Representation Theory of Symmetric Groups

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Contents

1 Introduction

1.1 Motivation

- Representation theory of finite groups: active area of research
- Many open problems, e.g. Local-Global Conjectures

Definition. *Let G be a finite group, p a prime. Then we let*

- $\mathrm{Irr}(G) := \{$*irreducible characters of* $G\}$,
- $\mathrm{Irr}_{p'}(G) := \{\chi \in \mathrm{Irr}(G) \mid p \nmid \chi(1)\}$.

Conjecture (McKay 1972)**.** Let G be a finite group, p a prime, P a Sylow p-subgroup of G. Then

$$|\,\mathrm{Irr}_{p'}(G)| = |\,\mathrm{Irr}_{p'}(N_G(P))|.$$

The case $p = 2$ has been proved in 2016.

Theorem 1.1 (Olsson 1976)**.** *The McKay Conjecture holds for all symmetric groups S_n and all primes p.*

Outline of the course:

- Chapter 1: Introduction and background
- Chapter 2: Specht modules ([Jam78])
- Chapter 3: Character theory ([JK84])
- Chapter 4: McKay numbers ([Ols94])

1.2 Background

Notation.

- $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.
- If $n \in \mathbb{N}$, let $[n] := \{1, 2, \dots, n\}$.
- $\mathrm{Irr}(G)$ (or $\mathrm{Irr}_{\mathbb{F}}(G)$ to specify the field $\mathbb{F}$) is a complete set of irreducible representations of G over $\mathbb{F}$.

1.2.1 Representations & modules

$\mathbb{F}$ will denote an arbitrary field and G a finite group. All modules considered in this course will be finite-dimensional left modules.

A (finite-dimensional) *representation of G over $\mathbb{F}$* is a group homomorphism $\rho : G \to \mathrm{GL}(V)$, where V is a (finite-dimensional) vector space over $\mathbb{F}$. We write $g \cdot v$ for $\rho(g)(v)$. Equivalently a representation is an $\mathbb{F}G$-module. The *degree* or *dimension* of a representation is the dimension of the underlying vector space.

Example. The (one-dimensional) *trivial* representation of G is a one-dimensional vector space with trivial G-action. It will be denoted by $\mathbb{1}_G$.

Other concepts.

- *Subprepresentations* W of V, written $W \leq V$
- *Simple* or *irreducible* modules, i.e. those with no proper non-zero submodules.
- *Semisimple* or *completely reducible* modules, i.e. direct sums of simple modules.
- *Decomposable* modules, i.e. modules decomposing into a direct sum of proper submodules; opposite: *indecomposable.*
- *G-homomorphisms*: If V, W are G-modules, then an $\mathbb{F}$-linear map $\theta : V \to W$ is a G-homomorphism if $g \cdot \theta(v) = \theta(g \cdot v)$ for all $g \in G, v \in V$.

Useful results.

Lemma 1.2 (Schur's Lemma)**.** *Let V, W be simple G-modules, $\theta : V \to W$ a G-homomorphism. Then $\theta = 0$ or θ is an isomorphism. If $\mathbb{F} = \mathbb{F}^{\mathrm{alg}}$ and $V = W$, then $\theta = c\,\mathrm{id}_V$ for some $c \in \mathbb{F}$, i.e. $\mathrm{End}_{\mathbb{F}G}(V) \cong \mathbb{F}$.*

Example. The (left) *regular module* of G is $\mathbb{F}G$ viewed as a left module over itself. If $\mathrm{Irr}_{\mathbb{F}}(G) = \{S_i \mid i \in I\}$ and $\mathrm{char}\,\mathbb{F} = 0$, then

$$\mathbb{F}G \cong \bigoplus_{i \in I} S_i^{\oplus \dim_{\mathbb{F}} S_i}$$

as G-modules.

Theorem 1.3 (Maschke's Theorem)**.** *Suppose* $\mathrm{char}\,\mathbb{F} \nmid |G|$. *If $U \leq V$ are G-modules, then there is a G-submodule $W \leq V$ such that $V = U \oplus W$.*

Corollary 1.4. *Every finite-dimensional representation of a finite group G over $\mathbb{F}$ where* $\mathrm{char}\,\mathbb{F} \nmid |G|$ *is semisimple.*

Common constructions.

- *Tensor products:* If V, W are G-modules, then $V \otimes_{\mathbb{F}} W$ becomes a G-module via $g \cdot (v \otimes w) = (gv) \otimes (gw)$ for all $g \in G, v \in V, w \in W$.

- *Restriction:* If $H \leq G$, V is a G-module, then we can also view V as an H-module, written $V\big\downarrow_H^G$, $V\big\downarrow_H$, V_H or $\mathrm{Res}_H^G(V)$.
- *Induction:* If $H \leq G$, U is an H-module, we can get a G-module out of it. Let $\{t_i \mid i \in I\}$ be a set of left coset representatives of H in G. Then the induction of U from H to G is the vector space direct sum

  $$\bigoplus_{i \in I} (t_i \otimes U) =: U\big\uparrow_H^G,\ U\big\uparrow^G \text{ or } U^G,$$

  where $t_i \otimes U = \{t_i \otimes u \mid u \in U\}$, and the G-action is as follows: $g \cdot (t_i \otimes u) := t_j \otimes (t_j^{-1} g t_i) u$ where given $g \in G, i \in I$, then $j \in I$ is the unique index such that $g t_i \in t_j H$. Equivalently, we can define the induction as $U\big\uparrow_H^G = \mathbb{F}G \otimes_{\mathbb{F}H} U$, see Example Sheet 1, Question 1.
- *Permutation modules:* A G-module with a permutation basis B, i.e. $g \cdot b \in B$ for all $g \in G, b \in B$. E.g. the left regular module $\mathbb{F}G$ is a permutation module with basis $B = G$.

Lemma 1.5. *Suppose G acts transitively on a set Ω. Let M be the corresponding permutation module. Then $M \cong \mathbb{1}_H\big\uparrow^G$, where $H = \mathrm{Stab}_G(\omega)$ for any $\omega \in \Omega$.*

Proof. Special case of Example Sheet 1, Question 2. □

1.2.2 Some Linear Algebra

- Recall that if M is a (finite-dimensional) $\mathbb{F}$-vector space, $M* = \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F})$ is again an $\mathbb{F}$-vector space. If $e_1, \dots, e_k$ is a basis of M, then the dual basis $\varepsilon_1, \dots, \varepsilon_k \in M^*$ is defined by $\varepsilon_i(e_j) = \delta_{ij}$.
- Let M be a G-module, the dual M^* of M carries the G action $(g \cdot \phi)(v) = \phi(g^{-1} \cdot v)$.
- Suppose we have a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on some finite-dimensional $\mathbb{F}$-vector space M. For a vector subspace V of M define

  $$V^\perp = \{m \in M \mid \langle v, m \rangle = 0 \,\forall v \in V\}.$$

  Consider the linear map $\phi : M \to M^*, m \mapsto \langle \cdot, m \rangle$. Note even if $\langle \cdot, \cdot \rangle$ is non-singular, i.e. $\ker \phi = M^\perp = 0$, we could have $V^\perp \cap V \neq 0$.

  We can describe how large this is using a basis of V. Let $e_1, \dots, e_k$ of V. The Gram matrix of V w.r.t. this basis be the matrix A with $A_{ij} = \langle e_i, e_j \rangle$.

  Lemma 1.6. *We have that* $\dim_{\mathbb{F}} V/(V \cap V^\perp) = \mathrm{rank}\, A$.

  Proof. Consider $\varphi : V \to V^*, v \mapsto \langle \cdot, v \rangle$. Let $\varepsilon_1, \dots, \varepsilon_k$ be the basis of V^* dual to $e_1, \dots, e_k$. Then $\varphi(e_i) = \sum_{j=1}^k \langle e_j, e_i \rangle \varepsilon_j$. So the Gram matrix A is the matrix of φ with repsect to the basis $e_1, \dots, e_k$ and $\varepsilon_1, \dots, \varepsilon_k$. Clearly $\ker \varphi = V \cap V^\perp$, and so $\dim V/(V \cap V^\perp) = \dim V - \dim \ker \varphi = \mathrm{rank}\, A$. □

1.2.3 Character Theory

In this subsection, $\mathbb{F} = \mathbb{C}$. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of the finite group G over some finite-dimensional $\mathbb{C}$-vector space V. Recall that this representation *affords* the *character* $\chi_V : G \to \mathbb{C}$, $g \mapsto \operatorname{tr} \rho(g)$.

Theorem 1.7. *$\mathbb{C}G$-modules U, V are isomorphic iff $\chi_U = \chi_V$.*

Useful facts.

- There is an inner product on class functions on G given by

$$\langle \chi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1})\phi(g) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)}\phi(g).$$

- $\mathrm{Irr}(G)$ is an orthonormal basis for the space of class functions w.r.t. $\langle \cdot, \cdot \rangle$, in particular $|\mathrm{Irr}(G)|$ is the number of conjugacy classes of G.
- Characters of the usual constructions:
  - Direct sum: $\chi_{U \oplus V} = \chi_U + \chi_V$.
  - Tensor product: $\chi_{U \otimes V} = \chi_U \chi_V$.
  - Permutation modules: If V is a permutation module with permutation basis B, then $\chi(g) = |\{b \in B \mid gb = b\}|$ is the number of fixed points of g.
  - Restriction: If $H \leq G$ is a subgroup and V a representation of G, then $\chi_V \big\downarrow_H := \chi_{V \downarrow_H} = \chi_V|_H$.
- *Frobenius reciprocity:* If χ is a character of G, θ a character of H, then

$$\langle \chi \big\downarrow_H, \theta \rangle = \langle \chi, \theta \big\uparrow^G \rangle.$$

- *Mackey's theorem*: For $H, K \leq G$, ϕ a character of H, we can compute $(\phi \big\uparrow_H^G) \big\downarrow_K$ by decomposing it as a sum of characters indexed by a set of double coset representations of K, H in G. (See handout for details)

2 Specht Modules

Let $\mathbb{F}$ be an arbitrary field.

2.1 The Symmetric Group

Let Ω be a finite set. Call the symmetric group on Ω, $\mathrm{Sym}(\Omega)$. When $\Omega = [n]$, write S_n for $\mathrm{Sym}(\Omega)$.

Conventions:

- $(123)(12) = (13)$ (i.e. composition from right to left)
- $S_0 = \mathrm{Sym}(\emptyset) =$ trivial group

Some representations of S_n:

- *Trivial representation* of S_n, $\mathbb{1}_{S_n}$.
- *Sign representation* of S_n, $\mathrm{sgn}_{S_n} : \rho : S_n \to \mathbb{F}^*$, $g \mapsto \mathrm{sgn}(g)$.
- *Natural permutation module* V_n with permutation basis $[n]$.

  Note $V_n \cong \mathbb{1}_{S_{n-1}}\big\uparrow^{S_n}$, because $\mathrm{Stab}(n) = S_{n-1}$.

  Also $V_n\big\downarrow_{S_{n-1}} \cong V_{n-1} \oplus \mathbb{1}_{S_{n-1}}$.

Definition. *A* partition λ *of* n*, written* $\lambda \vdash n$*, is a non-increasing sequence of positive integers which sum to* n*, i.e.* $\lambda = (\lambda_1, \dots, \lambda_k)$ *with* $\lambda_i \in \mathbb{N}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ *and* $\sum_{i=1}^{k} \lambda_i = n$*. We call*

- λ_i *the* parts *of the partition,*
- n *the* size *of* λ *(also denoted* $|\lambda|$*),*
- k *the* length *of* λ *(also denoted* $\ell(\lambda)$*).*

The set $\{\lambda \mid \lambda \vdash n\}$ *of all partitions of* n *will be denoted by* $\wp(n)$*.*

We can extend this notion to 0 *by convention: the only partition of* 0 *is the empty sequence, i.e.* $\wp(0) = \{\emptyset\}$*.*

Short notation: $\lambda = (4, 3, 3, 1) = (4, 3^2, 1) \vdash 11$.

Definition. *Let* $\lambda = (\lambda_1, \dots, \lambda_k)$ *be a partition. The* Young diagram of λ *is*

$$Y(\lambda) = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \le i \le k, 1 \le j \le \lambda_i\}.$$

Typically, Young diagrams are drawn using boxes rather than points, e.g.:

$$\wp(4) = \left\{ \begin{array}{c} (4) \\ \ydiagram{4} \end{array}, \begin{array}{c} (3,1) \\ \ydiagram{3,1} \end{array}, \begin{array}{c} (2,2) \\ \ydiagram{2,2} \end{array}, \begin{array}{c} (2,1^2) \\ \ydiagram{2,1,1} \end{array}, \begin{array}{c} (1^4) \\ \ydiagram{1,1,1,1} \end{array} \right\}.$$

The rows and columns are numbered as in a matrix.

Definition. *Let* $\lambda = (\lambda_1, \dots, \lambda_k)$ *be a partition. The* conjugate partition of λ *is the partition* λ' *such that* $Y(\lambda')$ *is the transpose of* $Y(\lambda)$*. Explicitly,* $\lambda' = (\mu_1, \dots, \mu_{\lambda_1})$ *where* $\mu_j = \#\{i \in [k] \mid \lambda_i \ge j\}$*. Note* $|\lambda'| = |\lambda|$ *and* $(\lambda')' = \lambda$*.*

Example. Consider $\lambda = (4, 3, 1) \vdash 8$. Then

$$Y(\lambda) = \ydiagram{4,3,1},$$

and so

$$Y(\lambda') = \ydiagram{3,2,2,1},$$

i.e. $\lambda' = (3, 2, 2, 1)$.

Definition. *Let* $\lambda = (\lambda_1, \dots, \lambda_k)$ *and* $\mu = (\mu_1, \dots, \mu_s)$ *be two partitions of* $n \in \mathbb{N}$*. Then we say that* λ dominates μ*, written* $\lambda \trianglerighteq \mu$ *or* $\mu \trianglelefteq \lambda$*, if* $\sum_{i=1}^{l} \lambda_i \ge \sum_{i=1}^{l} \mu_i$ *for all* $l \in \{1, 2, \dots, \min(k, s)\}$*.*

Example. Take $n = 4$. Then $(4) \trianglerighteq (3, 1) \trianglerighteq (2, 2) \trianglerighteq (2, 1^2) \trianglerighteq (1^4)$.

However, in general, dominance is only a partial order, for example $(4, 3, 1) \not\trianglerighteq (5, 1^3)$ and $(5, 1^3) \not\trianglerighteq (4, 3, 1)$.

Dominance can be extended to a total ordering on $\wp(n)$, e.g. the lexicographic ordering: If $\lambda \ne \mu$, we say $\lambda > \mu$ if $\lambda_i > \mu_i$ where $i = \min\{j \in \mathbb{N} \mid \lambda_j \ne \mu_j\}$.

Definition. *Let* λ *be a partition of* n*. A* λ-tableau*, or* Young tableau of shape λ*, is a bijection* $t : Y(\lambda) \to [n]$*. The set of all* λ*-tableaux will be denoted by* Δ^λ*.*

We usually write the values of a Young tableau t in the boxes of the Young diagram $Y(\lambda)$.

Example. Take $\lambda = (3, 1) \vdash 4$, so $Y(\lambda) = \ydiagram{3,1}$. Consider the tableau $t : Y(\lambda) \to [4], (1, 1) \mapsto 2, (1, 2) \mapsto 3, (1, 3) \mapsto 4, (2, 1) \mapsto 1$. Then we write this as a labelled Young diagram, namely

$$t = \begin{ytableau} 2 & 3 & 4 \\ 1 \end{ytableau}.$$

The natural permutation action of S_n on $[n]$ extends to a permutation action on Δ^λ:

$$(g \cdot t)(i,j) = g(t(i,j)) \text{ for } (i,j) \in Y(\lambda),\ t \in \Delta^\lambda,$$

i.e. we just apply g to each entry of t.

To continue the example above, take $g = (123) \in S_4$. Then

$$g \cdot t = g \cdot \begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 3&1&4\\\hline 2\\\cline{1-1}\end{array}.$$

Definition. *Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition and $t \in \Delta^\lambda$. For each $1 \le i \le k$, define*

$$R_i(t) := \{t(i,j) \mid 1 \le j \le \lambda_i\}$$

and for each $1 \le j \le \lambda_1$, define

$$C_j(t) = \{t(i,j) \mid 1 \le i \le (\lambda')_j\},$$

i.e. $R_i(t)$, $C_j(t)$ are the sets of entries in the i-th row, resp. j-th column of t.

Definition. *Let $\lambda \vdash n$ and $t, s \in \Delta^\lambda$. We say that t and s are* row-equivalent*, written $t \sim_R s$, if $R_i(t) = R_i(s)$ for all i. Note that $\sim_R$ is an equivalence relation on Δ^λ, we will denote the equivalence classes by $\Omega^\lambda := \Delta^\lambda/\!\sim_R$. Each element of Ω^λ (i.e. equivalence class) will be called a* λ-tabloid*. We write $\{t\}$ for the equivalence class containing $t \in \Delta^\lambda$.*

Example. Consider $\lambda = (3,2) \vdash 5$ and $t = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$, $s = \begin{array}{|c|c|c|}\hline 2&3&1\\\hline 5&4\\\cline{1-2}\end{array}$. Clearly $\{t\} = \{s\}$.

To denote tabloids, we omit the vertical bars, i.e. we write

$$\{t\} = \begin{array}{ccc}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} = \begin{array}{ccc}\hline 2&3&1\\\hline 5&4\\\cline{1-2}\end{array} = \{s\}.$$

The natural permutation of S_n on Δ^λ descends to a well-defined action on Ω^λ.

Definition. *Let $\lambda \vdash n$. The* λ-Young permutation module *M^λ is the S_n-module with permutation basis Ω^λ.*

Lemma 2.1. *Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$. Then $M^\lambda \cong \mathbb{1}_{S_\lambda}\big\uparrow^{S_n}$ where $S_\lambda \cong S_{\lambda_1} \times \cdots \times S_{\lambda_k}$.*

Proof. S_n acts transitively on $[n]$ and so acts transitively on Ω^λ. For $t \in \Delta^\lambda$,

$$\begin{aligned} S_\lambda := \mathrm{Stab}_{S_n}(\{t\}) = \{g \in S_n \mid gR_i(t) = R_i(t)\, \forall i\} &= \mathrm{Sym}(R_1(t)) \times \cdots \times \mathrm{Sym}(R_k(t)) \\ &\cong S_{\lambda_1} \times \dots S_{\lambda_k}. \end{aligned}$$

The claim then follows from Lemma 1.5. $\square$

Remark. The subgroup S_λ of S_n above is called a *Young subgroup of type λ*. There is a Young subgroup of type λ for each set partitions of $[n]$ into subsets of sizes $\lambda_1, \dots, \lambda_k$ and for fixed λ they are all conjugate to each other in S_n, and all isomorphic to $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$.

Example. Take $n = 9$, $\lambda = (4, 3, 2)$. There are $\binom{9}{4}\binom{5}{3}\binom{2}{2} = 1260$ many Young subgroups of type λ.

Examples.

(a) Let $\lambda = (n)$. Then

$$\Omega^\lambda = \{\overline{\underline{1\ 2\ \cdots\ n}}\},$$

and S_n acts trivially on this single λ-tabloid. Then $S_\lambda = S_n$ and $M^{(n)} \cong \mathbb{1}_{S_n}$.

(b) Let $\lambda = (n-1, 1) \vdash n$, for $n \geq 2$. Then

$$\Omega^\lambda = \left\{ \begin{array}{l} \overline{\underline{1\ 2\ \cdots\ i{-}1\ i{+}1\ \cdots\ n}} \\ \underline{j} \end{array} \middle| 1 \leq i \leq n \right\}.$$

Then $S_\lambda \cong S_{n-1} \times S_1 \cong S_{n-1}$, hence $M^{(n-1,1)} \cong \mathbb{1}_{S_{n-1}}\big\uparrow^{S_n} \cong V_n$, the natural permutation representation.

(c) Let $\lambda = (1^n) \vdash n$. Then $\{t\} = \{s\}$ iff $t = s$ for $t, s \in \Delta^\lambda$. So S_λ is trivial and so $M^{(1^n)} \cong \mathbb{1}_1\big\uparrow^{S_n}$ is the regular module $\mathbb{F}S_n$.

Definition. *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ and $t \in \Delta^\lambda$.*

(i) The row stabiliser of t *is*

$$R(t) := \{g \in S_n \mid gR_i(t) = R_i(t)\ \forall i\},$$

and similarly define the column stabiliser $C(t)$.

(ii) The column symmetriser of t *is*

$$\mathfrak{b}_t := \sum_{g \in C(t)} \operatorname{sgn}(g) g \in \mathbb{F}S_n.$$

(iii) The polytabloid corresponding to t, *or* t-polytabloid, *is*

$$e(t) := \mathfrak{b}_t \cdot \{t\} = \sum_{g \in C(t)} \operatorname{sgn}(g) g \cdot \{t\} \in M^\lambda.$$

Note that $e(t)$ depends on the tableau t, not just the tabloid $\{t\}$.

Example. Let $\lambda = (2, 1) \vdash 3$. Then

$$e\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right) = \begin{array}{l}\overline{\underline{1\ 2}} \\ \underline{3}\end{array} - \begin{array}{l}\overline{\underline{3\ 2}} \\ \underline{1}\end{array} \neq \begin{array}{l}\overline{\underline{2\ 1}} \\ \underline{3}\end{array} - \begin{array}{l}\overline{\underline{3\ 1}} \\ \underline{2}\end{array} = e\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right).$$

Definition. *Let $\lambda \vdash n$. The* λ-Specht module *is defined as*

$$\mathcal{S}^\lambda := \langle e(t) \mid t \in \Delta^\lambda \rangle_{\mathbb{F}} \subseteq M^\lambda,$$

i.e. $\mathcal{S}^\lambda$ is the $\mathbb{F}$-vector space spanned by polytabloids corresponding to tableaux of shape λ.

The next lemma shows that $\mathcal{S}^\lambda$ is indeed a module over S_n.

Lemma 2.2. *Let $\lambda \vdash n$ and $t \in \Delta^\lambda$.*

(1) $e(t) \neq 0$

(2) $\forall g \in S_n$, $g \cdot e(t) = e(g \cdot t)$

(3) $\forall g \in C(t)$, $g \cdot e(t) = \operatorname{sgn}(g)e(t)$

(4) $\mathcal{S}^\lambda$ *is a cyclic submodule of* M^λ*, in particular* $\mathcal{S}^\lambda = \mathbb{F}S_n \cdot e(u)$ *for any* $u \in \Delta^\lambda$.

Proof.

(1) Observe that $R(t) \cap C(t) = 1$, and so if $g \in C(t)$ and $g \cdot \{t\} = \{t\}$, then $g = 1$.

It follows that the coefficient of $\{t\}$ in $e(t)$ is $\operatorname{sgn}(1) = 1 \neq 0$, hence $e(t) \neq 0$.

[In fact, $R(t) \cap C(t) = 1$ implies that $e(t)$ is a signed sum of $|C(t)|$ distinct λ-tabloids]

(2) Observe that $C(g \cdot t) = gC(t)g^{-1}$, and so

$$\begin{aligned} g \cdot e(t) &= g \sum_{h \in C(t)} \operatorname{sgn}(h) h \cdot \{t\} \\ &= \sum_{h \in C(t)} \operatorname{sgn}(h)\{gh \cdot t\} \\ &= \sum_{h \in C(t)} \operatorname{sgn}(ghg^{-1}) ghg^{-1} \cdot \{g \cdot t\} \\ &= \sum_{x \in C(g \cdot t)} \operatorname{sgn}(x) x \cdot \{g \cdot t\} = e(g \cdot t). \end{aligned}$$

(3) If $g \in C(t)$, then

$$g \cdot e(t) = \sum_{h \in C(t)} \operatorname{sgn}(h)\{gh \cdot t\} = \sum_{y \in C(t)} \operatorname{sgn}(g^{-1}y)\{y \cdot t\} = \operatorname{sgn}(g)e(t).$$

(4) That $\mathcal{S}^\lambda$ is an S_n-submodule of M^λ follows from (2)

That $\mathcal{S}^\lambda$ can be generated as an $\mathbb{F}S_n$-module by $e(u)$ for any $u \in \Delta^\lambda$ also follows from (2) and the fact that S_n acts transitively on Δ^λ.

□

Examples.

(a) Let $\lambda = (n)$. We have by (1) and (4) of the lemma that $0 \neq \mathcal{S}^\lambda \leq M^\lambda$. But in a previous example we showed that $M^{(n)} \cong \mathbb{1}_{S_n}$. Hence $\mathcal{S}^{(n)} \cong \mathbb{1}_{S_n}$ also.

(b) Let $\lambda = (1^n) \vdash n$. Then $C(t) = S_n$ and thus by the lemma, $g \cdot e(t) = \operatorname{sgn}(g)e(t)$ for all $g \in S_n$, for any $t \in \Delta^\lambda$. Thus, $\dim_{\mathbb{F}}(\mathcal{S}^{(1^n)}) = 1$ and $\mathcal{S}^{(1^n)} = \mathbb{F}S_n \cdot e(t) \cong \operatorname{sgn}_{S_n}$.

(c) Let $\lambda = (2,1) \vdash 3$. Then

$$\mathcal{S}^\lambda = \left\langle e\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\right), e\left(\begin{array}{|c|c|}\hline 2&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\right), e\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\right), e\left(\begin{array}{|c|c|}\hline 3&1\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\right), e\left(\begin{array}{|c|c|}\hline 2&3\\\hline 1&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\right), e\left(\begin{array}{|c|c|}\hline 3&2\\\hline 1&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\right)\right\rangle_{\mathbb{F}}.$$

By (iii) of the lemma,

$$\mathcal{S}^\lambda = \left\langle \alpha := e\left(\begin{array}{|c|c|}\hline 2&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\right), \beta := e\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\right), \gamma := e\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\right)\right\rangle_{\mathbb{F}},$$

since e.g. $e\left(\begin{array}{|c|c|}\hline 3&2\\\hline 1&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\right) = -\alpha$. Moreover,

$$\begin{aligned}
\alpha &= \begin{array}{c}\hline 1\ 2\\\hline 3\\\hline\end{array} - \begin{array}{c}\hline 3\ 2\\\hline 1\\\hline\end{array},\\
\beta &= \begin{array}{c}\hline 2\ 1\\\hline 3\\\hline\end{array} - \begin{array}{c}\hline 3\ 1\\\hline 2\\\hline\end{array},\\
\gamma &= \begin{array}{c}\hline 1\ 3\\\hline 2\\\hline\end{array} - \begin{array}{c}\hline 2\ 3\\\hline 1\\\hline\end{array},
\end{aligned}$$

so $\alpha = \beta + \gamma$. Since β, γ are linearly independent, $\dim \mathcal{S}^\lambda = 2$ for all fields $\mathbb{F}$. See Exercise Sheet 1, Question 4 for more.

2.2 Irreducible modules

Goal: If $\operatorname{char}\mathbb{F} = 0$, then $\{\mathcal{S}^\lambda \mid \lambda \vdash n\}$ is a full set of irreducible $\mathbb{F}S_n$-modules.

Definition. *Let $\lambda \vdash n$. Define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on M^λ via*

$$\langle \{t\}, \{s\} \rangle = \begin{cases} 1 & \textit{if } \{t\} = \{s\}, \\ 0 & \textit{otherwise}, \end{cases}$$

for $t, s \in \Delta^\lambda$ and then extend linearly, i.e. we take the tabloids to be an "orthonormal basis".

We will always take the orthogonal complement $U^\perp$ of a subspace U with respect to this bilinear form.

Lemma 2.3. *Let $\lambda \vdash n$.*

(1) The form $\langle \cdot, \cdot \rangle$ is S_n-invariant, i.e. $\langle gx, gy \rangle = \langle x, y \rangle$ for all $x, y, \in M^\lambda, g \in S_n$.

(2) If U is an S_n-submodule of M^λ, then so is $U^\perp$.

Proof.

(1) This is clearly true for $x = \{t\}, y = \{s\}$, where $t, s \in \Delta^\lambda$, then follows by bilinearity.

(2) This follows from (1): For $x \in U^\perp, g \in S_n$ we have $\langle gx, u\rangle = \langle x, g^{-1}u\rangle = 0$ for all $u \in U$, so $gx \in U^\perp$.

$\square$

Plan:

- *James's Submodule Theorem:* If $U \leq M^\lambda$, then $U \geq \mathcal{S}^\lambda$ or $U \leq (\mathcal{S}^\lambda)^\perp$.
- JST $\implies$ certain quotients of $\mathcal{S}^\lambda$ are irreducible.

This will give us the first part of our goal: $\mathcal{S}^\lambda$ is irreducible when $\operatorname{char}\mathbb{F} = 0$.

Then the second part will be to show that they are pairwise non-isomorphic.

Proposition 2.4. *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$. Suppose $t, u \in \Delta^\lambda$ satisfy $\mathfrak{b}_t \cdot \{u\} \neq 0$. Then*

(1) $\exists h \in C(t)$ such that $h \cdot \{t\} = \{u\}$,

(2) $\mathfrak{b}_t \cdot \{u\} = \pm e(t)$,

(3) $\mathfrak{b}_t \cdot M^\lambda = \mathbb{F}e(t)$.

Proof.

(1) We want to construct $h \in C(t)$ such that $R_i(h \cdot t) = R_i(u)$ for all i.

Claim: $\mathfrak{b}_t \cdot \{u\} \neq 0 \implies$ if $x \neq y$ are the numbers appearing in the same row of u, then they appear in different columns of t.

Proof of claim: Suppose not, so $(xy) \in C(t)$. Take Z to be a set of left coset representatives of $\langle (xy)\rangle$ in $C(t)$, i.e. $C(t) = Z \dot\cup Z(xy)$.

Then $\mathfrak{b}_t = \sum_{g \in C(t)} \operatorname{sgn}(g)g = \sum_{z\in Z} \operatorname{sgn}(z)z(1 - (xy))$. But then

$$\mathfrak{b}_t \cdot \{u\} = \sum_{z \in Z} \operatorname{sgn}(z)z(\{u\} - (xy)\cdot\{u\}) = 0,$$

since $(xy) \in R(u)$ as x, y belong to the same row in u. This concludes the proof of the claim.

Returning to the proof of (1), let $R_1(u) = \{x_1, x_2, \ldots, x_{\lambda_1}\}$. Suppose x_r belongs to column j_r of t, for each $r \in [\lambda_1]$. By the claim the j_r are pairwise distinct. Let $y_r = t((1, j_r))$.

Define $h_1 = \prod_{\substack{r\in[\lambda_1]\\ x_r \neq y_r}} (x_r y_r) \in C(t)$. Then

$$R_1(h_1 \cdot t) = \{h_1(y_1), \ldots, h_1(y_{\lambda_1})\} = \{x_1, \ldots, x_{\lambda_1}\} = R_1(u).$$

Since $h_1 \in C(t)$, then $C(h_1 \cdot t) = h_1 C(t) h_1^{-1} = C(t)$. Thus $\mathfrak{b}_t = \mathfrak{b}_{h_1 t}$, and so $\mathfrak{b}_{h_1 t} \cdot \{u\} \neq 0$.

Let $R_2(u) = \{x'_1, \ldots, x'_{\lambda_2}\}$. Suppose x'_r belongs to column j'_r of $t' = h_1 \cdot t$. By the claim, the j'_r are pairwise distinct. Let $y'_r = t'((2, j'_r))$. Define $h_2 = \prod_{\substack{r\in[\lambda_2] \\ x'_r \neq y'_r}} (x'_r y'_r) \in C(t') = C(t)$. Observe $R_2(h_2 \cdot t') = R_2(u)$ and $R_1(h_2 \cdot t') = R_1(t') = R_1(u)$. That is: $R_i(h_2 h_1 \cdot t) = R_i(u)$ for all $i \in \{1, 2\}$.

Iteratively, we construct for each $m \in \{3, 4, \ldots, k\}$ an element $h_m \in C(t)$ such that $R_i(h_m h_{m-1} \cdots h_1 \cdot t) = R_i(u)$ for all $i \in [m]$. For $m = k$ we get what we want by taking $h = h_k \cdots h_2 h_1$.

(2) Let h be as in (1). Then $\mathfrak{b}_t \cdot \{u\} = \mathfrak{b}_t h \cdot \{t\} = \operatorname{sgn}(h)\mathfrak{b}_t \cdot \{t\} = \operatorname{sgn}(h)e(t)$.

(3) For all $\{u\} \in M^\lambda$ we have either $\mathfrak{b}_t \cdot \{u\} = 0$ or $\mathfrak{b}_t \cdot \{u\} = \pm\{u\}$ by (2), hence $\mathfrak{b}_t \cdot M^\lambda \subseteq \mathbb{F}e(t)$ and equality holds as $\mathfrak{b}_t\{t\} = e(t)$.

□

Theorem 2.5 (James's Submodule Theorem)**.** *Let $\lambda \vdash n$, $U \leq M^\lambda$. Then either $U \geq \mathcal{S}^\lambda$ or $U \leq (\mathcal{S}^\lambda)^\perp$.*

Proof. Suppose $U \not\leq (\mathcal{S}^\lambda)^\perp$, then there exists $x \in U$ and $t \in \Delta^\lambda$ such that $\langle x, e(t)\rangle \neq 0$. Then

$$0 \neq \langle x, e(t)\rangle = \sum_{g\in C(t)} \operatorname{sgn}(g)\langle g^{-1}x, \{t\}\rangle = \langle \mathfrak{b}_t \cdot x, \{t\}\rangle,$$

so in particular $\mathfrak{b}_t \cdot x \neq 0$. By the proposition we have $\mathfrak{b}_t \cdot x = ce(t)$ for some $c \in \mathbb{F}^*$. So from $\mathfrak{b}_t \cdot x \in U$ we get $e(t) \in U$ and thus $\mathcal{S}^\lambda = \mathbb{F}S_n e(t) \subseteq U$. □

Remark. By JST, if we decompose M^λ into a direct sum of indecomposable modules, then there is a unique summand that contains $\mathcal{S}^\lambda$. This module is denoted Y^λ, and called the *Young module* corresponding to λ (more later).

Corollary 2.6. *Let $\lambda \vdash n$. Then $\mathcal{S}^\lambda/(\mathcal{S}^\lambda \cap (\mathcal{S}^\lambda)^\perp)$ is either 0 or irreducible.*

Proof. If $\mathcal{S}^\lambda \leq (\mathcal{S}^\lambda)^\perp$, then the quotient is zero, so now suppose $\mathcal{S}^\lambda \cap (\mathcal{S}^\lambda)^\perp$ is a proper submodule of $\mathcal{S}^\lambda$. Let $U \leq \mathcal{S}^\lambda$. Then $U \leq M^\lambda$, so by JST we have $U = \mathcal{S}^\lambda$ or $U \leq \mathcal{S}^\lambda \cap (\mathcal{S}^\lambda)^\perp$. This tells us that $\mathcal{S}^\lambda/(\mathcal{S}^\lambda \cap (\mathcal{S}^\lambda)^\perp)$ is irreducible. □

Definition. *A representation $\rho : G \to \mathrm{GL}_n(\mathbb{F})$ is* absolutely irreducible *if for any field extension $\mathbb{K}$ of $\mathbb{F}$, the corresponding representation $\bar\rho : G \to \mathrm{GL}_n(\mathbb{K})$ is irreducible.*

Example. Let $G = C_4 = \langle g\rangle$. The representation

$$\rho : G \to \mathrm{GL}_2(\mathbb{Q}),\ \rho(g) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is irreducible, since it has no 1-dimensional submodules (i.e. eigenspaces of $\rho(g)$) when we work over $\mathbb{Q}$. However, it is not absolutely irreducible: $\bar\rho : G \to \mathrm{GL}_2(\mathbb{Q}(i))$ is a direct sum of two 1-dimensional submodules (because the eigenvalues of $\bar\rho(g)$ are $\pm i$).

Theorem 2.7. *Let $\lambda \vdash n$. Then $\mathcal{S}^\lambda/(\mathcal{S}^\lambda \cap (\mathcal{S}^\lambda)^\perp)$ is either 0 or absolutely irreducible.*

Proof. We can extract a basis $e_1, \dots, e_k$ of $\mathcal{S}^\lambda$ consisting of polytabloids. By Lemma 1.6,

$$\dim_{\mathbb{F}} \mathcal{S}^\lambda/(\mathcal{S}^\lambda \cap (\mathcal{S}^\lambda)^\perp) = \operatorname{rank} A$$

where A is the Gram matrix corresponding to $e_1, \dots, e_k$. But $A_{ij} = \langle e_i, e_j\rangle$ belongs to the prime subfield of $\mathbb{F}$ (i.e. $\mathbb{Q}$ or $\mathbb{F}_p$) and so the dimension of $\mathcal{S}^\lambda/(\mathcal{S}^\lambda \cap (\mathcal{S}^\lambda)^\perp)$ doesn't change when we extend $\mathbb{F}$. Since over any field $\mathcal{S}^\lambda/(\mathcal{S}^\lambda \cap (\mathcal{S}^\lambda)^\perp)$ is either 0 or irreducible by our previous result, it is either 0 or absolutely irreducible. $\square$

Corollary 2.8. *If $\operatorname{char} \mathbb{F} = 0$, then $\mathcal{S}^\lambda$ is irreducible for all partitions λ.*

Proof. Over $\mathbb{Q}$ the form $\langle \cdot, \cdot \rangle$ satisfies $\langle u, u\rangle \geq 0$ for all $u \in M^\lambda_{\mathbb{Q}}$, with equality iff $u = 0$. Hence $\mathcal{S}^\lambda_{\mathbb{Q}} \cap (\mathcal{S}^\lambda_{\mathbb{Q}})^\perp = 0$. Thus $\mathcal{S}^\lambda_{\mathbb{Q}}$ is absolutely irreducible by the theorem. Hence $\mathcal{S}^\lambda_{\mathbb{F}}$ is irreducible since $\mathbb{F}$ extends $\mathbb{Q}$. $\square$

Proposition 2.9. *Let $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_s)$ be two partitions of n. Suppose $t \in \Delta^\lambda$ and $u \in \Delta^\mu$ with $\mathfrak{b}_t \cdot \{u\} \neq 0$. Then*

(1) $\exists h \in C(t)$ such that for all $l \in \{1, 2, \dots, \min(k, s)\}$ we have

$$\bigsqcup_{i=1}^{l} R_i(u) \subseteq \bigsqcup_{i=1}^{l} R_i(h \cdot t).$$

(2) $\lambda \trianglerighteq \mu$.

Proof.

(1) Arguing as in the claim in the proof of Proposition 2.4 we have that if $x \neq y$ appear in the same row of u, then they appear different columns of t.

Let $R_1(u) = \{x_1, x_2, \dots, x_{\mu_1}\}$. Suppose x_r lies in column j_r of t, so the j_r are pairwise distinct. Let $y_r = t((1, j_r))$.

Define $h_1 = \prod_{\substack{r \in [\mu_1] \\ x_r \neq y_r}} (x_r y_r) \in C(t)$. Then

$$R_1(u) = \{x_1, \dots, x_{\mu_1}\} = \{h_1(y_1), \dots, h_1(y_{\mu_1})\} \subseteq R_1(h_1 \cdot t).$$

Since $C(h_1 \cdot t) = h_1 C(t) h_1^{-1} = C(t)$, so $\mathfrak{b}_{h_1 \cdot t} = \mathfrak{b}_t$, so $\mathfrak{b}_{h_1 \cdot t} \cdot \{u\} \neq 0$.

Let $R_2(u) = \{x'_1, x'_2, \dots, x'_{\mu_2}\}$ and $t' = h_1 \cdot t$. Suppose $t'((i'_r, j'_r)) = x'_r$. If $i'_r \geq 2$, then let $y'_r = t'((2, j'_r))$. Define $h_2 = \prod_{\substack{r \in [\mu_2] \\ i'_r \geq 2 \\ x'_r \neq y'_r}} (x'_r y'_r) \in C(t') = C(t)$. Then

$$R_2(u) = \{x'_1, \dots, x'_{\mu_2}\} = \{x'_r \mid i'_r \geq 2\} \sqcup \{x'_r \mid i'_r = 1\}$$

$$
\begin{aligned}
&= \{h_2(y'_r)\} \sqcup \{x'_r \mid i'_r = 1\} \\
&\subseteq R_2(h_2 \cdot t') \sqcup R_1(h_1 \cdot t')
\end{aligned}
$$

Also $R_1(u) \subseteq R_1(t') = R_1(h_2 \cdot t')$. Therefore $\bigsqcup_{i=1}^{l} R_i(u) \subseteq \bigsqcup_{i=1}^{l} R_i(h_2 h_1 \cdot t)$ for all $l \in \{1, 2\}$. Now induct.

(2) By (1),

$$
\sum_{i=1}^{l} \mu_i = \sum_{i=1}^{l} |R_i(u)| \leq \sum_{i=1}^{l} |R_i(h \cdot t)| = \sum_{i=1}^{l} \lambda_i,
$$

for all $l = 1, \ldots, \min(k, s)$.

$\square$

Theorem 2.10. *Let $\lambda, \mu \vdash n$. Suppose $0 \neq \phi \in \operatorname{Hom}_{\mathbb{F}S_n}(\mathcal{S}^\lambda, M^\mu)$. If there exists $\widetilde{\phi} \in \operatorname{Hom}_{\mathbb{F}S_n}(M^\lambda, M^\mu)$ extending ϕ, then $\lambda \trianglerighteq \mu$.*

Proof. Since $\mathcal{S}^\lambda = \mathbb{F}S_n \cdot e(t)$ for any $t \in \Delta^\lambda$, then $\phi(e(t)) \neq 0$ as $\phi \neq 0$. Fix any $t \in \Delta^\lambda$. Then $0 \neq \phi(e(t)) = \widetilde{\phi}(e(t)) = \widetilde{\phi}(\mathfrak{b}_t \cdot \{t\}) = \mathfrak{b}_t \cdot \widetilde{\phi}(\{t\})$. Writing $\phi(\{t\})$ as a sum of μ-tabloids, we see that there is $u \in \Delta^\mu$ such that $\mathfrak{b}_t \cdot \{u\} \neq 0$, so we are done by the proposition. $\square$

Example. Let $\operatorname{char} \mathbb{F} = 2$, $n = 2$, $\lambda = (1^2)$, $\mu = (2)$. Then $\mathcal{S}^{(1^2)} \cong \operatorname{sgn}_{S_2} \cong \mathbb{1}_{S_2} \cong M^{(2)}$, and so $\operatorname{Hom}_{\mathbb{F}S_2}(\mathcal{S}^\lambda, M^\mu) \neq 0$, in particular, it contains isomorphisms.

On the other hand, $M^\lambda = \left\langle \frac{\overline{1}}{\overline{2}}, \frac{\overline{2}}{\overline{1}} \right\rangle$ and if $\theta : M^\lambda \to M^\mu$ is $\mathbb{F}S_2$ linear, then $\theta(\frac{\overline{1}}{\overline{2}}) = (12)\theta(\frac{\overline{2}}{\overline{1}}) = \theta(\frac{\overline{2}}{\overline{1}})$. In particular $\theta(e(\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}})) = \theta(\frac{\overline{1}}{\overline{2}}) - \theta(\frac{\overline{2}}{\overline{1}}) = 0$. So for any $\theta \in \operatorname{Hom}_{\mathbb{F}S_2}(M^\lambda, M^\mu)$ we have $\theta|_{\mathcal{S}^\lambda} = 0$, in particular not all $\phi \in \operatorname{Hom}_{\mathbb{F}S_2}(\mathcal{S}^\lambda, M^\mu)$ have extensions to M^λ.

Corollary 2.11. *If $\operatorname{char} \mathbb{F} = 0$, $\lambda, \mu \vdash n$, then $\mathcal{S}^\lambda \cong \mathcal{S}^\mu$ iff $\lambda = \mu$.*

Proof. Suppose $\mathcal{S}^\lambda \cong \mathcal{S}^\mu$, take an isomorphism $\mathcal{S}^\lambda \to \mathcal{S}^\mu$ and compose this with the natural inclusion $\mathcal{S}^\mu \to M^\mu$ to get $0 \neq \phi \in \operatorname{Hom}_{\mathbb{F}S_n}(\mathcal{S}^\lambda, M^\mu)$. By Maschke's Theorem there exists $V \leq M^\lambda$ such that $M^\lambda = \mathcal{S}^\lambda \oplus V$. And so we can extend ϕ to $\widetilde{\phi} \in \operatorname{Hom}_{\mathbb{F}S_n}(M^\lambda, M^\mu)$ by setting $\widetilde{\phi}|_V = 0$, so $\lambda \trianglerighteq \mu$ by the theorem. By symmetry we also have $\mu \trianglerighteq \lambda$, so $\lambda = \mu$. $\square$

So far we showed: If $\operatorname{char} \mathbb{F} = 0$, then

- each $\mathcal{S}^\lambda$ is irreducible,
- the $\mathcal{S}^\lambda$ are pairwise non-isomorphic.

If $\mathbb{F} = \mathbb{C}$, then $|\operatorname{Irr}_{\mathbb{C}}(S_n)| = \#$conjugacy classes of $S_n = |\wp(n)|$, so

$$
\operatorname{Irr}_{\mathbb{C}}(S_n) = \{\mathcal{S}^\lambda_{\mathbb{C}} \mid \lambda \vdash n\}.
$$

We now extend this to arbitrary fields of characteristic 0.

Theorem 2.12. *If* $\operatorname{char}\mathbb{F} = 0$*, then* $\operatorname{Irr}_{\mathbb{F}}(S_n) = \{\mathcal{S}^\lambda_{\mathbb{F}} \mid \lambda \vdash n\}$*.*

We already know that $|\operatorname{Irr}_{\mathbb{F}}(S_n)| \geq |\wp(n)|$. We now want to prove the reverse inequality.

Definition. $\mathbb{F}$ *is a splitting field for the finite group* G *if every irreducible* $\mathbb{F}G$*-representation is absolutely irreducible.*

Fact. If $\mathbb{F} = \mathbb{F}^{\mathrm{alg}}$, then $\mathbb{F}$ is a splitting field. See [Isa76, Corollary 9.4]

Theorem 2.13. *If* $\mathbb{F}$ *is a splitting field for* G*, and* $\mathbb{K}$ *a field extension of* $\mathbb{F}$*, then* $\mathbb{K}$ *is also a splitting field for* G*, and* $|\operatorname{Irr}_{\mathbb{K}}(G)| = |\operatorname{Irr}_{\mathbb{F}}(G)|$*.*

Proof. See [Isa76, Corollary 9.8]. □

Fact. Every field is a splitting field for S_n. See [JK84, Theorem 2.1.12] and [CR62].

So in particular, $\mathbb{Q}$ is a splitting field for S_n. Hence $|\operatorname{Irr}_{\mathbb{F}}(S_n)| = |\operatorname{Irr}_{\mathbb{Q}}(S_n)| = |\operatorname{Irr}_{\mathbb{C}}(S_n)| = |\wp(n)|$.

Alternatively, one can use the following:

Theorem 2.14. *Let* $\mathbb{K}$ *be a field with* $\operatorname{char}\mathbb{K} \nmid |G|$*. Then* $|\operatorname{Irr}_{\mathbb{K}}(G)| \leq \#$*conjugacy classes of* G*. If* $\mathbb{K} = \mathbb{K}^{\mathrm{alg}}$*, then equality holds.*

Proof. See Moodle for a sketch using the Artin-Wedderburn theorem. □

Proof of Theorem 2.12. Corollary 2.8 and Corollary 2.11 show that the $\mathcal{S}^\lambda$ are pairwise distinct and irreducible. Then the claim follows either from Theorem 2.13 and the fact or from Theorem 2.14. □

Remarks.

- Modular representation theory: $\operatorname{char} = p > 0$, ordinary representation theory: $\operatorname{char} = 0$.
- If $\operatorname{char} = p > 0$, but $p \nmid |G|$, then the situation is similar to $\operatorname{char} = 0$.
- If $\operatorname{char} = p \mid |G|$, the situation is very different.
- For $\operatorname{char}(\mathbb{F}) = p > 0$:

$$\operatorname{Irr}_{\mathbb{F}}(S_n) = \left\{ \frac{\mathcal{S}^\lambda}{\mathcal{S}^\lambda \cap (\mathcal{S}^\lambda)^\perp} \middle| \lambda \vdash n \text{ is “}p\text{-regular”} \right\}.$$

Theorem 2.15 (Brauer)**.** *Suppose* $\operatorname{char}\mathbb{F} = p > 0$*. Then the number of isomorphism classes of absolutely irreducible* $\mathbb{F}G$*-modules is at most the number of* p*-regular conjugacy classes of* G*. If* $\mathbb{F}$ *is a splitting field for* G*, then equality holds.*

Proof. See [CR62, pp. 82.6, 83.6] □

Definition. *Let p be a prime.*

*(i) A partition $\lambda = (\lambda_1, \dots, \lambda_k)$ is p-*singular* if it has at least p equal parts, i.e. there exists $i \in [k-p+1]$ such that $\lambda_i = \lambda_{i+1} = \dots = \lambda_{i+p-1}$. Otherwise, λ is called p-*regular*.*

*(ii) An element $g \in G$ is p-*regular* if $p \nmid \operatorname{ord} g$. A conjugacy class of G is p-*regular* if its elements are p-regular.*

If $g \in S_n$, then g is p-regular iff in its disjoint cycle decomposition, no cycle has length divisible by p.

Proposition 2.16. *Let p be a prime, $n \in \mathbb{N}$. Then*

$$\#\{p\text{-regular } \lambda \vdash n\} = \#\{\lambda \vdash n \mid p \nmid \lambda_i \, \forall i\}.$$

Proof. **Proof 1.** The generating function for all partitions is

$$G(x) = \sum_{n \geq 0} |\wp(n)| x^n = \prod_{i \in \mathbb{N}} (1 + x^i + x^{2i} + \dots) = \prod_{i \in \mathbb{N}} \frac{1}{1 - x^i}$$

where a partition with a_i many parts of size i corresponds to choosing the x^{ia_i} term from the i-th bracket when we multiply out. The generating function for p-regular partitions is

$$\begin{aligned} F(x) = \sum_{n \geq 0} \#\{p\text{-regular } \lambda \vdash n\} x^n &= \prod_{i \in \mathbb{N}} (1 + x^i + \dots + x^{(p-1)i}) \\ &= \prod_{i \in \mathbb{N}} \frac{1 - x^{pi}}{1 - x^i} \\ &= \prod_{i \in \mathbb{N},\, p \nmid i} \frac{1}{1 - x^i} \\ &= \sum_{n \geq 0} \#\{\lambda \vdash n \mid p \nmid \lambda_i \, \forall i\}. \end{aligned}$$

Proof 2. Consider

$$\{p\text{-regular } \lambda \vdash n\} \underset{\varphi}{\overset{\theta}{\rightleftarrows}} \{\lambda \vdash n \mid p \nmid \lambda_i \, \forall i\}$$

where θ, φ are as follows:

- θ: If λ has a part of size divisibly by p, break it into p equal parts; repeat until there are no more parts of size divisible by p.
- φ: For each s, suppose λ has $\sum_{i \geq 0} a_i p^i$ parts of size s where $0 \leq a_i \leq p-1$. Glue them together to form a_i many parts of size sp^i for each i.

Then check that θ, φ are inverses. ☐

In fact, the proposition and both proofs hold for all $p \in \mathbb{N}$ (not necessarily prime), provided we extend the definition accordingly.

2.3 Standard Basis Theorem

We have $\mathcal{S}^\lambda = \langle e(t) \mid t \in \Delta^\lambda \rangle_{\mathbb{F}}$. Our goal for this section is to extract a basis of polytabloids for $\mathcal{S}^\lambda$, uniform over all $\mathbb{F}$, thereby computing $\dim \mathcal{S}^\lambda$ (independently of $\mathbb{F}$).

Definition. *Let $\lambda \vdash n$, $t \in \Delta^\lambda$. Then we say*

- *t is* row-standard *if the entries of t increase along rows from left to right, i.e. $t((i,j)) < t((i,j+1))$ for all $i \in [\ell(\lambda)], j \in [\lambda_i - 1]$,*
- *t is* column-standard *if the entries of t increase along columns from top to bottom, i.e. $t((i,j)) < t((i+1,j))$ for all $j \in [\lambda_1], i \in [(\lambda')_j - 1]$,*
- *t is* standard *if it is both row- and column-standard.*

Define $\operatorname{std}(\lambda) = \{t \in \Delta^\lambda \mid t$ *is standard*$\}$. *We say a polytabloid $e(t)$ is standard if t is.*

Examples.

- Let $\lambda = (n)$, so $\dim \mathcal{S}^\lambda = 1$ and $\operatorname{std}(\lambda) = \{\begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n \\ \hline\end{array}\}$.
- Let $\lambda = (1^n) \vdash n$, so $\dim \mathcal{S}^\lambda = 1$ and $\operatorname{std}(\lambda) = \{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \cdots \\ \hline n \\ \hline\end{array}\}$.
- Let $\lambda = (2,1)$. We have seen earlier that then $\dim \mathcal{S}^\lambda = 2$. Then

$$\operatorname{std}(\lambda) = \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \right\}.$$

- More generally, let $\lambda = (n-1,1)$ with $n \geq 2$. Then $\dim \mathcal{S}^\lambda = n-1$ by Example Sheet 1, Question 5, and

$$\operatorname{std}(\lambda) = \left\{ \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & \cdots & \widehat{j} & \cdots & n \\ \hline j \\ \cline{1-1}\end{array} \,\middle|\, 2 \leq j \leq n \right\}$$

Our aim will be to show that $\{e(t) \mid t \in \operatorname{std}(\lambda)\}$ is an $\mathbb{F}$-basis for $\mathcal{S}^\lambda$.

For linear independence, we begin by putting a total order on Ω^λ, the set of all tableaux of shape λ.

Definition. *Let $\lambda \vdash n$, $t, u \in \Delta^\lambda$. Let*

$$\begin{aligned} A &= \{\textit{numbers that don't appear in the same row of } t \textit{ and } u\} \\ &= [n] \setminus \bigcup_{i=1}^{\ell(\lambda)} R_i(t) \cap R_i(u). \end{aligned}$$

If $\{t\} \neq \{u\}$, equivalently $A \neq \emptyset$, then let $y = \max(A)$. We say $\{t\} > \{u\}$ if $y \in R_i(t) \cap R_j(u)$ where $i > j$.

Remark. Note that $>$ is a total order on Ω^λ; it is equivalent to a total order on the set of all row-standard λ-tableaux. The maximal element w.r.t. $>$ is

$$\begin{array}{l}\hline 1\ \ 2\ \ 3\ \cdots \lambda_1 \\ \hline \lambda_1{+}1\ \ \cdots\ \lambda_1{+}\lambda_2 \\ \hline \vdots \\ \hline \cdots\ n \\ \hline \end{array}$$

Small example: Take $\lambda = (3,2)$, $t = \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5\\ \cline{1-2}\end{array}$, $u = \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5\\ \cline{1-2}\end{array}$. Then $A = \{3,4\}$, so $y = 4$ and $\{t\} > \{u\}$.

Lemma 2.17. *Let $\lambda \vdash n$, $t \in \Delta^\lambda$ column-standard. Let $h \in C(t)\setminus\{1\}$. Then $\{h\cdot t\} < \{t\}$.*

Proof. Since $h \neq 1$ and $R(t) \cap C(t) = \{1\}$, then $\{h \cdot t\} \neq \{t\}$. Then

$$y := \max\Big([n] \setminus \bigcup_{i=1}^{\ell(\lambda)} R_i(t) \cap R_i(h \cdot h)\Big)$$

exists. Suppose $y = t((i,j))$. Where is y in $h \cdot t$? Since $h \in C(t)$, then $y \in C_j(h \cdot t)$, say $y \in R_{i'}(h \cdot t)$. First, $i' \neq i$ by definition of y. But also, $i' \not> i$ since the entries in column j below row i must match exactly in t and $h \cdot t$ by maximality of y and column-standardness of t. Hence $i' < i$, so $\{h \cdot t\} < \{t\}$. $\square$

Proposition 2.18. *Let $\lambda \vdash n$. Then the $e(t)$ with $t \in \mathrm{std}(\lambda)$ are linearly independent.*

Proof. Suppose not. Then there exists $\emptyset \neq \Delta \subseteq \mathrm{std}(\lambda)$ such that $\sum_{t\in\Delta} \alpha_t e(t) = 0$ where $\alpha_t \in \mathbb{F}^\times$. For $t, u \in \mathrm{std}(\lambda)$, we have $\{t\} = \{u\}$ iff $t = u$. So there is a unique $m \in \Delta$ such that $\{m\} > \{t\}$ for all $t \in \Delta, t \neq m$. For $t \in \Delta^\lambda$, recall $e(t) = \sum_{g\in C(t)} \mathrm{sgn}(g) g \cdot \{t\}$, so by the lemma,

$$e(t) = \{t\} + (\text{a signed sum of tabloids} < \{t\}).$$

Therefore,

$$0 = \alpha_m e(m) + \sum_{\substack{t\in\Delta\\ t\neq m}} \alpha_t e(t) = \alpha_m\{m\} + X \in M^\lambda,$$

where X is a linear combination of tabloids $< \{m\}$. Hence $\alpha_m = 0$, a contradiction. $\square$

To show that the $e(t)$ for $t \in \mathrm{std}(\lambda)$ span $\mathcal{S}^\lambda$, we want to find elements of $\mathbb{F}S_n$ that annihilate a given $e(t)$.

Definition. *Let $\lambda \vdash n$, $t \in \Delta^\lambda$. Let $X \subseteq C_j(t)$ and $Y \subseteq C_{j+1}(t)$ for some $j \in [\lambda_1 - 1]$. Then choose T a set of left coset representatives for $S_X \times S_Y$ in $S_{X\sqcup Y}$ where we abbreviate* $\mathrm{Sym}(X) =: S_X$*, etc. Define the* Garnir element $G_{X,Y} := \sum_{g\in T} \mathrm{sgn}(g) g \in \mathbb{F}S_n$.

Example. Let $\lambda = (2,1)$, $t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, $j = 1$, $X = \{1,3\}$, $Y = \{2\}$. Then choose $T = \{1, (12), (23)\}$ for $S_X \times S_Y = \langle(13)\rangle \times 1$ in S_3. Then $G_{X,Y} = 1 - (12) - (23)$. Observe

$$\begin{aligned} G_{X,Y}e(t) &= (1-(12)-(23))\left(\begin{array}{l}\hline 1\ 2 \\ \hline 3 \\ \hline\end{array} - \begin{array}{l}\hline 3\ 2 \\ \hline 1 \\ \hline\end{array}\right) \\ &= \left(\begin{array}{l}\hline 1\ 2 \\ \hline 3 \\ \hline\end{array} - \begin{array}{l}\hline 3\ 2 \\ \hline 1 \\ \hline\end{array}\right) - \left(\begin{array}{l}\hline 2\ 1 \\ \hline 3 \\ \hline\end{array} - \begin{array}{l}\hline 3\ 1 \\ \hline 2 \\ \hline\end{array}\right) - \left(\begin{array}{l}\hline 1\ 3 \\ \hline 2 \\ \hline\end{array} - \begin{array}{l}\hline 2\ 3 \\ \hline 1 \\ \hline\end{array}\right) \\ &= 0. \end{aligned}$$

Proposition 2.19. *Let $\lambda \vdash n$, $t \in \Delta^\lambda$, $j \in [\lambda_1 - 1]$, $X \subseteq C_j(t)$, $Y \subseteq C_{j+1}(t)$. Choose a set T of left coset representatives for $S_X \times S_Y$ in $S_{X \sqcup Y}$. Then if $|X| + |Y| > (\lambda')_j$, the length of the j-th column of $Y(\lambda)$, then $G_{X,Y} \cdot e(t) = 0$*

Proof. Consider $G_{X\sqcup Y} := \sum_{\rho \in S_{X\sqcup Y}} \operatorname{sgn}(\rho)\rho \in \mathbb{F}S_n$. Then

$$G_{X\sqcup Y} = \sum_{g\in T}\sum_{h\in S_X}\sum_{k \in S_Y} \operatorname{sgn}(ghk)ghk = \underbrace{\Big(\sum_{g\in T}\operatorname{sgn}(g)g\Big)}_{=G_{X,Y}}\Big(\sum_{h\in S_X}\operatorname{sgn}(h)h\Big)\Big(\sum_{k\in S_Y}\operatorname{sgn}(k)k\Big).$$

Recall from Lemma 2.2: For $\sigma \in C(t)$, $\sigma \cdot e(t) = \operatorname{sgn}(\sigma)e(t)$, and note $S_X, S_Y \subseteq C(t)$, so

$$G_{X\sqcup Y} \cdot e(t) = G_{X,Y}|S_X||S_Y|e(t) = |X|!|Y|!(G_{X,Y} \cdot e(t)).$$

We will show $G_{X\sqcup Y} \cdot e(t) = 0$. If $\operatorname{char}\mathbb{F} = 0$, then we immediately deduce $G_{X,Y} \cdot e(t) = 0$, but in positive characteristic we could have $|X|!|Y|! = 0$. But once we have that $G_{X,Y} \cdot e(t) = 0$ holds in characteristic 0, then $G_{X,Y} \cdot e(t)$ is just an integer linear combination of tabloids, so we can reduce the coefficients mod p to obtain $G_{X,Y} \cdot e(t) = 0$, viewed as an $\mathbb{F}_p$-linear combination. Hence we have $G_{X,Y} \cdot e(t) = 0$ for all fields.

It remains to show $G_{X\sqcup Y} \cdot e(t) = 0$. For $\sigma \in C(t)$, since $|X| + |Y| > (\lambda')_j$, there exist $x_\sigma \in X, y_\sigma \in Y$ such that x_σ, y_σ lie in the same row of $\sigma \cdot t$, i.e. $(x_\sigma y_\sigma) \cdot \{\sigma \cdot t\} = \{\sigma \cdot t\}$. Let Z be a set of left coset representatives for $\langle(x_\sigma y_\sigma)\rangle$ in $S_{X\sqcup Y}$, i.e. $S_{X\sqcup Y} = Z \sqcup Z(x_\sigma y_\sigma)$. Then

$$G_{X\sqcup Y} \cdot \{\sigma \cdot t\} = \sum_{z\in Z}\operatorname{sgn}(z)z(1-(x_\sigma y_\sigma)) \cdot \{\sigma \cdot t\} = 0.$$

Thus

$$G_{X\sqcup Y} \cdot e(t) = \sum_{\sigma \in C(t)} \operatorname{sgn}(\sigma) G_{X\sqcup Y} \cdot \{\sigma \cdot t\} = 0.$$

□

Definition. *Let $\lambda \vdash n$, $t, u \in \Delta^\lambda$ column-standard. Let*

$$B = \{\textit{numbers not in the same column of } t \textit{ and } u\}$$

$$= [n] \setminus \bigcup_{j=1}^{\lambda_1} C_j(t) \cap C_j(u).$$

If for all $\sigma \in C(t)$, $\sigma \cdot t \neq u$, then $B \neq \emptyset$, so $\max B =: x$ exists. In this case, we say $t \gg u$ if $x \in C_i(t) \cap C_j(u)$ where $i > j$.

Remark. This is the column analogue of the ordering $>$ defined earlier, except we defined it on tabloids earlier. The maximal column standard tableau w.r.t. $\gg$ is

$$\begin{array}{|c|c|c|c|}
\hline
1 & \lambda_1'{+}1 & \dots & \\
\hline
2 & \lambda_1'{+}2 & \dots & n \\
\hline
\vdots & \vdots \\
\cline{1-2}
\lambda_1' \\
\cline{1-1}
\end{array}$$

Note that this tableau is standard.

Proposition 2.20. *Let $\lambda \vdash n$, $v \in \Delta^\lambda$ column-standard. Then $e(v) \in \langle e(t) \mid t \in \operatorname{std}(\lambda)\rangle_{\mathbb{F}}$.*

Proof. Let $W = \langle e(t) \mid t \in \operatorname{std}(\lambda)\rangle_{\mathbb{F}}$. Let the column-standard λ-tableaux be $t_1 \gg t_2 \gg t_3 \gg \dots$. We prove by induction on r that $e(t_r) \in W$.

Base case $r = 1$: t_1 is standard, see the remark above, so $e(t_1) \in W$.

Inductive step: Suppose $t = t_r$ where we have already shown that $e(t_s) \in W$ for all $s < r$, i.e. whenever u is column-standard and $u \gg t$, then $e(u) \in W$. Then we want to show $e(t) \in W$. If t is row-standard, then t is standard and so $e(t) \in W$. Otherwise, $t((i,j)) > t((i,j+1))$ for some $i \in [\ell(\lambda)], j \in [\lambda_i - 1]$. Define $X = \{t((l,j)) \mid i \leq j \leq (\lambda')_j\}$ and $Y = \{t((l,j+1)) \mid 1 \leq l \leq i\}$. Then $G_{X,Y} \cdot e(t) = 0$ by Proposition 2.19, where $G_{X,Y}$ is defined w.r.t. any set T of coset representatives of $S_X \times S_Y$ in $S_{X \sqcup Y}$. Choose $1 \in T$. Then

$$0 = G_{X,Y} \cdot e(t) = e(t) + \sum_{g \in T \setminus \{1\}} \operatorname{sgn}(g) g \cdot e(t).$$

We will prove that $e(g \cdot t) \in W$ for all $g \in T \setminus \{1\}$. Then we also get $e(t) \in W$ from this relation. Fix $g \in T \setminus \{1\}$. Since $g \notin S_X \times S_Y$, we must have some $y \in Y$ such that $g(y) \in X$. Hence $A := \{g(y) \mid y \in Y,\, g(y) \in X\} \neq \emptyset$. It is easy to see that $A = X \cap C_{j+1}(g \cdot t)$.

Consider $B := [n] \setminus \bigcup_{l=1}^{\lambda_1} C_l(t) \cap C_l(g \cdot t) \subseteq X \sqcup Y$. Moreover,

$$\begin{aligned} B &= \{x \in X \mid x \in C_{j+1}(g \cdot t)\} \sqcup \{y \in Y \mid y \in C_j(g \cdot t)\} \\ &= \underbrace{(X \cap C_{j+1}(g \cdot t))}_{=A \neq \emptyset} \sqcup (Y \cap C_j(g \cdot t)) \end{aligned}$$

Therefore $\max(B) = \max(A) \in X \cap C_{j+1}(g \cdot t)$ (using that t is column-standard and $t((i,j)) > t((i,j+1))$). Let u be the unique column-standard λ-tableau such that $C_l(u) =$

$C_l(g \cdot t)$ for all l. Then $B = [n] \setminus \bigcup_{l=1}^{\lambda_1} C_l(t) \cap C_l(u)$. We have shown that $\max(B) \in X \cap C_{j+1}(g \cdot t) \subseteq C_j(t) \cap C_{j+1}(u)$, hence $u \gg t$, so $e(u) \in W$ by inductive hypothesis. There exists $\sigma \in C(u)$ such that $\sigma \cdot u = g \cdot t$, and so $e(g \cdot t) = e(\sigma \cdot u) = \sigma \cdot e(u) = \pm e(u)$. Therefore, $e(g \cdot t) \in W$ as desired. □

Theorem 2.21 (Standard Basis Theorem)**.** *Let* $\lambda \vdash n$, $\mathbb{F}$ *any field. Then* $\{e(t) \mid t \in \mathrm{std}(\lambda)\}$ *is a basis for* $\mathcal{S}^\lambda$, *called the* standard basis.

Proof. Linear independence holds by Proposition 2.18. For span, let $v \in \Delta^\lambda$. Then there is a $g \in C(v)$ such that $u := g \cdot v$ is column standard. By Proposition 2.20, $e(u) \in \langle e(t) \mid t \in \mathrm{std}(\lambda) \rangle_{\mathbb{F}}$. But $e(u) = \pm e(v)$, so we are done. □

Note that the standard basis is not a permutation basis in general: $g \cdot e(t) = e(g \cdot t)$ for all $g \in S_n, t \in \Delta^\lambda$. But there are many g, t such that $t \in \mathrm{std}(\lambda)$, but $g \cdot t$ is not.

Corollary 2.22. *For* $\lambda \vdash n$, *any field* $\mathbb{F}$,

$$\dim_{\mathbb{F}} \mathcal{S}^\lambda = \#\textit{standard } \lambda\textit{-tableaux}.$$

3 Character Theory

From now on, $\mathbb{F} = \mathbb{C}$, unless otherwise stated.

Notation. Let $\lambda \vdash n$. We will let χ^λ denote the character of the irreducible λ-Specht module.

3.1 Hook Length Formula

Goal. Prove the hook length formula, a closed formula for calculating $\dim \mathcal{S}^\lambda = \chi^\lambda(1)$.

Definition. *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$. Write $\lambda' = (\mu_1, \ldots, \mu_{\lambda_1})$.*

*(i) For a box $(i,j) \in Y(\lambda)$, the (i,j)-*hook* of λ is*

$$H_{i,j}(\lambda) := \{(i,j)\} \sqcup \underbrace{\{(i,y) \mid j < y \le \lambda_i\}}_{\text{arm}} \sqcup \underbrace{\{(x,j) \mid i < x \le \mu_j\}}_{\text{leg}}$$

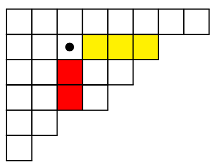
(ii) The arm *of $H_{i,j}(\lambda)$ is $\{(i,j) \mid j < y \le \lambda_i\}$, the* leg *is $\{(x,j) \mid i < x \le \mu_j\}$.*

(iii) The hand *of $H_{i,j}(\lambda)$ is the box (i, λ_i), the* foot *is (μ_i, j).*

(iv) The hook length *corresponding to (i,j) is $|H_{i,j}(\lambda)| =: h_{i,j}(\lambda)$.*

(v) Let $\mathcal{H}(\lambda) = \{h_{i,j}(\lambda) \mid (i,j) \in Y(\lambda)\}$ be the multiset *of hook lengths of λ (i.e. we also count repetitions of the same hook length).*

Example. Take $\lambda = (8,6,5,4,2,1) \vdash 26$, $(i,j) = (2,3)$. Then the hook is $\{\bullet\} \sqcup \text{arm} \sqcup \text{leg}$ as indicated in the diagram.



Theorem 3.1 (Hook Length Formula)**.** *Let $\lambda \vdash n$. Then*

$$\chi^\lambda(1) = \frac{n!}{\prod_{h \in \mathcal{H}(\lambda)} h}.$$

Examples.

(a) Let $\lambda = (n)$. List the hook lengths in $Y(\lambda)$: $\begin{array}{|c|c|c|c|}\hline n & \cdots & 2 & 1 \\ \hline \end{array}$. So $\chi^\lambda(1) = \frac{n!}{n!} = 1$. This is not unexpected as we already knew that $\mathcal{S}^\lambda \cong \mathbb{1}_{S_n}$.

(b) Let $\lambda = (3, 2) \vdash 5$. Then

$$\operatorname{std}(\lambda) = \left\{ \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4 \\ \cline{1-2}\end{array} \right\},$$

so $\chi^\lambda(1) = 5$ by the standard basis theorem. This is consistent with the hook length formula. Indeed, the hook lengths are $\begin{array}{|c|c|c|}\hline 4&3&1\\ \hline 2&1 \\ \cline{1-2}\end{array}$, so $\chi^\lambda(1) = \frac{5!}{4 \cdot 3 \cdot 2} = 5$.

(c) Let $\lambda = (6, 4, 4, 3, 2, 1, 1) \vdash 21$. Then the hook lengths are

$$\begin{array}{|c|c|c|c|c|c|}\hline 12&9&7&5&2&1\\ \hline 9&6&4&2 \\ \cline{1-4} 8&5&3&1 \\ \cline{1-4} 6&3&1 \\ \cline{1-3} 4&1 \\ \cline{1-2} 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

Therefore

$$\chi^\lambda(1) = \frac{21!}{\prod_{h \in \mathcal{H}(\lambda)} h} = 905304400.$$

We give a probabilistic proof of the hook length formula due to Greene, Nijenhuis and Wilf (1979). Another proof will be on the example sheets. The proof will be by induction on n.

Definition. *By a* composition *of n, we mean a sequence of non-negative integers which sum to n, written $\lambda \models n$.*

Define a function F on $\{\lambda \mid \lambda \models n\}$ as follows:

$$F(\lambda) = \begin{cases} \frac{n!}{\prod_{h \in \mathcal{H}(\lambda)} h} & \text{if } \lambda \vdash n, \\ 0 & \text{otherwise.} \end{cases}$$

If $\lambda = (\lambda_1, \dots, \lambda_k) \models n$, we want the inductive step to look like

$$F(\lambda) = \sum_{i=1}^{k} F(\underbrace{(\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots, \lambda_k)}_{\models n-1 \text{ if } \lambda_i \geq 1}).$$

Definition. *Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$. Define*

$$\begin{aligned} \lambda^- :&= \{\mu \vdash n-1 \mid Y(\mu) \textit{ can be obtained from } Y(\lambda) \textit{ by removing one box}\} \\ &= \{(\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots, \lambda_k) \mid i \in [k] \textit{ such that } \lambda_i - 1 \geq \lambda_{i+1}\}. \end{aligned}$$

(Here we treat $\lambda_{k+1} = 0$.)

We say the box (i, j) of $Y(\lambda)$ is removable *if $Y(\lambda) \setminus \{(i, j)\} = Y(\mu)$ for some $\mu \in \lambda^-$.*

Example. Let $\lambda = (3,3,1) \vdash 7$, so $Y(\lambda) = \ydiagram{3,3,1}$. Then

$$\lambda^- = \left\{ \overset{(3,3)}{\ydiagram{3,3}}\ ,\ \overset{(3,2,1)}{\ydiagram{3,2,1}} \right\}.$$

Observe that $\chi^\lambda(1) = \sum_{\mu\in\lambda^-} \chi^\mu(1)$. This follows from the Standard Basis Theorem. Indeed, $\chi^\lambda(1) = |\operatorname{std}(\lambda)|$ and in a standard λ-tableau, $\lambda \vdash n$, the number n must appear in a removable box, which when removed, leaves a standard μ-tableau for some $\mu \in \lambda^-$.

We would be able to prove Theorem 3.1 by induction on n if we can show

$$F(\lambda) = \sum_{\mu\in\lambda^-} F(\mu),$$

because we would have $\sum_{\mu\in\lambda^-} F(\mu) = \sum_{\mu\in\lambda^-} \chi^\mu(1)$ by the inductive hypothesis.

We will in fact show that $1 = \sum_{\mu\in\lambda^-} \frac{F(\mu)}{F(\lambda)}$ by interpreting $\frac{F(\mu)}{F(\lambda)}$ as probabilities. For the rest of this section, fix $\lambda \vdash n$, and abbreviate $H_{i,j}(\lambda) = H_{i,j}$ and $h_{i,j}(\lambda) = h_{i,j}$.

Consider the following probabilistic process on $Y(\lambda)$:

- **Step 1.** Pick a box of $Y(\lambda)$ uniformly at random (probability $= \frac{1}{n}$).
- **Step 2.** Suppose that (i,j) is the currently chosen box. If (i,j) is removable, equivalently $h_{i,j} = 1$, then terminate the process. Otherwise, choose $(i',j') \in H_{i,j} \setminus \{(i,j)\}$ (probability $= \frac{1}{h_{i,j}-1}$).
- **Step 3.** Repeat Step 2 until we terminate.

We will call each run of the process a *trial.*

Definition. *For $(\alpha,\beta) \in Y(\lambda)$, let $\mathbb{P}(\alpha,\beta)$ be the probability that a trial terminates at (α,β).*

Our aim is to show that $\mathbb{P}(\alpha,\beta) = \frac{F(\mu)}{F(\lambda)}$ where $\mu \in \lambda^-$ and $Y(\mu) = Y(\lambda) \setminus \{(\alpha,\beta)\}$ (note that if a trial terminates at (α,β), then this is necessarily a removable box, so this makes sense).

Definition. *Let $\pi : (a_1,b_1) \to (a_2,b_2) \to \cdots \to (a_m,b_m)$ be a trial of the process. Define $A_\pi = \{a_1,\dots,a_m\}$, the set of* horizontal projections *of π. Analogously, let $B_\pi = \{b_1,\dots,b_m\}$, the set of* vertical projections *of π.*

Example. Let $\lambda = (4,4,3,3,2)$. We could have the trial

$$\begin{ytableau} \ & \ & \ & \ \\ 1 & 2 & \ & \ \\ \ & \ & \ \\ \ & 3 & 4 \\ \ & \ \end{ytableau}$$

where we indicate the box we are in at time t by t. So $\pi : (2,1) \to (2,2) \to (4,2) \to (4,3)$. Then

$$A_\pi = \{2,4\}, \quad B_\pi = \{1,2,3\}.$$

Observe that for $\pi : (a_1, b_1) \to \cdots \to (a_m, b_m)$,

- the starting box (a_1, b_1) must equal $(\min A_\pi, \min B_\pi)$.
- the last box (a_m, b_m) must equal $(\max A_\pi, \max B_\pi)$.
- for each $i \in [n-1]$, either $a_i < a_{i+1}$ and $b_i = b_{i+1}$ (step down), or $a_i = a_{i+1}$ and $b_i < b_{i+1}$ (step right). So $m = |A_\pi| + |B_\pi| - 1$.

Definition. *Given $(a,b) \in Y(\lambda)$, $A, B \subseteq \mathbb{N}$, define $\mathbb{P}(A, B \mid a, b)$ to be the probability that a trial π starting at box (a,b) has $A_\pi = A, B_\pi = B$.*

Outline of proof of the hook length formula:

- We will calculate $\mathbb{P}(A, B \mid a, b)$ in terms of $\frac{1}{h_{ij}-1}$ for various i, j.
- For $\mu \in \lambda^-$, we will calculate $\frac{F(\mu)}{F(\lambda)}$ as a product of terms of the form $\frac{1}{h_{i,j}-1}$, and interpret the terms in the expansion as probabilities of the form $\mathbb{P}(A, B \mid a, b)$.
- We will show $\mathbb{P}(\alpha, \beta)$, the probability that a trial terminates at (α, β), is

$$\sum_{\substack{\text{possible projections} \\ A,B}} \sum_{\substack{\text{starting box} \\ (a,b)}} \mathbb{P}(A, B \mid a, b)$$

  to conclude $\mathbb{P}(\alpha, \beta) = \frac{F(\mu)}{F(\lambda)}$, where $\mu \in \lambda^-$ satisfies $Y(\mu) = Y(\lambda) \setminus \{(\alpha, \beta)\}$.

Lemma 3.2. *Let $(\alpha, \beta) \in Y(\lambda)$ be removable. Let $A = \{a_1, \ldots, a_t\}, B = \{b_1, \ldots, b_u\} \subseteq \mathbb{N}$, where $a_1 < a_2 < \cdots < a_t = \alpha$, $b_1 < b_2 < \cdots < b_u = \beta$. Then*

$$\mathbb{P}(A, B \mid a_1, b_1) = \prod_{x \in A \setminus \{\alpha\}} \frac{1}{h_{x,\beta} - 1} \prod_{y \in B \setminus \{\beta\}} \frac{1}{h_{\alpha,y} - 1}.$$

Proof. Induct on $t + u = |A| + |B|$. Base case $t + u = 2$, then $A = \{a_1 = \alpha\}$ and $B = \{b_1 = \beta\}$. Then $\mathbb{P}(A, B \mid a, b) = 1$ which is also the value of the RHS which is an empty product. For the inductive step now suppose $t + u > 2$, and so $(a_1, b_1) \neq (\alpha, \beta)$. Condition on the second box in the trial:

$$\mathbb{P}(A, B \mid a_1, b_1) = \sum_{\substack{(a',b') \\ \in H_{a_1,b_1} \setminus \{(a_1,b_1)\}}} \Bigg[\mathbb{P}\left(\begin{matrix} \text{proj. sets} \\ = A, B \end{matrix} \;\middle|\; \begin{matrix} \text{first box is } (a_1, b_1) \text{ and} \\ \text{second box is } (a', b') \end{matrix} \right)$$
$$\cdot \mathbb{P}(\text{second box is } (a', b') \mid \text{first box is } (a_1, b_1)) \Bigg]$$

$$= \sum_{(a',b')\in \text{arm of } H_{a_1,b_1}} + \sum_{(a',b')\in \text{leg of } H_{a_1,b_1}}$$

$$= \sum_{b_1<b'\leq\lambda_{a_1}} \mathbb{P}(A, B\setminus\{b_1\} \mid a_1, b')\mathbb{P}\left(\begin{matrix}\text{second box}\\ =(a_1,b')\end{matrix} \;\middle|\; \begin{matrix}\text{first box}\\ =(a_1,b_1)\end{matrix} \right)$$

$$+ \sum_{a_1<a'\leq(\lambda')_{b_1}} \mathbb{P}(A\setminus\{a_1\}, B \mid a', b_1)\mathbb{P}\left(\begin{matrix}\text{second box}\\ =(a',b_1)\end{matrix} \;\middle|\; \begin{matrix}\text{first box}\\ =(a_1,b_1)\end{matrix} \right)$$

$$= \frac{1}{h_{a_1,b_1}-1}\left(\sum_{b_1<b'\leq\lambda_{a_1}} \mathbb{P}(A, B\setminus\{b_1\} \mid a_1, b') + \sum_{a_1<a'\leq(\lambda')_{b_1}} \mathbb{P}(A\setminus\{a_1\}, B \mid a', b_1)\right)$$

Note that $\mathbb{P}(A, B\setminus\{b_1\} \mid a_1, b') = 0$ unless $b' = b_2$. Indeed, if $b' \neq b_2$ in a trial, then

- either $b_1 < b' < b_2$: b' is in the vertical projection set, but $b' \notin B\setminus\{b_1\}$.
- or $b' > b_2$: b' is in the vertical projection set, but b_2 is not in the vertical projection set.

Similarly, $\mathbb{P}(A\setminus\{a_1\}, B \mid a', b_1) = 0$ unless $a' = a_2$. Therefore,

$$\mathbb{P}(A, B \mid a_1, b_1) = \frac{1}{h_{a_1,b_1}-1}\Big(\mathbb{P}(A, B\setminus\{b_1\} \mid a_1, b_2) + \mathbb{P}(A\setminus\{a_1\}, B \mid a_2, b_1)\Big).$$

If one of u, t is 1, we simply omit the corresponding term. By the induction hypothesis, this is

$$\frac{1}{h_{a_1,b_1}-1}\Big(\prod_{x\in A\setminus\{\alpha\}} \frac{1}{h_{x,\beta}-1} \prod_{y\in B\setminus\{\beta,b_1\}} \frac{1}{h_{\alpha,y}-1} + \prod_{x\in A\setminus\{\alpha,a_1\}} \frac{1}{h_{x,\beta}-1} \prod_{y\in B\setminus\{\beta\}} \frac{1}{h_{\alpha,y}-1}\Big)$$

$$= \frac{(h_{\alpha,b_1}-1)+(h_{a_1,\beta}-1)}{h_{a_1,b_1}-1} \prod_{x\in A\setminus\{\alpha\}} \frac{1}{h_{x,\beta}-1} \prod_{y\in B\setminus\{\beta\}} \frac{1}{h_{\alpha,y}-1}$$

Now draw a picture to see why $(h_{\alpha,b_1}-1)+(h_{a_1,\beta}-1) = h_{a_1,b_1}-1$, so the first term disappears and we are done. $\square$

Proposition 3.3. *Let $(\alpha,\beta) \in Y(\lambda)$ be a removable box. Suppose $\mu \in \lambda^-$ is such that $Y(\mu) = Y(\lambda)\setminus\{(\alpha,\beta)\}$. Then*

$$\mathbb{P}(\alpha,\beta) = \frac{F(\mu)}{F(\lambda)}.$$

Proof. Observe that

- $h_{x,y}(\mu) = h_{x,y}(\lambda)$ if $x \neq \alpha$ and $y \neq \beta$,

- $h_{\alpha,y}(\mu) = h_{\alpha,y}(\lambda) - 1$ if $y \neq \beta$,
- $h_{x,\beta}(\mu) = h_{x,\beta}(\lambda) - 1$ if $x \neq \alpha$.

Thus,

$$\begin{aligned}\frac{F(\mu)}{F(\lambda)} &= \frac{\prod_{h\in\mathcal{H}(\lambda)} h}{n!} \frac{(n-1)!}{\prod_{h\in\mathcal{H}(\mu)} h} \\ &= \frac{1}{n} \prod_{1\leq x<\alpha} \frac{h_{x,\beta}}{h_{x,\beta}-1} \prod_{1\leq y<\beta} \frac{h_{\alpha,y}}{h_{\alpha,y}-1} \\ &= \frac{1}{n} \prod_{1\leq x<\alpha} \Big(1 + \frac{1}{h_{x,\beta}-1}\Big) \prod_{1\leq y<\beta} \Big(1 + \frac{1}{h_{\alpha,y}-1}\Big)\end{aligned}$$

We want to interpret the terms in the expansion as the probabilities that a trial terminating at (α, β) has certain horizontal and vertical projections. We have

$$\begin{aligned}\prod_{1\leq x<\alpha} \Big(1 + \frac{1}{h_{x,\beta}-1}\Big) &= \Big(1 + \frac{1}{h_{1,\beta}-1}\Big)\Big(1 + \frac{1}{h_{2,\beta}-1}\Big) \cdots \Big(1 + \frac{1}{h_{\alpha-1,\beta}-1}\Big) \\ &= \sum_{\substack{A\subseteq[\alpha] \\ \alpha\in A}} \prod_{x\in A\setminus\{\alpha\}} \frac{1}{h_{x,\beta}-1}\end{aligned}$$

and similarly

$$\prod_{1\leq y<\beta} \Big(1 + \frac{1}{h_{\alpha,y}-1}\Big) = \sum_{\substack{B\subseteq[\beta] \\ \beta\in B}} \prod_{y\in B\setminus\{\beta\}} \frac{1}{h_{\alpha,y}-1}.$$

Then

$$\begin{aligned}\frac{F(\mu)}{F(\lambda)} &= \frac{1}{n} \sum_{\substack{A\subseteq[\alpha],\alpha\in A \\ B\subseteq[\beta],\beta\in B}} \prod_{x\in A\setminus\{\alpha\}} \frac{1}{h_{x,\beta}-1} \prod_{y\in B\setminus\{\beta\}} \frac{1}{h_{\alpha,y}-1} \\ &= \frac{1}{n} \sum_{\substack{A\subseteq[\alpha],\alpha\in A \\ B\subseteq[\beta],\beta\in B}} \mathbb{P}(A, B \mid \min(A), \min(B)).\end{aligned}$$

Also, $\mathbb{P}(\alpha, \beta)$, the probability of terminating at (α, β), is

$$\begin{aligned}&\sum_{(a,b)\in Y(\lambda)} \mathbb{P}\Big(\begin{matrix}\text{terminate at} \\ (\alpha,\beta)\end{matrix} \Bigm| \begin{matrix}\text{start at} \\ (a,b)\end{matrix} \Big) \cdot \mathbb{P}(\text{start at } (a,b)) \\ &= \frac{1}{n} \sum_{(a,b)\in Y(\lambda)} \mathbb{P}\Big(\begin{matrix}\text{terminate at} \\ (\alpha,\beta)\end{matrix} \Bigm| \begin{matrix}\text{start at} \\ (a,b)\end{matrix} \Big) \\ &= \frac{1}{n} \sum_{(a,b)\in Y(\lambda)} \sum_{A',B'} \mathbb{P}(A', B' \mid a, b)\end{aligned}$$

where the second sum runs over $A' \subseteq [\alpha], B' \subseteq [\beta]$ such that $\alpha = \max A', a = \min A'$, $\beta = \max B', b = \min B'$. We conclude $\mathbb{P}(\alpha, \beta) = \frac{F(\mu)}{F(\lambda)}$. □

Proof of Theorem 3.1. Since a trial must terminate at a removable box,

$$1 = \sum_{(\alpha,\beta) \text{ removable}} \mathbb{P}(\alpha, \beta) = \sum_{\mu \in \lambda^-} \frac{F(\mu)}{F(\lambda)}.$$

So we are done by induction on n, as previously described. □

3.2 The Determinantal Form

Applications.

- Recall the permutation module $M^\lambda \cong \mathbb{1}_{S_\lambda}\uparrow^{S_n}$, see Lemma 2.1. In a direct sum decomposition of M^λ into irreducibles, how many times do we get S^μ? $\rightsquigarrow$ Young's Rule, Theorem 3.11 and Corollary 3.19.
- We have a nested structure: $S_1 < S_2 < \cdots < S_{n-1} < S_n < \ldots$. How do S_n-modules relate to S_{n-1}-modules?

  E.g. $V_n\downarrow^{S_n}_{S_{n-1}} \cong V_{n-1} \oplus \mathbb{1}_{S_{n-1}}$ where V_n is the natural permutation module of S_n. What is $\mathcal{S}^\lambda\downarrow^{S_n}_{S_{n-1}}$? $\rightsquigarrow$ Branching Rule, Theorem 3.22.
- What is $\chi^\lambda(g)$ for all $g \in S_n$? $\rightsquigarrow$ Murnaghan-Nakayama Rule, Theorem 3.25.
- And more:
  - e.g. Branching Rule describes $\mathcal{S}^\lambda\downarrow^{S_n}_{S_{n-1}\times S_1}$. What about $\mathcal{S}^\lambda\downarrow^{S_n}_{S_{n-m}\times S_m}$? $\rightsquigarrow$ Littlewood-Richardson Rule.
  - e.g. another proof of the hook length formula, see Example Sheet 2.

Notation.

- S_n is the symmetric group, S_λ Young subroups
- Before: $\mathcal{S}^\mu$ were Specht modules. For the rest of this chapter we use $[\mu]$ to replace $\mathcal{S}^\mu$ to denote the μ-Specht module. When it is clear from context, for $\mu = (m)$, we abbreviate $[\mu] = [(m)]$ to $[m]$.
- Let ξ^λ be the character of M^λ.

Definition.

- *Let G, H be finite groups, V a G-module, W an H-module. Then $V \otimes W$ can be into a $(G \times H)$-module via*

$$(g, h) \cdot (v \otimes w) = (gv) \otimes (hw)$$

for all $g \in G, h \in H$, $v \in V, w \in W$. The resulting $(G \times H)$-module is the (outer) tensor product *of V and W, which we will denote by $V\#W$. If V affords χ, and W affords ϕ, then $V\#W$ affords $\chi\#\phi$ where*

$$(\chi\#\phi)((g,h)) = \chi(g)\phi(h)$$

for all $g \in G, h \in H$.

- *Let $m, n \in \mathbb{N}$, $\alpha \vdash m, \beta \vdash n$. Then $\chi^\alpha\#\chi^\beta \in \mathrm{Irr}(S_m \times S_n)$ since $\chi^\alpha \in \mathrm{Irr}(S_m), \chi^\beta \in \mathrm{Irr}(S_n)$. Note that $S_m \times S_n$ naturally embeds inside S_{m+n} as $\mathrm{Sym}\{1,2,\dots,m\} \times \mathrm{Sym}\{m+1,\dots,m+n\}$. Then the* outer product *of $[\alpha]$ and $[\beta]$ is defined as*

$$[\alpha][\beta] = [\alpha]\#[\beta]\big\uparrow_{S_m \times S_n}^{S_{m+n}}.$$

Remarks.

(i) The outer product is associative and commutative.

(ii) Let $H \leq G$, $x \in G$. Then $\mathbb{1}_H\big\uparrow^G \cong \mathbb{1}_{xHx^{-1}}\big\uparrow G$. Suppose that $\lambda = (\lambda_1,\dots,\lambda_k) \vdash n$. Recall $S_\lambda \cong S_{\lambda_1} \times \cdots \times S_{\lambda_k}$. We may fix $S_\lambda = \mathrm{Sym}\{1,\dots,\lambda_1\} \times \mathrm{Sym}\{\lambda_1+1,\dots,\lambda_1+\lambda_2\} \times \cdots \times \mathrm{Sym}\{\sum_{i=1}^{k-1}\lambda_i+1,\dots,n\}$ when we consider $M^\lambda \cong \mathbb{1}_{S_\lambda}\big\uparrow^{S_n}$, since all Young subgroups of type λ are conjugate to this one.

Also,
$$M^\lambda \cong \mathbb{1}_{S_\lambda}\big\uparrow^{S_n} = \mathbb{1}_{S_{\lambda_1}}\#\mathbb{1}_{S_{\lambda_2}}\#\dots\#\mathbb{1}_{S_{\lambda_k}}\big\uparrow^{S_n} = [\lambda_1][\lambda_2]\dots[\lambda_k],$$
and so $[\lambda_1][\lambda_2]\dots[\lambda_k]$ has character ξ^λ.

Example. For $\lambda = (\lambda_1,\dots,\lambda_k) \vdash n$, consider the $k \times k$-matrix $\mathcal{D}_\lambda$ whose (i,j)-entry is the module $[\lambda_i - i + j]$ where we interpret $[l]$ as the zero module when $l < 0$.

(a) Let $\lambda = (n-1,1)$. Then, using the outer product to multiply modules,

$$\det \mathcal{D}_\lambda = \det \begin{pmatrix} [n-1] & [n] \\ [0] & [1] \end{pmatrix} = [n-1][n] - [n][0].$$

This (virtual) representation has (virtual) character

$$\xi^{(n-1,1)} - \xi^{(n)} = \xi^{(n-1,1)} - \chi^{(n)} = \chi^{(n-1,1)} = \chi^\lambda$$

by Example Sheet 1, Question 5.

(b) Let $\lambda = (3,1^2) \vdash 5$. Then

$$\begin{aligned}\det \mathcal{D}_\lambda = \det \begin{pmatrix} [3] & [4] & [5] \\ [0] & [1] & [2] \\ 0 & [0] & [1] \end{pmatrix} &= [3]\det\begin{pmatrix} [1] & [2] \\ [0] & [1] \end{pmatrix} - [0]\det\begin{pmatrix} [4] & [5] \\ [0] & [1] \end{pmatrix} \\ &= [3][1][1] - [3][2][0] - [4][1][0] + [5][0][0]\end{aligned}$$

which has (virtual) character

$$\xi^{(3,1^2)} - \xi^{(3,2)} - \xi^{(4,1)} + \xi^{(5)} = \xi^{(3,1^2)} = \xi^\lambda.$$

Definition. *A* virtual *character of G is a $\mathbb{Z}$-linear combination of irreducible characters of G.*

Definition. *Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$. Let $\mathcal{D}_\lambda$ be the $k \times k$ matrix whose (i,j)-entry is the module $[\lambda_i - i + j]$ (i.e. as in the example above).*

We could in fact have defined $\mathcal{D}_\lambda = [\lambda_i - i + j]_{ij}$ to be $k' \times k'$ for any $k' \geq k$, and the determinant remains unchanged. E.g. for $\lambda = (3,1,1)$,

$$\det \begin{pmatrix} [3] & [4] & [5] \\ [0] & [1] & [2] \\ 0 & [0] & [1] \end{pmatrix} = \det \begin{pmatrix} [3] & [4] & [5] & [6] \\ [0] & [1] & [2] & [3] \\ 0 & [0] & [1] & [2] \\ 0 & 0 & 0 & [0] \end{pmatrix},$$

viewing $(3,1,1) = (3,1,1,0,0,\dots)$.

Goal. Prove that $\det \mathcal{D}_\lambda$ has character χ^λ for all $\lambda \vdash n$.

For the rest of this chapter, we will work with $\mathbb{Z}^{\mathbb{N}}$, the set of sequences with integer entries, under pointwise addition.

Let $n \in \mathbb{N}$. Summary:

Term	Notation	Def.: $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathbb{Z}^{\mathbb{N}}$ s.t. $\sum_i \lambda_i = n$ and
partition of n	$\lambda \vdash n$	$\lambda_1 \geq \lambda_2 \geq \dots$ and $\lambda_i \in \mathbb{N}_0$ for all i
composition of n	$\lambda \models n$	$\lambda_i \in \mathbb{N}_0$ for all i
integer composition of n	$\lambda \Vdash n$	only finitely many λ_i are non-zero

(a) Recall $S_n = \mathrm{Sym}\{1, 2, \dots, n\}$. Define $S_{\mathbb{N}} = \bigcup_{n \in \mathbb{N}} S_n$.

- For $\pi \in S_{\mathbb{N}}$, we can view it as an element of $\mathbb{Z}^{\mathbb{N}}$ via $\pi = (\pi^{-1}(1), \pi^{-1}(2), \dots)$. Note that π does not have finite support, but $\pi^{-1}(i) = i$ for all sufficiently large i. In particular, the identity of $S_{\mathbb{N}}$ is $\mathrm{id} = (1,2,3,\dots)$.
- For $\pi \in S_{\mathbb{N}}$ and $\lambda \in \mathbb{Z}^{\mathbb{N}}$, we define $\pi \cdot \lambda := (\lambda_{\pi^{-1}(1)}, \lambda_{\pi^{-1}(2)}, \dots)$. Then $\pi \cdot \mathrm{id} = \mathrm{id} \cdot \pi = \pi$, $\pi \cdot \pi^{-1} = \pi^{-1} \cdot \pi = \mathrm{id}$, and $\tau \cdot (\pi \cdot \lambda) = (\tau\pi) \cdot \lambda$.
- For $\pi \in S_{\mathbb{N}}$ and $\lambda \Vdash n$, observe that $\pi \cdot \lambda \Vdash n$. Also $\lambda - \mathrm{id} + \pi = (\lambda_1 - 1 + \pi^{-1}(1), \lambda_2 - 2 + \pi^{-1}(2), \dots) \Vdash n$.

(b) In the above, we let λ_j be the j-th entry of λ as usual. If λ has finite support, we can define $\ell(\lambda) = \max\{i \in \mathbb{N} \mid \lambda_i \neq 0\}$. We may write $(\lambda_1, \dots, \lambda_{\ell(\lambda)})$ and $(\lambda_1, \dots, \lambda_{\ell(\lambda)}, 0, 0, \dots)$ interchangeably.

(c) We can extend Young subgroups to have type given by compositions, not just partitions; these will be conjugate to Young subgroups of type given by partitions. E.g. $S_{(1,0,0,2,0,0,\dots)} = S_{(1,2)} = \mathrm{Sym}\{1\} \times \mathrm{Sym}\{2,3\}$ is conjugate to $S_{(2,1)} = \mathrm{Sym}\{1,2\} \times \mathrm{Sym}\{3\}$.

(d) We can extend ξ^λ to be defined for all integer compositions $\lambda \models n$ by:

$$\xi^\lambda = \begin{cases} \mathbb{1}_{S_\lambda}\big\uparrow^{S_n} & \text{if } \lambda \models n, \\ 0 & \text{otherwise.} \end{cases}$$

So for all $\lambda \models n$, $[\lambda_1][\lambda_2]\dots[\lambda_{\ell(\lambda)}]$ has character ξ^λ, since $[r] = 0$ if $r < 0$.

(e) We could e.g. dominance partial ordering to $\lambda \models n$, Young diagrams, $\mathcal{D}_\lambda$, etc.

Definition. *For $\lambda \models n$, define*

$$\psi^\lambda = \sum_{\pi \in S_{\mathbb{N}}} \operatorname{sgn}(\pi) \xi^{\lambda - \mathrm{id} + \pi},$$

it is a virtual character of S_n.

Lemma 3.4. *Let $\lambda \models n$.*

(i) Only finitely many terms in the sum defining ψ^λ are non-zero.

(ii) The virtual character afforded by $\det \mathcal{D}_\lambda = \det([\lambda_i - i + j]_{ij})$ is ψ^λ.

Proof.

(i) Since λ has finite support, $k = \ell(\lambda)$ is defined. Let $\pi \in S_{\mathbb{N}} \setminus S_k$. We claim that $\lambda - \mathrm{id} + \pi$ has a negative entry. Indeed, let $m := \max\{i \mid \pi^{-1}(i) \neq i\}$. Since $\pi \notin S_k$, we must have $m > k$. By maximality of m, we must have $\pi^{-1}(m) < m$. Then $(\lambda - \mathrm{id} + \pi)_m = \lambda - m + \pi^{-1}(m) = \pi^{-1}(m) - m < 0$. So $\xi^{\lambda - \mathrm{id} + \pi} = 0$ for such π, and so $\psi^\lambda = \sum_{\pi \in S_k} \operatorname{sgn}(\pi) \xi^{\lambda - \mathrm{id} + \pi}$ is a finite sum.

(ii) Recall that the determinant of a $k \times k$ matrix D is given by

$$\det D = \sum_{\pi \in S_k} \operatorname{sgn} \pi \prod_{i=1}^{k} D_{i, \pi(i)}.$$

The claim follows since $[\alpha_1][\alpha_2]\dots[\alpha_{\ell(\alpha)}]$ has character ξ^α for all $\alpha \models n$.

□

So our goal is to show $\psi^\lambda = \chi^\lambda$ for all $\lambda \vdash n$.

Lemma 3.5. *Let $\lambda \models n$. Let $i \in \mathbb{N}$ and suppose that $\mu \models n$ satisfies $\mu - \mathrm{id} = (i\ i+1) \cdot (\lambda - \mathrm{id})$, i.e.*

$$\mu = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1} - 1, \lambda_i + 1, \lambda_{i+2}, \dots).$$

Then $\psi^\mu = -\psi^\lambda$. In particular, if $\lambda_i - i = \lambda_{i+1} - (i+1)$ for some $i \in \mathbb{N}$, then $\psi^\lambda = 0$.

Proof. Let $\tau = (i\ i+1)$. Then $\mu - \mathrm{id} = \tau \cdot (\lambda - \mathrm{id})$, so $\mu - \mathrm{id} + \tau\pi = \tau \cdot (\lambda - \mathrm{id} + \pi)$ for any $\pi \in S_{\mathbb{N}}$. Hence

$$\begin{aligned}\psi^\lambda &= \sum_{\pi \in S_{\mathbb{N}}} \mathrm{sgn}(\pi)\xi^{\lambda - \mathrm{id} + \pi} = \sum_{\pi \in S_{\mathbb{N}}} \mathrm{sgn}(\pi)\xi^{\tau\cdot(\lambda - \mathrm{id} + \pi)} \\ &= \sum_{\pi \in S_{\mathbb{N}}} \mathrm{sgn}(\pi)\xi^{\mu - \mathrm{id} + \tau\pi} = -\sum_{\pi \in S_n} \mathrm{sgn}(\tau\pi)\xi^{\mu - \mathrm{id} + \tau\pi} = -\psi^\mu.\end{aligned}$$

If $\lambda_i - i = \lambda_{i+1} - (i+1)$, then $\mu = \lambda$, and so $\psi^\lambda = -\psi^\lambda$, so $\psi^\lambda = 0$. □

Next we look at $\xi^\lambda\downarrow_{S_m \times S_k}$ where $\lambda \models n = m + k$. Note that $\xi^\lambda\downarrow_{S_m\times S_k} = \mathbb{1}_{S_\lambda}\uparrow^{S_n}\downarrow_{S_m\times S_k}$, so we will use Mackey's theorem. For this we will need to know the double cosets of S_λ, $S_m \times S_k$ in S_n.

Proposition 3.6. *Let $\lambda, \mu \vdash n$. There is a bijection between the set of double cosets of S_λ and S_μ in S_n, and the set of $\ell(\lambda) \times \ell(\mu)$-matrices with entries in $\mathbb{N}_0$ whose row sums are λ, and column sums are μ.*

Proof. Write $S_\lambda = S_{A_1} \times S_{A_1} \times \cdots \times S_{A_{\ell(\lambda)}}$ where $A_1 = [\lambda_1], A_2 = \{\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2\}, \ldots$, and $S_\mu = S_{B_1} \times S_{B_2} \times \cdots \times S_{B_{\ell(\mu)}}$ similarly.

For each $\sigma \in S_n$, define a matrix $Z(\sigma)$ via $Z(\sigma)_{ij} := |A_i \cap \sigma(B_j)|$ for all i, j. Note that the i-th row sum is

$$\sum_j |A_i \cap \sigma(B_j)| = |A_i \cap \bigcup_j \sigma(B_j)| = |A_i \cap [n]| = |A_i| = \lambda_i,$$

and similarly the j-th column sum is

$$\sum_i |A_i \cap \sigma(B_j)| = |\sigma(B_j)| = |B_j| = \mu_j.$$

Conversely, any matrix in the set described in the proposition is $Z(\sigma)$ for some $\sigma \in S_n$ (exercise).

Now we claim that for $\sigma, \tau \in S_n$, we have $Z(\sigma) = Z(\tau)$ iff $S_\lambda \sigma S_\mu = S_\lambda \tau S_\mu$. First, suppose $\tau = h\sigma k$ for some $h \in S_\lambda, k \in S_\mu$. Then $Z(\tau)_{ij} = |A_i \cap \tau(B_j)| = |A_i \cap h\sigma k(B_j)| = |A_i \cap h\sigma(B_j)|$ since $k \in S_\mu$, so that $k(B_j) = B_j$ for all j. Similarly, $h^{-1}(A_i) = A_i$, so $Z(\tau)_{ij} = |h^{-1}(A_i) \cap \sigma(B_j)| = |A_i \cap \sigma(B_j)| = Z(\sigma)_{ij}$. Conversely, suppose that $|A_i \cap \sigma(B_j)| = |A_i \cap \tau(B_j)|$ for all i, j. For each fixed i, $\{A_i \cap \sigma(B_j)\}_j$ and $\{A_i \cap \tau(B_j)\}_j$ are both partitions of the set A_i. But $|A_i \cap \sigma(B_j)| = |A_i \cap \tau(B_j)|$ for all j, so there exists $h_i \in S_{A_i}$ such that $h_i(A_i \cap \sigma(B_j)) = A_i \cap \tau(B_j)$ for all j. Then $h := h_1 \cdot h_2 \cdots h_{\ell(\lambda)} \in S_{A_1} \times \cdots \times S_{A_{\ell(\lambda)}} = S_\lambda$ satisfies $h(\sigma(B_j)) = \tau(B_j)$ for all j. Therefore $\tau^{-1}h\sigma(B_j) = B_j$ for all j, and so $\tau^{-1}h\sigma \in S_\mu$. Say $\tau^{-1}h\sigma = k^{-1}$, then $\tau = h\sigma k$ where $h \in S_\lambda, k \in S_\mu$.

Thus $S_\lambda \sigma S_\mu \mapsto Z(\sigma)$ is a well-defined bijection between the two sets in the proposition. □

Lemma 3.7. *Let $\lambda \models n = m + k$, $m, k \in \mathbb{N}_0$. Then*

(i) $\xi^\lambda\downarrow_{S_m \times S_k} = \sum_{\mu \models k} \xi^{\lambda-\mu} \# \xi^\mu$,

(ii) $\psi^\lambda\downarrow_{S_m \times S_k} = \sum_{\mu \models k} \psi^{\lambda-\mu} \# \xi^\mu$.

Proof.

(i) Both sides of (i) are equal to zero if $\lambda \not\models n$. So we may now assume that $\lambda \models n$. Also, note that the sum over $\mu \models k$ is finite since $\xi^{\lambda-\mu} = 0$ unless $\lambda - \mu \models m$, meaning we need $0 \leq \mu_i \leq \lambda - I$ for all i.

By Mackey:

$$\begin{aligned}\xi^\lambda\downarrow_{S_m \times S_k} &= \mathbb{1}_{S_\lambda}\uparrow^{S_n}\downarrow_{S_m \times S_k} \\ &= \sum_{\sigma \in S_m \times S_k \backslash S_n / S_\lambda} \mathbb{1}\uparrow^{S_m \times S_k}_{\sigma S_\lambda \sigma^{-1} \cap (S_m \times S_k)}.\end{aligned}$$

By Proposition 3.6 there is a bijection between $(S_m \times S_k) - S_\lambda$ double cosets in S_n and $2 \times \ell(\lambda)$ matrices over $\mathbb{N}_0$ with row sums (m, k) and column sums λ. Specifically, if $A_1 = [m], A_2 = [m+1, \ldots, m+k]$, $B_1 = [\lambda_1], B_2 = \{\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2\}$, etc., then the double coset $(S_m \times S_k)\sigma S_\lambda$ corresponds to $Z(\sigma)$ where $Z(\sigma)_{1j} = |A_i \cap \sigma(B_j)|$ and $Z(\sigma)_{2j} = |A_2 \cap \sigma(B_j)|$. Since $Z(\sigma)_{1j} + Z(\sigma)_{2j} = \lambda_j$ for all j, this matrix is in fact determined by just its second row, say, which we will call $\mu := (|A_2 \cap \sigma(B_1)|, \ldots, |A_2 \cap \sigma(B_{\ell(\lambda)})|) \models k$. In particular, the first row is then $\lambda - \mu$ and note $0 \leq \mu_i \leq \lambda_i$ for all i.

Observe

$$\sigma S_\lambda \sigma^{-1} = \sigma(S_{B_1} \times \cdots \times S_{B_{\ell(\lambda)}})\sigma^{-1} = S_{\sigma(B_1)} \times \cdots \times S_{\sigma(B_{\ell(\lambda)})}$$

and hence $\sigma S_\lambda \sigma^{-1} \cap (S_m \times S_k)$ is conjugate to $S_{\lambda-\mu} \times S_\mu$. Then

$$\begin{aligned}\mathbb{1}\uparrow^{S_m \times S_k}_{\sigma S_\lambda \sigma^{-1} \cap (S_m \times S_k)} &= \mathbb{1}_{S_{\lambda-\mu}}\uparrow^{S_m \times S_k} \\ &= \mathbb{1}_{S_{\lambda-\mu}}\uparrow^{S_m} \# \mathbb{1}_{S_\mu}\uparrow^{S_k} \\ &= \xi^{\lambda-\mu} \# \xi^\mu.\end{aligned}$$

This finishes the proof of (i).

(ii) We have by (i),

$$\begin{aligned}\psi^\lambda\downarrow_{S_m \times S_k} &= \sum_{\pi \in S_\mathbb{N}} \operatorname{sgn}(\pi) \xi^{\lambda - \mathrm{id} + \pi}\downarrow_{S_m \times S_k} \\ &= \sum_{\pi \in S_\mathbb{N}} \operatorname{sgn}(\pi) \sum_{\mu \models k} \xi^{\lambda - \mathrm{id} + \pi - \mu} \# \xi^\mu\end{aligned}$$

$$= \sum_{\mu \models k} \Big(\sum_{\pi \in S_{\mathbb{N}}} \operatorname{sgn}(\pi) \xi^{(\lambda-\mu)-\mathrm{id}+\pi} \Big) \# \xi^{\mu}$$
$$= \sum_{\mu \models k} \psi^{\lambda-\mu} \# \xi^{\mu}.$$

□

Definition. *Let* $0 \le k \le n$, $\lambda \models\!\!\models n$, $\mu \models k$. *Define*

$$\psi^{\lambda\backslash\mu} := \sum_{\pi \in S_{\mathbb{N}}} \operatorname{sgn}(\pi) \xi^{\lambda-\mathrm{id}-\pi\cdot(\mu-\mathrm{id})},$$

it is a virtual character of S_{n-k}.

Note. If $k = 0$, then $\mu = (0, 0, \dots)$; and $\psi^{\lambda\backslash\mu} = \psi^{\lambda}$.

We will informally call the $\psi^{\lambda\backslash\mu}$ *skew characters*, one can also define skew diagrams, etc.

We have the following analogue of Lemma 3.4

Lemma 3.8. *Let* $0 \le k \le n$, $\lambda \models\!\!\models n$, $\mu \models k$.

(i) Only finitely many terms in the sum defining $\psi^{\lambda\backslash\mu}$ *are non-zero.*

(ii) The virtual character afforded by the determinant $\det([\lambda_i - i - (\mu_j - j)])_{ij}$ *is* $\psi^{\lambda\backslash\mu}$.

Proof. Very similar as the proof of Lemma 3.4, see Example Sheet 2, Question 5. □

Lemma 3.9. *Let* $\lambda \models\!\!\models m + k$, $m, k \in \mathbb{N}_0$. *Then*

$$\psi^{\lambda}\downarrow_{S_m \times S_k} = \sum_{\beta \vdash k} \psi^{\lambda\backslash\beta} \# \psi^{\beta}.$$

Proof. All sums involved will be finite. First, from Lemma 3.7 we have

$$\begin{aligned}
\psi^{\lambda}\downarrow_{S_m \times S_k} &= \sum_{\pi \in S_{\mathbb{N}}} \operatorname{sgn}(\pi) \sum_{\mu \models k} \xi^{\lambda-\mathrm{id}+\pi-\mu} \# \xi^{\mu} \\
&= \sum_{\pi \in S_{\mathbb{N}}} \operatorname{sgn}(\pi) \sum_{\nu \models k} \xi^{\lambda-\mathrm{id}+\pi-\pi\circ\nu} \# \xi^{\pi\circ\nu} \qquad \nu = \pi^{-1} \circ \mu \\
&= \sum_{\pi \in S_{\mathbb{N}}} \operatorname{sgn}(\pi) \sum_{\nu \models k} \xi^{\lambda-\mathrm{id}-\pi\circ(\nu-\mathrm{id})} \# \xi^{\nu} \qquad (*)
\end{aligned}$$

On the other hand,

$$\sum_{\beta \vdash k} \psi^{\lambda\backslash\beta} \# \psi^{\beta} = \sum_{\beta \vdash k} \Big(\sum_{\pi \in S_{\mathbb{N}}} \operatorname{sgn}(\pi) \xi^{\lambda-\mathrm{id}-\pi\circ(\beta-\mathrm{id})} \Big) \# \Big(\sum_{\tau \in S_k} \operatorname{sgn}(\tau) \xi^{\beta-\mathrm{id}+\tau} \Big)$$

Note that if $\beta \vdash k$, then $\ell(\beta) \leq k$, so in the last sum we can sum over $\tau \in S_k$ instead of $S_{\mathbb{N}}$. Then

$$\begin{aligned}
\sum_{\beta\vdash k} \psi^{\lambda\backslash\beta} \# \psi^{\beta} &= \sum_{\beta\vdash k} \sum_{\pi\in S_{\mathbb{N}}} \sum_{\tau\in S_k} \operatorname{sgn}(\pi\tau)\xi^{\lambda-\mathrm{id}-\pi\circ(\beta-\mathrm{id})} \# \xi^{\tau^{-1}\circ(\beta-\mathrm{id}+\tau)} \\
&= \sum_{\beta\vdash k} \sum_{\tau\in S_k} \sum_{\rho\in S_{\mathbb{N}}} \operatorname{sgn}(\rho)\xi^{\lambda-\mathrm{id}-\rho\tau^{-1}\circ(\beta-\mathrm{id})} \# \xi^{\overbrace{\tau^{-1}\circ(\beta-\mathrm{id})+\mathrm{id}}^{=:\mu}} \qquad \rho = \pi\tau \\
&= \sum_{\rho\in S_{\mathbb{N}}} \sum_{\substack{\mu\models k \text{ such that} \\ \mu=\tau^{-1}\circ(\beta-\mathrm{id})+\mathrm{id} \\ \text{for some } \tau\in S_k,\beta\vdash k}} \operatorname{sgn}(\rho)\xi^{\lambda-\mathrm{id}-\rho\circ(\mu-\mathrm{id})} \# \xi^{\mu} \\
&= \sum_{\rho\in S_{\mathbb{N}}} \sum_{\substack{\mu\models k \text{ such that} \\ \mu=\tau^{-1}\circ(\beta-\mathrm{id})+\mathrm{id} \\ \text{for some } \tau\in S_k,\beta\vdash k}} \operatorname{sgn}(\rho)\xi^{\lambda-\mathrm{id}-\rho\circ(\mu-\mathrm{id})} \# \xi^{\mu} \qquad (**)
\end{aligned}$$

Note that we may replace $\sum_{\beta}\sum_{\tau}$ by $\sum_{\mu \text{ s.t.}...}$ because: if $\tau^{-1}\circ(\beta-\mathrm{id})+\mathrm{id} = \widetilde{\tau}^{-1}\circ(\widetilde{\beta}-\mathrm{id})+\mathrm{id}$ for some $\tau,\widetilde{\tau}\in S_k$, $\beta,\widetilde{\beta}\vdash k$, then $\beta-\mathrm{id} = (\tau\circ\widetilde{\tau}^{-1})\circ(\widetilde{\beta}-\mathrm{id})$. Since $\beta\vdash k$, $\beta_i \geq \beta_{i+1}$ for all i. But then $\beta_i - i > \beta_{i+1}-(i+1)$ for all i, i.e. $\beta-\mathrm{id}$ is strictly decreasing. Similarly for $\widetilde{\beta}-\mathrm{id}$. Therefore, $\tau\circ\widetilde{\tau}^{-1}=1$, i.e. $\tau=\widetilde{\tau}$ and then also $\beta=\widetilde{\beta}$.

We want to show $(*) = (**)$. For this we have to show that the restriction in $(**)$ can be removed.

First, we claim that

$$\begin{aligned}
&\{\mu\models k \mid \mu = \tau^{-1}\circ(\beta-\mathrm{id})+\mathrm{id} \text{ for some } \tau\in S_k,\beta\vdash k\} \\
=&\{\mu\models k \mid \mu_i = 0 \text{ for all } i>k,\ \mu_i - i \text{ are distinct for all } i\}
\end{aligned}$$

To see $\subseteq$: Take τ,β, define $\mu = \tau^{-1}\circ(\beta-\mathrm{id})+\mathrm{id}$. Then

- $|\mu| = |\beta| = k$,
- since $\beta\vdash k$, then $\beta-\mathrm{id}$ is strictly increasing, and so the $\mu_i - i$ are distinct for all i.
- since $\tau\in S_k$ and $\beta_i = 0$ for all $i>k$, then $\mu_i = 0$ for all $i>k$.

To see $\supseteq$: given $\mu\models k$ such that $\mu_i=0$ for all $i>k$, $\mu_i - i$ are distinct for all i, we will construct $\tau\in S_k,\beta\vdash k$ as follows:

Since $\mu_i = 0$ for all $i>k$, $\mu_i - i = -i$ for all $i>k$. Since the $\mu_i - i$ are distinct, $\mu_i - i \geq -k$ for all $i\leq k$. Moreover, we can order the $\mu_i - i$ and then define uniquely define $\tau\in S_k$ by

$$\mu_{\tau^{-1}(1)} - \tau^{-1}(1) > \mu_{\tau^{-1}(2)} - \tau^{-1}(2) > \cdots > \mu_{\tau^{-1}(k)} - \tau^{-1}(k) > -(k+1) > -(k+2) > \ldots$$

Then define $\beta := \tau\circ(\mu-\mathrm{id})+\mathrm{id}$. Then we get $\mu = \tau^{-1}\circ(\beta-\mathrm{id})+\mathrm{id}$, so we only have to check that $\beta\vdash k$. We have $|\beta| = |\mu| = k$. By construction, $\beta-\mathrm{id}$ is strictly decreasing, therefore $\beta_i \geq \beta_{i+1}$ for all i. Since $\tau \in S_k$, $\mu_i = 0$ for all $i > k$, then $\beta_i = 0$ for all $i > k$. Hence $\beta \vdash k$.

Second, we claim that

$$\begin{aligned}&\{\mu \models k \mid \mu_i = 0 \text{ for all } i > k, \text{ the } \mu_i - i \text{ are distinct for all } i\}\\ =&\{\mu \models k \mid \mu_i - i \text{ are distinct for all } i\}\end{aligned}$$

See Example Sheet 2.

Hence $(**)$ becomes

$$\sum_{\beta \vdash k} \psi^{\lambda\backslash\beta} \# \psi^{\beta} = \sum_{\rho \in S_{\mathbb{N}}} \sum_{\substack{\mu \models k \text{ such that} \\ \mu_i - i \text{ distinct } \forall i}} \operatorname{sgn}(\rho) \xi^{\lambda - \mathrm{id} - \rho \circ (\mu - \mathrm{id})} \# \xi^{\mu}$$

Finally, if $\mu \models k$ is such that $\mu_i - i = \mu_j - j$ for some $i \neq j$, then

$$\begin{aligned}
&\sum_{\rho \in S_{\mathbb{N}}} \operatorname{sgn}(\rho) \xi^{\lambda - \mathrm{id} - \rho(\mu - \mathrm{id})} \# \xi^{\mu} \\
&\quad = \frac{1}{2} \sum_{\sigma \in S_{\mathbb{N}}} \left[\operatorname{sgn}(\sigma) \xi^{\lambda - \mathrm{id} - \sigma \circ (\mu - \mathrm{id})} \# \xi^{\mu} + \operatorname{sgn}(\sigma \circ (ij)) \xi^{\lambda - \mathrm{id} - \sigma \circ \overbrace{(ij) \circ (\mu - \mathrm{id})}^{\mu - \mathrm{id}}} \# \xi^{\mu} \right] \\
&\quad = \frac{1}{2} \sum_{\sigma \in S_{\mathbb{N}}} \left[\operatorname{sgn}(\sigma) \xi^{\lambda - \mathrm{id} - \sigma \circ (\mu - \mathrm{id})} \# \xi^{\mu} - \operatorname{sgn}(\sigma) \xi^{\lambda - \mathrm{id} - \sigma \circ (\mu - \mathrm{id})} \# \xi^{\mu} \right] \\
&\quad = 0
\end{aligned}$$

Then

$$\sum_{\beta \vdash k} \psi^{\lambda\backslash\beta} \# \psi^{\beta} = \sum_{\rho \in S_{\mathbb{N}}} \sum_{\mu \models k} \operatorname{sgn}(\rho) \xi^{\lambda - \mathrm{id} - \rho \circ (\mu - \mathrm{id})} \# \xi^{\mu} = (*) = \psi^{\lambda} \big\downarrow_{S_m \times S_k}.$$

□

Theorem 3.10. *Let $0 \leq k \leq n$, $\alpha \vdash n$, $\beta \vdash k$.*

(i) If $\psi^{\alpha\backslash\beta} \neq 0$, then $\alpha_i \geq \beta_i$ for all i,

(ii) $\langle \psi^{\alpha\backslash\beta}, \xi^{(n-k)} \rangle = \begin{cases} 1 & \text{if } \alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \dots, \\ 0 & \text{otherwise.} \end{cases}$

If $\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \dots$, we say that α and β *intertwine.*

Proof.

(i) Recall from Lemma 3.8 that $\psi^{\alpha\backslash\beta}$ is the character of the determinant of the matrix A where $A_{ij} = [\alpha_i - i - (\beta_j - j)]$. Note that since α, β are partitions, $\alpha - \mathrm{id}, \beta - \mathrm{id}$ are strictly decreasing. If A_{ij} is zero (in other words, $\alpha_i - i - (\beta_j - j) < 0$), then all entries to its left and below are zero. Thus the determinant vanishes if a diagonal entry is zero. So if $\psi^{\alpha\backslash\beta} \neq 0$, we must have $\alpha_i - i - (\beta_i - i) \geq 0$, i.e. $\alpha_i \geq \beta_i$ for all i.

(ii) For $\lambda \models\!\!\models n-k$, recall $\xi^\lambda = 0$ if $\lambda \not\models n-k$. If $\lambda \models n-k$, then

$$\langle \xi^\lambda, \xi^{(n-k)} \rangle = \langle \mathbb{1}_{S_\lambda} \big\uparrow^{S_{n-k}}, \mathbb{1}_{S_{n-k}} \rangle \overset{\text{Frobenius reciprocity}}{=} \langle \mathbb{1}_{S_\lambda}, \mathbb{1}_{S_\lambda} \rangle = 1.$$

Thus

$$\langle \psi^{\alpha\backslash\beta}, \xi^{(n-k)} \rangle = \sum_{\pi \in S_{\mathbb{N}}} \operatorname{sgn}\pi \langle \xi^{\alpha - \mathrm{id} - \pi\circ(\beta - \mathrm{id})}, \xi^{(n-k)} \rangle = \sum_{\pi \in S_{\mathbb{N}}} (\operatorname{sgn}\pi) \delta_{\{\alpha - \mathrm{id} - \pi\circ(\beta - \mathrm{id}) \models n-k\}}$$

This is the determinant of M where $M_{ij} = \delta_{\{\alpha_i - i - (\beta_j - j) \geq 0\}}$. Note if $M_{ij} = 0$, then all entries to its left and below are also zero. Also, M only has $0-1$ entries. If $\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \dots$, then $M_{ii} = 1$ and $M_{i+1\ i} = 0$ for all i, and so $\det M = 1$. Otherwise, $M_{ii} = 0$ for some i, or $M_{i+1\ i} = 1$ for some i, but then M must have a column of all 0's, or have two equal columns; and therefore $\det M = 0$.

$\square$

Theorem 3.11 (Young's Rule)**.** *Let $\lambda \models n$ with $\ell(\lambda) \leq n$. Let $\alpha \vdash n$. Then $\langle \psi^\alpha, \xi^\lambda \rangle$ is equal to the number of tuples of partitions $(\beta^{(1)}, \beta^{(2)}, \dots, \beta^{(n-1)})$ satisfying*

(i) $\beta^{(i)} \vdash \sum_{j=1}^{i} \lambda_j$ *for all* $i \in [n-1]$,

(ii) $0 \leq \beta_j^{(1)} \leq \beta_j^{(2)} \leq \dots \leq \beta_j^{(n-1)} \leq \alpha_j$ *for all* $j \in [n]$,

(iii) $\beta_j^{(i)} \leq \beta_{j-1}^{(i-1)}$ *for all* $j > 1$, $i \geq 1$, *where we treat* $\beta^{(0)} = (0, 0, \dots)$ *and* $\beta^{(n)} = \alpha$.

Once we have proved $\psi^\alpha = \chi^\alpha$, then Young's Rule will tell us the multiplicity of the Spect module $[\alpha]$ in a direct sum decomposition of M^λ into irreducibles.

Example. Let $n = 5$, $\alpha = (3, 2)$.

(i) Let $\lambda = (2, 0, 1, 2) \models 5$. Then

$$\begin{array}{rccccccccl}
\beta^{(0)} & = & (& 0, & & 0, & & 0, & \dots\ ) & \\
 & & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & \\
\beta^{(1)} & = & (& \beta_1^{(1)}, & & \beta_2^{(1)}, & & \beta_3^{(1)}, & \dots\ ) & \vdash 2 \\
 & & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & \\
\beta^{(2)} & = & (& \beta_1^{(2)}, & & \beta_2^{(2)}, & & \beta_3^{(2)}, & \dots\ ) & \vdash 2+0=2 \\
 & & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & \\
\beta^{(3)} & = & (& \beta_1^{(3)}, & & \beta_2^{(3)}, & & \beta_3^{(3)}, & \dots\ ) & \vdash 2+0+1=3 \\
 & & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & \\
\beta^{(4)} & = & (& \beta_1^{(4)}, & & \beta_2^{(4)}, & & \beta_3^{(4)}, & \dots\ ) & \vdash 2+0+1+2=5 \\
 & & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & \\
\alpha = \beta^{(5)} & = & (& 3, & & 2, & & 0, & \dots\ ) &
\end{array}$$

where an arrow $a \to b$ indicates that $a \le b$. The yellow arrows are by (ii) and the violet arrows by condition (iii). We see that most entries are uniquely determined as follows:

$$\begin{array}{rcccccccccl}
\beta^{(0)} & = & (& 0, & & 0, & & 0, & \dots &) & \\
& & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\beta^{(1)} & = & (& 2, & & 0, & & 0, & \dots &) & \vdash 2 \\
& & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\beta^{(2)} & = & (& 2, & & 0, & & 0, & \dots &) & \vdash 2+0=2 \\
& & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\beta^{(3)} & = & (& \beta_1^{(3)}, & & \beta_2^{(3)}, & & 0, & \dots &) & \vdash 2+0+1=3 \\
& & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\beta^{(4)} & = & (& 3, & & 2, & & 0, & \dots &) & \vdash 2+0+1+2=5 \\
& & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\alpha = \beta^{(5)} & = & (& 3, & & 2, & & 0, & \dots &) &
\end{array}$$

We can have $\beta^{(3)} = (3, 0, \dots)$ or $(2, 1, 0, \dots)$. Therefore $\langle \psi^\alpha, \xi^\lambda \rangle = 2$.

(ii) Let $\lambda = (0, 2, 2, 0, 1) \models 5$. [Since $\xi^{(2,0,1,2)} = \xi^{(0,2,2,0,1)}$, we expect again two tuples]

$$\begin{array}{rcccccccccl}
\beta^{(0)} & = & (& 0, & & 0, & & 0, & \dots &) & \\
& & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\beta^{(1)} & = & (& \beta_1^{(1)}, & & \beta_2^{(1)}, & & \beta_3^{(1)}, & \dots &) & \vdash 0 \\
& & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\beta^{(2)} & = & (& \beta_1^{(2)}, & & \beta_2^{(2)}, & & \beta_3^{(2)}, & \dots &) & \vdash 0+2=2 \\
& & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\beta^{(3)} & = & (& \beta_1^{(3)}, & & \beta_2^{(3)}, & & \beta_3^{(3)}, & \dots &) & \vdash 0+2+2=4 \\
& & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\beta^{(4)} & = & (& \beta_1^{(4)}, & & \beta_2^{(4)}, & & \beta_3^{(4)}, & \dots &) & \vdash 0+2+2+0=4 \\
& & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\alpha = \beta^{(5)} & = & (& 3, & & 2, & & 0, & \dots &) &
\end{array}$$

Again we see that most entries are uniquely determined:

$$
\begin{array}{rcccccccccl}
\beta^{(0)} & = & (& 0, & & 0, & & 0, & \dots &) & \\
 & & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\beta^{(1)} & = & (& 0, & & 0, & & 0, & \dots &) & \vdash 0 \\
 & & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\beta^{(2)} & = & (& 2, & & 0, & & 0, & \dots &) & \vdash 0+2=2 \\
 & & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\beta^{(3)} & = & (& 2/3, & & 2/1, & & 0, & \dots &) & \vdash 0+2+2=4 \\
 & & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\beta^{(4)} & = & (& 2/3, & & 2/1, & & 0, & \dots &) & \vdash 0+2+2+0=4 \\
 & & & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & & & \\
\alpha = \beta^{(5)} & = & (& 3, & & 2, & & 0, & \dots &) &
\end{array}
$$

So we can have either $\beta^{(3)} = \beta^{(4)} = (2,2,0,\dots)$ or $\beta^{(3)} = \beta^{(4)} = (3,1,0,\dots)$. So we again get $\langle \psi^\alpha, \xi^\lambda \rangle = 2$.

Proof of Theorem 3.11. We have

$$
\begin{aligned}
\langle \psi^\alpha, \xi^\lambda \rangle &= \langle \psi^\alpha, \mathbb{1}_{S_\lambda} \uparrow^{S_n} \rangle \overset{\text{F.R.}}{=} \langle \psi^\alpha \downarrow^{S_n}_{S_\lambda}, \mathbb{1}_{S_\lambda} \rangle \\
&= \langle \Big(\psi^\alpha \downarrow^{S_n}_{S_{\lambda_n} \times S_{\lambda_{n-1}+\dots+\lambda_1}} \Big) \downarrow_{S_{\lambda_n} \times S_{\lambda_{n-1}} \times \dots \times S_{\lambda_1}}, \xi^{(\lambda_n)} \# \dots \# \xi^{(\lambda_1)} \rangle \\
&\overset{\text{Lemma 3.9}}{=} \langle \sum_{\beta^{(n-1)} \vdash \sum_{j=1}^{n-1} \lambda_j} \psi^{\alpha \backslash \beta^{(n-1)}} \# (\psi^{\beta^{(n-1)}}) \downarrow^{S_{\lambda_{n-1}+\dots+\lambda_1}}_{S_{\lambda_{n-1}} \times \dots \times S_{\lambda_1}}, \xi^{(\lambda_n)} \# \dots \# \xi^{(\lambda_1)} \rangle \\
&= \sum_{\beta^{(n-1)} \vdash \sum_{j=1}^{n-1} \lambda_j} \langle \psi^{\alpha \backslash \beta^{(n-1)}}, \xi^{(\lambda_n)} \rangle \cdot \langle \psi^{\beta^{(n-1)}} \downarrow^{S_{\lambda_{n-1}+\dots+\lambda_1}}_{S_{\lambda_{n-1}} \times \dots \times S_{\lambda_1}}, \xi^{(\lambda_{n-1})} \# \dots \# \xi^{(\lambda_1)} \rangle \\
&\overset{\text{Theorem 3.10}}{=} \sum_{\beta^{(n-1)}} \langle \psi^{\beta^{(n-1)}} \downarrow^{S_{\lambda_{n-1}+\dots+\lambda_1}}_{S_{\lambda_{n-1}} \times \dots \times S_{\lambda_1}}, \xi^{(\lambda_{n-1})} \# \dots \# \xi^{(\lambda_1)} \rangle
\end{aligned}
$$

where we sum over $\beta^{(n-1)} \vdash \sum_{j=1}^{n-1} \lambda_j$ such that α and $\beta^{(n-1)}$ intertwine. Iteratively applying Lemma 3.9 and Theorem 3.10, we get

$$
\begin{aligned}
\langle \psi^\alpha, \xi^\lambda \rangle &= \sum_{\beta^{(n-1)}} \sum_{\beta^{(n-2)}} \cdots \sum_{\beta^{(1)}} \langle \psi^{\beta^{(1)}} \downarrow^{S_{\lambda_1}}_{S_{\lambda_1}}, \xi^{(\lambda_1)} \rangle \\
&= \sum_{\beta^{(n-1)}, \beta^{(n-2)}, \dots, \beta^{(1)}} 1
\end{aligned}
$$

where we sum over $\beta^{(i)} \vdash \sum_{j=1}^{i} \lambda_j$ (this is condition (i)) such that $\beta^{(i)}$ and $\beta^{(i-1)}$ intertwine for all $i \in [n]$ (this gives conditions (ii) and (iii)). $\square$

Lemma 3.12. *Let $\alpha, \beta \vdash n$. If $\langle \xi^\alpha, \chi^\beta \rangle > 0$, then $\beta \trianglerighteq \alpha$.*

Proof. Since $\langle \xi^\alpha, \chi^\beta \rangle > 0$, we have $\mathrm{Hom}_{\mathbb{C}S_n}([\beta], M^\alpha) \neq 0$. Since $\operatorname{char} \mathbb{C} = 0$, Maschke's theorem gives a complement of $[\beta]$ in M^β, so we can extend any $\phi \in \mathrm{Hom}_{\mathbb{C}S_n}([\beta], M^\alpha)$ to $\widetilde{\phi} \in \mathrm{Hom}_{\mathbb{C}S_n}(M^\beta, M^\alpha)$. Then $\beta \trianglerighteq \alpha$ by Theorem 2.10. $\square$

Remarks.

- We can't use Theorem 3.11 to prove the lemma, since we don't have $\chi^\beta = \psi^\beta$ yet.
- The converse holds, see Lemma 3.20.

Theorem 3.13. *Let $\alpha \vdash n$. Then $\psi^\alpha = \chi^\alpha$. In particular, the irreducible representation $[\alpha]$ has determinantal form* $\det([\alpha_i - i + j])_{ij}$.

Proof.

- **Step 1.** We first show that if $\lambda \models n$ with $\ell(\lambda) \leq n$ and $\langle \xi^\lambda, \psi^\alpha \rangle > 0$, then $\alpha \trianglerighteq \lambda$.

  Proof. Suppose $\langle \xi^\lambda, \psi^\alpha \rangle > 0$. Then there exists $(\beta^{(1)}, \dots, \beta^{(n-1)})$ satisfying Theorem 3.11 (i), (ii), (iii). By (iii),

  $$0 = \beta_1^{(0)} \geq \beta_2^{(1)} \geq \beta_3^{(3)} \geq \cdots \geq 0,$$

  so $\ell(\beta^{(i)}) \leq i$ for all i. Now $\beta^{(i)} = (\beta_1^{(i)}, \beta_2^{(i)}, \dots, \beta_i^{(i)}, 0, \dots) \vdash \sum_{j=1}^i \lambda_j$ by (i), and $\alpha_j \geq \beta_j^{(i)}$ for all j by (ii). So

  $$\alpha_1 + \alpha_2 + \cdots + \alpha_i \geq \beta_1^{(i)} + \beta_2^{(i)} + \cdots + \beta_i^{(i)} = \lambda_1 + \lambda_2 + \cdots + \lambda_i,$$

  for all i, in other words, $\alpha \trianglerighteq \lambda$. $\square$

- **Step 2.** We show $\langle \psi^\alpha, \xi^\alpha \rangle = 1$.

  Proof. Observe that $(\beta^{(1)}, \dots, \beta^{(n-1)})$ with $\beta^{(i)} = (\alpha_1, \alpha_2, \dots, \alpha_i)$ satisfies the conditions in Theorem 3.11, and so $\langle \psi^\alpha, \xi^\alpha \rangle \geq 1$. Conversely, suppose $(\beta^{(1)}, \dots, \beta^{(n-1)})$ satisfies (i), (ii), (iii) in Theorem 3.11 with $\lambda = \alpha$. Then, as in Step 1, we obtain $\ell(\beta^{(i)}) \leq i$, $\alpha_j \geq \beta_j^{(i)}$ for all j, and $\alpha_1 + \cdots + \alpha_i \geq \beta_1^{(i)} + \cdots + \beta_i^{(i)} = \alpha_1 + \cdots + \alpha_i$ for all i. Hence, we must have equality in $\alpha_j \geq \beta_j^{(i)}$ for $j = 1, \dots, i$, so $\beta^{(i)} = (\alpha_1, \dots, \alpha_i)$. So there is only one such tuple $(\beta^{(1)}, \dots, \beta^{(n-1)})$ and therefore $\langle \psi^\alpha, \xi^\alpha \rangle = 1$. $\square$

- **Step 3.** We show $\langle \psi^\alpha, \psi^\alpha \rangle = 1$.

  Proof. First, for any $\pi \in S_{\mathbb{N}}$, $\alpha - \mathrm{id} + \pi \trianglerighteq \alpha$ since for all i,

  $$\pi^{-1}(1) + \pi^{-1}(2) + \cdots + \pi^{-1}(i) \geq 1 + 2 + \cdots + i,$$

  so

  $$(\alpha_1 - 1 + \pi^{-1}(1)) + (\alpha_2 - 2 + \pi^{-1}(2)) + \cdots + (\alpha_i - i + \pi^{-1}(i)) \geq \alpha_1 + \alpha_2 + \cdots + \alpha_i.$$

On the other hand, if $\pi \in S_n$ and $\langle \xi^{\alpha - \mathrm{id} + \pi}, \psi^\alpha \rangle > 0$, then $\alpha \trianglerighteq \alpha - \mathrm{id} + \pi$ by Step 1 (because $\alpha - \mathrm{id} + \pi \models n$, else $\xi^{\alpha - \mathrm{id} + \pi} = 0$, and $\alpha \vdash n$, so $\ell(\alpha) \leq n$, so $\ell(\alpha - \mathrm{id} + \pi) \leq n$). Hence $\alpha \trianglerighteq \alpha - \mathrm{id} + \pi \trianglerighteq \alpha$, so $\pi = \mathrm{id}$. Thus,

$$\begin{aligned} \langle \psi^\alpha, \psi^\alpha \rangle &= \sum_{\pi \in S_n} \operatorname{sgn} \pi \langle \xi^{\alpha - \mathrm{id} + \pi}, \psi^\alpha \rangle \\ &= \langle \xi^\alpha, \psi^\alpha \rangle \\ &= 1. \end{aligned}$$

□

We can now prove $\psi^\alpha = \chi^\alpha$. Since $\langle \psi^\alpha, \psi^\alpha \rangle = 1$, we have $\psi^\alpha = \pm \phi$ for some $\phi \in \mathrm{Irr}(S_n)$. Since also $\chi^\alpha \in \mathrm{Irr}(S_n)$, it thus suffices to prove $\langle \psi^\alpha, \chi^\alpha \rangle > 0$. Next, if $\lambda \models n$ such that $\langle \xi^\lambda, \chi^\alpha \rangle > 0$, then $\langle \xi^\beta, \chi^\alpha \rangle > 0$ where $\beta \vdash n$ is obtained from λ permuting its parts. By Lemma 3.12, $\alpha \trianglerighteq \beta$, but also clearly $\beta \trianglerighteq \lambda$. Therefore $\alpha \trianglerighteq \lambda$. So if $\pi \in S_n$, and $\langle \xi^{\alpha - \mathrm{id} + \pi}, \chi^\alpha \rangle > 0$, then $\alpha \trianglerighteq \alpha - \mathrm{id} + \pi \trianglerighteq \alpha$, i.e. $\pi = \mathrm{id}$. Thus

$$\langle \psi^\alpha, \chi^\alpha \rangle = \sum_{\pi \in S_n} \operatorname{sgn} \pi \langle \xi^{\alpha - \mathrm{id} + \pi}, \chi^\alpha \rangle = \langle \xi^\alpha, \chi^\alpha \rangle.$$

This is > 0, since $[\alpha] \leq M^\alpha$.

Therefore $\langle \psi^\alpha, \psi^\alpha \rangle = 1$ and $\langle \psi^\alpha, \chi^\alpha \rangle > 0$. These imply $\psi^\alpha = \chi^\alpha$. □

3.3 Applications

3.3.1 Young's Rule Revisited

Corollary 3.14. *Let $\alpha \vdash n$. Then $\langle \chi^\alpha, \xi^\alpha \rangle = 1$.*

Proof.

- Either from James Submodule Theorem and complete reducibility in char 0,
- or use Theorem 3.13 and Step 2 in its proof.

□

Corollary 3.15. *The permutation characters $\{\xi^\alpha \mid \alpha \vdash n\}$ gives a basis of the $\mathbb{C}$-vector space of class functions of S_n. In particular, the change of basis matrix to $\mathrm{Irr}(S_n) = \{\chi^\beta \mid \beta \vdash n\}$ is $\mathbb{Z}$-valued, and unitriangular if we order the partitions in a way that extends the dominance partial ordering.*

Proof. From the definition of ψ^β and the fact that $\psi^\beta = \chi^\beta$, it is clear that the χ^β are $\mathbb{Z}$-linear combinations of the permutation characters. Conversely, it is clear that the permutation characters are $\mathbb{Z}$-linear combinations of the χ^α. From Lemma 3.12 it follows that the matrix is triangular and Corollary 3.14 gives that the diagonal entries are 1. □

Remark. Young's Rule tells us the multiplicity of $[\alpha]_{\mathbb{C}}$ in a direct sum decomposition of $M^\lambda_{\mathbb{C}}$ into irreducibles. Over an arbitrary field $\mathbb{F}$, $M^\lambda_{\mathbb{F}}$ decomposes as a direct sum of indecomposables: We saw from James' Submodule Theorem that there is a unique summand containing $[\lambda]_{\mathbb{F}}$, which we called the Young module $Y^\lambda_{\mathbb{F}}$.

In general, Young modules for S_n are defined as the indecomposable summands of $M^\lambda_{\mathbb{F}}$ for some $\lambda \vdash n$. It turns out that isomorphism classes are indexed by $\wp(n)$.

Fact. $M^\lambda_{\mathbb{F}}$ can be decomposed as a direct sum of S_n-modules each of which is isomorphic to $Y^\mu_{\mathbb{F}}$ for some $\mu \trianglerighteq \lambda$, and $Y^\lambda_{\mathbb{F}}$ appears exactly once.

If $\operatorname{char}\mathbb{F} = 0$, then indecomposable = irreducible, and ew havae proven this fact (then $Y^\lambda_{\mathbb{C}} = [\lambda]_{\mathbb{C}}$).

In general, $Y^\lambda_{\mathbb{F}} \not\cong [\lambda]_{\mathbb{F}}$, e.g. in Example Sheet, Question 5, we saw that $[(n-1,1)]_{\mathbb{F}}$ was a submodule, but not a direct summand of $M^{(n-1,1)}_{\mathbb{F}}$ in the case $\operatorname{char}\mathbb{F} \mid n$.

If $\operatorname{char}\mathbb{F} > 2$, then it is known that Specht modules are always indecomposable. In $\operatorname{char}\mathbb{F} = 2$, this is still an open problem.

Next, we work towards another combinatorial way to interpret Young's Rule.

Lemma 3.16. *Let $m, k \in \mathbb{N}$, let $\alpha \vdash m+k, \beta \vdash k, \gamma \vdash m$. Then*

$$\langle \psi^{\alpha\backslash\beta}, \chi^\gamma \rangle = \langle \chi^\alpha {\downarrow}_{S_m \times S_k}, \chi^\gamma \# \chi^\beta \rangle.$$

Moreover, $\langle \psi^{\alpha\backslash\beta}, \chi^\gamma \rangle = \langle \psi^{\alpha\backslash\gamma}, \chi^\beta \rangle$.

Letting γ vary, this shows that $\psi^{\alpha\backslash\beta}$ is a genuine character.

Proof. We have

$$\begin{aligned}
\langle \chi^\alpha {\downarrow}_{S_m \times S_k}, \chi^\gamma \# \chi^\beta \rangle &= \langle \psi^\alpha |_{S_m \times S_k}, \chi^\gamma \# \chi^\beta \rangle \\
&= \sum_{\delta \vdash k} \langle \psi^{\alpha\backslash\delta} \# \psi^\delta, \chi^\gamma \# \chi^\beta \rangle \\
&= \sum_{\delta \vdash k} \langle \psi^{\alpha\backslash\delta}, \chi^\gamma \rangle \cdot \langle \psi^\delta, \chi^\beta \rangle \\
&= \sum_{\delta \vdash k} \langle \psi^{\alpha\backslash\delta}, \chi^\gamma \rangle \cdot \langle \chi^\delta, \chi^\beta \rangle \\
&= \langle \psi^{\alpha\backslash\beta}, \chi^\gamma \rangle.
\end{aligned}$$

The last part follows from $\langle \chi^\alpha {\downarrow}_{S_m \times S_k}, \chi^\gamma \# \chi^\beta \rangle = \langle \chi^\alpha {\downarrow}_{S_k \times S_m}, \chi^\beta \# \chi^\gamma \rangle$. $\square$

Remark. Multiplicities of the form $\langle \chi^\alpha {\downarrow}_{S_m \times S_k}, \chi^\gamma \# \chi^\beta \rangle$ are called *Littlewood-Richardson coefficients*, also denoted $c^\alpha_{\gamma,\beta}$, and they occur in many different contexts, e.g. symmetric functions and algebraic combinatorics, representation theory of algebraic groups, etc.

Lemma 3.17. *Let $m, k \in \mathbb{N}$, $\alpha \vdash m$. Then*

$$\chi^\alpha \# \chi^{(k)} \big\uparrow_{S_m \times S_k}^{S_{m+k}} = \sum \chi^\gamma$$

where the sum runs over $\gamma \vdash m+k$ such that $\alpha_i \leq \gamma_i \leq \alpha_{i-1}$ for all i, treating $\alpha_0 = \infty$.

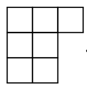
Proof. Let $\gamma \vdash m+k$. Then

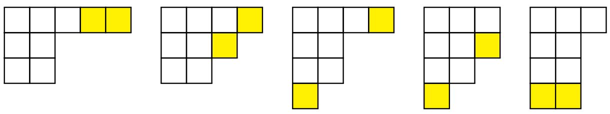
$$\begin{aligned}
\langle \chi^\gamma, \chi^\alpha \# \chi^{(k)} \big\uparrow^{S_{m+k}} \rangle &= \langle \chi^\gamma \big\downarrow_{S_m \times S_k}, \chi^\alpha \# \chi^{(k)} \rangle \\
&= \langle \psi^{\gamma \setminus \alpha}, \chi^{(k)} \rangle \\
&= \langle \psi^{\gamma \setminus \alpha}, \xi^{(k)} \rangle \\
&\overset{\text{Theorem 3.10}}{=} \begin{cases} 1 & \text{if } \gamma_1 \geq \alpha_1 \geq \gamma_2 \geq \alpha_2 \geq \dots, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$\square$

Corollary 3.18. *Notation as in Lemma 3.17. Then the Young diagrams $Y(\gamma)$ can be obtained from $Y(\alpha)$ by adding k many boxes in all possible ways such that no two of the newly added boxes lie in the same column*

Proof. Since $\gamma_i \geq \alpha_i$ for all i, we can certainly view $Y(\gamma)$ as a superset of $Y(\alpha)$. The condition $\gamma_i \leq \alpha_{i-1}$ corresponds to the assertion that no two boxes in $Y(\gamma) \setminus Y(\alpha)$ lie in the same column. $\square$

Example. Let $\alpha = (3, 2, 2) \vdash 7$, $k = 2$. Then $Y(\alpha) =$ . We have the following possible $Y(\gamma)$:



Therefore

$$\begin{aligned}
[\alpha][k] &= [5, 2^2] \oplus [4, 3, 2] \oplus [4, 2^2, 1] \oplus [3^2, 2, 1] \oplus [3, 2^3] \\
\xi^{(\alpha,k)} &= \chi^{(5,2^2)} + \chi^{(4,3,2)} + \chi^{(4,2^2,1)} + \chi^{(3^2,2,1)} + \chi^{(3,2^3)}
\end{aligned}$$ [1]

We can use Corollary 3.18 repeatedly to decompose $M^\alpha \cong [\alpha_1][\alpha_2] \cdots [\alpha_{\ell(\alpha)}]$ into irreducibles.

[1] Remark by L.T.: I believe on the LHS it should not be $\xi^{(\alpha,k)}$. Correct would be the character of $[\alpha][k]$ and this module does not coincide with $M^{(\alpha,k)} = [\alpha_1][\alpha_2][\alpha_3][k]$. E.g. we have $\dim M^{(\alpha,k)} = \frac{9!}{3!2!2!2!} = 7560$, but using the hook length formula we calculate $\dim[\alpha][k] = [S_9 : S_7 \times S_2] \dim[\alpha]\#[k] = \frac{9!}{7!2!} \dim[\alpha] \dim[k] = \frac{9!}{7!2!} \frac{7!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 2} \cdot 1 = 756$.

Example. Let $\alpha = (3,2,1) \vdash 6$. First,

$$[3][2] = \begin{ytableau} 1 & 1 & 1 & *(yellow) 2 & *(yellow) 2 \end{ytableau} \oplus \begin{ytableau} 1 & 1 & 1 & *(yellow) 2 \\ *(yellow) 2 \end{ytableau} \oplus \begin{ytableau} 1 & 1 & 1 \\ *(yellow) 2 & *(yellow) 2 \end{ytableau}$$

Here we label the original boxes with 1 and the new boxes with 2. Then,

$$\begin{aligned} [3][2][1] = {} & \begin{ytableau} 1 & 1 & 1 & 2 & 2 & *(yellow) 3 \end{ytableau} \oplus \begin{ytableau} 1 & 1 & 1 & 2 & 2 \\ *(yellow) 3 \end{ytableau} \\ & \oplus \begin{ytableau} 1 & 1 & 1 & 2 & *(yellow) 3 \\ 2 \end{ytableau} \oplus \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 & *(yellow) 3 \end{ytableau} \oplus \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 \\ *(yellow) 3 \end{ytableau} \\ & \oplus \begin{ytableau} 1 & 1 & 1 & *(yellow) 3 \\ 2 & 2 \end{ytableau} \oplus \begin{ytableau} 1 & 1 & 1 \\ 2 & 2 & *(yellow) 3 \end{ytableau} \oplus \begin{ytableau} 1 & 1 & 1 \\ 2 & 2 \\ *(yellow) 3 \end{ytableau} \end{aligned}$$

So

$$\xi^{(3,2,1)} = \chi^{(6)} + 2\chi^{(5,1)} + 2\chi^{(4,2)} + \chi^{(4,1,1)} + \chi^{(3,3)} + \chi^{(3,2,1)}.$$

Definition.

(i) A generalised Young tableau of shape $\alpha \vdash n$ and content (or weight, type) $\lambda \models n$ *is a filling of $Y(\alpha)$ with positive integers such that i appears exactly λ_i many times for all i.*

(ii) A generalised Young tableau is semistandard *if its entries weakly increase left to right along rows, but strictly increase down columns.*

We will abbreviate semistandard tableaux to SSYT.

Example. $\begin{ytableau} 2 & 1 & 1 & 4 & 2 & 2 \\ 4 & 1 \end{ytableau}$ has shape $(6,2)$, content $(3,3,0,2)$. The semistandard Young tableaux of shape this shape and content are

$$\begin{ytableau} 1 & 1 & 1 & 2 & 2 & 2 \\ 4 & 4 \end{ytableau} \qquad \begin{ytableau} 1 & 1 & 1 & 2 & 2 & 4 \\ 2 & 4 \end{ytableau} \qquad \begin{ytableau} 1 & 1 & 1 & 2 & 4 & 4 \\ 2 & 2 \end{ytableau}$$

Young tableaux from before are just generalised Young tableaux of content (1^n).

Using SSYT we can generalise the above example, determining $\xi^{(3,2,1)}$, and reformulate Young's Rule.

Corollary 3.19. *Let $\alpha \vdash n, \lambda \models n$. Then $\langle \xi^\lambda, \chi^\alpha \rangle$ is the number of SSYT of shape α and content λ.*

Note that unlike in Theorem 3.11 we don't require $\ell(\lambda) \leq n$.

Proof. Apply Corollary 3.18 and note that $M^\lambda \cong [\lambda_1][\lambda_2] \dots [\lambda_{\ell(\lambda)}]$ has character ξ^λ. $\square$

Example 1. We revisit the example after Theorem 3.11 where we showed that $\langle \chi^\alpha, \xi^\lambda \rangle = \langle \psi^\alpha, \xi^\lambda \rangle = 2$ for $\alpha = (3,2)$ and $\lambda = (2,0,1,2)$ or $\lambda = (0,2,2,0,1)$. The SSYT of shape α and content λ are:

$$\lambda = (2,0,1,2) \qquad \begin{ytableau} 1 & 1 & 3 \\ 4 & 4 \end{ytableau}, \quad \begin{ytableau} 1 & 1 & 4 \\ 3 & 4 \end{ytableau}$$

$$\lambda = (0, 2, 2, 0, 1) \qquad \begin{array}{|c|c|c|}\hline 2 & 2 & 3 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 2 & 2 & 5 \\ \hline 3 & 3 \\ \cline{1-2}\end{array}$$

Recall Lemma 3.12: If $\alpha, \beta \vdash n$ with $\langle \xi^\alpha, \chi^\beta \rangle > 0$, then $\alpha \trianglelefteq \beta$. The converse also holds.

Lemma 3.20. *Suppose $\alpha, \beta \vdash n$ with $\alpha \trianglelefteq \beta$. Then $\langle \xi^\alpha, \chi^\beta \rangle > 0$.*

Proof. Example Sheet 3. □

Remark. The number of SSYT of shape α and content λ is often denoted by $K_{\alpha,\lambda}$. Such quantities are known as *Kostka numbers.*

3.3.2 Branching Rule

We investigate restriction from S_n to $S_{n-1} \cong S_{n-1} \times S_1$. Note that this is a special case of $S_m \times S_k$.

Definition. *Let $\lambda \models n$, $i \in \mathbb{N}$. Define $\lambda^{i-} \models n - 1$ and $\lambda^{i+} \models n + 1$ via $\lambda^{i-} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots)$ and $\lambda^{i+1} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots)$.*

Lemma 3.21. *Let $\lambda \models n$. Then $\xi^\lambda\big\downarrow_{S_{n-1}} = \sum_{i=1}^{\infty} \xi^{\lambda^{i-}}$.*

Proof. First note that the RHS sum is finite since $\lambda^{i-1} \not\models n - 1$ for all $i > \ell(\lambda)$, whence $\xi^{\lambda^{i-1}} = 0$. Now, by Lemma 3.7,

$$\xi^\lambda\big\downarrow_{S_{n-1}} = \xi^\lambda\big\downarrow_{S_{n-1}\times S_1} = \sum_{\mu \models 1} \xi^{\lambda-\mu} \# \xi^\mu.$$

But $\xi^\mu = \mathbb{1}_{S_1}$ and $\mu = (0, \dots, 0, 1, 0, \dots)$ where the 1 is in the i-th position, so $\lambda - \mu = \lambda^{i-}$. □

Recall we defined α^-, where $\alpha \vdash n$, and removable boxes in Section 3.1. Observe

$$\alpha^- = \{\beta \vdash n - 1 \mid \beta = \alpha^{i-} \text{ for some } i \in \mathbb{N}\} = \{\alpha^{i-} \mid \alpha_i > \alpha_{i+1}\}.$$

Definition. *Let $\alpha \vdash n$. We define*

$$\alpha^+ := \{\beta \vdash n + 1 \mid \beta = \alpha^{i+1} \textit{ for some } i \in \mathbb{N}\} = \{\alpha^{i+} \mid \alpha_i < \alpha_{i+1}\},$$

where we treat $\alpha_0 = \infty$. In other words, α^+ is the set of all partitions β such that $Y(\beta)$ can be obtained from $Y(\alpha)$ by adding a single box.

We will call (i, j) addable *to α if $(i, j) \notin Y(\alpha)$ and $Y(\alpha) \cup \{(i, j)\} = Y(\beta)$ for some $\beta \in \alpha^+$.*

Example. Let $\alpha = (4,2,2,1) \vdash 9$. Then $Y(\alpha) = \begin{array}{ccccc} \square & \square & \square & \boxed{-} & + \\ \square & \square & + & & \\ \square & \boxed{-} & & & \\ \boxed{-} & + & & & \\ + & & & & \end{array}$ where the removable resp. addable boxes are marked with a $-$ resp. $+$. So

$$\begin{aligned}
\alpha^- &= \{(3,2^2,1),(4,2,1^1),(4,2^2)\},\\
\alpha^+ &= \{(5,2^2,1),(4,3,2,1),(4,2^3),(4,2^2,1^2)\}.
\end{aligned}$$

Theorem 3.22 (Branching Rule - restriction). *Let $\alpha \vdash n$. Then $\chi^\alpha\downarrow_{S_{n-1}} = \sum_{\beta\in\alpha^-}\chi^\beta$.*

Proof. We have

$$\begin{aligned}
\chi^\alpha\downarrow_{S_{n-1}} = \psi^\alpha\downarrow_{S_{n-1}} &= \sum_\pi \operatorname{sgn}\pi\xi^{\alpha-\operatorname{id}+\pi}\downarrow_{S_{n-1}}\\
&= \sum_\pi \operatorname{sgn}\pi \sum_{i\in\mathbb{N}} \xi^{(\alpha-\operatorname{id}+\pi)^{i-}}\\
&= \sum_{i\in\mathbb{N}}\sum_\pi (\operatorname{sgn}\pi)\xi^{\alpha^{i-}-\operatorname{id}+\pi}\\
&= \sum_{i\in\mathbb{N}} \psi^{\alpha^{i-}}
\end{aligned}$$

Now if $\psi^{\alpha^{i-}} \neq 0$, then $\alpha_i^{i-} - i \neq \alpha_{i+1}^{i-} - (i+1)$ by Lemma 3.5, so $\alpha_i - 1 - i \neq \alpha_{i+1} - (i+1)$ and so $\alpha_i \neq \alpha_{i+1}$, then $\alpha^{i-} \in \alpha^-$. $\square$

Corollary 3.23 (Branching Rule - induction). *Let $\alpha \vdash n$. Then $\chi^\alpha\uparrow^{S_{n+1}} = \sum_{\beta\in\alpha^+}\chi^\beta$.*

Proof. This follows from Theorem 3.22 and Frobenius reciprocity noting that $\beta \in \alpha^+$ iff $\alpha \in \beta^-$. $\square$

Example. Let $\alpha = (4,2^2,1) \vdash 9$. Then

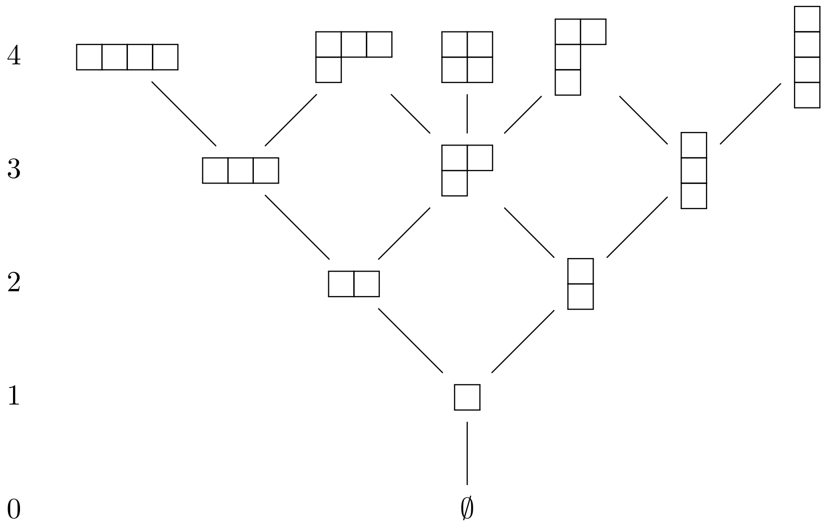
$$\begin{aligned}
\chi^\alpha\downarrow_{S_8} &= \chi^{(3,2^2,1)} + \chi^{(4,2,1^2)} + \chi^{(4,2^2)},\\
\chi^\alpha\uparrow^{S_{10}} &= \chi^{(5,2^2,1)} + \chi^{(4,3,2,1)} + \chi^{(4,2^3)} + \chi^{(4,2^2,1^2)}.
\end{aligned}$$

Definition. *The* Young (branching) graph $\mathbb{Y}$ *is the graph with*

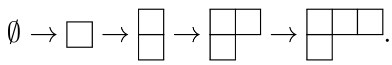
- *vertex set $\bigcup_{n\in\mathbb{N}_0}\wp(n)$,*
- *edge set $\{(\lambda,\mu) \mid \mu\in\lambda^-\}$.*

We will call $\wp(n)$ th n-th layer *or* level *of $\mathbb{Y}$.*

Here are the first five layers of $\mathbb{Y}$:



For each partition λ, there is a natural bijection between $\mathrm{std}(\lambda)$ and the set of upwards-directed paths from $\emptyset$ to λ in $\mathbb{Y}$. Indeed, given such a path, we construct the standard λ-tableau by putting in the layer number in each newly added box in the path. E.g. consider $\lambda = (3, 1)$ and the path



Then we get the sequence of tableaux

$$\emptyset \to \begin{array}{|c|}\hline 1 \\ \hline \end{array} \to \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \to \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c|}{3} \\ \hline 2 & \\ \cline{1-1} \end{array} \to \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}.$$

Now $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$ is the standard tableau corresponding to this path.

3.3.3 Murnaghan-Nakayama Rule

Definition. *Let $\lambda \vdash n$, $(i, j) \in Y(\lambda)$.*

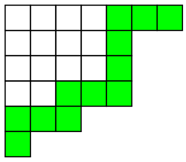
(i) The rim *of λ is $R(\lambda) := \{(x, y) \in Y(\lambda) \mid (x + 1, y + 1) \notin Y(\lambda)\}$.*

*(ii) The (i, j)-*rim hook *of λ is $R_{i,j}(\lambda) = \{(x, y) \in R(\lambda) \mid x \geq i$ and $y \geq j\}$. Its* hand *is (i, λ_i) and its* foot *is (λ'_j, j), the same as for $H_{i,j}(\lambda)$.*

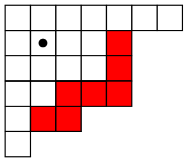
(iii) The leg length *of $R_{i,j}(\lambda)$ is $\lambda'_i - j$, and* arm length *$\lambda_i - -j$, same as for $H_{i,j}(\lambda)$.*

Note that for both the hook and the rim hook the leg (resp. arm) length is the number of rows (resp. columns) occupied by the hook, minus one.

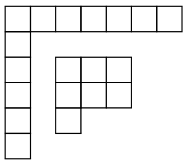
Example. Let $\lambda = (7, 5^3, 3, 1) \vdash 26$. Then the boxes in the rim $R(\lambda)$ are highlighted green.

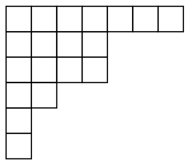


Take $(i, j) = (2, 2)$. The rim hook is highlighted red.



Removing $H_{2,2}(\lambda)$ (resp. $R_{2,2}(\lambda)$) leaves





$(7, 1^5), (3, 3, 2)$ $\qquad\qquad$ $(7, 4^2, 2, 1^2)$

Note that if we merge the two components obtained by removing $H_{2,2}(\lambda)$ get precisely what is left after removing $R_{2,2}(\lambda)$.

Lemma 3.24. *Let $\lambda \vdash n$, $(i, j) \in Y(\lambda)$.*

(i) $|R_{i,j}(\lambda)| = |H_{i,j}(\lambda)| = h_{i,j}(\lambda)$,

(ii) Removing $H_{i,j}(\lambda)$ from $Y(\lambda)$, and then sliding the lower-right component (if the result was disconnected) up and left one unit each, gives $Y(\lambda) \setminus R_{i,j}(\lambda)$.

Proof. Consider a walk along each hook, one box at a time traversing from the hand to the foot. Then the claims follow since

- $H_{i,j}(\lambda)$ and $R_{i,j}(\lambda)$ have the same hands and feet,
- we only move left or down at each step,
- we use the same number of leftward steps (namely the common arm length $\lambda_i - j$), and downward steps (by length $\lambda'_j - i$).

$\square$

Definition. *Let $\lambda \vdash n$, $(i, j) \in Y(\lambda)$. Define $\lambda \setminus H_{i,j}(\lambda)$ to be the partition of $|\lambda| - h_{i,j}(\lambda)$ such that $Y(\lambda \setminus H_{i,j}(\lambda)) = Y(\lambda) \setminus R_{i,j}(\lambda)$. Explicitly, letting $a = \lambda_i - j$ be the arm length, and $b = \lambda'_j - i$ be the leg length, then*

$$\lambda \setminus H_{i,j}(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1} - 1, \lambda_{i+2} - 1, \ldots, \lambda_{i+b} - 1, j - 1, \lambda_{i+b+1}, \lambda_{i+b+2}, \ldots).$$

Note that $h_{i,j}(\lambda) = 1 + a + b$, so $j - 1 = \lambda_i - a - 1 = \lambda_i - h_{i,j}(\lambda) + b$.

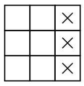
- From now on, when we remove a hook from λ, we mean to get $\lambda \setminus H_{i,j}(\lambda)$ for some $(i, j) \in Y(\lambda)$.
- If μ is obtained from λ by removing a hook, then we let $LL(\lambda \setminus \mu)$ denote the leg length of the removed hook. That is, $\mu = \lambda \setminus H_{i,j}(\lambda)$ for some $(i, j) \in Y(\lambda)$, and $LL(\lambda \setminus \mu)$ is the leg length of $H_{i,j}(\lambda)$, equivalently of $R_{i,j}(\lambda)$.

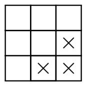
Theorem 3.25 (Murnaghan-Nakayama Rule)**.** *Let $\alpha \vdash n$, $k \in [n]$. Let $\pi \in S_n$, and suppose that it has a k-cycle in its disjoint cycle decomposition. Let $\rho \in S_{n-k}$ have the same cycle type as π but with one fewer k-cycle. Then*

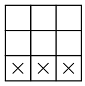
$$\chi^\alpha(\pi) = \sum_\beta (-1)^{LL(\alpha\setminus\beta)} \chi^\beta(\rho),$$

where the sum runs over partitions β obtained from α by removing a hook of size k.

Example. Let $\alpha = (3^3) \vdash 9$, $\pi = (1234)(56)(789)$. We take $k = 3$ and $\rho = (1234)(56)$. What are the possible hooks of size 3 we can remove?


$LL = 2$

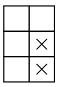

$LL = 1$

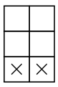

$LL = 0$

Then

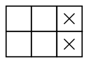
$$\chi^\alpha(\pi) = \big(\chi^{(2^3)} - \chi^{(3,2,1)} + \chi^{(3^2)}\big)(\mu)$$

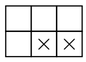
We repeat this with $n = 6, k = 2$. So this is


$LL = 1$

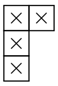

$LL = 0$

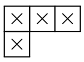
no removable hooks of size 2 from $(3, 2, 1)$


$LL = 1$


$LL = 0$

$$\begin{aligned}
&= \Big(-\chi^{(2,1^2)} + \chi^{(2^2)} + 0 - \chi^{(2^2)} + \chi^{(3,1)} \Big)((1234)) \\
&= (-\chi^{(2,1^2)} + \chi^{(3,1)})((1234))
\end{aligned}$$


$LL = 2$


$LL = 2$

$$\begin{aligned}
&= \big(-\chi^{\emptyset} - \chi^{\emptyset} \big)(e) \\
&= -1 - 1 = -2
\end{aligned}$$

Proof of Theorem 3.25. Since characters are class functions, we may assume that $\pi = \rho\sigma$ for some k-cycle σ disjoint from ρ. For $\mu \models k$, recall $\xi^\mu = \mathbb{1}_{S_\mu}\big\uparrow^{S_k}$, so $\xi^\mu(\sigma) = \frac{1}{|S_\mu|} \sum_{\substack{x \in S_k \\ x\sigma x^{-1} \in S_\mu}} 1$. But since σ is a k-cycle, σ belongs to a conjugate of S_μ if and only if

$\mu = (\dots, 0, k, 0, \dots)$. So if μ is not of this form, then $\xi^\mu(\sigma) = 0$. On the other hand, if $\mu = (\dots, 0, k, 0, \dots)$, then $\xi^\mu = \mathbb{1}_{S_{(k)}}\big\uparrow^{S_k} = \mathbb{1}_{S_k}$, and so $\xi^\mu(\sigma) = 1$. Therefore

$$\begin{aligned}
\chi^\alpha(\pi) = \psi^\alpha(\pi) = \psi^\alpha\big\downarrow_{S_{n-k}\times S_k}(\rho\sigma) \\
&\overset{\text{Lemma 3.7}}{=} \sum_{\mu \models k} \left(\psi^{\alpha-\mu} \# \xi^\mu\right)(\rho\sigma) \\
&= \sum_{\mu\models k} \psi^{\alpha-\mu}(\rho)\xi^\mu(\sigma) \\
&= \sum_{i=1}^{\infty} \psi^{(\alpha_1,\alpha_2,\dots,\alpha_{i-1},\alpha_i-k,\alpha_{i+1},\dots)}(\rho) \\
&= \sum_{i\in\mathbb{N}} \psi^{\beta_{i,0}}(\rho) \qquad (*)
\end{aligned}$$

where we let $\beta_{i,0} = (\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_i - k, \alpha_{i+1}, \dots) \models n-k$.

Recall from Lemma 3.5 that if $\gamma - \mathrm{id} = (j\ j{+}1) \circ (\lambda - \mathrm{id})$, then $\psi^\gamma = -\psi^\lambda$. Fix $i \in \mathbb{N}$, define $\beta_{i,m} \models n-k$ via $\beta_{i,m} - \mathrm{id} = (i+m\ i+m-1 \dots i+2\ i+1\ i) \circ (\beta_{i,0} - \mathrm{id})$, for each $m \in \mathbb{N}_0$. Explicitly, $\beta_{i,m} = (\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_{i+1}-1, \dots, \alpha_{i+m}-1, \alpha_i - k + m, \alpha_{i+m+1}, \dots)$. Since $(i+m\ i+m-1 \dots i+2\ i+1\ i) = (i+m\ i+m-1)\cdots(i+2\ i+1)(i+1\ i)$, we can apply Lemma 3.5 repeatedly to get

$$\psi^{\beta_{i,0}} = (-1)^m \psi^{\beta_{i,m}}.$$

We will see that

- *if* there exists $m \in \mathbb{N}_0$ such that $\beta_{i,m}$ is a partition, then m is unique, and we will relate $\beta_{i,m}$ to α by removing an appropiate hook,

† while if there does not exist such an m, then we will show that $\psi^{i,m} = 0$.

For $(i,j) \in Y(\alpha)$, letting b be the leg length of $H_{i,j}(\alpha)$, we recall that $\alpha \setminus H_{i,j}(\alpha) = (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}-1, \alpha_{i+2}-1, \dots, \alpha_{i+k}-1, \alpha_i - h_{i,j}(\alpha) + b, \alpha_{i+k+1}, \alpha_{i+k+2}, \dots)$. Compare this with $\beta_{i,m}$. For any given i, we see that the following are equivalent:

- the existence of an $m \in \mathbb{N}_0$ such that $\beta_{i,m}$ is a partition,
- the existence of a rim hook $R_{i,j}(\alpha)$ of size k, for some $j \in [\lambda_i]$.

The highest row occupied by this hook is row i. In particular, $i \le \ell(\alpha)$. A rim hook is uniquely determined by its highest row and size. In particular, there is at most one m for each i, and when this exists, m is uniquely determined as the leg length of the hook.

Notice if $i > \ell(\alpha)$, then for all $m \in \mathbb{N}_0$, $\beta_{i,m}$ has a negative part:

$$\begin{aligned}
\beta_{i,0} &= (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)}, 0, \dots, 0, -k, 0\dots) \\
\beta_{i,1} &= (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)}, 0, \dots, 0, -1, -k+1, 0\dots) \\
\beta_{i,2} &= (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)}, 0, \dots, 0, -1, -1, -k+2, 0\dots)
\end{aligned}$$

So we never talk about a hook length in row i unless i is a genuine row in $Y(\alpha)$.

Once we prove the claim †, then $(*)$ gives

$$\begin{aligned}
\chi^\alpha(\pi) &= \sum_{i\in\mathbb{N}} \psi^{\beta_{i,0}}(\rho) \\
&= \sum_{\substack{i\in\mathbb{N} \text{ such that} \\ \exists m \in \mathbb{N}_0\colon\ \beta_{i,m} \text{ is a partition}}} (-1)^m \psi^{\beta_{i,m}}(\rho) \\
&= \sum_{\substack{\beta\vdash n-k \\ \text{obtained from } \alpha \\ \text{by removing a hook} \\ \text{of size } k}} (-1)^{LL(\alpha\setminus\beta)} \psi^\beta(\rho) \\
&= \sum_\beta \chi^\beta(\rho).
\end{aligned}$$

So it remains to prove †. Fix $i \in \mathbb{N}$ and suppose $\beta_{i,m} \not\vdash n-k$ for all $m \in \mathbb{N}_0$. Observe

$$\begin{aligned}
\beta_{i,m} - \mathrm{id} = (&\alpha_1 - 1, \alpha_2 - 2, \dots, \alpha_{i-1} - (i-1), \alpha_{i+1} - (i+1), \dots, \alpha_{i+m} - (i+m), \\
&\alpha_i - i - k, \alpha_{i+m+1} - (i+m+1), \dots)
\end{aligned}$$

Since α is a partition, $\alpha - \mathrm{id}$ is strictly descreasing. Since $\alpha_i - i \geq \alpha_i - i - k \geq \alpha_{i+k} - (i+k)$, there exists a unique $t \in \mathbb{N}_0$ such that $\alpha_{i+t} - (i+t) \geq \alpha_i - i - k > \alpha_{i+t+1} - (i+t+1)$. If $\alpha_{i+t} - (i+t) = \alpha_i - i - k$, then $\beta_{i,t} - \mathrm{id}$ has two adjacent terms equal. But then $\psi^{\beta_{i,t}} = 0$ by Lemma 3.5, hence $\psi^{\beta_{i,0}} = (-1)^t \psi^{\beta_{i,t}} = 0$. Otherwise, $\alpha_{i+t} - (i+t) > \alpha_i - i - k$. But that means $\beta_{i,t}$ is weakly decreasing. Also $(\beta_{i,t})_j = \alpha_j$ for all $j \geq i+t+1$ and $\alpha_j \geq 0$ for all $j \in \mathbb{N}$. Also α has finite support, hence so does $\beta_{i,t}$, thus $\beta_{i,t}$ is a partition, contradicting our assumption. This proves † and hence the proof of the theorem. □

4 McKay Numbers

In this chapter we go back to partitions, and continue with $\mathbb{F} = \mathbb{C}$.

Main goal. Describe $\mathrm{Irr}_{p'}(S_n)$ and work towards understanding the techniques in Olsson's proof of the McKay Conjecture for symmetric groups.

4.1 James's Abacus

Example. Let $\lambda = (7, 5^3, 3, 1) \vdash 26$, and consider $H_{2,2}(\lambda), R_{2,2}(\lambda)$. Write $1, 2, \ldots, h_{2,2}(\lambda)$ into $R_{2,2}(\lambda)$ from hand to foot. For those numbers in boxes at the bottom of their column, write them in $H_{2,2}(\lambda)$ in the same column. For the rest, write them in to $H_{2,2}(\lambda)$ in the row below.

$$\begin{ytableau} \ & \ & \ & \ & \ & \ & \ \\ \ & \ & \ & \ & 1 \\ \ & \ & \ & \ & 2 \\ \ & \ & 5 & 4 & 3 \\ \ & 7 & 6 \\ \ \end{ytableau} \quad \rightsquigarrow \quad \begin{ytableau} \ & \ & \ & \ & \ & \ & \ \\ \ & 7 & 6 & 4 & 3 \\ \ & 1 & \ & \ & \ \\ \ & 2 & \ & \ & \ \\ \ & 5 & \ \\ \ \end{ytableau}$$

Observe

$$\begin{aligned} 1 &= 7 - 6 = h_{2,2}(\lambda) - h_{3,2}(\lambda) \\ 2 &= 7 - 5 = h_{2,2}(\lambda) - h_{4,2}(\lambda) \\ 5 &= 7 - 2 = h_{2,2}(\lambda) - h_{5,2}(\lambda) \end{aligned}$$

Lemma 4.1. *Let $\lambda \vdash n$, $(i, j) \in Y(\lambda)$. Then*

$$\{1, 2, \ldots, h_{i,j}(\lambda)\} = \{h_{i,y}(\lambda) \mid j \leq y \leq \lambda_i\} \sqcup \{h_{i,j}(\lambda) - h_{x,j}(\lambda) \mid i < x \leq \lambda'_j\}$$

Proof. We omit (λ) from the notation. Let $A = \{(u, v) \in R_{i,j} \mid u = \lambda'_v\} = \{(\lambda'_y, y) \mid j \leq y \leq \lambda_i\}$ and $B = \{(u, v) \in R_{i,j} \mid u \neq \lambda'_v\} = \{(x - 1, \lambda_x) \mid i < x \leq \lambda'_j\}$.

By Lemma 3.24, $|R_{i,j}| = h_{i,j}$, so we may fill the numbers $1, 2, \ldots, h_{i,j}$ into $R_{i,j}$ one number in each box from hand to foot. We claim that A is filled with $\{h_{i,y} \mid j \leq y \leq \lambda_i\}$ and B with $\{h_{i,j} - h_{x,j} \mid i < x \leq \lambda'_j\}$, whence the lemma follows. Consider (λ'_y, y) in A. It is filled with

$$\begin{aligned} &1 + \#\text{left steps} + \#\text{down steps} \\ =\, &1 + \text{arm length of } H_{i,y} + \text{leg length of } H_{i,y} = h_{i,y} \end{aligned}$$

Consider $(x-1, \lambda_x) \in B$. It is filled with

$$\begin{aligned} & 1 + \#\text{left steps} + \#\text{down steps} \\ = & 1 + (\lambda_i - \lambda_x) + (x - 1 - i) = (1 + \lambda_i - j + \lambda'_j - i) - (1 + \lambda_x - j + \lambda'_j - x) \\ = & h_{i,j} - h_{x,j}. \end{aligned}$$

$\square$

Definition. *Let $\lambda \vdash n$, $m = \ell(\lambda)$.*

(i) Let $X_\lambda = \{h_{1,1}(\lambda), h_{2,1}(\lambda), \ldots, h_{m,1}(\lambda)\}$ be the set of first column hook lengths of λ.

(ii) For each $i \in [m]$, let $\mathcal{H}_i(\lambda) = \{h_{i,j}(\lambda) \mid j \in [\lambda_i]\}$ be the set of row i hook lengths of λ.

Note that $\mathcal{H}_i(\lambda) = \{1, 2, \ldots, h_{i,1}(\lambda)\} \setminus \{h_{1,1}(\lambda) - h_{x,1}(\lambda) \mid i < x \leq m\}$ by Lemma 4.1.

Convention: If $i > m$, then $\mathcal{H}_i(\lambda) = \emptyset$.

Notice that X_λ determines λ: If we know that $\{h_1, h_2, \ldots, h_m\}$ where $h_1 > h_2 > \cdots > h_m$, is the set of first column hook lengths for some partition λ, then λ must be $\lambda = (h_1 - (m-1), \ldots, h_{m-1} - 1, h_m - 0)$.

Idea. We represent partitions using beads on an abacus.

- Info about hook lengths is encoded into the bead positions
- given an arrangement of beads, we will be able to reconstruct the partition using observations like the above.
- advantages: operations on partitions (e.g. hook removal) are easy to describe.

Definition. *A β-*set *X is a finite subset $\{h_1, \ldots, h_m\}$ of $\mathbb{N}_0$. Convention: $h_1 > h_2 > \cdots > h_m$.*

*For a β-set $X = \{h_1, \ldots, h_m\}$ and $l \in \mathbb{N}_0$, we define X^{+l}, the l-*shift *of X, as follows:*

- *$X^{+0} = X$,*
- *if $l > 0$, then $X^{+l} = \{h_1 + l, h_2 + l, \ldots, h_m + l\} \cup \{l-1, l-2, \ldots, 1, 0\}$.*

We define the partition corresponding to X to be $\mathcal{P}(X) = (h_1 - (m-1), h_2 - (m-2), \ldots, h_{m-1} - 1, h_m - 0)$. This expression for $\mathcal{P}(X)$ may have trailing zeros, which can be removed.

Example. Let $X = \{4, 2\}$. Then $\mathcal{P}(X) = (4-1, 2-0) = (3, 2)$. And $X^{+2} = \{6, 4, 1, 0\}$ and $\mathcal{P}(X^{+2}) = (6-3, 4-2, 1-1, 0-0) = (3, 2)$.

Lemma 4.2. *Let $\lambda \vdash n$ and X a β-set. Then X is a β-set for λ, meaning $\mathcal{P}(X) = \lambda$, if and only if $X \in \{X_\lambda^{+l} \mid l \in \mathbb{N}_0\}$.*

Proof. Let $X = \{h_1, h_2, \ldots, h_m\}$ and $t = \ell(\lambda)$. Then

$$\begin{aligned}
\mathcal{P}(X) = \lambda &\iff (h_1 - (m-1), \ldots, h_{m-1} - 1, h_m - 0) = (\lambda_1, \lambda_2, \ldots, \lambda_t) \\
&\iff m \geq t \text{ and } \begin{cases} h_j - (m-j) = \lambda_j & \text{if } j \leq t, \\ h_j - (m-j) = 0 & \text{if } j > t \end{cases} \\
&\iff m \geq t \text{ and } h_j = \begin{cases} \lambda_j + (t-j) + (m-t) & \text{if } j \leq t, \\ m - j & \text{if } j > t \end{cases} \\
&\iff m - t \in \mathbb{N}_0 \text{ and } X = X_\lambda^{+(m-t)}
\end{aligned}$$

□

Definition. *Let* $e \in \mathbb{N}$. James's e-abacus *consists of* e runners *(drawn as columns) labelled* $0, 1, 2, \ldots, e-1$ *from left to right, with rows labelled by* $\mathbb{N}_0$ *increasing downwards. The positions are labelled by* $\mathbb{N}_0$*, with that in row* a *and runner* i *labelled by* $ae + i$*.*

Given a β*-set* X*, the* e-abacus configuration *corresponding to* X *has beads precisely in positions given by the elements of* X*. We call the configuration* A_X*. Conversely, given an* e*-abacus configuration* A*, i.e. a finite set of beads in the* e*-abacus, define the corresponding* β*-set* X_A *to be the set of position labels of the beads. We define the corresponding partition to be* $\mathcal{P}(A) := \mathcal{P}(X_A)$*.*

Also, if $X = X_\lambda$*, then abbreviate* $A_{X_\lambda} = A_\lambda$*.*

Clearly,

$$\begin{aligned}
\{e\text{-abacus configurations}\} &\xleftrightarrow{1-1} \{\beta\text{-sets}\} \\
A_X &\longleftarrow\!\shortmid X \\
A &\longmapsto X_A \\
\text{bead positions} &\longleftrightarrow \{h_1, \ldots, h_m\}
\end{aligned}$$

e-abacus:

$$\begin{array}{c|ccccc}
 & 0 & 1 & 2 & \cdots & e-1 \\
\hline
0 & 0 & 1 & 2 & \cdots & e-1 \\
1 & e & e+1 & e+2 & \cdots & 2e-1 \\
2 & 2e & 2e+1 & 2e+2 & \cdots & 3e-1 \\
\vdots & & & \vdots & &
\end{array}$$

Examples.

(i) Let $e \geq 2$, $X = \{2e, e+1, 2\}$. On an e-abacus we have

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & \cdots & e-1 \\ \hline 0 & 0 & 1 & 2 & \cdots & e-1 \\ 1 & e & e+1 & e+2 & \cdots & 2e-1 \\ 2 & 2e & 2e+1 & 2e+2 & \cdots & 3e-1 \\ \vdots & & & \vdots & & \end{array}$$

and $\mathcal{P}(X) = (2e-2, e, 2)$.

(ii) Consider the 3-abacus configuration A given by

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 3 & 4 & 5 \\ 2 & 6 & 7 & 8 \\ 3 & 9 & 10 & 11 \end{array}$$

so $X_A = \{11, 9, 5, 3, 2, 1\}$ and $\mathcal{P}(A) = \mathcal{P}(X_A) = (6, 5, 2, 1^3) \vdash 16$.

Let $X = \{6, 4, 1, 0\}$, $e = 3$ or $e = 4$. Then

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 3 & 4 & 5 \\ 2 & 6 & 7 & 8 \end{array} \qquad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 3 \\ 1 & 4 & 5 & 6 & 7 \end{array}$$

We have $\mathcal{P}(X) = (3, 2)$.

Lemma 4.3. *Let $e \in \mathbb{N}$. Given an e-abacus configuration A, with beads at $h_1 > h_2 > \cdots > h_m$, then $\mathcal{P}(A) = (a_1, a_2, \ldots, a_m)$ where a_j is the number of gaps, i.e. empty positions, i such that $0 \leq i < h_j$.*

Proof. By definition, $\mathcal{P}(A) = (h_1 - (m-1), \ldots, h_m - 0)$. But there are h_j positions before h_j, of which $m - j$ have beads, namely $h_{j+1}, \ldots, h_m$. □

Definition. *Let $X = \{h_1, \ldots, h_m\}$ be a β-set. For $i \in [m]$, define $\mathcal{H}_i(X) = \{1, 2, \ldots, h_i\} \setminus \{h_i - h_j \mid i < j \leq m\}$.*

Lemma 4.4. *Let $\lambda \vdash n$ and X a β-set for λ. If $X = \{h_1, \ldots, h_m\}$, then $\mathcal{H}_i(X) = \mathcal{H}_i(\lambda)$ for all $i \in [m]$.*

Proof. We have $X = X_\lambda^{+(m-\ell(\lambda))}$ from the proof of Lemma 4.2. If $i > \ell(\lambda)$, then $|\mathcal{H}_i(X)| = h_i - (m-i) = 0$, so $\mathcal{H}_i(X) = \mathcal{H}_i(\lambda) = \emptyset$. If $i \leq \ell(\lambda)$, then $\mathcal{H}_i(\lambda) = \{1, 2, \ldots, h_{i,1}(\lambda)\} \setminus$

$\{h_{1,1}(\lambda) - h_{j,1}(\lambda) \mid i < j \leq \ell(\lambda)\}$ and so clearly $\mathcal{H}_i(\lambda) = \mathcal{H}_i(X_\lambda)$. So it remains to check $\mathcal{H}_i(X) = \mathcal{H}_i(X^{+1})$. We have $X^{+1} = \{h_1+1, h_2+1, \ldots, h_m+1, 0\}$, so

$$\begin{aligned}\mathcal{H}_i(X^{+1}) &= \{1,2,\ldots,h_i+1\} \setminus (\{(h_i+1)-(h_j+1) \mid i<j\leq m\} \cup \{h_i+1)-0\}) \\ &= \{1,2,\ldots,h_i\} \setminus \{h_i - h_j \mid i<j\leq m\} = \mathcal{H}_i(X).\end{aligned}$$

□

Corollary 4.5. *Let $\lambda \vdash n$ and $X = \{h_1, \ldots, h_m\}$ be a β-set for λ. Let $h \in \mathbb{N}_0$. Then $h \in \mathcal{H}_i(\lambda)$ iff $h_i - h \geq 0$ and $h_i - h \notin X$, for any $i \in [m]$.*

Proof. The claim is clear if $i > \ell(\lambda)$ (since then $\mathcal{H}_i(\lambda) = \emptyset$), or if $h = 0$. So we may assume that $i \leq \ell(\lambda)$ and $h > 0$. If $h > h_i$, then $h > \max \mathcal{H}_i(X) = \max \mathcal{H}_i(\lambda)$, so $h \in \mathcal{H}_i(\lambda)$. Otherwise, $h \leq h_i$. Recall $\mathcal{H}_i(\lambda) = \mathcal{H}_i(X) = \{1,2,\ldots,h_i\} \setminus \{h_i - h_j \mid i < j \leq m\}$. So

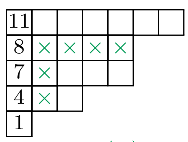
$$\begin{aligned}h \notin \mathcal{H}_i(\lambda) &\iff h = h_i - h_j \text{ for some } i < j \leq m \\ &\iff h_i - h \in X\end{aligned}$$

□

Corollary 4.6. *Let $\lambda \vdash n$ and suppose $ef \in \mathcal{H}(\lambda)$ for some $e, f \in \mathbb{N}$. Then $e \in \mathcal{H}(\lambda)$.*

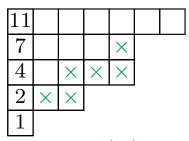
Proof. Let $X = X_\lambda = \{h_1, h_2, \ldots, h_m\}$. Since $ef \in \mathcal{H}(\lambda)$, then $ef \in \mathcal{H}_i(\lambda)$ for some $i \in [m]$. By Corollary 4.5, $0 \leq h_i - ef \notin X$. But $h_i \in X$, so there exists $l \in \{0, 1, \ldots, f-1\}$ such that $0 \leq h_i - e(l+1) \notin X$, but $h_i - el \in X$. This means $h_i - el = h_k$ for some $i \leq k \leq m$. But then $0 \leq h_k - e = h_i - e(l+1) \notin X$, hence by Corollary 4.5 again, $e \in H_k(\lambda)$. □

Example. Let $\lambda = (7, 5^2, 3, 1) \vdash 21$. So:



$H_{2,2}(\lambda)$

$X_\lambda = \{11, 8, 7, 4, 1\}$



$R_{2,2}(\lambda)$

$X_{\lambda \setminus H_{2,2}(\lambda)} = \{11, 7, 4, 2, 1\}$

Note that $X_{\lambda\setminus H_{2,2}(\lambda)} = (X_\lambda \setminus \{8\}) \sqcup \{8 - h_{2,2}(\lambda)\}$ and 8 is the second element in X_λ.

Proposition 4.7. *Let $\lambda \vdash n$, $X = \{h_1, h_2, \ldots, h_m\}$ be a β-set for λ. Let $(i,j) \in Y(\lambda)$. Then*

(i) $0 \leq h_i - h_{i,j}(\lambda) \notin X$,

(ii) $Z := (X \setminus \{h_i\}) \sqcup \{h_i - h_{i,j}(\lambda)\}$ *is a β-set for $\lambda \setminus H_{i,j}(\lambda)$*

Proof.

(i) Immediate from Corollary 4.5.

(ii) Since β-sets are determined up to shift, and $Z^{+l} = (X^{+l} \setminus \{h_i+l\}) \sqcup \{(h_i+l) - h_{i,j}(\lambda)\}$, then it is enough to prove (ii) for $X = X_\lambda$. So now assume $X = X_\lambda$, $m = \ell(\lambda)$, $h_i = h_{i,j}(\lambda)$. Let $\mu = \lambda \setminus H_{i,j}(\lambda)$. Recall that if b is the leg length of $H_{i,j}(\lambda)$, then $\mu = (\lambda_1, \lambda_2, \dots, \lambda_{i-1}, \lambda_{i+1} - 1, \dots, \lambda_{i+b} - 1, j-1, \lambda_{i+b+1}, \dots)$. Let Z' be the β-set for μ such that $|Z'| = m$. This does exist, since $\ell(\mu) \le \ell(\lambda) = m$, so in particular, Z' is just $X_\mu^{+(m-\ell(\mu))}$. Let $Z' = \{k_1, \dots, k_m\}$. We compute Z':

- For $s < i$, then $k_s = \mu_s + (\ell(\mu) - s) + (m - \ell(\mu)) = \lambda_s + m - s = h_{s,1}(\lambda) = h_s$.
- For $s \in \{0, 1, \dots, b-1\}$, $k_{i+s} = \mu_{i+s} + (\ell(\mu) - (i+s)) + (m - \ell(\mu)) = (\lambda_{i+s+1} - 1) + m - (i+s) = \lambda_{i+s+1} + m - (i+s+1) = h_{i+s+1}$.
- $k_{i+b} = \mu_{i+b} + (\ell(\mu) - (i+b)) + (m - \ell(\mu)) = j - 1 + m - i - b$.
- For $s \ge i+b+1$, $k_s = \mu_s + m - s = \lambda_s + m - s = h_s$.

So $Z' = (X \setminus \{h_i\}) \sqcup \{j - 1 + m - i - b\}$. But $h_i - h_{i,j} = h_{i,1}(\lambda) - h_{i,j}(\lambda) = (\lambda_i + m - i) - (1 + \lambda_i - j + b) = j - 1 + m - i - b$. So $Z' = Z$.

□

Corollary 4.8. *Let $e \in \mathbb{N}$, $\lambda \vdash n$, $X = \{h_1, \dots, h_m\}$ a β-set for λ, $i \in [m]$. Write $h_i = ae + j$ for $a \in \mathbb{N}_0$ and $j \in \{0, 1, \dots, e-1\}$. Then the following are equivalent:*

- *There exists $y \in [\lambda_i]$ such that $h_{i,y}(\lambda) = e$.*
- *$a \ge 1$ and $(a-1)e$ is an empty position in the e-abacus configuration A_X.*

When these hold, y is unique.

Moreover, the e-abacus configuration A' obtained from A_X by sliding the bead in position h_i to position $h_i - e$ has $\mathcal{P}(A') = \lambda \setminus H_{i,y}(\lambda)$.

In other words, removing a hook of size e is the same as sliding a bead up one row on an e-abacus.

Proof. By Corollary 4.5,

$$\begin{aligned} e \in \mathcal{H}_i(\lambda) &\iff 0 \le h_i - e \notin X \\ &\iff a \ge 1, \text{ and } (a-1)e + j \notin X. \end{aligned}$$

Hence the equivalence. For the second part, clearly $X_{A'} = (X \setminus \{h_i\}) \cup \{h_i - e\}$, but this is a β-set for $\lambda \setminus H_{i,y}(\lambda)$ by Proposition 4.7. □

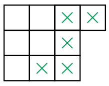
Remark. Recall the proof of Corollary 4.6 - we had $0 \le h_i - ef \notin X$, $h_i \in X$. The existence of l in the proof is equivalent to there being a bead immediately below a gap somewhere on this runner between h_i and $h_i - ef$. By Corollary 4.8, this corresponds to a hook of length e.

Just as we have the division algorithm for integers, giving quotients and remainders when we divide by e, we can do something similar for partitions, giving "e-quotients", and "e-cores".

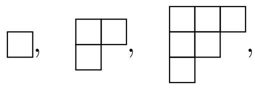
Definition. *Let $e \in \mathbb{N}$, $\lambda \vdash n$. We say that λ is an* e-core partition *if $e \notin \mathcal{H}(\lambda)$. The empty partition $\emptyset$ is always an e-core for any e.*

Example.

(i) Suppose $|\lambda| < e$. Then λ is an e-core partition.

(ii)  We can see that $(4, 3, 3)$ is not a 5 core, but $(2, 2, 1)$ is.

(iii) Let $e = 2$. Hooks of size 2 are always "dominoes" (i.e. 2×1 or 1×2 rectangles). So the 2-core partitions are precisely

$\emptyset$,  $\ldots$

i.e. $\emptyset$ and $(t, t-1, \ldots, 2, 1)$ for $t \in \mathbb{N}$.

Definition. *Let $e \in \mathbb{N}$, $\lambda \vdash n$, X a β-set for λ.*

(i) For $i \in \{0, 1, \ldots, e-1\}$, define $X_i^{(e)} = \{a \in \mathbb{N}_0 \mid ae + i \in X\}$. That is, $X_i^{(e)}$ is the set of row labels of beads on runner i of the e-abacus configuration A_X.

(ii) The e-quotient of λ *is $Q_e(\lambda) := (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(e-1)})$ where $\lambda^{(i)} = \mathcal{P}(X_i^{(e)})$. That is, $\lambda^{(i)}$ is the partition corresponding to the runner i of A_X viewed as a 1-abacus.*

(iii) Define $X_{(e)} = \bigsqcup_{i=0}^{e-1} \{ae + i \mid 0 \leq a \leq |X_i^{(e)}| - 1\}$.

(iv) The e-core of λ *is $C_e(\lambda) := \mathcal{P}(X_{(e)})$.*

The e-abacus configuration $A_{X_{(e)}}$ is obtained from A_X by sliding beads up as high as possible. The description of $A_{X_{(e)}}$ and Corollary 4.8 imply that $C_e(\lambda)$ is indeed an e-core partition.

Lemma 4.9. *Let $e \in \mathbb{N}$, $\lambda \vdash n$, X a β-set for λ.*

(i) For $i \in \{1, 2, \ldots, e-1\}$, $(X^{+1})_i^{(e)} = X_{i-1}^{(e)}$.

(ii) $(X^{+1})_0^{(e)} = (X_{e-1}^{(e)})^{+1}$.

(iii) For $i \in \{0, 1, \ldots, e-1\}$, $\mathcal{P}((X^{+1})_i^{(e)}) = \mathcal{P}(X_{i-1}^{(e)})$ where $i-1$ is taken mod e

(iv) $(X^{+1})_{(e)} = (X_{(e)})^{+1}$

(v) $\mathcal{P}((X^{+1})_{(e)}) = \mathcal{P}(X_{(e)})$

Proof. Example Sheet 3. □

Remarks.

- Lemma 4.9 (iv), (v) and Lemma 4.2 show that $C_e(\lambda)$ just depends only on e and λ, but not the choice of β-set X for λ.
- Lemma 4.9 (i), (ii) and (iii) show that if we shift X to X^{+1}, we induce a cyclic shift of the components of $Q_e(\lambda) = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(e-1)})$. So far, $Q_e(\lambda)$ therefore still depends on the choice of X. But X and X^{+e} give the same cyclic shift of $\lambda^{(i)}$, and $|X^{+l}| = |X| + l$, so to fix an ordering of the components of $Q_e(\lambda)$ and thereby specifying $Q_e(\lambda)$ uniquely from now on, we will always choose β-sets X such that $|X|$ is a multiple of e when calculating e-quotients.

Example. Let $e = 3$, $\lambda = (6,5,2,1^3) \vdash 16$. Then $X_\lambda = \{11, 9, 5, 3, 2, 1\}$. Note that $3 \mid |X_\lambda|$. Let $X = X_\lambda$. Then

A_λ

	0	1	2
0	0	1	2
1	3	4	5
2	6	7	8
3	9	10	11

$A_{X_{(3)}}$

	0	1	2
0	0	1	2
1	3	4	5
2	6	7	8
3	9	10	11

So $C_3(\lambda) = (3,1)$,

$$X_0^{(3)} = \{3,1\},$$
$$X_1^{(3)} = \{0\},$$
$$X_2^{(3)} = \{3,1,0\}$$

and $Q_3(\lambda) = ((2,1), \emptyset, (1))$.

Note that in total we moved four beads up when going from A_X to $A_{X_{(3)}}$. This could correspond to removing rim hooks as follows (order indicated by number)

$$\begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & 2 & 1 & 1 \\ \hline \ & 4 & 2 & 2 & 1 \\ \cline{1-5} 4 & 4 \\ \cline{1-2} 3 \\ \cline{1-1} 3 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \quad \text{or} \quad \begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & 3 & 3 & 3 \\ \hline \ & 4 & 2 & 2 & 2 \\ \cline{1-5} 4 & 4 \\ \cline{1-2} 1 \\ \cline{1-1} 1 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \quad \text{etc.}$$

Definition. *Let $e \in \mathbb{N}$. An* e-hook *is a hook of size exactly e.*

Theorem 4.10. *Let $e \in \mathbb{N}$, $\lambda \vdash n$. Then $C_e(\lambda)$ is the unique e-core partition we obtain by successively removing e-hooks from λ until we cannot remove any more. In particular, this is independent of the order in which we removed the hooks.*

Proof. Let X be a β-set for λ. Let γ be an e-core partition obtained from λ by removing some e-hooks. By Corollary 4.8, there exists a β-set Z for γ such that the e-abacus configuration A_Z is obtained from A_X by sliding all beads up as far as possible. But then clearly $Z = X_{(e)}$, and so $\gamma = \mathcal{P}(Z) = \mathcal{P}(X_{(e)}) = C_e(\lambda)$. $\square$

Definition. *Let $e \in \mathbb{N}$, $\lambda \vdash n$. Consider $Q_(\lambda) = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(e-1)})$. We say that H is a* hook of $Q_e(\lambda)$ *if $H = H_{i,j}(\lambda^{(s)})$ for some $s = 0, \dots, e-1$ and $(i,j) \in Y(\lambda^{(s)})$. Moreover, we define $Q_e(\lambda) \setminus H := (\lambda^{(0)}, \dots, \lambda^{(s-1)}, \lambda^{(s)} \setminus H, \lambda^{(s+1)}, \dots, \lambda^{(e-1)})$. When we refer to a hook H of $Q_e(\lambda)$, it is considered to carry both the information of which component $\lambda^{(s)}$ it came from, as well as the box (i,j).*

Theorem 4.11. *Let $e \in \mathbb{N}$, $\lambda \vdash n$. There is a bijection*

$$f : \{H_{i,j}(\lambda) \text{ s.t. } e \mid h_{i,j}(\lambda)\} \to \{\text{hooks of } Q_e(\lambda)\}$$

such that if $H = H_{i,j}(\lambda)$ with $e \mid h_{i,j}(\lambda)$, then $|H| = e|f(H)|$ and $Q_e(\lambda \setminus H) = Q_e(\lambda) \setminus f(H)$.

Proof. Let $X = \{h_1, h_2, \dots, h_m\}$ be a β-set for λ with $e \mid m$. Recall from Corollary 4.5 that for $i \in [m]$ and $h \in \mathbb{N}_0$,

$$h \in \mathcal{H}_i(\lambda) \iff 0 \le h_i - h \notin X.$$

So we get a bijection

$$\{H_{i,j}(\lambda) \text{ s.t. } (i,j) \in Y(\lambda)\} \to \{(b,g) \in \mathbb{N}_0^2 \mid b > g, b \in X, g \notin X\},$$

i.e. pairs of positions (b,g) in the e-abacus configuration A_X such that b is a bead, g is a gap and $b > g$. If $H_{i,j}(\lambda) \mapsto (b,g)$, then $h_i = b$ and $h_i - h_{i,j}(\lambda) = g$. In particular, $h_{i,j}(\lambda) = b - g$. So this restricts to a bijection

$$F : \{H_{i,j}(\lambda) \text{ s.t. } e \mid h_{i,j}(\lambda)\} \to \{(b,g) \in \mathbb{N}_0^2 \mid b > g, b \in X, g \notin X, b \equiv g \bmod e\}$$

If $b \equiv g \bmod e$, then $b = b'e + s$ and $g = g'e + s$ for some $s \in \{0, 1, \dots, e-1\}$ and some $b' > g' \in \mathbb{N}_0$. Again by Corollary 4.5, since $Q_e(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(e-1)})$ has $\lambda^{(s)} = \mathcal{P}(X_s^{(e)})$, and $X_s^{(e)} = \{a \in \mathbb{N}_0 \mid ae + s \in X\}$, we have bijections

$$f_s : \{H_{i,j}(\lambda^{(s)}) \text{ s.t. } (i,j) \in Y(\lambda^{(s)})\} \to \{(b',g') \in \mathbb{N}_0^2 \mid b' > g', b' \in X_s^{(e)}, g' \notin X_s^{(e)}\}$$

And as before, if $H_{i,j}(\lambda^{(s)}) \mapsto (b',g')$, then $h_{i,j}(\lambda^{(s)}) = b' - g'$. The bijection f that we seek follows from composing F with the inverses of $f_0, f_1, \dots, f_{e-1}$, noting that

$$\{(b,g) \mid b > g, b \in X, g \notin X, b \equiv g \bmod e\} \overset{1-1}{\longleftrightarrow} \bigsqcup_{s=0}^{e-1} \{(b',g') \mid b' > g', b' \in X_s^{(e)}, g' \notin X_s^{(e)}\}.$$

Moreover, $b - g = e(b' - g')$ gives $|H| = e|f(H)|$.

To see that $Q_e(\lambda \setminus H) = Q_e(\lambda) \setminus f(H)$ when $H = H_{i,j}(\lambda)$ with $e \mid h_{i,j}(\lambda)$: from Proposition 4.7, we know that Z is a β-set for $\lambda \setminus H$, where

$$Z = (X \setminus \{h_i\}) \sqcup \{h_i - h_{i,j}(\lambda)\} = (X \setminus \{b'e + s\}) \sqcup \{g'e + s\}$$

Note $e \mid |X| = |Z|$, so we can use Z to calculate $Q_e(\lambda \setminus H)$: $Z_t^{(e)} = X_t^{(e)}$ for all $t \in \{0, 1, \dots, e-1\} \setminus \{s\}$, and $Z_s^{(e)} = (X_s^{(e)} \setminus \{b'\}) \sqcup \{g'\}$. So $Z_s^{(e)}$ is a β-set for $\lambda^{(s)} \setminus f(H)$, hence $Q_e(\lambda \setminus H) = (\lambda^{(0)}, \dots, \lambda^{(s-1)}, \lambda^{(s)} \setminus f(H), \lambda^{(s+1)}, \dots, \lambda^{(e-1)}) =: Q_e(\lambda) \setminus f(H)$. $\square$

Example. Continue the example from before, so let $e = 3$, $\lambda = (6, 5, 2, 1^3) \vdash 16$. Then $X_\lambda = \{11, 9, 5, 3, 2, 1\}$.

hook lengths

$$\begin{array}{|c|c|c|c|c|c|}\hline 11 & 7 & 5 & 4 & 3 & 1 \\ \hline 9 & 5 & 3 & 2 & 1 \\ \cline{1-5} 5 & 1 \\ \cline{1-2} 3 \\ \cline{1-1} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}$$

A_λ

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 3 & 4 & 5 \\ 2 & 6 & 7 & 8 \\ 3 & 9 & 10 & 11 \end{array}$$

3-quotient

$$Q_3(\lambda) = ((2,1), \emptyset, (1))$$
$$= \left(\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}, \emptyset, \boxed{1} \right)$$

$h_{1,5}(\lambda) = 3$ row 1	$11 \to 8$ $h_1 \to h_1 - h_{1,5}(\lambda)$	$H_{1,2}(\lambda) \overset{f}{\longmapsto} H_{1,1}(\lambda^{(2)})$ $11 \equiv 2 \bmod 3$
$h_{2,1}(\lambda) = 9$	$9 \to 0$	$H_{2,1}(\lambda) \overset{f}{\longmapsto} H_{1,1}(\lambda^{(0)})$
$h_{2,3}(\lambda) = 3$	$9 \to 6$	$H_{2,3}(\lambda) \overset{f}{\longmapsto} H_{1,2}(\lambda^{(0)})$
$h_{4,1}(\lambda) = 3$	$3 \to 0$	$H_{4,1}(\lambda) \overset{f}{\longmapsto} H_{2,1}(\lambda^{(0)})$

To see that e.g. $H_{2,3}(\lambda) \overset{f}{\longmapsto} H_{1,2}(\lambda^{(0)})$ note that the runner 0 of the abacus goes from

$$\begin{array}{c} 0 \\ \hline 0 \\ 3 \\ 6 \\ 9 \end{array} \quad \text{to} \quad \begin{array}{c} 0 \\ \hline 0 \\ 3 \\ 6 \\ 9 \end{array}$$

which has partition $(1,1) = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} = \lambda^{(0)} \setminus H_{1,2}(\lambda^{(0)})$ by Lemma 4.3.

Definition. *Let $e \in \mathbb{N}$, $\lambda \vdash n$. Then the e-weight of λ is $w_e(\lambda) := |Q_e(\lambda)| := \sum_{i=0}^{e-1} |\lambda^{(i)}|$.*

Proposition 4.12. *Let $e \in \mathbb{N}$, $\lambda \vdash n$. Then*

(i) $w_e(\lambda)$ is the number of e-hooks we need to remove to get from λ to $C_e(\lambda)$.

(ii) $|\lambda| = |C_e(\lambda)| + e|Q_e(\lambda)|$.

(iii) $w_e(\lambda)$ is the number of hooks of λ of size divisible by e.

Proof.

(i) Induct on $w_e(\lambda)$. If $w_e(\lambda) = |Q_e(\lambda)| = 0$, then by Theorem 4.11, λ has no e-hooks, and so $\lambda = C_e(\lambda)$. Now suppose $w_e(\lambda) > 0$. Then by the same theorem, λ has a hook length divisible by e. So there also exists a hook H of λ of size exactly e, by Corollary 4.6 or also Theorem 4.11. Recall $Q_e(\lambda \setminus H) = Q_e(\lambda) \setminus f(H)$ and $|f(H)| = 1$, so $w_e(\lambda) = |Q_e(\lambda)| = 1 + |Q_e(\lambda) \setminus f(H)| = 1 + |Q_e(\lambda \setminus H)| = 1 + w_e(\lambda \setminus H)$, so the claim follows from the inductive hypothesis since we removed one e-hook to get from λ to $\lambda \setminus H$ and $C_e(\lambda) = C_e(\lambda \setminus H)$.

(ii) Immediate from (i).

(iii) Follows from Theorem 4.11 as $|Q_e(\lambda)|$ is the number of hooks of $Q_e(\lambda)$.

□

Theorem 4.13. *Let $e \in \mathbb{N}$, $n \in \mathbb{N}_0$, and define*

$$B(n) := \left\{ (\gamma; \rho^0, \rho^1, \ldots, \rho^{e-1}) \;\middle|\; \begin{array}{c} \gamma \text{ is an } e\text{-core partition, } \rho^i \text{ is a partition for all } i \\ \text{and } |\gamma| + e\sum_{i=0}^{e-1} |\rho^i| = n \end{array} \right\}.$$

Then

$$\begin{aligned} g : \wp(n) &\longrightarrow B(n), \\ \lambda &\longmapsto (C_e(\lambda); Q_e(\lambda)) \end{aligned}$$

is a bijection. In other words, a partition is uniquely determined by its e-core and e-quotient.

Proof.

- By Proposition 4.12, $n = |\lambda| = |C_e(\lambda)| + e|Q_e(\lambda)|$, so $g(\lambda) \in B(n)$ and g is well-defined.
- g is surjective: Let $(\gamma; \underline{\rho}) \in B(n)$, where $\underline{\rho} = (\rho^0, \rho^1, \ldots, \rho^{e-1})$. Let X be a β-set for γ such that $e \mid |X|$ and $|X_i^{(e)}| \geq \ell(\rho^i)$ for all i. Then define Z_i to be the β-set for ρ^i such that $|Z_i| = |X_i^{(e)}|$ for all i, and set $Z := \bigsqcup_{i=0}^{e-1} \{ae + i \mid a \in Z_i\}$. Let $\lambda = \mathcal{P}(Z)$. Since γ is an e-core, $X = X_{(e)}$ and so we have $Z_{(e)} = X_{(e)} = X$. Hence $C_e(\lambda) = \mathcal{P}(Z_{(e)}) = \mathcal{P}(X) = \gamma$. Next, $e \mid |X| = |Z|$, so $Q_e(\lambda) = (\lambda^{(0)}, \ldots, \lambda^{(e-1)})$ with $\lambda^{(i)} = \mathcal{P}(Z_i^{(e)}) = \mathcal{P}(Z_i) = \rho^i$. Finally, by Proposition 4.12, $|\lambda| = |C_e(\lambda)| + e|Q_e(\lambda)| = n$ since $(\gamma; \underline{\rho}) \in B(n)$. So $g(\lambda) = (\gamma; \underline{\rho})$ with $\lambda \vdash n$.
- g is injective: notation as above, suppose $g(\mu) = (\gamma; \underline{\rho})$, for some $\mu \vdash n$. Since $C_e(\mu) = \gamma$, there exists a unique β-set W for μ such that $|W| = |X|$. Now $|W_{(e)}| = |W| = |X|$, and $\mathcal{P}(W_{(e)}) = \gamma = \mathcal{P}(X)$. Hence $W_{(e)} = X$ by Lemma 4.2. Also,

$|W_i^{(e)}| = |(W_{(e)})_i^{(e)}| = |X_i^{(e)}| = |Z_i^{(e)}|$, and $\mathcal{P}(W_i^{(e)}) = \rho^i$ since $g(\mu) = (\gamma; \underline{\rho})$ noting that $e \mid |X| = |W|$. But also $\rho^i = \mathcal{P}(Z_i^{(e)})$, hence $W_i^{(e)} = Z_i^{(e)}$ for all i again from Lemma 4.2.

Thus $W_{(e)} = X = Z_{(e)}$ and $W_i^{(e)} = Z_i^{(e)}$ for all i, so $W = Z$, so $\mu = \mathcal{P}(W) = \mathcal{P}(Z) = \lambda$.

□

Example. How do we reconstruct λ, given $C_e(\lambda)$ and $Q_e(\lambda)$? Let $e = 3$ and $(\gamma; \underline{\rho}) = ((3,1); (2,1), \emptyset, (1)) \in B(16)$. We expect $\lambda = (6,5,2,1^3) \vdash 16$.

- **Step 1.** Start with A_γ,

$$\begin{array}{c|ccc}
 & 0 & 1 & 2 \\ \hline
0 & 0 & 1 & 2 \\
1 & 3 & 4 & 5 \\
2 & 6 & 7 & 8 \\
 & & \vdots &
\end{array}$$

- **Step 2.** Shift to get $e \mid |X|$,

$$\begin{array}{c|ccc}
 & 0 & 1 & 2 \\ \hline
0 & 0 & 1 & 2 \\
1 & 3 & & 5 \\
2 & 6 & 7 & 8 \\
 & & \vdots &
\end{array}$$

- **Step 3.** Add enough full rows of beads, i.e. shift enough by multiples of e, to get $|X_i^{(e)}| \geq \ell(\rho^i)$ for all i,

$$\begin{array}{c|ccc}
 & 0 & 1 & 2 \\ \hline
0 & 0 & 1 & 2 \\
\vdots & & \vdots & \\ \hdashline
\# & \# & \# & \# \\
\# & \# & \# & \# \\
\# & \# & \# & \# \\
\# & \# & \# & \# \\
\vdots & & \vdots &
\end{array}$$

- **Step 4.** Slide down to get ρ^i on runner i for all i.

$$
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
\vdots & & \vdots & \\
\hdashline
\# & \# & \# & \# \\
\# & \# & \# & \# \\
\# & \# & \# & \# \\
\# & \# & \# & \# \\
\vdots & & \vdots &
\end{array}
$$

Now this is an abacus configuration A for λ. We can now shift back and start numbering after the green dashed line. So we get the β-set $\{11, 9, 5, 3, 2, 1\}$. So $\lambda = \mathcal{P}(A) = (6, 5, 2, 1^3)$.

4.2 Towers

Just as the division algorithm for integers gives us base e expansion, we can use Theorem 4.13 to give "e-adic expansion" for partitions.

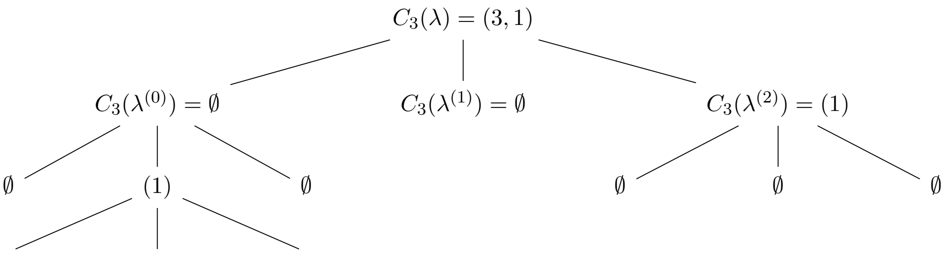
Example. Let $e = 3$, $\lambda = (6, 5, 2, 1^3) \vdash 16$. Then $C_3(\lambda) = (3, 1)$, $Q_3(\lambda) = ((2, 1), \emptyset, (1))$.

$$
\begin{array}{ccc}
\lambda^{(0)} = (2,1) & \lambda^{(1)} = \emptyset & \lambda^{(1)} = (1) \\
A_{(X_{\lambda^{(0)}})^{+1}} = & A_{X_{\lambda^{(1)}}} = & A_{(X_{\lambda^{(1)}})^{+2}} = \\
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 3 & 4 & 5 \\
2 & 6 & 7 & 8 \\
 & & \vdots &
\end{array}
&
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 3 & 4 & 5 \\
2 & 6 & 7 & 8 \\
 & & \vdots &
\end{array}
&
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 3 & 4 & 5 \\
2 & 6 & 7 & 8 \\
 & & \vdots &
\end{array}
\\
C_3(\lambda^{(0)}) = \emptyset & C_3(\lambda^{(1)}) = \emptyset & C_3(\lambda^{(2)}) = (1) \\
Q_3(\lambda^{(0)}) = (\emptyset, (1), \emptyset) & Q_3(\lambda^{(1)}) = (\emptyset, \emptyset, \emptyset) & Q_3(\lambda^{(2)}) = (\emptyset, \emptyset, \emptyset)
\end{array}
$$

We get the sequence of quotients as follows:

- $\lambda = (6, 5, 2, 1^3)$
  - $\lambda^{(0)} = (2, 1)$
    - $\lambda^{(0)(0)} = \emptyset$
    - $\lambda^{(0)(1)} = (1)$
    - $\lambda^{(0)(2)} = \emptyset$
  - $\lambda^{(1)} = \emptyset$
  - $\lambda^{(2)} = (1)$
    - $\lambda^{(2)(0)} = \emptyset$
    - $\lambda^{(2)(1)} = \emptyset$
    - $\lambda^{(2)(2)} = \emptyset$

The 3-cores are



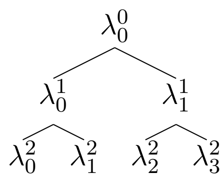
Definition. *Let $e \in \mathbb{N}$. An* e-tower *is an infinite sequence $T = (T_0, T_1, T_2, \dots)$ such that each T_j is a sequence of e^j many partitions, $T_j = (\lambda_0^j, \lambda_1^j, \dots, \lambda_{e^j-1}^j)$.*

- *The T_j are the* layers *or* rows *of T, define $|T_j| := \sum_{j=0}^{e^j-1} |\lambda_i^j|$.*
- *The* depth *of T is* $\operatorname{depth}(T) = \sup\{k \in \mathbb{N}_0 \mid |T_k| \neq \emptyset\}$*. We will call the depth of the empty tower -1.*
- *We say T is an* e-core tower *if* $\operatorname{depth}(T) < \infty$ *and λ_i^j is an e-core partition for all i, j.*

As we saw in the example above, we can visualise e-towers using graphs.

- vertices: λ_i^j,
- edges: μ, ν are joined if $\mu = \lambda_i^j$ and $\nu = \lambda_{ie+t}^{j+1}$ for some $j \in \mathbb{N}_0, i \in \{0, 1, \dots, e^j - 1\}, t \in \{0, 1, \dots, e-1\}$.

e.g. for $e = 2$,



These graphs are rooted, ordered, full e-ary trees. When we use graphs to describe e-towers, we always mean trees like this.

Notation. Let $e \in \mathbb{N}$

- $[\overline{e}] := \{0, 1, \dots, e-1\}$ (residues mod e)
- For each $x \in [\overline{e}]$, write $Q_e(\lambda^{(x)}) = (\lambda^{(x,0)}, \lambda^{(x,1)}, \dots, \lambda^{(x,e-1)})$, instead of $\lambda^{(x)(0)}, \lambda^{(x)(1)}$, etc.
- Similarly, for all $r \in \mathbb{N}$, and for all $\underline{i} = (i_1, i_2, \dots, i_r) \in [\overline{e}]^r$, will write $Q_e(\lambda^{\underline{i}}) = (\lambda^{(i_1,i_2,\dots,i_r,0)}, \dots, \lambda^{(i_1,\dots,i_r,e-1)})$.

Definition. *Let $e \in \mathbb{N}$, $\lambda \vdash n$. The* e-quotient tower of λ *is the e-tower $T^Q(\lambda)$ with*

- $T^Q(\lambda)_0 = (\lambda)$

- $T^Q(\lambda)_1 = Q_e(\lambda) = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(e-1)})$.
- *For all* $j \in \mathbb{N}$, $T^Q(\lambda)_j = (\lambda^{\underline{i}})_{\underline{i} \in [\overline{e}]^j}$, *lexicographically ordered.*

Lemma 4.14. *Let* $e \in \mathbb{N}$, $\lambda \vdash n$, $T^Q(\lambda)$ *the* e*-quotient tower. Suppose* $e \geq 2$, *then* $\mathrm{depth}(T^Q(\lambda)) < \infty$.

Proof. From Proposition 4.12, $|\lambda| = |C_e(\lambda)| + e|Q_e(\lambda)| \geq |Q_e(\lambda)|$ with equality iff $|C_e(\lambda)| = |Q_e(\lambda)| = 0$, since $e \geq 2$. By Theorem 4.13, equality holds iff $\lambda = \emptyset$. Hence $|T^Q(\lambda)_j| > |T^Q(\lambda)_{j+1}|$ unless $T^Q(\lambda_j) = (\emptyset, \dots, \emptyset)$. □

Remark. $Q_1(\lambda) = (\lambda^{(0)}) = (\lambda)$, so the 1-quotient tower $T^Q(\lambda)$ has all layers equal to (λ). So its depth is -1 if $\lambda = \emptyset$, and ∞ otherwise.

Definition. *Let* $e \in \mathbb{N}$, $\lambda \vdash n$. *The* e-core tower *of* λ *is the* e*-tower* $T^C(\lambda)$ *obtain from the* e*-quotient tower* $T^Q(\lambda)$ *by replacing every vertex with its* e*-core. That is,* $T^C(\lambda)_j = (C_e(\lambda^{\underline{i}}))_{\underline{i} \in [\overline{e}]^j}$, *lexicographically ordered.*

When $e \geq 2$, $\mathrm{depth}(T^C(\lambda)) < \infty$ since $\mathrm{depth}(T^Q(\lambda)) < \infty$. When $e = 1$, hen $T^C(\lambda)$ is empty, so also $\mathrm{depth}(T^C(\lambda)) < \infty$. So $T^C(\lambda)$ is indeed an e-core tower.

Lemma 4.15. *Let* $e \in \mathbb{N}$, $\lambda \vdash n$. *For* $x \in [\overline{e}]$, *the subtree of* $T^C(\lambda)$ *rooted at* $C_e(\lambda^{(x)})$ *is the* e*-core tower of* $\lambda^{(x)}$, *so the* $(j+1)$*-th layer of* $T^C(\lambda)$ *is the concatenation of the* j*-th layers of* $T^C(\lambda^{(0)}), T^C(\lambda^{(1)}), \dots, T^C(\lambda^{(e-1)})$. *That is,* $T^C(\lambda^{(x)})_j = (C_e(\lambda^{(x,\underline{i})}))_{i \in [\overline{e}]^j}$ *and* $T^C(\lambda)_{j+1}) = (T^C(\lambda^{(0)})_j, T^C(\lambda^{(1)})_j, \dots, T^C(\lambda^{(e-1)})_j)$.

Proof. The subtree of $T^Q(\lambda)$ rooted at $\lambda^{(x)}$ is $T^Q(\lambda^{(x)})$. □

Theorem 4.16. *Let* $e \in \mathbb{N}$, $e \geq 2$, *let* $n \in \mathbb{N}_0$. *Define*

$$\theta(n) := \{e\text{-core towers } T \text{ such that } \sum_{j=0}^{\infty} |T_j| e^j = n\}.$$

Then

$$\begin{aligned} h : \wp(n) &\longrightarrow \theta(n), \\ \lambda &\longmapsto e\text{-core tower } T^C(\lambda) \end{aligned}$$

is a bijection.

Proof. First, we check $\sum_{j=0}^{\infty} |T^C(\lambda)_j| e^j = n$, by induction on n. The base case $n = 0$ is clear since then $\lambda = \emptyset$. Now suppose $n > 0$. Then $n = |\lambda| = |C_e(\lambda)| + e \sum_{x=0}^{e-1} |\lambda^{(x)}| = |T^C(\lambda)_0| + e \sum_{x=0}^{e-1} \sum_{j=0}^{\infty} |T^C(\lambda^{(x)})_j| e^j$ by the inductive hypothesis, since ≥ 2 means $|\lambda^{(x)}| < |\lambda|$. This is

$$|T^C(\lambda)_0| + \sum_{j=0}^{\infty} \Big(\sum_{x=0}^{e-1} |T^C(\lambda^{(x)}_j)| \Big) e^{j+1} = |T^C(\lambda)_0| + \sum_{j=0}^{\infty} |T^C(\lambda)_{j+1}| e^{j+1} = \sum_{j=0}^{\infty} |T^C(\lambda)_j| e^j.$$

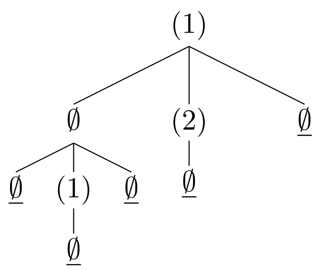
Next, to prove that h is a bijection, we show for all $T \in \theta(n)$ that there exists a unique $\lambda \vdash n$ such that $T^C(\lambda) = T$. For $x \in [\overline{e}]$, let $S(x)$ be the subtree of T rooted at λ^1_x, where $T = (T_0, T_1, T_2, \dots)$, $T_j = (\lambda^j_0, \lambda^j_1, \dots, \lambda^j_{e^j-1})$. Since T is an e-core tower, so is $S(x)$. Then since $n_x := \sum_{j=0}^{\infty} |S(x)_j| e^j \leq \sum_{j=0}^{\infty} |T_{j+1}| e^j < |T_0| + \sum_{j=0}^{\infty} |T_{j+1}| e^{j+1} = n$, we can use inductive hypothesis to see that there is a unique $\mu_x \vdash n_x$ such that $T^C(\mu_x) = S(x)$. By Theorem 4.13 there is a unique partition λ such that $C_e(\lambda) = \lambda^0_0$ and $Q_e(\lambda) = (\mu_0, \mu_1, \dots, \mu_{e-1})$. Observe $T^C(\lambda) = T$ since $T^C(\mu_x) = S(x) = T^C(\lambda^{(x)})$, i.e. $T^C(\mu_x)$ is the subtree of $T^C(\lambda)$ rooted at $\lambda^1_x = C_e(\lambda^{(x)}) = C_e(\mu_x)$. To check $|\lambda| = n$:

$$\begin{aligned}
|\lambda| &= |C_e(\lambda)| + e \sum_{x=0}^{e-1} |\mu_x| \\
&= |T^C(\lambda)_0| + e \sum_{x=0}^{e-1} \sum_{j=0}^{\infty} |S(x)_j| e^j \\
&= |\lambda^0_0| + \sum_{j=0}^{\infty} \Big(\sum_{x=0}^{e-1} |S(x)_j| \Big) e^{j+1} \\
&= \sum_{j=0}^{\infty} |T_j| e^j = n
\end{aligned}$$

Uniqueness of λ is also clear from this argument. $\square$

Remark. This is not a bijection when $e = 1$ since then $T^C(\lambda)$ is empty for all λ.

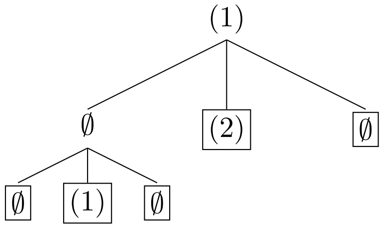
Example. Given $T \in \theta(n)$, how to compute $\lambda = h^{-1}(T)$? Let $e = 3$, $T =$

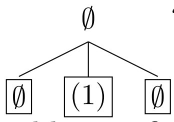


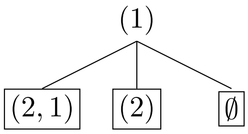
where $\underline{\emptyset}$ means that from that vertex onwards there are only empty partitions. We have $n = 1 \cdot 3^0 + 2 \cdot 3^1 + 1 \cdot 3^2 = 16$.

- When we see a subtree rooted at an e-core partition γ with all empty below, this subtree is the e-core tower of γ because $C_e(\gamma) = \gamma$, $Q_e(\gamma) = (\emptyset, \dots, \emptyset)$.
- We will draw boxes to replace subtrees by the partition whose e-core tower is that

subtree.



- Work up the layers: What μ has $T^C(\mu) =$  ? It is the partition μ with $C_3(\mu) = \emptyset$ and $Q_3(\mu) = (\emptyset, (1), \emptyset)$. We showed how to find this in the example after Theorem 4.13. In this case we get $\mu = (2, 1)$. Then we get

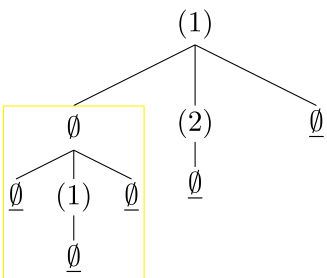
$T =$ 

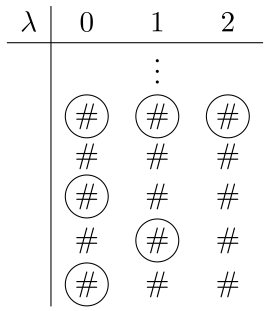
So $T = T^C(\lambda)$ where $C_3(\lambda) = (1)$ and $Q_3(\lambda) = ((2, 1), (2), \emptyset)$. We find that $\lambda = (7, 6, 3)$.

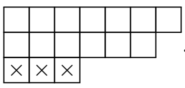
Example. How does hook removal interact with core towers? Let $e = 3$, $\lambda = (7, 6, 3)$,

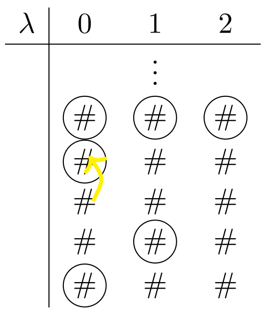
$T^C(\lambda) =$

$T^C((2,1)) =$

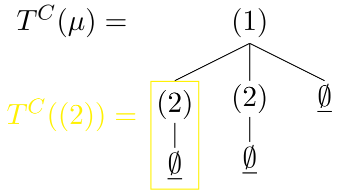


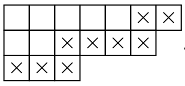


(a) Remove the 3-hook marked in . So let $\mu = \lambda \setminus H$, where $H = H_{3,1}(\lambda)$, $h_{3,1}(\lambda) = 3$. On the abacus:



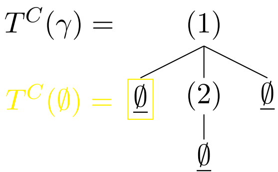
So $C_3(\mu) = (1) = C_3(\lambda)$, $Q_3(\mu) = Q_3(\lambda) \setminus f(H) = (\lambda^{(0)} \setminus f(H), \lambda^{(1)}, \lambda^{(2)}) = ((2),(2),\emptyset)$. We have



(b) Remove the 9-hook marked in  . Let $\gamma = \lambda \setminus K$, where $K = H_{1,1}(\lambda)$, $h_{1,1}(\lambda) = 9$.

λ	0	1	2
		$\vdots$	
	#	#	#
	#	#	#
	#	#	#
	#	#	#
	#	#	#

So $C_3(\gamma) = (1) = C_3(\gamma)$ and $Q_3(\gamma) = Q_3(\gamma) \setminus f(K) = (\lambda^{(0)} \setminus f(K), \lambda^{(1)}, \lambda^{(2)}) = (\emptyset, (2), \emptyset)$. So



Proposition 4.17. *Let $e \in \mathbb{N}$, let $k, n \in \mathbb{N}_0$ with $n < e^{k+1}$. Let $\lambda \vdash n$ and $\mu = C_{e^k}(\lambda)$. Then the e-core tower $T^C(\mu)$ of μ is obtained from the e-core tower $T^C(\lambda)$ by replacing every partition in the k-th layer by the empty partition. That is,* $T^C(\lambda)_j = \begin{cases} T^C(\lambda)_j & \text{if } j \neq k, \\ (\emptyset, \emptyset, \ldots, \emptyset) & \text{if } j = k. \end{cases}$

Part (b) of the example above is an example for this proposition.

Proof. Example Sheet 4. $\square$

Definition. *Let p be a prime. The* p-adic valuation *$v_p : \mathbb{N} \to \mathbb{N}_0$ is defined as $v_p(n) = \max\{k \in \mathbb{N}_0$ s.t. $p^k \mid n\}$.*

Theorem 4.18. *Let p be prime, $n \in \mathbb{N}_0$ with p-adic expansion $n = \sum_{r=0}^{\infty} \alpha_r p^r$, i.e. $\alpha_r \in \{0, 1, \ldots, p-1\}$ for all $r \in \mathbb{N}_0$. Let $\lambda \vdash n$. Then*

$$v_p(\chi^\lambda(1)) = \frac{\sum_{r=0}^{\infty} |T^c(\lambda)_r| - \sum_{r=0}^{\infty} \alpha_r}{p-1},$$

where $T^C(\lambda)$ is the p-core tower of λ.

Proof. Recall the hook length formula, Theorem 3.1: We get

$$v_p(\chi^\lambda(1)) = v_p\Big(\frac{n!}{\prod_{n\in\mathcal{H}(\lambda)} h}\Big) = v_p(n!) - v_p\Big(\prod_{h\in\mathcal{H}(\lambda)} h\Big).$$

- **Step 1.** We compute $v_p(n!)$. Observe that

$$\begin{aligned}
v_p(n!) = \sum_{r=1}^{\infty}\lfloor\frac{n}{p^r}\rfloor &= \sum_{r=1}^{\infty}(\alpha_r + \alpha_{r+1}p + \alpha_{r+2}p^2 + \dots) \\
&= \sum_{r=1}^{\infty}\alpha_r(1 + p + p^2 + \cdots + p^{r-1}) \\
&= \sum_{r=1}\alpha_r\frac{p^r-1}{p-1} \\
&= \frac{1}{p-1}\Big(\sum_{r=1}^{\infty}\alpha_r p^r - \sum_{r=1}^{\infty}\alpha_r\Big) \\
&= \frac{1}{p-1}\Big(\sum_{r=0}^{\infty}\alpha_r p^r - \sum_{r=0}^{\infty}\alpha_r\Big) \\
&= \frac{n - \sum_{r=0}^{\infty}\alpha_r}{p-1}.
\end{aligned}$$

- **Step 2.** We claim that $v_p(\prod_{h\in\mathcal{H}(\lambda)} h) = \sum_{r=1}^{\infty}|T^Q(\lambda)_r|$, where $T^Q(\lambda)$ is the p-quotient tower of λ. We prove this by induction on n. The base case $n = 0$ is clear since $v_p(1) = 0$. Now suppose $n > 0$. We write $\mathcal{H}(Q_p(\lambda))$ for the multiset of hook lengths of $Q_p(\lambda)$. Then

$$\begin{aligned}
v_p\Big(\prod_{h\in\mathcal{H}(\lambda)} h\Big) &= v_p\Big(\prod_{\substack{h\in\mathcal{H}(\lambda)\\ p\mid h}} hv) \\
&\overset{\text{Theorem 4.11}}{=} v_p\Big(\prod_{h\in\mathcal{H}(Q_p(\lambda))} ph\Big) \\
&= |Q_p(\lambda)| + v_p\Big(\prod_{h\in\mathcal{H}(Q_p(\lambda))} h\Big) \\
&= |Q_p(\lambda)| + v_p\Big(\prod_{x=0}^{p-1}\prod_{h\in\mathcal{H}(\lambda^{(x)})} h\Big) \\
&= |T^Q(\lambda)_1| + \sum_{x=0}^{p-1} v_p\Big(\prod_{h\in\mathcal{H}(\lambda^{(x)})} h\Big)
\end{aligned}$$

$$\begin{aligned}
&\overset{\text{ind. hypothesis}}{=} |T^Q(\lambda)_1| + \sum_{x=0}^{p-1}\sum_{r=1}^{\infty} |T^Q(\lambda^{(x)})_r| \\
&= |T^Q(\lambda)_1| + \sum_{r=1}^{\infty}\sum_{x=0}^{p-1} |T^Q(\lambda^{(x)})_r| \\
&= |T^Q(\lambda)_1| + \sum_{r=1}^{\infty} |T^Q(\lambda)_{r+1}| \\
&= \sum_{r=1}^{\infty} |T^Q(\lambda)_r|
\end{aligned}$$

- **Step 3.** By Proposition 4.12 for all $r \in \mathbb{N}_0$, $\underline{i} \in [\overline{p}]^r$,

$$|\lambda^{\underline{i}}| = |C_p(\lambda^{\underline{i}})| + p|Q_p(\lambda^{\underline{i}})|.$$

Summing over $\underline{i} \in [\overline{p}]^r$, we get

$$|T^Q(\lambda)_r| = |T^C(\lambda)_r| + p|T^Q(\lambda)_{r+1}|.$$

Therefore,

$$\begin{aligned}
n = |\lambda| = |T^Q(\lambda)_0| \\
&= \sum_{r=0}^{\infty} |T^Q(\lambda)_r| - \sum_{r=1}^{\infty} |T^Q(\lambda)_r| \\
&= \Big(\sum_{r=0}^{\infty} |T^C(\lambda)_r| - p\sum_{r=0}^{\infty} |T^Q(\lambda)_{r+1}|\Big) - \sum_{r=1}^{\infty} |T^Q(\lambda)_r| \\
&= \sum_{r=0}^{\infty} |T^C(\lambda)_r| + (p-1)\sum_{r=1}^{\infty} |T^Q(\lambda)_r|.
\end{aligned}$$

Hence

$$\begin{aligned}
v_p(\chi^\lambda(1)) &= v_p(n!) - v_p\Big(\prod_{h\in\mathcal{H}(\lambda)} h\Big) \\
&= \frac{1}{p-1}\Big(n - \sum_{r=0}^{\infty}\alpha_r\Big) - \sum_{r=1}^{\infty} |T^Q(\lambda)_r| \\
&= \frac{1}{p-1}\Big(\sum_{r=0}^{\infty} |T^C(\lambda)_r| + (p-1)\sum_{r=1}^{\infty} |T^Q(\lambda)_r| - \sum_{r=0}^{\infty}\alpha_r\Big) - \sum_{r=1}^{\infty} |T^Q(\lambda)_r| \\
&= \frac{1}{p-1}\Big(\sum_{r=0}^{\infty} |T^C(\lambda)_r| - \sum_{r=0}^{\infty}\alpha_r\Big).
\end{aligned}$$

□

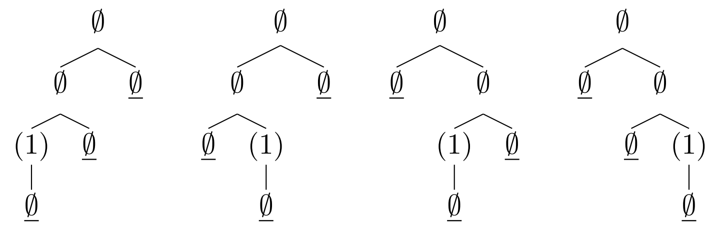
Corollary 4.19. *Let p be prime, $n \in \mathbb{N}_0$ with p-adic expansion $n = \sum_{r=0}^{\infty} \alpha_r p^r$. Let $\lambda \vdash n$. Then $v_p(\chi^\lambda(1)) = 0$ iff $|T^C(\lambda)_r| = \alpha_r$ for all $r \in \mathbb{N}_0$, where $T^C(\lambda)$ is the p-core tower of λ.*

Proof. "if" is clear from the theorem. For "only if" the theorem gives us $\sum_{r=0}^{\infty} |T^C(\lambda)_r| = \sum_{r=0}^{\infty} \alpha_r$. Also note that $\sum_{r=0}^{\infty} |T^C(\lambda)_r| p^r = n$. Let $\beta_r = |T^C(\lambda)_r|$. So we have

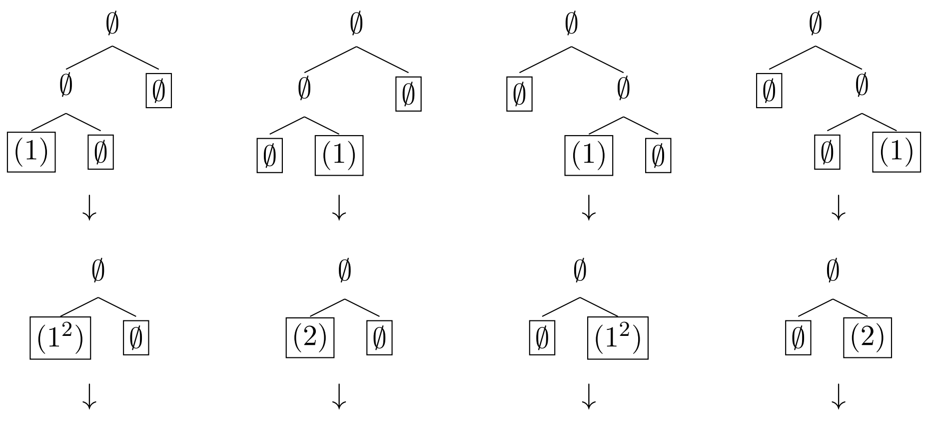
$$\sum_{r\geq 0} \alpha_r = \sum_{r\geq 0} \beta_r$$
$$\sum_{r\geq 0} \alpha_r p^r = \sum_{r\geq 0} \beta_r p^r$$

We show that $\alpha_r = \beta_r$ for all $r \in \mathbb{N}_0$. First, $\beta_0 \equiv \alpha_0 \bmod p$. Hence we can write $\beta_0 = \alpha_0 + m_1 p$, for some $m_1 \in \mathbb{N}_0$. Since $\beta_0 \in \mathbb{N}_0$ and $\alpha_0 \in \{0, 1, \ldots, p-1\}$. Thus $\sum_{r\geq 0} \beta_r p^r = \sum_{r\geq 2} \beta_r p^r + (\beta_1 + m_1)p + \alpha_0 = \sum_{r\geq 0} \alpha_r p^r$. Then $\beta_1 + m_1 \equiv \alpha_1 \bmod p$, so $\beta_1 + m_1 = \alpha_1 + m_2 p$ for some $m_2 \in \mathbb{N}_0$. Iterating, $\beta_r + m_r = \alpha_r + m_{r+1} p$ for all $r \in \mathbb{N}_0$ where $m_r \in \mathbb{N}_0$ and $m_0 = 0$. Then $\sum_{r\geq 0} \alpha_r = \sum_{r\geq 0} \beta_r = \sum_{r\geq 0} \alpha_r + (p-1)\sum_{r\geq 0} m_r$, hence $m_r = 0$ for all r and so $\alpha_r = \beta_r$. $\square$

Example. We compute $\mathrm{Irr}_{2'}(S_4)$. By Theorem 4.16 there is a bijection between partitions and 2-core towers. By the corollary, for $\lambda \vdash 4 = 1 \cdot 2^2$, we have $\chi^\lambda \in \mathrm{Irr}_{2'}(S_4)$ iff $|T^C(\lambda)_2| = 1$ and $|T^C(\lambda)_r| = 0$ for all $r \neq 2$. So we already see that $|\mathrm{Irr}_{2'}(S_4)| = 4$. The towers are:



As in the example after Theorem 4.16 we compute:



$\lambda = (1^4)$ $\qquad \lambda = (3,1)$ $\qquad \lambda = (2,1)$ $\qquad \lambda = (4)$

Hence we see that

$$\operatorname{Irr}_{2'}(S_4) = \{\chi^\lambda \text{ s.t. } \lambda \in \{(4), (3,1), (2,1^2), (1^4)\}\}.$$

Note that these partitions are exactly the hooks of size 4.

4.3 The McKay Conjecture

Recall the McKay Conjecture: Let G be a finite group, p a prime, P a Sylow p-subgroup of G. Then

$$|\operatorname{Irr}_{p'}(G)| = |\operatorname{Irr}_{p'}(N_G(P))|.$$

Definition. *Let G be a finite group, p a prime. The* McKay numbers *of G are*

$$m_i(p, G) = |\{\chi \in \operatorname{Irr}(G) \text{ s.t. } v_p(\chi(1)) = i\}|,$$

for $i \in \mathbb{N}_0$.

So we are interested in $m_0(p, G)$ for $G = S_n$ (and $G = N_{S_n}(P)$).

Corollary 4.20. *Let $n \in \mathbb{N}$ with binary expansion $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_t}$, i.e. $t \in \mathbb{N}$, and the $n_i \in \mathbb{N}_0$ are distinct. Then*

$$m_0(2, S_n) = |\operatorname{Irr}_{2'}(S_n)| = 2^{n_1+n_2+\cdots+n_t}.$$

Proof. By Theorem 4.16, we have a bijection

$$\begin{aligned} h : \wp(n) &\longrightarrow \theta(n) \\ \lambda &\longmapsto \text{2-core tower } T^C(\lambda) \end{aligned}$$

By Corollary 4.19, for $\lambda \vdash n$ we have $\chi^\lambda \in \operatorname{Irr}_{2'}(S_n)$ iff

$$|T^C(\lambda)_r| = \begin{cases} 1 & \text{if } r \in \{n_1, \dots, n_t\}, \\ 0 & \text{otherwise.} \end{cases}$$

But $|T^C(\lambda)_r| = 1$ means $T^C(\lambda)_r$ is a sequence of 2^r many partitions, exactly one of which is (1), the rest $\emptyset$. So the number of such 2-core towers is $2^{n_1} \cdot 2^{n_2} \cdots 2^{n_t}$. $\square$

Corollary 4.21. *Let p be a prime, $n \in \mathbb{N}$ with p-adic expansion $n = \sum_{r \geq 0} \alpha_r p^r$. Then*

$$m_0(p, S_n) = |\operatorname{Irr}_{p'}(S_n)| = \prod_{r \geq 0} k_p(p^r, \alpha_r),$$

where $k_p(l, m)$ is the number of tuples of partitions $(\gamma^1, \dots, \gamma^l)$ such that each γ^i is a p-core partition and $\sum_{i=1}^{l} |\gamma^i| = m$.

Proof. The same as the previous corollary, use Theorem 4.16 and Corollary 4.19. $\square$

Sketch towards the McKay conjecture. We need some group theoretic facts.

Suppose $p = 2$.

- Let $P_n \in \mathrm{Syl}_2(S_n)$ be a Sylow 2-subgroup of S_n. Then $N_{S_n}(P_n) = P_n$.
- For $n = 2^k$, $\mathrm{Irr}_{2'}(N_{S_n}(P_n)) = \mathrm{Irr}_{2'}(P_n) = \{\text{degree 1 characters of } P_n\}$ as the degree of any irreducible character divides the group order. But now the degree 1 characters of any group H are in bijection with $\mathrm{Irr}(H/H')$ where H' is the commutator subgroup. If $H = P_{2^k}$, then $H/H' \cong C_2^{\times k}$, hence $|\,\mathrm{Irr}_{2'}(P_{2^k})| = |\,\mathrm{Irr}(C_2^{\times k})| = 2^k$.
- For general $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_t}$, count the number of factors of $p = 2$ in $|S_n| = n!$ to see that
$$P_n \cong P_{2^{n_1}} \times P_{2^{n_2}} \times \cdots \times P_{2^{n_t}}.$$
Then
$$|\,\mathrm{Irr}_{2'}(N_{S_n}(P_n))| = |\,\mathrm{Irr}_{2'}(P_n)| = \prod_{i=1}^{s} |\,\mathrm{Irr}_{2'}(P_{2^{n_i}})| = \prod_{i=1}^{b} 2^{n_i} = m_0(2, S_n).$$

For $p > 2$ the first point need no longer be true.

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