$

see [KW87]. Now one can either be satisfied with always arguing that the relevant spectra have $K_*(-)$ concentrated in even degrees, or let it take values in $\mathbb{Z}/2$-graded chain complexes with a symmetric monoidal structure mimicking the above formula to make $K_*(-)$ a symmetric monoidal functor.[a]

[a] I want to thank Nicholas Kuhn for teaching me the latter approach.

1.1 Morava K-homology of connective covers of BU and BO, and the connection to (real) k-structures

In this Section we recall the concept of *k-structures* on a formal group law and review some results connecting the complex oriented homology of connective covers of BU and BO to them. This definition is borrowed from [AHS01], compare also [AHS99] and [KL02]. For a good overview, which also includes the relevant results of the unpublished [AHS99], see [Pet19, ch. 5].

We first note that the spaces BU, BSU, $BU\langle 6\rangle$, BSO, $BSpin$ and $BString$ all have Morava K-homology concentrated in even degrees. For the first three can be found in [AHS01], for the last three in [KLW04].

Definition 1.8. Let F be a formal group law on a ring R and $k \geq 1$. A *k-structure* on an R-algebra S is a power series $f \in S[\![x_1, \dots, x_k]\!]$ such that

- $f(0, x_2, \dots, x_k) = 1$
- $f(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = f(x_1, \dots, x_k)$ for all $\sigma \in \Sigma_k$
- for $k \geq 2$:
$$\frac{f(x_1, x_2, \dots, x_k)}{f(x_0, x_1, x_3, \dots, x_k)} = \frac{f(x_0 +_F x_1, x_2, \dots, x_k)}{f(x_0, x_1 +_F x_2, \dots, x_k)}.$$

A real 1-structure is a 1-structure such that $f(x) = f([-1](x))$, a real 2-structure is a 2-structure such that $f(x, y) = f([-1](x), [-1](y))$.
Given a k-structure f, we can produce a $(k+1)$-structure

$$\delta f(x_1, \dots, x_{k+1}) = \frac{f(x_1, x_3, \dots, x_{k+1}) f(x_2, x_3, \dots, x_{k+1})}{f(x_1 +_F x_2, x_3, \dots, x_{k+1})}.$$

If f was a real 1-structure, then δf will be a real 2-structure.
All of these are corepresentable by R-algebras $C_k F$, $C_1^r F$, $C_2^r F$ (and maps between them) which are natural in both F and R, and the natural maps $C_2 F \to C_2^r F$ and $C_1 F \to C_1^r F$ are surjective.

Theorem 1.9 ([AHS01, thm. 2.31])**.** *Let E be an even periodic complex oriented homotopy commutative ring spectrum with associated formal group law F over E_0. Then there are, for $k \leq 3$, natural isomorphisms*

$$C_k F \to E_0 BU\langle 2k\rangle$$

classifying the power series associated to the bundle

$$[L_1 - 1] \otimes \ldots \otimes [L_k - 1] : (\mathbb{C}P^\infty)^k \to BU\langle 2k\rangle.$$

The maps corepresenting δ correspond to the covering maps

$$BU\langle 2k+2\rangle \to BU\langle 2k\rangle.$$

Theorem 1.10 ([AHS99, thm. 1.6], see also [Pet19, thm. 5.6.16])**.** *Let K be an even periodic complex oriented Morava K-theory at a prime p and height n. Consider the natural maps*

$$C_2 F \simeq K_0 BSU \to E_0 BSpin \text{ and } C_1 F \simeq K_0 BU \to K_0 BSO.$$

If p is odd, they exhibit their targets as $C_2^r F$ and $C_1^r F$. If $p = 2$, this works for the first map at heights 1 and 2, and for the second map at height 1.

Theorem 1.11 ([KL02, thm. 1.1]). *For F the Honda formal group law over $\mathbb{F}_2$ (or if 2 is invertible in the coefficients) the map corepresenting δ gives an isomorphism*

$$C_2^r F \to C_1^r F.$$

By definition a 1-structure is a power series $f(x)$ such that $f(0) = 1$. Thus, $C_1 F \cong R[b_1, b_2, \ldots]$ and the universal example is $f(x) = 1 + \sum_{k \geq 1} b_k x^k$. Note that the choice of the b_i is natural in F and R. $C_1^r F$ is then a quotient of this polynomial algebra, but in many cases it is itself polynomial.

Proposition 1.12 (Compare [KL02, prop. 2.5, 2.7]). *Let E be a complex oriented Morava E-theory of finite height with quotient field k of characteristic 2, and let F be the associated formal group law. Then*

$$(C_1^r F)^\wedge_{\mathfrak{m}} \cong (E_0[b_2, b_4, b_6, \ldots])^\wedge_{\mathfrak{m}}.$$

Proof. As in the proof of Proposition 2.5 in [KL02] we see that

$$(E_0[b_2, b_4, b_6, \ldots])/\mathfrak{m}^n \to C_1^r F/\mathfrak{m}^n$$

is surjective for $n = 1$, and then for $n \geq 1$ by Nakayama's lemma.
Inspecting the proof of Proposition 2.7 in [KL02] we are reduced to proving that the classes

$$(-1)^k \left(b_k^2 + 2 \sum_{j=1}^{k} b_{k+j} b_{k-j} \right) + \text{ terms of degree less than } 2k$$

are algebraically independent in $E_0/\mathfrak{m}^n[b_1, b_2, \ldots]$, where we give b_i degree i.
The b_k^2 parts are clearly algebraically independent, and adding multiples of 2 to them doesn't change that since 2 is nilpotent in $E_0/\mathfrak{m}^n$. Adding the terms of lower degree also doesn't change the algebraic independence by looking at the top degree part.
We saw that the map is an isomorphism after quotienting out finite powers of $\mathfrak{m}$, thus it is an isomorphism after completion. □

Remark 1.13. The above also suggests a way to determine how to write b_{2k+1} in terms of $b_2, \ldots, b_{2k}$: since there is no 2-torsion in $(C_1^r F)^\wedge_{\mathfrak{m}}$ it is enough to look at the coefficients of x^{2k+1} in

$$2f(x) = f(x) + f([-1](x)).$$

This expresses $2b_{2k+1}$ as a linear combination of $b_1, \ldots, b_{2k}$.

Proposition 1.14. *Let E be a complex oriented Morava E-theory of finite height with quotient field k of characteristic 2, and let F be the associated formal group law. Then the map corepresenting δ gives an isomorphism*

$$(C_2^r F)^\wedge_{\mathfrak{m}} \to (C_1^r F)^\wedge_{\mathfrak{m}}.$$

Proof. In the algebraic closure of k the formal group law F is isomorphic to the Honda formal group law. So, by naturality in the formal group law, Theorem 1.11, and that field extensions are faithfully flat, we find that

$$(C_2^r F)/\mathfrak{m} \to (C_1^r F)/\mathfrak{m}$$

is an isomorphism. By Nakayama's lemma we find that

$$(C_2^r F)/\mathfrak{m}^n \to (C_1^r F)/\mathfrak{m}^n$$

is surjective for all $n \geq 1$.
Denote the kernel of the above map by K_n. We saw that $K_1 = 0$. In the previous proposition we have also seen that $(C_1^r F)/\mathfrak{m}^n$ is a free $E_0/\mathfrak{m}^n$ module, so in particular it is flat. A diagram chase now shows that $K_n = K_{n+1}/\mathfrak{m}^n$. Since $\mathfrak{m}^n$ is a nilpotent ideal in $E_0/\mathfrak{m}^{n+1}$, we can use Nakayama's lemma again to show that all K_n vanish. □

2 Compact p-adic analytic Lie groups

Let's recall some definitions and results from the theory of profinite groups. This will be relevant since the Morava stabilizer group $\mathbb{G}$ we will encounter in Section 3.2 is a compact p-adic analytic Lie group, and we want to make use of that property.

Definition 2.1.

1. A topological group G is called *profinite* if it is compact, Hausdorff, and the set of normal open subgroups $U \subset G$ is a basis of neighborhoods of $e \in G$.
2. A profinite group G is called *finitely generated* if there exists a finite subset $S \subset G$ such that G is the closure of the subgroup generated by S.
3. A profinite group G is called a *pro-p group* if every open subgroup $U \subset G$ has index a power of p: $[G : U] = p^i$ for some i.
4. A pro-p group G is called *powerful* if $[G, G] \subset \overline{\langle g^p \mid g \in G\rangle}$ for p odd, while for $p = 2$ we require $[G, G] \subset \overline{\langle g^4 \mid g \in G\rangle}$.
5. The *lower p-series* of a pro-p group G is the filtration given by $P_1(G) = G$ and $P_{i+1}(G) = \overline{P_i(G)^p[P_i(G), G]}$.
6. A pro-p group is called *uniform* if it is finitely generated, powerful, and for all $i \geq 1$ we have $[P_i(G) : P_{i+1}(G)] = [P_{i+1}(G) : P_{i+2}(G)]$.
7. Finally, we call a profinite group G a *compact p-adic analytic Lie group* if it has an open normal subgroup which is a uniform pro-p group of finite index in G.

Remark 2.2. A theorem of Lazard [Laz65] tells us that the above definition of a compact p-adic analytic Lie group agrees with a different one in terms of p-adic analytic atlases, hence the name. It also tells us that a homomorphism between them is locally analytic iff it is continuous.

Theorem 2.3 ([Dix+99, thm. 4.5])**.** *Let G be a finitely generated powerful pro-p group. Then G is uniform if and only if G is torsion free.*

Theorem 2.4 ([Dix+99, thm. 4.9])**.** *Let G be a uniform pro-p group with topological generating set $\{g_1, \ldots, g_d\}$ of minimal cardinality. Then the map*

$$\mathbb{Z}_p^d \to G,\ (n_1, \ldots, n_d) \mapsto g_1^{n_1} \ldots g_d^{n_d}$$

is a homeomorphism.

A classical result of Serre tells us that every finite index subgroup of a finitely generated pro-p group is open. This has been generalized by Nikolov and Segal to all finitely generated profinite groups [NS07], using the classification of finite simple groups. Combining this with Lazard's result gives the following theorem.

Theorem 2.5 ([Laz65],[NS07])**.** *Let G be a compact p-adic analytic Lie group. Then a subgroup $H \subset G$ is open iff it has finite index. Moreover, the forgetful functor from compact p-adic analytic Lie groups and locally analytic homomorphisms to groups is fully faithful.*

3 Morava E-theories

For a good overview, see [BB20, § 5.3.2]. Fix a finite field $\mathbb{F}$ of characteristic p and cardinality q, and let G_0 be a formal group of finite positive height n over $\mathbb{F}$. Goerss-Hopkins-Miller ([GH04, § 7]), and later Lurie ([Lur18, § 5], see also [Dav25, thm. 1.11, prop. 6.1]), gave a functorial construction of complex periodic $\mathbb{E}_\infty$-ring spectra $E = E(\mathbb{F}, G_0)$ such that:

- E_0 is a complete local ring with maximal ideal $\mathfrak{m}$, and there are natural isomorphisms $E_0/\mathfrak{m} \simeq \mathbb{F}$ and of the mod $\mathfrak{m}$ reduction of the formal group G_E^Q to G_0, exhibiting G_E^Q as a universal deformation of G_0.

- $\pi_* E$ is concentrated in even degrees, and the even homotopy groups $\pi_{2k} E$ are naturally equivalent to $\omega_{G_E^Q}^{\otimes k}$.
- The endomorphism space $CAlg(E, E)$ is equivalent to the discrete set of automorphisms of the pair $(\mathbb{F}, G_0)$.

These are called Morava E-theories, and the group $\mathbb{G} = Aut(\mathbb{F}, G_0)$ the associated Morava stabilizer group. As E_0 carries a universal deformation of G_0, it is isomorphic to the Lubin-Tate ring

$$E_0 \simeq W(\mathbb{F})[\![u_1, \ldots, u_{n-1}]\!]$$

with maximal ideal $(p, u_1, \ldots, u_{n-1})$, though the isomorphism is not canonical.

Notation 3.1. We shall denote by $\mathfrak{m}$ the maximal ideal of E_0, by K the associated Morava K-theory $E/\mathfrak{m}$, and for $g \in \mathbb{G}$ the associated map of $\mathbb{E}_\infty$-rings by $\psi^g : E \to E$. Also, we denote by $\mathbb{S}$ the kernel of the canonical map $\mathbb{G} \to Gal(\mathbb{F}, \mathbb{F}_p)$.

Remark 3.2. The spectrum E is $K(n)$-local, and the Bousfield class of K equals that of $K(n)$.

The action of $\mathbb{G}$ on E_0 and $E^0 BU(1)$ is, in general, quite complicated. At height 1 the comparison with the Adams operations on KU gives a full understanding, while at height 2 recent work of Salch gives explicit formulas, see [Sal25]. Working mod $\mathfrak{m}$ we have the following answer:

Proposition 3.3. *The action of $\mathbb{G}$ fixes the ideal $\mathfrak{m}$ in $E^0 BU(1)$, and on*

$$E^0 BU(1)/\mathfrak{m} \simeq K^0 BU(1) \simeq \mathcal{O}_{G_0}$$

agrees with the canonical action of $Aut(\mathbb{F}, G_0)$.

Definition 3.4. Let G_0 be as above, and z a coordinate on G_0 such that the associated formal group law F_0 is already defined over $\mathbb{F}_p$. Furthermore, assume that there exists a p-adic unit u such that $[up](z) = z^q$. We shall call the triple (G_0, z, u) an *adapted*[a] formal group.

[a]We don't know if there is a standard name for such a structure in the literature, but since we will only discuss adapted formal groups from now, we felt the need to give it *some* name.

Remark 3.5. Note that not every formal group G_0 over $\mathbb{F}$ admits the structure of an adapted formal group. For example, G_0 might not admit a lift to $\mathbb{F}_p$. Moreover, an adapted formal group of height n will always have $q = p^n$. It also implies that every endomorphism of F_0 over $\overline{\mathbb{F}_p}$ is already defined over $\mathbb{F}$.

For the remainder of this Section, let (G_0, z, u) be a fixed adapted formal group over $\mathbb{F}$ with associated formal group law F_0.

3.1 Complex Orientations of Morva E-theories

We now want to discuss the existence of coordinates z_E on G_E^Q such that the associated map $MUP \to E$ admits the structure of an $\mathbb{E}_\infty$-ring map. A first question might be to ask when it has the property of being an H_∞-ring map:

Theorem 3.6 ([Zhu20, prop. 8.17, cor. 8.20], [AHS04, prop. 6.1])**.** *For each coordinate z_0 on G_0 there exists a unique coordinate z_E on G_E^Q lifting z_0 such that the associated map $MUP \to E$ is an H_∞-map.*

Remark 3.7. Such coordinates can be characterized by the purely algebraic property of being *norm coherent*, see [Zhu20, def. 6.21, prop. 6.25].

At low heights, this is sufficient to lift this to $\mathbb{E}_\infty$ complex orientations:

Theorem 3.8 ([Sen23, thm. 1.4])**.** *If the formal group G_0 has height ≤ 2, then the map $MU \to MUP \to E$ associated to a norm coherent coordinate z_E admits the structure of an $\mathbb{E}_\infty$-ring map, unique up to homotopy.*

We also have the following results due to Balderrama:

Theorem 3.9 ([Bal23, thm. 6.5.3])**.**

- *The map $MUP \to KU$ associated to the coordinate $z_{KU} = [L-1]$ admits a unique structure of a map of $\mathbb{E}_\infty$-rings.*
- *Let G_0 have height 2. For every norm coherent coordinate z_E on G_E^Q, the associated map $MUP \to E$ admits a structure of a map of $\mathbb{E}_\infty$-rings.*

Remark 3.10. In the above discussion, we have not really used that (G_0, z, u) is adapted, but only that $\mathbb{F}/\mathbb{F}_p$ is an algebraic extension.

Remark 3.11. The above also shows that the coordinate $-z = [1-L]$ of $E = KU_2^\wedge$ can't be norm coherent, since z and $-z$ reduce to the same coordinate over $\mathbb{F}_2$.

At heights ≥ 3, not much is known in general, though the results of [BSY22] imply that there are $\mathbb{E}_\infty$-ring maps $MUP \to E(L, H_n)$ for some algebraically closed field L of characteristic p and H_n the Honda formal group.

3.2 The strucutre of $\mathbb{G}$

The material here can, for the Honda formal groups, be found in [Rav86, A 2], a good review is [Hen17, § 3.4]. For a reference that also treats other formal groups, see [Buj12].

Let (G_0, z, u) be an adapted formal group of height n over $\mathbb{F} \simeq \mathbb{F}_{p^n}$. We collect the statements about the structure of the associated Morava stabilizer group we need in the following omnibus theorem:

Theorem 3.12.

1. *The map $\mathbb{G} = Aut(\mathbb{F}, G_0) \to Gal(\mathbb{F}, \mathbb{F}_p)$, which sends an automorphism of the pair to the underlying automorphism of $\mathbb{F}$, is split surjective. The preferred splitting sends an automorphism ϕ of $\mathbb{F}$ to the automorphism (ϕ, id) of the pair.*
2. *The ring $End(F_0)$ of endomorphism of the formal group law F_0 over $\mathbb{F}$ is isomorphic to*
$$End(F_0) \simeq W(\mathbb{F})\langle S\rangle/(S^n = up, Sw = w^\sigma S \text{ for } w \in W(\mathbb{F}))$$
where $W(\mathbb{F})$ is the ring of Witt vectors, the angled brackets indicate a noncommutative polynomial ring, and w^σ is the result of applying the canonical lift of the Frobenius to the Witt vector w. We shall describe a choice of isomorphism in Construction 3.13.
3. *The z-adic topology on $\mathbb{F}[\![z]\!]$ equips the group $\mathbb{S}$ with the structure of a profinite group. As such, it is a compact p-adic analytic Lie group of dimension n^2.*
4. *The map $\mathbb{S} \to \mathbb{F}^\times$ sending an automorphism of F_0 to its leading coefficient is split surjective and its kernel is a pro-p group. There exists a splitting which commutes with the natural actions of $Gal(\mathbb{F}, \mathbb{F}_p)$.*

Construction 3.13. Choose a lift F of F_0 to $W(\mathbb{F}_p)^a$. The $[up]$-series of F then equals z^q mod p. The Lubin-Tate lifting lemma ([Pst21, lem. 14.10, rmk. 14.12]) then tells us that

- There is a unique formal group law $\Gamma_n(u)$ over $W(\mathbb{F}_p)$ such that its $[up]$-series is $upz + z^q$, and
- there is a unique strict isomorphism $\phi : F \to \Gamma_n(u)$ over $W(\mathbb{F}_p)$.

Let $\gamma_n(u)$ be the reduction mod p. Then ϕ induces a strict isomorphism $F_0 \to \gamma_n(u)$ defined over $\mathbb{F}_p$, and so an isomorphism $End(F_0/\mathbb{F}) \simeq End(\gamma_n(u)/\mathbb{F})$ which commutes with the action of $Gal(\mathbb{F}, \mathbb{F}_p)$.

An element $A \in W(\mathbb{F})\langle S\rangle/(S^n = up, Sw = w^\sigma S \text{ for } w \in W(\mathbb{F}))$ can be uniquely written as a power series

$$A = a_0 + a_1 S + a_2 S^2 + \dots$$

with the a_i Teichmüller lifts of elements in $\mathbb{F}$, that is $a_i^q = a_i$. There is a canonical isomorphism

$$W(\mathbb{F})\langle S\rangle/(S^n = up, Sw = w^\sigma S \text{ for } w \in W(\mathbb{F})) \xrightarrow{\sim} End(\gamma_n(u)/\mathbb{F})$$

sending S to z^p and the Teichmüller lift of $a \in \mathbb{F}$ to az, see [Pst21, thm. 16.6]. Note that the composite isomorphism does depend on the choice of F, but for any choice S will be sent to the endomorphism z^p.

[a] A lift always exists (since the Lazard ring is polynomial), but may be hard to specify in practice. This is related to the non-canonicity of the isomorphism $E_0 \simeq W(\mathbb{F})[\![u_1, \dots, u_{n-1}]\!]$.

A choice of isomorphism as in Construction 3.13 exhibits $End(F_0)$ a free left $W(\mathbb{F})$-module of rank n, with a basis given by $(S^0, \dots, S^{n-1})$. Multiplying by an element of $\mathbb{S}$ from the right and taking the determinant of the resulting $n \times n$ matrix defines a homomorphism

$$\det_F : \mathbb{S} \to W(\mathbb{F})^\times.$$

In fact, this homomorphism already lands in $\mathbb{Z}_p^\times$ and is equivariant with respect to the action of $Gal(\mathbb{F}, \mathbb{F}_p)$, so extends canonically to $\mathbb{G}$ by sending automorphisms of the form (ϕ, id) to 1. As the determinant of $1 + S$ is $1 + up$, the image of $\det_F$ always contains $(1 + p\mathbb{Z}_p^\wedge) \subset \mathbb{Z}_p^\times$. Since the exact sequence

$$0 \to (1 + p\mathbb{Z}_p^\wedge) \to \mathbb{Z}_p^\times \to \mathbb{F}_p^\times \to 0$$

is canonically split by the Teichmüller map $\mathbb{F}_p^\times \to \mathbb{Z}_p^\times$, we get a surjective homomorphism

$$N_F : \mathbb{S} \to (1 + p\mathbb{Z}_p^\wedge)$$

called the *norm*. For any choice of F the norm is split by sending the generator $1 + up$ to the automorphism $F_0(z, z^p)$, which also shows that N_F does not depend on the choice of F.

The ring $End(F_0)$ has a natural complete filtration by the ideals (S^k) for $k \geq 0$. This induces a filtration of $\mathbb{S}$ by open normal subgroups

$$F_{k/n}\mathbb{S} = \{g \in \mathbb{S} \mid g \equiv 1 \bmod (S^k)\}.$$

The indexing is chosen such that the automorphism $1 + p$ has filtration 1.

Finally, $F_{1/n}\mathbb{S}$ is the kernel of the natural map $\mathbb{S} \to \mathbb{F}^\times$ and a pro-p group, and any choice of Galois equivariant section yields an isomorphism

$$\mathbb{G} \simeq F_{1/n}\mathbb{S} \rtimes \left(\mathbb{F}^\times \rtimes Gal(\mathbb{F}, \mathbb{F}_p)\right).$$

3.3 Completed E-homology

Notation 3.14. Let X be a spectrum. We denote the homotopy groups $\pi_* L_K (E \otimes X)$ by $E_*^\vee X$ and call them the *completed* E-homology groups of X.

Note that $E_*^\vee(-)$ is, in general, not a homology theory.[4] The properties of this construction have been thoroughly studied by Hovey and Strickland in [HS99b], and a lot of interesting results are in [BH16]. We recall here some key properties, and in Appendix C state some more technical lemmas about *continuous* duality between $E_*^\vee X$ and $E^* X$.

[4] The exceptions are the height 0 and 1 cases, see [Hov08a].

Definition 3.15. Let $\mathcal{M}$ be the category of E_0-modules, and denote by L_0 the 0-th left derived functor of the completion functor $M \mapsto M^{\wedge}_{\mathfrak{m}}$. We call an E_0-module M *L-complete* if the natural comparison map $M \to L_0 M$ is an equivalence, and denote the full subcategory on L-complete modules by $L\mathcal{M}$. We call an E_0-module which is isomorphic to the $\mathfrak{m}$-completion of a free module *pro-free*, and for any choice of such an isomorphism we call the collection of the images of the basis a *pro-basis*.
Also, we apply the same terminology for graded E_*-modules.

Theorem 3.16.

1. *Every $\mathfrak{m}$-complete E_0-module is L-complete.*
2. *For every E_0-module M, $L_0 M$ is L-complete.*
3. *The functor $L_0 : \mathcal{M} \to L\mathcal{M}$ is left adjoint to the inclusion.*
4. *$L\mathcal{M}$ is an Abelian subcategory of $\mathcal{M}$ closed under extensions.*
5. *$M \in L\mathcal{M}$ is projective iff it is pro-free.*
6. *If $M \in \mathcal{M}$ is projective or finitely generated, then the natural map $L_0 M \to M^{\wedge}_{\mathfrak{m}}$ is an isomorphism.*
7. *The construction $(M, N) \to M \overline{\otimes} N := L_0(M \otimes_{E_0} N)$ equips $L\mathcal{M}$ with a symmetric monoidal structure.*

Theorem 3.17. *Let M be a K-local E-module. Then each $\pi_k M$ is an L-complete E_0-module. In particular, for any spectrum X, both $E^* X$ and $E^{\vee}_* X$ are L-complete E_*-modules. Let X be a spectrum such that $K_* X$ is concentrated in even degrees. Then the following statements hold:*

1. *$E^{\vee}_* X$ is concentrated in even degrees.*
2. *The natural map $E^{\vee}_0 X \to K_0 X$ is surjective with kernel $\mathfrak{m} E^{\vee}_0 X$.*
3. *$E^* X$ and $K^* X$ are concentrated in even degrees, and the natural map $E^* X \to K^* X$ is surjective with kernel $\mathfrak{m} E^* X$.*
4. *The natural map $E_0 X \to E^{\vee}_0 X$ exhibits its target as the completion $(E_0 X)^{\wedge}_{\mathfrak{m}}$.*
5. *$E^{\vee}_0 X$ is pro-free.*

Conversely, if $E^{\vee}_* X$ is pro-free, then $K_* X = (E^{\vee}_* X)/\mathfrak{m}$.
Finally, let $f : X \to Y$ be a map between spectra whose K-homology is concentrated in even degrees. Then f is a K-equivalence iff it induces an isomorphism

$$E^{\vee}_0 X \to E^{\vee}_0 Y.$$

3.4 The Galois action of $\mathbb{G}$ on E

In [DH04] Devinatz and Hopkins develop a theory for taking homotopy fixed points of E with respect to any closed subgroup of $\mathbb{G}$ which agrees with the usual notion if the subgroup is finite. Note that, while they only develop this for the Honda formal group of height n over $\mathbb{F}_{p^n}$, their approach works whenever every automorphism of the formal group over the algebraic closure is already defined over the field one is working with. In particular, it works for every adapted formal group over $\mathbb{F}$. For a thorough review, see [Bar+22, § 2], and see [Mor23, § 2] for a discussion which also includes the non-Honda case. Let us collect what we need about these fixed point spectra in the following theorem.

Theorem 3.18 ([DH04, thm. 1],[Rog08, thm. 5.4.4, 5.4.9])**.**

1. *The unit map* $\mathbb{S} \to E^{h\mathbb{G}}$ *is a* $K(n)$*-localization.*
2. *For each pair of closed subgroups* $H \subset K \subset \mathbb{G}$ *with* H *normal and of finite index in* K *the map* $E^{hH} \to E^{hK}$ *is a faithful* $K(n)$*-local Galois extension for the group* K/H*.*
3. *For any closed subgroup* $H \subset \mathbb{G}$ *the map*

$$E_*^\vee E^{hH} \to C^0\left(\mathbb{G}/H, E_*\right), \, x \mapsto \left([g] \mapsto \langle x, \psi^g \circ \iota \rangle\right),$$

where $\iota : E^{hH} \to E$ *is the inclusion of the fixed points and* E_* *carries the* $\mathfrak{m}$*-adic topology, is well-defined and an isomorphism.*

4. *For each closed subgroup* $H \subset \mathbb{G}$ *the spectrum* E^{hH} *is* K*-local.*

Lemma 3.19. *Let* $A, B \in \mathbb{G}$ *be closed subgroups, and let* $H \subset A \cap B$ *be a closed subgroup such that the map*

$$\mathbb{G}/H \to \mathbb{G}/A \times \mathbb{G}/B,$$

given by the projections, is a bijection. Then the map

$$E^{hA} \otimes E^{hB} \xrightarrow{\iota \otimes \iota} E^{hH} \otimes E^{hH} \xrightarrow{\mu} E^{hH}$$

is a K*-equivalence.*

Proof. Applying K_* and the Künneth isomorphism we need to show that the map

$$C^0(\mathbb{G}/A, \mathbb{F}) \otimes_{\mathbb{F}} C^0(\mathbb{G}/B, \mathbb{F}) \simeq C^0(\mathbb{G}/A \times \mathbb{G}/B, \mathbb{F}) \to C^0(\mathbb{G}/H, \mathbb{F})$$

is an isomorphism, but this follows directly from our assumption. □

Lemma 3.20. *Let* X *be a spectrum with* K_*X *concentrated in even degrees. Then the map*

$$E_*^\vee(E \otimes X) \to C^0(\mathbb{G}, E_*^\vee X), \, x \mapsto \left(g \mapsto (\mu \otimes \mathrm{id}_X) \circ (\mathrm{id}_E \otimes \psi^g \otimes \mathrm{id}_X)(x)\right),$$

where $E_*^\vee X$ *carries the* $\mathfrak{m}$*-adic topology, is well-defined and an isomorphism.*

Proof. First note that, by the Künneth isomorphism, we have

$$K_*(E \otimes X) \simeq K_*(E) \otimes_{K_*} K_*(X)$$

so that also $E \otimes X$ has K-homology concentrated in even degrees. Theorem 3.17 then tells us that $E_*^\vee(E \otimes X)$ is concentrated in even degrees, too, and that $E_0^\vee(E \otimes X)$ is pro-free.
Our strategy will be as follows:

- Show that $C^0(\mathbb{G}, E_0^\vee X)$ is pro-free.
- Define a map on suitable choices of pro-bases which is an isomorphism by construction.
- Show that it agrees with the map in question.

Theorem 3.17 tells us that $E_0^\vee X$ is pro-free, say with pro-basis $\{m_i\}_{i \in I}$. So we have an isomorphism

$$\left(\bigoplus_{i \in I} E_0\right)_{\mathfrak{m}}^\wedge \xrightarrow{\sim} E_0^\vee X.$$

Combining Theorem 3.18 and Theorem 3.17 we also see that $C^0(\mathbb{G}, E_0)$ is pro-free, say with pro-basis $\{f_j\}_{j \in J}$. From Lemma C.7 we know that there are canonical equivalences

$$C^0(\mathbb{G}, E_0/\mathfrak{m}^n) \simeq C^0(\mathbb{G}, E_0)/\mathfrak{m}^n.$$

We now calculate

$$C^0(\mathbb{G}, E_0^\vee X) = C^0\left(\mathbb{G}, \lim_n \bigoplus_{i\in I} E_0/\mathfrak{m}^n\right) = \lim_n C^0\left(\mathbb{G}, \bigoplus_{i\in I} E_0/\mathfrak{m}^n\right) = \lim_n \bigoplus_{i\in I} C^0\left(\mathbb{G}, E_0/\mathfrak{m}^n\right)$$
$$= \lim_n \bigoplus_{i\in I} C^0(\mathbb{G}, E_0)/\mathfrak{m}^n = \lim_n \left(\bigoplus_{i\in I, j\in J} E_0\right)/\mathfrak{m}^n$$

where in the third step we used that $\mathbb{G}$ is a compact space. This shows that $C^0(\mathbb{G}, E_0^\vee X)$ is pro-free, and a pro-basis is given by the functions

$$\{g \mapsto f_j(g)m_i\}_{i\in I, j\in J}.$$

Now let $\{\alpha_{i,j}\}_{i\in I, j\in J}$ be the family of elements in $E_0^\vee(E \otimes X)$ defined by

$$\alpha_{i,j} := (\mu \otimes \mathrm{id}_E \otimes \mathrm{id}_X) \circ (\mathrm{id} \otimes \mathrm{swap} \otimes \mathrm{id})(f_j \otimes m_i).$$

We claim that this is a pro-basis. For this it to suffice to show that it maps to a basis of the $\mathbb{F}$-vector spaces $K_0(E \otimes X)$. Applying the Künneth isomorphism shows that $\alpha_{i,j}$ is mapped to $[f_j] \otimes [m_i] \in K_0 E \otimes_{\mathbb{F}} K_0 X$, which forms a basis by construction.
Let $h : E_0^\vee(E \otimes X) \to C^0(\mathbb{G}, E_0^\vee X)$ be the unique isomorphism such that

$$h(\alpha_{i,j}) = (g \mapsto f_j(g)m_i).$$

We need to show that, for every $x \in E_0^\vee(E \otimes X)$ we have

$$h(x)(g) = (\mu \otimes \mathrm{id}_X) \circ (\mathrm{id}_E \otimes \psi^g \otimes \mathrm{id}_X)(x).$$

Since $C^0(\mathbb{G}, E_0^\vee X) \to \mathit{Set}(\mathbb{G}, E_0^\vee X)$ is injective and the target is still L-complete, it suffices to check this on the pro-basis $\alpha_{i,j}$. This now follows from a quick diagram chase.

□

4 Discrete twisted $\mathbb{G}$-modules and K_0E-comodules

Fix an adapted formal group (G_0, z, u) over $\mathbb{F}$, let E be the associated Morava E-theory, and let $\mathbb{G}$ be the associated stabilizer group. Since E_* is Landweber exact, there is a natural way to equip (E_*, E_*E) with the structure of a flat Hopf algebroid, and to lift the functor $E_*(-)$ to take values in (left) E_*E-comodules, see [Rav86, prop. 2.2.8]. Very importantly, the ideal $\mathfrak{m} \subset E_*$ is an *invariant* ideal for this Hopf algebroid structure, so that $(E_*/\mathfrak{m}, E_*E/\mathfrak{m}E_*E)$ inherits the structure of a flat Hopf algebroid, and the functor $E_*(-)/\mathfrak{m}E_*(-)$ naturally takes values in left comodules over it.

Notation 4.1. We denote the full subcategory of spectra so that their K-homology is concentrated in even degrees by $\mathcal{C}$, and we denote the degree zero part of the above Hopf algebroid $(E_0/\mathfrak{m}, E_0E/\mathfrak{m}E_0E)$ by Σ.

Construction 4.2. Restricted to the category $\mathcal{C}$ the functor $K_0(-)$ naturally takes values in Σ-comodules. In fact, for $X \in \mathcal{C}$ the canonical map $E_0(X)/\mathfrak{m}E_0(X) \to K_0(X)$ is an equivalence, so it follows from the discussion above.

The canonical projection $\mathbb{G} \to \mathit{Gal}(\mathbb{F}, \mathbb{F}_p)$ defines an action of $\mathbb{G}$ on $\mathbb{F}$. Note that the canonical action of $\mathbb{G}$ on E_0 fixes the maximal ideal $\mathfrak{m}$, and on the quotient $E_0/\mathfrak{m} \simeq \mathbb{F}$ these actions agree.

Proposition 4.3. *The map from part 3 of Theorem 3.18 induces an equivalence*

$$\Sigma \simeq (\mathbb{F}, C^0(\mathbb{G}, \mathbb{F}))$$

where $\mathbb{F}$ is given the discrete topology and the Hopf algebroid structure is explained as follows:

1. *The left unit η_L maps $a \in \mathbb{F}$ to the constant function with value a.*
2. *The right unit η_R maps $a \in \mathbb{F}$ to the map sending $g \in \mathbb{G}$ to $g.a$.*
3. *The algebra structure is given by point wise multiplication.*
4. *The counit ϵ is given by evaluation in $1 \in \mathbb{G}$.*
5. *The conjugation c acts on $f \in C^0(\mathbb{G}, \mathbb{F})$ as $(cf)(g) = g.(f(g^{-1}))$.*
6. *The comultiplication Δ is given by composing*

$$C^0(\mathbb{G}, \mathbb{F}) \xrightarrow{pr_2^*} C^0(\mathbb{G} \times \mathbb{G}, \mathbb{F})$$

with the inverse of the equivalence

$$C^0(\mathbb{G}, \mathbb{F}) \otimes_{\mathbb{F}} C^0(\mathbb{G}, \mathbb{F}) \to C^0(\mathbb{G} \times \mathbb{G}, \mathbb{F}),\ \alpha \otimes \beta \mapsto \left((g,h) \mapsto \alpha(g)g.(\beta(g^{-1}h))\right).$$

For $X \in \mathcal{C}$ the coaction is explained by composing

$$K_0X \xrightarrow{const.} C^0(\mathbb{G}, K_0X)$$

(where K_0X is given the discrete topology) with the inverse of the equivalence

$$C_0(\mathbb{G}, \mathbb{F}) \otimes_{\mathbb{F}} K_0X \to C^0(\mathbb{G}, K_0X),\ f \otimes m \mapsto (g \mapsto f(g)g.m)$$

where $g.m = [(\psi^g \otimes \mathrm{id}_X)(m')]$ for any lift $m' \in E_0^\vee X$ of m.

Proof. Most of the statements are straightforward to show, so let us focus on the statement about the coaction (from which the statement about the comultiplication follows).
Let $m \in K_0X$ and choose a lift $m' \in E_0^\vee X$. To define the coaction, as in [Rav86, prop. 2.2.8], we need to first determine

$$(\mathrm{id}_E \otimes \eta_E \otimes \mathrm{id}_X)(m') \in E_0^\vee(E \otimes X).$$

Under the isomorphism from Lemma 3.20 this is mapped to the constant function with value m'. A quick diagram chase now confirms that

$$C_0(\mathbb{G}, \mathbb{F}) \otimes_{\mathbb{F}} K_0X \to C^0(\mathbb{G}, K_0X),\ f \otimes m \mapsto (g \mapsto f(g)g.m)$$

is indeed the map one gets when modding out $\mathfrak{m}$ from the canonical equivalence

$$(E_*E) \otimes_{E_*} E_*X \to E_*(E \otimes X),$$

and we are done. $\square$

Remark 4.4. When talking about Σ, we sometimes write formulas in terms of δ *functions*

$$\delta_g : \mathbb{G} \to \mathbb{F},\ h \mapsto 1 \text{ if } h = g,\ 0 \text{ else}.$$

In that case, these formulas should be interpreted in $Set(\mathbb{G}, \mathbb{F})$, and it will be implicit that in every step we actually define elements of $C^0(\mathbb{G}, \mathbb{F})$. For example, we might write the coaction as

$$K_0X \to \Sigma \otimes K_0X,\ x \mapsto \sum_{g \in \mathbb{G}} \delta_g \otimes g^{-1}.x.$$

Construction 4.5. Let Σ' be the Hopf algebra $(\mathbb{F}, C^0(\mathbb{S}, \mathbb{F}))$. The restriction along $\mathbb{S} \subset \mathbb{G}$ defines a map of Hopf algebroids $\Sigma \to \Sigma'$, so that K_0X is naturally a Σ'-comodule for $X \in \mathcal{C}$.

Remark 4.6. As a coalgebra Σ' is the *Iwasawa coalgebra* of $\mathbb{S}$ we describe in Appendix B. As $\mathbb{S} \simeq F_{1/n}\mathbb{S} \rtimes \mathbb{F}^\times$ with $F_{1/n}\mathbb{S}$ a pro-p group, this coalgebra is pointed and we have the Milnor-Moore argument of Appendix A available.
Also note that Σ' is exactly the *associated Hopf algebra* from [Rav86, A1.1.9].

Let us now introduce the Milnor-Moore argument we will use throughout. For a lenghtier discussion and proofs, see Appendix A.

Definition 4.7. Let $\mathbb{F}$ be a finite field and C be a Hopf algebra over $\mathbb{F}$ which, as a coalgebra, is pointed. Denote by G the subset of group like elements in C. Let M be an $\mathbb{F}[G]$-module in C-comodules. Consider the natural maps

$$q_n : F_n(M)/F_{n-1}(M) \to F_n(C)/F_{n-1}(C) \otimes P_1(M),\ [m] \mapsto \sum_{i,g} [c_i] \otimes g^{-1}.m_{i,g}$$

for $\psi(m) \equiv \sum_i c_i \otimes m_i \mod F_{n-1}(C) \otimes F_n(M)$ with $c_i \in F_n(C)$ and $m_i \in F_0(M)$. We call M *splittable* if these are surjective for all $n \geq 0$.

Theorem 4.8. *Let C be a Hopf algebra over $\mathbb{F}$ which, as a coalgebra, is pointed, and let $G \subset C$ be the set of group-like elements. Let M be an $\mathbb{F}[G]$-module in C-comodules which is splittable. Then, for any $\mathbb{F}$-linear retraction $r : M \to \mathbb{F}[G] \otimes P_1(M)$ of the natural map $\mathbb{F}[G] \otimes P_1(M) \simeq F_0(M) \subset M$, the map*

$$h_r : M \xrightarrow{\psi} C \otimes M \xrightarrow{\mathrm{id} \otimes r} C \otimes \mathbb{F}[G] \otimes P_1(M) \xrightarrow{\mathrm{id} \otimes \epsilon \otimes \mathrm{id}} C \otimes P_1(M)$$

is an isomorphism of C-comodules and exhibits M as a (co)free $\mathbb{F}[G]$-module in C-comodules.

Proposition 4.9. *Let M be an $\mathbb{F}[G]$-algebra in C-comodules, and assume there exists a map $s : C \to M$ of comodules with $s(1) = 1$. Then M is splittable.*

There is another perspective on Σ-comodules which is often useful:

Definition 4.10. A *twisted* $\mathbb{G}$*-module* over $\mathbb{F}$ is a $\mathbb{F}$-vector space M and an $\mathbb{F}_p$ linear action of $\mathbb{G}$ on M that is compatible with the action on $\mathbb{F}$, that is for all $g \in \mathbb{G}, a \in \mathbb{F}$, and $m \in M$ we have $g.(am) = (g.a)(g.m)$.
A morphism $\phi : M \to N$ of twisted $\mathbb{G}$-modules is an $\mathbb{F}$-linear map that intertwines the actions of $\mathbb{G}$.
Finally, we call a twisted $\mathbb{G}$-module M *discrete* if the stabilizer of every element of M is an open subgroup of $\mathbb{G}$. This is equivalent to the action map

$$\mathbb{G} \times M \to M$$

being continuous when we give M the discrete topology. We denote the category of twisted $\mathbb{G}$-modules by $\mathrm{Mod}_{\mathbb{G}}$ and the full subcategory of discrete twisted $\mathbb{G}$-modules by $\mathrm{Mod}_{\mathbb{G}}^d$.

The proof of Proposition B.8 is easily generalized to show that the category of Σ-comodules is equivalent to the category of discrete twisted $\mathbb{G}$-modules.

Proposition 4.11. *The category of discrete twisted $\mathbb{G}$-modules is equivalent to the category of pairs (M, τ) of discrete $\mathbb{F}[\![\mathbb{S}]\!]$-modules M equipped with an $\mathbb{F}_p$-linear automorphism τ such that:*

1. *$\tau^n = \mathrm{id}_M$.*
2. *For any $a \in \mathbb{F}$ and $m \in M$ we have $\tau(am) = a^p\tau(m)$.*
3. *For any $g \in \mathbb{S}$ and $m \in M$ we have that $\tau(g.m) = g^\sigma.\tau(m)$ where $\sigma \in Gal(\mathbb{F}, \mathbb{F}_p)$ is the Frobenius and $g^\sigma = \sigma g \sigma^{-1} \in \mathbb{S}$.*

Proof. Let M be a discrete twisted $\mathbb{G}$-module. Then restricting to $\mathbb{S}$ yields a discrete $\mathbb{F}[\![\mathbb{S}]\!]$-module, and we let $\tau(m) := \sigma.m$.
In the other direction, let M be a discrete $\mathbb{F}[\![\mathbb{S}]\!]$-module and τ as above. The canonical semidirect product decomposition $\mathbb{G} = \mathbb{S} \rtimes Gal(\mathbb{F}, \mathbb{F}_p)$ lets us define an action of $\mathbb{G}$ on M as follows: Write $g \in \mathbb{G}$ as $h\sigma^k$ for $h \in \mathbb{S}$ and define

$$g.m := h.\tau^k(m).$$

These constructions define an equivalence of categories and commute with the forgetful functors to $\mathbb{F}$-vector spaces. $\square$

When trying to proof that a given Σ-comodule M is cofree, we will employ the following strategy. Assume that M is a splittable $\mathbb{F}[\mathbb{F}^\times]$-algebra in Σ'-comodules. Then by Theorem A.10 any retraction $r : M \to \mathbb{F}[\mathbb{F}^\times] \otimes M^{\mathbb{S}}$ gives us an equivalence

$$h_r : M \xrightarrow{\psi} \Sigma' \otimes M \xrightarrow{\mathrm{id} \otimes r} \Sigma' \otimes \mathbb{F}[\mathbb{F}^\times] \otimes M^{\mathbb{S}} \xrightarrow{\mathrm{id} \otimes \epsilon \otimes \mathrm{id}} \Sigma' \otimes M^{\mathbb{S}}.$$

If we can choose r so that the action of σ 'looks' like that on a direct sum of Σs we are done. For that we need to understand how σ acts on Σ.

Construction 4.12. The map

$$\Sigma' \otimes_{\mathbb{F}_p} \mathbb{F} \to \Sigma, \; f \otimes x \mapsto \left(\gamma\sigma^k \mapsto f(\gamma)\sigma^k.x\right)$$

is an isomorphism of Σ'-comodules, and τ acts on the source as

$$\tau(f \otimes x) = (\sigma \circ f \circ Ad_{\sigma^{-1}}) \otimes x.$$

Moreover, the composite

$$K_0 E^{hGal(\mathbb{F},\mathbb{F}_p)} \to K_0 E = \Sigma \to \Sigma'$$

is an isomorphism of Σ-comodules, so that the 'natural' action of τ on Σ' is

$$\tau(f) = \sigma \circ f \circ Ad_{\sigma^{-1}}.$$

Part II

$MSpin^c$ and friends at height 1

In this Part we will recover the $K(1)$-local versions of the well-known 2-local splittings by Anderson-Brown-Peterson of the spectra $MSpin$ and $MSpin^c$, and by Pengelley of MSU, see [ABP67] and [Pen82]. We have three reasons to include this discussion:

- The proofs we provide are independent of the known 2-local results, and in that sense 'new'.
- The techniques here will be mostly the same as the ones we employ to study $MString$ at the height 2, but the technical details are much easier. Moreover, we can completely answer many questions which remain open for $MString$.
- Especially the cases of $MSpin$ and $MSpin^c$ showcase many phenomena which we will reencounter when discussing $MString$. We hope that the reader will then be well-equipped to understand that more technical part.

To be precise, for $MSpin^c$ we can fully recover the $K(1)$-local Anderson-Brown-Peterson splitting, while for $MSpin$ we can only show that it splits into KOs, but can't pinpoint the projections. Finally, for MSU we are only able to show that it splits into KUs after tensoring with a C_2-Galois extension of the $K(1)$-local sphere. We discuss what it would take to complete these results in Sections 9.3 and 10.2.

5 The choice of E at height 1

Consider the formal multiplicative group law $F_0(x,y) = x + y + xy$ over $\mathbb{F}_2$, and let G_0 be the underlying formal group with coordinate z. The [2]-series of F_0 is given by z^2, so that $(G_0, z, 1)$ is an adapted formal group of height 1.

From [Dav25, prop. 6.6] we know that a convenient choice for $E(\mathbb{F}_2, G_0)$ is given by $E = KU_2^\wedge$: Let $z = [L-1] : BU(1) \to KU \to E$. We have that $E_0 = \mathbb{Z}_2^\wedge = W(\mathbb{F}_2)$ and the formal group law F from the coordinate z on G_E^Q is $F(x,y) = x + y + xy$, which is obviously a lift of F_0 to $W(\mathbb{F}_2)$. Since the relevant Galois group is trivial, we find that $\mathbb{G} = \mathbb{S} = F_1\mathbb{S} \simeq \mathbb{Z}_2^\times$ and the action of $g \in \mathbb{Z}_2^\times$ coincides with the Adams operations on $E = KU_2^\wedge$, see [Dav25, prop. 6.21]. Furthermore, with the above choice of F, the determinant map equals the identity $\det_F = \mathrm{id} : \mathbb{Z}_2^\times \to \mathbb{Z}_2^\times$.

Let $u \in \pi_{-2}E$ be the unit associated to the coordinate z. That ψ^g coincides with the Adams operations gives us that

$$g.u = u/g \text{ and } g.z = (z+1)^g - 1.$$

We will be interested in two specific subgroups of $\mathbb{G}$: Let $H = \overline{\langle 5 \rangle} \subset \mathbb{Z}_2^\times$ be the closed subgroup generated by 5 (or -3), and let $C_2 = \{\pm 1\} \subset \mathbb{Z}_2^\times$. Then H is an open subgroup of $\mathbb{G}$, and we have that $\mathbb{G} \simeq H \times C_2$.

Since limits commute with limits, the homotopy fixed points $ER := E^{hC_2}$ coincide with $KO_2^\wedge$.

6 Orientations and characteristic classes for KU

There is a whole zoo of $\mathbb{E}_\infty$-orientations related to KU:

$$\begin{tikzcd}
 & MSpin \arrow[d, dashed, "Th(i_1)"] \arrow[dr, "\tau_O"] & \\
MSU \arrow[ur, "Th(re)"] \arrow[d, "Th(pr)"'] & MSpin^c \arrow[dr, "\tau_c"] & KO \arrow[d, "cx"] \\
MU \arrow[ur, dashed, "Th(a)"] \arrow[rr, "\tau_{\mathbb{C}}"] & & KU
\end{tikzcd}$$

For an overview, see [HK24]. The orientations τ_O and τ_c are known as the Atiyah-Bott-Shapiro orientations, while $\tau_{\mathbb{C}}$ is the usual complex orientation. We also have that both $\tau_{\mathbb{C}}$ and τ_c are equivariant with respect to the usual C_2-action by complex conjugation.

Let us now discuss the other maps appearing in the diagram. Per definition, $BSpin^c$ features in pullback squares of infinite loop spaces

$$\begin{tikzcd} BU(1) \arrow[r, "i_2"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & BSpin^c \arrow[d, "pr_1"] \\ * \arrow[r] & BSO \end{tikzcd} \qquad \begin{tikzcd} BSpin^c \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\lrcorner"] & BU(1) \arrow[d] \\ BSO \arrow[r, "w_2"] & B^2\mathbb{Z}/2 \end{tikzcd} \qquad \begin{tikzcd} BSpin \arrow[r, "i_1"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & BSpin^c \arrow[d, "pr_2"] \\ * \arrow[r] & BU(1) \end{tikzcd}$$

Moreover, these maps and the map $a : BU \to BSpin^c$ fit into commutative diagrams of infinite loop spaces

$$\begin{tikzcd} BU \arrow[r, "\det"] \arrow[d, "re"'] \arrow[dr, "a"] & BU(1) \\ BSO & BSpin^c \arrow[l, "pr_1"] \arrow[u, "pr_2"'] \end{tikzcd} \qquad \begin{tikzcd} BU(1) \arrow[r, "i_2"] \arrow[dr, "{[2]}"'] & BSpin^c \arrow[d, "pr_2"] \\ & BU(1) \end{tikzcd} \qquad \begin{tikzcd} BSpin \arrow[r, "i_1"] \arrow[dr, "pr"'] & BSpin^c \arrow[d, "pr_1"] \\ & BSO \end{tikzcd}$$

It is well known that the K-homologies of the spaces BU, BSU, $BU(1)$, BSO, and $BSpin$ are concentrated in even degrees, see [KLW04] for example. Working with these spaces is made a lot easier by the following lemmas:

Lemma 6.1. *The maps*

$$BSpin \xrightarrow{pr} BSO \text{ and } BSpin^c \xrightarrow{pr_1 \times pr_2} BSO \times BU(1)$$

induce equivalences in K-homology. In particular, K_*BSpin^c is concentrated in even degrees.

Proof. For the first map, this is exactly the statement of [Sna75, prop. 8.11].
Now regarding the second map: By construction it commutes with the projections to $BU(1)$, and on the fiber it induces the map $pr : BSpin \to BSO$. Since $BU(1)$ is connected and the map on the fiber is a K-equivalence, also the map of total spaces is a K-equivalence. □

We also have the following extremely useful theorem of Anderson, related to the Atiyah-Segal completion theorem:

Theorem 6.2 ([And64])**.** *Let G be a compact connected Lie group, and let RO, RU, and RSp denote the real, complex, and quaternionic representation rings and module of G. Then we have*

$$KU^0(BG) = RU^\wedge,\ KU^1(BG) = 0$$

and

$$KO^0(BG) = RO^\wedge,\ KO^1(BG) = 0,\ KO^2(BG) = \frac{RU^\wedge}{\mathrm{im}(RO^\wedge)},\ KO^3(BG) = \frac{RSp^\wedge}{\mathrm{im}(RU^\wedge)},$$

$$KO^4(BG) = RSp^\wedge,\ KO^5(BG) = 0,\ KO^6(BG) = \frac{RU^\wedge}{\mathrm{im}(RSp^\wedge)},\ KO^7(BG) = \frac{RO^\wedge}{\mathrm{im}(RU^\wedge)}$$

where the hat indicates completion at the augmentation ideal, and the maps are extension and restriction of scalars along $\mathbb{R} \to \mathbb{C}$ and $\mathbb{C} \to \mathbb{H}$.

Proposition 6.3. *There are classes $\pi_k \in KO^0BSO$ for $k \geq 0$, with $\pi_0 = 1$, such that:*
1. *$KO^0BSO = \mathbb{Z}[\![\pi_1, \pi_2, \ldots]\!]$.*
2. *Restricted to $BSO(2) = BU(1)$ we have that $\mathrm{cx}(\pi_t) = \sum_{k\geq 0} \mathrm{cx}(\pi_k)t^k = 1 + t\frac{z^2}{1+z}$.*
3. *$(\mu_{BSO})^*\pi_t = (\mu_{KO})_*\pi_t \otimes \pi_t$ (Cartan formula).*

Proof. The construction of the classes π_k with properties (2) and (3) is due to Anderson-Brown-Peterson, see [ABP66, § 4], while the first statement is proven in [Lau16, § 4]. $\square$

Lemma 6.4. *The complexification map*

$$KO^0BSO \xrightarrow{\mathrm{cx}} KU^0BSO$$

is an isomorphism.

Proof. Let us first consider $BSO(2n+1)$ for $n \geq 0$. Then by [BD85, p. VI.5.4.vi] the complexification map

$$RO(SO(2n+1)) \to RU(SO(2n+1))$$

from the real to the complex representation ring is surjective. Since it is always injective, it is an isomorphism.
The Atiyah-Segal completion theorem the lets us deduce that also

$$KO^0BSO(2n+1) \to KU^0BSO(2n+1)$$

is an isomorphism.
Finally, by taking the limit $n \to \infty$, we need to show that the $\lim^1$ terms vanish. For KU^0BSO this is clear, because $KU^{-1}BSO(2n+1) = 0$ for each n. Note that also the $\lim^1$ term for KU^1BSO vanishes, since the maps $RU(SO(2n+3)) \to RU(SO(2n+1))$ are surjective (which follows from [BD85, pp. VI.5.4.iii, iv]).
For KO^0BSO, assume we had a map $f : BSO_+ \to KO$ which is zero when restricted to each $BSO(2n+1)$. Then $\mathrm{cx} \circ f = 0$, so by Wood's theorem there is a $g : BSO_+ \to \Sigma KO$ such that $f = \eta g$. The composite $\mathrm{cx} \circ g = 0$, so that also g factors as ηh.
From Theorem 6.2 we know that, restricted to each $BSO(2n+1)$, h must lie in the image of the boundary map $\delta = \mathrm{re} \circ 1/u : KU^0BSO(2n+1) \to KO^2BSO(2n+1)$, say with preimage h'. Since h' is the complexification of some class $h'' \in KO^0BSO(2n+1)$ by our previous discussion, we see that

$$\mathrm{cx} \circ h = \mathrm{cx} \circ \delta \circ \mathrm{cx} \circ h'' = 0 = (\mathrm{id} + \psi^{-1}) \circ 1/u \circ \mathrm{cx} \circ h'' = 0$$

restricted to each $BSO(2n+1)$.
Since the $\lim^1$ terms for KU^*BSO vanish, we thus must already have that $\mathrm{cx} \circ h = 0$, thus f is divisible by $\eta^3 = 0$, and we are done. $\square$

Lemma 6.5. *For any set S the map*

$$\pi_0 \mathrm{map}\left(BSO_+, L_K \bigoplus_S ER\right) \to \pi_0 \mathrm{map}\left(BSO_+, L_K \bigoplus_S E\right)$$

is an isomorphism.

Proof. The case $S = \emptyset$ is trivial. Consider the case $S = \{*\}$. By Lemma 3.3 the homotopy fixed point spectral

$$H^s\left(C_2, \pi_t \mathrm{map}\left(BSO_+, E\right)\right) \implies \pi_{t-s} \mathrm{map}\left(BSO_+, ER\right)$$

is a 'free module' on permanent cycles in bidegree $(0,0)$ over the homotopy fixed point spectral sequence

$$H^s(C_2, \pi_t E) \implies \pi_{t-s} ER.$$

Since in the latter spectral sequence only the bidegree $(0,0)$ contributes to π_0, and all classes in that bidegree a permanent cycles, this is also the case for the former, and we are done.
For general S, consider first the situation with the sum replaced with a product over S. Then the homotopy fixed point spectral sequence is a product over S of the one above, and we get the right behaviour. By [HS99b, prop. A.13] the map

$$L_K \bigoplus_S E \to \prod_S E$$

induces a split injection of E_*-modules. We conclude that the same is true in every bidegree of every page for the comparison map of the homotopy fixed point spectral sequences. This allows us to transport the differentials from the case with the product to the case with the sum, which thus also has the right behaviour, and we conclude. $\square$

The results from Section 1.1 now tell us everything we want to know about characteristic classes in this context, and how the Morava stabilizer group acts on them.

6.1 Cannibalistic classes for KU

To understand the action of the Morava stabilizer group on $E_*^\vee MSpin^c$ and co we will need to understand how it acts on the orientations.

Definition 6.6. Let $g \in \mathbb{Z}_2^\times$. We define the *cannibalistic classes* by

$$\Theta^g_{\mathbb{C}} = g.\tau_{\mathbb{C}}/\tau_{\mathbb{C}} \in E^0BU,\ \Theta^g_c = g.\tau_c/\tau_c \in E^0BSpin^c,\ \text{and } \Theta^g_O = g.\tau_O/\tau_O \in ER^0BSpin.$$

Remark 6.7. The adjective 'cannibalistic' for these kinds of classes was introduced by Frank Adams, the reason being that they take vector bundles on X (maps $X \to BU$) and turn them into (p-adic) vector bundles (classes in E^0X).

These are canonically commutative algebra maps and we have equalities

$$a^*\Theta^u_c = \Theta^u_{\mathbb{C}} \text{ and } i_1^*\Theta^u_c = cx(\Theta^u_O).$$

Proposition 6.8. *Let $f : BU(1) \to BU$ be the map classifying $[L-1]$. The class $\Theta^g_{\mathbb{C}}$ is determined by the* 1*-structure*

$$f^*\Theta^g_{\mathbb{C}} = \frac{[g](z)}{gz}.$$

Proof. Per definition, we have a commutative diagram

$$\begin{tikzcd}
E & E \arrow[l, "\psi^g"'] & MU \arrow[l, "\tau_{\mathbb{C}}"'] \arrow[dl, "\tilde{\Delta}"] & Th(L-1) \arrow[l, "Th(f)"'] \arrow[d, "\tilde{\Delta}"] & BU(1)_+ \arrow[l, "s_0"'] \arrow[d, "\Delta"] \\
& MU \otimes BU_+ \arrow[ul, "\tau_c \times \Theta^g_{\mathbb{C}}"] & & Th(L-1) \otimes BU(1)_+ \arrow[ll, "Th(f)\otimes f"] & BU(1)_+ \otimes BU(1)_+ \arrow[l, "s_0 \otimes \mathrm{id}"]
\end{tikzcd}$$

This witnesses the equality

$$g.x_E = x_E f^*\Theta^g_{\mathbb{C}}$$

and the proposition follows. $\square$

Determining Θ^g_c is a bit more tricky. We start with some lemmas:

Lemma 6.9. *Let $g \in \mathbb{Z}_2^\times$. There are unique maps of commutative algebras*

$$\chi^g : BSO_+ \to E \text{ and } \gamma^g : BU(1)_+ \to E$$

such that $\Theta^g_c = pr_1^*\chi^g pr_2^*\gamma^g$.

Proof. From Lemma 6.1 we know that

$$L_KE \otimes BSpin^c_+ \xrightarrow{pr_1 \times pr_2} L_KE \otimes BSO_+ \otimes BU(1)_+$$

is an equivalence of K-local commutative E-algebras. Since the target of the above map is the coproduct of $L_KE \otimes BSO_+$ and $L_KE \otimes BU(1)_+$ and we have an adjunction

$$CAlg_E(L_KE \otimes (-), E) \simeq CAlg(-, E),$$

the statement follows. $\square$

Lemma 6.10. *We have $\chi^{-1} = 1$ and $\Theta_c^{-1} = pr_2^* \frac{1}{1+z} = pr_2^*\left(\overline{L}\right)$.*

Proof. Since C_2 acts trivially on ER we have that $\Theta_O^{-1} = 1$, so $pr^*\chi^{-1} = i_1^*\Theta_c^{-1} = 1$. Since pr^* is an equivalence, we have that $\chi^{-1} = 1$.
Then we have that $\Theta_{\mathbb{C}}^{-1} = a^*\Theta_c^{-1} = a^* pr_2^*\gamma^{-1}$. Let $f : BU(1) \to BU$ be that map that classifies $[L-1]$. The composite $pr_2 \circ a \circ f$ equals $\mathrm{id}_{BU(1)}$, so by Proposition 6.8 we find

$$\frac{[-1](z)}{-z} = f^*\Theta_{\mathbb{C}}^{-1} = \gamma^{-1},$$

and we are done. $\square$

Lemma 6.11. *Let V be the virtual vector bundle on $BU(1)$ classified by the map*

$$BU(1)_+ \xrightarrow{Th(i_2)} MSpin^c \xrightarrow{\tau_c} KU.$$

For every $g \in \mathbb{Z}_2^\times$ we have

$$\psi^g(V) = V \cdot [2]^*\gamma^g.$$

Proof. By definition, we have a commutative diagram

$$\begin{tikzcd}
E & E \arrow[l, "\psi^g"'] & MSpin^c \arrow[l, "\tau_c"'] \arrow[dl, "\tilde{\Delta}"] & BU(1)_+ \arrow[l, "Th(i_2)"'] \arrow[d, "\Delta"] \\
 & MSpin^c \otimes BSpin^c_+ \arrow[ul, "\tau_c \times \Theta_c^g"] & & BU(1)_+ \otimes BU(1)_+ \arrow[ll, "Th(i_2)\otimes i_2"]
\end{tikzcd}$$

This witnesses the equality

$$\psi^g(V) = V \cdot i_2^*\Theta_c^g = V \cdot [2]^*\gamma^g.$$

$\square$

Proposition 6.12. *The bundle V from the previous lemma equals $[L]$.*

Proof. Since V, by construction, is classified by a map of commutative algebras, there is some $n \in \mathbb{Z}$ such that $V = [L^{\otimes n}]$. Combining the two previous lemmas we get the equation

$$\left[L^{\otimes -n}\right] = \left[L^{\otimes n-2}\right]$$

which implies $n = 1$. $\square$

Remark 6.13. By a theorem of Snaith ([Sna81, thm. 2.12], see also [Lur18, § 6.5]) there is a certain class $\beta \in \pi_2\Sigma^\infty BU(1)_+$ so that the map $[L] : BU(1)_+ \to KU$ witnesses KU as the localization $\Sigma^\infty BU(1)_+[\beta^{-1}]$. Let $\alpha = Th(i_2)(\beta)$. Then we have that $MSpin^c[\alpha^{-1}]$ is an augmented commutative KU-algebra, isomorphic to $KU \otimes MSpin$. Note that, $K(1)$-locally, the canonical map $MSpin^c \to MSpin^c[\alpha^{-1}]$ is an equivalence, since multiplication with $u\alpha = Th(i_2)_*\beta_1$ induces the identity on $K_0 MSpin^c$. It would be interesting to study how this interacts with the results of [HK24], which tell us that the natural comparison maps between $MSpin$ and various forms of C_2-fixed points of $MSpin^c$ are not equivalences.

Since $[2]^*$ is injective we find:

Corollary 6.14. *For every $g \in \mathbb{Z}_2^\times$ we have*

$$\gamma^g = [L]^{(g-1)/2}.$$

7 A factorization Lemma

In this Section we want to prove that $K_0(E^{hH} \otimes MSpin^c)$ and friends are splittable K_0E-comodules, see Definition A.8. As we show in Proposition A.13, a sufficient condition to be splittable is to admit a unital map of K_0E-comodules from K_0E.

Proposition 7.1. *Let $f : C\eta \to BP$ be any extension of the unit $\mathbb{S} \to BP$. Then the map*

$$C\eta \otimes MSU \xrightarrow{f \otimes \tau_{BP}} BP \otimes BP \to BP \to ku$$

admits a $K(1)$-local unital section, where we view the projection $\mathbb{S} \to C\eta$ as the unit of $C\eta$.

Proof. By [HR95, thm. 2.1] The spectrum $C\eta \otimes MSU$ splits 2-locally as a direct sum of suspensions of BP. Inspecting the proof we see that one can choose

$$C\eta \otimes MSU \xrightarrow{f \otimes \tau_{BP}} BP \otimes BP \to BP$$

as the projection onto the bottom summand, and it thus admits a 2-local unital section.
Following [HS99a, thm. 4.3] and the remark following its proof, and using that the 2-typical complex orientation $BP \to ku$ exhibits ku as a form of $BP\langle 1\rangle$, we see that $BP \to ku$ admits a $K(1)$-local unital section.
Composing these sections proves the proposition. □

Recall that the action of $g \in \mathbb{Z}_2^\times$ on $E^*BU(1)$ is given by

$$g.u = u/g, \; g.z = (z+1)^g - 1.$$

Lemma 7.2. *Let f be the restriction of $uz \in \tilde{E}^2BU(1)$ to $\mathbb{C}P^2 \simeq \Sigma^2 C\eta$. The induced map $K_0C\eta \to K_0E$ factors uniquely through the inclusion of the fixed points $K_0E^{hH} \to K_0E$. The factorizing map yields a unital isomorphism $K_0C\eta \simeq K_0E^{hH}$ of K_0E-comodules.*

Proof. Since uz, uz^2 form a basis of $K^0C\eta$, their duals form a basis of $K_0C\eta$. The above calculation then shows that the image of this basis under K_0f is given by

$$g \mapsto 1 \text{ and } g \mapsto \frac{g-1}{2} \in C^0(\mathbb{Z}_2^\times, \mathbb{F}_2) = K_0E.$$

Since both of these are invariant under multiplication by 5 from the right, the first part follows. The factorizing map is then automatically a unital morphism of K_0E-comodules, it is injective by construction, and both the source and the target are 2-dimensional $\mathbb{F}_2$-vector spaces, so it must be an isomorphism. □

Remark 7.3. This is a purely algebraic statement, there is no unital equivalence $L_KC\eta \simeq E^{hH}$.

Corollary 7.4. *$K_0\left(E^{hH} \otimes MSU\right)$, $K_0\left(E^{hH} \otimes MSpin\right)$, and $K_0\left(E^{hH} \otimes MSpin^c\right)$ are splittable K_0E-comodules.*

Proof. By Proposition A.13 we need to produce unital maps

$$K_0E \to K_0\left(E^{hH} \otimes MSU\right), K_0E \to K_0\left(E^{hH} \otimes MSpin\right), K_0E \to KK_0\left(E^{hH} \otimes MSUpin^c\right)$$

of K_0E-comodules. From Proposition 7.1 we know that there exists a unital map

$$E \to L_K\left(C\eta \otimes MSU\right).$$

Applying K_0 to this and composing with the factorization from Lemma 7.2 gives the desired map. Composing this with the ring maps $MSU \to MSpin$ and $MSU \to MSpin^c$ also gives the desired maps for these cases. □

8 Lifting an algebraic to a spectral splitting

In the previous Section we have seen that all of

$$K_0E^{hH}\otimes MSU,\ K_0E^{hH}\otimes MSpin,\ \text{and } K_0E^{hH}\otimes MSpin^c$$

are splittable K_0E-comodules.

Remark 8.1. We'll focus on the case of MSU here, but note that analogous constructions work for $MSpin$ and $MSpin^c$ as well.

We now apply Theorem A.10. Thus, for any $\mathbb{F}_2$-linear retraction

$$r : K_0E^{hH}\otimes MSU \to P_1\left(K_0E^{hH}\otimes MSU\right),$$

the map

$$h_r : K_0E^{hH}\otimes MSU \xrightarrow{(\mathrm{id}\otimes r)\circ\psi} K_0E\otimes_{\mathbb{F}_2} P_1\left(K_0E^{hH}\otimes MSU\right)$$

is an isomorphism of K_0E-comodules.

By Lemma C.5 we can lift this to a K-equivalence of spectra

$$E^{hH}\otimes MSU \xrightarrow{f_r} L_K\bigoplus_{i\in I} E$$

such that $K_0f_r = h_r$, where I is an indexing set of an $\mathbb{F}_2$-basis of $P_1\left(K_0E^{hH}\otimes MSU\right)$.

Now assume that the map h_r is K_0E^{hH}-linear. Restricting the map f_r to MSU and then freely E^{hH}-linearizing it again then gives another lift of h_r. So we may assume that f_r is E^{hH}-linear.

Construction 8.2. From Lemma B.9 we know that

$$P_1(K_0E^{hH}\otimes MSU) = \left(K_0E^{hH}\otimes MSU\right)^{\mathbb{Z}_2^\times} \simeq C^0(\mathbb{Z}_2^\times/H, K_0MSU)^{\mathbb{Z}_2^\times} \simeq (K_0MSU)^H$$

where the composite evaluates the tensor factor in $K_0E^{hH}\simeq C^0(\mathbb{Z}_2^\times/H\mathbb{F}_2)$ in [1]. Let $\tilde r : K_0MSU \to (K_0MSU)^H$ be a retraction, and set

$$r : K_0E^{hH}\otimes MSU \to (K_0MSU)^H,\ f\otimes m \mapsto f([1])\tilde r(m).$$

For this choice of r the map h_r is K_0E^{hH}-linear. The same construction also works for $MSpin$ and $MSpin^c$ instead of MSU.

Corollary 8.3. *There are maps*

$$A : MSU \to L_K\bigoplus_{i\in I}E,\ B : MSpin \to L_K\bigoplus_{j\in J}E,\ \text{and } C : MSpin^c \to L_K\bigoplus_{j\in J'}E$$

such that their E^{hH}-linearizations are K-equivalences.

Remark 8.4. Let $\{m_i\}_{i\in I}$ be a basis of $(K_0MSU)^H$, let λ_i be the dual projections, and denote the projection mod $\mathfrak{m}$ of $A_i = pr_i\circ A \in E^0MSU$ by $a_i\in K^0MSU$. Lemma C.5 and the remark following it show that, for every $m\in K_0MSU$, we have

$$\langle m, a_i\rangle = \epsilon\circ(\mathrm{id}\otimes\lambda_i)\circ h_r(m) = \lambda_i(\tilde r(m)),$$

which determines a_i uniquely by duality.
Of course, this works just as well for $MSpin$ and $MSpin^c$.

9 Removing the Galois extension

In this Section we want to show that all of MSU, $MSpin$, and $MSpin^c$ split K-locally as a direct sum of fixed points of E with respect to finite subgroups of $\mathbb{G}$. There are exactly two such subgroups, namely $\{e\}$ and C_2, with fixed points E and ER. A necessary condition is the following:

Proposition 9.1. *Let X be a spectrum that admits a K-local equivalence to*

$$\bigoplus_{IO} ER \oplus \bigoplus_{IU} E$$

for some index sets IO and IU. Then $(K_0X)^H$ is a direct sum of a trivial and a free $\mathbb{F}_2[C_2]$ module, and the inclusion into K_0X admits a C_2-equivariant retraction.

Proof. It suffices to prove this for $X = E$ and $X = ER$. For these the relevant inclusions are, using Theorem 3.18,

$$C^0(H\backslash\mathbb{G}, \mathbb{F}_2) \subset C^0(\mathbb{G}, \mathbb{F}_2) \text{ and } C^0(H\backslash\mathbb{G}/C_2) \subset C^0(\mathbb{G}/C_2, \mathbb{F}_2),$$

and both allow equivariant retractions since $\mathbb{G} = H \times C_2$. □

While this is, of course, rather straightforward, it gives a good hint about what needs to be done to prove such splittings. If we already know that $E^{hH} \otimes X$ splits E^{hH}-linearly into a direct sum of Es, and we are able to prove the above necessary condition, the remaining problem is to lift the projections corresponding to the trivial summand through $ER \to E^{hH} \otimes ER \simeq E$ and those corresponding to the free summand through $E \to E^{hH} \otimes E \simeq E \oplus E$.

9.1 $MSpin$

Let us first focus on the case of $MSpin$, which is the most straight forward.

Theorem 9.2. *There is a K-local equivalence*

$$B' : MSpin \to L_K \bigoplus_{j \in J} ER$$

Proof. Consider the map

$$B : MSpin \to L_K \bigoplus_{j \in J} KU$$

from Corollary 8.3. By the Thom isomorphism this is the product of the Atiyah-Bott-Shapiro orientation $cx \circ \tau_O$ with some class

$$b \in \pi_0 \mathrm{map}\left(BSpin_+, L_K \bigoplus_{j \in J} E\right).$$

By Lemma 6.4 we have that $b = b' \circ pr$ for a unique

$$b' \in \pi_0 \mathrm{map}\left(BSO_+, L_K \bigoplus_{j \in J} E\right).$$

By Lemma 6.5 we have $b' = cx \circ b''$ for a unique

$$b'' \in \pi_0 \mathrm{map}\left(BSO_+, L_K \bigoplus_{j \in J} ER\right).$$

Since Atiyah-Bott-Shapiro orientation factors over ER this shows that the map B factors as

$$MSpin \xrightarrow{B'} L_K \bigoplus_{j \in J} ER \to L_K \bigoplus_{j \in J} E.$$

The map

$$\mathrm{id} \otimes B' : E^{hH} \otimes MSpin \to L_K \bigoplus_{j \in J} E^{hH} \otimes KO$$

agrees with the E^{hH}-linearization of B and is thus a K-equivalence. Since E^{hH} faithful in the K-local category by [Rog08, prop. 5.4.9(b)], also B' is a K-equivalence. □

9.2 $MSpin^c$

Note that $MSpin^c$ is complex orientable, so we could immediately conclude that $K_0 MSpin^c$ is splittable and call it a day. Instead, we will proceed in a way that demonstrates a technique we will see again when studying $MString$ at the height 2.

Consider the fiber sequence of infinite loop spaces

$$BSpin \xrightarrow{i_1} BSpin^c \xrightarrow{pr_2} BU(1).$$

We have seen (Lemma 6.1) that this gives us an equivalence

$$K^* BSpin^c \simeq K^* BSO[\![r-1]\!]$$

where $r = pr_2^*(z+1)$.

Proposition 9.3. *The map*

$$K_0 BSpin^c \to C^0(\mathbb{Z}_2^\wedge, K_0 BSO),\ m \mapsto \big(k \mapsto (pr_1)_*(m \frown r^k)\big)$$

is a well-defined isomorphism of $\mathbb{F}_2$-vector spaces. Moreover, it commutes with the action of $\mathbb{Z}_2^\times$ where we give the target the standard mapping space action and act on $\mathbb{Z}_2^\wedge$ by multiplication.

Proof. The proof is a lot like the one of Lemma 3.20. Let $\{\alpha_i\}_{i\in I}$ be a basis of $K_0 BSO$. By a result of Mahler, see [CC16, thm. 11], the target has a basis of the form

$$\phi_j \alpha_i : k \mapsto \binom{k}{j} \alpha_i$$

for $i \in I$ and $j \geq 0$. By the previous discussion the source has a basis of the form $\alpha_i \otimes \beta_j$ with $\beta_j \in K_0 BU(1)$ the duals of z^j. Sending $\alpha_i \otimes \beta_j$ to $\phi_j \alpha_i$ clearly gives an isomorphism of $\mathbb{F}_2$-vector spaces, and agrees with the map described in the proposition.
For the 'moreover' part, let $g \in \mathbb{Z}_2^\times$. We find that

$$(pr_1)_*((g.m) \frown) r^k = g.(pr_1)_*(m \frown (g^{-1}.r)^k).$$

Since $r = pr_2^*(z+1)$ we have that $g^{-1}.r = r^{1/g}$ so that

$$(pr_1)_*((g.m) \frown r^k) = g.(pr_1)_*(m \frown r^{k/g}),$$

and we are done. $\square$

Transitioning to the Thom spectrum $MSpin^c$ has a very interesting effect: it changes the domain to $\mathbb{Z}_2^\times$ and twists the action on $K_0 BSO$!

Construction 9.4. Let $N_\chi = K_0 BSO$ and define an action of $g \in \mathbb{Z}_2^\times$ on $n \in N_\chi$ by

$$g \triangleright n := (g.n) \frown \left(g.\chi^{g^{-1}}\right)$$

where . indicates the usual action. This is indeed an action:

$$\begin{aligned} a \triangleright (b \triangleright n) &= (a.b.n) \frown \left(a.b.\chi^{b^{-1}}\right) \frown \left(a.\chi^{a^{-1}}\right) = ((ab).n) \frown \left((ab).\left(\chi^{b^{-1}} b^{-1}.\chi^{a^{-1}}\right)\right) \\ &= ((ab).n) \frown \left((ab).\chi^{(ab)^{-1}}\right) \end{aligned}$$

where in the last step we used the formula

$$\Theta^{gh} = \Theta^g g.\Theta^h$$

valid for all kinds of cannibalistic classes.

Proposition 9.5. *Let $t_*: K_0MSpin^c \to K_0BSpin^c$ be the Thom isomorphism associated to τ_c and N_χ the $\mathbb{Z}_2^\times$-module from above. Then the map*

$$K_0MSpin^c \to C^0(\mathbb{Z}_2^\times, N_\chi),\, m \mapsto \left(a \mapsto (pr_1)_*\left(t_*(m) \frown r^{(a-1)/2}\right)\right)$$

is an isomorphism and commutes with the action of $\mathbb{Z}_2^\times$ when we give the target the usual mapping space action.

Proof. The isomorphism part directly follows from the Proposition above and that $a \mapsto (a-1)/2$ is a homeomorphism $\mathbb{Z}_2^\times \to \mathbb{Z}_2^\wedge$.
To show that it commutes with the action, let $a \in \mathbb{Z}_2^\times, m \in K_0MSpin^c$, and $T \in K^0BSO$. We calculate:

$$\begin{aligned}\langle(pr_1)_*\left(t_*(g.m) \frown r^{(a-1)/2}\right), T\rangle &= \langle t_*(g.m), r^{(a-1)/2}pr_1^*T\rangle = \langle g.m, \tau_c r^{(a-1)/2}pr_1^*T\rangle \\ &= \langle m, \tau_c pr_1^*\chi^{g^{-1}}r^{(a-1)/(2g)+(1/g-1)/2}pr_1^*(g^{-1}.T)\rangle \\ &= \langle g.\left((pr_1)_*\left(t_*(m) \frown r^{(a/g-1)/2}\right)\right) \frown g.\chi^{g^{-1}}, T\rangle \\ &= \langle g \triangleright \left((pr_1)_*\left(t_*(m) \frown r^{(a/g-1)/2}\right)\right), T\rangle\end{aligned}$$

Here we used the duality between the homological and cohomological Thom isomorphism, Lemma 6.9, and Corollary 6.14. Since this holds for any $T \in K^0BSO$ we conclude by duality. ☐

We will use the previous two propositions to freely view elements of K_0BSpin^c and K_0MSpin^c as functions with values in K_0BSO.

Lemma 9.6. *The map*

$$\tilde{r}: K_0MSpin^c \to K_0BSO \oplus K_0BSO,\, m \mapsto (m(1), m(-1))$$

is a retraction of the inclusion

$$(K_0MSpin^c)^H \subset K_0MSpin^c.$$

Moreover, it is equivariant for the action of $C_2 \subset \mathbb{Z}_2^\times$ if we let -1 act on the target by

$$-1.(a,b) = (-1.b, -1.a).$$

Proof. This follows from the fact that H acts freely on $\mathbb{Z}_2^\times$ with two orbits, generated by 1 and -1, and the fact that $\chi^{-1} = 1$. ☐

Pick a basis $\{n_j\}_{j\in J}$ of K_0BSO with dual projections λ_j. Then a basis of $K_0BSO \oplus K_0BSO$ is given by

$$\{s.(n_j, 0)\}_{(s,j)\in C_2\times J}.$$

Let $\{\lambda_{s,j}\}_{(s,j)\in C_2\times J}$ denote the dual projections. Applying Construction 8.2 to this we obtain a map

$$C: MSpin^c \to L_K \bigoplus_{C_2\times J} E.$$

Denote the image of $pr_{\pm j} \circ C \in E^0MSpin^c$ in K^0Mspin^c by $c_{\pm j}$. From Remark 8.4 we know that, for every $m \in K_0MSpin^c$ we have

$$\langle m, c_{\pm j}\rangle = \lambda_{\pm j}(\tilde{r}(m)) = \lambda_{\pm j}(m(1), m(-1)) = \lambda_j((pr_1)_*(t_*(\pm 1.m))).$$

Let $q_j \in K^0BSO$ be the unique class such that $\langle -, q_j\rangle = \lambda_j(-)$. Then $c_{\pm j}$ is given by

$$c_{\pm j} = \pm 1.(\tau_c pr_1^* q_j).$$

Theorem 9.7. *Let C^{ev} be the composite*

$$MSpin^c \xrightarrow{C} L_K \bigoplus_{C_2\times J} E \simeq L_K \bigoplus_{+J} E \oplus L_K \bigoplus_{-J} E \to L_K \bigoplus_{+J} E$$

where the last map projects the '$-J$'-components to zero. Then C^{ev} is a K-equivalence.

Proof. By [Rog08, prop. 5.4.9(b)] E^{hH} is faithful in the K-local category, so it suffices to show that

$$\mathrm{id}\otimes C^{ev} : L_K\left(E^{hH}\otimes MSpin^c\right) \to L_K\left(E^{hH}\otimes \bigoplus_{+J} E\right) \simeq L_K \bigoplus_{+J} E^{hH}\otimes E$$

is an equivalence.
The map

$$E^{hH}\otimes E \xrightarrow{\mu\circ(\iota\otimes\mathrm{id})\oplus\mu\circ(\iota\otimes\psi^{-1})} E\oplus E$$

is an E^{hH}-linear K-equivalence. Composing with this and restricting to $MSpin^c$ we need to show that the free E^{hH}-linearization of

$$MSpin^c \xrightarrow{C^{ev}\oplus -1.C^{ev}} L_K \bigoplus_{+J}(E\oplus E)$$

is a K-equivalence. But, by construction, this map agrees with C mod $\mathfrak{m}$ (and up to reindexing the direct sum), so we are done. □

9.3 MSU

As the cases of $MSpin$ and $MSpin^c$ have demonstrated, for a spectrum X with K-homology concentrated in even degrees, a K-local summand ER in X induces a summand of the trivial C_2-representation $\mathbb{F}_2$ in $(K_0X)^H$, while a K-local summand E in X induces a summand of the permutation module $\mathbb{F}_2[C_2]$ in $(K_0X)^H$.

So a first step towards a splitting would be to show that the C_2-representation $(K_0MSU)^H$ is a direct sum of trivial and permutation modules. If we could then pick a C_2-equivariant retraction

$$\tilde{r} : K_0MSU \to (K_0MSU)^H$$

the problem would then reduce to lifting the projections

$$MSU \to E$$

corresponding to the C_2-trivial summand in $(K_0MSU)^H$ to ER.

Though we have rather complete calculations, and the lifting problem mentioned above is mostly a representation theoretic one, the combinatorics seem too hard. But combining Pengelley's 2-local splitting of MSU, a result of Wilson on the loop spaces of BoP, and the Bousfield-Kuhn functor, it follows that it should be possible:

Proposition 9.8. *There is a K-local equivalence*

$$MSU \simeq \bigoplus_{IO} KO \oplus \bigoplus_{IU} KU$$

for some indexing sets IO and IU.

Proof. From [Pen82] we know that, 2-locally, MSU is a direct sum of suspensions of BP and a spectrum he called BoP. By [HS99a, thm. 4.3] the summands BP turn into direct sums of KU upon K-localization, so that is fine.
Focusing on the summands BoP, Wilson proved in [Wil19, p. 2.9] that the underlying space of

$\Omega^\infty BoP$ is given by

$$\Omega^\infty BoP \simeq \Omega^\infty bo \times \prod_{2^{k-2}>u\geq 0} \Omega^\infty \Sigma^{2^{k+1}+8u-2} BP\langle k\rangle.$$

Since there are, for every n, only finitely many factors which are not n-connective, the right-hand side of this is the infinite loop space of the spectrum

$$bo \oplus \bigoplus_{2^{k-2}>u\geq 0} \Sigma^{2^{k+1}+8u-2} BP\langle k\rangle.$$

Since K-localization factors as $\Phi \circ \Omega^\infty$, where Φ is the Bousfield-Kuhn functor [Kuh89, thm. 1.1], we find that there is an equivalence

$$L_K BoP \simeq L_K \left(bo \oplus \bigoplus_{2^{k-2}>u\geq 0} \Sigma^{2^{k+1}+8u-2} L_K BP\langle k\rangle \right).$$

Since L_K is insensitive to connective covers, we have $L_K bo = L_K KO = ER$. The other summands are even suspensions of $BP\langle k\rangle$ for $k \geq 2$. Since each $BP\langle k\rangle$ admits a unital map from BP, and $K_0 BP$ admits a unital map from $K_0 E$ of $K_0 E$-comodules, $K_0 BP\langle k\rangle$ is a splittable $K_0 E$-comodule. We conclude that there is an equivalence

$$L_K BoP \simeq ER \oplus L_K \bigoplus_J E$$

for some indexing set J. As only eightfold suspensions of BoP appear in Pengelley's splitting, $\Sigma^8 ER \simeq ER$, and $K_* MSU$ is concentrated in even degrees, we conclude. $\square$

10 Determining the projections

To close this part, we want to show that the projections appearing in Theorem 9.7 can be chosen to agree with those appearing in the Anderson-Brown-Peterson splitting.

For the case of $MSpin$ we indicate what would need to be done to determine the projections, but the combinatorics turn out to be quite complicated.

10.1 $MSpin^c$

We have seen that $MSpin^c$ is split K-locally by any set of classes $\{C_j \in E^0 BSO\}_{j\in J}$ such that

- they fulfill the finiteness condition of Lemma C.4, and
- their reductions mod $\mathfrak{m}$ form a 'dual basis' of $K_0 BSO$.

From Proposition 6.3 it is clear that the natural candidate for such a collection is given by $\{\pi^\lambda\}_{\lambda\in P}$ where P is the set of all partitions, and

$$\pi^\lambda := \prod_{k=1}^{l(\lambda)} \pi_k^{\lambda_k}.$$

This collection, basically per definition, fulfills the second of the two conditions above. Let us now prove that it also fulfills the first.

Lemma 10.1. *The collection*

$$\{\pi^\lambda\}_{\lambda \in P} \in \prod_P E^0 BSO$$

admits a unique lift to

$$\pi_0 \mathrm{map}\left(BSO_+, L_K \bigoplus_P E\right).$$

Proof. Existence and uniqueness follow directly from Lemma C.4 if we can show:

$$\forall x \in E_0^\vee BSO, \forall n \geq 0 : \langle x, \pi^\lambda \rangle \in \mathfrak{m}^n \text{ for all but finitely many } \lambda \in P.$$

As $E_0^\vee BSO$ is pro-free, it suffices to show this on a pro-basis. By Proposition 1.12 and Theorem 1.10 a pro-basis is given by the monomials

$$\mathrm{re}_* \left(b_{2k_1} \cdots b_{2k_n}\right).$$

A calculation using the Cartan formula from Proposition 6.3 shows that pairing this with π^λ is zero if $l(\lambda) > n\max(k_1, \ldots, k_n)$ or λ contains a number larger than n. This leaves only finitely many non-zero options, and we are done. □

Remark 10.2. Of course, one could also use, as Anderson-Brown-Peterson show, that the π^λ lift to suitable connective covers of KU, so that the direct sum of these connective covers agrees with the product.

Corollary 10.3. *Let P be the set of partitions. Then there is a K-equivalence*

$$C : MSpin^c \to L_K \bigoplus_P KU$$

such that $pr_\lambda \circ C = \tau_c pr_1^* \pi^\lambda$.

10.2 *MSpin*

This case is a bit more complicated. From the Anderson-Brown-Peterson result we expect the projections to be given by π^λ for λ not containing 1. From the above we know that this collection of elements in $E^0 BSO$ fulfills the finiteness condition of Lemma C.4, too.

This leaves us with the task of showing that the collection $\{\pi^\lambda\}_{1 \notin \lambda}$ forms a 'dual basis' for N_χ^H. I believe that, in principle, this could be done by some kind of induction argument on the length of the partitions involved, since we have quite some control over the action of H on N_χ. A similar approach, which connects back to Remark 6.13, is the following. There is a K-equivalence $KU \otimes MSpin \simeq MSpin^c$. To show that the collection $\{\pi^\lambda\}_{1 \notin \lambda}$ implements a splitting it would suffice to show that there exists a KU-linear equivalence making the diagram

$$\begin{tikzcd} KU \otimes MSpin \arrow[r] \arrow[d, "\simeq"'] & \bigoplus_{1 \notin \lambda} KU \otimes KO \arrow[d, dashed] \\ MSpin^c \arrow[r, "\simeq"'] & \bigoplus_\lambda KU \end{tikzcd}$$

commute. In particular, one would need to show that every class π^μ can be written as a linear combination of classes of the form $\chi^u u.\pi^\lambda$ with $1 \notin \lambda$ and $u \in H$. This is, basically, the same problem as showing that the collection $\{\pi^\lambda\}_{1 \notin \lambda}$ forms a 'dual basis'.

Part III

$MU\langle 6\rangle$ and $MString$ at height 2

We now turn to the main act. As already discussed in the introduction, we will see how to split finite Galois extensions of $L_{K(2)}MString$ into sums of fixed points of Morava E-theories. The general strategy will be the same as in Part II, but the details are more involved.

11 The choice of E at height 2

We adopt the notation of [Bea17] for most things concerning E, see Sections 2 and 3 therein, but we drop the subscript C from the Morava stabilizer groups.

The Morava E-theory we consider is an elliptic spectrum E with coefficient ring

$$E_* = W(\mathbb{F}_4)[\![u_1]\!][u^{\pm}],\ |u_1| = 0,\ |u| = -2$$

and Weierstrass elliptic curve $C_E : y^2 + 3u_1xy + (u_1^3 - 1)y = x^3$. In the coordinate $\widetilde{z_E} = -x/y$ the Hazewinkel generators are given by

$$v_1 = 3u_1/\tilde{u} \text{ and } v_2 = (u_1^3 - 1)/\tilde{u}^3.$$

It carries the universal deformation of the formal group associated to the supersingular curve

$$C_0 : y^2 + y = x^3$$

over $\mathbb{F}_4$, so by the Serre-Tate Theorem C_E itself is the universal deformation of C_0. Importantly, in the coordinate $z = x/y$ on C_0 the -2-series is given by $[-2](z) = z^4$, so that $(\widehat{C_0}, z, -1)$ is an adapted formal group.

The above choice of universal deformation of C_0 is due to Strickland. A more straightforward choice might be

$$C_{E'} : y^2 + axy + y = x^3$$

over the ring $E_0' = W(\mathbb{F}_4)[\![a]\!]$. As in [Bea17, rmk. 2.1.2] there is an isomorphism of pairs

$$(E_0, C_E) \simeq (E_0', C_{E'})$$

over $(\mathbb{F}_4, C_0)$ given by the $W(\mathbb{F}_4)$-linear map

$$\phi : E_0' \to E_0,\ a \mapsto -3u_1(1 - u_1^3)^{-1/3}$$

and the map of Weierstrass curves

$$q : C_E \to \phi^* C_{E'},\ q^*(x, y) = \Big((u_1^3 - 1)^{-2/3}x, (u_1^3 - 1)^{-1}y\Big).$$

This induces an equivalence of $\mathbb{E}_\infty$-rings $E' \to E$. From [Zhu20, ex. 7.20] we know that the unique norm coherent lift of the coordinate x/y on C_0 to $C_{E'}$ is $z_{E'} = -x/y$. Thus, the unique norm coherent lift of x/y to C_E is

$$z_E = q^*(-x/y) = (u_1^3 - 1)^{1/3}\widetilde{z_E}.$$

Since these differ by a linear factor, we have that $uz_E = \tilde{u}\widetilde{z_E}$, and thus the associated Hazewinkel generators agree.

11.1 The stabilizer group and generators of certain subgroups

Fix a generator ζ of $\mathbb{F}_4$ over $\mathbb{F}_2$, and denote a Teichmüller lift to the Witt vectors by the same letter. The endomorphism ring of the formal group law F_{C_0} associated to C_0 in the coordinate $t = -x/y$ is given by

$$\mathrm{End}(F_{C_0}) \simeq W(\mathbb{F}_4)\langle T\rangle/(T^2 + 2, \zeta T - T\zeta^2)$$

where $\zeta(t) = \zeta t$ and $T(t) = t^2$. Then every endomorphism can be written uniquely as a power series

$$\gamma = \sum_{n \geq 0} a_n T^n$$

with $a_n \in \{0, 1, \zeta, \zeta^2\}$.

The restricted Morava stabilizer group is given by the units $\mathbb{S} = \mathrm{End}(F_C)^\times$ and the full Morava stabilizer group is its extension by the Galois group

$$\mathbb{G} = \mathrm{End}(F_C)^\times \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2).$$

Remark 11.1. When viewing $\zeta \in W(\mathbb{F}_4)^\times \subset \mathbb{G}$ as an element of the Morava stabilizer group we will often denote it by ω to make the distinction between it and the element $\zeta \in E_0 = W(\mathbb{F}_4)[\![u_1]\!]$ clearer.

Denote by σ the generator of the Galois group, and for $\gamma \in \mathbb{S}$ let $\overline{\gamma} = \sigma\gamma\sigma$. There are two more or less natural structures on $\mathbb{S}$ which will play an important role later on.

Definition 11.2. Let $\gamma = x + yT$ be an endomorphism, with $x, y \in W(\mathbb{F}_4)$. The *determinant* of γ is defined to be

$$|\gamma| = x\overline{x} + 2y\overline{y}.$$

This extends to a morphism of groups $\mathbb{G} \to \mathbb{Z}_2^\times$ via $|\sigma| = 1$.

Definition 11.3. Let $F_{n/2}\mathbb{S} \subset \mathbb{S}$ be the normal subgroup of automorphisms of the form $1 + xT^n$. We will often shorten the notation to just $F_{n/2}$.

Consider the elements

$$\pi = 1 + 2\zeta \text{ and } \alpha = (1 - 2\zeta)/\sqrt{-7}$$

with $\sqrt{-7} \equiv 1 \bmod (4)$. Their determinants are 3 and -1, showing that the determinant is a surjective map $\mathbb{G} \to \mathbb{Z}_2^\times$. We have that π^2, α^2, and $\pi\alpha$ all lie in $F_{3/2}$. Let G_{24} be the group of automorphisms of the elliptic curve C_0 over $\mathbb{F}_4$ preserving the base point, which maps injectively into the automorphisms of the associated formal group law, and let G_{48} be the extension of G_{24} by the Galois group. As C_0 is already defined over $\mathbb{F}_2$, the group of automorphisms of the pair $(\mathbb{F}_4, C_0)$ is naturally equivalent to G_{48}. The group G_{24} is isomorphic to $Q_8 \rtimes \mathbb{F}_4^\times$, with generators i and ω, and we let $j = \omega i \omega^2, k = \omega j \omega^2$. Explicitly, in Weierstrass coordinates, i acts as

$$i^*x = x + 1,\ i^*y = y + x + \zeta$$

and ω acts as

$$\omega^*x = \zeta x,\ \omega^*y = y.$$

Then $\mathbb{S}$ is given as

$$\mathbb{S} = \overline{\langle F_{3/2}, \pi \rangle} \rtimes G_{24}$$

where the overline denotes closure in the natural profinite topology. All elements of G_{24} have determinant 1. We will later have use for the following statement:

Lemma 11.4. *The group $F'_{3/2} = \{\gamma \in F_{3/2} \mid |\gamma| = 1\}$ is topologically generated by α^2, $[i, \alpha]$, and $[j, \alpha]$.*

Proof of 11.4. In this proof we will implicitly use the isomorphism of endomorphism algebras of the elliptic and the Honda formal group laws described in Lemma 3.1.2 of [Bea17].
By Lemma 2.2.1 and the proof of Proposition 2.5.3 in [Bea15] we know that $F_{4/2}$ is a powerful pro-2 group topologically generated by any set of elements generating all of $F_{4/2}/F_{6/2}$. The map

$$\mathbb{F}_4 \times \mathbb{F}_4 \to F_{4/2}/F_{6/2}, (a, b) \mapsto [1 + aS^4 + bS^5]$$

is an isomorphism of groups. Direct computation shows that a minimal set of generators of

$F_{4/2}/F_{6/2}$, and thus of $F_{4/2}$, is given by

$$\alpha^2, [i, \alpha]^2, [j, \alpha]^2, \pi/\alpha.$$

$F_{4/2}$ is also uniform: Since we already know it to be finitely generated powerful we only need to show that it is free of torsion by [Dix+99, thm. 4.5]. For $n \geq 4$ consider an element in $F_{4/2}$ of the form

$$1 + aS^n + xS^{n+1}$$

with $a \in \{1, \zeta, \zeta^2\}$. Its square is given by

$$1 + aS^{n+2} + xS^{n+3} + aa^{\sigma^n} S^{2n} + xx^{\sigma^{n+1}} S^{2n+2} + (ax^{\sigma^n} + xa^{\sigma^{n+1}}) S^{2n+1}.$$

For this to be zero we would need $a = 0$, a contradiction.
By [Dix+99, thm. 4.9] any element of $F_{4/2}$ can now uniquely be written as

$$\alpha^{2a}[i, \alpha]^{2b}[j, \alpha]^{2c}(\pi/\alpha)^d$$

for $a, b, c, d \in \mathbb{Z}_2$, which has determinant $(-3)^d$. Thus, $F'_{4/2} = \overline{\langle \alpha^2, [i, \alpha]^2, [j, \alpha]^2 \rangle}$. Finally, the quotient $F'_{3/2}/F'_{4/2} = F_{3/2}/F_{4/2}$ is generated by $[i, \alpha]$ and $[j, \alpha]$, whence the claim. $\square$

11.2 The connection to TMF with level structure

Let us discuss how various fixed points of E relate to K-localizations of TMF with level structure. We include this discussion since we find the available sources stating the content of Proposition 11.6 not very elucidating, but claim no originality. In fact, the following was already known to Hopkins and Mahowald.

Let R be the ring

$$R := \frac{\mathbb{Z}\left[\frac{1}{3}, B, C, \Delta^{-1}\right]}{(B^3 = (B + C)^3)},$$

with Δ the discriminant of the Weierstrass curve

$$C_3 : y^2 + (3C - 1)xy + (-3C^2 - B - 3BC) = x^3$$

and let P and Q be the points

$$P = (0, 0),\ Q = (C, B + C).$$

Then (P, Q) is a full level 3 structure on C_3, and the classifying map

$$(C_3, P, Q) : \mathrm{Spec}(R) \to \mathcal{M}(3)$$

is an equivalence, see [KM85, § 2.2.10]. This gives us that $TMF(3)$ is even periodic and Landweber exact, and yields preferred isomorphisms

$$\pi_0 TMF(3) \simeq R \text{ and } G^Q_{TMF(3)} \simeq \widehat{C_3}.$$

The points $P_0 = (0, 0)$ and $Q_0 = (1, \zeta)$ equip C_0 with a full level 3 structure, and the associated map $R \to \mathbb{F}_4$ sends C to 1 and B to ζ^2.

Since the map $\mathcal{M}(3) \to \mathcal{M}_{Ell}[\frac{1}{3}]$ is finite étale, the space of dotted arrows in the diagram

$$\begin{tikzcd}
\mathrm{Spec}(\mathbb{F}_4) \arrow[r, "{(C_0, P_0, Q_0)}"] \arrow[d] & \mathcal{M}(3) \arrow[d] \\
\mathrm{Spf}(E_0, \mathfrak{m}) \arrow[r, "C_E"'] \arrow[ur, dashed] & \mathcal{M}_{Ell}[\frac{1}{3}]
\end{tikzcd}$$

is contractible. Denote the associated full level 3 structure on C_E by (P_E, Q_E).

Proposition 11.5. *The map*
$$TMF(3) \to E$$
associated to the above full level 3 structure is a K-equivalence.

Proof. Let us first determine $\pi_* L_K TMF(3)$. As $TMF(3)$ is even periodic and Landweber exact, we have
$$\pi_* L_K TMF(3) \simeq (v_2 u^3)^{-1} R^{\wedge}_{(2,v_1 u)}[u^{\pm}]$$
for some choice of unit $u \in \pi_{-2} TMF(3)$. Using the coordinate $-x/y$ on C_3 we find
$$uv_1 = 3C - 1 \text{ and } u^3 v_2 = -3C^2 - B - 3BC.$$
Mod $(2, uv_1)$ both $u^3 v_2$ and Δ are congruent to 1, so we can leave out the localizations. Setting $B' = B/(uv_1 + 1)$ we find
$$\pi_0 L_K TMF(3) = \left(\mathbb{Z}_2^{\wedge}[B']/(B'^2 + B' + 1/3)\right) [\![uv_1]\!].$$
By Hensel's lemma there is a unique $b \in W(\mathbb{F}_4) \subset W(\mathbb{F}_4)[\![u_1]\!]$ such that
$$b^2 + b + 1/3 = 0 \text{ and } b \equiv \zeta^2 \bmod \mathfrak{m},$$
so the map to $\pi_0 E$ must send B' to b. In particular, since b agrees with a Teichmüller lift of ζ up to multiples of 2, the map to E induces an equivalence
$$\mathbb{Z}_2^{\wedge}[B']/(B'^2 + B' + 1/3) \xrightarrow{\sim} W(\mathbb{F}_4).$$
Finally, since $TMF(3) \to E$ is a ring map, the image of uv_1 agrees with the Hazewinkel generator $3u_1$ up to multiples of 2, so the map is an equivalence. ☐

There is a natural action of $GL_2(\mathbb{Z}/3)$ on $\mathcal{M}(3)$, given by:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .(C, P, Q) = (C, aP + bQ, cP + dQ).$$

By functoriality this acts on $TMF(3)$ by $\mathbb{E}_\infty$-ring maps, and so by the previous Proposition also on E. Since $CAlg(E, E)$ is the discrete set $\mathbb{G}$, this action is completely determined by a group homomorphism
$$GL_2(\mathbb{Z}/3) \to \mathbb{G}.$$

The abelian group $C_0(\mathbb{F}_4)$ is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$ on the generators P_0 and Q_0, and the action of the G_{48} on $(\mathbb{F}_4, C_0)$ induces an isomorphism
$$G_{48} \simeq GL_2(\mathbb{Z}/3).$$

The map $\mathrm{Spec}(\mathbb{F}_4) \to \mathcal{M}(3)$ of stacks over $\mathcal{M}_{Ell}[\frac{1}{3}]$ classifying (C_0, P_0, Q_0) is equivariant with respect to these actions. This shows that the map $GL_2(\mathbb{Z}/3) \to \mathbb{G}$ is given by
$$GL_2(\mathbb{Z}/3) \simeq G_{48} \subset \mathbb{G},$$
where the last map is the inclusion discussed in the previous Section.

Proposition 11.6. *The above discussion provides canonical equivalences*
- $L_K TMF(3) \simeq E$,
- $L_K TMF_1(3) \simeq E^{h\langle \omega, \sigma \rangle}$,
- $L_K TMF_0(3) \simeq E^{h\langle -1, \omega, \sigma \rangle}$, *and*
- $L_K TMF \simeq E^{hG_{48}}$.

Proof. An explanation of the equivalences
$$TMF_1(3) \simeq TMF(3)^{h\langle \omega, \sigma \rangle}, TMF_0(3) \simeq TMF(3)^{h\langle -1, \omega, \sigma \rangle}, \text{ and } TMF[1/3] \simeq TMF(3)^{hG_{48}}$$

can be found in the introduction of [MR09].
Let $G \subset GL_2(\mathbb{Z}/3)$ be a subgroup. We only need to show that the natural map

$$L_K\left(TMF(3)^{hG}\right) \to \left(L_K TMF(3)\right)^{hG}$$

is an equivalence. By [HS99b, prop. 7.10(e)] it suffices to show that $TMF(3)^{\wedge}_2$ is $E(2)$-local, since on such spectra L_K is computed by a limit, so that it commutes with limits.
From [LN12, thm. 1.1] we know that $tmf_1(3)_{(2)}$ is a form of $BP\langle 2\rangle$, so that $v_2^{-1}tmf_1(3)_{(2)}$ is a form of $E(2)$. Note that the Hazewinkel v_2 is a unit in $TMF(3)_*$ since it divides the discriminant of C_3, so that we get a map of $\mathbb{E}_\infty$-rings

$$v_2^{-1}tmf_1(3)_{(2)} \to TMF(3)^{\wedge}_2.$$

So $TMF(3)^{\wedge}_2$ is an $E(2)$-module, and thus $E(2)$-local. □

Remark 11.7. As the equivalence $(E_0, C_E) \simeq (E_0', C_{E'})$commutes with the maps from $(\mathbb{F}_4, C_0)$, the above discussion also shows that $L_K TMF(3) \simeq E'$, and the equivalence $E' \to E$ is equivariant with respect to the action of $GL_2(\mathbb{Z}/3)$.

12 Orientations and characteristic classes for $TMF(3)$

In [AHR10] the authors construct an $\mathbb{E}_\infty$-map $\tau_W : MString \to tmf$ lifting the Witten genus, which is called the Ando-Hopkins-Rezk orientation. Denote its restriction to $MU\langle 6\rangle$ by τ_σ. We also have an $\mathbb{E}_\infty$ complex orientation $\tau_{\mathbb{C}} : MU \to E$, associated to the norm coherent coordinate z_E on C_E. A key difference to the height 1 case is that the diagram

$$\begin{tikzcd} MU\langle 6\rangle \arrow[r, "\tau_\sigma"] \arrow[d] & tmf \arrow[d] \\ MU \arrow[r, "\tau_{\mathbb{C}}"] & E \end{tikzcd}$$

does *not* commute! Denote the difference class by $r_U = \tau_W/\tau_{\mathbb{C}} \in E^0BU\langle 6\rangle$. In [LO16] and [LO18], building on the work [Lau16], the authors used this class to calculate the cohomology of $BString$ with respect to various forms of TMF with level structure, and give some first calculations of cannibalistic classes in this context.

Let us recall this work now. We have the following commutative diagram of infinite loop spaces

$$\begin{tikzcd} K(\mathbb{Z},3) \arrow[r, "\mathrm{id}"] \arrow[d, "j"'] & K(\mathbb{Z},3) \arrow[r, "{[2]}"] \arrow[d, "i"'] & K(\mathbb{Z},3) \arrow[d, "j"'] \\ BU\langle 6\rangle \arrow[r, "\mathrm{re}"] \arrow[d, "pr"'] & BString \arrow[r, "\mathrm{cx}"] \arrow[d, "pr"'] & BU\langle 6\rangle \arrow[d, "pr"'] \\ BSU \arrow[r, "\mathrm{re}"] & BSpin \arrow[r, "\mathrm{cx}"] & BSU \end{tikzcd}$$

where i and j are the natural inclusions of the fibers of the maps to BSU and $BSpin$, and re and cx are the realfication and complexification maps. Let $r = \mathrm{cx}^* r_U$ and $r_K = j^* r_U$. By construction r_U, r, and r_K are maps of $\mathbb{E}_\infty$-rings.

We then get isomorphisms

$$E^*BSpin[\![r-1]\!] \xrightarrow{\sim} E^*BString \text{ and } E^*[\![r_K - 1]\!] \xrightarrow{\sim} E^*K(\mathbb{Z},3).$$

Construction 12.1. Consider the composite

$$E^0BSpin \xrightarrow{pr^*} E^0BString \to E^0BString/(r-1).$$

It is a bijection by the above discussion, and a continuous map between compact Hausdorff spaces if we give E^0BSpin and $E^0BString$ the natural topology of Section 11 in [HS99b], thus a homeomorphism. Composing the projection with the inverse of the composite we obtain a continuous retraction of pr^* we will call 'evaluation at $r = 1$' and denote by

$$x \mapsto x|_{r=1}.$$

Lemma C.2 further furnishes a section of pr_* we will denote by

$$s : E_0^\vee BSpin \to E_0^\vee BString.$$

Modding out $\mathfrak{m}$ we also obtain analogous maps between the K-cohomologies and homologies.

Lemma 12.2. *The maps $T_k : E^0BString \to E^0BSpin$ sending a power series in $(r-1)$ to the coefficient of $(r-1)^k$ are continuous with respect to the natural topology. In particular, they assemble to a homeomorphism*

$$E^0BString \xrightarrow{\sim} \prod_{k\geq 0} E^0BSpin.$$

The analogous statement for $E^0K(\mathbb{Z},3)$ and the coefficients of $(r_K - 1)^k$ holds as well.

Proof. We focus on the first statement, the proof for $E^0K(\mathbb{Z},3)$ being much the same. We have already constructed $T_0 = (x \mapsto x|_{r=1})$ as a continuous map in Construction 12.1. Consider the ideal $(r-1)E^0BString \subset E^0BString$. As the image of multiplication by $r-1$ it is closed, and thus compact Hausdorff. Since $E^0BString$ has no $(r-1)$-torsion we get that

$$E^0BString \to (r-1)E^0BString,\, x \mapsto (r-1)x$$

is a continuous bijection between compact Hausdorff spaces and thus a homeomorphism. Denote its inverse by f. We then have the recursive formula

$$T_{k+1}(x) = T_0\left(f^{k+1}\left(x - (r-1)^k T_k(x) - \ldots - T_0(x)\right)\right)$$

writing every T_k as a composite of continuous functions, showing that they are continuous themselves.
Finally, the map from $E^0BString$ to the countable product of E^0BSpin induced by the T_k is a continuous bijection between compact Hausdorff spaces. ☐

Proposition 12.3. *In completed Morava E-homology we have an isomorphism*

$$E_0^\vee BString \xrightarrow{\sim} C^0(\mathbb{Z}_2, E_0^\vee BSpin),\, x \mapsto (n \mapsto pr_*(x \frown r^n))$$

where we give $E_0^\vee BSpin$ the $\mathfrak{m}$-adic topology.

Proof. The proof is much like the proof of Lemma 3.20. Our strategy will be to define a map on a pro-basis of the source which, by construction, is an isomorphism. A direct computation will then show that it agrees with the above map.
Let us first show that the target is an L-complete and, in fact, pro-free E_0-module. Let $\{y_i\}_{i\in I}$ be a pro-basis of $E_0^\vee BSpin$. Also note that $C^0(\mathbb{Z}_2, E_0)$ is pro-free with pro-basis given by the functions

$$\phi_k : \mathbb{Z}_2 \to E_0,\, n \mapsto \binom{n}{k},$$

for $k \geq 0$, by a result of Mahler, see [CC16, thm. 11]. By Lemma C.7 also have that

$C^0(\mathbb{Z}_2, E_0)/\mathfrak{m}^n = C^0(\mathbb{Z}_2, E_0/\mathfrak{m}^n)$. We find that

$$\begin{aligned}
C^0(\mathbb{Z}_2, E_0^\vee BSpin) &= C^0\left(\mathbb{Z}_2, \lim_n \bigoplus_{i\in I} E_0/\mathfrak{m}^n\right) = \lim_n C^0\left(\mathbb{Z}_2, \bigoplus_{i\in I} E_0/m^n\right) \\
&= \lim_n \bigoplus_{i\in I} C^0\left(\mathbb{Z}_2, E_0/m^n\right) = \lim_n \bigoplus_{i\in I}\bigoplus_{k\geq 0} E_0/\mathfrak{m}^n \\
&= \left(\bigoplus_{i\in I, k\geq 0} E_0\right)^\wedge_{\mathfrak{m}}
\end{aligned}$$

where we used that $\mathbb{Z}_2$ is a compact space. So we see that $C^0\left(\mathbb{Z}_2, E_0^\vee BSpin\right)$ is pro-free with a pro-basis given by $n \mapsto \phi_k(n)y_i$.
Using Lemma C.2, Remark C.3, and Lemma 12.2 we define a pro-basis

$$\{\alpha_{k,i}\}_{k\geq 0, i\in I}$$

of $E_0^\vee BString$ such that, for all $c \in E^0 BString$, we have

$$\langle \alpha_{k,i}, c\rangle = \langle \alpha_{k,i}, \sum_{n\geq 0}(r-1)^n pr^* c_n\rangle = \langle y_i, c_k\rangle.$$

For any $n \in \mathbb{Z}_2$ and $T \in E^0 BSpin$ we have

$$\langle pr_*\left(\alpha_{k,i} \frown r^n\right), T\rangle = \langle \alpha_{k,i}, r^n pr^* T\rangle = \left\langle y_i, \binom{n}{k} T\right\rangle$$

so that

$$pr_*\left(\alpha_{k,i} \frown r^n\right) = \phi_k(n) y_i.$$

The map

$$E_0^\vee BString \to C^0\left(\mathbb{Z}_2, E_0^\vee BSpin\right),\ \alpha_{k,i} \mapsto \phi_k y_i$$

is an isomorphism as it sends a pro-basis to a pro-basis, and it agrees on this pro-basis (and thus on all elements by $\mathfrak{m}$-adic convergence) with the map in question. □

Remark 12.4. One can directly show that for every $x \in E_0^\vee BString$ the map

$$\mathbb{Z}_2 \to E_0^\vee BSpin,\ n \mapsto pr_*(x \frown r^n)$$

is continuous if we give the target the natural topology. Using the proposition one can then show that it does not matter whether we use the natural or the $\mathfrak{m}$-adic topology in the proposition.

Remark 12.5. Using that the integers are dense in $\mathbb{Z}_2$ and that r is a ring map one can show that the map we just constructed is an isomorphism of rings if we give $C^0\left(\mathbb{Z}_2, E_0^\vee BSpin\right)$ the point wise ring structure. We also see that under this isomorphism pr_* corresponds to evaluation at 0 and s corresponds to the inclusion of the constant functions, thus s is a ring map. Dualizing then gives that $x \mapsto x|_{r=1}$ is a map of coalgebras.

Proposition 12.6 ([LO16, thm. 1.1], [Lau16, thm. 1.2])**.** *There are classes $p_i \in E^{4i}BSpin$ for $i \geq 0$, with $p_0 = 1$, such that*

1. *we have $E^*BSpin = E^*[\![p_1, p_2, \ldots]\!]$, and*
2. *on each maximal torus $U(1)^m$ embedded in $Spin(2m)$, the power series $p_t = \sum_{i\geq 0} p_i t^i$ restricts to*

$$\prod_{k=1}^{m}\left(1 - t\rho^*(c_1^E(L_k)c_1^E(\overline{L_k}))\right)$$

where ρ is the double covering onto the image in $SO(2m)$, which is again a torus of rank m, and the $c_1^E = uz_E \in E^2BU(1)$ are the first Chern classes.

Furthermore, each $\pi_i = p_i/u^{18i}$ lifts to $TMF_0(3)^{-32i}BSpin$.

Remark 12.7. Note that Laures and Olbermann prove this, essentially, for the Morava E-theory E' with the coordinate $z_{E'}$. The above statement follows by noticing that in E' we have $v_2 = 1/(u')^3$ and that the equivalence $E' \to E$ maps $z_{E'} \to z_E$ and u' to u.

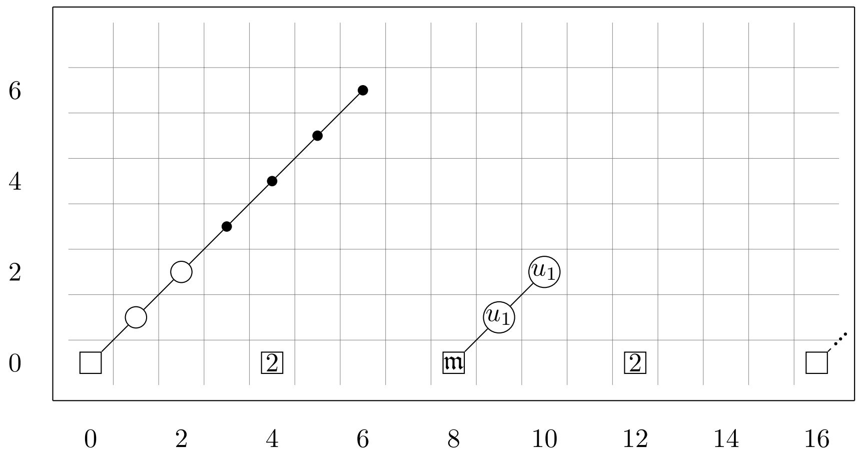
We close with two lemmas which, essentially, follow from the calculation of $TMF_0(3)^*BSpin$ in [LO16] and the determination of the homotopy fixed point spectral sequence for $TMF_0(3)$ in [MR09].

Lemma 12.8. *Let $ER = E^{h\{\pm 1\}}$. Then for any set S and any integer k the natural map*

$$\pi_{16k} map\left(BSpin_+, L_K \bigoplus_S ER\right) \to \pi_{16k} map\left(BSpin_+, L_K \bigoplus_S E\right)$$

is an isomorphism.

Proof. The proof proceeds by looking at the homotopy fixed point spectral sequence (HFPSS). Let us first assume that $S = \{*\}$.
Comparing with [MR09, § 4] we can deduce the whole HFPSS for ER. Note that also that computation compares most naturally to the computation of $E'R$, but by Remark 11.7 it also carries over to ER. The E_2-page is given by $E'_0[u'^{\pm 2}, c]/(2c)$ where E'_0 lies in bidegree $(0,0)$, u'^2 lies in $(-4,0)$, and c lies in $(1,1)$, where we use Adams grading. For degree reasons this is also the E_3-page, and d_3 is determined by $d_3(a) = 0$, $d_3(c) = 0$, and $d_3(1/u'^2) = ac^3$.
Next, the E_4-page equals the E_7-page for degree reasons, and there is a further differential $d_7(1/u'^4) = c^7$. Finally, the spectral sequence collapses at the E_8-page with finite vanishing line, and the same holds for ER:



In this picture a box stands for E_0, a circle for $E_0/2$, and a dot for $E_0/\mathfrak{m}$, while a box with a 2 inside stands for the ideal $(2) \subset E_0$, and so on. We see that the class u^8 lifts (uniquely) to ER, so that ER is 16-periodic, and that the map $\pi_{16k}ER \to \pi_{16k}E$ is an isomorphism.
By [LO16, thm. 1.1] and [Lau16, thm. 1.2(ii)] there are classes

$$\pi_i \in TMF_0(3)^{-32i} BSpin$$

such that $E^*BSpin = E^*[\![\pi_1, \pi_2, \ldots]\!]$. Let $\tilde{\pi}_i = u^{16i}\pi_i \in E^0BSpin$. By the above discussion we see that $\tilde{\pi}_i$ lifts to ER^0BSpin and thus is a permanent cycle in the HFPSS for ER^*BSpin, as are all monomials in the $\tilde{\pi}_i$. The HFPSS for ER^*BSpin thus looks like a product of copies of the one for ER, and we find

$$ER^*BSpin = ER^*[\![\tilde{\pi}_1, \tilde{\pi}_2, \ldots]\!].$$

This finishes the case $S = \{*\}$.
For general S first consider the case where we take the product over S instead of the sum. Here the HFPSS is then also just a product over the above HFPSS for $S = \{*\}$. On coefficients the map

$$L_K \bigoplus_S E \to \prod_S E$$

induces a split injection of E_*-modules by [HS99b, prop. A.13]. Since the differentials in the HFPSS for ER are all given by multiplication with elements in E_0, the comparison map between the spectral sequences for the sum and product is a split injection on every page and bidegree as well. This allows us to transport the differentials to the case at hand, proving the lemma. $\square$

Lemma 12.9. *For any set S and any $n_s \in \{0, 16, 32\}$ the natural map*

$$\pi_0 Map\left(BSpin_+, L_K \bigoplus_{s\in S} \Sigma^{n_s} TMF_0(3)\right) \to \prod_{s\in S} TMF_0(3)^{n_s} BSpin$$

is injective.

Proof. Combining Lemma C.4 and Lemma 12.8 shows that the statement is true for $TMF_0(3)$ replaced with ER. The homotopy fixed point spectral sequence for the action of ω collapses on the E_2-page and is concentrated on the 0-line, thus the statement is also true for $E^{h\langle -1,\omega\rangle}$. Finally, we have an equivalence

$$TMF_0(3) \oplus TMF_0(3) \xrightarrow{(1,\zeta)} E^{h\langle -1,\omega\rangle}$$

so the statement follows. $\square$

12.1 Cannibalistic classes

As before, we need to understand the difference classes for the action of $\mathbb{G}$ on τ_W and $\tau_{\mathbb{C}}$ to investigate the K_0E-comodule structures of $K_0MU\langle 6\rangle$ and $K_0MString$.

Definition 12.10. Let $g \in \mathbb{G}$. We define the *cannibalistic classes* by

$$\Theta_W^g = g.\tau_W/\tau_W \in E^0 BString \text{ and } \Theta_{\mathbb{C}}^g = g.\tau_{\mathbb{C}}/\tau_{\mathbb{C}} \in E^0 BU.$$

We also define $\chi^g \in E^0 BSpin$ as $\Theta_W^g|_{r=1}$.

By construction Θ_W^g and $\Theta_{\mathbb{C}}^g$ are naturally $\mathbb{E}_\infty$-ring maps, and χ^g is a group like element in $E^0 BSpin$.

It is in general not easy at all to determine the classes Θ_W^g in terms of more geometrically defined classes. See [LO18] for an extensive discussion of these classes and their properties.

Nevertheless, we have the following result:

Lemma 12.11. *In $E^0 BString$ we have the equality $\Theta_W^g = r^{(|g|-1)/2} pr^* \chi^g$.*

Proof. Let $\{y_i\}_{i\in I}$ be a pro-basis of $E_0^\vee BSpin$ and let $\alpha_j \in E_0^\vee K(\mathbb{Z},3)$ be the elements which correspond, under Remark C.3, to projecting onto the coefficient of $(r_K - 1)^j$. Then $\{i_*\alpha_{2k} s y_i\}_{k\geq 0, i\in I}$ is a pro-basis of $E_0^\vee BString$: under the isomorphism of Proposition 12.3 they send $n \in \mathbb{Z}$ to

$$pr_*\left((i_*\alpha_{2k} s y_i) \frown r^n\right) = pr_*(i_*(\alpha_{2k} \frown r_K^{2n})) pr_*(s y_i \frown r^n) = \binom{2n}{2k} y_i,$$

thus they agree, mod $\mathfrak{m}$, with the elements $\alpha_{k,i}$ considered in the proof of Proposition 12.3.
Using that r, Θ_W^g, and χ^g are group-like elements, and that $i^*\Theta_W^g = r_K^{|g|-1}$ (see [LO18] above Theorem 5.4), we find

$$\begin{aligned}
\langle i_*\alpha_{2k} s y_i, \Theta_W^g\rangle &= \langle i_*\alpha_{2k}, \Theta_W^g\rangle \langle s y_i, \Theta_W^g\rangle = \langle \alpha_{2k}, i^*\Theta_W^g\rangle \langle y_i, \chi^g\rangle \\
&= \langle \alpha_{2k}, r_K^{|g|-1}\rangle \langle y_i, (pr^*\chi^g)|_{r=1}\rangle \\
&= \langle i_*\alpha_{2k}, r^{\frac{|g|-1}{2}}\rangle \langle s y_i, pr^*\chi^g\rangle = \langle i_*\alpha_{2k} s y_i, r^{\frac{|g|-1}{2}} pr^*\chi^g\rangle
\end{aligned}$$

and conclude by duality. $\square$

We will also need to give a name to the following classes:

Definition 12.12. For $g \in \mathbb{G}$ consider the element $g.r_U/r_U^{|g|} \in E^0BU\langle 6\rangle$. By [LO18, thm. 3.8] it admits a unique lift to E^0BSU we will denote by q_0^g.

Finally, since χ^g is group-like, it corresponds to a real 2-structure for the formal group law $E^0BU(1)$ in the coordinate z. The following gives a (fairly inefficient) way to determine its image in K^0BSpin:

Proposition 12.13. *Let $g \in \mathbb{S}$. There is a unique real 1-structure $l_g(z)$ on the formal group law of K such that, as 3-structures, we have*

$$\delta(\mathrm{re}^*\chi^g)(x,y,z) = \frac{s(g(x),g(y),g(z))\,\delta^2\left(\frac{g}{g'(0)}\right)(x,y,z)}{s(x,y,z)^{\frac{|g|+1}{2}}s([-1](x),[-1](y),[-1](z))^{\frac{|g|-1}{2}}}$$

$$= \delta^2 l_g(x,y,z)$$

where s is the standard 3-structure on the formal group law associated to the Weierstrass curve $y^2+y=x^3$, see [AHS01, § B] or [LO16, § 3]. In particular, there is a canonical lift of $\mathrm{re}^*\chi^g + \mathfrak{m}$ to a group-like class l_g in K^0BU.

Proof. The first equality follows from

$$\mathrm{re}^*pr^*\chi^g = \frac{g.r_U}{r_U^{\frac{|g|+1}{2}}\overline{r_U}^{\frac{|g|-1}{2}}}\frac{g.\tau_{\mathbb{C}}}{\tau_{\mathbb{C}}}$$

and identifying the 3-structure associated to the right-hand side. That this is given by δ^2 of some real 1-structure comes from the equivalence $C_r^2F \to C_r^1F$. Uniqueness now follows from $K^0BSpin \to K^0BString$ being injective and $K^0BString \to K^0BU\langle 6\rangle$ being injective on group-like elements.
Finally, the real 1-structure l_g determines an algebra morphism $K_0BU \to K_0$ sending b_i to the coefficient of z^i in $l_g(z)$, which by duality is determined by a class in K^0BU we shall also denote by l_g. □

Since we know formulas for the 3-structre s, the -1-series, and the formal group law, we can approximate l_g for the generators of $F'_{3/2}$. By direct computation[5] we find:

Proposition 12.14. *The real 1-structures l_g for the generators of $F'_{3/2}$ are*

$$\begin{aligned} l_{\alpha^2}(z) &= 1 + z^6 + (z^8) \\ l_{[i,\alpha]}(z) &= 1 + z^2 + z^4 + z^5 + z^6 + (z^8) \\ l_{[j,\alpha]}(z) &= 1 + \zeta z^2 + \zeta^2 z^4 + \zeta z^5 + z^6 + (z^8). \end{aligned}$$

13 A harder factorization Lemma

In this Section we want to show that the Σ'-comodules

$$K_0\left(E^{hF_{3/2}} \otimes MU\langle 6\rangle\right) \text{ and } K_0\left(E^{hF_{3/2}} \otimes MString\right)$$

are splittable, see Definition A.8. As we show in Proposition A.13 a sufficient condition is to admit a unital map of Σ'-comodules from Σ'.

[5] To determine these approximations one needs to calculate the expression of Proposition 12.13 roughly up to order 80 in (x,y,z) which we did using the SageMath software [The24].

Remark 13.1. For the definition of a splittable comodule to apply, we need them to be $\mathbb{F}_4[G]$-modules, where G is the set of group like elements in Σ'. In our case, group like elements correspond to group homomorphisms $\mathbb{S} \to \mathbb{F}_4^\times$, all of which must contain $F_{3/2}$ in their kernel, so that already $K_0E^{hF_{3/2}}$ is an $\mathbb{F}_4[G]$-module in Σ'-comodules.

Let $DA(1)$ be the even 2-local finite complex called Z in [HR95], see also [Mat15, § 4.1], and choose a 'unit' map $\mathbb{S} \to DA(1)$ inducing an isomorphism in zeroth homotopy.

Lemma 13.2. *The map $DA(1) \otimes MU\langle 6\rangle \to BP \to tmf_1(3)$ admits a unital section after $K(2)$-localization.*

Proof. The first part of the composite is the external product of any fixed choice of unital map $DA(1) \to BP$ and of the canonical map $MU\langle 6\rangle \to MU \to BP$, the second is the 2-typicalization of the standard complex orientation of $tmf_1(3)$.[a]
By [HR95, cor. 2.2] we have that $DA(1) \otimes MU\langle 6\rangle$ splits 2-locally as a direct sum of suspensions of BP. Inspecting the proof of [HR95, thm. 2.1] we see that this splitting is obtained by choosing maps

$$x_i : DA(1) \otimes MU\langle 6\rangle \to \Sigma^{|x_i|}\mathbb{F}_2$$

forming a basis of $\mathbb{F}_2^*\left(DA(1) \otimes MU\langle 6\rangle\right)$ over $\mathbb{F}_2^*BP$ and lifting them through the ring map

$$BP \to \mathbb{F}_2$$

(using that $DA(1) \otimes MU\langle 6\rangle$ is even). This shows that the first map in the composite can appear as one of the 2-local projection maps, and thus admits a 2-local section. Since the map

$$DA(1) \otimes MU\langle 6\rangle \to BP$$

induces an isomorphism on π_0, the section can be chosen to be unital.
Following [HS99a, thm. 4.1] and the remark after its proof, and using that the 2-typical complex orientation $BP \to tmf_1(3)$ exhibits $tmf_1(3)$ as a form of $BP\langle 2\rangle$ (see [LN12, thm. 1.1]), we find that the second map in the composite admits a unital section after $K(2)$-localization. □

[a] This orientation maps along the canonical map $tmf_1(3) \to E$ to uz_E.

Lemma 13.3. *The map $\left(\mathbb{S}^0 \oplus \mathbb{S}^0\right) \otimes \left(\mathbb{S}^0 \oplus \mathbb{S}^{-2} \oplus \mathbb{S}^{-4}\right) \otimes DA(1) \otimes MU\langle 6\rangle \to E$, given by the external product of the map from Lemma 13.2 and the map sending the spheres to the elements $1, \zeta, 1, u, u^2$ in π_*E, admits a $K(2)$-local unital section.*

Proof. In the $K(2)$-local category we have an equivalence

$$\left(\mathbb{S}^0 \oplus \mathbb{S}^0\right) \otimes \left(\mathbb{S}^0 \oplus \mathbb{S}^{-2} \oplus \mathbb{S}^{-4}\right) \otimes tmf_1(3) \to E$$

given by sending the spheres to the elements $1, \zeta, 1, u, u^2 \in \pi_*E$ (in that order). We can now take the section of Lemma 13.2 and tensor it with the identity on the spheres. □

Lemma 13.4. *The map*

$$K_0\left(\left(\mathbb{S}^0 \oplus \mathbb{S}^0\right) \otimes \left(\mathbb{S}^0 \oplus \mathbb{S}^{-2} \oplus \mathbb{S}^{-4}\right) \otimes DA(1)\right) \to K_0E$$

factors, in K_0E-comodules, through the injective map $K_0E^{hF_{3/2}} \to K_0E$.

Proof. Since $K_0E^{hF_{3/2}} \to K_0E$ is injective we only need to show that it factors as a map of sets. First note that, for dimensional and connectivity reasons, the map $DA(1) \to BP$ factors through $T(2) \to BP$, where $T(2)$ is the 2-local summand of Ravenel's $X(4)$, see [Dev24, thm. 3.1.5]. The inclusion map identifies $K_*T(2)$ with the subset

$$K_*[t_1, t_2] \subset K_*[t_1, t_2, \ldots] = K_*BP.$$

So the image of the map in question is included in the sub-$\mathbb{F}_4$-algebra generated by

$$\eta_R(\zeta), \eta_R(u)/\eta_L(u), ut_1, \text{ and } u^3t_2.$$

Via the isomorphism $K_0E = C^0(\mathbb{G}, \mathbb{F}_4)$ we can identify these with the following functions: Let the element $\gamma \in \mathbb{S}$ be presented as a power series

$$\gamma = \sum_{i \geq 0} a_i T^i$$

with $a_i \in \{0, 1, \zeta, \zeta^2\}$ and $a_0 \neq 0$, and let $\epsilon \in \{0, 1\}$. Then we have, by Proposition 4.3 and [Bea17, thm. 6.2.2],

$$\eta_R(\zeta)(\gamma\sigma^\epsilon) = \sigma^\epsilon(\zeta) \text{ and } (\eta_R(u)/\eta_L(u))(\gamma\sigma^\epsilon) = a_0.$$

To get formulas for ut_1 and u^3t_2 we will use the calculation of the action of $\mathbb{S}$ on E_* of [Bea17, § 6], though we will denote by $f_i(\gamma)$ what she calls $t_i(\gamma)$ to avoid confusion with the elements $t_i \in BP_*BP$.
Recall that in BP_*BP we have the relations

$$\begin{aligned} 2t_1 &= \eta_R(v_1) - v_1 \\ 2t_2 &= \eta_R(v_2) - v_2 + 3v_1^2 t_1 + 5v_1 t_1^2 + 4t_1^3 \end{aligned}$$

where we use Hazewinkel generators. In $E_* = \mathbb{W}(\mathbb{F}_4)[[u_1]][u^\pm]$ we have that $uv_1 = 3u_1$ and $u^3v_2 = u_1^3 - 1$. Mapping these relations through

$$BP_*BP \to E_*^\vee E = C^0(\mathbb{G}, E_*)$$

we thus find that

$$\begin{aligned} 2t_1(\gamma\sigma^\epsilon) &= 3\frac{\gamma.u_1}{\gamma.u} - 3\frac{u_1}{u} = \frac{2}{u} f_1(\gamma)/f_0(\gamma)^2 \\ \implies ut_1(\gamma\sigma^\epsilon) &= f_1(\gamma)/f_0(\gamma)^2. \end{aligned}$$

For u^3t_2 we have to work a bit harder:

$$\begin{aligned} 2u^3t_2 &= u^3\frac{(\gamma.u_1)^3 - 1}{(\gamma.u)^3} - u_1^3 + 1 + 27u_1^2 f_1/f_0^2 + 15u_1 f_1^2/f_0^4 + 4f_1^3/f_0^6 \\ &= 1 - 1/f_0^3 + 29u_1^2 f_1/f_0^2 + (4/3 + 15)u_1 f_1^2/f_0^4 + (8/27 + 4)f_1^3/f_0^6 \\ &= 1 - 1/f_0^3 - u_1^2 f_1/f_0^2 - u_1 f_1^2/f_0^4 + (4) \\ &= (f_0^4 - f_0 - u_1^2 f_1 f_0^2 - u_1 f_1^2)/f_0^4 + (4) \\ &\overset{\star}{=} (f_0 - 2f_3 - 3u_1 f_1^2 - 2u_1 f_0 f_2 - 3u_1^2 f_0^2 f_1 - f_0 - u_1^2 f_1 f_0^2 - u_1 f_1^2)/f_0^4 + (4) \\ &= 2f_3/f_0^4 + (4, 2u_1) \end{aligned}$$

where we suppressed the evaluation at γ and in the step marked with $\star$ used the formula [Bea17, prop. 6.3.3]:

$$f_0^4 \equiv f_0 - 2f_3 - 3u_1 f_1^2 - 2u_1 f_0 f_2 - 3u_1^2 f_0^2 f_1 + (4).$$

Now we can divide by 2:

$$u^3t_2(\gamma\sigma^\epsilon) = f_3(\gamma)/f_0(\gamma)^4 + (2, u_1).$$

By a small variation of [Bea17, cor. 6.3.5] we now find that, in K_0E, we have

$$(ut_1)(\gamma\sigma^\epsilon) = a_1/(a_0)^2, (u^3t_2)(\gamma\sigma^\epsilon) = a_2/a_0.$$

Since all of the functions only depend on ϵ, a_0, a_1, a_2 they factor over $\mathbb{G}/F_{3/2}$, so they already lie in the image of $K_0E^{hF_{3/2}}$. □

Corollary 13.5. *The Σ'-comodules $K_0\left(E^{hF_{3/2}} \otimes MU\langle 6\rangle\right)$ and $K_0\left(E^{hF_{3/2}} \otimes MString\right)$ are splittable.*

Proof. We apply Proposition A.13 using the section and factorization we have just constructed. □

14 Lifting an algebraic to a spectral splitting, reprise

Theorem 14.1. *There exists a map*

$$D : MU\langle 6\rangle \to L_K \bigoplus_{i\in I} E$$

such that the free $E^{hF_{3/2}}$-linearization of D induces a K-equivalence

$$E^{hF_{3/2}} \otimes MU\langle 6\rangle \xrightarrow{\sim} L_K \bigoplus_{i\in I} E.$$

Proof. Corollary 13.5 shows that

$$K_0\left(E^{hF_{3/2}} \otimes MU\langle 6\rangle\right)$$

is a splittable Σ'-comodule. By Lemma B.9 the primitive elements are given by

$$P_1\left(K_0\left(E^{hF_{3/2}} \otimes MU\langle 6\rangle\right)\right) = \left(K_0\left(E^{hF_{3/2}} \otimes MU\langle 6\rangle\right)\right)^{\mathbb{S}}.$$

Denoting $K_0 MU\langle 6\rangle$ by M we find that the invariants are given by

$$C^0\left(\mathbb{G}/F_{3/2}, M\right)^{\mathbb{S}} \xrightarrow{\sim} M^{F_{3/2}} \oplus M^{F_{3/2}},\ f \mapsto (f([e]), f([\sigma])).$$

Under this equivalence and the one from Lemma A.7 the inclusion of the zeroth filtration is given by

$$\begin{aligned} i : C^0(\mathbb{F}_4^\times, \mathbb{F}_4) \otimes \left(M^{F_{3/2}} \oplus M^{F_{3/2}}\right) &\to C^0(\mathbb{G}/F_{3/2}, \mathbb{F}_4) \otimes M \\ s \otimes (m_e, m_\sigma) &\mapsto \left([\gamma\sigma^\epsilon] \mapsto s([\gamma])\gamma.(m_{\sigma^\epsilon})\right). \end{aligned}$$

Let $r_M : M \to M^{F_{3/2}}$ be a retraction and define

$$\begin{aligned} &r' : C^0(\mathbb{G}/F_{3/2}, \mathbb{F}_4) \otimes M \to C^0(\mathbb{F}_4^\times, \mathbb{F}_4) \otimes \left(M^{F_{3/2}} \oplus M^{F_{3/2}}\right), \\ &f \otimes m \mapsto (x \mapsto f([x])) \otimes (r_M(m), 0) + (x \mapsto f([x\sigma])) \otimes (0, \sigma.r_M(\sigma.m)). \end{aligned}$$

The composite $r' \circ i$ is an equivalence

$$r'(i(s \otimes (a, b))) = \sum_{x\in\mathbb{F}_4^\times} s(x)\delta_x \otimes (x.a, x.b)$$

and we define $r = (r' \circ i)^{-1} \circ r'$. Also note that $\epsilon \otimes \mathrm{id} \circ (r' \circ i)^{-1} = \epsilon \otimes \mathrm{id}$ so that we may use r' instead of r in the formula for h_r:

$$h_r(f \otimes m) = \sum_{\gamma\in\mathbb{S}} \delta_\gamma \otimes \left(f([\gamma])r_M(\gamma^{-1}.m), f([\gamma\sigma])\sigma.r_M(\sigma\gamma^{-1}.m)\right)$$

Now we can see how the splitting interacts with the involution:

$$\begin{aligned} h_r(\sigma.(f \otimes m)) &= \sum_{g\in\mathbb{S}} \delta_g \otimes \left(\sigma\left(f([\bar{g}\sigma])r_M^\sigma(\bar{g}^{-1}.m)\right), \sigma\left(f([\bar{g}])r_M(\bar{g}.m)\right)\right) \\ &= \sum_{g\in\mathbb{S}} \delta_{\bar{g}} \otimes \left(\sigma\left(f([g\sigma])r_M^\sigma(g^{-1}.m)\right), \sigma\left(f([g])r_M(g.m)\right)\right) \end{aligned}$$

so we must have

$$\tau(\delta_\gamma \otimes (a, b)) = \delta_{\bar{\gamma}} \otimes (\sigma.b, \sigma.a).$$

Let $\{x_i\}_{i\in I}$ be a basis of $M^{F_{3/2}}$. Then the elements $y_i = \sigma.x_i$ form a basis, too. The map

$$\Sigma' \otimes \left(M^{F_{3/2}} \oplus M^{F_{3/2}}\right) \to \Sigma \otimes M^{F_{3/2}},\ \delta_\gamma \otimes (ax_i, by_j) \mapsto \delta_\gamma \otimes ax_i + \delta_{\gamma\sigma} \otimes bx_j$$

is an isomorphism of Σ'-comodules with involution (where we view the target as a direct sum over the basis $\{x_i\}_{i\in I}$), and thus an isomorphism of Σ-comodules by Proposition 4.11.
Finally, after choosing an isomorphism of Σ-comodules

$$\Sigma \otimes M^{F_{3/2}} \to K_0 \bigoplus_{i\in I} E,$$

which amounts to a choice of basis of $M^{F_{3/2}}$, we can lift this isomorphism to the spectrum level as by Lemma C.5, yielding a $K(2)$-local equivalence

$$D' : E^{hF_{3/2}} \otimes MU\langle 6\rangle \to L_K \bigoplus_{i\in I} E.$$

Let $\lambda_i : M^{F_{3/2}} \to \mathbb{F}_4$ be the projections such that

$$\forall m \in M^{F_{3/2}} : m = \sum \lambda_i(m)x_i$$

and denote by $d_i \in K^0 MU\langle 6\rangle$ the unique classes such that $\lambda_i(r_M(m)) = \langle m, d_i\rangle$. We find that, in total, the isomorphism sends

$$K_0\left(E^{hF_{3/2}} \otimes MU\langle 6\rangle\right) \to \Sigma \otimes M^{F_{3/2}},\ f \otimes m \mapsto \sum_i (g \mapsto f([g])\langle m, g.d_i\rangle) \otimes x_i$$

thus is $K_0E^{hF_{3/2}}$-linear. The $E^{hF_{3/2}}$-linearization of the restriction of D' to $MU\langle 6\rangle$ will induce the same map in K_0 as D', so it is still be a $K(2)$-local equivalence. Thus, we can take D to be the restriction of D' to $MU\langle 6\rangle$. $\square$

Remark 14.2. The proof of the above also shows that the projections $D_i = pr_i \circ D$ are given mod $\mathfrak{m}$ by the unique classes $d_i \in K^0 MU\langle 6\rangle$ such that

$$\langle m, d_i\rangle = \lambda_i(r_M(m))$$

for all $m \in K_0 MU\langle 6\rangle$.

The exact same proof also goes through for $MString$ in place of $MU\langle 6\rangle$, so we get:

Theorem 14.3. *There exists a map*

$$C : MString \to L_K \bigoplus_{i\in I} E$$

such that the free $E^{hF_{3/2}}$-linearization of C induces a K-equivalence

$$E^{hF_{3/2}} \otimes MString \xrightarrow{\sim} \bigoplus_{i\in I} E.$$

15 Reducing the Galois extension for $MString$

To proceed we need a more detailed understanding of the comodule structure of $K_0MString$. In the following, let $t_* : K_0MString \to K_0BString$ be the Thom isomorphism coming from the Ando-Hopkins-Rezk orientation.

Proposition 15.1. *Under the isomorphism*

$$K_0MString \xrightarrow{\sim} C^0\left(\mathbb{Z}_2^\times, K_0BSpin\right),\ m \mapsto u \mapsto pr_*\left((t_*x) \frown r^{(u-1)/2}\right)$$

the Morava stabilizer group acts by

$$(g.m)(u) = g.\left(m(u/|g|) \frown \left(\chi^{g^{-1}}\left(\mathrm{cx}^* q_0^{g^{-1}}\right)^{(u-1)/2}\right)\right).$$

Proof. This follows from the map $\mathbb{Z}_2^\times \to \mathbb{Z}_2$, $u \mapsto (u-1)/2$ being a homeomorphism, Proposition 12.3, and direct computation using the duality of the homological and cohomological Thom isomorphisms. □

In light of this result we will think of $K_0MString$ as the space of continuous sections of a $\mathbb{G}$-equivariant vector bundle on $\mathbb{Z}_2^\times$ with constant fiber K_0BSpin, where $\mathbb{G}$ acts on the base $\mathbb{Z}_2^\times$ by multiplication with the determinant.

Remark 15.2. Going through the same motions for $MU\langle 6\rangle$, the base space is again $\mathbb{Z}_2^\wedge$ where $\mathbb{G}$ acts via multiplication by the determinant. This action is not transitive, in contrast to the case of $MString$, and so the following considerations would be more complicated.

Recall that the element $\pi = 1 + 2\zeta \in \mathbb{G}$ has determinant 3.

Proposition 15.3. *Let $M = K_0MString$ and $N = K_0BSpin$. Then the map*

$$M \to N \oplus N,\ m \mapsto (m(1), (\pi.m)(1))$$

restricts to an equivalence

$$M^{F_{3/2}} \cong N_1 \oplus N_1$$

where $N_1 = \{m(1) \mid m \in M^{F_{3/2}}\}$.

Proof. First of all, $(\pi.m)(1)$ sits in N_1 as well, since π normalizes $F_{3/2}$.
For injectivity note that

$$(\pi.m)(1) = \pi.\left(m(1/3) \frown \chi^{\pi^{-1}}\right),$$

so we may extract the value of m at 1 and at $1/3$. Since

$$\mathbb{Z}_2^\times = (1 + 4\mathbb{Z}_2) \times \{\pm 1\},$$

the subgroup $F_{3/2} \subset \mathbb{G}$ acts transitively on $(1 + 4\mathbb{Z}_2)$, and

$$1 \in (1 + 4\mathbb{Z}_2) \times \{+1\},\ 1/3 \in (1 + 4\mathbb{Z}_2) \times \{-1\}$$

we can use the $F_{3/2}$-invariance of m to reconstruct its value on all of $\mathbb{Z}_2^\times$.
For surjectivity consider $(a(1), b(1)) \in N_1 \oplus N_1$ for some $a, b \in M^{F_{3/2}}$. Define another continuous section c by

$$c(u) := \begin{cases} a(u) \text{ for } u \in (1 + 4\mathbb{Z}_2) \times \{+1\} \\ (\pi^{-1}.b)(u) \text{ for } u \in (1 + 4\mathbb{Z}_2) \times \{-1\} \end{cases}.$$

This section satisfies $c(1) = a(1)$ and $(\pi.c)(1) = b(1)$, and is $F_{3/2}$-invariant since the action does not mix the $+1$ and -1 components of $\mathbb{Z}_2^\times$. □

It turns out that N_1 coincides with the fixed points of the action of

$$F'_{3/2} = \{g \in F_{3/2} \mid |g| = 1\}$$

on N given by

$$g \triangleright n := g.n \frown 1/\chi^g.$$

Note that, while the formula makes sense for any $g \in \mathbb{G}$, it only determines a group action on the elements with determinant 1.

Proposition 15.4. *We have that*

$$N_1 = \{n \in N \mid \forall g \in F'_{3/2} : n = g \triangleright n\}.$$

Proof. This follows from $F'_{3/2}$ being the stabilizer of $1 \in \mathbb{Z}_2^\times$, the action described above being the action on N viewed as the fiber over 1, and $F_{3/2}$ acting transitively on $1 + 4\mathbb{Z}_2$. $\square$

Let us now prove that the splitting of Theorem 14.3 is already determined by spin characteristic classes, and that it descends to the fixed points under the slightly larger subgroup $K = \langle F_{3/2}, \pi\rangle$.

Theorem 15.5. *There exists a map*

$$Q' : BSpin_+ \to L_{K(2)} \bigoplus_{j \in J} E$$

such that the E^{hK}-linear map

$$E^{hK} \otimes MString \xrightarrow{\tau_W pr^* Q'} L_K \bigoplus_{j \in J} E$$

is a K-equivalence.

Proof. Let $M = K_0 MString$ and $N = K_0 BSpin$. Starting as in the proof of Theorem 14.1 we now choose a particular retraction: First choose a retraction r_N of $N_1 \subset N$. Choose a basis $\{x_j\}_{j\in J}$ of N_1, let $\rho_j : N_1 \to \mathbb{F}_4$ be the associated projections, and let $q_j \in K^0 BSpin$ be the unique class such that

$$\forall x \in K_0 BSpin : \rho_j(r_N(x)) = \langle x, q_j\rangle.$$

By Proposition 15.3 we obtain a retraction of the form

$$\begin{array}{ccc} M & \xrightarrow{ev_1 \oplus ev_1 \circ \pi} & N \oplus N \\ \uparrow & & \downarrow{\scriptstyle r_N \oplus r_N} \\ M^{F_{3/2}} & \xrightarrow[\simeq]{} & N_1 \oplus N_1 \end{array}$$

The classes in $K^0 MString$ such that pairing $m \in K_0 MString$ with them equals $\rho_j(r_N(ev_1(m)))$ and $\rho_j(r_N(ev_1(\pi.m)))$ are given by

$$\{\tau_W pr^* q_j, \tau_W \Theta_W^{\pi^{-1}} pr^* \pi^{-1}.q_j\}_{j\in J}$$

where we identified I with $J \coprod J$ in a suitable way. Lifting these as above and projecting to the first half of the J's we get a map

$$\tau_W C : MString \to L_{K(2)} \bigoplus_{j\in J} E$$

such that $C_j = pr_j \circ C \equiv pr^* q_j \bmod \mathfrak{m}$. Expanding the C_j as

$$C_j = \sum_{k \geq 0} (r-1)^k pr^* C_j^{(k)}$$

shows that $C_j^{(0)} \equiv q_j \bmod \mathfrak{m}$.

We now want to show that already the collection $\tau_W pr^* C_j^{(0)}$ satisfies the finiteness condition of Lemma C.4. It suffices to show this on the pro-basis $\alpha_{i,k}$ considered in the proof of Proposition 12.3:

$$\langle \alpha_{i,k}, C_j\rangle = \langle y_i, C_j^{(k)}\rangle.$$

Leaving out the summands $(r-1)^k pr^* C_j^{(k)}$ for $k > 0$ in C_j will result in more of these pairings being zero, but will not change the values of the non-zero pairings. Thus, also the collection $\tau_W pr^* C_j^{(0)}$ satisfies the finiteness condition of Lemma C.4 and assembles to a map we denote

$$\tau_W pr^* C^{(0)} : MString \to L_K \bigoplus_{j \in J} E.$$

We claim that the E^{hK}-linearization of this map induces a $K(2)$-equivalence. In fact, since the extension $E^{hK} \to E^{hF_{3/2}}$ is faithful by [Rog08, prop. 5.4.9(b)], we may show that the E^{hK}-linear map is an equivalence after tensoring up along this extension. Consider the $E^{hF_{3/2}}$-linear equivalence

$$E^{hF_{3/2}} \otimes_{E^{hK}} E \to E \oplus E$$

which is induced by

$$(\mathrm{id}, \psi^{\pi^{-1}}) : E \to E \oplus E.$$

We obtain an $E^{hF_{3/2}}$-linear map

$$E^{hF_{3/2}} \otimes MString \to L_{K(2)} \bigoplus_{j \in J} (E \oplus E)$$

such that the jth projection, restricted to $MString$, is given by

$$\left(\tau_W pr^* C_j^{(0)}, \tau_W \Theta_W^{\pi^{-1}} pr^* \pi^{-1}.C_j^{(0)} \right),$$

so it is a lift of the map resulting from the Milnor-Moore argument applied to the retraction we chose, and thus an equivalence. This shows that we can take $C^{(0)}$ to be the map Q'. □

Having this in our hands we can now descend along the ladder of extensions

$$K \subset \langle K, -1 \rangle \subset \langle K, -1, \omega \rangle \subset \langle K, -1, \omega, \sigma \rangle = H.$$

Theorem 15.6. *There exists a map*

$$Q : BSpin_+ \to L_{K(2)} \bigoplus_{j \in J} \Sigma^{n_j} TMF_0(3),$$

with $n_j \in \{0, 16, 32\}$*, such that the* E^{hH}*-linear map*

$$E^{hH} \otimes MString \xrightarrow{\tau_W pr^* Q} \bigoplus_{j \in J} \Sigma^{n_j} TMF_0(3)$$

is a $K(2)$*-local equivalence.*

Proof. The proof works by observing that, for suitable choices of retraction and basis, the map Q' constructed in Theorem 15.5 can be made to factor as claimed.

Let us begin by choosing a particular retraction $r_N : N \to N_1$ and basis of N_1. Since $\mathbb{F}_4$ contains a third root of unity ζ we can decompose N_1 as

$$N_1 = L_0 \oplus L_8 \oplus L_{16},\ L_i = \{x \in N_1 \mid \omega.x = \zeta^i x\}$$

and similarly for $N = G_0 \oplus G_8 \oplus G_{16}$; the choice of $\{0, 8, 16\}$ instead of $\{0, 1, 2\}$ will allow us to use Lemma 12.8 below. Since the action of σ is $\mathbb{F}_4$-antilinear and we have that $\sigma\omega\sigma = \omega^2$, L_i and G_i are σ-invariant subspaces. Thus, there are canonical equivalences

$$L_i = \mathbb{F}_4 \otimes_{\mathbb{F}_2} L_i^\sigma \text{ and } G_i = \mathbb{F}_4 \otimes_{\mathbb{F}_2} G_i^\sigma.$$

Now choose retractions $r_i : G_i^\sigma \to L_i^\sigma$ and let $r_N = \mathrm{id}_{\mathbb{F}_4} \otimes_{\mathbb{F}_2} (r_0 \oplus r_8 \oplus r_{16})$, so that r_N intertwines the action of ω and σ. Also choose a basis $\{x_j\}_{j \in J}$ of N_1 such that each x_j lies in one of the L_i^σ, say with $i = m_j \in \{0, 8, 16\}$.

Let $\rho_j : N_1 \to \mathbb{F}_4$ be the projections such that

$$\forall x \in N_1 : x = \sum_{j \in J} \rho_j(x) x_j$$

and $q_j \in K^0 BSpin$ the unique class such that

$$\forall x \in N : \rho_j(r_N(x)) = \langle x, q_j \rangle.$$

We have that $\omega.q_j = \zeta^{2m_j} q_j$ and $\sigma.q_j = q_j$, so that the classes $u^{m_j} q_j \in K^{2m_j} BSpin$ are invariant under the action of ω and σ.
In Theorem 15.5 we constructed a map

$$Q' : BSpin_+ \to L_{K(2)} \bigoplus_{j \in J} E$$

such that $pr_j \circ Q' \equiv q_j \mod \mathfrak{m}$. Composing with the direct sum of maps $u^{m_j} : E \to \Sigma^{2m_j} E$ we obtain a map

$$Q'' : BSpin_+ \to L_{K(2)} \bigoplus_{j \in J} \Sigma^{2m_j} E$$

such that $pr_j \circ Q'' = u^{m_j} q_j$. By Lemma 12.8 there now exists a unique map

$$Q''' : BSpin_+ \to L_{K(2)} \bigoplus_{j \in J} \Sigma^{2m_j} ER$$

lifting Q''. Consider the map

$$\frac{Q''' + \omega.Q''' + \omega^2.Q'''}{3} : BSpin_+ \to L_{K(2)} \bigoplus_{j \in J} \Sigma^{2m_j} ER.$$

Since 3 is invertible the homotopy fixed point spectral sequence for the action of ω collapses at the E_2-page and is concentrated on the zeroth line. Thus, the above map, which is ω-invariant by construction, admits a unique lift to

$$Q'''' : BSpin_+ \to L_{K(2)} \bigoplus_{j \in J} \Sigma^{2m_j} E^{h\langle -1, \omega \rangle}.$$

Finally, we have a commutative square

$$\begin{array}{ccc} TMF_0(3) \oplus TMF_0(3) & \xrightarrow{(1,\zeta)} & E^{h\langle -1,\omega \rangle} \\ \downarrow & & \downarrow \\ E^{h\langle \sigma \rangle} \oplus E^{h\langle \sigma \rangle} & \xrightarrow{(1,\zeta)} & E \end{array}$$

where the horizontal maps are equivalences and the vertical maps inclusions of fixed points. Denote by $\alpha : E \to E^{h\langle \sigma \rangle}$ and $\beta : E^{h\langle -1,\omega \rangle} \to TMF_0(3)$ the projections to the first summands in the above diagram, which are retractions of the inclusions $\iota_\alpha : E^{h\langle \sigma \rangle} \to E$ and $\iota_\beta : TMF_0(3) \to E^{h\langle -1,\omega \rangle}$.
Let Q be the composite of Q'''' with a direct sum of shifts of β. We claim that the E^{hH}-linear map

$$E^{hH} \otimes MString \xrightarrow{\tau_W pr^* Q} L_{K(2)} \bigoplus_{j \in J} \Sigma^{2m_j} TMF_0(3)$$

is a $K(2)$-local equivalence. It is enough to show that it is an equivalence after tensoring up along the extension $E^{hH} \to E^{hK}$, which is faithful by [Rog08, prop. 5.4.9(b)]. Note that we have an E^{hK}-linear $K(2)$-local equivalence

$$E^{hK} \otimes_{E^{hH}} TMF_0(3) \xrightarrow{\sim} E$$

which is induced by the inclusion of fixed points $\iota : TMF_0(3) \to E$. So, after tensoring up the above map will be induced by the composite

$$BSpin_+ \xrightarrow{Q} L_{K(2)} \bigoplus_{j \in J} \Sigma^{2m_j} TMF_0(3) \xrightarrow{\iota} L_{K(2)} \bigoplus_{j \in J} \Sigma^{2m_j} E.$$

Tracing through the construction we obtain

$$\begin{aligned} pr_j \circ \iota \circ Q &= \iota_\alpha \circ \alpha \circ \frac{\mathrm{id} + \psi^\omega + \psi^{\omega^2}}{3} \circ u^{m_j} \circ pr_j \circ Q' \\ &\equiv \iota_\alpha \circ \alpha \circ \frac{\mathrm{id} + \psi^\omega + \psi^{\omega^2}}{3} \circ u^{m_j} q_j + \mathfrak{m} \\ &\equiv \iota_\alpha \circ \alpha \circ u^{m_j} q_j + \mathfrak{m} \\ &\equiv u^{m_j} q_j + \mathfrak{m} \end{aligned}$$

where we used in the third step that $u^{m_j} q_j$ is ω-invariant and in the fourth step that it is σ-invariant so that the idempotent $\iota_\alpha \circ \alpha$ acts trivially on it. Thus, after tensoring up and shifting the summands, Q induces the same map as Q' in K_0, thus it induces a $K(2)$-local equivalence. $\square$

16 Determining some projections

In this Section we want to show that the maps

$$MString \xrightarrow{\tau_W} TMF \to E^{hH} \otimes TMF \simeq TMF_0(3)$$

and

$$\begin{aligned} MString \xrightarrow{\tau_W pr^* p} & \Sigma^{16} TMF_0(3) \to E^{hH} \otimes \Sigma^{16} TMF_0(3) \\ & \simeq \Sigma^{16} \left(TMF_0(3) \oplus TMF_0(3) \oplus \Sigma^{32} TMF_0(3) \oplus \Sigma^{16} TMF_0(3) \right) \end{aligned}$$

appear in the splitting of Theorem 15.6, for a suitable choice of Q, where $p = \pi_4/\Delta^6 \in TMF_0(3)^{16} BSpin$. This gives a further hint that tmf and $\Sigma^{16} tmf_0(3)$ should be 2-local summands of $MString$, as already suggested in [MR09, § 7], and proves that the map

$$L_K \left(E^{hH} \otimes MString \right) \xrightarrow{\mathrm{id} \otimes \tau_W} L_K \left(E^{hH} \otimes TMF \right)$$

admits an E^{hH}-linear unital section.

16.1 Calculating and detecting N_1

First, we need to determine some classes in $N_1 \subset K_0 BSpin$, as well as cohomology classes detecting them.

Recall the discussion about k-structure from Section 1.1. The situation can be summarized using the following diagram:

$$\begin{tikzcd}
(C_2F)^\wedge_\mathfrak{m} \arrow[rr] \arrow[dr, "\simeq"] \arrow[dd] & & (C_2^r F)^\wedge_\mathfrak{m} \arrow[dr, "\simeq"] \arrow[dd] & & \\
& E_0^\vee BSU \arrow[rr, two heads] \arrow[dd] & & E_0^\vee BSpin \arrow[dl, dashed, "\simeq"] \arrow[ddr, "pr_*"] & \\
(C_1F)^\wedge_\mathfrak{m} \arrow[rr, two heads] \arrow[dr, "\simeq"] & & (C_1^r F)^\wedge_\mathfrak{m} \arrow[drr, hook] & & \\
& E_0^\vee BU \arrow[ur, dashed, two heads] \arrow[rrr, "\mathrm{re}_*"] & & & E_0^\vee BSO
\end{tikzcd}$$

Moreover, there are unique classes $\widetilde{b_{2i}} \in E_0^\vee BSpin$ such that $pr_* \widetilde{b_{2i}} = \mathrm{re}_* b_{2i}$, and $E_0^\vee BSpin$ is a completed polynomial algebra on these classes. Notice that the map $(C_1^r F)^\wedge_\mathfrak{m} \to E_0^\vee BSO$ is also injective mod $\mathfrak{m}$, which follows directly from the proof of Proposition 2.7 in [KL02].

Proposition 16.1. *Consider the map*

$$U(1) \xrightarrow{z \mapsto 1/z} U(1) = Spin(2) \longrightarrow Spin$$

and denote by $e_i \in E_0^\vee BSpin$ the image of $\beta_i \in E_0^\vee BU(1)$ under it. Then

$$pr_* e_i \equiv \begin{cases} \mathrm{re}_* b_{i/4} + \mathfrak{m} & \text{if } 4|i \\ 0 + \mathfrak{m} & \text{else.} \end{cases}$$

In particular,

$$E_0^\vee BSpin = (E_0[e_8, e_{16}, e_{24}, \ldots])^\wedge_{\mathfrak{m}}.$$

Proof. Denote the map $U(1) \to U(1)$, $z \mapsto z^{-2}$ by g. Since the projection

$$Spin \longrightarrow SO$$

induces the squaring map

$$U(1) = Spin(2) \longrightarrow SO(2) = U(1),\ z \mapsto z^2$$

we find that $pr_* e_i = \mathrm{re}_* g_* \beta_i$. Up to $\mathfrak{m}$ we know that the (-2)-series of the formal group law is given by $[-2](x) = x^4$. We calculate

$$\begin{aligned} f_*\beta_i &= \sum_{j \geq 0} \beta_j \langle f_*\beta_i, z^j \rangle = \sum_{j \geq 0} \beta_j \langle \beta_i, [-2](z)^j \rangle \\ &= \sum_{j \geq 0} \beta_j \langle \beta_i, z^{4j} \rangle = \sum_{j \geq 0} \beta_j \delta_i^{4j} \end{aligned}$$

which shows the first claim. The second now follows since $e_{8i} = \widetilde{b_{2i}} + \mathfrak{m}$. □

Recall that the action of $F'_{3/2}$ we are interested in is the one twisted by the elements $\chi^g \in E^0 BSpin$ such that

$$\Theta_W^g \equiv r^{\frac{|g|-1}{2}} pr^* \chi^g.$$

Proposition 16.2. *The action of elements $g \in \mathbb{S}$ with determinant 1 on the $\widetilde{b_{2i}} \in K_0 BSpin$ is determined by the following formula:*

$$pr_* \left(g^{-1} \triangleright \widetilde{b_{2i}} \right) = \mathrm{re}_* \left(\mathrm{coeff}_{z^{2i}}\, l_g(z) l_b(g(z)) \right).$$

Here l_g are the classes from 12.13, and l_b is the power series

$$l_b(z) = 1 + \sum_{i \geq 1} b_i z^i.$$

Proof. First of all note that, when $|g| = 1$, we have an equality $\chi^g g.\chi^{g^{-1}} = 1$. This allows us to rewrite the action as

$$g^{-1} \triangleright n = g^{-1}.\left(n \frown \chi^g\right).$$

Since $K_0 BSU \to K_0 BSpin$ is surjective, we can pick a class $x \in K_0 BSU$ such that $\mathrm{re}_* x = \widetilde{b_{2i}}$. This class will then, after mapping down to $K_0 BU$, agree with b_{2i} up to elements in the kernel of $\mathrm{re}_* : K_0 BU \to K_0 BSO$. We can now calculate

$$\begin{aligned} pr_* \left(g^{-1} \triangleright \widetilde{b_{2i}} \right) &= pr_* \left(g^{-1}.\left(\mathrm{re}_* x \frown \chi^g\right) \right) = pr_* \mathrm{re}_* \left(g^{-1}.\left(x \frown \mathrm{re}^* \chi^g\right) \right) \\ &= \mathrm{re}_* pr_* \left(g^{-1}.\left(x \frown pr^* l_g\right) \right) = \mathrm{re}_* \left(g^{-1}.\left(pr_* x \frown l_g\right) \right). \end{aligned}$$

Now we observe that both acting with g^{-1} and capping with l_g preserve the kernel of the map

$\mathrm{re}_* : K_0BU \to K_0BSO$, so we may replace x by b_{2i}:

$$pr_*\left(g^{-1} \triangleright \widetilde{b_{2i}}\right) = \mathrm{re}_*\left(g^{-1}.\left(b_{2i} \frown l_g\right)\right).$$

It will be useful to consider a generating function. In the following let

$$f : BU(1) \longrightarrow BU$$

be the usual inclusion map so that $b_i = f_*\beta_i$.

$$\begin{aligned}\sum_{i\geq 0} t^i\left(g^{-1}.\left(b_i \frown l_g\right)\right) &= \sum_{i\geq 0} t^i f_*\left(\sum_{j\geq 0}\beta_j\langle g^{-1}.\left(\beta_i \frown f^*l_g\right), z^j\rangle\right)\\ &= \sum_{i\geq 0} t^i f_*\left(\sum_{j\geq 0}\beta_j\langle \beta_i, l_g(z)g(z)^j\rangle\right)\\ &= \sum_{j\geq 0} b_j l_g(t)g(t)^j = l_g(t)l_b(g(t))\end{aligned}$$

where we used that the action of g on $z \in K^0BU(1)$ is given by the underlying power series of $g \in \mathbb{S} \subset \mathbb{F}_4[\![t]\!]$ by Proposition 3.3. $\square$

By the nature of the action of $F'_{3/2}$ on N the subrings

$$\mathbb{F}_4\left[\widetilde{b_{2i}} \mid 1 \leq i \leq n\right] \subset N$$

are invariant subspaces. With the computation in Proposition 12.14 of the real 1-structures for the generators of $F'_{3/2}$ we can now determine the action on $\mathbb{F}_4\left[\widetilde{b_2}, \widetilde{b_4}, \widetilde{b_6}\right]$:

$\triangleright$	$\widetilde{b_2}$	$\widetilde{b_4}$	$\widetilde{b_6}$
α^{-2}	$\widetilde{b_2}$	$\widetilde{b_4}$	$\widetilde{b_6}+1$
$[i,\alpha]^{-1}$	$\widetilde{b_2}+1$	$\widetilde{b_4}+\widetilde{b_2}+1$	$\widetilde{b_6}+\widetilde{b_4}+\widetilde{b_2}+1$
$[j,\alpha]^{-1}$	$\widetilde{b_2}+\zeta$	$\widetilde{b_4}+\zeta\widetilde{b_2}+\zeta^2$	$\widetilde{b_6}+\zeta\widetilde{b_4}+\zeta^2\widetilde{b_2}+1$

Proposition 16.3. *Let x, y, z denote $\widetilde{b_2}, \widetilde{b_4}, \widetilde{b_6}$. Then the subring of $F'_{3/2}$-invariants in $\mathbb{F}_4[x,y,z]$ is a polynomial $\mathbb{F}_4$-algebra on the elements*

$$\begin{aligned}c =& x^4 + x,\\ d =& y^4 + y + x^5 + x^2, \text{ and}\\ e =& z^2 + z + y^2x^2 + yx + x^6 + x^3.\end{aligned}$$

Proof. The claim that these elements are invariant follows from direct computation, that they are algebraically independent from looking at their leading terms. It remains to show that the above elements do generate all invariant elements.

Let $p = x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots$ be an invariant polynomial only involving x. We want to show that its degree must be a multiple of 4. Acting with $[i,\alpha]$ we get the equation $p_1 = n + p_1$, thus n must be even, say $n = 2m$. Acting with both $[i,\alpha]$ and $[j,\alpha]$ we now find the equations

$$p_2 = m + p_1 + p_2 = \omega^2 m + \omega p_1 + p_2$$

so that also m must be even. Since the degree of any invariant polynomial in x is a multiple of 4 and $c = x^4 + \ldots$ we can write it as a polynomial in c.

Let $p = y^n p_0(x) + y^{n-1}p_1(x) + y^{n-1}p_2(x) + \ldots$ be an invariant polynomial in x and y. We want to show that p_0 is invariant and that n is a multiple of 4. Acting with $[i,\alpha]$ and $[j,\alpha]$ and comparing coefficients of y we find that p_0 must be invariant, and we get the equation

$$p_1(x) = p_1(x+1) + n(x+1)p_0(x+1).$$

Using that p_0 is already known to be invariant one finds

$$n(x+1)p_0(x) = p_1(x) + p_1(x+1) = nxp_0(x)$$

showing that $n = 2m$ must be even, and that p_1 must be invariant as well.
Assuming m is odd and looking at the coefficient of y^{n-2} a similar trick gives $p_0 = p_1$ and $p_0 = \zeta p_1$, so that $p_0 = 0$, a contradiction. Thus, p is a polynomial in c and d.
Finally, let $p = z^n p_0(y,x) + z^{n-1}p_1(y,x) + \dots$ be invariant. We want to show that n is even and that p_0 is invariant. Acting with α^2 and looking at the coefficient of z^{n-1} we get the equation

$$p_1 = p_1 + np_0$$

so that n must be even. Invariance of p_0 again follows from looking at leading coefficients after acting with $[i,\alpha]$ and $[j,\alpha]$. □

Recall from Proposition 12.6 that, restricted to $Spin(2) = U(1)$, we have

$$\sum_{k\geq 0} t^k p_k = 1 - tu^2[2](z_E)[-2](z_E)$$

since ρ is a double covering.

Proposition 16.4. *The only non-zero pairings between the polynomial generators $\widetilde{b_{2i}}$ and the images of the Pontryagin classes p_j under the projection $E^*BSpin \to K^*BSpin$ (which we will also denote by p_j) are*

$$\langle \widetilde{b_0}, p_0\rangle = \langle \widetilde{b_2}, u^{-2}p_1\rangle = 1.$$

Proof. The first pairing is simply the pairing $\langle 1,1\rangle = 1$. For the second pairing we have

$$\begin{aligned}\langle \widetilde{b_{2i}}, u^{-2}p_1\rangle &= \langle \beta_{8i}, [2](z)[-2](z)\rangle \\ &= \langle \beta_{8i}, z^4 \sum_{k\geq 0} \left(z^4\right)^{2^k 3-2}\rangle \\ &= \delta_{i,1}\end{aligned}$$

since the only multiple-of-eight power on the right is z^8. All other pairings are zero for degree reasons. □

Proposition 16.5. *Pairing with the classes $1, u^{-8}p_4, u^{-8}i.p_4, u^{-8}j.p_4, u^{-8}k.p_4$ induces an isomorphism*

$$\langle 1, cd^3, i.(cd^3), j.(cd^3), k.(cd^3)\rangle \longrightarrow (\mathbb{F}_4)^5.$$

Proof. Direct computation shows that the five mentioned classes are mapped to a basis of $(\mathbb{F}_4)^5$. □

Remark 16.6. Identifying $N_1 \oplus N_1$ with the fixed points $(K_0 MString)^{F_{3/2}}$ as in Proposition 15.3 we get a residual action of $\mathbb{G}/F_{3/2}$ on it. Since the element $\pi \in \mathbb{G}$ swaps the two summands and $-1 \in \mathbb{G}$ acts trivially we find that we are really looking at an action of $G_{48}/\{\pm 1\}$ on N_1. For an element $g \in G_{48} \subset \mathbb{G}$ the cannibalistic class Θ^g_W equals 1 and the determinant equals 1, showing that this residual action agrees with the natural action on $K_0 BSpin$. That is also why we used . to denote the action instead of $\triangleright$ in the above proposition.

16.2 Manipulating Q

By Lemma C.4 and Lemma 12.9 it is relatively easy to determine finitely many of the $Q_j = pr_j \circ Q$. This we will do with the following strategy:

Construction 16.7. Let $V \subset N_1$ be a k-dimensional subspace, and let c_i be classes in K^0BSpin such that

$$\epsilon : N \xrightarrow{x \mapsto \langle x, c_i \rangle} \bigoplus_{i=1}^{k} \mathbb{F}_4$$

is an isomorphism when restricted to V. Then we have natural isomorphisms

$$N = V \oplus \ker \epsilon, \; N_1 = V \oplus \ker \epsilon|_{N_1}$$

and we may choose a retraction of the form $r_N = \mathrm{id}_V \oplus r_{\ker}$. It follows that, if we choose a basis of N_1 as a basis of V dual to the c_i plus a basis of $\ker \epsilon|_{N_1}$, the Q_j constructed in Theorem 15.6 associated to the basis elements of V are lifts of the c_i, and since they are finitely many[a] we may replace them by any other lifts of the c_i. Also note that the other Q_j must pair to 0 mod $\mathfrak{m}$ with all elements in V. Of course, in all of this we must make sure to satisfy the invariance conditions on r_N and the basis of N_1 we used in the proof of Theorem 15.6. This will be the case whenever the classes c_i are σ-invariant and satisfy $\omega.c_i = \zeta^{2m_i} c_i$.

[a]For infinitely many we would need a version of Lemma C.4 for $TMF_0(3)$ and be able to verify the corresponding finiteness conditions.

This turns determining (finitely many of) the Q_j into a game of listing elements in N_1 and detecting them with suitable classes in $TMF_0(3)^*BSpin$.

Proposition 16.8. *In the setting of Theorem 15.6 we may choose one of the Q_j to be $1 \in TMF_0(3)^0BSpin$, and the other Q_js to be reduced classes.*

Proof. Since the Morava stabilizer group acts on E by ring maps, it fixes the unit

$$1 \in M^{F_{3/2}} \subset K_0MString.$$

In particular, as $1 = pr_*1$, we also have that $1 \in N_1 \subset K_0BSpin$, which we may detect with $1 \in TMF_0(3)^0BSpin$. By the discussion in Construction 16.7 we get from (the proof of) Theorem 15.6 an E^{hH} linear equivalence

$$E^{hH} \otimes MString \to \bigoplus_{j \in J} \Sigma^{n_j} TMF_0(3)$$

such that, projecting onto the summand corresponding to the unit, we are looking at the map

$$E^{hH} \otimes MString \xrightarrow{\mathrm{id} \otimes \tau_W} E^{hH} \otimes TMF = TMF_0(3)$$

which we identify with the Ando-Hopkins-Rezk orientation $MString \to TMF$ tensored with the identity on E^{hH}. This already shows that, after tensoring with E^{hH}, the Ando-Hopkins-Rezk orientation admits an E^{hH}-linear section, by including the summand back into the sum and applying the inverse equivalence.
By Lemma 12.9 the section is then unital if and only if all the other Q_j are reduced classes. Looking at the process we went through to extract Q out of Q' we see that it is enough to make sure that the other Q'_j are reduced classes. This can be arranged: give the 'first' summand the index $0 \in J$. Then the collection $\{Q'_j|_{pt} \in E^0\}_{j \neq 0}$ also satisfies the finiteness condition of Lemma C.4, and so does the collection $\{Q'_j - Q'_j|_{pt}\}_{j \neq 0}$. So we may replace all other Q'_j by their reduced counterparts. Since we have chosen $q_0 = 1 \in K^0BSpin$ the construction of the discussion above tells us that the other q_j are reduced, so the reductions of the other Q'_j are still lifts of the q_j. $\square$

We now want to discuss what summands of $\Sigma^{2m}TMF_0(3)$ in $MString$ would look like in this splitting. The inclusion map $\Sigma^{2m}TMF_0(3) \to MString$ would map the unit of $TMF_0(3)$ to a class $a \in N_1$ such that $\omega.a = \zeta^m a$, $\sigma.a = a$, and such that $\{a, i.a, j.a, k.a\}$ are linearly independent.

Changing the basis a bit and denoting

$$x = i + j + k, y = i + \zeta j + \zeta^2 k, z = i + \zeta^2 j + \zeta k \in \mathbb{F}_4[Q_8/\{\pm 1\}]$$

this is equivalent to asking for $\{a, x.a, y.a, z.a\}$ to be linearly independent. This choice is convenient since, in the terminology of the proof of Theorem 15.6, we then have that

$$a \in L^\sigma_m, x.a \in L^\sigma_m, y.a \in L^\sigma_{m+2}, \text{ and } z.a \in L^\sigma_{m+1}.$$

On the other hand, after tensoring with E^{hH}, $TMF_0(3)$ splits as four copies of $TMF_0(3)$:

$$E^{hH} \otimes TMF_0(3) \xrightarrow{\sim} TMF_0(3) \oplus TMF_0(3) \oplus \Sigma^{32} TMF_0(3) \oplus \Sigma^{16} TMF_0(3)$$

where the equivalence is E^{hH}-linear and the projection maps are, after restricting to $TMF_0(3)$ in the source and including as fixed points into E in the target, given by

$$\iota, (\psi^i + \psi^j + \psi^k) \circ \iota, u^{16}(\psi^i + \zeta\psi^j + \zeta^2\psi^k) \circ \iota, \text{ and } u^8(\psi^i + \zeta^2\psi^j + \zeta\psi^k) \circ \iota.$$

Looking for such summands corresponds to looking for classes $a \in L^\sigma_m$ and $p \in TMF_0(3)^{2m} BSpin$ such that pairing the classes $\{a, x.a, y.a, z.a\}$ with the classes $\{p, x.p, y.p, z.p\}$ gives an invertible matrix (where we abused notation to identify p with its image in $K^0 BSpin$). A little computation shows that the determinant of the matrix is given by

$$\det\left(\langle \{a, x.a, y.a, z.a\}, \{p, x.p, y.p, z.p\}\rangle\right) = \langle a, p\rangle + \langle x.a, p\rangle.$$

Proposition 16.9. *In the setting of Theorem 15.6 one can choose Q such that the map*

$$BSpin \xrightarrow{1 \oplus p} TMF \oplus \Sigma^{16} TMF_0(3)$$

composed with the unit $\mathbb{S} \to E^{hH}$ appears in the splitting, where

$$p = \pi_4/\Delta^6 \in TMF_0(3)^{16} BSpin$$

and we implicitly fix an E^{hH}-linear splitting of $E^{hH} \otimes (TMF \oplus \Sigma^{16} TMF_0(3))$ as a sum of shifts of $TMF_0(3)$. Thus, the map

$$E^{hH} \otimes MString \xrightarrow{\tau_W \oplus \tau_W pr^* p} E^{hH} \otimes (TMF \oplus \Sigma^{16} TMF_0(3))$$

admits an E^{hH}-linear section, and restricted to $E^{hH} \otimes TMF$ this section can be made unital.

Proof. With the strategy described in Construction 16.7, consider the subspace

$$V = \operatorname{span}\{1, cd^3, i.(cd^3), j.(cd^3), k.(cd^3)\} \subset N_1$$

and the detecting classes $1, p, x.p, y.p, z.p$. The resulting map

$$Q : BSpin_+ \to L_{K(2)} \bigoplus_{j \in J} \Sigma^{n_j} TMF_0(3)$$

will then have $Q_0, \ldots, Q_4$ as lifts of $1, p, x.p, y.p, z.p$, and by the discussion above we may replace them by any other lifts. By the discussion about the splitting of $E^{hH} \otimes TMF_0(3)$ and Proposition 16.8 we thus see that we may choose these as the composite

$$\begin{aligned} BSpin_+ &\xrightarrow{1 \oplus \pi_4/\Delta^6} TMF \oplus \Sigma^{16} TMF_0(3) \to E^{hH} \otimes \left(TMF \oplus \Sigma^{16} TMF_0(3)\right) \\ &\xrightarrow{\sim} TMF_0(3) \oplus \Sigma^{16} TMF_0(3) \oplus \Sigma^{16} TMF_0(3) \oplus \Sigma^{48} TMF_0(3) \oplus \Sigma^{32} TMF_0(3). \end{aligned}$$

We then, by the same argument as in the proof of Proposition 16.8, make sure that all classes but Q_0 are reduced, which does not affect the classes coming from π_4/Δ^6 since they are already reduced. ∎

Appendix

A The Milnor-Moore argument

In this Section we recall some basic definitions around coalgebras and their comodules, and prove a slight strengthening of the Minor-Moore type argument introduced in [LS19].

Let $\mathbb{F}$ be a field, C be a coalgebra over $\mathbb{F}$, and M a (left) C-comodule.

Definition A.1. The *coradical* $F_0(C)$ of C is defined to be the sum of all simple subcoalgebras of C. We say that C is *pointed* if each simple subcoalgebra is one dimensional over $\mathbb{F}$.
The *coradical filtration* $F_n(C)$ of C is inductively defined by

$$F_{n+1}(C) := \Delta^{-1}\left(F_n(C) \otimes C + C \otimes F_0(C)\right).$$

This also induces a natural filtration of the comodule M:

$$F_n(M) := \psi^{-1}\left(F_n(C) \otimes M\right).$$

Proposition A.2 ([Swe69, cor. 9.0.4, cor. 9.1.7],)**.** *The coradical filtration is exhaustive, that is*

$$C = \bigcup_{n \geq 0} F_n(C).$$

Moreover, this makes C into a filtered coalgebra, that is

$$\Delta(F_n(C)) \subset \sum_{k=0}^{n} F_{n-k}(C) \otimes F_k(C).$$

Remark A.3. This also shows that the coradical filtration of C coincides with the natural filtration as a comodule over itself.

Before we proceed, we will need to recall some constructions around the *dual algebra* of C.

Construction A.4. Let $C^\vee = \mathrm{Hom}_{\mathbb{F}}(C, \mathbb{F})$ and equip it with the $\mathbb{F}$-algebra structure

$$\phi \star \psi(c) := (\phi \otimes \psi)(\Delta(c)),\ 1 := \epsilon.$$

We also equip it with the topology described in [Die73, § I.2]. For linear subspaces $L \subset C$ and $\Phi \subset C^\vee$, we call

$$L^\perp = \{\phi \in C^\vee \mid \phi|_L = 0\} \text{ and } \Phi^\perp = \{c \in C \mid ev_c|_\Phi = 0\}$$

the *annihilator spaces* of L and Φ. Then $L^\perp$ is closed, $L = (L^\perp)^\perp$, and $(\Phi^\perp)^\perp$ is the closure of Φ in $C^\vee$. This gives an inclusion reversing bijection between subspaces of C and closed subspaces of $C^\vee$, and restricts to an inclusion reversing bijection between subcoalgebras and ideals.
Note that a maximal closed ideal is a maximal ideal in $C^\vee$: if I contains a maximal closed ideal $\mathfrak{m}$, then $C^\vee/I$ is a quotient of $C^\vee/\mathfrak{m}$. Per definition the projection $C^\vee \to C^\vee/\mathfrak{m}$ is continuous for the discrete topology on the target. Thus, also the projection $C^\vee \to C^\vee/I$ is continuous, showing that I is closed.
Let $J \subset C^\vee$ be the Jacobson radical, which is the intersection of all maximal closed ideals by the previous paragraph, so itself and all its powers are closed. We also have that $F_n(C) = (J^{n+1})^\perp$.
Consider the right action of $C^\vee$ on M given by

$$m.\phi := (\phi \otimes \mathrm{id})(\psi(m)).$$

We can then identify $F_n(M)$ with the J^{n+1}-torsion submodule $M[J^{n+1}]$.

Lemma A.5. *Let M be a C-comodule and $m \in F_n(M)$. Then*

$$\psi(m) \in \sum_{k=0}^{n} F_{n-k}(C) \otimes F_k(M).$$

Proof. Let $\{c_i^n \in F_n(C)\}_{i \in I_n, n \geq 0}$ be such that $\{c_i^k \mid k \leq n\}$ is a basis of $F_n(C)$, and similarly $\{m_j^n \in F_n(M)\}_{j \in J_n, n \geq 0}$. Let also ϕ_i^n be the linear forms dual to the c_i^n. In particular, since ϕ_i^n vanishes on $F_{n-1}(C)$, it lies in $J^n \subset C^\vee$. Write $\psi(m)$ as

$$\psi(m) = \sum_{a,i,b,j} D(m,a,i,b,j) c_i^a \otimes m_j^b.$$

This shows that

$$m.\phi_i^a = \sum_{b,j} D(m,a,i,b,j) m_j^b.$$

As m was J^{n+1}-torsion by assumption, we must have that $m.\phi_i^a$ is J^{n+1-a}-torsion, so the coefficients $D(m,a,i,b,j)$ with $a+b > n$ must vanish, and we are done. □

From now on, let C be a Hopf algebra over $\mathbb{F}$, which, as a coalgebra, is pointed. Let $G \subset C$ be the subset of group-like elements of C. Then the multiplication of C equips this with the structure of a group, and pointedness implies that $F_0(C) \simeq \mathbb{F}[G]$ as Hopf algebras.

Definition A.6. The primitive elements of M are defined as

$$P_1(M) := \{m \in M \mid \psi(m) = 1 \otimes M\}.$$

Lemma A.7. *Let M be an $F_0(C) = \mathbb{F}[G]$-module in C-comodules. Then we have that the map*

$$F_0(C) \otimes P_1(M) \to F_0(M),\ g \otimes m \mapsto g.m$$

is an equivalence.

Proof. We explicitly construct an inverse: for $m \in F_0(M)$ write $\psi(m) = \sum_g g \otimes m_g$. By coassociativity we have that $\psi(m_g) = g \otimes m_g$, so that $g^{-1}.m_g \in P_1(M)$. The inverse then maps

$$m \mapsto \sum_g g \otimes g^{-1}.m_g.$$

□

Definition A.8. Let M be an $\mathbb{F}[G]$-module in C-comodules. Consider the natural maps

$$q_n : F_n(M)/F_{n-1}(M) \to F_n(C)/F_{n-1}(C) \otimes P_1(M),\ [m] \mapsto \sum_{i,g} [c_i] \otimes g^{-1}.m_{i,g}$$

for $\psi(m) \equiv \sum_i c_i \otimes m_i \bmod F_{n-1}(C) \otimes F_n(M)$ with $c_i \in F_n(C)$ and $m_i \in F_0(M)$. We call M *splittable* if these are surjective for all $n \geq 0$.

Lemma A.9. *Let M be an $\mathbb{F}[G]$-module in C-comodules. Then the natural maps*

$$q_n : F_n(M)/F_{n-1}(M) \to F_n(C)/F_{n-1}(C) \otimes P_1(M)$$

are injective for all $n \geq 0$.

Proof. The case $n = 0$ follows from Lemma A.12. For $n \geq 1$, consider $m \in F_n(M)$ and write $\psi(m) \equiv \sum_{i,g} c_i \otimes m_{i,g} \mod F_{n-1}(C) \otimes F_n(M)$ and $\Delta(c_i) \equiv \sum_g c_{i,g} \otimes g \mod F_{n-1}(C) \otimes F_n(C)$. By coassociativity the elements $c_{i,g}$ fulfill

$$\Delta(c_{i,g}) \equiv c_{i,g} \otimes g \mod F_{n-1}(C) \otimes F_n(C).$$

Again by coassociativity we find that

$$\sum_{i,g} c_i \otimes g \otimes m_{i,g} \equiv \sum_{i,g,h} c_{i,h} \otimes h \otimes m_{i,g} \mod F_{n-1}(C) \otimes F_0(C) \otimes F_0(M).$$

Projecting to the terms of the form $A \otimes h \otimes B$ for a fixed $h \in G$ we get

$$\forall h \in G : \sum_i c_i \otimes m_{i,h} \equiv \sum_{i,g} c_{i,h} \otimes m_{i,g} \mod F_{n-1}(C) \otimes F_0(M).$$

By the equivalence of Lemma A.12 this implies

$$\circledast \quad \forall h \in G : \sum_i c_i \otimes m_{i,h} \equiv \sum_i c_{i,h} \otimes m_{i,h} \mod F_{n-1}(C) \otimes F_0(M).$$

Now assume that

$$q_n(m) = \sum_{i,g} [c_{i,g}] \otimes g^{-1}.m_{i,g} = 0 \iff \sum_{i,g} c_{i,g} \otimes g^{-1}.m_{i,g} \in F_{n-1}(C) \otimes P_1(M).$$

Applying $\Delta \otimes \mathrm{id}$ this gives

$$\sum_{i,g} c_{i,g} \otimes g \otimes g^{-1}.m_{i,g} \in F_{n-1}(C) \otimes F_0(C) \otimes P_1(M).$$

Now Lemma A.12 implies that

$$\sum_{i,g} c_{i,g} \otimes m_{i,g} \in F_{n-1}(C) \otimes F_0(M).$$

Using the congruence $\circledast$ this shows

$$\psi(m) \equiv \sum_i c_i \otimes m_i \equiv \sum_{i,g} c_{i,g} \otimes m_{i,g} \equiv 0 \mod F_{n-1}(C) \otimes F_n(M).$$

Thus m is already in $F_{n-1}(M)$, and we are done. □

We now prove a slight strengthening of the Milnor-Moore type argument introduced in [LS19].

Theorem A.10. *Let C be a Hopf algebra over $\mathbb{F}$ which, as a coalgebra, is pointed, and let $G \subset C$ be the set of group-like elements. Let M be an $\mathbb{F}[G]$-module in C-comodules which is splittable. Then, for any $\mathbb{F}$-linear retraction $r : M \to \mathbb{F}[G] \otimes P_1(M)$ of the natural map $\mathbb{F}[G] \otimes P_1(M) \simeq F_0(M) \subset M$, the map*

$$h_r : M \xrightarrow{\psi} C \otimes M \xrightarrow{\mathrm{id} \otimes r} C \otimes \mathbb{F}[G] \otimes P_1(M) \xrightarrow{\mathrm{id} \otimes \epsilon \otimes \mathrm{id}} C \otimes P_1(M)$$

is an isomorphism of C-comodules and exhibits M as a (co)free $\mathbb{F}[G]$-module in C-comodules.

Proof. First of all, h_r is a map of comodules since already $\psi : M \to C \otimes M$ is a map of comodules when we give the target the comodule structure $\Delta \otimes \mathrm{id}$.
Since the coradical filtration of C, and thus also the natural comodule filtration induced by it, is exhaustive, it suffices to show that $F_n(h_r)$ is an isomorphism for all $n \geq 0$. We will do so by induction in n.
For $n = 0$ the map $F_0(h_r)$ equals the map considered in Lemma A.12 since r is a retraction, so it

is an isomorphism.
Now assume that we have already established that $F_n(h_r)$ is an isomorphism. Consider the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & F_n(M) \arrow[r] \arrow[d, "F_n(h_r)"'] & F_{n+1}(M) \arrow[r] \arrow[d, "F_{n+1}(h_r)"'] & F_{n+1}(M)/F_n(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_n(C) \otimes P_1(M) \arrow[r] & F_{n+1}(C) \otimes P_1(M) \arrow[r] & F_{n+1}(C)/F_n(C) \otimes P_1(M) \arrow[r] & 0
\end{tikzcd}$$

We claim that the right vertical map is exactly the natural map q_{n+1}. Indeed, consider an element $m \in F_{n+1}(m)$ and write $\psi(m) \equiv \sum c_i \otimes m_i \bmod F_n(C) \otimes F_{n+1}(M)$. Then $h_r(m)$ is given by

$$h_r(m) \equiv \sum_{i,g} c_i \otimes g^{-1}.m_{i,g} \bmod F_n(C) \otimes P_1(M)$$

since r is a retraction. So the image of m in the bottom right corner is

$$\sum_{i,g} [c_i] \otimes g^{-1}.m_{i,g} = q_{n+1}([m]).$$

Now by Lemma A.14 and the assumption that M is splittable q_{n+1} is an isomorphism, so that also $F_{n+1}(h_r)$ is an isomorphism. $\square$

Let us now connect this with the version stated in [LS19].

Definition A.11. Let M be an $\mathbb{F}[G]$-algbra in C-comodules. We define, for $n \geq 0$, the *n-th graded right primitives* of M as

$$\overline{P}_1 Gr_n(M) := \{m \in F_n(M) \mid \psi(m) \equiv c \otimes 1 \bmod F_{n-1}(M) \otimes F_n(M)\}.$$

We call a map of $\mathbb{F}[G]$-algebras in C-comodules $A : M \to N$ *$\star$-surjective* if it induces surjections on all graded right primitives.

Proposition A.12. *Let M be an $\mathbb{F}[G]$-algebra in C-comodules which admits a $\star$-surjective map $A : M \to C$. Then M is splittable.*

Proof. We need to prove that the natural maps

$$q_n : F_n(M)/F_{n-1}(M) \to F_n(C)/F_{n-1}(C) \otimes P_1(M)$$

are surjective. Since every element $[c] \in F_n(C)/F_{n-1}(C)$ can be written as a sum $[c] = \sum_g [c_g]$ with $\Delta(c_g) \equiv c_g \otimes g \bmod F_{n-1}(C) \otimes F_n(C)$ we only need to show that elements of the form $[c_g] \otimes m$ are in the image. By assumption, $c_g g^{-1}$ is an n-th graded right primitive of C. By $\star$-surjectivity there exists an n-th graded right primitive a of M such that $A(a) = c_g g^{-1}$. Since A is a map of comodules, we find that $\psi(a) \equiv c_g g^{-1} \otimes 1 \bmod F_{n-1}(C) \otimes F_n(M)$. Then, using the algebra structure of M, we calculate $\psi(agm) \equiv c_g \otimes gm \bmod F_{n-1}(C) \otimes F_n(M)$. This shows that $q_n(agm) = [c_g] \otimes m$ and we are done. $\square$

Finally, let us state another criterion for $\mathbb{F}[G]$-algebras in C-comodules to be splittable

Proposition A.13. *Let M be an $\mathbb{F}[G]$-algebra in C-comodules, and assume there exists a map $s : C \to M$ of comodules with $s(1) = 1$. Then M is splittable.*

Proof. We need to prove that the natural maps

$$q_n : F_n(M)/F_{n-1}(M) \to F_n(C)/F_{n-1}(C) \otimes P_1(M)$$

are surjective. Since every element $[c] \in F_n(C)/F_{n-1}(C)$ can be written as a sum $[c] = \sum_g [c_g]$ with $\Delta(c_g) \equiv c_g \otimes g \bmod F_{n-1}(C) \otimes F_n(C)$, we only need to show that elements of the form

$[c_g] \otimes m$ are in the image. Since s is a map of comodules with $s(1) = 1$ we have

$$\psi(s(c_g g^{-1})) = (\mathrm{id} \otimes s)(\Delta(c_g g^{-1})) \equiv (\mathrm{id} \otimes s)(c_g g^{-1} \otimes 1) \equiv s(c_g g^{-1}) \otimes 1 \bmod F_{n-1}(C) \otimes F_n(M).$$

Using the algebras structure of M we calculate

$$\psi(s(c_g g^{-1})gm) \equiv c_g \otimes gm \bmod F_{n-1}(C) \otimes F_n(M).$$

This shows that $q_n([s(c_g g^{-1})gm]) = [c_g] \otimes m$ and we are done. □

Remark A.14. In the other direction, assume that M is an $\mathbb{F}[G]$-algebra in C-comodules and splittable. Then by Theorem A.9 we have $h_r : M \simeq C \otimes P_1(M)$. Since $1 \in P_1(M)$ the map

$$s_r : C \to M,\ c \mapsto h_r^{-1}(c \otimes 1)$$

gives a map as in the above lemma.

B Iwasawa (co)algebras and their (co)modules

Let $\mathbb{F}$ be a finite field of characteristic p and cardinality $q = p^n$, and G a compact p-adic analytic group.

Definition B.1. The *Iwasawa algebra* $\mathbb{F}[\![G]\!]$ of G over $\mathbb{F}$ is defined to be the limit

$$\mathbb{F}[\![G]\!] := \lim_U \mathbb{F}[G/U]$$

over all open normal subgroups U of G. We equip it with the limit topology by giving the terms $\mathbb{F}[G/U]$ the discrete topology.
The associated *Iwasawa coalgebra* is defined to be the colimit

$$C^0(G, \mathbb{F}) := \operatorname*{colim}_U \mathrm{Set}(G/U, \mathbb{F})$$

over all open normal subgroups U of G.

Remark B.2. As G is compact p-adic analytic, a subgroup of G is open iff it is of finite index in G, see Theorem 2.5.

Lemma B.3. *An ideal of $\mathbb{F}[\![G]\!]$ is open iff it is of finite codimension.*

Proof. First, assume that I is open. By definition that means that I contains the kernel of $\mathbb{F}[\![G]\!] \to \mathbb{F}[G/U]$ for some open normal subgroup $U \subset G$. As G/U is finite this augmentation ideal is already of finite codimension, so I must be of finite codimension, too.
In the other direction, assume that I has finite codimension, and denote the quotient algebra by A. Since $\mathbb{F}$ is finite and A is finite dimensional over $\mathbb{F}$, the group of units $A^\times$ is finite. Now the kernel U of the induced $G \to A^\times$ is normal of finite index in G, and thus is open (since G is p-adic analytic). Thus, I contains the kernel of $\mathbb{F}[\![G]\!] \to \mathbb{F}[G/U]$, which is an open ideal, thus I must be open itself. □

Lemma B.4. *There are natural isomorphisms*

$$C^0(G, \mathbb{F})^\vee \simeq \mathbb{F}[\![G]\!] \text{ and } \mathbb{F}[\![G]\!]^\circ \simeq C^0(G, \mathbb{F})$$

of (co)algebras, where $\vee$ denotes the $\mathbb{F}$-linear dual and $\circ$ the restricted dual.

Proof. For the first statement we calculate

$$C^0(G, \mathbb{F})^\vee = \left(\operatorname*{colim}_U C^0(G/U, \mathbb{F})\right)^\vee = \lim_U C^0(G/U, \mathbb{F})^\vee = \lim_U \mathbb{F}[G/U] = \mathbb{F}[\![G]\!].$$

For the second statement, recall that for an $\mathbb{F}$-algebra R the restricted dual is defined to be the subspace $R^\circ \subset R^\vee$ on those linear functionals whose kernel contains an ideal of finite codimension. Since those are exactly the open ideals, we can identify the restricted dual with the continuous dual, which is $C^0(G, \mathbb{F})$. $\square$

Lemma B.5. *Let $I \subset \mathbb{F}[\![G]\!]$ be an ideal of finite codimension. Then the annihilator $I^\perp \subset C^0(G, \mathbb{F})$ can be identified with $(\mathbb{F}[\![G]\!]/I)^\vee$.*

Proof. An element $f \in C^0(G, \mathbb{F})$ lies in $I^\perp$ iff evaluation at f is zero for all elements in I. Since I is of finite codimension, the inclusion of $(\mathbb{F}[\![G]\!]/I)^\vee$ into the double dual of $C^0(G, \mathbb{F})$ factors over $C^0(G, \mathbb{F})$. Since their equality is evident after including both into the double dual, we are done. $\square$

Lemma B.6. *Let G have the form $S \rtimes \mathbb{F}^\times$ with S an open normal pro-p group. Then the Jacobson radical of $\mathbb{F}[\![G]\!]$ is the augmentation ideal*

$$J = I_S^G = \ker \mathbb{F}[\![G]\!] \to \mathbb{F}[\mathbb{F}^\times]$$

and $\mathbb{F}[\![G]\!]$ is complete with respect to J.

Proof. From [NSW08, prop. 5.2.16(iii)] we know that $\mathbb{F}[\![S]\!]$ is local with maximal ideal the augmentation ideal I_S^S. By [Pas89, thm. 4.2] we then get that the Jacobson radical J of $\mathbb{F}[\![G]\!]$ is $I_S^S \star \mathbb{F}^\times = I_S^G$.
Since the coradical filtration on $C^0(G, \mathbb{F})$ is exhaustive, we have that

$$\mathbb{F}[\![G]\!] = C^0(G, \mathbb{F})^\vee = \lim_n \left(F_n(C^0(G, \mathbb{F}))\right)^\vee = \lim_n C^0(G, \mathbb{F})^\vee / J^{n+1},$$

where we used the identification $F_n(C)^\perp = J^{n+1}$ from Construction A.4. Thus, $\mathbb{F}[\![G]\!]$ is complete with respect to J $\square$

Lemma B.7. *Let G have the form $S \rtimes \mathbb{F}^\times$ with S a pro-p group. Then the Iwasawa coalgebra $C^0(G, \mathbb{F})$ is pointed, and the subcoalgebra $F_0(C^0(G, \mathbb{F}))$ is given by $C^0(\mathbb{F}^\times, \mathbb{F})$.*

Proof. Let $C \subset C^0(G, \mathbb{F})$ be a simple subcoalgebra. Dualizing this we get a simple quotient $\mathbb{F}[\![G]\!] \to C^\vee$, meaning that the kernel is a maximal ideal. It thus contains the Jacobson radical, so we see that $C^\vee$ is also a simple quotient of $\mathbb{F}[\![G]\!]/J = \mathbb{F}[\mathbb{F}^\times]$. Since $\mathbb{F}^\times \simeq C_{q-1}$ and the polynomial $x^{q-1} - 1$ splits into linear factors over $\mathbb{F}$, we see that we must have an isomorphism $\mathbb{F} \simeq C^\vee$. Thus, C is one dimensional over $\mathbb{F}$, showing that $C^0(G, \mathbb{F})$ is pointed.
The group-like elements are exactly the group homomorphisms $G \to \mathbb{F}^\times$. Since S is a pro-p group and $\mathbb{F}^\times \simeq C_{q-1}$ every such homomorphism must factor over $G/S \simeq \mathbb{F}^\times$. $\square$

The categories of comodules of the Iwasawa coalgebra and modules of the Iwasawa algebra are closely linked:

Proposition B.8. *The category of (left) $C^0(G, \mathbb{F})$-comodules is equivalent to the category of discrete (left) $\mathbb{F}[\![G]\!]$-modules.*

Proof. Given a comodule M we define an action of G on M by

$$G \times M \to M,\ (g, m) \mapsto (ev_{g^{-1}} \otimes \mathrm{id})(\psi(m)).$$

That this defines a discrete action follows from writing

$$\psi(m) = \sum_{k=1}^{N} f_k \otimes m_k$$

and observing that, since each f_k is continuous, they must all evaluate to 1 on some open normal subgroup U_k. Thus, the stabilizer of m must contain the intersection of the U_k and thus must be open itself.
In the other direction, assume that M is a discrete module and define a coaction by

$$M \mapsto C^0(G, M) \simeq C^0(G, \mathbb{F}) \otimes M,\ m \mapsto (g \mapsto g^{-1}.m).$$

It is evident that these define functors and implement an equivalence between the categories. □

Lemma B.9. *Under the equivalence of Proposition B.8, we have an equality*

$$P_1(M) = M^G$$

of the primitives of M as a $C^0(G, \mathbb{F})$-comodule and the invariants of M as an $\mathbb{F}[\![G]\!]$-module.

Proof. This follows directly from Definition A.5 of the primitive elements of M and the formula for the action of G on M from Proposition B.8. □

C Continuous duality in Morava E-theory

Throughout, let E be a Morava E-theory associated to an adapted formal group (see Definition 3.4), and let $K = E/\mathfrak{m}$ be the associated Morava K-theory. We make this assumption since then all of the machinery developed in [HS99b] applies rather straightforwardly.

Lemma C.1. *Let X be a spectrum such that K_*X is concentrated in even degrees. Then the evaluation maps induce an equivalence*

$$E^0X \xrightarrow{\sim} Hom^{\text{cont.}}_{E_0}\left(E_0^\vee X, E_0\right) = Hom_{E_0}\left(E_0^\vee X, E_0\right),\ f \mapsto (x \mapsto \langle x, f\rangle)$$

where $E_0^\vee X$ and E_0 are given the 'natural topology' introduced in Section 11 of [HS99b].

Proof. Since $E_*^\vee X$ is even and pro-free we get that the evaluations assemble to a bijection

$$E^0X \xrightarrow{\sim} Hom_{E_0}\left(E_0^\vee X, E_0\right).$$

It follows from 11.1-11.4 in [HS99b] that the above map factors through the inclusion of the continuous linear maps, which also shows that all E_0-linear maps from $E_0^\vee X$ to E_0 are continuous. □

Lemma C.2. *Let X be a spectrum such that K_*X is concentrated in even degrees. Then the evaluation maps induce an equivalence*

$$E_0^\vee X \xrightarrow{\sim} Hom^{\text{cont.}}_{E_0}\left(E^0X, E_0\right),\ x \mapsto (f \mapsto \langle x, f\rangle).$$

Proof. Let $\{x_i\}_{i\in I}$ be a pro-basis of $E_0^\vee X$. By C.1 the evaluations at x_i give a bijection

$$E^0X \xrightarrow{\sim} \prod_{i\in I} E_0.$$

This is a continuous bijection between compact Hausdorff spaces by 11.1-11.5 in [HS99b], and thus a homeomorphism. Under this homeomorphism evaluation at x_i goes to projection to the ith component.
Consider the map

$$ev : Hom^{\text{cont.}}_{E_0}\left(\prod_{i\in I} E_0, E_0\right) \to \prod_{i\in I} E_0,\ f \mapsto (i \mapsto f(\delta_i)).$$

We claim that ev is injective: in fact, let $ev(f) = 0$ and F be the directed set of finite subsets of

I. For any $c \in \prod_{i\in I} E_0$ consider the net

$$N_C : A \to \prod_{i\in I} E_0,\ J \mapsto \sum_{j\in J} c_j\delta_j$$

which converges to c in every projection, thus converges to c. By linearity, we find that

$$f(N_c(J)) = \sum_{j\in J} c_j ev(f)_j = 0,$$

by continuity of f we have $f(c) = 0$, and thus $f = 0$.
We also claim that ev already lands in the submodule

$$L_0 \bigoplus_{i\in I} E_0 \subset \prod_{i\in I} E_0,$$

that is for all $n \geq 0$ and for all but finitely many $i \in I$ we have

$$ev(f)_i \in \mathfrak{m}^n.$$

Since the natural topology on E_0 coincides with the $\mathfrak{m}$-adic one by [HS99b, prop. 11.9] we find that $f^{-1}(\mathfrak{m}^n)$ contains a neighborhood U of 0, showing that there are n_i, with all but finitely many equal to 0, such that

$$\prod_{i\in I} \mathfrak{m}^{n_i} \subset f^{-1}(\mathfrak{m}^n).$$

Thus, for those i with $n_i = 0$ we have $f(\delta_i) \in \mathfrak{m}^n$, which is exactly the claim.
Now the composite

$$E_0^\vee X \to Hom_{E_0}^{\text{cont.}}\left(\prod_{i\in I} E_0, E_0\right) \to L_0 \bigoplus_{i\in I} E_0$$

sends x_i to the ith pro-basis element, so it is an isomorphism. Since the second map is injective, both maps have to be bijections, completing the proof. □

Remark C.3. The same argument also shows that, for any homeomorphism

$$E^0X \xrightarrow{\sim} \prod_{j\in J} E_0,$$

the elements $x_j \in E_0^\vee X$ corresponding to the jth projection via the previous lemma form a pro-basis of $E_0^\vee X$.

Lemma C.4. *Let X be a spectrum such that K_*X is concentrated in even degrees. Then the map*

$$\pi_0 Map\left(X, L_{K(2)} \bigoplus_{i\in I} E\right) \to \prod_{i\in I} E^0X$$

is injective, and its image consists of those collections $\{q_i \in E^0X\}_{i\in I}$ fulfilling the following finiteness condition: for all $n \geq 0$ and $x \in E_0^\vee X$, we have that

$$\langle x, q_i\rangle \in \mathfrak{m}^n$$

for all but finitely many $i \in I$.

Proof. Using the universal coefficient theorem for $K(2)$-local E-modules [Hov, thm. 4.1] and since our assumption implies that $E_*^\vee X$ is projective in L-complete E_*-modules, the above map can be identified with

$$Hom_{E_0}\left(E_0^\vee X, \left(\bigoplus_{i\in I} E_0\right)^\wedge_{\mathfrak{m}}\right) \to Hom_{E_0}\left(E_0^\vee X, \prod_{i\in I} E_0\right).$$

Thus, the injectivity statement and the identification of the image follow from the analogous statements for the map

$$\left(\bigoplus_{i\in I} E_0\right)^\wedge_{\mathfrak{m}} \to \prod_{i\in I} E_0.$$

□

Lemma C.5. *Let X be a spectrum such that K_*X is concentrated in even degrees, and let*

$$\alpha : K_0X \to \bigoplus_{i\in I} K_0E$$

be an isomorphism of K_0E-comodules. Then there exists a map

$$\beta : X \to L_K \bigoplus_{i\in I} E$$

such that $K_0\beta = \alpha$. In particular, β is a K-equivalence of spectra.

Proof. Denote $L_K \bigoplus_{i\in I} E$ by F. By assumption $E_0^\vee X$ is pro-free, so that we can choose a lift $\tilde{\alpha}$ in E_0-modules such that

$$\begin{tikzcd} E_0^\vee X \arrow[r, dashed, "\tilde{\alpha}"] \arrow[d, two heads] & E_0^\vee F \arrow[d, two heads] \\ K_0X \arrow[r, "\alpha"] & K_0F \end{tikzcd}$$

commutes. Using the universal coefficient theorem for K-local E-modules [Hov, thm. 4.1], and since both $E_0^\vee F$ and $E_0^\vee X$ are, by assumption, projective in L-complete E_0-modules, we get a sequence of isomorphisms

$$F^0F \simeq \mathrm{Hom}_{E_0}(E_0^\vee F, F_0) \xrightarrow{\tilde{\alpha}^*} \mathrm{Hom}_{E_0}(E_0^\vee X, F_0) \simeq F^0X.$$

Let β be the image of id_F under this. We now want to show that $K_0\beta = \alpha$. For this it suffices to show that, for all $i \in I$, we have $K_0(pr_i \circ \beta) = pr_i \circ \alpha$. These are both maps $K_0X \to K_0E$ of K_0E-comodules and, by construction, agree when composed with the counit $K_0E \to K_0$. Dualizing this, they both send the unit of K^0E to the same element of K^0X. Since both maps are K^0E-linear they must agree on all of K^0E, and we are done. □

Remark C.6. Let $q_i \in K^0X$ be the unique class such that

$$\langle x, q_i\rangle = \epsilon(pr_i(\alpha(x))) \text{ for all } x \in K_0X.$$

The above also shows that $\beta_i = pr_i \circ \beta \in E^0X$ maps to q_i in K^0X.

We close with a lemma about mapping spaces into E_0:

Lemma C.7. *Let X be a topological space, $\mathbb{F}$ a finite field of characteristic p, and let*

$$R_d = W(\mathbb{F})[\![x_1, \dots, x_{d-1}]\!]$$

where $d \geq 1$. Let $\mathfrak{m}_d = (p, x_1, \dots, x_{d-1})$ be the maximal ideal and give R_d the $\mathfrak{m}_d$-adic topology. Then the natural map

$$C^0(X, R_d) \to C^0(X, R_d/\mathfrak{m}_d^n)$$

exhibits the target as $C^0(X, R_d)/\mathfrak{m}_d^n$.

Proof. To begin, note that R_d is a Noetherian regular local ring with finite quotient field, so each of the quotients $R_d/\mathfrak{m}_d^n$ is finite. This shows that R_d is compact Hausdorff in the $\mathfrak{m}_d$-adic topology. We also have that the associated graded $Gr_*R_d = \mathfrak{m}_d^*/\mathfrak{m}_d^{*+1}$ is a polynomial ring over $\mathbb{F}$ in d generators in degree 1.

Let us now show surjectivity. As $R_d/\mathfrak{m}_d^n$ is discrete any choice of section s of $R_d \to R_d/\mathfrak{m}_d^n$ will be continuous, and the map

$$C^0(X, R_d/\mathfrak{m}_d) \to C^0(X, R_d),\ f \mapsto s \circ f$$

gives a section of $C^0(X, R_d) \to C^0(X, R_d/\mathfrak{m}_d^n)$, so the latter must be surjective.
We now identify the kernel. It always contains $\mathfrak{m}_d^n C^0(X, R_d)$, and we need to show the other inclusion. The proof proceeds by induction in $d \geq 1$.
For the base case $d = 1$, let f be in the kernel of $C^0(X, W(\mathbb{F})) \to C^0(X, W(\mathbb{F})/p^n)$. This means that, for every $x \in X$, p^n divides $f(x)$, that is f factors over the closed subspace $p^n W(\mathbb{F})$. Since $W(\mathbb{F})$ is a domain the map

$$W(\mathbb{F}) \xrightarrow{\cdot p^n} p^n W(\mathbb{F})$$

is a continuous bijection between compact Hausdorff spaces, thus a homeomorphism. Denote its inverse by T. Then $T \circ f \in C^0(X, W(\mathbb{F}))$ and $p^n(T \circ f) = f$, so that $f \in p^n C^0(X, W(\mathbb{F}))$.
For the induction step from d to $d+1$, consider the composite

$$R_d \xrightarrow{i} R_{d+1} \xrightarrow{pr} R_{d+1}/(x_d),$$

where i and pr are the standard inclusion and projection, which are continuous. The quotient topology on $R_{d+1}/(x_d)$ coincides with the $\mathfrak{m}_{d+1}/(x_d)$-adic topology, and the composite is a continuous bijection between compact Hausdorff spaces, thus a homeomorphism. Denote its inverse by T.
Let f be in the kernel of $C^0(X, R_{d+1}) \to C^0(X, R_{d+1}/\mathfrak{m}_{d+1}^n)$. By the inductive assumptions, we have that $T \circ pr \circ f$ is in $m_d^n C^0(X, R_d)$, so $f' = i \circ T \circ pr \circ f$ is in $(p, x_1, \ldots, x_{d-1})^n C^0(X, R_{d+1})$ and $pr \circ f = pr \circ f'$.
For every $x \in X$, we have $(f - f')(x) \in (x_d)$. Since division by x_d is continuous (which can be shown as for p^n above) there is a unique $f'' \in C^0(X, R_{d+1})$ with $x_d f'' = (f - f')$. Now let's do an induction in $n \geq 1$.
For the base case $n = 1$ the above discussion shows that

$$f = f' + x_d f'' \in (p, x_1, \ldots, x_{d-1}, x_d) C^0(X, R_{d+1}),$$

and we are done.
For the induction step from $n-1$ to n, note that the above discussion implies $x_d f''(x) \in \mathfrak{m}_{d+1}^n$ for all $x \in X$. Since the associated graded has no zero divisors this shows that already $f''(x) \in \mathfrak{m}_{d+1}^{n-1}$ for all $x \in X$, so that $f'' \in \mathfrak{m}_{d+1}^{n-1} C^0(X, R_{d+1})$ by the inductive hypothesis. We thus have

$$f = f' + x_d f'' \in \left((p, x_1, \ldots, x_{d-1})^n + x_d \mathfrak{m}_{d+1}^{n-1}\right) C^0(X, R_{d+1}) = \mathfrak{m}_{d+1}^n C^0(X, R_{d+1})$$

and we are done. $\square$

References

[ABP66] D. W. Anderson, E. H. Brown Jr., and F. P. Peterson. “SU-cobordism, KO-characteristic numbers, and the Kervaire invariant”. In: *Ann. of Math. (2)* 83 (1966), pp. 54–67. ISSN: 0003-486X. DOI: 10.2307/1970470.

[ABP67] D. W. Anderson, E. H. Brown Jr., and F. P. Peterson. “The structure of the Spin cobordism ring”. In: *Ann. of Math. (2)* 86 (1967), pp. 271–298. ISSN: 0003-486X. DOI: 10.2307/1970690.

[Ada95] J. F. Adams. *Stable homotopy and generalised homology.* Chicago Lectures in Mathematics. Reprint of the 1974 original. University of Chicago Press, Chicago, IL, 1995, pp. x+373. ISBN: 0-226-00524-0.

[AHR10] M. Ando, M. J. Hopkins, and C. Rezk. “Multiplicative Orientations of KO-theory and of the Spectrum of Topological Modular Forms”. 2010. URL: https://rezk.web.illinois.edu/koandtmf.pdf (visited on 07/02/2024).

[AHS01] M. Ando, M. J. Hopkins, and N. P. Strickland. “Elliptic spectra, the Witten genus and the theorem of the cube”. In: *Invent. Math.* 146.3 (2001), pp. 595–687. ISSN: 0020-9910. DOI: 10.1007/s002220100175.

[AHS04] M. Ando, M. J. Hopkins, and N. P. Strickland. “The sigma orientation is an H_∞ map”. In: *Amer. J. Math.* 126.2 (2004), pp. 247–334. ISSN: 0002-9327. DOI: 10.1353/ajm.2004.0008.

[AHS99] M. Ando, M. J. Hopkins, and N. P. Strickland. “Elliptic Homology of $BO{<}8{>}$ in positive characteristic”. 1999.

[And64] D. W. Anderson. “The Real K-Theory of Classifying Spaces”. In: *Proc. N. A. S.* 51.4 (1964), pp. 634–636. ISSN: 00278424, 10916490. URL: http://www.jstor.org/stable/71980 (visited on 05/26/2025).

[AS63] M. F. Atiyah and I. M. Singer. “The index of elliptic operators on compact manifolds”. In: *Bull. Amer. Math. Soc.* 69 (1963), pp. 422–433. ISSN: 0002-9904. DOI: 10.1090/S0002-[]9904-[]1963-[]10957-[]X.

[Bal23] W. Balderrama. “Algebraic theories of power operations”. In: *J. Topol.* 16.4 (2023), pp. 1543–1640. ISSN: 1753-8416. DOI: 10.1112/topo.12318.

[Bar+22] T. Barthel, A. Beaudry, P. G. Goerss, and V. Stojanoska. “Constructing the determinant sphere using a Tate twist”. In: *Math. Z.* 301.1 (2022), pp. 255–274. ISSN: 0025-5874. DOI: 10.1007/s00209-[]021-[]02864-[]x.

[BB20] T. Barthel and A. Beaudry. “Chromatic structures in stable homotopy theory”. In: *Handbook of homotopy theory.* CRC Press/Chapman Hall Handb. Math. Ser. CRC Press, Boca Raton, FL, 2020, pp. 163–220.

[BD85] T. Bröcker and T. tom Dieck. *Representations of compact Lie groups.* Vol. 98. Graduate Texts in Mathematics. Springer-Verlag, New York, 1985, pp. x+313. ISBN: 0-387-13678-9. DOI: 10.1007/978-[]3-[]662-[]12918-[]0.

[Bea15] A. Beaudry. “The algebraic duality resolution at $p = 2$”. In: *Algebr. Geom. Topol.* 15.6 (2015), pp. 3653–3705. ISSN: 1472-2747. DOI: 10.2140/agt.2015.15.3653.

[Bea17] A. Beaudry. “Towards the homotopy of the $K(2)$-local Moore spectrum at $p = 2$”. In: *Adv. Math.* 306 (2017), pp. 722–788. ISSN: 0001-8708. DOI: 10.1016/j.aim.2016.10.020.

[Ber24] D. Berwick-Evans. *Elliptic cohomology and quantum field theory.* 2024. arXiv: 2408.07693 [math.AT].

[BH16] T. Barthel and D. Heard. “The E_2-term of the $K(n)$-local E_n-Adams spectral sequence”. In: *Topology Appl.* 206 (2016), pp. 190–214. ISSN: 0166-8641. DOI: 10.1016/j.topol.2016.03.024.

[BSY22] R. Burklund, T. M. Schlank, and A. Yuan. *The Chromatic Nullstellensatz.* 2022. arXiv: 2207.09929 [math.AT].

[Buj12] C. Bujard. *Finite subgroups of extended Morava stabilizer groups.* 2012. arXiv: 1206.1951 [math.AT].

[CC16] P.-J. Cahen and J.-L. Chabert. "What you should know about integer-valued polynomials". In: *Amer. Math. Monthly* 123.4 (2016), pp. 311–337. ISSN: 0002-9890. DOI: 10.4169/amer.math.monthly.123.4.311.

[CDN24] C. Carrick, J. M. Davies, and S. van Nigtevecht. *The descent spectral sequence for topological modular forms.* 2024. arXiv: 2412.01640 [math.AT].

[Dav25] J. M. Davies. "On Lurie's theorem and applications". In: *Math. Z.* 309.2 (2025), Paper No. 24, 53. ISSN: 0025-5874. DOI: 10.1007/s00209-[]024-[]03626-[]1.

[Dav82] D. M. Davis. "On the cohomology of $M\mathrm{O}\langle 8\rangle$". In: *Symposium on Algebraic Topology in honor of José Adem (Oaxtepec, 1981).* Vol. 12. Contemp. Math. Amer. Math. Soc., Providence, RI, 1982, pp. 91–104.

[Dev20] S. K. Devalapurkar. *The Ando-Hopkins-Rezk orientation is surjective.* 2020. arXiv: 1911.10534 [math.AT].

[Dev24] S. K. Devalapurkar. "Higher chromatic Thom spectra via unstable homotopy theory". In: *Algebr. Geom. Topol.* 24.1 (2024), pp. 49–108. ISSN: 1472-2747. DOI: 10.2140/agt.2024.24.49.

[DH04] E. S. Devinatz and M. J. Hopkins. "Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups". In: *Topology* 43.1 (2004), pp. 1–47. ISSN: 0040-9383. DOI: 10.1016/S0040-[]9383(03)00029-[]6.

[Die73] J. Dieudonné. *Introduction to the theory of formal groups.* Vol. 20. Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1973, pp. xii+265.

[Dix+99] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal. *Analytic pro-p groups.* Second. Vol. 61. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1999, pp. xviii+368. ISBN: 0-521-65011-9. DOI: 10.1017/CBO9780511470882.

[DM82] D. M. Davis and M. Mahowald. "The nonrealizability of the quotient $A//A_2$ of the Steenrod algebra". In: *Amer. J. Math.* 104.6 (1982), pp. 1211–1216. ISSN: 0002-9327. DOI: 10.2307/2374058.

[GH04] P. G. Goerss and M. J. Hopkins. "Moduli spaces of commutative ring spectra". In: *Structured ring spectra.* Vol. 315. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2004, pp. 151–200. DOI: 10.1017/CBO9780511529955.009.

[Hen17] H.-W. Henn. "A mini-course on Morava stabilizer groups and their cohomology". In: *Algebraic topology.* Vol. 2194. Lecture Notes in Math. Springer, Cham, 2017, pp. 149–178.

[Hir53] F. Hirzebruch. "On Steenrod's reduced powers, the index of inertia, and the Todd genus". In: *Proceedings of the National Academy of Sciences of the United States of America* 39 (1953). DOI: 10.1073/pnas.39.9.951.

[HK24] Z. Halladay and Y. Kamel. *Real spin bordism and orientations of topological* K*-theory.* 2024. arXiv: 2405.00963 [math.AT].

[Hop14] M. J. Hopkins. "$K(1)$-local E_∞-ring spectra". In: *Topological modular forms.* Ed. by C. L. Douglas, J. Francis, A. G. Henriques, and M. A. Hill. Vol. 201. Mathematical Surveys and Monographs. 2014. DOI: 10.1090/surv/201.

[Hov] M. Hovey. "Some spectral sequences in Morava E-theory". URL: https://folk.ntnu.no/drewkh/hovey_ss.pdf (visited on 07/02/2024).

[Hov08a] M. Hovey. "Morava E-theory of filtered colimits". In: *Trans. Amer. Math. Soc.* 360.1 (2008), pp. 369–382. ISSN: 0002-9947. DOI: 10.1090/S0002-[]9947-[]07-[]04298-[]5.

[Hov08b] M. Hovey. "The homotopy of $M\,\mathrm{String}$ and $M\,\mathrm{U}\langle 6\rangle$ at large primes". In: *Algebr. Geom. Topol.* 8.4 (2008), pp. 2401–2414. ISSN: 1472-2747. DOI: 10.2140/agt.2008.8.2401.

[HR95] M. A. Hovey and D. C. Ravenel. "The 7-connected cobordism ring at $p = 3$". In: *Trans. Amer. Math. Soc.* 347.9 (1995), pp. 3473–3502. ISSN: 0002-9947. DOI: 10.2307/2155020.

[HS99a] M. Hovey and H. Sadofsky. "Invertible spectra in the $E(n)$-local stable homotopy category". In: *J. London Math. Soc. (2)* 60.1 (1999), pp. 284–302. ISSN: 0024-6107. DOI: 10.1112/S0024610799007784.

[HS99b] M. Hovey and N. P. Strickland. "Morava K-theories and localisation". In: *Mem. Amer. Math. Soc.* 139.666 (1999), pp. viii+100. ISSN: 0065-9266. DOI: 10.1090/memo/0666.

[KL02] N. Kitchloo and G. Laures. "Real structures and Morava K-theories". In: *K-Theory* 25.3 (2002), pp. 201–214. ISSN: 0920-3036. DOI: 10.1023/A:1015683917463.

[KLW04] N. Kitchloo, G. Laures, and W. S. Wilson. "The Morava K-theory of spaces related to BO". In: *Adv. Math.* 189.1 (2004), pp. 192–236. ISSN: 0001-8708. DOI: 10.1016/j.aim.2003.10.008.

[KM85] N. M. Katz and B. Mazur. *Arithmetic moduli of elliptic curves.* Vol. 108. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1985, pp. xiv+514. ISBN: 0-691-08349-5; 0-691-08352-5. DOI: 10.1515/9781400881710.

[Kuh89] N. J. Kuhn. "Morava K-theories and infinite loop spaces". In: *Algebraic Topology.* Ed. by Gunnar Carlsson, Ralph Cohen, Haynes Miller, and Douglas Ravenel. Berlin, Heidelberg: Springer Berlin Heidelberg, 1989, pp. 243–257. ISBN: 978-3-540-46160-9.

[KW87] R. Kultze and U. Würgler. "A note on the algebra $P(n)_*(P(n))$ for the prime 2". In: *Manuscripta Math.* 57.2 (1987), pp. 195–203. ISSN: 0025-2611. DOI: 10.1007/BF02218080.

[Lau03] G. Laures. "An E_∞ splitting of spin bordism". In: *Amer. J. Math.* 125.5 (2003), pp. 977–1027. ISSN: 0002-9327. DOI: 10.1353/ajm.2003.0032.

[Lau16] G. Laures. "Characteristic classes in TMF of level $\Gamma_1(3)$". In: *Trans. Amer. Math. Soc.* 368.10 (2016), pp. 7339–7357. ISSN: 0002-9947. DOI: 10.1090/tran/6575.

[Laz65] M. Lazard. "Groupes analytiques p-adiques". In: *Inst. Hautes Études Sci. Publ. Math.* 26 (1965), pp. 389–603. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1965__26__389_0.

[LN12] T. Lawson and N. Naumann. "Commutativity conditions for truncated Brown-Peterson spectra of height 2". In: *J. Topol.* 5.1 (2012), pp. 137–168. ISSN: 1753-8416. DOI: 10.1112/jtopol/jtr030.

[LO16] G. Laures and M. Olbermann. "$TMF_0(3)$-characteristic classes for string bundles". In: *Math. Z.* 282.1-2 (2016), pp. 511–533. ISSN: 0025-5874. DOI: 10.1007/s00209-[]015-[]1551-[]3.

[LO18] G. Laures and M. Olbermann. "Cannibalistic classes of string bundles". In: *Manuscripta Math.* 156.3-4 (2018), pp. 273–298. ISSN: 0025-2611. DOI: 10.1007/s00229-[]017-[]0978-[]8.

[LS19] G. Laures and B. Schuster. "Towards a splitting of the $K(2)$-local string bordism spectrum". In: *Proc. Amer. Math. Soc.* 147.1 (2019), pp. 399–410. ISSN: 0002-9939. DOI: 10.1090/proc/14190.

[Lur09a] J. Lurie. "A survey of elliptic cohomology". In: *Algebraic topology.* Vol. 4. Abel Symp. Springer, Berlin, 2009. DOI: 10.1007/978-[]3-[]642-[]01200-[]6_9.

[Lur09b] J. Lurie. *Higher Topos Theory.* Princeton University Press, 2009. ISBN: 9780691140490.

[Lur17] J. Lurie. "Higher Algebra". 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf (visited on 06/15/2025).

[Lur18] J. Lurie. "Elliptic Cohomology II: Orientations". 2018. URL: https://www.math.ias.edu/~lurie/papers/Elliptic-[]II.pdf (visited on 05/08/2025).

[Mat15] A. Mathew. *The homology of* tmf. 2015. arXiv: 1305.6100 [math.AT].

[McT18] C. McTague. "tmf is not a ring spectrum quotient of string bordism". In: *New directions in homotopy theory.* Vol. 707. Contemp. Math. Amer. Math. Soc., [Providence], RI, 2018, pp. 67–76. DOI: 10.1090/conm/707/14254.

[Mor23] I. Mor. *Picard and Brauer groups of $K(n)$-local spectra via profinite Galois descent.* 2023. arXiv: 2306.05393 [math.AT].

[MR09] M. Mahowald and C. Rezk. "Topological modular forms of level 3". In: *Pure Appl. Math. Q.* 5.2, Special Issue: In honor of Friedrich Hirzebruch. Part 1 (2009), pp. 853–872. ISSN: 1558-8599. DOI: 10.4310/PAMQ.2009.v5.n2.a9.

[NS07] N. Nikolov and D. Segal. “On finitely generated profinite groups. I. Strong completeness and uniform bounds”. In: *Ann. of Math. (2)* 165.1 (2007), pp. 171–238. ISSN: 0003-486X. DOI: 10.4007/annals.2007.165.171.

[NSW08] J. Neukirch, A. Schmidt, and K. Wingberg. *Cohomology of number fields*. Second. Vol. 323. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin, 2008, pp. xvi+825. ISBN: 978-3-540-37888-4. DOI: 10.1007/978-[]3-[]540-[]37889-[]1.

[Pas89] D. S. Passman. *Infinite crossed products*. Vol. 135. Pure and Applied Mathematics. Academic Press, Inc., Boston, MA, 1989, pp. xii+468. ISBN: 0-12-546390-1.

[Pen82] D. J. Pengelley. “The homotopy type of MSU”. In: *Amer. J. Math.* 104.5 (1982), pp. 1101–1123. ISSN: 0002-9327. DOI: 10.2307/2374085.

[Pen83] D. J. Pengelley. “$H^*(M\mathrm{O}{<}8{>}; \mathbf{Z}/2)$ is an extended A_2^*-coalgebra”. In: *Proc. Amer. Math. Soc.* 87.2 (1983), pp. 355–356. ISSN: 0002-9939. DOI: 10.2307/2043716.

[Pet19] E. Peterson. *Formal geometry and bordism operations*. Vol. 177. Cambridge Studies in Advanced Mathematics. With a foreword by Matthew Ando. Cambridge University Press, Cambridge, 2019, pp. xiv+405. ISBN: 978-1-108-42803-3.

[Pst21] P. Pstrągowski. “Finite height chromatic homotopy theory”. 2021. URL: https://www.kurims.kyoto-[]u.ac.jp/~piotr/252y_notes.pdf (visited on 05/08/2025).

[Rav86] D. C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*. Vol. 121. Pure and Applied Mathematics. Academic Press, Inc., Orlando, FL, 1986, pp. xx+413. ISBN: 0-12-583430-6; 0-12-583431-4.

[Rog08] J. Rognes. “Galois extensions of structured ring spectra. Stably dualizable groups”. In: *Mem. Amer. Math. Soc.* 192.898 (2008), pp. viii+137. ISSN: 0065-9266. DOI: 10.1090/memo/0898.

[Sal25] A. Salch. *The action of the Morava stabilizer group on the coefficients of Morava E-theory at height 2*. 2025. arXiv: 2503.04686 [math.AT].

[Sen23] A. Senger. “Obstruction theory and the level n elliptic genus”. In: *Compos. Math.* 159.9 (2023), pp. 2000–2021. ISSN: 0010-437X. DOI: 10.1112/S0010437X23007406.

[Sna75] V. P. Snaith. “Dyer-Lashof operations in K-theory”. In: *Topics in K-theory. Two independent contributions*. Lecture Notes in Math., Vol. 496. Springer, Berlin-New York, 1975, pp. 103–294.

[Sna81] V. Snaith. “Localized stable homotopy of some classifying spaces”. In: *Math. Proc. Cambridge Philos. Soc.* 89.2 (1981), pp. 325–330. ISSN: 0305-0041. DOI: 10.1017/S0305004100058205.

[Swe69] M. E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.

[The24] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 10.2)*. https://www.sagemath.org. 2024.

[Tho54] R. Thom. “Quelques propriétés globales des variétés différentiables”. In: *Comment. Math. Helv.* 28 (1954), pp. 17–86. ISSN: 0010-2571. DOI: 10.1007/BF02566923.

[Tok24] Leonard Tokic. *$K(2)$-local splittings of finite Galois extensions of $MU{<}6{>}$ and $MString$*. 2024. arXiv: 2411.16280 [math.AT].

[Wal60] C. T. C. Wall. “Determination of the cobordism ring”. In: *Ann. of Math. (2)* 72 (1960), pp. 292–311. ISSN: 0003-486X. DOI: 10.2307/1970136.

[Wil19] W. S. Wilson. *The Omega spectrum for Pengelley’s BoP*. 2019. arXiv: 1807.03738 [math.AT].

[Wit88] E. Witten. “The index of the Dirac operator in loop space”. In: *Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986)*. Vol. 1326. Lecture Notes in Math. Springer, Berlin, 1988, pp. 161–181. DOI: 10.1007/BFb0078045.

[Zhu20] Y. Zhu. “Norm coherence for descent of level structures on formal deformations”. In: *J. Pure Appl. Algebra* 224.10 (2020), pp. 106382, 35. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2020.106382.

Eigenständigkeitserklärung

Ich versichere an Eides statt, dass ich die eingereichte Dissertation selbstständig und ohne unzulässige fremde Hilfe verfasst, andere als die in ihr angegebene Literatur nicht benutzt und dass ich alle ganz oder annähernd übernommenen Textstellen sowie verwendete Grafiken, Tabellen und Auswertungsprogramme kenntlich gemacht habe.

Des Weiteren versichere ich, dass die vorgelegte elektronische mit der schriftlichen Version der Dissertation übereinstimmt und die Abhandlung in dieser oder ähnlicher Form noch nicht anderweitig als Promotionsleistung vorgelegt und bewertet wurde.

Bochum, den 16.06.2025